UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SEMILINEAR ELLIPTIC PROBLEMS VIA REDUCTION METHODS

TÉSIS PARA OPTAR AL TÍTULO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

OSCAR IVAN AGUDELO RICO

PROFESOR GUÍA:
MANUEL DEL PINO MANRESA.

MIEMBROS DE LA COMISIÓN:
JUAN DÁVILA BONCZOS
MANUEL ELGUETA DEDES
MICHAL KOWALCZYK
MONICA MUSSO

SANTIAGO DE CHILE
SEPTIEMBRE 2012

RESUMEN

Este trabajo se concentra principalmente en estudiar el método de reducción de *Lyapunov-Schmidt* y sus aplicaciones al estudio de existencia de soluciones a problemas semilineales elípticos. En particular, utilizamos exitosamente este método para estudiar la ecuación de Allen-Cahn

$$\Delta u + u(1 - u^2) = 0, \quad \text{en } \mathbb{R}^N$$

en diferentes contextos. La geometría de los conjuntos de nivel de soluciones enteras de esta ecuación, presenta una estructura variada y compleja. En particular, esta ecuación esta presente en la famosa conjetura de E. De Giorgi, la cual afirma que si la dimensión del espacio es tal que $2 \leq N \leq 8$, las soluciones acotadas de esta ecuación que son monótonas en una dirección, tienen por conjuntos nivel a una familia de hiperplanos paralelos entre si, es decir, la solución depende sólo de una variable. Gran progreso se ha alcanzado en la demostración de esta conjetura durante las últimas decadas. La monotonía de las soluciones esta relacionada con sus propiedades de estabilidad. En el programa de entender el conjunto de soluciones enteras de esta ecuación, es interesante estudiar soluciones que tienes índice de Morse finito, de las cuales para nuestro conocimiento, pocos ejemplos se conocen hasta ahora.

En la primera parte de esta investigación, utilizamos el método de reducción, en esencia no variacional, para construir una familia de soluciones acotadas axialmente simétricas a la ecuación de Allen-Cahn en $\mathbb{R}^3$, con la propiedad de tener multiples transiciones sobre una dilatación grande de una catenoide. De nuestro desarrollo, se evidencia contundentemente que estas soluciones tienen indice de Morse grande a medida que la catenoide se vuelve más y más dilatada.

Motivados por este descubrimiento y utilizando el mismo método, continuamos este trabajo construyendo una nueva familia de soluciones axialmente simétricas a la ecuación de Allen-Cahn en $\mathbb{R}^3$, cuyo conjunto nodal consiste en dos componentes conexas que provienen del grafico y su reflexión respecto al eje z, de una solución suave y radialmente simétrica de la ecuación de Liouville en $\mathbb{R}^2$. De igual forma, encontramos fuerte evidencia para afirmar que el índice de Morse de esta familia de soluciones es finito.

Luego, presentamos el estudio de la ecuación no homogénea de Allen-Cahn en $\mathbb{R}^2$, en la cual presentamos otra aplicación del método reducción construyendo, bajo ciertas condiciones geométricas, una familia de soluciones cuyos conjuntos nodales, fuera de una bola grande de $\mathbb{R}^2$, tienen dos componentes conexas que son asintóticamente semirectas no paralelas entre si.

Finalmente, y en contraste, consideramos el contexto variacional presentando resultados de existencia de multiples soluciones para un sistema elíptico de ecuaciones con un acoplamiento simétrico. La aplicación del método de reducción variacional, permite luego aplicar de forma clásica el teorema de paso de montaña simétrico. La importancia del método de reducción, en este caso, radica en que las propiedades de simétria del sistema de ecuaciones, las cuales provienen de la forma del sistema, en lugar de las nolinealidades, son heredadas por ecuación reducida.

ABSTRACT

This work mainly focuses in studying the *Lyapunov-Schmidt* reduction method and its applications to the study of existence of solutions to semilinear elliptic problems. In particular, we use this method successfully to study the Allen-Cahn equation

$$\Delta u + u(1-u^2) = 0, \quad \text{en } \mathbb{R}^N$$

in different settings. The geometry of the level sets of entire solutions to this equation presents a very rich and complex structure. In particular, this equation is present in the famous conjecture due to E. De Giorgi, which says that, if the dimension of the ambient space is such that $2 \leq N \leq 8$, then entire bounded solutions which are monotone in one direction, must have a family of parallel hyperplanes by level sets, in other words, the solution must depend on one variable. Great progress has been achieved in the proof of this conjecture in the last decades. Monotonicity of solutions is related with their stability properties. In the program of understanding the set of entire solutions to this equation, it is interesting to study entire solutions with finite Morse index, of which, to our knowledge, few examples are known so far.

In the first part of this research, we use the reduction method, in a non-variational scheme, to construct a family of of bounded and axially symmetric solutions to the Allen-Cahn equation in $\mathbb{R}^3$, having multiple transition layers over a large dilation of a catenoid. From our developments, we find strong evidence to claim that these solutions have large Morse index, as the catenoid we consider becomes more and more dilated.

Motivated by this finding and using the same method, we continue this research constructing a new family of bounded axially symmetric solutions to the Allen-Cahn equation in $\mathbb{R}^3$, with nodal set having two connected components being the graph and its reflection against the z axis, of a entire smooth axially symmetric solution to the liouville equation in $\mathbb{R}^2$. Likewise, we find strong evidence to say that the Morse index of this family of solutions is finite.

Next, we present the study of the Inhomogeneuos Allen-Cahn equation in $\mathbb{R}^2$, in which we present another application of the reduction method by constructing, under certain geometrical conditions, a family of solutions whose nodal sets, outside a large ball in $\mathbb{R}^2$, has two connected components which are asymptotically two non-parallel half lines.

Finally, and in contrast, we consider the variational setting, presenting results on existence of multiple solutions of a elliptic system with a symmetric coupling. The application of the variational reduction method, allows us to apply in a standard fashion the symmetric mountain pass theorem. The importance of the reduction method in this case, is that the properties of symmetry of the elliptic system , which come from the form of the system rather than the nonlinearities, are inherited by the reduced equation.

Dedicado a mi madre, con todo mi amor

AGRADECIMIENTOS

Este trabajo no hubiese sido culminado sin la colaboración de muchas personas que me brindaron su ayuda, su paciencia, sus conocimientos y principalmente su apoyo, y a las cuales aprovecho para expresarles mis más sinceros agradecmientos.

Agradezco a mi profesor guía, el Dr. Manuel del Pino por haber confiado en mi como estudiante, por su constante motivación, por su paciencia, por muchas discusiones sobre matemáticas y sobre la vida. Por la dirección de este trabajo y por enseñarme la importancia de seguir intentando a pesar de las dificultades. Ha sido un privilegio ser su estudiante de doctorado.

A todos los profesores que tuve durante los cursos del programa de doctorado, les doy las gracias por haber preparado un terreno firme. A los profesores que estuvieron involucrados durante el desarrollo de este trabajo, les agradezco por sus multiples discusiones. En especial, quiero mencionar a los profesores Juan Dávila, Michal Kowalczyk y Juncheng Wei por las multiples y provechosas conversaciones que tuvimos sobre matemáticas. De ustedes he aprendido cosas fundamentales que me ayudaron mucho en el desarollo de este trabajo. A la comisión evaluadora, quiero agradecer su interés en este trabajo así como sus valiosos comentarios y sugerencias.

A Carlos Vélez por ser mi mentor en matemáticas y mi amigo. Por motivarme y acompañarme durante mi incipiente camino en la investigación. Sin la ayuda que me brindaste, esto hubiese sido sólo una posibilidad y no una realidad. Al Profesor Bernhard Ruf, le debo una mención muy importante y afectuosa por su hospitalidad durante mi estadía en Europa y por su frecuente ánimo a hacer matemáticas. Como parte de este proceso, también quiero agradecer a Andrés Zuñiga por que fue un gusto hacer matemáticas contigo y en el camino encontrar un gran amigo.

Agradezco profundamente a mi madre Gloria por su continuo aliento y su incondicional apoyo en todas las travesías en las que me he embarcado. Su motivación y sus consejos siempre estuvieron cuando los necesite. Este trabajo esta dedicado a ti con todo mi amor.

Quiero agradecer también a Carlos García Ángel, por apoyarnos a mi madre y a mi durante todo este tiempo.

A la familia Jiménez Ramírez por haberme acogido como si fuese un miembro más de los suyos y brindarme apoyo incondicional en momentos de crisis y felicidad, les estare por siempre agradecido. En particular a Andrea Jiménez, por que fuiste un apoyo muy grande desde el primer momento de mi vida en Chile.

A mis amigos de Colombia, Julio, Ricardo, Liliana, Carolina, Nelson por que siempre creyeron en mi y me ayudaron a recargar baterías para continuar con esta investigación. A mis amigos Maria Clara , Natalia, Luis y Juan Carlos Lopez y Duver Quintero les agradezco por los momentos de alegría y de goce. Fueron un gran apoyo durante todo el doctorado.

Un afectuoso agradecimiento merecen todos los funcionarios del DIM, en especial Éterin Jana ,Óscar Mori, Luis Mella, Silvia Mariano, Maria Cecilia Cea y Gladys Cavallone por su infinita disposición y paciencia. Tambien quiero agradecer a Regina Mateluna por la ayuda cuando iniciaba mi vida en Chile.

Finálmente, agradezco a los proyectos MECESUP UCH0607, al Progetto UNIALA otorgado por la fundacion CARIPLO y los proyectos de Fondecyt 1070389 y 1110181 y finalmente al Anillo CAPDE AC715 que financiaron esta investigación.

Contents

Chapter 1

Introduction

This thesis work is mainly devoted to study existence and asymptotic behavior of solutions to the classic semilinear elliptic equation

$$\Delta u - F'(u) \,=\, 0, \quad \text{in } \mathbb{R}^N \tag{1.1}$$

where the function F is the balanced and bi-stable twin-pit

$$F(u) := \frac{1}{4}(1-u^2)^2, \quad -F'(u) = u - u^3. \tag{1.2}$$

Equation (1.1) is known as the *A*llen-Cahn equation and it arises in the gradient theory of phase transitions by Allen and Cahn [1], where (1.1) is the prototype equation for the continuous modeling of *phase transition phenomena* finding applications on material sciences, superconductivity, population dynamics and biological patterns formation, see for instance [40]. In this physical model, the function u is meant to represent the phase of a material in a given point of $\mathbb{R}^N$.

Equation (1.1) is also related to the energy functional

$$J_\alpha(v) \;=\; \int_\Omega \frac{\alpha}{2}\,|\nabla v|^2 \;+\; \frac{1}{\alpha}\,F(v) \tag{1.3}$$

whose Euler-Lagrange equation corresponds exactly to the equation

$$\alpha^2 \Delta v - F'(v) = 0, \quad \text{in } \Omega.$$

Let us assume for the moment that $\Omega \subset \mathbb{R}^N$, is an open set containing the origin and $N \geq 2$. From (1.2), we observe that the constant functions $v = \pm 1$, minimize J_α in Ω. They corresponds to stables phases of a material placed in the region Ω. It is of interest to analyze configurations where two phases of a material, say $+1$ and -1, coexist in this region Ω, and which are separated by an interface $(N-1)-$dimensional. The phase is idealized as an $\alpha-$regularization of a discrete function having the form

$$v_* = \chi_\Lambda - \chi_{\Omega - \Lambda} \tag{1.4}$$

where $\Lambda \subset \Omega$ and $\partial\Lambda = M$ corresponds to the idealized interface separating both phases. Observe that any function having the form (1.4) minimizes the second term in (1.3), while the gradient term makes an $\alpha-$regularization of v_* a test function for which the energy is bounded and proportional to the area of the interface M, so that, in addition to minimizing approximately the second term in (1.3), stationary configurations v_α, should also select asymptotically interfaces M that are stationary for surface area, namely, *minimal hypersurfaces*. This intuition on the Allen-Cahn equation gave an important impulse to the calculus of variations, motivating the development of the theory of the $\Gamma-$convergence in the 1970's. Modica [32], proved that a family of local minimizers v_α of J_α, with uniformly bounded energy, must converge in $L^1_{loc}-$sense to a function of the form (1.4), where $M = \partial\Lambda$ minimizes perimeter and hence, being a generalized minimal hypersurface. Modica's result is based upon the intuition that, if M happens to be a smooth orientable surface, then the transition from the equilibria -1 to 1, of v_α, should take place along the normal direction to M_α and where v_α should take the approximate form $v_\alpha(x) = w(z)$, where z corresponds to the normal direction to M. Consequently, the function w should solve the ODE problem

$$w'' - F'(w) \,=\, 0, \quad \text{in } \mathbb{R}, \quad w(\pm\infty) = \pm 1. \tag{1.5}$$

A solution to (1.5) indeed exists. Even more this solution is strictly increasing and uniquely determined up to translations by

$$w(t) = \tanh\left(\frac{t}{\sqrt{2}}\right), \quad t \in \mathbb{R}.$$

This convergence shows an important connection between solutions to equation (1.1) and the theory of minimal hypersurfaces.

If we take such a critical point v_α, and we scale it around $0 \in \Omega$ by setting $u_\alpha(x) = v_\alpha(\alpha\, x)$, we see that u_α satisfies equation

$$\Delta u_\alpha \,+\, u_\alpha(1 - u_\alpha)^2$$

in the expanding domain $\alpha^{-1}\Omega$, so that proceeding formally and letting $\alpha \to 0$, we end up with equation (1.1) in the entire space $\mathbb{R}^N$. The Interface for u_α should thus be around the asymptotically flat minimal surface $M_\alpha = \alpha^{-1}M$. From the fact that $v_\alpha(x) = w(z)$ and for $\alpha > 0$ small, observe that

$$J_\alpha(v_\alpha) \,\approx\, Area(M) \int_{\mathbb{R}} [\frac{1}{2}\,|w'|^2 \,+\, F(w)] \tag{1.6}$$

which is what makes plausible that M is a minimal for the Area functional. Results similar to Modica's hold true for critical points not necessarily minimizers, see [37], [41], and for stronger notions of convergence, see [4], [5]. To be more precise, the condition of local minimizers can be relaxed to a family of critical points with uniformly bounded energy, as was proved in [27]. In this case, the authors showed that the convergence of the interface remains under an integer multiplicity, which takes into account the possibility of multiple transitions layers converging to the same set of minimal perimeter.

The considerations mentioned above, led E. De Giorgi to formulate in 1978 the following celebrated conjecture concerning entire solutions to the equation (1.1), which is in parallel to Bernstein's conjecture theorem for minimal hypersurfaces.

De Giorgi's Conjecture: *The level sets of a bounded entire solution u to* (1.1)*, which is in addition monotone in one direction, must be hyperplanes, at least for dimension $N \leq 8$.*

This conjecture, basically states that, up to translations and rotations of $\mathbb{R}^N$, $u(x) = w(x_N)$, where w is determined by (1.5). The conjecture was proved in dimension $N = 2$ by Ghoussoub and Gui, see [21] in dimension $N = 3$ by Ambrosio and Cabré, see [2], and in dimensions $4 \leq N \leq 8$ by Savin, under the additional assumption

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1$$

see [38]. Recently in [16], it was constructed a counter-example to this conjecture in dimension $N \geq 9$, using Infinite dimensional Lyapunov-Schmidt reduction, monotone in the x_N direction and whose zero level set is close to a large dilation of the Bombieri-De Giorgi-Giusti minimal graph that disproves Bernstein's conjecture in high dimensions, see [3]. On the other hand, the monotonicity of u implies that the scaled function $u(\alpha^{-1}x)$ are in some suitable sense, local minimizers of J_α. Even more, the level sets of u are all graphs. Indeed, without loss of generality assume $\partial_{x_N} u > 0$, in $\mathbb{R}^N$, then it is not hard to check that the linearized operator $L := \Delta - F''(u)$ satisfies maximum principle. This implies stability of u in the sense that the quadratic form

$$\mathtt{B}(\psi, \psi) := \int_{\mathbb{R}^N} |\nabla \psi|^2 + F''(u)\psi^2$$

is positive, for all $\psi \in C_c^\infty(\mathbb{R}^N)$. Let us remark that stability is at the core of the proof of the conjecture in dimensions $N = 2, 3$, where is used to control at infinity the Dirichlet integral, actually it turns out that

$$\int_{B_R(0)} |\nabla u|^2 = \mathcal{O}(R^2) \tag{1.7}$$

which intuitively says that the level sets of u must have a finite number of components outside a large ball, which are all asymptotically flat. The question if stability is sufficient to conclude (1.7), remains open. Actually, it is believed that property (1.7) is equivalent to Finite Morse Index of the solution u. The Morse index of a entire bounded solution u to (1.1) is defined as the maximal dimension of a vector space E of compactly supported functions such that

$$\mathtt{B}(\psi, \psi) < 0, \quad \forall\, \psi \in E - \{0\}.$$

Strikingly, there are basically no examples of finite Morse index solutions to (1.1) in dimension 3, and the connection between Allen-Cahn equation and the theory of minimal surfaces has only been partially explored to produced more examples of finite Morse index solutions.

As remarked in [11], Morse index is a natural element regarding classification of entire bounded solutions to (1.1). This is of course, the natural step to follow beyond De Giorgi's conjecture, towards the understanding of the geometrical structure of solutions to (1.1).

From the comments made above and highlighting relation (1.6), we are led to the question of existence of minimal hypersurfaces with finite Morse index. Let us restrict ourselves to dimension $N = 3$ and to minimal surfaces with finite total Gaussian curvature. For more than a century there were only two known examples of minimal surfaces of finite total curvature, namely the catenoid and the helicoid. The first nontrivial example was found by Costa in 1981, see [10], [23]. The Costa surface is a genus one, minimal, complete and properly embedded surface. It has three connected components outside some compact set, say a large ball, for which two of these components are asymptotically catenoids with the same axis of symmetry, while the remaining one is asymptotically a plane, perpendicular to the axis of symmetry of the catenoidal ends. Later, Hoffman and Meeks generalized Costa's construction by exhibiting a genus k, embedded, minimal surface with three ends and with the same look as the Costa genus one surface outside a large ball, see [24], [25], [26]. Many other examples of this kind of surfaces, with multiple connected components outside a compact set, either asymptotically catenoidal or flat, have been found, see for instance [28], [30] and references there in.

