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MODELOS DE OPTIMIZACIÓN DE EQUILIBRIO PARA CONTRATOS OPTIMALES GARANTIZANDO UN NIVEL DE SERVICIO EN MERCADOS DE TECNOLOGÍAS DE INFORMACIÓN

MEMORIA PARA OPTAR AL TITULO DE INGENIERO CIVIL MATEMÁTICO

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RESUMEN DE LA MEMORIA
PARA OPTAR AL TITULO DE
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"MODELOS DE OPTIMIZACIÓN DE EQUILIBRIO PARA CONTRATOS OPTIMALES GARANTIZANDO UN NIVEL DE SERVICIO EN MERCADOS DE TECNOLOGÍAS DE INFORMACIÓN"

El objetivo de este trabajo es desarrollar mediante el uso de teoría de juegos un modelo que ayude a entender el proceso de negociación que enfrenta Hewlett Packard al ofrecer distintos servicios, como por ejemplo cálculo de alto desempeño. De esta manera se podrá tener una herramienta que ayude a comprender mejor el proceso de negociación, permitiendo así el eventual desarrollo de nuevas políticas de negociación. Un objetivo secundario es el desarrollo de una aplicación que sea capaz de encontrar equilibrios para el juego modelado, y ayude a comprender mejor el proceso de negociación.

Actualmente HP Labs entrega estos servicios basándose en plantillas previamente definidas según distintos niveles de exigencia de los clientes. Sin embargo, estas plantillas pueden no adaptarse a las necesidades de un cliente particular por lo que un proceso de negociación puede ser necesario para satisfacer de manera más apropiada los requerimientos de este cliente.

Se realizaron diversos modelos apuntando a este problema, y se evaluaron cada uno en términos de los resultados obtenidos por cada modelo. De esta manera se fueron mejorando los modelos para que se asemejaran cada vez más al proceso real. Para cada modelo se obtuvieron proposiciones sobre la dinámica del juego en cuestión, llegando a desarrollar una aplicación que ilustrara y encontrara un equilibrio en cada caso. Este trabajo involucra métodos numéricos, considera a la vez factores psicológicos y elementos de teoría de juegos para desarrollar los modelos.

En un sentido cualitativo todos los modelos eran capaces de replicar el proceso de negociación bajo distintas circunstancias, y el modelo utilizado para la aplicación entregaba resultados que satisfacen la intuición económica. Finalmente se entregan diversos puntos sobre los que se puede profundizar para calibrar, o más bien refinar, el modelo desarrollado.

Este trabajo fue desarrollado en conjunto entre HP Laboratories y el Centro de Modelación Matemática durante un semestre. Es presentado en inglés pues este es el idioma del articulo original presentado a HP Labs, salvo por la introducción y las conclusiones que se presentan en español.

Contents

1 Introducción

Descifrar las leyes de la naturaleza ha sido un trabajo constante de la ciencia. Muchas veces para lograr esto se simplifica la realidad y se describe aquella realidad simplificada mediante modelos matemáticos basados en supuestos. De esta manera es como asumiendo que el calor viaja de lugares de mayor temperatura a lugares de menor temperatura a una cierta tasa que depende del material es posible deducir la ecuación del calor; lo que no significa que la naturaleza siga la ecuación del calor, sino que la realidad simplificada sigue la ecuación del calor. Así es como muchos modelos matemáticos han surgido en diversas areas tal como ecología (e.g. La ecuación de Lotka-Volterra), economía (e.g. el modelo de Ramsey), finanzas (e.g. La ecuación de Black Scholes), física, etc.

Gracias a la versatilidad de la matemáticas muchas empresas han pedido mejorar su eficiencia por medio de la modelación de distintos procesos para entender mejor complejos procedimientos tales como la asignación de recursos computacionales, la predicción de demanda por parte de los clientes, o el calentamiento en los *data centers*. Este trabajo se enfocará en modelar mediante modelos matemáticos el proceso de negociación que ocurre en la industria computacional en donde un proveedor de servicios y un cliente negocian con el objeto de lograr diseñar un acuerdo de servicios que los satisfaga a ambos. Un acuerdo de servicios (SLA por sus siglas en inglés *service level agreement*) es un acuerdo que se realiza entre el proveedor y el consumidor que está compuesto por objetivos de servicio que garantizan una calidad de servicio (tal como disponibilidad, rendimiento y fiabilidad), una promesa de pago y penalidades a imponer en caso de que los objetivos no sean alcanzados. El estudio de estos contratos esta siendo cada vez más importante con el aumento progresivo de subcontratación en los mercados de tecnologías de información, que ha alcanzado 56 mil millones de dólares el 2000 y se esperaba que alcanzara 100 mil millones el 2005 (Dermikan et al. 2005). Mientras que la práctica original de la subcontratación en estos mercados involucraba complicadas medidas para proteger los intereses del cliente contra potenciales accidentes, un enfoque más moderno utiliza penalidades y recompensas basadas en la calidad observada, compensando al cliente en caso de que el servicio fuese suboptimal (Dermikan et al. 2005).

El diseño de un SLA requiere de la interacción entre el proveedor y el consumidor, puesto que aunque el proveedor conoce el nivel de servicio que puede ser entregado solo el cliente sabe el nivel de servicio requerido, por esto es natural que el diseño de un SLA sea producto de una negociación entre el proveedor y el cliente. Por supuesto el desarrollo de esta negociación es complejo y eventualmente puede llevar a un SLA que no satisfaga a ninguna de las partes.

El objetivo de este trabajo es desarrollar un modelo matemático para el proceso de negociación previo a acordar un SLA. Los modelos estará basado en teoría de juegos[5]. La idea es tratar de capturar el proceso de negociación que ocurre cuando a los clientes no se les ofrece simplemente un contrato ”tómalo o déjalo” sino que también la oportunidad de expresar sus preferencias mediante una contraoferta. La idea es encontrar un equilibrio para los modelos, estas son situaciones donde ninguna de las partes desea cambiar su estrategia. Los modelos desarrollados son juegos de señales, en estos casos se buscaran equilibrios que cumplen la regla de Bayes[1]. Esto significa que en un equilibrio dada las estrategias de los jugadores, no se actualizaran las creencias y por tanto no habrá incentivo a cambiar las estrategias.

Para empezar en la sección 2 se describirá el problema que se desea modelar. En la sección 3 se presentará el juego que sirve de base para los modelos, y en la sección 4 se describirán diversos modelos, ejemplos de aplicación de los modelos y otras características de los modelos. Luego en la sección 5, se describen otros trabajos y enfoques en esta area. Finalmente en la sección 6 se recapitularán los resultados que establece el modelo y comentarios sobre el contraste de los modelos con la realidad.

Este es un trabajo conjunto entre Hewlett Packard Laboratories en Palo Alto, California y el Centro de Modelamiento Matemático localizado en Santiago de Chile. Este trabajo fue desarrollado en un semestre, 4 meses en California y 2 en Chile. Debido a esto se presenta el trabajo en el idioma original de su concepción, inglés a excepción de la introducción y las conclusiones.

[1] La razón por la que se requiere esto es que si no sigue la regla de Bayes significa que el jugador no actualiza sus creencias de manera racional.

2 The Problem

We will consider that a client goes to the service provider looking for some kind of service such as storage, web services, or high-performance computing (HPC). The service provider initially has some SLA (Service Level Agreement) that the client may look at, for example if the client is looking for HPC the service provider has different menus stating a price for a calculation time. Since different clients have different needs, clients will ask to modify some of the contracts in order to adjust it to their needs, following the example if a contract states a price of 200 USD for 10 hrs calculation time, may be the client needs the result in 8 hrs and is willing to pay 400 USD.

Sometimes these counteroffers do not comply with the service provider's standards, so the service provider by looking at the counteroffer gets an idea of the client's preferences and decides to make a final offer.

This is the problem we are looking forward to model, of course reality is much more complex since the service provider not only has to solve the negotiation problem with the client, but also think on how to efficiently allocate resources with every new client that comes along. Likewise the client may be negotiating with one or more companies for the same service. Of course every client is different, even the same client at different stages in time might behave completely different, although as a company gets to know different kinds of clients this company may be able to classify its clients based on common properties that the clients have. For example, big companies usually ask for greater quality while small and medium business wouldn't appreciate high quality as much.

The problem presented involves many factors difficult to control. These are mostly related to social/psychological issues in the negotiation, for example, factors such as charisma play an important role in the negotiation process. Those kind of factors add an extra difficulty to the modelling that does not occur in other disciplines such as physics or biology, and social factors are the main reason why results may not be exactly like reality but similar. There are also other factors that, even though they cannot be measured exactly, are easily perceived such as reputation. This also plays a role in the negotiation and is hard to model, in [1] a reputation based theory of bargaining can be found. Also who makes the first move as well as number of times they've negotiated and the difference on information about the other party that they handle are key factors in a negotiation.

3 Approach: 2-Stage Bargaining: General Model

Since we want the model to capture strategic behavior, game theory was used as the tool to model the problem. The basic model will be a 2-stage game where a player starts with an offer then the other counteroffers and the last player may accept or reject. The payoff involved would be the utility that each player gets if the offer is accepted or their reservation value in case is not.

The idea to introduce an infinite stage game was considered, but given the experience of Akhil Sahai, our HP advisor, negotiations didn't last long so a 2-stage model may be able to recreate the situation. The two stage model as presented before was too simple for the purpose of this work and since a key issue in negotiations in the information technology market is knowledge about the parties, therefore we decided to introduce a signal that the parties try to recognize given the offers.

The approach for all the models developed will be based in the same game: The signaling game. To model the problem we will consider a signaling game approach as described in Fudenberg-Tirole's Game Theory[2]. Let's describe the general model.

There are 2 players and the first player has private information. Player 1 is called the leader (or sender, as he sends a signal) , and player 2 is the follower (or receiver). Player 1 has private information about his type $\theta \in \Theta$ and chooses action $a_1 \in A_1$. Player 2 whose type is common knowledge, observes a_1 and chooses $a_2 \in A_2$. The spaces of mixed actions are $\mathcal{A}_1$ and $\mathcal{A}_2$ with elements α_1 and α_2. Players i's payoff is denoted $u_i(\alpha_1, \alpha_2, \theta)$. Before the game begins, it is common knowledge that player 2 has prior beliefs $\{p(\theta)\}_{\theta \in \Theta}$ about player 1's type. A strategy for player 1 prescribes a probability distribution $\sigma_1(\cdot|\theta)$ over actions a_1 for each type θ. A strategy for player 2 prescribes a probability distribution $\sigma_2(\cdot|a_1)$ over actions a_2 for each action a_1. Type θ's payoff to strategy $\sigma_1(\cdot|\theta)$ when player 2 plays $\sigma_2(\cdot|a_1)$ is:

$$u_1(\sigma_1, \sigma_2, \theta) = \sum_{a_1} \sum_{a_2} \sigma(a_1|\theta)\sigma_2(a_2|a_1)u_1(a_1, a_2, \theta). \tag{1}$$

Player 2's *ex ante* payoff to strategy $\sigma_2(\cdot|a_1)$ when player 1 plays $\sigma_1(\cdot|\theta)$ is

$$u_2(\sigma_1, \sigma_2, \theta) = \sum_{\theta} p(\theta) \sum_{a_1} \sum_{a_2} \sigma(a_1|\theta)\sigma_2(a_2|a_1)u_2(a_1, a_2, \theta). \tag{2}$$

Player 2, who observes player 1's move before choosing her own action, should update her beliefs about θ and base her choice of a_2 on the posterior distribution $\mu(\cdot|a_1)$ over Θ. In our concept of equilibrium this posterior belief should be consequent with Bayes' rule, given player 1's strategy. And given this posterior belief player 2 should maximize her payoff conditional on a_1, for each a_1,

[2]Drew Fudenberg, Jean Tirole; *Game Theory*, MIT Press

where the conditional payoff to strategy $\sigma_2(\cdot|a_1)$ is

$$\sum_{\theta} \mu(\theta|a_1) u_2(a_1, \sigma_2(\cdot|a_1), \theta) = \sum_{\theta} \sum_{a_2} \mu(\theta|a_1) \sigma_2(a_2|a_1) u_2(a_1, a_2, \theta). \tag{3}$$

So the concept of equilibrium in this framework is the *perfect Bayesian equilibrium* or PBE. A PBE $(\mu, \sigma_1^*, \sigma_2^*)$ satisfies:

$$(P1) \quad \forall \theta, \sigma_1^* \in \arg\max_{\alpha_1} u_1(\alpha_1, \sigma_2^*, \theta), \tag{4}$$

$$(P2) \quad \forall a_1, \sigma_2^*(\cdot|a_1) \in \arg\max_{\alpha_2} \sum_{\theta} \mu(\theta|a_1) u_2(a_1, \alpha_2, \theta), \tag{5}$$

$$(B) \quad \text{If } \sum_{\theta} p(\theta') \sigma_1^*(a_1|\theta') > 0, \mu(\theta|a_1) = \frac{p(\theta)\sigma_1^*(a_1|\theta)}{\sum_{\theta'} p(\theta')\sigma_1^*(a_1|\theta')}. \tag{6}$$

(P1) and (P2) mean players have no incentive to change their strategies since they are maximizing their utility. (B) means that given the actions of player 1, player 2 doesn't change her beliefs about player's 1 type.