Recently, a new family of finite Morse index bounded solutions of equation (1.1) in $\mathbb{R}^3$, was found in [14] . Each one of these solutions has the property that its nodal set is close to a large dilation of a fixed, complete, embedded and nondegenerate minimal surface and along the normal direction of this large dilation of the surface it has the one dimensional profile of the heteroclinic solution w, to (1.5). Their Morse index coincides with the index of the surface, which is counted as $i = 2l - 1$, where $l \in \mathbb{N}$ is the genus of the surface. In this regard solutions with Morse index 1, associated to the catenoid and Morse index k for $k \geq 3$, associated to the Costa-Hoffman-Meeks surface do exist.

A natural question that rises is wether the construction of solutions to (1.1) with multiple transitions "close" to a complete embedded minimal surfaces of finite total curvature, can be carried out, under the same conditions as in [14]. One of the goals of this thesis is to give a partial answer to this question by constructing a family of bounded solution to problem (1.1) with an arbitrary finite number of transitions layers near a large dilation of a catenoid in $\mathbb{R}^3$.

We make use of an *Infinite Dimensional Lyapunov Schmidt Reduction*, in the spirit of the pioneering work due to Floer and Weinstein, see [20]. As we will see throughout the construction, this solution is expected to have large Morse index. Taking into account the result in [1], no gap condition is required.

Entire solutions with multiple transition layers to (1.1) in $\mathbb{R}^2$ were found in [12]. In this case the nodal set of the solutions consists on multiple asymptotically straight lines, not intersecting themselves, whose locations are governed by the Toda system of ODEs.

As a byproduct of this result, we also present a new familiy of solutions to equation (1.1) in $\mathbb{R}^3$, with the property that its zero level set, outside a large ball, has four logarithmical divergent connected components. The interfaces of this solution take places near the graph of a radially symmetric solution to the Toda System in $\mathbb{R}^2$ and expected to have Morse index 2. This represents the missing Morse index mentioned above.

In this work we also consider the following variations of equation (1.1). We consider the problem of finding bounded solutions to

$$\alpha^2 div(a(x)\nabla u) - F'(u) = 0, \quad \text{in} \quad \mathbb{R}^2 \tag{1.8}$$

where where a is a smooth positive potential, bounded away from zero and $\alpha > 0$ is a small parameter. The function F is as in (1.2). The potential $a(x)$ can be thought as the square root of a nontrivial metric in $\mathbb{R}^2$, hence endowing the space with geodesics that may not be straight line segments. In this regard, there are some related results for the equation

$$\alpha^2 \Delta_g u - V(z)F'(u) = 0, \quad \text{in} \quad M$$

where M is a smooth riemannian manifold and N is a minimal submanifold of M.

In [36], Pacard and Ritoré consider the case $V(z) = 1$, and the setting where M is a compact manifold, establishing that, associated to a non-degenerate minimal submanifolds a solution with a single interface exists. Existence of a solution with multiple interfaces was found in [15], but the nature of this solution differs drastically from that one in [36]. While the solution found by Pacard and Ritoré exists for every $\alpha > 0$ small enough, the solution with arbitrary multiple transitions exists for $\alpha > 0$ small enough but away from certain values where a shift on index occurs.

We also mention the work done by B.Lai and Z.Du in [19] where a family of solutions with a single transition is constructed. Additionally L.Wang and Z.Du dealt in [18] with the same problem, considering multiple transitions this time. In both works minimality and nondegeneracy properties of N, are with respect to the weighted area functional $\int_M V^{1/2}$. In the same line, it is worth to mention a recent work due to Z.Du and C.Gui [17] where they build a smooth solution to the Neumann problem

$$\alpha^2 \Delta u - V(z)F'(u) = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

having a single transition near a smooth closed curve $\Gamma \subset \Omega$, nondegenerate geodesic relative to the arclength $\int_\Gamma V^{1/2}$. Here, Ω is a smooth bounded domain in $\mathbb{R}^2$, and V is an uniformly positive smooth potential.

Restricting ourselves to the case where $M = \mathbb{R}^2$ and using the same scheme as in [16], we present, for every $\alpha > 0$ small, a family of solutions to equation (1.8) having a single transition layer near a planar curve Γ, which is minimal and nondegenerate respect to the length functional $\int_\Gamma a(x)$ and which in addition, outside a large ball, it has two asymptotically a half lines as connected components.

We remark that, up to this point, all of our contributions make use of the Lyapunov-Schmidt reduction method in a non-variational essence. We finish this thesis work by presenting existence of solutions multiple solutions to a system of PDEs with symmetric coupling using a variational reduction technique together with symmetric mountain pass theorem, in the same spirit of the works of [7],[9],[8].

It is well-known that a symmetry in a differential equation generates the existence of multiple solutions. Consider e.g. the superlinear and subcritical equation

$$-\Delta u = f(u) \ , \quad \text{in } \ \Omega \ , \ u|_{\partial\Omega} = 0 \ , \tag{1.9}$$

where $f \in C(\mathbb{R})$ is a superlinear and subcritical nonlinearity. If $f(u)$ is an odd function, then the equation has the symmetry $u \mapsto -u$. Index theories (e.g. the Krasnoselskii genus), show that this symmetry implies the existence of infinitely many solutions for this equation. We consider systems of the following form

$$\begin{cases} -\Delta u + g(v) &= 0 \\ -\Delta v + g(u) &= 0 \qquad \text{in } \ \Omega, \\ \qquad u = v &= 0 \qquad \text{on } \ \partial\Omega, \end{cases} \tag{1.10}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1-function satisfying some assumptions to be specified later, but is not required to be odd. Note that this system allows the following symmetry:

$$T_1 : (u, v) \mapsto (v, u).$$

Thus, one may try to proceed similarly as for equation (1.9) by defining a suitable index. However, one encounters two major problems. First, the functional associated to the system (1.10) is strongly indefinite. Second, the group T has an infinite-dimensional fixed point space, given by the pairs of functions of the form $\{(u, u)\}$. We overcome these difficulties by performing an infinite dimensional Lyapunov-Schmidt reduction (following Castro-Lazer [7]). Surprisingly, the resulting reduced functional has the classical $\mathbb{Z}_2$-symmetry $\{id, -id\}$ and so classical variational methods for the existence of multiple solutions can be employed.

The structure of the thesis is as follows. Chapter two is concerned with the construction of solutions to equation (1.1) with multiple catenoidal transitions. Chapter 3 continues with the construction of solution to (1.1) having two transitions near a solution of the Toda system of PDEs in $\mathbb{R}^2$. Next, in chapter 4, we present the predicted existence result for equation (1.8).

We devote also one appendix to a detailed discussion on the variational reduction method applied to a system of PDEs.

Chapter 2

Multiple Catenoidal Ended Solutions to the Allen-Cahn Equation in $\mathbb{R}^3$

In this chapter we consider bounded, entire solutions to the Allen-Cahn equation

$$\Delta u + u(1-u^2) = 0, \quad \text{in } \mathbb{R}^3. \tag{2.1}$$

Consider a catenoid M in $\mathbb{R}^3$, which is a minimal surface in $\mathbb{R}^3$. We prove that for every $\alpha > 0$ small enough and every integer $m \geq 2$, there exists a bounded solution u_α in $\mathbb{R}^3$, having m transitions layers diverging logarithmically from M. These solutions inherit also the axial symmetry of the catenoid M. Our construction is a first generalization of the construction done in [14].

2.1 Statement of the main result

In what follows we denote $x = (x_1, x_2, x_3)$ points in $x \in \mathbb{R}^3$, and for such points, we write

$$R(x) = |(x_1, x_2)| = \sqrt{x_1^2 + x_2^2}.$$

Let M denote a catenoid in $\mathbb{R}^3$, which is the surface of revolution with the catenary curve C as profile curve. We observe that M divides $\mathbb{R}^3$ into two connected components, say S^+ and S^-, where we choose S^+ to be the component containing the axis of symmetry, namely the x_3-axis.

The mapping $Y : \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^3$, defined by

$$Y(y, \theta) := (\sqrt{1+y^2}\cos\theta, \sqrt{1+y^2}\sin\theta, \log(y + \sqrt{1+y^2})).$$

gives coordinates on the catenoid in terms of the angle of rotation, and the signed arch-length variable of the catenary curve.

The unit normal vector to M, pointing towards S^-, is then given by

$$\nu(y,\theta) = \frac{1}{\sqrt{1+y^2}}\,(\cos\theta, \sin\theta, -y).$$

Let us now consider a large dilation of the catenoid M, given by

$$M_\alpha = \alpha^{-1}M$$

for any small positive number α. We parameterize M_α by $Y_\alpha : (y,\theta) \mapsto \alpha^{-1}Y(\alpha y,\theta)$ and we define associated local Fermi coordinates in $\mathbb{R}^3$,

$$X_\alpha(y,\theta,z) = \alpha^{-1}Y(\alpha y,\theta) + z\nu(\alpha y,\theta), \quad |z| < \frac{\eta}{\alpha} + \frac{1}{2\alpha}\log(1+y^2).$$

The result we prove in this chapter is the following:

Theorem 2.1.1. *Let $N = 3$ and M be a catenoid in $\mathbb{R}^3$. Then for all sufficiently small $\alpha > 0$ there exists a bounded solution u_α to problem* (1.1) *such that*

$$u_\alpha(x) = \sum_{j=1}^{m}(-1)^{j-1}\,w\,(z\,-\,h_j(\alpha y))\,+\,\frac{(-1)^{m-1}\,-\,1}{2}\,+\,o(1), \quad \text{as } \alpha\to 0$$

for $x = X_\alpha(y,\theta,z)$, $\;|z| < \frac{\eta}{\alpha} + \frac{1}{2\alpha}\log(1+y^2)$. These solutions have the additional properties that they are axially symmetric and they converge to ± 1 away from M_α, i.e

$$u_\alpha(x) = u_\alpha(R(x),x_3), \quad u_\alpha^2(x)\to 1, \quad \text{as } dist(x,M_\alpha)\to\infty, \quad \text{for } x=(x_1,x_2,x_3).$$

In addition, the location of the interfaces $h_j's$ is governed by the Jacobi-Toda system of PDEs on M,

$$\alpha^2\left(\Delta_M h_j + |A_M|^2 h_j\right)\,-\,a_0\left[e^{-\sqrt{2}(h_j-h_{j-1})}\,-\,e^{-\sqrt{2}(h_{j+1}-h_j)}\right]\,=\,0$$

where $a_0 > 0$ is a constant and

$$h_{j+1}-h_j \geq \log\left(\frac{1}{\alpha}\right) + \log\left(1+(\alpha y)\right).$$

Remark 2.1.1: The proof of Theorem 2.1.1, as mentioned before, relies on an infinite dimensional reduction procedure, for which the choice of a "good" approximation to a solution is of vital importance. The proof also combines elements from the analysis made in [14] and [16] for one transition in a noncompact setting and [15] for multiple transition for the compact setting. We remark that, contrary to the compact case treated [15], no gap condition is required in this setting. This is due to the fact that we are looking for solutions with high symmetry. From the proof we will see that there is high evidence that this solutions have finite Morse index which goes to infinity as $\alpha > 0$ goes to zero.

Remark 2.1.2: Another important ingredient in the proof of Theorem 2.1.1, is the nondegeneracy of the catenoid. To make this more precise, let us consider the Jacobi operator of the catenoid

$$\mathcal{J}(h) = \Delta_M h + |A_M|^2 h,$$

where $|A_M^2| = -2K_M$ is the euclidean norm of the second fundamental form of M. M is nondegenerate, in the sense that the bounded kernel of $\mathcal{J}$ consists exactly on the jacobi fields z_1, z_2, z_3, associated to the translation along the coordinates axis, where

$$z_i(x) = \nu(x) \cdot e_i, \quad \text{for every } x \in M, \quad i = 1, 2, 3.$$

When working in the space of function in M which depend only on $R(x)$ and with derivatives decaying as $R(x) \to \infty$, it turns out that $\mathcal{J}$ is then invertible. So M is isolated in a smooth topology.

This kind of nondegeneracy is expected to hold true for complete embedded minimal surfaces with finite total curvature, but it is known to hold true not only for the catenoid, but for some other important cases, such as the Costa-Hoffman-Meeks surface of genus k. Nondegeneracy has been a used as a tool to construct new minimal surfaces, see for instance [22], [31], and also to construct solutions to the Allen-Cahn equation over compact manifold, see [36].

2.2 Geometrical setting near a dilated catenoid

In this section we compute the euclidean Laplacian in $\mathbb{R}^3$, in a neighborhood of the dilated catenoid M_α.

Let C denote the catenary curve in $\mathbb{R}^2$, which is the parameterized curve

$$\gamma(s) = (\cosh(s), s), \quad s \in \mathbb{R}$$

and for which we can compute explicitly the corresponding signed arch-length variable as

$$y(s) = \int_0^s \|\gamma'(\zeta)\| d\zeta = \sinh(s).$$

Setting $s(y) = \log(y + \sqrt{1+y^2})$, for $y \in \mathbb{R}$, we can parameterize C by

$$\gamma(s(y)) = \left(\sqrt{1+y^2}, \log\left(y + \sqrt{1+y^2}\right)\right), \quad y \in \mathbb{R}.$$

Let us now consider the catenoid M in $\mathbb{R}^3$, with C as profile curve. The mapping $Y : \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^3$ defined by

$$Y(y, \theta) := \left(\sqrt{1+y^2}\cos\theta\,,\ \sqrt{1+y^2}\sin\theta\,,\ \log(y + \sqrt{1+y^2})\right),$$

gives local coordinates on M in terms of the signed arch-length variable of the curve C and the angle of rotations around the x_3-axis which, in our setting, corresponds to the axis of symmetry of M. Observe also that, for $\mathrm{y} = (\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3) = Y(y, \theta) \in M$,

$$r(\mathrm{y}) := |(\mathrm{y}_1, \mathrm{y}_2)| = \sqrt{1+y^2}.$$

We introduce local Fermi coordinates

$$X(y, \theta, z) = Y(y, \theta) + z\nu(y, \theta), \quad y \in \mathbb{R},\ \theta \in (0, 2\pi),\ z \in \mathbb{R}.$$

This map defines a smooth local change of variables onto the open neighborhood of M, given by

$$\mathcal{N} := \left\{ Y(y,\theta) + z\nu(y,\theta) \ : \ |z| < \eta + \frac{1}{2}\log(1+y^2) \right\}$$

for some small, but fixed $\eta > 0$. Observe that $|z| = dist(x, M)$, for every $x \in \mathcal{N}$ with $x = X(y, \theta, z)$.

Let us compute the euclidean Laplacian in $\mathcal{N}$, in terms of these local coordinates from the formula

$$\Delta_X = \frac{1}{\sqrt{\det(g)}} \partial_i (\sqrt{\det(g)} g^{ij} \partial_j), \quad i,j = y, \theta, z$$

where $g_{ij} = \partial_i X \cdot \partial_j X$ corresponds to the ij-th entry of the metric g on $\mathcal{N}$ and $g^{ij} = (g^{-1})_{ij}$.

Computing the metric g, we find that

$$g \ = \ \begin{bmatrix} g_{yy} & 0 & 0 \\ 0 & g_{\theta\theta} & 0 \\ 0 & 0 & g_{zz} \end{bmatrix} \ = \ \begin{bmatrix} \left(1 - \frac{z}{1+y^2}\right)^2 & 0 & 0 \\ 0 & (1+y^2)\left(1 + \frac{z}{1+y^2}\right)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that

$$\sqrt{\det(g)} = \sqrt{1+y^2}\left(1 - \frac{z^2}{(1+y^2)^2}\right).$$

Since

$$\Delta_X = \frac{1}{\sqrt{\det(g)}} \left[\partial_y (\sqrt{\det(g)} g_{yy}^{-1} \partial_y) + \partial_\theta (\sqrt{\det(g)} g_{\theta\theta}^{-1} \partial_\theta) + \partial_z (\sqrt{\det(g)} \partial_z) \right]$$

we find by a direct computation that

$$\Delta_X \ = \partial_{zz} + \partial_{yy} + \frac{y}{1+y^2}\,\partial_y + \frac{1}{1+y^2}\,\partial_{\theta\theta} - \frac{2z}{(1+y^2)^2}\,\partial_z + D \tag{2.1}$$

where

$$D \ = \ z\, a_1(y,z)\, \partial_{yy} \ + \ z\, a_2(y,z)\, \partial_{\theta\theta} \ + \ z\, b_1(y,z)\, \partial_y \ + \ z^3\, b_2(y,z)\, \partial_z$$

and the functions $a_i(y,z)$, $b_i(y,z)$ are smooth with

$$|a_i| \ + \ |y\, D_y a_i| \ = \ O(|y|^{-2}), \quad |b_1| \ + \ |y\, D_y b_1| \ = \ O(|y|^{-3})$$

$$|b_2| \ + \ |y\, D_y b_2| \ = \ O(|y|^{-8}),$$

as $|y| \to \infty$, uniformly on z in the neighborhood $\mathcal{N}$ of M. Actually, it is not hard to check that, inside $\mathcal{N}$ and for $i = 1, 2$, it holds that

$$a_i(y,z) \ = \ a_{i,0}(y) \ + \ z\, a_{i,1}(y,z), \quad b_1(y,z) \ = \ b_{1,0}(y) \ + \ z\, b_{1,1}(y,z),$$

$$b_2(y,z) \ = \ b_{2,0}(y) \ + \ z^2\, b_{2,1}(y,z),$$

where

$$a_{i,0}(y) \ = \ \frac{(-2)^{i-1}}{(1+y^2)^i}, \quad b_{1,0}(y) \ = \ -\frac{2y}{(1+y^2)^2}, \quad b_{2,0}(y) \ = \ -\frac{2}{(1+y^2)^4},$$

and

$$|a_{i,1}| + |y\, D_y a_{i,1}| = O(|y|^{-(4+2i)}), \quad |b_{1,1}| + |y\, D_y b_{1,1}| = O(|y|^{-5})$$

$$|b_{2,1}| + |y\, D_y b_{2,1}| = O(|y|^{-12}).$$

Let us now consider a large dilation of the catenoid M, given by

$$M_\alpha = \alpha^{-1} M$$

for a small positive number α. We parameterize M_α by $Y_\alpha : (y,\theta) \mapsto \alpha^{-1} Y(\alpha y, \theta)$ and define associated local Fermi coordinates

$$X_\alpha(y,\theta,z) = \alpha^{-1} Y(\alpha y,\theta) + z\nu(\alpha y,\theta)$$

on the neighborhood $\mathcal{N}_\alpha = \alpha^{-1}\mathcal{N}$ of M_α. Observe that

$$\mathcal{N}_\alpha = \left\{ Y_\alpha(y,\theta) + z\nu(\alpha y,\theta) \;:\; |z| < \frac{\eta}{\alpha} + \frac{1}{2\alpha}\log(1 + (\alpha y)^2) \right\}$$

Scaling formula (2.1) we find that

$$\Delta_{X_\alpha} = \partial_{zz} + \partial_{yy} + \frac{\alpha^2 y}{1+(\alpha y)^2}\,\partial_y + \frac{\alpha^2}{1+(\alpha y)^2}\,\partial_{\theta\theta} - \frac{2\alpha^2 z}{(1+(\alpha y)^2)^2}\,\partial_z + D_\alpha \tag{2.2}$$

where

$$D_\alpha \;=\; \alpha\, z\, a_1(\alpha y, \alpha z)\,\partial_{yy} \;+\; \alpha^3\, z\, a_2(\alpha y,\alpha z)\,\partial_{\theta\theta} \;+\; \alpha^2\, z\, b_1(\alpha y,\alpha z)\,\partial_y \;+\; \alpha^4\, z^3\, b_2(\alpha y,\alpha z)\,\partial_z.$$

Let us consider next an arbitrary smooth function $h : \mathbb{R} \to \mathbb{R}$ and local coordinates near M_α, defined as

$$X_{\alpha,h}(y,\theta,t) = \alpha^{-1} Y(\alpha y,\theta) + (t + h(\alpha y))\,\nu(\alpha y,\theta)$$

onto the region $\mathcal{N}_\alpha$, which can be described as

$$\mathcal{N}_\alpha = \left\{ X_{\alpha,h}(y,\theta,t) \;/\; |t + h(\alpha y)| \le \frac{\eta}{\alpha} + \frac{1}{\alpha}\log(\sqrt{1+(\alpha y)^2}\;) \right\}.$$

Observe that for $x \in \mathcal{N}_\alpha$, we have $x = X_\alpha(y,\theta,z) = X_{\alpha,h}(y,\theta,t)$, which means $t = z - h(\alpha y)$. We will often emphasize the description of the region $\mathcal{N}_\alpha$ in terms of the local coordinates $X_{\alpha,h}$ by writing $\mathcal{N}_{\alpha,h}$.

We compute directly, from expression (2.2), the Euclidean Laplacian in these new coordinates.

Lemma 2.2.1. *On the open neighborhood $\mathcal{N}_{\alpha,h}$ of M_α in $\mathbb{R}^3$, in the coordinates $x = X_{\alpha,h}(y,\theta,t)$, the Euclidean Laplacian has the following expression:*

$$\begin{aligned}
\Delta_{X_{\alpha,h}} \;&=\; \partial_{tt} \,+\, \partial_{yy} \,+\, \frac{\alpha^2 y}{1+(\alpha y)^2}\,\partial_y \,+\, \frac{\alpha^2}{1+(\alpha y)^2}\,\partial_{\theta\theta} \\
&-\; \alpha^2 \left\{ h''(\alpha y) \,+\, \frac{\alpha y}{1+(\alpha y)^2} h'(\alpha y) + \frac{2(t+h)}{(1+(\alpha y)^2)^2} \right\} \partial_t \\
&-\; 2\alpha h'(\alpha y)\,\partial_{ty} \,+\, \alpha^2 [h'(\alpha y)]^2\,\partial_{tt} \,+\, D_{\alpha,h}
\end{aligned} \tag{2.3}$$

where

$$
\begin{aligned}
D_{\alpha,h} &= \alpha(t+h)a_1(\alpha y,\alpha(t+h))\left(\partial_{yy}-2\alpha h'(\alpha y)\partial_{yt}-\alpha^2 h''(\alpha y)\partial_t+\alpha^2[h'(\alpha y)]^2\partial_{tt}\right)\\
&+ \alpha^3(t+h)a_2(\alpha y,\alpha(t+h))\partial_{\theta\theta}\\
&+ \alpha^2(t+h)b_1(\alpha y,\alpha(t+h))\left(\partial_y-\alpha h'(\alpha y)\partial_t\right)\\
&+ \alpha^4(t+h)^3 b_2(\alpha y,\alpha(t+h))\,\partial_t. \qquad (2.4)
\end{aligned}
$$

Proof. Set $z = t + h(\alpha y)$ and consider a function $U \in C^2(\mathcal{N}_{\alpha,h})$. From the previous comments, we know that U can be expressed in the coordinates $X_{\alpha,h}$ as well as in the coordinates X_α. So, setting

$$U(X_\alpha(y,\theta,z)) = u(y,\theta,z) \quad \text{and } U(X_{\alpha,h}(y,\theta,t)) = v(y,\theta,t)$$

and from the definition of $X_{\alpha,h}$, we see that $u(y,\theta,z) = v(y,\theta,z-h(\alpha y))$.

From this and formula (2.2), to compute the Euclidean Laplacian in the local coordinates $X_{\alpha,h}$, it remains to express the partial derivatives of u, in terms of the partial derivatives of v. We directly compute

$$
\begin{aligned}
\partial_z u = \partial_t v, &\quad \partial_{zz}u = \partial_{tt}v\\
\partial_\theta u = \partial_\theta v, &\quad \partial_{\theta\theta}w = \partial_{\theta\theta}v\\
\partial_y u &= \partial_y v - \alpha h'(\alpha y)\partial_t v\\
\partial_{yy}u = \partial_{yy}v - 2\alpha h'(\alpha y)&\partial_{ty}v - \alpha^2 h''(\alpha y)\partial_t v + \alpha^2[h'(\alpha y)]^2\partial_{tt}v.
\end{aligned}
$$

Substituting these partial derivatives into formula (2.2) and using that $z = t + h$, we get expression (2.3). ☐

Remark 2.2.1: The Laplace-Beltrami operator of the dilated catenoid M_α, in the coordinates $Y_\alpha(y,\theta)$, corresponds to the differential operator

$$\Delta_{M_\alpha} = \partial_{yy} + \frac{\alpha^2 y}{1+(\alpha y)^2}\,\partial_y + \frac{\alpha^2}{1+(\alpha y)^2}\,\partial_{\theta\theta}$$

with the convention that $M = M_1$. On the other hand, since each one of these dilated catenoids is a minimal surface, we have that the Gaussian curvature, K_{M_α} of M_α, is given by the relation

$$2K_{M_\alpha}(y) = -\frac{2\,\alpha^2}{(1+(\alpha y)^2)^2} = -|A_M(\alpha y)|^2, \quad y \in \mathbb{R}$$

where $|A_M(y)|$ is the norm of the second fundamental form of the catenoid M.

With this comments, we can write the euclidean Laplacian in expression (2.3), as follows

$$
\begin{aligned}
\Delta_{X_{\alpha,h}} &= \partial_{tt} + \Delta_{M_\alpha} - \alpha^2\left\{\Delta_M h + (t+h)|A_M|^2\right\}\partial_t\\
&\quad -2\alpha h'(\alpha y)\,\partial_{ty} + \alpha^2[h'(\alpha y)]^2\,\partial_{tt} + D_{\alpha,h} \qquad (2.5)
\end{aligned}
$$

where the functions h, $\Delta_M h$, $|A_M|^2$ are evaluated in αy.

2.3 The Jacobi-Toda system on the catenoid

In this section, we study solvability of the nonlinear system

$$\alpha^2\left(\Delta_M h_j + |A_M|^2\, h_j\right) - a_0[e^{-\sqrt{2}(h_j-h_{j-1})} \;-\; e^{-\sqrt{2}(h_{j+1}-h_j)}] = \alpha^2 g_j, \quad \text{in } M, \quad j=1,\dots,m \tag{2.1}$$

in the class of axially symmetric functions on M, where $\alpha > 0$ is a small parameter and a_0 a positive constant. We also consider axially symmetric even right-hand sides g_j satisfying

$$\|\, g_j \,\|_{p,\mu} := \|(1+r(\mathrm{y})^{\mu})g_j\|_{L^p(M)} < \infty, \quad 1 < p \le \infty. \tag{2.2}$$

The strategy to solve nonlinear problem (2.1) is to look for

$$h = (h_1, h_2, \dots, h_m)$$

with the form

$$h_j(\mathrm{y}) = \left(j - \frac{m+1}{2}\right)\sigma_\alpha \;+\; q_j(\mathrm{y}), \quad j = 1,\dots,m \tag{2.3}$$

where the constant $\sigma = \sigma_\alpha$ solves the algebraic equation

$$\alpha^2\sigma \;=\; a_0\, e^{-\sqrt{2}\,\sigma}$$

so that, σ_α is a smooth function of α, satisfying the asymptotic expansion

$$\sigma_\alpha = \log\left(\frac{\sqrt{2}\,a_0}{\alpha^2}\right) - \log\left(\log\left(\frac{\sqrt{2}\,a_0}{\alpha^2}\right)\right) + \mathcal{O}\left(\frac{\log\log\log\frac{1}{\alpha^2}}{\log\log\frac{1}{\alpha^2}}\right). \tag{2.4}$$

In what follows, we omit the explicit dependence of σ respect to α, and so we write σ instead of σ_α.

Setting $\delta = \sigma_\alpha^{-1}$, plugging expression (2.3) into (2.1) and dividing by σ_α, we find that system (2.22) becomes

$$\delta\left(\Delta_M q_j + |A_M|^2\, q_j\right) - \left[e^{-\sqrt{2}(q_j-q_{j-1})} \;-\; e^{-\sqrt{2}(q_{j+1}-q_j)}\right] + \left(j - \frac{m+1}{2}\right)|A_M|^2 \;=\; \delta\, g_j, \quad \text{in } M \tag{2.5}$$

where we make the convention that

$$-\infty = q_0 \;<\; q_1 \;<\; q_2 \;<\; \cdots \;<\; q_m \;<\; q_{m+1} = +\infty.$$

We decoupled system (2.5) by considering the auxiliary functions

$$v_j = \left(q_{j+1} - q_j\right), \quad j = 1,\dots,m-1, \quad v_m = \sum_{i=1}^{m} q_j, \quad v_0 = v_{m+1} = +\infty. \tag{2.6}$$

Let us denote

$$\mathrm{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_{m-1} \end{pmatrix}, \quad e^{\mathrm{v}} = \begin{pmatrix} e^{v_1} \\ \vdots \\ e^{v_{m-1}} \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

With this definitions and notations, system (2.5) can be written as

$$\delta\,\left(\Delta_M \mathrm{v} + |A_M|^2\,\mathrm{v}\right) + \mathbf{C}\cdot e^{-\sqrt{2}\,\mathrm{v}} \;+\; |A_M|^2\,\mathbb{1} = \delta\,q, \quad \text{in } M \tag{2.7}$$

$$\Delta_M v_m \;+\; |A_M|^2\,v_m \;=\; q_m, \quad \text{in } M, \tag{2.8}$$

where $\mathbf{C}$ is the constant, invertible matrix

$$\mathbf{C} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & \cdots & 1 & -2 \end{pmatrix}.$$

and where we have denoted

$$q_j = g_{j+1} - q_j, \quad j = 1,\ldots,m-1, \quad q_m = \sum_{i=1}^{m} g_j, \quad q = \begin{pmatrix} q_1 \\ \vdots \\ q_{m-1} \end{pmatrix}.$$

Since we want v_m to be bounded and small, invertibility theory for equation (2.8) is required. Regarding this matter, we state the following lemma.

Proposition 1. *Le q be an axially symmetric even function such that $\|\,q\,\|_{p,\mu} < \infty$, for $2 \le p \le \infty$ and $2 < \mu < 3$. Then, there exists an axially symmetric even function v, solving*

$$\Delta_M\,v + |A_M|^2\,v \;=\; q, \quad \textit{in } M$$

and satisfying the following estimate

$$\|v\|_{2,p,\mu} \le C\,\|q\|_{p,\mu} \tag{2.9}$$

where

$$\|v\|_{2,p,\mu} := \|v\|_{L^\infty(M)} + \|r^{\beta-1}(y)Dv\|_{L^\infty(M)} + \|D^2 v\|_{p,\mu}.$$

In order to keep the presentation as clear as possible, we postpone the proof of this proposition until section 2.4.

Observe that we only need to take care of system (2.7). In order to solve this system, we look for a solution v, having the particular form

$$\mathrm{v} = \omega(\mathrm{y},\delta) + \zeta$$

with

$$\omega(\mathrm{y},\delta) \;=\; \mathrm{v}^0(\mathrm{y}) + \delta\,\mathrm{v}^1(\mathrm{y}).$$

To find what v_0 and v^1 should be, we denote

$$E(\omega,\delta,\mathrm{y}) := \delta\,\left(\Delta_M \mathrm{v} + |A_M|^2\,\mathrm{v}\right) \;+\; \mathbf{C}\cdot\,e^{-\sqrt{2}\,\mathrm{v}} \;+\; |A_M|^2\,\mathbb{1} \tag{2.10}$$

and for $\omega = \mathrm{v}^0 + \delta\, \mathrm{v}^1$, we observe that expression (2.10) becomes

$$E(\omega,\delta,\mathrm{y}) \,=\, \mathbf{C}\cdot e^{-\sqrt{2}\,\mathrm{v}^0} \,+\, |A_M|^2\,\mathbb{1}$$

$$+\,\delta\,\left(\Delta_M\,\mathrm{v}^0 \,+\, |A_M|^2\,\mathrm{v}^0\right) \,+\, \delta\, D_{\mathrm{v}}\left(\mathbf{C}\cdot e^{-\sqrt{2}\,\mathrm{v}}\right)_{\mathrm{v}=\mathrm{v}^0}\cdot \mathrm{v}^1$$

$$+\,\delta^2\,\left(\Delta_M\,\mathrm{v}^1 \,+\, |A_M|^2\,\mathrm{v}^1\right)$$

$$+\,\mathbf{C}\cdot\left[e^{-\sqrt{2}\,(\mathrm{v}^0+\delta\mathrm{v}^1)} - e^{-\sqrt{2}\,\mathrm{v}^0} - \delta\, D_{\mathrm{v}}\left(e^{-\sqrt{2}\,\mathrm{v}}\right)_{\mathrm{v}=\mathrm{v}^0}\mathrm{v}^1\right]. \tag{2.11}$$

We want $E(\mathrm{v}^0+\delta\mathrm{v}^1,\delta,\mathrm{y})$ to be as close to zero as possible, so that, proceeding formally by taking $\delta\to 0$ in expression (2.11), we find that v^0 must be the solution to the algebraic equation

$$\mathbf{C}\cdot e^{-\sqrt{2}\,\mathrm{v}^0} \,+\, |A_M|^2\,\mathbb{1} \,=\, 0. \tag{2.12}$$

We recall that in coordinates

$$|A_M(\mathrm{y})|^2 = \frac{2}{(1+y^2)^2}, \quad \mathrm{y} = Y(y,\theta).$$

From this, we find that $\mathrm{v}^0 = (v^0_1,\ldots,v^0_{m-1})$ is given by

$$v^0_j(\mathrm{y}) = -\frac{1}{\sqrt{2}}\,\log\left(\frac{1}{2}|A_M(\mathrm{y})|^2(m-j)j\right), \quad 1\le j\le m-1.$$

Since

$$v^0_j(\mathrm{y}) = -\frac{1}{\sqrt{2}}\,\log\left((m-j)j\right) \,+\, \frac{1}{\sqrt{2}}\,\log\left(|A_M|^{-2}\right),$$

we can write

$$\mathrm{v}^0 \;=\; \frac{1}{\sqrt{2}}\,\log\left(|A_M|^{-2}\right)\,\mathbb{1} \,+\, c_0 \tag{2.13}$$

for some constant vector c_0. Observe that

$$\Delta_M\mathrm{v}^0 + |A_M|^2\mathrm{v}^0 = |A_M|^2\left(2\,\mathbb{1} \,+\, \mathrm{v}^0\right). \tag{2.14}$$

Next, with this choice of v_0, we divide expression (2.11) by δ and we take $\delta\to 0$ to obtain that v^1 must solve the algebraic equation

$$\left(\Delta_M\,\mathrm{v}^0 \,+\, |A_M|^2\,\mathrm{v}^0\right) \,+\, D_{\mathrm{v}}\left(\mathbf{C}\cdot e^{-\sqrt{2}\,\mathrm{v}}\right)_{\mathrm{v}=\mathrm{v}^0}\cdot \mathrm{v}^1 \,=\, 0. \tag{2.15}$$

Observe that

$$D_{\mathrm{v}}\left(\mathbf{C}\cdot e^{-\sqrt{2}\,\mathrm{v}}\right)_{\mathrm{v}=\mathrm{v}^0} = -\sqrt{2}\,|A_M|^2\,\mathbf{C}\cdot diag\left(\frac{(m-j)j}{2}\right)_{(m-1)\times(m-1)} \tag{2.16}$$

$$= \sqrt{2}\,|A_M|^2\begin{pmatrix} -2a_1 & a_2 & \ldots & \ldots & 0 & 0\\ a_1 & -2a_2 & \ldots & \ldots & 0 & 0\\ 0 & a_2 & 2a_3 & \ldots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots & 0\\ 0 & 0 & \ldots & a_{m-3} & -2a_{m-2} & a_{m-1}\\ 0 & 0 & \ldots & 0 & a_{m-2} & -2a_{m-1}\end{pmatrix}$$

where

$$a_j = \frac{(m-j)j}{2}, \quad j = 1, \dots, m-1.$$

It follows that

$$-\mathbf{C} \cdot diag\left(\frac{(m-j)j}{2}\right)_{(m-1)\times(m-1)} \mathbb{1} \;=\; \mathbb{1}.$$