In the following section we'll develop several models based in this game.

4 Incorporating the model in SLA negotiation

The main issue is to determine what the uncertainty will be. This uncertainty is what is referred as type in the model, and it can be a parameter of the utility function, may be related to risk aversion or clients preferences. Given the uncertainty we define the players, their strategies and the payoffs. Then we start looking for equilibria. There are many possibilities: service provider's utility function is uncertain, client's reservation utility is uncertain, service provider's reservation utility is uncertain, client's type/utility function is uncertain, etc. The service provider will be referred to as SP indistinctively.

4.1 First Model: SP's Utility Function is Uncertain

The client may not know how tight is the SP situation in terms of resources. The service provider will most of the time try to give the impression she is in a tight situation in order to get more profit as the SP may claim it's more costly for her to give quality given the resources. Let's define the game in this situation:

The player with private information is the SP then she will be player 1 and the client will be player 2. The strategies for both players could be various SLA's templates and the space of possible types could be a parameter of a particular function family or completely different functions. It is important to point out that even if we consider certain utility functions and assumptions, the client usually doesn't know the SP's utility function (Most of the time the SP doesn't know his own utility function) but he does know that if the resources are scarce it will be more costly, and has an idea of how costly it can be. So in reality the situation is similar, and the main results established apply in a certain degree.

4.2 Linear Example For The First Model

We start with a lineal example since linear models are able to capture most of the results of the nonlinear model in situations where ranges are not too different, for example if the negotiation starts in the order of thousands and ends in the order of millions that difference may be too high to be captured by a linear model, but if the negotiation always stays in the order of thousands then the linear model may be accurate enough depending on the convexity of the real utility functions. Another advantage of linear models is that their simplicity allows develop the model completely.

Consider that the possible utility functions for the SP may be $u((p,q),\theta) = p - \theta q, \quad \theta \in \Theta = \{\theta_1, \theta_2\}$, where $\theta_1 < \theta_2$; that is when the SP utility function is $u(\cdot, \theta_2)$ it means that the marginal cost[3] quality is higher than when her utility function is $u(\cdot,\theta_1)$. For simplicity let us consider that the action space of the service provider is $A_1 = \{(\underline{p},\underline{q}),(\overline{p},\overline{q})\}$ such that $\underline{p}-\theta_1\underline{q} > 0 > \underline{p}-\theta_2\underline{q}$ and $\overline{p}-\theta_1\overline{q} > \overline{p}-\theta_2\overline{q} > 0$; $(\underline{p},\underline{q})$ represents a contract that the SP is unable to accept when she is tight on resources. We consider that the probability that the SP is under the $u(\cdot,\theta_1)$ utility function is t, and that the service provider accepts any contract that gives her utility greater or equal than zero.

Let's assume that the utility for the client when accepting a contract (p,q) is $V(p,q) = q-p$, and that the client's budget bounded by p^*. At first it may be considered a certain constant multiplying quality in the function V, but since there are also constants in the SP utility function, and the relevance is on the relative appreciations the constant multiplying quality in V is not important and was normalized.

Now we can describe the payoff function of the game:

$$F: \begin{array}{ccc} A_1 \times \mathbb{R}^2 \times \Theta & \longrightarrow & \mathbb{R} \\ ((p,q),(r,s),\theta) & \longmapsto & \begin{pmatrix} \max\{0, u((r,s),\theta)\} \\ v(r,s)\mathbb{1}_{\{u(\cdot,\theta)>0\}}(r,s) \end{pmatrix} \end{array}$$

Initially the payoff function should also have a fourth component, the accept/reject component, but since we stated that any contract that gives utility greater than zero is accepted, the accept/reject stage is already incorporated in the payoff function as a maximum and an indicator function. The payoff for the service provider when accepting contract (p,q) is $F^1(\cdot,(p,q),\theta)$[4]. The payoff for the client is $F^2(\cdot,(p,q),\theta)$.

With everything defined we can describe the game in extensive form:

The payoff for player 1, as described before, is the first component of the vector located at the bottom of the figure, player 2's payoff is the second component .

4.2.1 Equilibria of the game

We would like to find $(\mu,\sigma_1^*,\sigma_2^*)$ PBE. Let's study the possibilities:

[3]The marginal cost is the cost of delivering 1 additional unit of quality.

[4]Even though it may appear that the initial offer has no relevance, the relevance of the SP initial offer is that it influences the beliefs that the client has about the resources availability of the SP

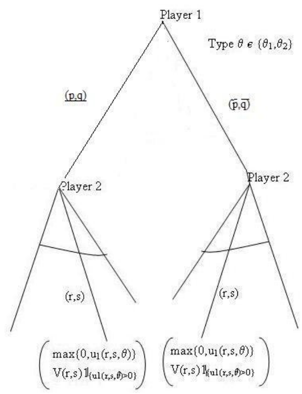

Figure 1: Game in extensive form.

$(P1)$ Establishes that the SP must maximize her expected utility. That is:

$$\sigma_1^* \in \arg\max_{\alpha_1} \sum_{(p,q)} \sum_{(r,s)} \alpha_1((p,q)|\theta)\sigma_2^*((r,s)|(p,q))(r-\theta s)^+$$

Which excluding all the contracts that the SP won't accept, can be rewritten as:

$$\sigma_1^* \in \arg\max_{\alpha_1} \sum_{(p,q)} \sum_{(r,s)\in\mathcal{F}_\theta} \alpha_1((p,q)|\theta)\sigma_2^*((r,s)|(p,q))(r-\theta s) \tag{7}$$

Where $\mathcal{F}_\theta$ denotes all the feasible contracts for the SP. That is $\mathcal{F}_\theta = \{(p,q)|p > \theta q\}$.

$(P2)$ Establishes that the client must maximize his expected utility given the beliefs for any action (p,q). That is:

$$\sigma_2^* \in \arg\max_{\alpha_2} \sum_{\theta} \mu(\theta|(p,q))F^2((p,q),\alpha_2,\theta)$$

Which can be rewritten as:

$$\sigma_2^* \in \arg\max_{\alpha_2} \mu(\theta_1|(p,q)) \sum_{(r,s)\in\mathcal{F}_{\theta_1}} \alpha_2((r,s)|(p,q))(s-r) + \mu(\theta_2|(p,q)) \sum_{(r,s)\in\mathcal{F}_{\theta_2}} \alpha_2((r,s)|(p,q))(s-r) \tag{8}$$

The last condition is that μ satisfies the Bayes' rule. As done in [5] according to the beliefs induced after the offer we divide the equilibria in separating, pooling and hybrid. A separating equilibrium represents the situation where the different types choose different actions, in this case:

$$\mu(\theta_1|(\underline{p},\underline{q})) = 1; \qquad \mu(\theta_2|(\overline{p},\overline{q})) = 1$$

This directly implies that the client will have complete information. A pooling equilibrium is where the different types choose the same action. In this case it means that the service provider will pretend to be tight on resources. In a pooling equilibrium the client will not be able to update the beliefs:

$$\mu(\theta_1|(\underline{p},\underline{q})) = t; \qquad \mu(\theta_2|(\overline{p},\overline{q})) = 1-t$$

There is also the hybrid equilibrium (or semi-separating equilibrium) that is somewhere in between

- **Separating Equilibria**

  A separating equilibrium is one that after the SP makes her move her type is revealed. Since there is a contract that is unfeasible for the SP under one type, there is only one reasonable possibility, that is:

  $$\mu(\theta_1|(\underline{p},\underline{q})) = 1; \qquad \mu(\theta_2|(\overline{p},\overline{q})) = 1$$

  What are the consequences/conditions for this type of equilibrium?

  Proposition 4.1. *Under the assumptions of this model there are no separating equilibria.*

  Proof. $(P2)$ establishes that the client should maximize his utility given the beliefs, therefore:

  $$\sigma_2^* \in \arg\max_{\alpha_2} \sum_{(r,s)\in\mathcal{F}_{\theta_1}} \alpha_2((r,s)|(\underline{p},\underline{q}))(s-r) \tag{9}$$

  $$\sigma_2^* \in \arg\max_{\alpha_2} \sum_{(r,s)\in\mathcal{F}_{\theta_2}} \alpha_2((r,s)|(\overline{p},\overline{q}))(s-r) \tag{10}$$

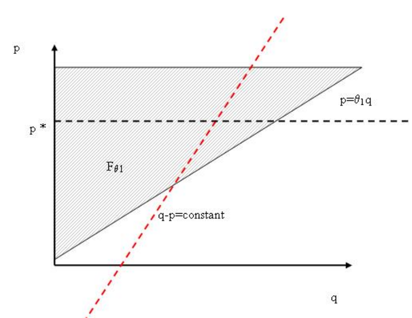

Figure 2: Maximization problem described by (9)

  Therefore we deduce that if $\theta_1 > 1$ then $\sigma_2^*((0,0)|(\overline{p},\overline{q})) = 1$ that is, there is no deal. We'll suppose from now on, that $\theta_1 < 1$. Then the previous condition states that $\sigma_2^*((p^*, \frac{p^*}{\theta_1})|(\underline{p}, \underline{q})) = 1$ and $\sigma_2^*((p^*, \frac{p^*}{\theta_2})|(\overline{p}, \overline{q})) = 1$.

Let's see the conditions $(P1)$ given this strategy σ_2^*:

$$\sigma_1^* \in \arg\max_{\alpha_1} \alpha_1((\overline{p}, \overline{q})|\theta_1)(p^* - \theta_1 \tfrac{p^*}{\theta_2}) + \alpha_1((\underline{p}, \underline{q})|\theta_1)(p^* - \theta_1 \tfrac{p^*}{\theta_1}) \tag{11}$$

$$\Rightarrow \qquad \sigma_1^*((\overline{p}, \overline{q})|\theta_1) = 1 \tag{12}$$

$$\sigma_1^* \in \arg\max_{\alpha_1} \alpha_1((\overline{p}, \overline{q})|\theta_2)(p^* - \theta_2 \tfrac{p^*}{\theta_2}) + \alpha_1((\underline{p}, \underline{q})|\theta_2) \underbrace{(p^* - \theta_2 \frac{p^*}{\theta_1})^+}_{0} \tag{13}$$

$$\Rightarrow \qquad \sigma_1^*((\overline{p}, \overline{q})|\theta_2) \in [0, 1] \tag{14}$$

Finally the Bayes' rule condition must be studied by cases:

If $\sum_i \sigma_1^*((\underline{p}, \underline{q})|\theta_i) > 0$ then (B) is satisfied only by the direct revelation strategy $\sigma_1^*((\underline{p}, \underline{q})|\theta_1) = 1, \sigma_1^*((\underline{p}, \underline{q})|\theta_2) = 0$. This goes against condition (12). Therefore there is no separating equilibrium such that $\sigma_1^*((\underline{p}, \underline{q})|\theta_1) > 0$ or $\sigma_1^*((\underline{p}, \underline{q})|\theta_1) > 0$.

If $\sum \sigma_1^*((\overline{p}, \overline{q})|\theta_i) > 0$ then is satisfied only by the direct revelation strategy $\sigma_1^*((\overline{p}, \overline{q})|\theta_2) = 1$ and $\sigma_1^*((\overline{p}, \overline{q})|\theta_1) = 0$, but this also goes against condition (12).

Then we conclude that there are no separating equilibria. □

This has an economic intuition behind: if the client will believe the SP, the SP provider will take advantage of this and bluff.