From this, we find that v^1 is given by

$$\sqrt{2}\,\mathbf{C} \cdot diag\left(\frac{(m-j)j}{2}\right)_{m-1} \mathrm{v}^1 \;=\; \left(2\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} + \mathrm{v}^0\right)$$

from where

$$\mathrm{v}^1 \;=\; -\sqrt{2}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} + \frac{1}{\sqrt{2}}\left[\mathbf{C} \cdot diag\left(\frac{(m-j)j}{2}\right)_{(m-1)\times(m-1)}\right]^{-1} \mathrm{v}^0$$

$$\mathrm{v}^1 = -\sqrt{2}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} - \frac{1}{2}\log\left(|A_M|^{-2}\right)\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} + c_1 \tag{2.17}$$

for some constant vector c_1. Thus, we have obtained that

$$\omega(\mathrm{y},\delta) \;=\; \frac{1}{\sqrt{2}}\left(1 - \frac{\delta}{\sqrt{2}}\right)\log\left(|A_M(\mathrm{y})|^{-2}\right)\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} + c_0 + \delta\, c_1$$

and observe that v^0 and v^1 were chosen in such a way that

$$E(w,\delta,\mathrm{y}) \;=\; \delta^2\,\left(\Delta_M\,\mathrm{v}_1 \;+\; |A_M|^2\,\mathrm{v}_1\right)$$

$$+\,\mathbf{C} \cdot \left[e^{-\sqrt{2}\,(\mathrm{v}^0+\delta\mathrm{v}^1)} - e^{-\sqrt{2}\,\mathrm{v}^0} - \delta\,D_{\mathrm{v}}\left(e^{-\sqrt{2}\mathrm{v}}\right)_{\mathrm{v}=\mathrm{v}^0}\delta\mathrm{v}^1\right]. \tag{2.18}$$

Hence, from (2.13), (2.17) and (2.18) and a direct computation we get the pointwise estimate in M

$$|E(w,\delta,\mathrm{y})| \;\leq\; C\delta^2|A_M|^{2(1-\delta)}\left[1 + |\log\left(|A_M|^2\right)| + \mathcal{O}(|\log\left(|A_M|^2\right)|^2)\right]$$

and consequently, for any $1 < p \leq \infty$ and any μ such that $2 < \mu < 4 - 4\delta$, we obtain that

$$\|E(w,\delta)\|_{p,\mu} \leq C\,\delta^2. \tag{2.19}$$

To verify this last claim, notice that $|A_M|^2 \sim \mathcal{O}(r(\mathrm{y})^{-4})$ and that

$$\left|e^{-\sqrt{2}\,(\mathrm{v}^0+\delta\mathrm{v}^1)} - e^{-\sqrt{2}\,\mathrm{v}^0} - \delta\,D_{\mathrm{v}}\left(e^{-\sqrt{2}\mathrm{v}}\right)_{\mathrm{v}=\mathrm{v}^0}\delta\mathrm{v}^1\right| \leq C\,\delta^2|A_M|^2|\mathrm{v}^1|^2$$

since we can use Taylor expansion, up to second derivatives, in the region of M where

$$\delta\,\log(|A_M|^2) \leq K_1$$

for some K_1 independent of δ and y. This actually occurs in the large region determined by

$$r(\mathrm{y}) \le e^{\frac{K_1}{4\delta}}, \quad \mathrm{y} \in M$$

while in the remaining part of M, we use the fast decay of $|A_M|^2$ to get that

$$\left| e^{-\sqrt{2}\,(\mathrm{v}^0+\delta \mathrm{v}^1)} - e^{-\sqrt{2}\,\mathrm{v}^0} - \delta\, D_{\mathrm{v}} \left(e^{-\sqrt{2}\mathrm{v}} \right)_{\mathrm{v}=\mathrm{v}^0} \delta \mathrm{v}^1 \right| \le C\, |A_M|^2 e^{\delta \log(r^4(\mathrm{y}))}.$$

$$\le C\, r(\mathrm{y})^{-2-\beta}\, e^{(-2+\beta+4\delta)\frac{K_1}{4\delta}} \le r(\mathrm{y})^{-2-\beta} e^{-\frac{c_1}{\delta}}$$

which is exponentially small in δ, provided that we choose β so that $2 < \mu < 4 - 4\delta$. Clearly, (2.19) follows at once from these remarks.

Next, we linearize system (2.7) around the approximate solution $\omega(\mathrm{y}, \delta)$, we have found above. Let us recall that

$$\omega(\mathrm{y}, \delta) \;=\; \frac{1}{\sqrt{2}} \left(1 - \frac{\delta}{\sqrt{2}} \right) \log\left(|A_M(\mathrm{y})|^{-2} \right) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + c_0 + \delta\, c_1 \tag{2.20}$$

and as stated above we look for a solution to (2.7) of the form

$$\mathrm{v} = \omega + \zeta.$$

Linearizing $E(\omega + \zeta, \delta, \mathrm{y})$ around $\omega(\mathrm{y}, \delta)$, we find that ζ must solve the system

$$\delta\, \left(\Delta_M \zeta + |A_M|^2\, \zeta \right) + D_{\mathrm{v}} \left[\mathbf{C} \cdot e^{-\sqrt{2}\,\mathrm{v}} \right]_{\mathrm{v}=\omega} \zeta \;=$$

$$-\; E(\omega, \delta) - \left(\mathbf{C} \cdot e^{-\sqrt{2}(\omega+\zeta)} - \mathbf{C} \cdot e^{-\sqrt{2}\,\omega} - D_{\mathrm{v}} \left[\mathbf{C} \cdot e^{-\sqrt{2}\,\mathrm{v}} \right]_{\mathrm{v}=\omega} \zeta \right) \;+\; \delta\, q, \quad \text{in } M. \tag{2.21}$$

Let us observe that

$$D_{\mathrm{v}} \left[\mathbf{C} \cdot e^{-\sqrt{2}\,\mathrm{v}} \right]_{\mathrm{v}=\mathrm{v}^0+\delta \mathrm{v}^1} = D_{\mathrm{v}} \left[\mathbf{C} \cdot e^{-\sqrt{2}\,\mathrm{v}} \right]_{\mathrm{v}=\mathrm{v}^0}$$
$$+\, \mathbf{C} \cdot \left(\left[D_{\mathrm{v}} e^{-\sqrt{2}\,\mathrm{v}} \right]_{\mathrm{v}=\mathrm{v}^0+\delta \mathrm{v}^1} - D_{\mathrm{v}} \left[e^{-\sqrt{2}\,\mathrm{v}} \right]_{\mathrm{v}=\mathrm{v}^0} \right).$$

Using (2.16) and that

$$\left\| \mathbf{C} \cdot \left(\left[D_{\mathrm{v}} e^{-\sqrt{2}\,\mathrm{v}} \right]_{\mathrm{v}=\mathrm{v}^0+\delta \mathrm{v}^1} - D_{\mathrm{v}} \left[e^{-\sqrt{2}\,\mathrm{v}} \right]_{\mathrm{v}=\mathrm{v}^0} \right) \right\|_{\infty,\mu} \le C\delta \tag{2.22}$$

we can write system (2.21) as

$$\mathcal{L}_\delta(\zeta) \;=\; -\, E(\omega, \delta) \;-\; \mathrm{Q}(\omega, \zeta) + \delta\, q, \quad \text{in } M. \tag{2.23}$$

where

$$\mathcal{L}_\delta(\zeta) \;:=\; \delta\; \left(\Delta_M \zeta + |A_M|^2\, \zeta \right) - \sqrt{2} |A_M|^2 \mathbf{C} \cdot A(\mathrm{y}, \delta)\, \zeta$$

$$A(\mathrm{y},0) := diag\left(\frac{(m-j)j}{2}\right)_{(m-1)\times(m-1)}$$

$$\|\,|A_M|^2\,(A(\cdot,\delta)-A(\cdot,0))\,\|_{\infty,\mu} \le C\,\delta, \quad 2<\mu\le 4-4\delta$$

and

$$\mathrm{Q}(\omega,\zeta) := \mathbf{C}\cdot e^{-\sqrt{2}(\omega+\zeta)} - \mathbf{C}\cdot e^{-\sqrt{2}\,\omega} - D_{\mathrm{v}}\left[\mathbf{C}\cdot e^{-\sqrt{2}\,\mathrm{v}}\right]_{\mathrm{v}=\omega}\zeta.$$

Hence, we need solvability theory for the linear equation

$$\mathcal{L}_\delta(\zeta) = q, \quad ,\text{in } M \tag{2.24}$$

in the class of axially symmetric even functions. The following proposition provides this suitable linear theory needed to solve system (2.23)

Proposition 2. *For every $\delta>0$ small enough and any given vector function q with*

$$\|\,q\,\|_{p,\mu} < \infty$$

for $2\le p\le\infty$ and $\mu>\frac{5}{2}$, there exists a unique solution ζ to system (2.24) *satisfying the estimate*

$$\|\zeta\|_{*,\delta} \le C\delta^{-\frac{3}{4}}\,\|q\|_{p,\mu} \tag{2.25}$$

where

$$\|\zeta\|_{*,\delta} := \delta\|D^2\zeta\|_{p,\beta} + \delta^{\frac{1}{4}}\|(1+r(y))\,D\,\zeta\|_{L^\infty(M)} + \|\log(\,r(y)+2\,)^{-1}\,\zeta\|_{L^\infty(M)}. \tag{2.26}$$

We remark that the constant $C>0$ in proposition 2 does not depend on δ but rather on $\mu>\frac{5}{2}$. We provide the proof of this result also in section 2.4.

With the aid of proposition 2 we can recast system(2.23) as a fixed point problem. For a given vector function q with

$$\|\,q\,\|_{p,\mu} < \infty, \quad 2\le p\le\infty, \quad \frac{5}{2}<\mu$$

let us denote $\zeta = T_\delta(q)$, the solution given in proposition 2, and let us write

$$R(\zeta) := T_\delta^{-1}\left[\delta\,q - E(\omega,\delta) - \mathrm{Q}(\omega,\zeta)\right]$$

so that (2.23) becomes the fixed point problem

$$\zeta = R(\zeta)$$

posed in the Banach space X of smooth vector functions ζ for which

$$\|\zeta\|_X := \|\zeta\|_{*,\delta} < \infty.$$

From (2.19), we obtain that

$$\|\,T_\delta[\,E(\omega,\delta)\,]\,\|_X \le C\,\delta^{\frac{5}{4}}.$$

On the other hand, proceeding as did to verify (2.19), for any $\frac{5}{2}<\mu<4-4\delta$, any $p\ge 2$ and any ζ such that

$$\|\zeta\|_X \le C\delta^{\frac{5}{4}} \tag{2.27}$$

we obtain that

$$\begin{aligned}\| T_\delta[\mathrm{Q}(\omega,\mu)] \|_X &\le C\delta^{-\frac{3}{4}} \| \mathrm{Q}(\omega,\zeta) \|_{p,\mu} \\ &\le C\delta^{-\frac{3}{4}} \left(\|\zeta\|_X^2 + \|(\log(r(\mathrm{y})+2))^{-1}\zeta\|_\infty \, \|(1+r(\mathrm{y}))D\zeta\|_\infty \right) \\ &= \mathcal{O}(\delta^{\frac{3}{2}}).\end{aligned}$$

This follows from the fact that

$$\|(1+r(\mathrm{y}))D\zeta\|_{L^\infty(M)} \le C\delta.$$

Finally, we check on the Lipschitz character of $\mathrm{Q}(\omega,\zeta)$, respect to ζ, we simply observe that for ζ_1, ζ_2 satisfying (2.27), we have

$$\mathrm{Q}(\omega,\zeta_1) - \mathrm{Q}(\omega\zeta_2) =$$

$$\mathbf{C} \cdot \left[e^{\sqrt{2}(\omega+\zeta_1)} - e^{\sqrt{2}(\omega+\zeta_2)} - D_{\mathrm{v}}(e^{-\sqrt{2}\,\mathrm{v}})_{\mathrm{v}=\omega}(\zeta_1-\zeta_2) \right] \mathcal{O}(|A_M|^{2(1-\delta)})$$

From this and proceeding again as in (2.19), we obtain that

$$\|\mathrm{Q}(\omega,\zeta_1) - \mathrm{Q}(\omega,\zeta_2)\|_{\infty,\mu} \le C\,\delta \|\zeta_1-\zeta_2\|_X. \tag{2.28}$$

This implies that

$$\|R(\zeta_1) - R(\zeta_2)\|_X \le C\delta^{-\frac{3}{4}} \|\mathrm{Q}(\omega,\zeta_1) - \mathrm{Q}(\omega,\zeta_2)\|_{\infty,\mu} \le C\,\delta^{\frac{1}{4}} \|\zeta_1-\zeta_2\|_X.$$

Hence a direct application of contraction mapping principle gives us the following proposition.

Proposition 3. *For every $\delta > 0$ small enough and every vector function q such that for $2 \le p \le \infty$ and μ such that $\frac{5}{2} < \mu < 3$ and $C > 0$ independent of δ*

$$\| q \|_{p,\mu} \le C\delta$$

there exists a unique axially symmetric even solution ζ to the system

$$\mathcal{L}_\delta(\zeta) \,=\, -\,E(\omega,\delta) \,-\, Q(\omega,\zeta) + \delta\, q, \quad \text{in } M$$

satisfying that

$$\|\zeta\|_{*,\delta} \le \widetilde{K}\,\delta^{\frac{5}{4}}$$

and

$$\|\zeta_1-\zeta_2\|_{*,\delta} \le \widetilde{K}\delta^{\frac{1}{4}} \|q_1-q_2\|_{p,\mu}$$

where we recall that

$$\|\zeta\|_{*,\delta} := \delta\|D^2\zeta\|_{p,\beta} \,+\, \delta^{\frac{1}{2}} \|(1+r(y))\,D\,\zeta\|_{L^\infty(M)} \,+\, \|\log(\,r(y)+2\,)^{-1}\,\zeta\|_{L^\infty(M)}.$$

2.4 The Jacobi operator and the linear Jacobi-Toda operator on the Catenoid.

This section is devoted to prove propositions 1 and 2. First, we develop solvability theory for the equation

$$\mathcal{J}_M(v) = \Delta_M v + |A_M|^2 v = q, \quad \text{in } M, \tag{2.1}$$

Operator $\mathcal{J}_M$ in equation (2.1) corresponds to the jacobi operator of the catenoid. We study this equation for functions v depending only on the arch-length variable of the catenary.

It is well known that the catenoid M is L^∞-nondegenerate, in the sense that the only bounded solutions to the equation

$$\mathcal{J}_M(v) = \Delta_M v + |A_M|^2 v = 0, \quad \text{in } M,$$

are the functions $z_i = \nu \cdot e_i$, for $i = 1, 2, 3$, where e_1, e_2, e_3 corresponds to the canonical basis in $\mathbb{R}^3$.

The functions z_1, z_2, z_3 corresponds to the bounded jacobi fields of the catenoid arising from translations. One can check directly that, among these bounded jacobi fields of M, $z_3(\mathrm{y})$ is the only one that is axially symmetric. We notice that, in the coordinates $\mathrm{y} = Y(y, \theta)$, z_3 has the explicit expression

$$z_3(y) = \frac{y}{\sqrt{1+y^2}}, \quad y \in \mathbb{R}.$$

One can easily find a logarithmic jacobi field with logarithmic growth, associated to the dilation of the catenoid M, namely

$$z_4(y) := Y(y, \theta) \cdot \nu(y, \theta), \quad y \in \mathbb{R}.$$

One can also deduce the existence of z_4, using the reduction of order formula with the ansatz

$$z_4(y) = 1 + s(y) z_3(y), \quad y \neq 0.$$

Either way, we find that

$$z_4(y) = 1 - \ln(y + \sqrt{1+y^2}) \frac{y}{\sqrt{1+y^2}}, \quad \text{in } \mathrm{y} \in \mathbb{R}.$$

We compute the derivatives of z_3 and z_4, respect to y, so we get

$$\partial_y \, z_3'(y) \; = \; -\frac{1}{(1+y^2)^{\frac{3}{2}}} \; = \; \mathcal{O}\left(|y|^{-3}\right) \tag{2.2}$$

$$\partial_y \, z_4'(y) \; = \; -\ln\left(y + \sqrt{1+y^2}\right)\left(1+y^2\right)^{-\frac{3}{2}} - \frac{\mathrm{y}}{1+y^2} = \mathcal{O}(|y|^{-1}). \tag{2.3}$$

Using the function z_3, z_4 and the variations of parameters formula, we can set one inverse to the equation (2.1) as follows. For any function q satisfying that

$$\|q\|_{p,\mu} := \|(1 + r(\mathrm{y})^\mu) q\|_{L^p(M)} < \infty$$

we define $\mathcal{J}^{-1}(q) := v$, where

$$v(y) := -z_3(y)\int_0^y \sqrt{1+\xi^2}q(\xi)z_4(\xi)d\xi + z_4(y)\int_{-\infty}^y \sqrt{1+\xi^2}q(\xi)z_3(\xi)d\xi. \tag{2.4}$$

Formula (2.4) defines a function v that solves equation (2.1). We next prove that, under the orthogonality condition

$$\int_{-\infty}^{\infty} \sqrt{1+\xi^2}q(\xi)z_3(\xi)d\xi = 0 \tag{2.5}$$

this solution is unique in the class of bounded functions with $v'(0) = 0$ and the following lemma gives us an estimate on the size of $\mathcal{J}^{-1}$.

Lemma 2.4.1. *Let q be an axially symmetric function satisfying condition (2.5), and such that $\|q\|_{p,\mu} < \infty$, for $2 \le p \le \infty$ and $2 < \mu < 3$. Then, the function v, given by formula (2.4), defines an axially symmetric solution to*

$$\Delta_M v + +|A_M|^2 v = q, \quad in\ M,$$

such that $v'(0) = 0$ and the following estimate holds true

$$\|v\|_{2,p,\mu} \le C\,\|q\|_{p,\mu} \tag{2.6}$$

where

$$\|v\|_{2,p,\mu} := \|v\|_{L^\infty(M)} + \|r^{\mu-1}(y)\nabla v\|_{L^\infty(M)} + \|D^2 v\|_{p,\mu}.$$

Proof. Take p, $p' > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. From Hölder inequality, we see that

$$\left|\int_0^y \sqrt{1+\xi^2}\,z_4(\xi)\,q(\xi)\,d\xi\right| \le \int_0^y \sqrt{1+\xi^2}\,|z_4(\xi)|\,|q(\xi)|\,d\xi \le C\,\|q\|_{p,\mu}\left(\int_0^y (1+|\xi|)^{(1-\mu p')}|z_4(\xi)|^{p'}d\xi\right)^{\frac{1}{p'}}.$$

Since, for any $\kappa_1, \kappa_2 > 0$ there exists a constant $C = C(\kappa_1,\kappa_2) > 0$, such that for every $y > 1$ we have that

$$\int_0^y (1+|\xi|)^{-\kappa_1}|z_4(\xi)|^{\kappa_2}d\xi \le C\left[1 + \int_1^y \xi^{-\kappa_1}\ln^{\kappa_2}(\xi)d\xi\right],$$

and $\ln(\xi)/\xi^{\frac{\kappa_1}{\kappa_2}} \to 0$, as $\xi \to \infty$, we obtain that for $p > 1$, $\mu > 2\left(1 - \frac{1}{p}\right)$ and some $\varepsilon > 0$ small enough such that $2 - \mu p' + \varepsilon < 0$, we obtain that

$$\begin{aligned}\left(\int_0^y (1+|\xi|)^{(1-\mu p')}|z_4(\xi)|^{p'}d\xi\right)^{\frac{1}{p'}} &\le C\left(\int_0^y (1+|\xi|)^{(1+\varepsilon-\mu p')}d\xi\right)^{\frac{1}{p'}}\\ &\le C\left(\int_0^\infty (1+|\xi|)^{(1+\varepsilon-\mu p')}d\xi\right)^{\frac{1}{p'}} < \infty.\end{aligned}$$

We conclude, changing C to a larger one if necessary, that

$$\left|\int_0^y \sqrt{1+\xi^2}\,z_4(\xi)\,q(\xi)\,d\xi\right| \le C\,\|q\|_{p,\mu}, \quad y \in \mathbb{R}$$

where the constant C clearly depends only on p and μ.

On the other hand, we observe that from the orthogonality condition (2.5) we have that

$$\int_y^\infty \sqrt{1+\xi^2}\, z_3(\xi)\, q(\xi)\, d\xi \;=\; -\int_{-\infty}^y \sqrt{1+\xi^2}\, z_3(\xi)\, q(\xi)\, d\xi$$

Hence, we estimate the integral

$$\int_y^\infty \sqrt{1+\xi^2} z_3(\xi) q(\xi) d\xi$$

only for positive values of the argument y. Observe that, proceeding as above and since we are assuming that $\mu p' > 2(1-\frac{1}{p})+1$, we obtain that

$$\begin{aligned}
\left|\int_y^\infty \sqrt{1+\xi^2}\, z_3(\xi)\, q(\xi)\, d\xi\right| &\leq C\, \|q\|_{p,\mu} \left(\int_y^\infty (1+|\xi|)^{(1-\mu p')} d\xi\right)^{\frac{1}{p'}} \\
&\leq C\, \|q\|_{p,\mu}\, (1+|y|)^{\frac{2-\mu p'}{p'}} \\
&\leq C\, \|q\|_{p,\mu}\, (1+|y|)^{2-\mu}.
\end{aligned}$$

From (2.2)-(2.3) and the estimates above, we conclude directly from formula (2.4) that

$$\|v\|_{L^\infty(M)} + \|(1+r(\mathrm{y})^{\mu-1})Dv\|_{L^\infty(M)} \leq C\|q\|_{p,\mu}.$$

To get the whole estimate, we simply notice that, since for $\mathrm{y} = Y(y,\theta)$ we have that $|A_M(\mathrm{y})|^2 \sim \mathcal{O}(|y|^{-4})$ as $|y| \to \infty$, we obtain that

$$\int_M (1+|\mathrm{y}|^\mu)^p\, |A_M(\mathrm{y})|^{2p}\, |v(\mathrm{y})|^p\, dV_M \leq C\|v\|^p_{L^\infty(M)} \int_M (1+r(\mathrm{y}))^{(\mu-4)p}.$$

Since $(\mu-4)p < -2$, we obtain that

$$\|\, |A_M|^2 v\, \|_{p,\mu} \leq C\, \|v\|_{L^\infty(M)} \leq C\, \|q\|_{p,\mu}.$$

Finally, to get the whole estimate, we see from the equation that

$$\Delta_M v = -|A_M|^2\, v + q, \quad \text{in } M$$

and so,

$$\|D^2 v\|_{p,\mu} \leq C\left(\|\, |A_M|^2 v\, \|_{p,\mu} \;+\; \|q\|_{p,\mu}\right) \leq C\, \|q\|_{p,\mu}.$$

This completes the proof of the estimate. □

Remark 2.4.1: To prove lemma 2.4.1, we simply notice that an even axially symmetric function q in $L^1(M)$, automatically satisfies the orthogonality condition (2.5). In such a case, formula (2.4) defines an even function.

Finally, in order to solve (2.23), we study the first the linear system

$$\delta\Delta_M\, \zeta \;+\; |A_M|^2(-\sqrt{2}\mathbf{C}\cdot A(\mathrm{y},0) + \delta\, \mathbf{I})\, \zeta \;=\; q, \quad \text{in } M \tag{2.7}$$

where we recall that

$$A(\mathrm{y},0) \;=\; diag\left(\frac{(m-j)j}{2}\right)_{(m-1)\times(m-1)}.$$

A direct computation shows that the numbers

$$1, \frac{1}{2}, \dots, \frac{m-1}{m}$$

are the $m-1$ eigenvalues of the matrix $-\mathbf{C}$, we observe that $-\mathbf{C}$ is symmetric and positive definite. Let us then, write

$$\zeta = [-\mathbf{C}]^{\frac{1}{2}}\psi, \quad q = [-\mathbf{C}]^{\frac{1}{2}}\tilde{q}.$$

We see that (2.7) becomes

$$\delta\Delta_M\,\psi \;+\; |A_M|^2(\delta\mathbf{I} \;+\; \mathtt{B})\,\psi \;=\; \tilde{q}, \quad \text{in } M. \tag{2.8}$$

where the matrix $\mathtt{B}$ is given by

$$\mathtt{B} \;=\; \frac{1}{\sqrt{2}}[-\mathbf{C}]^{\frac{1}{2}} diag\,((m-j)j)_{(m-1)\times(m-1)}\,[-\mathbf{C}]^{\frac{1}{2}}.$$

Next, we consider the eigenvectors $e_1, \dots, e_{m-1}$, of the matrix $\mathtt{B}$, i,e

$$\mathtt{B}\cdot e_i \;=\; \lambda_i \mathrm{e}_i, \quad i = 1, \dots, m-1$$

and we write

$$\psi \;=\; \sum_{i=1}^{m-1}\psi_i\,e_i, \quad \tilde{q} \;=\; \sum_{i=1}^{m-1}\tilde{q}_i\,e_i.$$

Hence, system (2.7) decouples into $m-1$ scalar equations, namley

$$\delta\Delta_M\,\psi_i \;+\; |A_M|^2(\lambda_i + \delta)\,\psi_i \;=\; \tilde{q}_i, \quad \text{in } M, \quad i = 1, \dots, m-1. \tag{2.9}$$

Of vital importance is the fact that the eigenvalues $\lambda_1, \dots, \lambda_{m-1}$ are indeed positive, a fact that makes invertibility of each equation in (2.9) a very delicate matter.

Hence, we study solvability theory for the model linear equation

$$L_\delta(\psi) \;=\; q, \quad \text{in } M \tag{2.10}$$

$$L_\delta(\psi) \;:=\; \delta\,\Delta_M\psi + \lambda|A_M|^2\,\psi, \quad \lambda > 0. \tag{2.11}$$

Since we are working in a symmetric class, we use the variations of parameters formula, for which precise information on the kernel of (2.11) is needed. We assume also without any lose of generality that $\lambda = 1$.

We study smooth axially symmetric solutions to the equation

$$L_\delta\,\psi \;=\; 0, \quad \text{in } M \tag{2.12}$$

So, we can thought of ψ as a function of the arc-length i.e for $\mathrm{y} \in M$

$$\psi(\mathrm{y}) = \psi(y), \quad \mathrm{y} = Y(y,\theta), \quad y \in \mathbb{R}$$

for which the (2.12) corresponds to the ODE

$$\delta\left(\psi''(y) + \frac{y}{1+y^2}\,\psi'(y)\right) + \frac{2}{(1+y^2)^2}\,\psi(y) = 0, \quad y \in \mathbb{R}.$$

We analyze this operator in a region where the operator is oscillatory in character, an effect coming from δ, and on another region where the small parameter δ has no effect at all so the operator resembles the euclidean radial Laplacian.

First, we denote $y_\delta > 0$ the real number such that $\sqrt{1+y_\delta^2} = \frac{1}{\sqrt{\delta}}$ and we pass to the sphere S^2 by making the change of variables

$$y = \tan(\theta), \quad \text{for } 0 < \theta < \theta_\delta$$

where the number $\theta < \theta_\delta$ is such that $y_\delta = \tan(\theta_\delta)$, $0 < \theta_\delta < \frac{\pi}{2}$. Next, we look for a solution to (2.12) with the form

$$\psi(y) \;=\; \varphi(\theta), \quad \text{for } 0 < \theta < \theta_\delta$$

so that the function φ solves the equation

$$\partial_{\theta\theta}\,\varphi(\theta) - \tan(\theta)\,\partial_\theta\,\varphi(\theta) + \frac{2}{\delta}\,\varphi(\theta) = 0. \tag{2.13}$$

In order to eliminate first derivative term in (2.13), we assume further that

$$\psi(y) = \varphi(\theta) = \frac{1}{\sqrt{\cos(\theta)}}\,\phi(\theta).$$

Hence, we find that ϕ must solve the ODE

$$\partial_{\theta\theta}\,\phi(\theta) + \left(\frac{2}{\delta} + \frac{1}{4}\left(1+\sec^2(\theta)\right)\right)\phi(\theta) = 0. \tag{2.14}$$

Finally, we make one more assumption on the form of ψ, namely

$$\psi(y) = \frac{1}{\sqrt{\cos(\theta)}}\,\phi(\theta) = \frac{1}{\sqrt{\cos(\theta)}}\,\gamma\left(\frac{\theta}{\sqrt{\delta}}\right), \quad \text{for } 0 < \theta < \theta_\delta. \tag{2.15}$$

Scaling equation (2.14), we obtain that $\gamma = \gamma(s)$, solves

$$\partial_{ss}\,\gamma(s) + \left(\left[1+\frac{\delta}{4}\right] + \frac{\delta}{4}\sec^2(\sqrt{\delta}\,s)\right)\gamma(s) = 0, \quad \text{for } 0 < s < \frac{\theta_\delta}{\sqrt{\delta}}.$$

Now, let us prove that $\gamma(s)$ and $\partial_s\,\gamma(s)$ are uniformly bounded. To do so, we consider the pointwise energy

$$J(s) := |\partial_s\,\gamma(s)|^2 + \left[1+\frac{\delta}{4}\right]|\gamma(s)|^2.$$

Observe that

$$\partial_s\,J(s) = -2\partial_s\,\gamma(s)\,\gamma(s)\,\frac{\delta}{4}\sec^2(\sqrt{\delta}\,s).$$

Hence, for constant $C > 0$ independent of $\delta > 0$, it follows that

$$|\partial_s J(s)| \leq C\, J(s)\, \frac{\delta}{4} \sec^2(\sqrt{\delta}\, s)$$

and consequently

$$0 \leq J(s) \leq J(0) + C\, \frac{\delta}{4} \int_0^s J(\xi)\, \sec^2(\sqrt{\delta}\, \xi)\, d\xi, \quad \text{for } 0 < s < s_\delta$$

where we have set s_δ by $\theta_\delta = \sqrt{\delta}\, s_\delta$. Using Gronwall's inequality, we find that

$$J(s) \leq J(0) \exp\left(C\, \frac{\delta}{4} \int_0^{s_\delta} \sec^2(\sqrt{\delta}\, \xi)\, d\xi \right). \tag{2.16}$$

We compute explicitly the integral in (2.16) to find that

$$\frac{\delta}{4} \int_0^{s_\delta} \sec^2(\sqrt{\delta}\, \xi)\, d\xi = \frac{\sqrt{\delta}}{4} \tan(\sqrt{\delta}\, s_\delta) = \frac{\sqrt{\delta}}{4} \tan(\theta_\delta) \leq c_0$$

where c_0 does not depend on $\delta > 0$. Hence we find that

$$J(s) := |\partial_s\, \gamma(s)|^2 + \left[1 + \frac{\delta}{4} \right] |\gamma(s)|^2 \leq C\, J(0), \quad 0 < s < \frac{\theta_\delta}{\sqrt{\delta}}.$$

Pulling back the change of variables, we obtain from (2.15) that

$$|\psi(y)| \leq C \frac{1}{\sqrt{\cos(\theta)}} = C \left(1 + y^2 \right)^{\frac{1}{4}} \leq \frac{C}{\delta^{\frac{1}{4}}}, \quad \text{for } 0 \leq \sqrt{1 + y^2} \leq \frac{1}{\sqrt{\delta}}. \tag{2.17}$$

In particular we find from (2.15) that $\psi(0) = \gamma(0)$. As for the derivatives of ψ, we compute explicitly again from (2.15), to find that

$$(1 + y^2)\, \partial_y\, \psi(y) = \frac{1}{\sqrt{\delta}} \frac{\partial_s\, \gamma\left(\frac{\theta}{\sqrt{\delta}}\right)}{\sqrt{\cos(\theta)}} + \frac{\sin(\theta)\, \gamma\left(\frac{\theta}{\sqrt{\delta}}\right)}{2\, \cos^{\frac{3}{2}}(\theta)}. \tag{2.18}$$

This implies that

$$(1 + y^2)\, |\partial_y\, \psi(y)|\, \leq\, \frac{C\, (1 + y^2)^{\frac{1}{4}}}{\sqrt{\delta}}\, (1 + |y|)\, . \tag{2.19}$$

In particular, we obtain from (2.18) that $\partial_y\, \psi(0) = \delta^{-\frac{1}{2}}\, \partial_s\, \gamma(0)$. We can take fundamental set of (2.12), $\psi_{1,1}(y)$, and $\psi_{1,2}(y)$, satisfying (2.17)-(2.19) and with wronski determinant given by

$$W(\psi_{1,1}, \psi_{1,2}) = \frac{\delta^{-\frac{1}{2}}}{\sqrt{1 + y^2}}, \quad \text{for } 0 < \sqrt{1 + y^2} \leq \frac{1}{\sqrt{\delta}}.$$

Next, in this inner region, we solve (2.10) by choosing $\psi_1(y)$ to be defined by the formula

$$\psi_1(y) = -\frac{1}{\sqrt{\delta}}\, \psi_{1,1}(y) \int_0^y \sqrt{1 + \xi^2}\, \psi_{1,2}(\xi)\, q(\xi) d\xi + \frac{1}{\sqrt{\delta}}\, \psi_{1,2}(y) \int_0^y \sqrt{1 + \xi^2}\, \psi_{1,1}(\xi)\, q(\xi) d\xi \tag{2.20}$$

Proceeding as in lemma (2.4.1), we observe that for $2 \le p \le \infty$ and $\mu > \frac{5}{2}$

$$\left|\int_0^y \sqrt{1+\xi^2}\,\psi_{1,i}(\xi)\,q(\xi)\,d\xi\right| \le \int_0^y \sqrt{1+\xi^2}\,|\psi_{1,i}(\xi)|\,|q(\xi)|\,d\xi \le C\,\|q\|_{p,\mu}\left(\int_0^y (1+|\xi|)^{(1+\frac{p'}{2}-\mu p')}d\xi\right)^{\frac{1}{p'}}.$$

Directly from this inequality and using (2.17)-(2.19), we find that

$$\left|\psi_{1,i}(y)\int_0^y \sqrt{1+\xi^2}\psi_{1,j}(\xi)\,q(\xi)d\xi\right| \le C\delta^{-\frac{1}{4}}\|q\|_{p,\mu}, \quad i,j=1,2, \quad i\neq j.$$

and consequently, since we are taking $\mu > \frac{5}{2}$, we get that

$$\delta^{\frac{1}{4}}\sqrt{1+y^2}\,|\psi_1'(y)| \,+\, |\psi_1(y)| \le C\delta^{-\frac{3}{4}}\,\|q\|_{p,\mu}, \quad |y| \le \frac{c_0}{\sqrt{\delta}}.$$

In particular, we observe that

$$\delta^{-\frac{1}{2}}\|(1+r(\mathrm{y}))D\psi_1\|_{L^\infty(M_\delta)} + \|\psi_1\|_{L^\infty(M_\delta)} \le C\delta^{-\frac{3}{4}}\|q\|_{p,\mu} \tag{2.21}$$

where

$$M_\delta := \left\{\mathrm{y} = Y(y,\theta) : \sqrt{1+y^2} \le \frac{c_0}{\sqrt{\delta}}, \quad \theta \in (0,2\pi)\right\}.$$

Concerning the outer region, let us consider the change of variables $y = \sinh(t)$ and let us choose $T_\delta > 0$ so that $\delta\,\cosh^2(T_\delta) = 2$. Hence looking for solutions to (2.12) such that $\psi(y) = \phi(t)$, we see that the function φ must satisfy equation

$$\partial_{tt}\phi + p_\delta(t)\,\phi = 0, \quad p_\delta(t) := 2\delta^{-1}\,\mathrm{sech}^2(t) \quad t > T_\delta \tag{2.22}$$

we state the following lemma

Lemma 2.4.2. *The linear ODE has two linearly independent solutions, $\phi_1(t)$, $\phi_2(t)$, satisfying that*

$$\phi_1(t) = 1 + o(1), \quad \partial_t\phi_1(t) = o(1), \quad \textit{for } t > T_\delta \tag{2.23}$$

$$\phi_2(t) = t + \mathcal{O}(1), \quad \partial_t\phi_2(t) = 1 + o(1), \quad \textit{for } t > T_\delta \tag{2.24}$$

provided δ is small enough, which amounts to the fact that T_δ is large enough.