- **Pooling Equilibria**

  Pooling equilibria are the ones that after the SP makes her move, the client does not update his beliefs. That is:

$$\mu(\theta_1|(\overline{p}, \overline{q})) = t; \qquad \mu(\theta_1|(\underline{p}, \underline{q})) = t.$$

Proposition 4.2. *Under the assumptions of this model the strategy of the client will be deterministic depending on the prior probability t. More specifically, $\sigma_2((p^*, \frac{p^*}{\theta_1})|(\overline{p}, \overline{q})) = \sigma_2((p^*, \frac{p^*}{\theta_1})|(\underline{p}, \underline{q})) = 1$ if $t > \frac{\frac{1}{\theta_2} - 1}{\frac{1}{\theta_1} - 1}$, and $\sigma_2((p^*, \frac{p^*}{\theta_2})|(\overline{p}, \overline{q})) = \sigma_2((p^*, \frac{p^*}{\theta_2})|(\underline{p}, \underline{q})) = 1$ otherwise.*

Proof. Let's study the conditions now, let's call $\xi(r, s)$ the variable representing $\alpha_2((r, s)|(\overline{p}, \overline{q}))$, then:

$$\sigma_2^* \in \arg\max_{\xi} t \sum_{(r,s) \in \mathcal{F}_{\theta_1}} \xi(r, s)(s - r) + (1 - t) \sum_{(r,s) \in \mathcal{F}_{\theta_2}} \xi(r, s)(s - r) \tag{15}$$

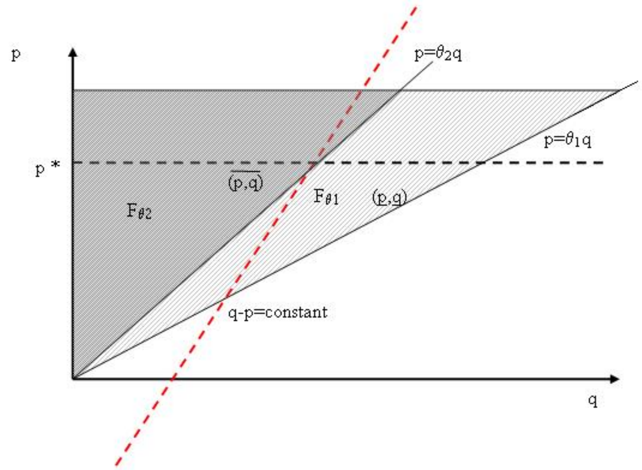

Figure 3: Maximization problem for the client

Linearity implies that the solution of the maximization problem (15) is such that $\xi(p^*, \frac{p^*}{\theta_1}) + \xi(p^*, \frac{p^*}{\theta_2}) = 1$ since other allocations give strictly less payoff. now the problem is to find the optimal distribution over these 2 points. Let $\phi = \xi(p^*, \frac{p^*}{\theta_1})$, then the maximization problem we have to solve is:

$$\max_{0\leq\phi\leq 1} t[(1-\phi)(\frac{p^*}{\theta_2} - p^*) + \phi(\frac{p^*}{\theta_1} - p^*)] + (1-t)(1-\phi)(\frac{p^*}{\theta_2} - p^*) \tag{16}$$

The solution to this problem is $\phi = 1$ if $t > \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1}$, and $\phi = 0$ otherwise.

Since the equations are the same for $\alpha_2((r,s)|(\underline{p},\underline{q}))$ we have the same result: the strategy is deterministic depending on t. □

This means that if the probability of a relaxed service provider is high enough the client will always counteroffer according to the relaxed type. This is consequent with the fact that the service providers signal does not allow the client to update his beliefs, therefore his decision is not based in the signal.

Let's see the other conditions that make this equilibrium sustainable.

Proposition 4.3. *The pooling equilibria of this game are infinitely many. More specifically* *given $s \in (0,1)$:*

$$\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \sigma_1^*((\overline{p},\overline{q})|\theta_2) = s \tag{17}$$

$$\sigma_2((p^*,\frac{p^*}{\theta_1})|(\underline{p},\underline{q})) = 1 \ \textit{if} \ t > \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1} \qquad \sigma_2((p^*,\frac{p^*}{\theta_2})|(\underline{p},\underline{q})) = 1 \ \textit{otherwise} \tag{18}$$

$$\sigma_2((p^*,\frac{p^*}{\theta_1})|(\overline{p},\overline{q})) = 1 \ \textit{if} \ t > \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1} \qquad \sigma_2((p^*,\frac{p^*}{\theta_2})|(\overline{p},\overline{q})) = 1 \ \textit{otherwise} \tag{19}$$

$$\mu(\theta_1|(\overline{p},\overline{q})) = t \qquad \mu(\theta_1|(\underline{p},\underline{q})) = t \tag{20}$$

is an equilibrium.

Proof. The characterization of σ_2 had already been proven, and the last equation is the definition of pooling equilibrium, let's see the characterization of σ_1.

Since player 2 strategy is deterministic, depending only on the prior probabilities it is not hard to see that if $\sigma_2(p^*,\frac{p^*}{\theta_1}) = 1$ the expected utility is always zero independent of the type (since when she accepts the counteroffer gets zero, and otherwise she rejects the counteroffer), and if $\sigma_2(p^*,\frac{p^*}{\theta_1}) = 0$ she gets zero in case her type is θ_2 and constant equal to $p^* - \frac{\theta_1}{\theta_2}p^*$ in case her type is θ_1. Then there are no restrictions for σ_1^* regarding $(P1)$.

The Bayes condition states that:

$$\begin{pmatrix} t & 1-t \\ t & 1-t \end{pmatrix} \begin{pmatrix} \sigma_1^*((\overline{p},\overline{q})|\theta_1) \\ \sigma_1^*((\overline{p},\overline{q})|\theta_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^*((\overline{p},\overline{q})|\theta_1) \\ \sigma_1^*((\overline{p},\overline{q})|\theta_2) \end{pmatrix} \tag{21}$$

That is SP's strategy must be an eigenvector associated to the eigenvalue 1 for the matrix above. Then to satisfy this system it is necessary that $\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \sigma_1^*((\overline{p},\overline{q})|\theta_2)$. Then we conclude it can be any strategy as long as it doesn't depend on the type. □

Since we can use any strategy as long as it doesn't depend on the type it would be nice to find the optimal for the service provider. But since the client does not care what the service providers signal, he only cares about the probability *ex ante*, the Service provider expected utility is $t(p^* - \frac{\theta_1}{\theta_2}p^*)$ only if $t < \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1}$ regardless of the strategy.

- **Semi-Pooling Equilibria**

  In the case that the client believe the service provider never offers a contract that does not give him positive utility, we have that:

$$\mu(\theta_1|(\overline{p},\overline{q})) = t; \qquad \mu(\theta_1|(\underline{p},\underline{q})) = 1$$

Proposition 4.4. *The semi pooling equilibria of this game are two. Given $s \in \{0,1\}$ are characterized by:*

$$\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \sigma_1^*((\overline{p},\overline{q})|\theta_2) = s \tag{22}$$

$$\sigma_2((p^*,\frac{p^*}{\theta_1})|(\underline{p},\underline{q})) = 1 \text{ if } t > \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1} \qquad \sigma_2((p^*,\frac{p^*}{\theta_2})|(\underline{p},\underline{q})) = 1 \text{ otherwise} \tag{23}$$

$$\sigma_2((p^*,\frac{p^*}{\theta_1})|(\overline{p},\overline{q})) = 1 \text{ if } t > \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1} \qquad \sigma_2((p^*,\frac{p^*}{\theta_2})|(\overline{p},\overline{q})) = 1 \text{ otherwise} \tag{24}$$

$$\mu(\theta_1|(\overline{p},\overline{q})) = t \qquad \mu(\theta_2|(\underline{p},\underline{q})) = 0 \tag{25}$$

Proof. (P2) Condition results for $\sigma_2^*((r,s)|(\overline{p},\overline{q}))$ are basically the same, the change is that now $\sigma_2^*((p^*,\frac{p^*}{\theta_1})|(\underline{p},\underline{q})) = 1$.

If $\sum \sigma_1^*((\underline{p},\underline{q})|\theta_i) > 0$ the Bayes' rule implies that $\sigma_1^*((\underline{p},\underline{q})|\theta_1) = 1$ which has as a consequence that $\sigma_1^*((\overline{p},\overline{q})|\theta_1) = 0$, also if $\sum \sigma_1^*((\overline{p},\overline{q})|\theta_i) > 0$, then Bayes' condition states that $\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \sigma_1^*((\overline{p},\overline{q})|\theta_2)$ therefore $\sigma_1^*((\overline{p},\overline{q})|\theta_2) = 0$ which contradicts $\sum \sigma_1^*((\overline{p},\overline{q})|\theta_i) > 0$. Then the service providers strategy will be to always offer $(\underline{p},\underline{q})$.

If $\sum \sigma_1^*((\underline{p},\underline{q})|\theta_i) = 0$, means that $\sigma_1^*((\overline{p},\overline{q})|\theta_i) = 1$ which satisfies Bayes' condition, then the other equilibrium is to always offer $(\overline{p},\overline{q})$. □

- **Hybrid Equilibria**

  First we will study the case when the client believes the service provider is not willing to lose money. In this context we have 1 extra parameter:

$$\mu(\theta_1|(\overline{p},\overline{q})) = \eta; \qquad \mu(\theta_1|(\underline{p},\underline{q})) = 1$$

Proposition 4.5. *There are no equilibria with the beliefs $\mu(\theta_1|(\overline{p},\overline{q})) = \eta;\quad \mu(\theta_1|(\underline{p},\underline{q})) = 1$ other than the case $t = \eta$*

Proof. Algebraically is the same problem for the client as the case before, thus $\sigma_2^*((p^*,\frac{p^*}{\theta_1})|(\overline{p},\overline{q})) = 1$ if $\eta > \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1}$, and $\sigma_2^*((p^*,\frac{p^*}{\theta_2})|(\overline{p},\overline{q})) = 1$ otherwise.

Since strategy of $(P2)$ is very similar as in the case before (algebraically the same), $(P1)$ does not add any restrictions just like in the case before.

The Hybrid equilibria desirable for the service provider are those where $\eta < \frac{\frac{1}{\theta_2}-1}{\frac{1}{\theta_1}-1}$. To achieve this Bayes' rule implies that:

$$\begin{pmatrix} \eta & \frac{1-t}{t}\eta \\ \frac{t}{1-t}(1-\eta) & 1-\eta \end{pmatrix} \begin{pmatrix} \sigma_1^*((\overline{p},\overline{q})|\theta_1) \\ \sigma_1^*((\overline{p},\overline{q})|\theta_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^*((\overline{p},\overline{q})|\theta_1) \\ \sigma_1^*((\overline{p},\overline{q})|\theta_2) \end{pmatrix}$$

Which implies that $\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \frac{(1-t)\eta}{t(1-\eta)}\sigma_1^*((\overline{p},\overline{q})|\theta_2)$

If $\sum \sigma_1^*((\underline{p},\underline{q})|\theta_i) > 0$, then Bayes' rule condition states that $\sigma_1^*((\underline{p},\underline{q})|\theta_1) = 1$ and $\sigma_1^*((\underline{p},\underline{q})|\theta_2) = 0$, therefore $1 = \sigma_1^*((\overline{p},\overline{q})|\theta_1) \neq \frac{(1-t)\eta}{t(1-\eta)}\sigma_1^*((\overline{p},\overline{q})|\theta_2) = 0$. Then this equilibrium is not sustainable.

If $\sigma_1^*((\underline{p},\underline{q})|\theta_1) = \sigma_1^*((\underline{p},\underline{q})|\theta_2) = 0$, then the Bayes' rule does not apply, but this directly implies that $\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \sigma_1^*((\overline{p},\overline{q})|\theta_2) = 1$. Which contradicts the previous Bayes' condition unless $t = \eta$. Then the equilibria is not sustainable. $\square$

Now in the more general case we have:

$$\mu(\theta_1|(\overline{p},\overline{q})) = \eta; \qquad \mu(\theta_1|(\underline{p},\underline{q})) = \pi$$

Theorem 4.6. *Given $\pi \in (0,1) \setminus \{t\}, \eta \in (0,1) \setminus \{t\}$ and $\pi \neq \eta$, the following $(\sigma_1^*, \sigma_2^*, \mu)$:*

$$\sigma_1^*((\overline{p},\overline{q})|\theta_2) = \frac{\frac{t}{1-t} - \frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta} - \frac{\pi}{1-\pi}} \tag{26}$$

$$\sigma_1^*((\underline{p},\underline{q})|\theta_2) = 1 - \frac{\frac{t}{1-t} - \frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta} - \frac{\pi}{1-\pi}} \tag{27}$$

$$\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \frac{(1-t)\eta}{t(1-\eta)} \frac{\frac{t}{1-t} - \frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta} - \frac{\pi}{1-\pi}} \tag{28}$$

$$\sigma_1^*((\underline{p},\underline{q})|\theta_1) = \frac{(1-t)\pi}{t(1-\pi)}(1 - \frac{\frac{t}{1-t} - \frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta} - \frac{\pi}{1-\pi}}) \tag{29}$$

$$\sigma_2^*((p^*, \frac{p^*}{\theta_2})|(\overline{p},\overline{q})) = 1 \qquad \textit{If } \frac{(1-\pi)+\eta}{\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}) \tag{30}$$

$$\sigma_2^*((p^*, \frac{p^*}{\theta_1})|(\overline{p},\overline{q})) = 1 \qquad \textit{If } \frac{(1-\pi)+\eta}{\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}) \tag{31}$$

$$\sigma_2^*((p^*, \frac{p^*}{\theta_2})|(\underline{p},\underline{q})) = 1 \qquad \textit{If } \frac{\pi+(1-\eta)}{1-\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)} \tag{32}$$

$$\sigma_2^*((p^*, \frac{p^*}{\theta_1})|(\underline{p},\underline{q})) = 1 \qquad \textit{If } \frac{\pi+(1-\eta)}{1-\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)} \tag{33}$$

$$\mu(\theta_1|(\overline{p},\overline{q})) = \eta \qquad \mu(\theta_1|(\underline{p},\underline{q})) = \pi \tag{34}$$

Is an equilibrium.