Proof. First let us look for a solution $\phi(t)$ to the equation having the form $\phi(t) = tv(t)$. Computing the equation for $v(t)$ we find that

$$\partial_t(t^2\,\partial_t v(t)) + p_\delta(t)t^2\,v(t) = 0.$$

Setting $z(t) = t^2\,\partial_t v(t)$, we obtain the first order system IVP for $z(t)$ and $v(t)$

$$\partial_t\,z(t) = -p_\delta(t)\,t^2 v(t), \quad \partial_t\,v(t) = \frac{1}{t^2}z(t), \quad z(T_\delta) = z_0, \quad v(T_\delta) = v_0.$$

Integrating each equation on the system, we find that

$$z(t) = z_0 - \int_{T_\delta}^t p_\delta(\tau)\tau^2 v(\tau)d\tau, \quad v(t) = v_0 + \int_{T_\delta}^t \frac{1}{\tau^2}z(\tau)d\tau.$$

Hence, using this integral formulas and Fubini's theorem, we obtain the integral representation for $z(t)$

$$z(t) = z_0 - v_0 \int_{T_\delta}^{t} p_\delta(\tau)\tau^2 d\tau - \int_{T_\delta}^{t} \frac{1}{\tau^2} z(\tau) \int_{\tau}^{t} p_\delta(s)s^2 ds\, d\tau$$

Next, we prove that $z(t)$ is bounded. First observe that

$$0 \leq \int_{T_\delta}^{t} p_\delta(\tau)\tau^2 d\tau \leq \int_{T_\delta}^{\infty} p_\delta(\tau)\tau^2 d\tau \leq C\delta^{-1} T_\delta^2 e^{-2T_\delta} \leq T_\delta^2$$

where $C > 0$ is independent of δ, provided $\delta > 0$ is small enough. On the other hand,

$$|z(t)| \leq C(|z_0| + \delta^{-1}|v_0|) + \int_{T_\delta}^{t} p_\delta(\tau)|z(\tau)|d\tau.$$

Directly from Gronwall inequality we obtain that,

$$|z(t)| \leq C(|z_0| + \delta^{-1}|v_0|) \exp\left(\int_{T_\delta}^{t} p_\delta(\tau)d\tau\right)$$

and since

$$\int_{T_\delta}^{\infty} p_\delta(\tau)d\tau \leq \frac{C}{\delta} e^{-2T_\delta}$$

then for T_δ large enough or equivalently, δ small enough, and taking $v_0 = 0$, we find that

$$|z(t)| \leq C|z_0|.$$

Plugging this into the integral formula for $z(t)$ we observe that

$$z(t) = z_0 + \int_{T_\delta}^{t} z(\tau)\frac{1}{\tau^2} \int_{\tau}^{t} p_\delta(s)s^2 ds\, d\tau.$$

Since $z(t)$ is bounded, we obtain that

$$z(\infty) = \lim_{t\to\infty} z(t) = z_0 + \int_{T_\delta}^{\infty} z(\tau)\frac{1}{\tau^2} \int_{\tau}^{\infty} p_\delta(s)s^2 ds\, d\tau$$

We write then, without any loss of generality

$$z(t) = 1 + \int_{t}^{\infty} z(\tau)\frac{1}{\tau^2} \int_{\tau}^{t} p_\delta(s)s^2 ds\, d\tau$$

from where we observe that

$$|z(t) - 1| \leq C\, p_\delta(t) \leq \frac{C}{\delta}\, e^{-2|t|}.$$

From the integral formula for $v(t)$ we obtain that

$$v(t) = v(\infty) + \int_{t}^{\infty} z(\tau)\frac{1}{\tau^2} d\tau = v(\infty) + \mathcal{O}\left(\frac{1}{t}\right).$$

So we observe that

$$\phi(t) = t + \mathcal{O}(1), \quad t > T_\delta, \quad \partial_t \phi(t) = v(t) + t\,\partial_t v(t) = 1 + \mathcal{O}\left(\frac{1}{t}\right).$$

Finally, from the reduction of order formula we find the second solution from $\phi(t)$, satisfying that

$$\widetilde{\phi}(t) = \phi(t)\int_t^\infty \frac{1}{\phi(\tau)^2}d\tau = 1 + \mathcal{O}\left(\frac{1}{t}\right), \quad \partial_t\widetilde{\phi}(t) = \partial_t\phi(t)\int_t^\infty \frac{1}{\phi(\tau)^2}d\tau + \frac{1}{\phi(t)} = \mathcal{O}\left(\frac{1}{t}\right).$$

This concludes the proof of the lemma. □

In order to find the exact behavior of the bounded solutions to equation (2.12) in the outer domain, we still need one more lemma.

Lemma 2.4.3. *Assume $\phi(t)$ is a bounded solution to the equation (2.4.2) for $t > T_\delta$, then, the following estimate holds true*

$$|\partial_t\phi(t)| \le C\|\phi\|_{L^\infty(T_\delta,\infty)}\, p_\delta(t), \quad t > T_\delta. \tag{2.25}$$

Proof. First observe that for $t > T_\delta$

$$\partial_t\phi(t) = \partial_t\phi(T_\delta) - \int_{T_\delta}^t p_\delta(\tau)\phi(\tau)d\tau.$$

Since $p_\delta(t)$ decays fast and $\phi(t)$ is uniformly bounded, $\partial_t\phi(\infty)$ exists, so we can actually write

$$\partial_t\phi(t) = \partial_t\phi(\infty) - \int_t^\infty p_\delta(\tau)\phi(\tau)d\tau.$$

Let us prove that $\partial_t\phi(\infty) = 0$. To see this, we simply integrate again to obtain

$$\phi(t) = \phi(T_\delta) \;+\; \partial_t\phi(\infty)(t - T_\delta) \;-\; \int_{T_\delta}^t\int_\tau^\infty p_\delta(s)\phi(s)ds\,d\tau.$$

Observe that

$$\left|\int_{T_\delta}^t\int_\tau^\infty p_\delta(s)\phi(s)ds\,d\tau\right| \le C\|\phi\|_{L^\infty(T_\delta,\infty)}\int_{T_\delta}^\infty p_\delta(\tau)d\tau \le C\|\phi\|_{L^\infty(T_\delta,\infty)}p_\delta(T_\delta).$$

Hence this estimate and the formula for $\phi(t)$ imply that $\partial_t\phi(\infty) = 0$. So we obtain that

$$\partial_t\phi(t) = -\int_t^\infty p_\delta(\tau)\phi(\tau)d\tau$$

from where

$$|\partial_t\phi(t)| \le C\|\phi\|_{L^\infty(T_\delta,\infty)}\, p_\delta(t), \quad \text{for } t > T_\delta.$$

□

Next, we solve (2.10) in remaining part of M, where it certainly resembles Poisson equation. Since we are working in the class of axially symmetric even functions, we must study the following IVP

$$L_\delta(\psi_2) = q, \quad \psi_2(y_\delta) = \psi_1(y_\delta), \quad \psi_2'(y_\delta) = \psi_1'(y_\delta), \quad \text{in } M - M_\delta \tag{2.26}$$

Hence, from lemmas (2.4.2) and (2.4.3) we can take two linearly independent elements of the kernel of $\mathcal{L}_\delta$ in $M - M_\delta$, say $\psi_{2,1}(y)$ and $\psi_{2,2}(y)$ satisfying that

$$\psi_{2,1}(y_\delta) = 1, \quad \partial_y \psi_{2,1}(y_\delta) = 0, \quad \psi_{2,2}(y_\delta) = 0, \quad \partial_y \psi_{2,2}(y_\delta) = \delta^{\frac{1}{4}}.$$

Observe that the wronskian of this fundamental set, $W := W(\psi_{2,1}, \psi_{2,2}) = \frac{\delta^{\frac{1}{4}}}{\sqrt{1+y^2}}$ and

$$|\psi_{2,i}(y)| \,+\, \delta(1+|y|)^3 |\partial_y \psi_{2,i}(y)| \le C\delta^{\frac{i-1}{4}} \ln\left(y + \sqrt{1+y^2}\right), \quad |y| \ge \frac{c_0}{\sqrt{\delta}}, \quad i = 1, 2.$$

Hence, we solve this problem by setting the variations of parameters formula as follows

$$\psi_2(y) \;=\; \psi_1(y_\delta)\psi_{2,1}(y) \,+\, \delta^{\frac{1}{4}}\psi_1'(y_\delta)\psi_{2,2}(y) +$$
$$-\,\delta^{-\frac{3}{4}}\,\psi_{2,1}(y)\int_{y_\delta}^{y} \sqrt{1+\xi^2}\,\psi_{2,2}(\xi)\,q(\xi)d\xi \,+\, \delta^{-\frac{3}{4}}\,\psi_{2,2}(y)\int_{y_\delta}^{y} \sqrt{1+\xi^2}\,\psi_{2,1}(\xi)\,q(\xi)d\xi.$$

Proceeding as above, we get that

$$\|(1+r(y))D\,\psi_2\|_{L^\infty(M-M_\delta)} + \|\log(2+|y|)^{-1}\,\psi_2\|_{L^\infty(M-M_\delta)} \le C\delta^{\frac{3}{4}}\|q\|_{p,\mu} \tag{2.27}$$

Writing $\psi = \chi_{M_\delta}\psi_1 + \chi_{M-M_\delta}\psi_2$ and putting together, estimate (2.21) and (2.27) we obtain that

$$\sqrt{\delta}\|(1+r(y))D\,\psi_2\|_{L^\infty(M)} + \|\log(2+|y|)^{-1}\,\psi_2\|_{L^\infty(M)} \le C\delta^{\frac{3}{4}}\|q\|_{p,\mu} \tag{2.28}$$

Finally, observe that for $2 \le p \le \infty$, $\mu < 3$ and some $\varepsilon > 0$ arbitrarily small, we have that

$$\int_M (1+|\mathrm{y}|^\mu)^p\,|A_M(\mathrm{y})|^{2p}\,|\psi(\mathrm{y})|^p\,dV_M \le C\|(\log(r(\mathrm{y})+2)^{-1}\,\psi\|_{L^\infty(M)}\int_M (1+|\mathrm{y}|)^{(\mu-4-\varepsilon)p}.$$

Since $(\mu-4)p < -2$, we obtain that

$$\|\,|A_M|^2\psi\,\|_{p,\mu} \le C\,\|(\log(r(\mathrm{y})+2)^{-1}\,\psi\|_{L^\infty(M_\delta)} \le C\,\delta^{-\frac{3}{4}}\|q\|_{p,\mu}.$$

and so, from (2.10)

$$\|\psi\|_{*,\delta} \le C\delta^{-\frac{3}{4}}\|q\|_{p,\mu}$$

where

$$\|\psi\|_{*,\delta} \;=\; \delta\|D^2\,\psi\|_{p,\mu} + \delta^{\frac{1}{2}}\|(1+r(\mathrm{y}))D\psi\|_{L^\infty(M)} + \|\log(2+r(\mathrm{y}))^{-1}\,\psi\|_{L^\infty(M)}.$$

Now, we prove Proposition 2. Since this linear equation can be written as the fixed point problem

$$\psi = L_\delta^{-1}\,[q] \,-\, L_\delta^{-1}\left[-|A_M|^2(A(\mathrm{y},\delta) - A(\mathrm{y},0)\,\psi)\right]$$

and as we observed before, it hold that

$$\left\|\,|A_M|^2\,(A(\cdot,\delta) - A(\cdot,0))\,\right\|_{p,\mu} \le C\delta.$$

then a direct application of the contraction mapping principle, in the norm (2.26) for ψ, completes the proof of the proposition 2.

Finally we estate an useful lemma that we borrow from section 8 in [14].

Lemma 2.4.4. *Assume $g(y,t)$ is a function defined in $M_\alpha \times \mathbb{R}$ and for which*

$$\sup_{(y,t)\in M_\alpha\times\mathbb{R}} (1 + r(\alpha y)^\mu) e^{\rho|t|} \|\psi\|_{L^p(B_1(y,t))} < \infty$$

for some $\rho, \mu > 0$ and $p > 2$. The function defined in M as $q(y) := \int_{\mathbb{R}} g\left(\frac{y}{\alpha}, t\right) w'(t)dt$ satisfies

$$\|q\|_{p,\beta} \le C \sup_{(y,t)\in M_\alpha\times\mathbb{R}} (1 + r(y)^\mu) e^{\rho|t|} \|\psi\|_{L^p(B_1(y,t))}$$

provided

$$\mu > \beta + \frac{2}{p}.$$

We refer the reader to lemma 8.1 in [14] for a detailed proof.

2.5 Approximation of the solution

In order to define the approximate solution to problem (2.1), we first observe that the heteroclinic solution to

$$w''(s) + f(w(s)) = 0, \quad t \in \mathbb{R}, \quad f(w) = w(1 - w^2)$$

is given explicitly by

$$w(s) \, = \, \tanh\left(\frac{s}{\sqrt{2}}\right), \quad s \in \mathbb{R}$$

has the asymptotic properties

$$\begin{aligned} w(s) &= \ \ 1 \, - \, 2\, e^{-\sqrt{2}\, s} \, + \mathcal{O}\left(e^{-2\sqrt{2}|s|}\right), & s > 1 \\ w(s) &= -1 \, + \, 2\, e^{-\sqrt{2}\, s} \, + \mathcal{O}\left(e^{-2\sqrt{2}|s|}\right), & s < -1 \\ w'(s) &= \ \ 2\sqrt{2}\, e^{-\sqrt{2}\,|s|} \, + \mathcal{O}\left(e^{-2\sqrt{2}|s|}\right), & |s| > 1. \end{aligned} \tag{2.1}$$

We assume that the location of these interfaces are determined by m embedded surfaces, each of which corresponds to the normal graph over M, of axially symmetric even smooth functions $h_i \in C^2(M)$, $i = 1, \ldots, m$. We write in coordinates $Y(y,\theta)$

$$h_j\left(Y(y,\theta)\right) \, = \, h_j(y), \quad y \in \mathbb{R}, \quad \theta \in (0, 2\pi), \quad j = 1, \ldots, m.$$

We assume further that

$$-\infty \equiv h_0 \, < \, h_1 \, < \, \ldots \, < \, h_m \, < \, h_{m+1} \equiv +\infty, \quad \text{in } M. \tag{2.2}$$

and that every h_j has the form

$$h_j(y) \, = \, \left(j - \frac{m+1}{2}\right)\left[\sigma \, + \, \sqrt{2}\left(1 - \frac{1}{\sigma}\right)\log\left(1 + y^2\right)\right] \, + \, \mathrm{v}_j(y), \quad y \in \mathbb{R} \tag{2.3}$$

for some functions $\mathrm{v}_1,\dots,\mathrm{v}_m \in C^2(M)$ satisfying the apriori estimate

$$\sigma^{-\frac{1}{2}}\|(1+r(\mathrm{y}))\,D\mathrm{v}_j\|_{L^\infty(M)} \;+\; \|(\,\log(2+r(\mathrm{y})\,)^{-1}\,\mathrm{v}_j\|_{L^\infty(M)} \le K\,\sigma^{\frac{5}{4}}, \quad j=1,\dots,m \tag{2.4}$$

for some universal constant K, that will be determined later, and where we recall that σ is the unique positive real number that solves the algebraic equation

$$\alpha^2\,\sigma \;=\; \sqrt{2}\,a_0\,e^{-\sqrt{2}\,\sigma}. \tag{2.5}$$

so that

$$\sigma = \log\left(\frac{\sqrt{2}\,a_0}{\alpha^2}\right) - \log\left(\log\left(\frac{\sqrt{2}\,a_0}{\alpha^2}\right)\right) + \mathcal{O}\left(\frac{\log\log\log\frac{1}{\alpha^2}}{\log\log\frac{1}{\alpha^2}}\right). \tag{2.6}$$

We observe from (2.2)-(2.3)-(2.4) that for every fixed $j=1,\dots,m-1$ and $\mathrm{y}\in M$,

$$h_{j+1}(y) - h_j(y) \;\ge\; \sigma \;+\; \sqrt{2}\left(1-\frac{1}{\sigma}\;-\mathcal{O}(\sigma^{-\frac{5}{4}})\right)\log\left(1+y^2\right), \quad \mathrm{y}=Y(y,\theta)\in M. \tag{2.7}$$

In the region $\mathcal{N}_\alpha$, using the local coordinates $x=X_\alpha(y,\theta,z)$, we consider as a first local approximation

$$U_0(x) \;=\; \sum_{j=1}^{m} w_j\big(\,z-h_j(\alpha y)\,\big) \;+\; \frac{(-1)^{m-1}-1}{2}, \quad w_j(s) \;=\; (-1)^{j-1}\,w(s). \tag{2.8}$$

Observe that, for points $x=X_\alpha(y,\theta,z)\in\mathcal{N}_\alpha$, for which z is close enough to $h_j(\alpha y)$, we have that

$$U_0(x) \approx w_j\big(\,z-h_j(\alpha y)\,\big).$$

For $l=1,\dots,m$ fixed, we consider the set

$$A_l = \left\{X_\alpha(y,\theta,z)\;:\;|z-h_l(\alpha y)| \le \frac{1}{2}\left[\sigma\;+\;\sqrt{2}\left(1-\frac{1}{\sigma}+\mathcal{O}(\sigma^{-\frac{5}{4}})\right)\log\left(1+(\alpha y)^2\right)\right]\right\}.$$

From (2.3)-(2.6) it is direct to check that $A_l\subset\mathcal{N}_\alpha$, for every $\alpha>0$ small enough. Setting $t = z-h_l(\alpha y)$, the set A_l can also be describe in terms of the local coordinates $X_{\alpha,h_l}(y,\theta,t)$ as

$$A_l \;=\; \left\{X_{\alpha,h_l}(y,\theta,t)\;:\;|t| \le \frac{1}{2}\left[\sigma\;+\;\sqrt{2}\left(1-\frac{1}{\sigma}+\mathcal{O}(\sigma^{-\frac{5}{4}})\right)\log\left(1+(\alpha y)^2\right)\right]\right\}.$$

Next, with the aid of lemma 2.2.1, we compute the error of the approximation defined in (2.8)

$$S(U_0) \;=\; \Delta U_0 + U_0(1-U_0^2), \quad \text{in } \mathcal{N}_\alpha.$$

We collect all the computations of the error in the following lemma.

Lemma 2.5.1. *For $l=1,\dots,m$ and every $x\in A_l$, $x=X_{\alpha,h_l}(y,\theta,t)$, we have that*

$$(-1)^{l-1}\,S(U_0) \;=\; -\,\alpha^2\left(\Delta_M h_l + |A_M|^2 h_l\right)w'(t) + 6\left(1-w^2(t)\right)\left[e^{-\sqrt{2}t}\,e^{-\sqrt{2}(h_l-h_{l-1})} \;-\; e^{\sqrt{2}t}\,e^{-\sqrt{2}(h_{l+1}-h_l)}\right]$$

$$-\;\alpha^2\,|A_M|^2\,t\,w'(t)\;+\;\alpha^2\,[\,h_l'\,]^2 w''(t)\;-\;\alpha^3(t+h_l)\,a_1(\alpha y,\alpha(t+h_l))\,h_l'' w'(t)$$

$$-\ \alpha^2\ \left(\Delta_M h_j - \alpha\,(t+h_l)\,a_1(\alpha y, \alpha(t+h_l))h_j''\right)\ w_j'(t+h_l-h_j) +$$

$$+\ \ \mathrm{R}_l(\alpha y, t, \mathrm{v}_1, \dots, \mathrm{v}_m, D\mathrm{v}_1, \dots, D\mathrm{v}_m) \tag{2.9}$$

where $\mathrm{R}_l = \mathrm{R}_l(\alpha y, t, p, q)$ *is smooth on its arguments and*

$$|D_p \mathrm{R}_l(\alpha y, t, p, q)|\ +\ |D_q \mathrm{R}_l(\alpha y, t, p, q)|\ +\ |\mathrm{R}_l(\alpha y, t, p, q)| \le C\,\alpha^{2+\tau}(1+|\alpha y|)^{-4} e^{-\rho\,|t|} \tag{2.10}$$

for some $\tau > 0$ *small and some* $0 < \rho < \sqrt{2}$ *and where*

$$p = (\mathrm{v}_1, \dots, \mathrm{v}_m), \quad q = (D\mathrm{v}_1, \dots, D\mathrm{v}_m).$$

Proof. We denote

$$E_1\ =\ f\left(\,(-1)^{l-1}\,U_0\,\right), \quad E_2\ =\ \Delta_{X_{\alpha,h_l}}\left[(-1)^{l-1}\,U_0(x)\right]$$

and we observe that, since $x = X_{\alpha,h_l}(y,\theta,t)$

$$\begin{aligned}(-1)^{l-1}U_0(x) \quad &= \quad -\,w_j\,(t+h_l-h_{l-1})\ +\ w(t)\ -\ w_j\,(t+h_l-h_{l+1}) \\ &\quad + \sum_{|j-l|\ge 2} w_j\,(t+h_l-h_j)\ +\ \frac{(-1)^{m-1}-1}{2}.\end{aligned}$$

We first compute E_1. We begin noticing that

$$\begin{aligned}(-1)^{l-1} f\,(U_0) \quad &= \quad (-1)^{l-1}\left[f\,(U_0(x)) - \sum_{j=1}^{m} f\,(\,w_j\,(t+h_l-h_j)\,)\right] \\ &\quad + \sum_{j=1}^{m}(-1)^{l-1} f\,(w_j\,(t+h_l-h_j))\,.\end{aligned}$$

Assume for the moment that $2 \le l \le m-1$ and observe that for $1 \le j < l$, it holds that

$$\begin{aligned}t + h_l(\alpha y) - h_j(\alpha y) \quad &\ge \quad (l-j)\left[\sigma + \sqrt{2}\left(1 - \frac{1}{\sigma} + \mathcal{O}(\sigma^{-\frac{5}{4}})\right)\log(1+(\alpha y)^2)\right] + t \\ &\ge \quad \frac{1}{2}\left[\sigma + \sqrt{2}\left(1 - \frac{1}{\sigma}\right)\log(1+(\alpha y)^2)\right]\end{aligned}$$

while for $l < j \le m$, it holds that

$$t + h_l(\alpha y) - h_j(\alpha y) \le -\frac{1}{2}\left[\sigma + \sqrt{2}\left(1 - \frac{1}{\sigma} + \mathcal{O}(\sigma^{-\frac{5}{4}})\right)\log(1+(\alpha y)^2)\right].$$

Using the asymptotic behavior of $w(s)$ from (2.1), we find that

$$w(t+h_l-h_j) = 1 - 2\,e^{-\sqrt{2}\,t}e^{-\sqrt{2}\,(h_l-h_j)} + \mathcal{O}\left(e^{-2\sqrt{2}\,|t+h_l-h_j|}\right), \quad 1 \le j < l$$

$$w(t+h_l-h_j) = -1 + 2\, e^{\sqrt{2}\,t} e^{\sqrt{2}\,(h_l-h_j)} + \mathcal{O}\left(e^{-2\sqrt{2}\,|t+h_l-hj|}\right), \quad l<j\leq m.$$

Notice also that for $|j-l|\geq 1$, we have that

$$|h_l-h_j| = |l-j| \left[\sigma + \sqrt{2}\left(1-\frac{1}{\sigma}+\mathcal{O}(\sigma^{-\frac{5}{4}})\right)\log(1+(\alpha y)^2)\right]$$

Hence, we obtain for $|j-l|\geq 2$, for some $\varepsilon\in(0,1)$ small, that

$$\begin{aligned}|t+h_l-h_j| &\geq |l-j| \left[\sigma + \sqrt{2}\left(1-\frac{1}{\sigma}+\mathcal{O}(\sigma^{-\frac{5}{4}})\right)\log(1+(\alpha y)^2)\right] - |t| \\ &\geq \left(2-\frac{1+\varepsilon}{2}\right)\left[\sigma + \sqrt{2}\left(1-\frac{1}{\sigma}\right)\log(1+(\alpha y)^2)\right] + \varepsilon\,|t|.\end{aligned}$$

On the other hand, since $f(s)=s(1-s^2)$, for $s\in\mathbb{R}$, it holds that

$$0\leq |f(s)|\leq |1-s||1+s|, \quad s\in[-1,1].$$

From this remarks, we conclude that there exist $\tau>0$ and $0<\rho<\sqrt{2}$ such that

$$\begin{aligned}0\leq |f(w_j(t+h_l-h_j))| &\leq C\,e^{-\sqrt{2}\,|t+h_l-h_j|} \\ &\leq C\alpha^{2+\tau}\,(1+|\alpha y|)^{-4}\,e^{-\rho\,|t|}.\end{aligned}$$

So, we get the estimate

$$\left|\sum_{|j-l|\geq 2} f(w_j(t+h_l-h_j))\right| \leq C \max_{|j-l|\geq 2} e^{-\sqrt{2}\,|t+h_l-h_j|} \leq C\alpha^{2+\tau}\,(1+|\alpha y|)^{-4}\,e^{-\varrho\,|t|}.$$

From (2.3) and the previous estimate, we observe that

$$(-1)^{l-1}\left[f\left(U_0(x)\right) - \sum_{j=1}^{m} f\left(\, w_j\left(t+h_l-h_j\right)\right)\right] =$$

$$\begin{aligned}&(-1)^{l-1} f\left(U_0(x)\right) \,+\, f(w(t+h_l-h_{l-1})) \,-\, f(w(t)) \,+\, f(w(t+h_l-h_{l+1}))\, + \\ &\qquad +\; R_{l,1}(\alpha y, t, \mathrm{v}_1,\ldots,\mathrm{v}_m)\end{aligned} \tag{2.11}$$

where

$$R_{l,1} = R_{l,1}(\alpha y,t,p), \quad |D_p\, R_{l,1}(\alpha y,t,p)| \,+\, |R_{l,1}(\alpha y,t,p)| \leq C\alpha^{2+\tau}\,(1+|\alpha y|)^{-4}\,e^{-\rho\,|t|}.$$

Let us now denote

$$a_1 = w(t+h_l-h_{l-1})-1, \quad a_2 = w(t+h_l-h_{l+1})+1.$$

From the mean value theorem, we can choose numbers $s_i = s_i(t,\alpha y,h_1,\ldots,h_m)\in(0,1)$, for $i=1,2$, such that

$$\begin{aligned}f(w(t+h_l-h_{l-1})) &= f(1) \,+\, f'(1)a_1 \,+\, \frac{1}{2}f''(1+s_1\,a_1)\,a_1^2 \\ f(w(t+h_l-h_{l+1})) &= f(-1) \,+\, f'(-1)a_2 \,+\, \frac{1}{2}f''(-1+s_2\,a_2)\,a_2^2.\end{aligned}$$

Proceeding in a similar fashion, there exists $s_3 \in (0,1)$, so that

$$\begin{aligned}(-1)^{l-1} f\left(U_0(x)\right) &= f(w) + f'(w)\left[(-1)^{l-1}U_0(x) - w(t)\right] \\ &+ \frac{1}{2} f''\left(w + s_3\left[(-1)^{l-1}U_0(x) - w(t)\right]\right)\left[(-1)^{l-1}U_0(x) - w(t)\right]^2 .\end{aligned}$$

Hence, we obtain that

$$(-1)^{l-1} f\left(U_0(x)\right) = f(w) - f'(w)(a_1 + a_2)$$

$$+ f'(w) \sum_{|j-l|\geq 2} (-1)^{j-l}[w(t + h_l - h_j) - sign(l-j)]$$

$$+ \frac{1}{2} f''[w + s_3\left((-1)^{l-1}U_0 - w\right)] \left(\sum_{|j-l|\geq 1} (-1)^{j+l} w(t + h_l - h_j) - sign(l-j)\right)^2 .$$

This means that we can write

$$(-1)^{l-1} f\left(U_0(x)\right) = f(w) - f'(w)(a_1 + a_2) + R_{l,2}(\alpha y, t, \mathrm{v}_1, \ldots, \mathrm{v}_m) \tag{2.12}$$

where

$$R_{l,2} = R_{l,2}(\alpha y, t, p), \quad |D_p R_{l,2}(\alpha y, t, p)| + |R_{l,2}(\alpha y, t, p)| \leq C\alpha^{2+\tau}\left(1 + |\alpha y|\right)^{-4} e^{-\rho|t|}. \tag{2.13}$$

Hence, putting together (2.11)-(2.13) and using that $F'(1) = F'(-1)$, we get that

$$\begin{aligned}(-1)^{l-1} f\left(U_0\right) &= \sum_{j=1}^{m} (-1)^{l-1} f\left(w_j\left(t + h_l - h_j\right)\right) + \left[f'(1) - F'(w)\right](a_1 + a_2) \\ &+ \frac{1}{2}\left[f''(1 + s_1 a_1)a_1^2 + F''(1 + s_2 a_2)a_2^2\right] + \\ &+ R_{l,1}(\alpha y, t, \mathrm{v}_1, \ldots, \mathrm{v}_m) + R_{l,2}(\alpha y, t, \mathrm{v}_1, \ldots, \mathrm{v}_m)\end{aligned}$$

from where we obtain that

$$(-1)^{l-1} f\left(U_0\right) = \sum_{j=1}^{m} (-1)^{l-1} f\left(w_j\left(t + h_l - h_j\right)\right) +$$

$$+ 6\left(1 - w^2(t)\right)\left[e^{-\sqrt{2}\,t} e^{-\sqrt{2}\,(h_l - h_{l-1})} - e^{\sqrt{2}\,t} e^{-\sqrt{2}\,(h_{l+1} - h_l)}\right]$$

$$+ R_l(\alpha y, t, \mathrm{v}_1, \ldots, \mathrm{v}_m) \tag{2.14}$$

where

$$R_l = R_l(\alpha y, t, p), \quad |D_p R_l(\alpha y, p)| + |R_l(\alpha y, t, p)| \leq C\alpha^{2+\tau}\left(1 + |\alpha y|\right)^{-4} e^{-\rho|t|}. \tag{2.15}$$

The remainder cases, namely $l = 1$ and $l = m$, are treated in an similar fashion, replacing the term

$$e^{-\sqrt{2}\,t} e^{-\sqrt{2}\,(h_l - h_{l-1})} - e^{\sqrt{2}\,t} e^{-\sqrt{2}\,(h_{l+1} - h_l)}$$

by the respective terms

$$-\,e^{\sqrt{2}\,t}e^{-\sqrt{2}\,(h_2-h_1)}, \quad l=1$$

and

$$e^{-\sqrt{2}\,t}e^{-\sqrt{2}\,(h_m-h_{m-1})}, \quad l=m.$$

So far, we have only written the term E_1 in a convenient way. We still have to compute E_2. In order to do so, we write

$$\begin{aligned} E_2 &= \Delta_{X_{\alpha,h_l}} w(t) \;+\; \sum_{|j-l|\geq 1} \Delta_{X_{\alpha,h_l}} \Big[(-1)^{l-1} w_j(t+h_l-h_j)\Big] \\ &= E_{21} \;+\; E_{22}. \end{aligned}$$

Directly from lemma 2.2.1, we obtain that

$$\begin{aligned} E_{21} \;=\; & w''(t) \;-\; \alpha^2 \left(\Delta_M h_l + |A_M|^2 h_l\right) w'(t) - \alpha^2 |A_M|^2 t\, w'(t) \;+\; \alpha^2 [\,h_l'\,]^2\, w''(t) \\ & -\alpha^3 (t+h_l)\, a_1(\alpha y, \alpha(t+h_l)) \left\{ \, h_l'' w'(t) - [\,h_l'\,]^2 w''(t) \, \right\} \\ & -\alpha^3 (t+h_l)\, b_1(\alpha y, \alpha(t+h_l))\, h_l'\, w'(t) - \alpha^4 (t+h_l)^3\, b_2(\alpha y, \alpha(t+h_l))\, w'(t). \end{aligned}$$

Notice that, using assumptions (2.3)-(2.4), we can write E_{21} as follows

$$\begin{aligned} E_{21} \;=\; & w''(t) \;-\; \alpha^2 \left(\Delta_M h_l + |A_M|^2 h_l\right) w'(t) - \alpha^2 |A_M|^2 t\, w'(t) \;+\; \alpha^2 [\,h_l'\,]^2\, w''(t) \\ & -\; \alpha^3 (t+h_l)\, a_1(\alpha y, \alpha(t+h_l))\, h_l'' w'(t) \;+\; \mathrm{Q}_{21}(\alpha y, t, \mathrm{v}_l, D\mathrm{v}_l) \end{aligned} \tag{2.16}$$

where

$$\mathrm{Q}_{21} = \mathrm{Q}_{21}(\alpha y, t, p, q)$$

and for some $0 < \rho\sqrt{2}$,

$$|D_p \mathrm{Q}_{21}(\alpha y, t, p, q)| \;+\; |D_q \mathrm{Q}_{21}(\alpha y, t, p, q)| \;+\; |\mathrm{Q}_{21}(\alpha y, t, p, q)| \leq C \alpha^3 (1+|\alpha y|)^{-4} e^{\rho\, t}. \tag{2.17}$$

Next, we compute E_{22}. Since we are using the local coordinates $x = X_{\alpha,h_l}(y,\theta,t)$, it holds that

$$\partial_t \left[w_j(t+h_l-h_j)\right] \;=\; w_j'(t+h_l-h_j), \quad \partial_{tt} \left[w_j(t+h_l-h_j)\right] \;=\; w_j''(t+h_l-h_j)$$

$$\partial_y \left[w_j(t+h_l-h_j)\right] \;=\; \alpha\, w_j'(t+h_l-h_j)\cdot(h_l'-h_j'), \quad \partial_{ty} \left[w_j(t+h_l-h_j)\right] \;=\; \alpha\, w_j''(t+h_l-h_j)\cdot(h_l'-h_j')$$

$$\partial_{yy} \left[w_j(t+h_l-h_j)\right] \;=\; \alpha^2\, w_j''(t+h_l-h_j)\cdot(h_l'-h_j')^2 + \alpha^2\, w_j'(t+h_l-h_j)\cdot(h_l''-h_j'').$$

Hence, by a direct computation and using the convention that we are summing over repeated indices, we find that

$$\begin{aligned} (-1)^{l-1}\, E_{22} \;=\; & w_j''(t+h_l-h_j) \\ -\; \alpha^2 \left[\, \Delta_M h_j \;+\; |A_M|^2 (h_l \;+\; t) \,\right] & w_j'(t+h_l-h_j) \;+\; \alpha^2 \left[\, h_j' \,\right]^2\; w_j''(t+h_l-h_j) \\ -\; \alpha^3\, (t+h_l)\, a_1(\alpha y, \alpha(t+h_l)) & \left[h_j''\, w_j'(t+h_l-h_j) - [\,h_j'\,]^2\, w_j''(t+h_l-h_j) \right] \\ -\; \alpha^3\, (t+h_l)\, b_1(\alpha y, \alpha(t+h_l))\, h_j'\, w_j' & \;+\; \alpha^4\, (t+h_l)^3\, b_2(\alpha y, \alpha(t+h_l))\, w_j'(t+h_l-h_j). \end{aligned}$$

Using the fact that, for $\varepsilon \in (0,1)$ and $|j-l| \geq 1$, so that

$$\begin{aligned}|t+h_l-h_j| &\geq |l-j|\left[\sigma+\sqrt{2}\left(1-\frac{1}{\sigma}+\mathcal{O}(\sigma^{-\frac{5}{4}})\right)\log(1+(\alpha y)^2)\right]-|t| \\ &\geq \left(1-\frac{1+\varepsilon}{2}\right)\left[\sigma+\sqrt{2}\left(1-\frac{1}{\sigma}\right)\log(1+(\alpha y)^2)\right]+\varepsilon\,|t|\end{aligned}$$

and proceeding as above, we can write E_{22} as follows

$$\begin{aligned}(-1)^{l-1}E_{22} \;=\; & w_j''(t+h_l-h_j)- \\ -\;\alpha^2\;&\left(\Delta_M h_j-\alpha\,(t+h_l)\,a_1(\alpha y,\alpha(t+h_l))h_j''\right)\;w_j'(t+h_l-h_j) \\ &+\;\mathrm{Q}_{22}(\alpha y,t,\mathrm{v}_1,\ldots,\mathrm{v}_m,D\mathrm{v}_1,\ldots,D\mathrm{v}_m)\end{aligned} \tag{2.18}$$

where

$$\mathrm{Q}_{22}=\mathrm{Q}_{22}(\alpha y,t,p,q)$$

and for some $0<\rho<\sqrt{2}$,

$$|D_pQ_{22}(\alpha y,t,v,q)|\;+\;|\nabla_q Q_{22}(\alpha y,t,p,q)|\;+\;|Q_{22}|\leq C\,\alpha^{2+\tau}(1+|\alpha y|)^{-4}e^{-\rho\,|t|}. \tag{2.19}$$