Proof. Bayes' rule condition (if the strategy is mixed) implies that:

$$\begin{pmatrix} \eta & \frac{1-t}{t}\eta \\ \frac{t}{1-t}(1-\eta) & 1-\eta \end{pmatrix} \begin{pmatrix} \sigma_1^*((\overline{p},\overline{q})|\theta_1) \\ \sigma_1^*((\overline{p},\overline{q})|\theta_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^*((\overline{p},\overline{q})|\theta_1) \\ \sigma_1^*((\overline{p},\overline{q})|\theta_2) \end{pmatrix} \tag{35}$$

$$\begin{pmatrix} \pi & \frac{1-t}{t}\pi \\ \frac{t}{1-t}(1-\pi) & 1-\pi \end{pmatrix} \begin{pmatrix} \sigma_1^*((\underline{p},\underline{q})|\theta_1) \\ \sigma_1^*((\underline{p},\underline{q})|\theta_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^*((\underline{p},\underline{q})|\theta_1) \\ \sigma_1^*((\underline{p},\underline{q})|\theta_2) \end{pmatrix} \tag{36}$$

These equations imply that $\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \frac{(1-t)\eta}{t(1-\eta)}\sigma_1^*((\overline{p},\overline{q})|\theta_2)$ and $\sigma_1^*((\underline{p},\underline{q})|\theta_1) = \frac{(1-t)\pi}{t(1-\pi)}\sigma_1^*((\underline{p},\underline{q})|\theta_2)$.

The solution to $(P2)$ maximization problem is:

If $\frac{(1-\pi)+\eta}{\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ then $\sigma_2^*((p^*, \frac{p^*}{\theta_2})|(\overline{p},\overline{q})) = 1$, otherwise $\sigma_2^*((p^*, \frac{p^*}{\theta_1})|(\overline{p},\overline{q})) = 1$, and if $\frac{\pi+(1-\eta)}{1-\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ then $\sigma_2^*((p^*, \frac{p^*}{\theta_2})|(\underline{p},\underline{q})) = 1$, otherwise $\sigma_2^*((p^*, \frac{p^*}{\theta_1})|(\underline{p},\underline{q})) = 1$.

Now instead of looking at condition $(P1)$ let's see how the utility function of the service provider changes as she tries to implement this equilibria. That is let's assume that the service provider reacts according to Bayes' rule, and see how much utility can she get depending on the beliefs.

Finally there is a restriction that completely defines σ_1^*:

$$\sigma_1^*((\overline{p},\overline{q})|\theta_1) + \sigma_1^*((\underline{p},\underline{q})|\theta_1) = 1 \tag{37}$$

$$\frac{(1-t)\eta}{t(1-\eta)}\sigma_1^*((\overline{p},\overline{q})|\theta_2) + \frac{(1-t)\pi}{t(1-\pi)}\sigma_1^*((\underline{p},\underline{q})|\theta_2) = 1 \tag{38}$$

$$\frac{(1-t)\eta}{t(1-\eta)}\sigma_1^*((\overline{p},\overline{q})|\theta_2) + \frac{(1-t)\pi}{t(1-\pi)}(1-\sigma_1^*((\overline{p},\overline{q})|\theta_2)) = 1 \tag{39}$$

The last equation implies that:

$$\sigma_1^*((\overline{p},\overline{q})|\theta_2) = \frac{\frac{t}{1-t} - \frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta} - \frac{\pi}{1-\pi}} \tag{40}$$

$$\sigma_1^*((\underline{p},\underline{q})|\theta_2) = 1 - \frac{\frac{t}{1-t} - \frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta} - \frac{\pi}{1-\pi}} \tag{41}$$

$$\sigma_1^*((\overline{p},\overline{q})|\theta_1) = \frac{(1-t)\eta}{t(1-\eta)}\frac{\frac{t}{1-t} - \frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta} - \frac{\pi}{1-\pi}} \tag{42}$$

$$\sigma_1^*((\underline{p},\underline{q})|\theta_1) = \frac{(1-t)\pi}{t(1-\pi)}(1 - \frac{\frac{t}{1-t} - \frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta} - \frac{\pi}{1-\pi}}) \tag{43}$$

□

Therefore, there are 4 cases:

If $\frac{(1-\pi)+\eta}{\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ then the SP's expected revenue is $t(p^* - \theta_1 \frac{p^*}{\theta_2})$.

If $\frac{(1-\pi)+\eta}{\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ then the SP's expected revenue is $t(p^* - \theta_1 \frac{p^*}{\theta_2}) \frac{(1-t)\eta}{t(1-\eta)} \frac{\frac{t}{1-t}-\frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta}-\frac{\pi}{1-\pi}}$.

If $\frac{(1-\pi)+\eta}{\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ then the SP's expected revenue is $t(p^* - \theta_1 \frac{p^*}{\theta_2}) \frac{(1-t)\pi}{t(1-\pi)} (1 - \frac{\frac{t}{1-t}-\frac{\pi}{1-\pi}}{\frac{\eta}{1-\eta}-\frac{\pi}{1-\pi}})$.

If $\frac{(1-\pi)+\eta}{\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ then the SP's expected revenue is 0.

Observation : Of course the service provider would like to achieve the first equilibrium, the one when $\frac{(1-\pi)+\eta}{\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$, to simplify notation let's consider $\frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)} = c \in [0,1]$. In order to satisfy these conditions it is necesary that :

$$(1-c)\eta + c\pi < c \quad \text{and} \quad 1-c < c\pi + (1-c)\eta \tag{44}$$

Rewritting this last equation we have:

$$c\pi + (1-c)\eta \in (c, 1-c) \tag{45}$$

To obtain this equilibrium it is necessary that $\frac{1}{2} < c$. We can do the same with the other equilibria:

To obtain the one when $\frac{(1-\pi)+\eta}{\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ we must have that:

$$c\pi + (1-c)\eta \in [0, \min\{c, 1-c\}) \tag{46}$$

To obtain the one when $\frac{(1-\pi)+\eta}{\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} > \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ we must have that:

$$c\pi + (1-c)\eta \in (\max\{c, 1-c\}, 1] \tag{47}$$

To obtain the one when $\frac{(1-\pi)+\eta}{\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ and $\frac{\pi+(1-\eta)}{1-\eta} < \frac{(\frac{1}{\theta_1}-1)}{(\frac{1}{\theta_2}-1)}$ we must have that:

$$c\pi + (1-c)\eta \in (c, 1-c) \tag{48}$$

To be able to achieve the last equilibrium it's necessary that $\frac{1}{2} > c$, [5]

[5]Inducing the belief $\eta = \pi = \frac{1}{2}$ can not be achieved unless $t = \frac{1}{2}$. Since it violates (37) -(39).

The case (46) and (47) are always accesible, and they give smaller revenue than the one that can be achieved if $\frac{1}{2} < c$.

So if $\frac{1}{2} < c$, we can obtain maximum revenue by finding feasible beliefs such that $c\pi + (1-c)\eta \in (c, 1-c)$. If $\frac{1}{2} > c$ the service provider has to solve a maximization problem that yields what beliefs are optimal.

4.3 Second Model: Client's type/utility function is uncertain

Now let's assume the service provider doesn't know a parameter of the client's utility function. For example if the client's utility function is $V((p,q),\lambda) = a_\lambda - c_\lambda e^{-\lambda q} - p$, where $\{a_\lambda, c_\lambda\}$ are known for each λ, and the SP is uncertain of what's the λ of the client he is negotiating with, although he might have an initial idea. We model this by assuming that there is a prior probability distribution over a set of posible values of λ ($\mathbb{P}(\lambda = \lambda_i) = r_i$), that is common knowledge for both, the client and the service provider.

The situation we are modelling is the following: The client goes to the service provider and asks her what are the possible contracts she has to offer. The client then chooses one of the contracts and modifies it a little. The service provider's looks at what the client chose as a contract and updates her beliefs about what kind of client she is dealing with, and then offers a final contract, taking into account the client's counteroffer. The client will then accept or reject this counteroffer.

The changes in the model are as follows:

Now the first player will be the client and his action space is restricted to:

$$A_1 = \{(p,q) | \exists (\tilde{p},\tilde{q}) \in \mathcal{M} \quad \|(\tilde{p},\tilde{q}) - (p,q)\| < \varepsilon\}.$$

Where $\mathcal{M}$ is the menu of contracts the service provider initially has to offer.

The counteroffer from the service provider when the client offers $(\hat{p},\hat{q})$ will also be restricted to the set:

$$A_2(\hat{p},\hat{q}) = \{(p,q) | \quad \|(\hat{p},\hat{q}) - (p,q)\| < \eta\}.$$

We'll consider ε, η proportional to the price of the contract being modified.
The proportionality constant will be referred to as the client's and service provider's negotiation factor respectively.

These changes were introduced since in the results from the previous model the counteroffers usually were really far from the previous offer, and the previous offer was not taken into account, it worked only as a signal. The idea behind the action spaces presented is that in a negotiation situation it is unusual to go very far from the offer presented since it is not commonly done. The idea of the negotiation factor and the action space form was developed by consulting with Akhil Sahai.

4.4 Linear Example For The Second Model

First let's consider as an example when the utility functions of the service provider and the client are linear. Then the client's utility function is $V((p,q),\lambda) = \lambda q - p$ where $\lambda \in \{\lambda_1, \lambda_2, \lambda_3\}$ (this is a simplification of the example mentioned before). The service provider utility function will be $u(p,q) = p - zq$ where z is some fixed constant. The service provider initially offers 3 menus: $(p_b, q_b), (p_s, q_s)$ and (p_g, q_g). Now let's describe the game:

First the client chooses (r,s) such that $\|(r,s) - (p_b, q_b)\| < \varepsilon$, $\|(r,s) - (p_s, q_s)\| < \varepsilon$ or $\|(r,s) - (p_g, q_g)\| < \varepsilon$. Of course, the client has many choices, and for simplicity we'll usually focus on 9 options: the ones that give maximum utility to the client for any kind of type.

Assumption 4.7. *By solving the maximization problem:*

$$\max \quad \lambda q - p \tag{49}$$

$$s.t. \quad \|(p,q) - (p_j, q_j)\| \leq \varepsilon \tag{50}$$

where $j \in \{b, s, g\}$, we obtain the different actions that the player may take.

The solution to this problem is:

$$p = p_j - \frac{\varepsilon}{(1+\lambda^2)^{\frac{1}{2}}} \tag{51}$$

$$q = q_j + \frac{\lambda\varepsilon}{(1+\lambda^2)^{\frac{1}{2}}} \tag{52}$$

Which is what can be expected, they counteroffer less price and more quality within the range.

Next the service provider will take that offer, update her beliefs, and make a counteroffer that will also be around the client's offer. That is, the counteroffer (p,q) is such that $\|(p,q) - (r,s)\| < \eta$. The payoffs are similar to the game described before. If no contract is accepted then they both get zero, and if the contract (p,q) is accepted the client type λ and the service provider get $V((p,q),\lambda)$ and $u(p,q)$ respectively.We will also assume that the client accepts any contract such that $V((p,q),\lambda) \geq 0$, the service provider will accept any contract such that $u((p,q)) \geq 0$.

Proposition 4.8. *Given that $z \neq \lambda_k, \quad k = 1,2,3$, the set of the SP actions when the client's offer* *is* (p^*, q^*) *is* $A_1 = X \cup Y$ *where:*

$$X = \{(p,q)|p = p^* + \frac{\eta}{(1+z^2)^{\frac{1}{2}}} \ \textit{and}\ q = q^* - \frac{z\eta}{(1+z^2)^{\frac{1}{2}}}\}$$

$$Y = \{(p,q)|q = \frac{(\lambda p^* + q^*) \pm \sqrt{(\lambda p^* + q^*)^2 - (1+\lambda^2)(p^{*2} + q^{*2} - \eta^2)}}{(1+\lambda^2)}$$

$$p = \lambda \frac{(\lambda p^* + q^*) \pm \sqrt{(\lambda p^* + q^*)^2 - (1+\lambda^2)(p^{*2} + q^{*2} - \eta^2)}}{(1+\lambda^2)}, \lambda \in \{\lambda_1, \lambda_2, \lambda_3\}\}$$

Proof. Since we know that the client's offers are limited we may do the same with the service provider. The service provider will face the problem:

$$\max \quad p - zq \tag{53}$$

$$\text{s.t.} \quad \|(p,q) - (p^*, q^*)\| \leq \eta \tag{54}$$

Where (p^*, q^*) is the client's offer. Then the solution to this problem will be:

$$p = p^* + \frac{\eta}{(1+z^2)^{\frac{1}{2}}} \tag{55}$$

$$q = q^* - \frac{z\eta}{(1+z^2)^{\frac{1}{2}}} \tag{56}$$

With this we have reduced the action space the players.

But there is a chance that the contract is not feasible for the client in that case the service provider will go for a contract that's good for him and also feasible for the client. A contract defined by:

$$\max \quad p - zq \tag{57}$$

$$\text{s.t.} \quad \|(p,q) - (p^*, q^*)\| \leq \eta \tag{58}$$

$$\lambda q - p \geq 0 \tag{59}$$

The solution to this problem is:

$$q = \frac{(\lambda p^* + q^*) \pm \sqrt{(\lambda p^* + q^*)^2 - (1+\lambda^2)(p^{*2} + q^{*2} - \eta^2)}}{(1+\lambda^2)} \tag{60}$$

$$p = \lambda q \tag{61}$$

Whether is plus or minus is determined by which one gives greater utility to the service provider. □

The game played will be $G = (A_1, A_2, F)$ where A_1 and A_2 represent action space for player 1 and player 2 respectively and F is the payoff function of the game. The action space for the client will be

$$A_1 = \{(p,q) \in \mathbb{R}^2 | \|(p,q) - (p_j, q_j)\| \leq p_j \varepsilon \text{ for some } (p_j, q_j) \in \mathcal{M}\}. \tag{62}$$

For the service provider or player 2, the action space will depend on the action taken by player 1:

$$A_2(p', q') = \{(p,q) \in \mathbb{R}^2 | \|(p,q) - (p', q')\| \leq p'\eta\}. \tag{63}$$

The payoff function $F : A_1 \times A_2 \times \Lambda \to \mathbb{R}^2$ is as follows:

$$F^1((p', q'), (p,q), \lambda_k) = (V((p,q), \lambda_k))^+ \tag{64}$$

$$F^2((p', q'), (p,q), \lambda_k) = \begin{cases} 0 & \text{Si } V((p,q), \lambda_k) < 0 \\ u(p,q) & \text{Si } V((p,q), \lambda_k) \geq 0 \end{cases} \tag{65}$$

4.4.1 Equilibria

Here we have more parameters than in the case before, let's just study the equilibria we'd think more likely: hybrid equilibria. Let:

$$\mu(\lambda_i | o_j) = a_{ij}$$

Where o_j is the optimal offer for client type $\lambda_{j-[\frac{j}{3}]\cdot 3}$ when choosing the $(p_{[\frac{j}{3}]+1}, q_{[\frac{j}{3}]+1})$ contract.

To study $(P2)$ condition let's recall what was mentioned earlier about the action space of the service provider. Now that we know the strategies, payoff and beliefs for the service provider. This show us that depending on the belief of the value of λ the service provider has 3 choices for each value, and 7 choices total. Considering this the solution to $(P2)$ is direct: The service provider will play the contract that gives him the most expected utility. Explicitly the condition is:

$$\sigma_2^*(\cdot | o_j) \in \arg\max_{\alpha} \sum_k [F^2(o_j, \varphi_k, \lambda_1) a_{1j} + F^2(o_j, \varphi_k, \lambda_2) a_{2j} + F^2(o_j, \varphi_k, \lambda_3) a_{3j}] \alpha(\varphi_k) \tag{66}$$

Where φ_k represents one of the possible offers made by the service provider, mentioned in the previous paragraph. As mentioned before the solution to this problem is $\sigma_2^*(\varphi_{k(j)} | o_j) = 1$, where $k(j)$ is such that $[F^2(o_j, \varphi_{k(j)}, \lambda_1) a_{1j} + F^2(o_j, \varphi_{k(j)}, \lambda_2) a_{2j} + F^2(o_j, \varphi_{k(j)}, \lambda_3) a_{3j}]$ is maximum.

Now with σ_2^* defined (with the beliefs as a parameter), we can find the client's expected payoff given an strategy. Given a strategy $\sigma_1(\cdot|\lambda_i)$ the client's payoff is:

$$F^1(\sigma_1(\cdot|\lambda_i), \sigma_2^*, \lambda_i) = \sum_j \sigma_1(o_j|\lambda_i) \sum_k \sigma_2^*(\varphi_k|o_j)(\lambda_i q_k - p_k) \tag{67}$$

$$= \sum_j \sigma_1(o_j|\lambda_i)(\lambda_i q_{k(j)} - p_{k(j)}) \tag{68}$$

Where q_k and p_k are the quality and price stated in contract φ_k. Therefore if the client is rational, given the beliefs he will maximize his payoff while trying to keep the beliefs.

Even though studying this game would be interesting we will focus on the next model, which is more general than this one.

4.5 Third Model: Elaboration Over The Second Model

We will model the problem as a dynamic game between the service provider and the client. Before describing the game development, let's describe the elements of the game. There are initial contracts $\mathcal{M} = \{(p_j, q_j) | j = 1, \ldots, m\}$, the client may have different utility functions based on its type, this type is denoted as $\lambda \in \Lambda = \{\lambda_i | i = 1, \ldots, n\}$, there are probabilities known to each of the players that refer to the client's type, these probabilities will be denoted as $\mathbb{P}(\lambda = \lambda_i) = r_i$ and finally there will be utility functions that describe the utility of committing to a contract: for the service provider the utility will be denoted as $u(p, q)$ and for the client of type λ will be denoted as $V((p, q), \lambda)$.

Next we describe the game and how it develops:

- The client approaches the service provider, chooses one of the contracts $(p, q) \in \mathcal{M}$ and modifies it to his advantage. We'll model this modification as choosing a contract (p', q') such that $\|(p', q') - (p, q)\| \leq \varepsilon p$ for a given parameter ε which will be referred to as *negotiation percentage.*
- The service provider receives this counteroffer and extracts some information about the client's type. He uses this information to update the initial beliefs about the client's type and give a final counteroffer. Following the previous idea if (p', q') is the client's counteroffer, the service provider chooses a contract (p^*, q^*) such that $\|(p^*, q^*) - (p', q')\| \leq \eta p'$, the parameter η will be referred to as the service provider *negotiation percentage.*

As in the previous model, player 2 will be the service provider with a utility function $u : \mathbb{R}^2 \to \mathbb{R}$ and $u((p, q))$ describes service provider's utility when the contract (p, q) is established. Player 1 would be the client with a utility function $V : \mathbb{R}^2 \times \Lambda \to \mathbb{R}$ and $V((p, q), \lambda)$ represents the client's type $\lambda \in \Lambda$ utility when the contract (p, q) is established. Here the usual hypothesis are assumed, that is, $u_p > 0, V_p < 0, u_q < 0, V_q > 0, u_{pp} \leq 0, u_{qq} \leq 0, u_{pq} = 0, V_{pp} \leq 0, V_{qq} \leq 0, V_{pq} = 0$. We will also assume that the client accepts any contract such that $V((p, q), \lambda) \geq 0$, the service provider will accept any contract such that $u((p, q)) \geq 0$. The game played will be $G = (A_1, A_2, F)$ where A_1 and A_2 represent action space for player 1 and player 2 respectively and F is the payoff function of the game. The action space for the client will be

$$A_1 = \{(p, q) \in \mathbb{R}^2 | \|(p, q) - (p_j, q_j)\| \leq p_j \varepsilon \text{ for some } (p_j, q_j) \in \mathcal{M}\}. \tag{69}$$

For the service provider or player 2, the action space will depend on the action taken by player 1:

$$A_2(p', q') = \{(p, q) \in \mathbb{R}^2 | \|(p, q) - (p', q')\| \leq p' \eta\}. \tag{70}$$

The payoff function $F : A_1 \times A_2 \times \Lambda \to \mathbb{R}^2$ is as follows:

$$F^1((p',q'),(p,q),\lambda_k) = (V((p,q),\lambda_k))^+ \tag{71}$$

$$F^2((p',q'),(p,q),\lambda_k) = \begin{cases} 0 & \text{Si } V((p,q),\lambda_k) < 0 \\ u(p,q) & \text{Si } V((p,q),\lambda_k) \geq 0 \end{cases} \tag{72}$$

Here we will add an additional assumption in order to simplify the subject:

Assumption 4.9. *Given the assumptions in section 4.5 then the non dominated strategies for the client are finite. Moreover there are at most nm possible counteroffers (one for each contract in the menu) and are solution to the problem:*

$$\max \quad V((p,q),\lambda) \tag{73}$$

$$s.t. \quad \|(p,q)-(p_j,q_j)\| \leq p_j\varepsilon \tag{74}$$

Here we consider the case when a client try to pretend to have another type[6]. A proposition follows:

Proposition 4.10. *Given the assumptions in section 4.5 then the non dominated strategies for the service provider given a counteroffer from the client (p',q') are finite. Moreover there are at most m possible final offers (one for each client type) and are solution to the problem:*

$$\max \quad u((p,q)) \tag{75}$$

$$s.t. \quad \|(p,q)-(p',q')\| \leq p'\eta \tag{76}$$

$$V((p,q),\lambda) \geq 0 \tag{77}$$

Proof. Proposition 4.10. The first thing is to recall the payoff for player 2 $F^2 : A_1 \times A_2 \times \Lambda \to \mathbb{R}$ defined by:

$$F^2((p',q'),(p,q),\lambda_k) = \begin{cases} 0 & \text{Si } V((p,q),\lambda_k) < 0 \\ u(p,q) & \text{Si } V((p,q),\lambda_k) \geq 0 \end{cases}$$

Which in its extended version is given by:

[6] This is not always the case since the client might choose a contract that is on the convex hull of his optimal contract and the other type optimal contract since that way the client is pretending and at the same time the restriction over the counteroffers of the service provider allows the client to not lose as much by pretending. There is a trade off: the more you pretend to be another type the more utility you are risking.

$$F^2((p',q'),\sigma_2((p,q)|(p',q'))) = \begin{cases} \int_{\mathbb{R}^2} \sum_k \sigma_2(\vec{\xi}|(p',q'))F((p',q'),\vec{\xi},\lambda_k)\mu(\lambda_k|(p',q'))d\vec{\xi} & \text{Si}|A_2| > \aleph_0 \\ \sum_j \sum_k \sigma_2((p_j,q_j)|(p',q'))F((p',q'),(p_j,q_j),\lambda_k)\mu(\lambda_k|(p',q')) & \text{Si}|A_2| \leq \aleph_0 \end{cases}$$

Then since the player is rational will be maximizing its expected payoff, by choosing an strategy:

$$\sigma_2^* \in \arg\max F^2((p',q'),\sigma_2((p,q)|(p',q')))$$

To prove the proposition consider an action (p,q) not characterized by the system on the proposition, and examine the cases.

Case 1: It's feasible but not maximum. It can be seen directly that by choosing an action that is maximal the payoff would be greater.

Case 2: It's maximal but does not satisfy the inequality for any client. The payoff will be zero, which is dominated by any action that's feasible for some client, or weakly dominated by any action.

Case 3: It's not on the negotiation area. In this case the action is not feasible

Then the action space is finite and characterized by the systems on the proposition. $\square$

The assumption and the proposition that followed make it much easier to study the game and therefore its equilibria. Now we extend the payoff function to the set of strategies:

$$\mathcal{A}_1 = \{\sigma_1 = (\sigma_1(\cdot|\lambda_i))_{i=1}^m | \sum_{(p,q)\in A_1} \sigma_1((p,q)|\lambda_i) = 1 \quad \forall i = 1,\ldots,m.\}. \tag{78}$$

$$\mathcal{A}_2 = \{\sigma_2 = (\sigma_2(\cdot|(p,q)))_{(p,q)\in A_1} | \sum_{(r,s)\in A_2} \sigma_2((r,s)|(p,q)) = 1 \quad \forall (p,q) \in A_1\}. \tag{79}$$

In the following way,$F : \mathcal{A}_1 \times \mathcal{A}_2 \times \Lambda \to \mathbb{R}^2$:

$$F^1(\sigma_1,\sigma_2,\lambda_k) = \sum_{(p,q)\in A_1} \sum_{(r,s)\in A_2(p,q)} \sigma_1((p,q)|\lambda_k)\sigma_2((r,s)|(p,q))F^1((p,q),(r,s),\lambda_k). \tag{80}$$

$$F^2(\sigma_1,\sigma_2,\lambda_k) = \sum_{(p,q)\in A_1} \sum_{(r,s)\in A_2(p,q)} \sigma_1((p,q)|\lambda_k)\sigma_2((r,s)|(p,q))F^2((p,q),(r,s),\lambda_k). \tag{81}$$

The idea is to find an equilibrium, that is a situation where neither the client nor the service provider would like to change their behavior. The concept of equilibrium in this framework is the *perfect bayesian equilibrium* or PBE. A PBE is a trio (beliefs, client's strategy, service provider strategy) $=(\mu, \sigma_1^*, \sigma_2^*)$ that satisfies[7]:

$$
\begin{aligned}
(P1) \quad & \forall i, \sigma_1^* \in \arg\max_{\alpha_1 \in \mathcal{A}_1} F^1(\alpha_1, \sigma_2^*, \lambda_i), \\
(P2) \quad & \forall a_1, \sigma_2^*(\cdot|a_1) \in \arg\max_{\alpha_2 \in \mathcal{A}_2} \sum_{i=1}^{m} \mu(\lambda_i|a_1) F^2(a_1, \alpha_2, \lambda_i), \\
(B) \quad & \text{If } \sum_{i=1}^{m} r_i \sigma_1^*(a_1|r_i) > 0, \mu(\lambda_j|a_1) = \frac{r_j \sigma_1^*(a_1|\lambda_j)}{\sum_{i=1}^{m} r_i \sigma_1^*(a_1|\lambda_i)}.
\end{aligned}
$$

(P1) and (P2) mean neither the service provider nor the client have incentive to change their strategies since they are maximizing their utility. (B) means that given the actions of the client, the service provider doesn't change her beliefs about the client's type.

Some results have been deduced for the model, such as:

Proposition 4.11. *Given the model described, any strategy that satisfies (P2) is such that:*

$$(\forall (p', q') \in A_1)(\exists (p, q) \in A_2(p', q')) \quad \sigma_2^*((p, q)|(p', q')) = 1$$

Whenever there is a feasible point that gives utility greater than zero.

Proof. Proposition 4.11. According to the previous proposition the payoff function for player 2 is:

$$F^2((p', q'), \sigma_2((p, q)|(p', q'))) = \sum_j \sum_k \sigma_2((p_j, q_j)|(p', q')) F((p', q'), (p_j, q_j), \lambda_k) \mu(\lambda_k|(p', q'))$$

Regrouping:

$$F^2((p', q'), \sigma_2((p, q)|(p', q'))) = \sum_j \sigma_2((p_j, q_j)|(p', q')) \{\sum_k F((p', q'), (p_j, q_j), \lambda_k) \mu(\lambda_k|(p', q'))\}$$

By calling $g((p_j, q_j)) = \sum_k F((p', q'), (p_j, q_j), \lambda_k) \mu(\lambda_k|(p', q'))$. If we prove that $\max_j g((p_j, q_j))$ is achieved by just one point, the result would be direct. Given the assumptions if the value is greater than zero, the result follows from the conditions of strict increase and convexity. $\square$

The previous proposition allows to identify an strategy form player 2 with a function $h : A_1 \to A_2$

[7] Here we considered the natural extension of the utility functions to mixed strategies

such that $h((p,q)) = (r,s)$ with (r,s) satisfying $\sigma_2^*((r,s)|(p,q)) = 1$.

4.6 Nonlinear Numeric Example: 3 Clients 3 Initial Contracts

The model considered for this example consists on a client's utility function $V((p,q),\lambda_i) = a_i - c_i e^{-\lambda_i q} - p$ where the constants $\{a_i\}_{i=1}^{m}$ and $\{c_i\}_{i=1}^{m}$ are given, and a service provider utility function:

$$F^2(\cdot,(p,q),\lambda) = \begin{cases} p - d - ce^{zq} & \text{If client type } \lambda \text{ accepts} \\ 0 & \text{Otherwise} \end{cases}$$

where the constants d, c and z are given. It was also considered $m = n = 3$, that is 3 types of client and 3 initial contracts.

Initially we have the clients and the service provider, with their respective utility functions. In figure 4 we show the contour of the utility function associated to the reservation utility, that is to say any contract with utility lower than the described is not acceptable, any contract with utility greater than the described is acceptable. The contracts with greater utility are in the direction of the arrow next to the utility function. For example for the service provider the contracts with greater utility are found by increasing the price or lowering the quality.

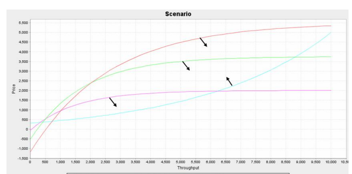

Figure 4: Utility functions of the players.

This allows to define negotiation areas, that is contracts that are rationally acceptable for both the service provider and the client. For example in figure 5 we see the negotiation area between the service provider and the client type 2.

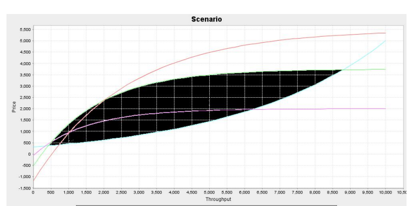

Figure 5: Negotiation area between the type 2 client and the service provider.

The same can be done for the type 3 client and the type 1 client.

Initially the service provider offers 3 contracts that can be located in the graph as seen in figure 7.

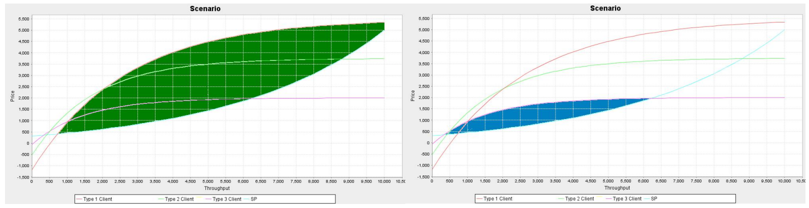

Figure 6: Negotiation areas between the type 1 and 3 client and the service provider.

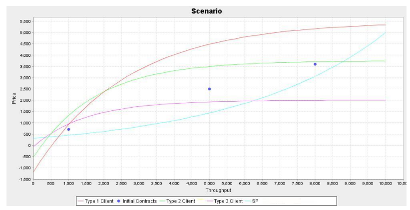

Figure 7: Location of the initial contracts.

The client has a certain area in which he can modify the contract, as seen in figure 8:

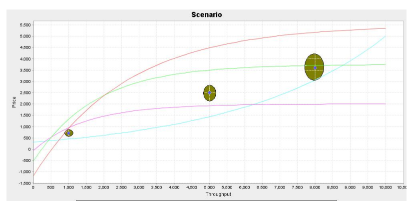

Figure 8: Area of posible replies from the client.

From within these contracts the client may choose his reply, of course this reply will have less price and more quality if possible. For example possible client replies might be like the ones described in figure 9.

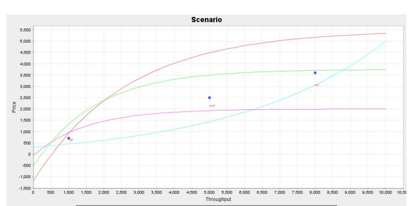

Figure 9: Possible client replies.

Now seeing this counter offer the service provider tries to infer information about the client and designs an optimal counteroffer within a certain negotiation range. In figure 10 we present a negotiation range.

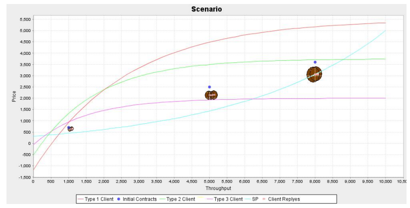

Figure 10: Service provider's negotiation range.

In figure 11 we present the final schema of the negotiation. This is the actual output for the application developed.

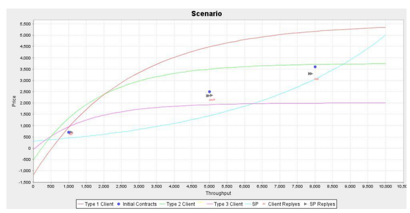

Figure 11: Service provider's negotiation range.

4.6.1 An Example Of Application

The application was developed as a Java Swing frame based application using Eclipse's Visual Editor. The applications first screen consist of a dialog where you input all the parameters for the problem: Negotiation parameters, utility function parameters (This may be estimated also) and the initial contracts.

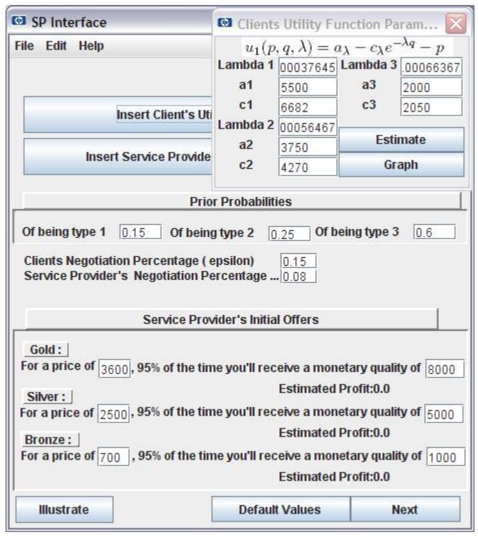

Figure 12: Application Screen 1.

The next screen requires as an input a maximum number of iterations, with default value as 100. These iterations are for computing the action space of the players. As we saw in proposition 4.9 the client's actions can be found by solving the problem (73), therefore according to Karush Kuhn Tucker conditions the solution to (73) is such that:

$$p = p_j - \frac{\varepsilon}{(e^{2\lambda_i q} + c_i^2 \lambda_i^2)^{\frac{1}{2}}} e^{\lambda_i q} \tag{82}$$

$$q = q_j + \frac{c_i \lambda_i \varepsilon}{(e^{2\lambda_i q} + c_i^2 \lambda_i^2)^{\frac{1}{2}}} \tag{83}$$

Therefore we used the following fixed point iteration to solve (83):

$$q_0 = q_j + \frac{c_i \lambda_i \varepsilon p_j}{(1 + c_i^2 \lambda_i^2)^{\frac{1}{2}}} \tag{84}$$

$$q_{n+1} = q_j + \frac{c_i \lambda_i \varepsilon p_j}{(e^{2\lambda_i q_n} + c_i^2 \lambda_i^2)^{\frac{1}{2}}} \tag{85}$$

Here q_0 which corresponds to the solution of the linearized problem[8] and the number of iterations made are defined by the input parameter. To obtain the value of p we just replace the obtained value of q in (82). In order to have convergence of the method it is enough that $|\frac{\epsilon p_j}{c_i^2 \lambda_i} e^{2\lambda_i q}| < 1$ in a neighborhood of the solution. with the parameters set as default values by the interface we have that $\frac{\epsilon p_j}{2c_i^2 \lambda_i} < 0.017$

In the second screen of the application the program prints the optimal replies for the client as well as a maximum error, this error is the difference:

$$Err = q_N - (q_j + \frac{c_i \lambda_i \varepsilon}{(e^{2\lambda_i q_N} + c_i^2 \lambda_i^2)^{\frac{1}{2}}})$$

where N is the maximum number of iterations, then the error is the difference between the last iteration and what would be the next iteration. The number printed is the maximum error among all the computations.

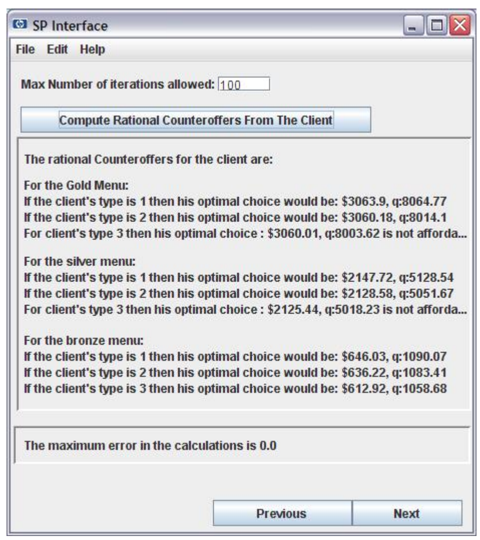

Figure 13: Application Screen 2.

After having the solution for the client's replies the next step in the program is to find the service provider replies. In order to do this the problem to solve is described by (75), to find the optimal solution to this problem we first found the solution to the following problems:

[8] Just linearizing the utility function.

$$
\begin{aligned}
(Pb1) \quad & \max_{(p,q)} \quad F^2((p',q'),(p,q),\lambda) && (86)\\
& \text{s.t.} \quad \|(p,q)-(p',q')\| \leq p'\eta && (87)\\
(Pb2) \quad & V((p,q),\lambda) = 0 && (88)\\
& \|(p,q)-(p',q')\| = p'\eta && (89)
\end{aligned}
$$

To solve (Pb1) we used an analogous method, obtained the Karush Kuhn Tucker Conditions and the followed the following fixed point schema in order to solve them:

$$
\begin{aligned}
q_0 &= q' - \frac{cz\eta p'}{(1+c^2z^2)^{\frac{1}{2}}} && (90)\\
q_{n+1} &= q' - \frac{cz\eta p'}{(e^{-2zq_n}+c^2z^2)^{\frac{1}{2}}} && (91)
\end{aligned}
$$

As before we obtain convergence if $|\frac{\eta p'}{c^2 z}e^{-2zq}| < 1$ To solve (Pb2) we used (88) to obtain a relationship $p = f(q) + p'$ for some function f that does not depend on p. and replaced on (89). The following equation was obtained after replacing:

$$(q-q')^2 = \eta^2 p'^2 - (a_i - c_i e^{-\lambda_i q} - p')^2$$

Here $f(q) = a_i - c_i e^{-\lambda_i q}$. This problem has two solutions, both may be feasible. Then we followed two fixed point schemas:

$$
\begin{aligned}
q_0^0 &= q' && (92)\\
q_{n+1}^0 &= q' - \sqrt{|\eta^2 p'^2 - (a_i - c_i e^{-\lambda_i q_n^0} - p')^2|} && (93)\\
q_0^1 &= q' && (94)\\
q_{n+1}^1 &= q' + \sqrt{|\eta^2 p'^2 - (a_i - c_i e^{-\lambda_i q_n^1} - p')^2|} && (95)
\end{aligned}
$$

Here in order to obtain convergence it is enough that $|\frac{c_i\lambda_i e^{-\lambda_i q}}{2\sqrt{|\eta^2p'^2-(a_i-c_ie^{-\lambda_i q}-p')^2|}}| < 1$

To illustrate the fixed points we are computing let's see figure 14

In figure 14 we represent the negotiation area of the service provider by the brown circle, the client's utility function by the green curve and the client's reply by the pink center of the circle. The solution to (91) is the blue square which would be the optimal without considering if the client accepts or not, the solution to (93) would be the black square, which would be a feasible solution for both the client and the service provider, and the other feasible solution would be (95). To obtain the optimal

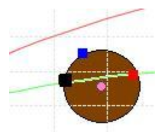

Figure 14: Fixed points solution.

reply given the type we consider out of these 3 problems (if applicable) which one gives the most utility.

Screens 3 and 4 of the application show the optimal reply of the service provider given each type.

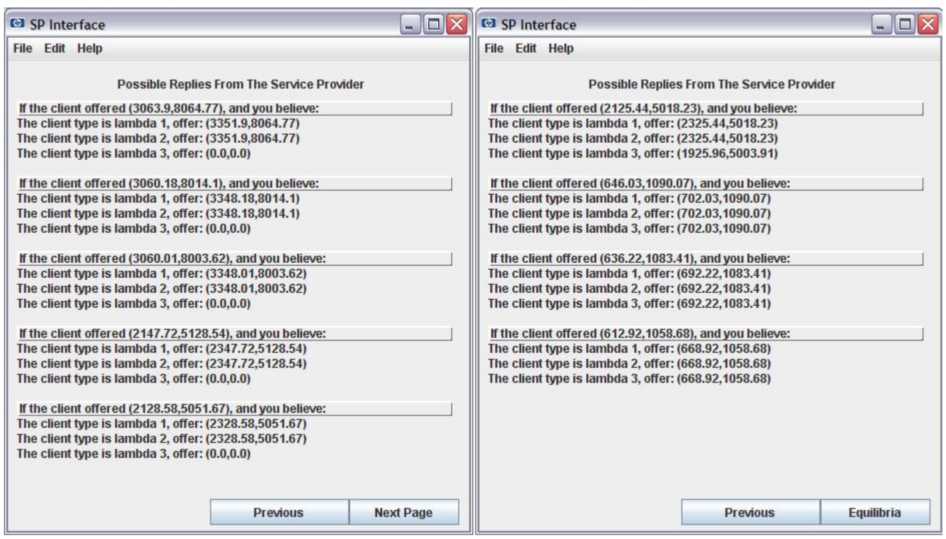

Figure 15: Application's screen 3 and 4.

Now that we the information regarding the actions of the player what's left is to find a perfect Bayesian equilibrium. Before explaining the methodology followed, first let's show the *natural* order of the problem:

- The client follows a strategy $\sigma_1(\cdot|\lambda_i)$

$\rightarrow$ That strategy induces some beliefs $\mu(\lambda_j|a_1) = \frac{r_j \sigma_1^*(a_1|\lambda_j)}{\sum_{i=1}^m r_i \sigma_1^*(a_1|\lambda_i)}$

$\rightarrow$ The service provider decides to play $\sigma_2^*(\cdot|a_1) \in \arg\max_{\alpha_2 \in \mathcal{A}_2} \sum_{i=1}^m \mu(\lambda_i|a_1) F^2(a_1, \alpha_2, \lambda_i)$

$\rightarrow$ The client obtains a utility $F^1(\sigma_1(\cdot|\lambda_i), \sigma_2^*, \lambda_i)$

$\rightarrow$ If the strategy is optimal for the client, we have reached an equilibrium.

Following this idea we decided to take an initial strategy for the client as an input, and use it to compute a utility $F^1(\sigma_1(\cdot|\lambda_i), \sigma_2^*, \lambda_i)$ following the schema mentioned above. Then by using as motivation replicator dynamics we updated the strategy of the client as follows:

$$\sigma_1^{n+1}((p,q)|\lambda) = \sigma_1^n((p,q)|\lambda)\frac{F^1((p,q),\sigma_2^*,\lambda)}{\sum_{(p'q')}\sigma_1^n((p',q')|\lambda)F^1((p',q'),\sigma_2^*,\lambda)} \tag{96}$$

This means that whatever action that gives greater utility is used more often that those that give less utility. Since the utility depends on the service provider response it's not clear that this dynamics converge to some value. We have the following proposition:

Proposition 4.12. *Consider the algorithm referenced above, when used over σ^0 such that $\sigma^0((p_j,q_j)|(p',q')) > 0 \quad \forall (p',q') \in A_1 \quad (p_j,q_j) \in A_2$, if $\sigma_1^n \to_n \hat{\sigma_1}$ then $(\hat{\sigma_1}, \sigma_2^*, \mu)$ with σ_2^*, μ obtained by the algorithm form a PBE, whenever σ_2^* is achieved on finite time, this means that after a finite number of iterations σ_2^N remains the same.*[9]

Proof. We must verify that when we have $\sigma_1^n \to_n \hat{\sigma_1}$ we are on a PBE. Given the algorithm, (B) and (P2) follow directly since it's the invariant of the algorithm. (P1) is still to be checked. Assume (P1) is not satisfied, that means:

$$\exists i, \sigma_1^*(\cdot|\lambda_i) \notin \arg\max_{\alpha_1 \in \mathcal{A}_1} u_1(\alpha_1, \sigma_2^*, \lambda_i)$$

If we have this it means that there is another strategy that delivers strictly more utility. Such strategy can not have the same support as $\hat{\sigma_1}$, if it had the same support and also the condition it means:

$$\begin{aligned}
F^1(\sigma_1^*, \sigma_2^*, \lambda_i) &> F^1(\sigma_1^n, \sigma_2^*, \lambda_i) \Leftrightarrow \\
\sum_{k\in A_1}\sum_{j\in A_2(p_k,q_k)} \sigma_1^*((p_k,q_k)|\lambda_i)F^1((p_k,q_k),(\hat{p}_j,\hat{q}_j),\lambda_i) &> \sum_{k\in A_1}\sum_{j\in A_2(p_k,q_k)} \sigma_1^n((p_k,q_k)|\lambda_i) \\
&\quad F^1((p_k,q_k),(\hat{p}_j,\hat{q}_j),\lambda_i) \Leftrightarrow_{\text{prop 4.11}} \\
\sum_{k\in A_1} \sigma_1^*((p_k,q_k)|\lambda_i)F^1((p_k,q_k),(\hat{p}_{g(k)},\hat{q}_{g(k)}),\lambda_i) &> \sum_{k\in A_1} \sigma_1^n((p_k,q_k)|\lambda_i)F^1((p_k,q_k),(\hat{p}_{g(k)},\hat{q}_{g(k)}),\lambda_i)
\end{aligned}$$

Since $\hat{\sigma_1}$ is the limit strategy using (96), it satisfies:

$$\hat{\sigma_1}((r,s)|\lambda_i) = \hat{\sigma_1}((r,s)|\lambda_i)\frac{F^1((r,s),\sigma_2^*,\lambda_i)}{F^1(\hat{\sigma_1},\sigma_2^*,\lambda_i)} \tag{97}$$

[9] This is a conjecture but has not been proven yet.

This implies that:

$$\hat{\sigma_1}((r,s)|\lambda_i)F^1(\hat{\sigma_1},\sigma_2^*,\lambda_i) = \hat{\sigma_1}((r,s)|\lambda_i)F^1((r,s),\sigma_2^*,\lambda_i) \tag{98}$$

Then:

$$\hat{\sigma_1}((r,s)|\lambda_i) > 0 \Rightarrow F^1(\hat{\sigma_1},\sigma_2^*,\lambda_i) = F^1((r,s),\sigma_2^*,\lambda_i). \tag{99}$$

So all actions played on the limit strategy pay the same value. Also there is no (p,q) such that $F^1((p,q),\sigma_2^*,\lambda_i) > F^1(\hat{\sigma_1},\sigma_2^*,\lambda_i)$, since as they have the same support this contradicts (99).

Actually the only relevant thing is that there is $(r,s) \in \arg\max F^1((r,s),\sigma_2^*,\lambda_i)$ such that $\hat{\sigma_1}(r,s)|\lambda_i) > 0$. This follows directly from the fact that σ_2^* is achieved on finite time, and in any finite number of iterations N $\sigma_1^0 > 0 \Leftrightarrow \sigma_1^N > 0$. □

The application screen 5 is where you input an initial strategy and compute the clients utilities, while application screen 6 shows the beliefs induced and the optimal reply by the service provider. In screen 6 there is a RD button that iterates with replicator dynamics, there is a graphic button which shows the final situation and there is a Txt button that creates and prints the results on the text file: results.txt.

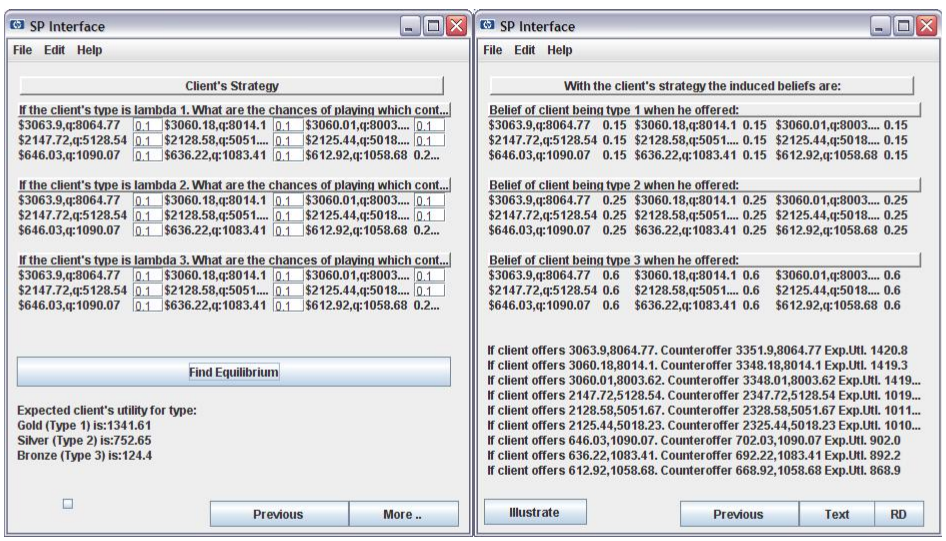

Figure 16: Application's screen 5 and 6.

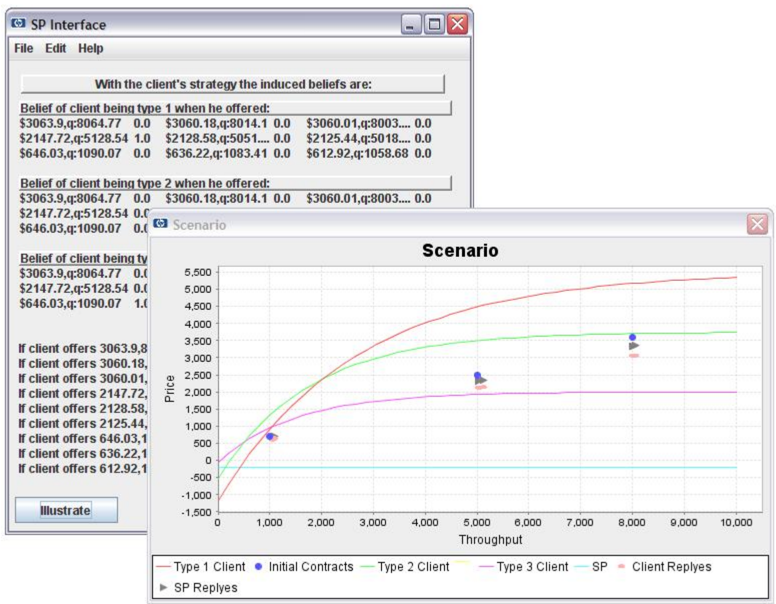

Figure 17: Application's final screen.

4.7 Fourth Model: Third Model Freed From 4.9

Here we will not consider assumption 4.9. And we will consider that the client knows the utility function of the service provider in order for him to play strategically. Since we don't have assumption 4.9 it directly means that we won't have proposition 4.10, so we will have to study the game with a continuos space of actions. However the payoff for a deterministic action will still be the same:

$$F: \quad A_1 \times A_2 \times \Lambda \longrightarrow \quad \mathbb{R}^2 \tag{100}$$

$$((p,q),(r,s),\lambda) \longmapsto \left(\begin{array}{c} \max\{V((r,s),\lambda_k),0\} \\ F^2((p,q),(r,s),\lambda_k) \end{array} \right) \tag{101}$$

Given that $(r,s) \in A_2(p,q)$. And F^2 is defined by:

$$F^2((p,q),(r,s),\lambda_k) = \begin{cases} 0 & \text{Si } V((r,s),\lambda_k) < 0 \\ u(r,s) & \text{Si } V((r,s),\lambda_k) \geq 0 \end{cases} \tag{102}$$

On the previous model the client was blind, since he didn't have enough knowledge to know about the reaction of the service provider given an action. Now the client will know exactly the payoff for the service provider and this will allow him to anticipate actions of the service provider. Now in order to extend the payoff function we must first describe the mixed strategy spaces:

$$\mathcal{A}_1 = \{f \in L^1 | \int_{A_1} f((p,q)|\lambda_i) d\pi(p,q) = 1 \quad \forall i = 1,\ldots,m.\}. \tag{103}$$

$$\mathcal{A}_2 = \{g \in L^1 | \int_{A_2(p,q)} g((r,s)|(p,q)) d\pi(r,s) = 1 \quad \forall (p,q) \in A_1\}. \tag{104}$$

And we extend it in the following way,$F : \mathcal{A}_1 \times \mathcal{A}_2 \times \Lambda \to \mathbb{R}^2$:

$$F^1(f,g,\lambda_k) = \int_{A_1} \int_{A_2(p,q)} f((p,q)|\lambda_k) g((r,s)|(p,q)) F^1((p,q),(r,s),\lambda_k) d\pi(r,s) d\pi(p,q). \tag{105}$$

$$F^2(f,g,\lambda_k) = \int_{A_1} \int_{A_2(p,q)} f((p,q)|\lambda_k) g((r,s)|(p,q)) F^2((p,q),(r,s),\lambda_k) d\pi(r,s) d\pi(p,q). \tag{106}$$

Here π represents Lebesque's measure.

For a moment let's leave aside the beliefs and focus on obtaining a Nash Equilibria, since we have two stages let's make it a Subgame Perfect Equilibrium.

4.7.1 Subgame Perfect Equilibrium

Here we will also consider player 2 maximizes expected payoff. First we solve the last stage:

$$g \in \arg\max \mathbb{E}_\lambda(F^2(f,g,\lambda)) \tag{107}$$

Here we can obtain a similar result as proposition 4.10. Player 2's problem is:

$$\max \sum_k r_k \int_{A_1} \int_{A_2(p,q)} f((p,q)|\lambda_k) g((r,s)|(p,q)) F^2((p,q),(r,s),\lambda_k) d\pi(r,s) d\pi(p,q). \tag{108}$$

Therefore it's enough to solve the problem:

$$\max \int_{A_2(p,q)} g((r,s)|(p,q)) \{\sum_k r_k F^2((p,q),(r,s),\lambda_k)\} d\pi(r,s) \quad \forall (p,q) \in A_1. \tag{109}$$

Therefore we deduce that

$$g((r,s)|(p,q)) = 0 \text{ where } \sup_{x \in A_2(p,q)} \sum_k r_k F^2((p,q),x,\lambda_k) \text{ is not achieved} \tag{110}$$

And,

$$\sup_{x \in A_2(p,q)} \sum_k r_k F^2((p,q),x,\lambda_k) = \max \int_{A_2(p,q)} g((r,s)|(p,q)) \{\sum_k r_k F^2((p,q),(r,s),\lambda_k)\} d\pi(r,s) \quad \forall (p,q) \in A_1. \tag{111}$$

Since the solution to this problem may not be unique there are two cases.

If the solution to this problem is unique player 1 will maximize her utility considering given the other player move.

If the solution to this problem is not unique, player 1 will have to maximize her expected utility given a certain distribution on the other player's move; this distribution is being considered exogenous by using for example previous games information.

Then we have the following proposition:

Proposition 4.13. *If player's 2 utility function u is strictly concave then the counteroffer (r,s) to the offer $(p,q)$is determined by:*

$$(r,s) \in arg\max \int_{A_2(p,q)} g((r,s)|(p,q)) \{\sum_k r_k F^2((p,q),(r,s),\lambda_k)\} d\pi(r,s) \quad \forall (p,q) \in A_1.$$

Corolary 4.14. *Given the assumptions of proposition 4.13 an equilibria of this game is a fixed point of the function:*

$$BR: \quad A_1 \times A_2 \longrightarrow \quad \mathbb{R}^2 \tag{112}$$

$$((p,q),(r,s)) \longmapsto \begin{pmatrix} BR^1((p,q),(r,s)) \\ BR^2((p,q),(r,s)) \end{pmatrix} \tag{113}$$

Where $BR^1((p,q),(r,s)) = arg\max_{\xi} F^1(\xi,(r,s),\lambda^*)$ *and*
$BR^2((p,q),(r,s)) =$

$$arg\max_x \int_{A_2(BR^1((p,q),(r,s)))} g(x|BR^1((p,q),(r,s)))\{\sum_k r_k F^2(BR^1((p,q),(r,s)),x,\lambda_k)\}d\pi(r,s).$$

4.7.2 Adding Bayesian Effect

Here we will add the consideration that whenever the first player does an action it unveils some information about him. As in a previous model we'll denote the beliefs by μ and they will satisfy the following relation:

$$\text{If } \sum_k f((p,q)|\lambda_k)r_k > 0, \qquad \mu(\lambda_j|(p,q)) = \frac{f((p,q)|\lambda_j)}{\sum_k f((p,q)|\lambda_k)r_k} \quad \forall j. \tag{114}$$

As before Player 2's problem is:

$$\max \int_{A_1}\int_{A_2(p,q)} \sum \mu(\lambda_k|(p,q))f((p,q)|\lambda_k)g((r,s)|(p,q))F^2((p,q),(r,s),\lambda_k)d\pi(r,s)d\pi(p,q). \tag{115}$$

Therefore it's enough to solve the problem:

$$\max \int_{A_2(p,q)} g((r,s)|(p,q))\{\sum_k \mu(\lambda_k|(p,q))F^2((p,q),(r,s),\lambda_k)\}d\pi(r,s) \quad \forall (p,q)\in A_1. \tag{116}$$

Therefore we deduce that

$$g((r,s)|(p,q)) = 0 \text{ where } \sup_{x\in A_2(p,q)} \sum_k \mu(\lambda_k|(p,q))F^2((p,q),x,\lambda_k) \text{ is not achieved} \tag{117}$$

And,

$\sup_{x\in A_2(p,q)} \sum_k \mu(\lambda_k|(p,q))F^2((p,q),x,\lambda_k) =$
$\max \int_{A_2(p,q)} g((r,s)|(p,q))\{\sum_k \mu(\lambda_k|(p,q))F^2((p,q),(r,s),\lambda_k)\}d\pi(r,s) \qquad \forall (p,q)\in A_1$

Here what follows is not as straight forward as before, since each move that the first player plays brings as a consequence different beliefs, with this beliefs the behavior of the other player is somehow determined as before, but given this behavior the first player may have taken some other move. It's basically the same loop as before and as before there may be multiple equilibria and different dynamics to achieve any of them.

5 Related Work

Bargaining has attracted a lot of interest from economists for a long time. The emphasis done here is based on signaling games as presented on [5], there is also a great development of this topic in [3] and [7]; The signaling game that is somewhat similar to the one described is the education game presented, where the degree acquired acted as a signal for the employer. Different approaches to bargaining in multiple stages have been made by different authors, for example [6] used fictitious play to replicate the bargaining process, which is an interesting dynamic that has many appreciable properties. [8] uses an evolutionary approach to bargaining such as the one used in the numerical implementation designed to find the PBE.

This work is also incomplete as it is of importance the resource allocation problem while bargaining with two or more clients, since if both clients accept and the resources aren't enough the lack of service may lead to money loses on penalization and reputation.

Also the design of the initial contracts is of importance since if they are optimal for the clients initially there may be no need for bargaining. Several approaches about designing optimal contracts can be seen in [4].

Experiments may be used to validate the model, although as seen in [2] designing an experiment is a very delicate issue, where "cleaning the test tubes" in order to isolate the effects is fundamental.

6 Conclusiones

El objetivo del trabajo era desarrollar un esquema de negociación basado en teoría de juegos para la negociación de SLA para estudiar las dinámicas del proceso de negociación. Se desarrollaron diversos modelos, desde modelos simples que lograron ser estudias en su completitud hasta modelos complejos donde solo se obtuvieron resultados parciales. La idea detrás de hacer los modelos mas complejos era ser capaz de reproducir la mayor cantidad de aspectos de la negociación real.

En el primer modelo desarrollado todos los equilibrios posibles fueron analizados, obteniendo el resultado usual de que dadas ciertas circunstancias la firma siempre pretenderá estar escasa en recursos. El primer modelo tenia como inconveniente que entregaba resultados donde la contraoferta tenia muy poca relación con la oferta inicial, transformando la oferta inicial en una señal sin efectos monetarios en la siguiente etapa, esto fue resuelto en los siguientes modelos.

El segundo modelo desarrollado fue un modelo relativamente sencillo que fue usado como un paso intermedio para el tercer modelo. Dado que la idea era desarrollar un modelo no lineal y un programa durante los cuatro meses, el tiempo asignado al segundo modelo fue mínimo. Para el tercer modelo las dinámicas y la implementación fueron el tópico principal, llevando a una proposición acerca del espacio de acciones del proveedor y el desarrollo de un programa capaz de encontrar un equilibrio perfecto Bayesiano.

Los modelos tienen dinámicas similares, esto entrega validación a los resultados al hacerlos más independientes del modelo. Es importante notar que los resultados tienen intuición económica, y son capaces de replicar en algún nivel la negociación cuando los jugadores tienen suficiente información uno del otro y acuerdan rangos de negociación. Estas hipótesis a pesar de sonar bastante restrictivas, son de sentido común en una negociación real, el problema es que los parámetros tales como el rango de negociación difieren entre jugadores y usualmente son ocultos. Cada modelo tiene parámetros que son exógenos, esto nos permite desarrollar mas resultados para el modelo, pero nos restringe ya que dichos resultados dependerían de los parámetros, aunque parámetros tales como el rango de negociación no afecta el resultado cualitativamente.

El principio de la revelación dice que esta negociación se podría haber hecho de manera optima mediante contratos "tómalo o déjalo"[4], sin embargo este acercamiento es muy estricto para una empresa real porque los contratos son completamente dependientes en las funciones de utilidad de los clientes, y la optimalidad esta sujeta al conocimiento de todos los tipos posibles de clientes. Por esta razón se consideró que un enfoque por medio de etapas podría ayudar a atenuar los efectos en la incertidumbre de las funciones de utilidad y tipos de clientes. Permitiendo así que si hay un tipo de cliente desconocido, o un cliente con diferente función de utilidad, este se regule mediante la modificación de contratos.

Luego de definir el modelo el siguiente paso era el desarrollo de un programa capaz de encontrar un equilibrio del modelo, esta aplicación esta desarrollado para asistir decisiones mas que para definir decisiones, esto dado que la gente no suele estar al tanto de sus funciones de utilidad y toman decisiones basándose en una noción sobre su utilidad. Este trabajo es un reporte interno técnico de HP y esta en proceso de patente. Con este trabajo en conjunto entre HP Labs y el Centro de Modelamiento Matemático (CMM) se ha decidido hacer un programa de investigación en conjunto entre estas instituciones.

6.1 Algunas Extensiones Posibles

Existen pequeños ajustes que se pueden hacer para reproducir resultados mas precisos, tales como considerar que el valor de reserva es $u(0,0,\lambda)$ en lugar de 0, esto es útil para representar que el valor de reserva de distintos clientes es distinto. La mejor forma de decidir que ajustes son necesarios es comparar los resultados del modelo con los de una negociación real, para hacer esto primero se necesita una etapa de calibración donde se buscan los parámetros para las funciones de utilidad, rangos de negociación y tipos de clientes distintos. Esto ultimo se podría encontrar por medio del uso de tests estadísticos tales como análisis factorial discriminante. Los parámetros pueden ser resultados de ajustes usando información anterior o puede provenir de expertos en el area. El modelo y el programa son parte de un proyecto mayor que como fue mencionado antes puede ser completado agregando métodos estadísticos para determinar los parámetros, refinamientos de los equilibrios, o incorporar efectos de saturación en la negociación con varios clientes simultáneos.

Considering the idea as a whole the model and the application developed are just a contribution to a bigger and more ambitious project that as mentioned before may be complete by adding some statistic methods for determining the parameters given data, and equilibrium refinements for example.

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A The Application

As an appendix there is a compact disc where the application is included along the source. Java runtime enviroment is required for the application to work.