Setting $\mathrm{R}_l=R_l\,+\,\mathrm{Q}_{21}+\mathrm{Q}_{22}$, we have that $\mathrm{R}_l=\mathrm{R}_l(\alpha y,t,p,q)$ is smooth on its arguments and

$$|D_p\mathrm{R}_l(\alpha y,t,p,q)|\;+\;|D_q\mathrm{R}_l(\alpha y,t,p,q)|\;+\;|\mathrm{R}_l(\alpha y,t,p,q)|\leq C\,\alpha^{2+\tau}(1+|\alpha y|)^{-4}e^{-\rho\,|t|}$$

for some $\tau>0$ small and some $0<\rho<\sqrt{2}$. Putting together (2.14)-(2.16)-(2.18), we obtain that

$$\begin{aligned}(-1)^{l-1}\,S(U_0)\;=\;&-\alpha^2\left(\Delta_M h_l+|A_M|^2h_l\right)w'(t)+6\left(1-w^2(t)\right)\left[e^{-\sqrt{2}t}\,e^{-\sqrt{2}(h_l-h_{l-1})}\;-\;e^{\sqrt{2}t}\,e^{-\sqrt{2}(h_{l+1}-h_l)}\right] \\ &-\;\alpha^2\,|A_M|^2\,t\,w'(t)\;+\;\alpha^2\,[h_l']^2w''(t)\;-\;\alpha^3(t+h_l)\,a_1(\alpha y,\alpha(t+h_l))\,h_l''w'(t) \\ &-\;\alpha^2\;\left(\Delta_M h_j-\alpha\,(t+h_l)\,a_1(\alpha y,\alpha(t+h_l))h_j''\right)\;w_j'(t+h_l-h_j)+ \\ &+\mathrm{R}_l(\alpha y,t,\mathrm{v}_1,\ldots,\mathrm{v}_m,D\mathrm{v}_1,\ldots,D\mathrm{v}_m)\end{aligned}$$

and the proof of the lemma is complete. □

The approximation U_0 is so far defined only on the neighborhood $\mathcal{N}_\alpha$ of M_α. Let us consider a non-negative function $\beta \in C^\infty(\mathbb{R})$ such that

$$\beta(s)\;=\;\begin{cases}1, & |s|\leq 1\\ 0, & |s|\geq 2\end{cases}$$

and consider the cut-off function defined by

$$\beta_\alpha(x)\;=\;\beta(|z|-\frac{\eta}{\alpha}-2\sqrt{2}\,(m+1)\,\log(r(\alpha y))+3),\quad x=X_\alpha(y,\theta,z)\in\mathcal{N}_\alpha.$$

We see that β_α is supported in a region that expands logarithmically in $r(\alpha y)$. With the aid of this function, we set up as approximation in $\mathbb{R}^3$, the function

$$\mathrm{w}(x)\;=\;\beta_\alpha(x)\,U_0+(1-\beta_\alpha(x))\,\frac{U_0}{|U_0|},\quad x\in\mathbb{R}^3. \tag{2.20}$$

Let $\mathbb{H}$ be the function

$$\mathbb{H}(x) \;=\; \begin{cases} 1, & x \in S_\alpha^+ \\ (-1)^m, & x \in S_\alpha^- \end{cases}$$

where $S_\alpha^\pm \;=\; \alpha^{-1} S^\pm$, $S^\pm$ being the two connected components of $\mathbb{R}^3 - M_\alpha$ for which S_α^+ is the component containing the x_3-axis. We compute the new error as follows

$$S(\mathrm{w}) \;=\; \Delta \mathrm{w} + F(\mathrm{w}) \;=\; \beta_\alpha(x)\, S(U_0) \;+\; E$$

where

$$E \;=\; 2\nabla\beta_\alpha \nabla U_0 \;+\; \Delta\beta_\alpha (U_0 - \mathbb{H}) \;+\; F(\beta_\alpha U_0 + (1-\beta_\alpha)\mathbb{H}) - \beta_\alpha F(U_0).$$

Due to the choice of $\beta_\alpha(x)$ and the explicit form the term E has, the error created only takes into account values of β_α for $x \in \mathbb{R}^3$ in the region

$$|z| \geq \frac{\eta}{\alpha} + 4\ln(r(\alpha y)) - 2, \quad x = X_\alpha(y, \theta, z)$$

and so, we get the following estimate for the term E

$$|D_y\, E| \;+\; |E| \leq C e^{-\frac{\eta}{\alpha}} r^{-4}(\alpha y).$$

We observe that the error E decays rapidly and is exponentially small in $\alpha > 0$, so that its contribution is negligible.

2.6 The Proof of theorem 2.1.1.

The proof of Theorem 1 is quite technical and so, we prefer to sketch the steps of the proof and leave the detailed proofs of the propositions and lemmas mentioned here, for subsequent sections.

First, we introduced the norms we will use to set up an appropriate functional analytic scheme for the proof of Theorem 2.1.1. Let us denote

$$R(x) = \sqrt{x_1^2 + x_2^2}, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3$$

and let us define for $\alpha > 0$, $\mu > 0$, $\sigma \in (0,1)$ and $g(x)$, defined in $\mathbb{R}^3$, the norm

$$\|g\|_{p,\mu,\sim} \;:=\; \sup_{x \in \mathbb{R}^3} (1 + R(\alpha x))^\mu \, \|g\|_{L^p(B_1(x))}. \tag{2.1}$$

We also consider for $\alpha > 0$, $0 < \rho < \sqrt{2}$, $\mu > 0$ and functions $g = g(\mathrm{y}, t)$ and $\phi = \phi(\mathrm{y}, t)$, defined for every $(\mathrm{y}, t) \in M_\alpha \times \mathbb{R}$, we define the norms

$$\|g\|_{p,\mu,\rho} := \sup_{(\mathrm{y},t) \in M_\alpha \times \mathbb{R}} (1 + r(\alpha \mathrm{y}))^\mu e^{\rho\,|t|} \|g\|_{L^p(B_1(\mathrm{y},t))} \tag{2.2}$$

$$\|\phi\|_{\infty,\mu,\rho} := \|(1 + r(\alpha y)^\mu) e^{\rho\,|t|} \phi\|_{L^\infty(M)} \tag{2.3}$$

$$\|\phi\|_{2,p,\mu,\rho} := \|D^2\, \phi\|_{p,\mu,\rho} \;+\; \|D\phi\|_{\infty,\mu,\rho} \;+\; \|\phi\|_{\infty,\mu,\rho}. \tag{2.4}$$

Finally, for functions v and q defined in M, recall that we are considering the norms

$$\|q\|_{p,\mu} := \|(1+r(\mathrm{y})^{\mu})q\|_{L^p(M)} \tag{2.5}$$

$$\|\mathrm{v}\|_{*,\delta} := \|D^2\mathrm{v}\|_{p,\mu} + \|(1+r(\mathrm{y}))\,D\,\mathrm{v}\|_{L^\infty(M)} + \|\log(\,r(\mathrm{y})+2\,)^{-1}\,\mathrm{v}\|_{L^\infty(M)}. \tag{2.6}$$

Now, in order to prove Theorem 2.1.1, let us look for a solution to equation (2.1) of the form

$$U(x) \,=\, \mathrm{w}(x) + \varphi(x)$$

where $\mathrm{w}(x)$ is the global approximation defined in (2.20) and ϕ is going to be chosen small in the norm (2.4). Hence, for $U(x)$ being a genuine solution to (2.1), we see that φ must solve the equation

$$\Delta\varphi \,+\, f'(\mathrm{w})\varphi \,+\, S(\mathrm{w}) + N(\varphi) \,=\, 0, \quad \text{in } \mathbb{R}^3.$$

or equivalently

$$\begin{aligned} \Delta\varphi \,+\, f'(\mathrm{w})\varphi \;&=\; -S(\mathrm{w}) - N(\varphi) \\ &=\; -\beta_\alpha\, S(U_0) - E - N(\varphi) \end{aligned} \tag{2.7}$$

where

$$N(\varphi) \,=\, f(\mathrm{w}+\varphi) - f(\mathrm{w}) - f'(\mathrm{w})\varphi.$$

To solve equation (2.7), we consider again the function $\beta(s)$ from the previous section, and we define for $l=1,\ldots,m$ and $n\in\mathbb{N}$, the cut off function for $x = X_{\alpha,h_l}(y,\theta,t)\in\mathcal{N}_{\alpha,h_l}$

$$\zeta_{l,n}\,(x) \,=\, \beta\left(|t| - \frac{1}{2}\left[\sigma \,+\, \sqrt{2}\left(1-\frac{1}{\sigma}\right)\log\left(1+(\alpha y)^2\right)\right] + n\right).$$

Observe that for $k\neq l$, $\zeta_{l,3}\cdot\zeta_{k,3}=0$. Now we look for a solution $\varphi(x)$ with the particular form

$$\varphi(x) \,=\, \sum_{j=1}^{m}\; \zeta_{j,3}\,(x)\,\varphi_j(\mathrm{y},z) \,+\, \psi(x)$$

where the functions $\varphi_j(\mathrm{y},z)$ are defined in $M_\alpha\times\mathbb{R}$ and the function $\psi(x)$ is defined in the whole $\mathbb{R}^3$. So, from equation (2.7) we find that

$$\sum_{j=1}^{m}\left[\zeta_{j,3}\,\Delta_{\mathcal{N}_\alpha}\varphi_j + 2\,\nabla\zeta_{j,3}\cdot\nabla_{\mathcal{N}_\alpha}\varphi_j + \phi_j\Delta\zeta_{j,3} \,+\, f'(\mathrm{w})\zeta_{j,3}\phi_j\right]$$

$$+\,\Delta\psi \,+\, f'(\mathrm{w})\psi + S(\mathrm{w}) + N\left[\psi + \sum_{j=1}^{m}\zeta_{j,3}\varphi_j\right] = 0.$$

Notice that $\zeta_{j,2}\cdot\zeta_{j,3}=\zeta_{j,3}$, so we can write the previous equality as

$$\sum_{j=1}^{m}\zeta_{j,3}\left[\Delta_{\mathcal{N}_\alpha}\varphi_j + f'(\zeta_{j,2}\mathrm{w})\varphi_j + \zeta_{j,2}S(\mathrm{w}) + \zeta_{j,2}N(\varphi_j+\psi) + \zeta_{j,2}(f'(\mathrm{w})+2)\psi\right]$$

$$+ \Delta\psi - [2 - (1 - \sum_{j=1}^{m} \zeta_{j,3})(f'(\mathrm{w}) + 2)]\psi + (1 - \sum_{j=1}^{m} \zeta_{j,3}) S(\mathrm{w})$$

$$+ \sum_{j=1}^{m} 2\nabla\zeta_{j,3} \cdot \nabla_{\mathcal{N}_\alpha}\varphi_j + \varphi_j \Delta\zeta_{j,3} + (1 - \zeta_{j,3}) N[\psi + \sum_{i=1}^{m} \zeta_{i,2}\varphi_i] = 0.$$

Hence, to construct a solution to (2.7), we need to solve the system of PDEs

$$\Delta_{\mathcal{N}_\alpha}\varphi_l + f'(\zeta_{l,2}\mathrm{w})\varphi_l = -\zeta_{l,2}S(\mathrm{w}) - \zeta_{l,2}N(\varphi_l + \psi) - \zeta_{l,2}(f'(\mathrm{w}) + 2)\psi, \quad \text{in } |t| \le \rho_\alpha(\mathrm{y}), \quad l = 1, \dots, m \tag{2.8}$$

$$\Delta\psi - \left[2 - \left(1 - \sum_{j=1}^{m} \zeta_{j,2}\right)(f'(\mathrm{w}) + 2)\right]\psi = -\left(1 - \sum_{j=1}^{m} \zeta_{j,2}\right) S(\mathrm{w}) - - \sum_{j=1}^{m} 2\,\nabla\zeta_{j,2} \cdot \nabla_{\mathcal{N}_\alpha}\varphi_j - \varphi_j \Delta\zeta_{j,2} - \left(1 - \sum_{j=1}^{m} \zeta_{j,3}\right) N\left[\sum_{i=1}^{m} \zeta_{i,2}\varphi_i + \psi\right], \quad \text{in } \mathbb{R}^3. \tag{2.9}$$

where

$$\rho_\alpha(\mathrm{y}) = \frac{1}{2}\left[\sigma_\alpha - \sqrt{2}\left(1 - \frac{1}{\sigma_\alpha}\right)\log\left(1 + (\alpha y)^2\right)\right], \quad \mathrm{y} = Y_\alpha(y, \theta) \in M_\alpha$$

Now, we extend equation (2.8) to the whole $M_\alpha \times \mathbb{R}$. Let us set for $l = 1, \dots, m$

$$B_l := \zeta_{l,2}[\Delta_{\mathcal{N}_{\alpha,h_l}} - \partial_{tt} - \Delta_{M_\alpha}]$$

where we recall that in the local coordinates $Y_\alpha(y, \theta)$, we have that

$$\Delta_{M_\alpha} = \partial_{yy} + \frac{\alpha^2 y}{1 + (\alpha y)^2}\partial_y + \frac{\alpha^2}{1 + (\alpha y)^2}\partial_{\theta\theta}.$$

Recall also that the differential operator Δ_{M_α} is nothing but the Laplace-Beltrami operator of the catenoid. Observe also that B_l vanishes in the domain

$$|t| \ge \frac{1}{2}\left[\sigma_\alpha + 2\left(1 - \frac{1}{\sigma_\alpha}\right)\ln\left(1 + (\alpha y)^2\right)\right] - 1.$$

Hence, instead of equation (2.8), we consider the equation

$$\partial_{tt}\phi_l + \Delta_{M_\alpha}\phi_l + f'(w_l(t))\phi_l = -S_l(\mathrm{w}) - B_l(\phi_l)$$

$$- [F'(\zeta_{l,2}\mathrm{w}) - F'(w_l(t))]\phi_l - \zeta_{l,2}(F'(\mathrm{w}) + 2)\psi - \zeta_{l,2}N(\phi_l + \psi), \quad \text{in } M_\alpha \times \mathbb{R} \tag{2.10}$$

where, setting $z = t + h_l$, we choose in the local coordinates

$$\phi_l(y, t) = \varphi_l(y, z), \quad x = X_\alpha(y, \theta, z) = X_{\alpha,h_l}(y, \theta, t)$$

and where we have denoted

$$(-1)^{l-1}S_l(\mathrm{w}) := -\alpha^2\left(\Delta_M h_l + |A_M|^2 h_l\right) w'(t)$$

$$+6\left(1-w^2(t)\right)\,\zeta_{l,2}\,\left[e^{-\sqrt{2}t}\,e^{-\sqrt{2}(h_l-h_{l-1})}\,-\,e^{\sqrt{2}t}\,e^{-\sqrt{2}(h_{l+1}-h_l)}\right]$$

$$-\,\alpha^2\,|A_M|^2\,t\,w'(t)\,+\,\alpha^2\,[\,h_l'\,]^2 w''(t)$$

$$+\,\zeta_{l,2}\,\big[-\alpha^3(t+h_l)\,a_1(\alpha y,\alpha(t+h_l))\,h_l''w'(t)-$$

$$-\,\alpha^2\,\left(\Delta_M h_j-\alpha\,(t+h_l)\,a_1(\alpha y,\alpha(t+h_l))h_j''\right)\,w_j'(t+h_l-h_j)\,+\,\mathrm{R}_l\big] \tag{2.11}$$

where

$$\mathrm{R}_l=\mathrm{R}_l(\alpha y,t,\mathrm{v}_1,\dots,\mathrm{v}_m,D\mathrm{v}_1,\dots,D\mathrm{v}_m)$$

and

$$|D_p\mathrm{R}(\alpha y,t,p,q)|\,+\,|D_q\mathrm{R}(\alpha y,t,p,q)|\,+\,|\mathrm{R}(y,t,p,q)|\le C\alpha^{2+\tau}r_\alpha(y)^{-4}e^{-\rho|t|}. \tag{2.12}$$

Observe that $S_l(\mathrm{w})$ coincides with $S(U_0)$ where $\zeta_{l,2}=1$, but we have basically cut-off the parts in $S(U_0)$ that, in the local coordinates X_{α,h_l}, are not defined for all $t\in\mathbb{R}$.

We decompose this error into two parts

$$S_l(\mathrm{w})\;=\;S_{l,1}(\alpha y)\,w'(t)+S_{l,2} \tag{2.13}$$

where

$$(-1)^{l-1}S_{l,1}\;=\;-\alpha^2\left(\Delta_M h_l+|A_M|^2h_l\right),\quad S_{l,2}\;=\;S_l(\mathrm{w})-S_{l,1}(\alpha y)w'(t).$$

Using 2.11 and 2.12, we compute directly the size of this error to obtain that for some $0<\rho<\sqrt{2}$ we have that

$$\|S_{l,2}\|_{p,2,\rho}\le C\alpha^{2-\tau} \tag{2.14}$$

for some universal constant $C>0$ and some $\tau>0$ but arbitrarily small. This is and easy computation since the support of $\zeta_{l,2}$ is contained in a region of the form

$$|t|\le\frac{1}{2}\left[\sigma_\alpha\,+\,2\left(1-\frac{1}{\sigma_\alpha}\right)\ln\left(1+(\alpha y)^2\right)\right].$$

We also obtain the same estimate for $S_l(\mathrm{w})$

$$\|S_l(\mathrm{w})\|_{p,2,\rho}\le C\alpha^2. \tag{2.15}$$

Hence we solve system (2.9)-(2.10). We first solve equation (2.9), using the fact that the potential $2-(1-\sum_{j=1}^m\zeta_{j3})(F'(\mathrm{w})+2)$ is uniformly positive, so that the linear operator there behaves like $\Delta_{\mathbb{R}^3}-2$. A solution $\psi=\Psi(\phi_1,\dots,\phi_m)$ is then found using contraction mapping principle. We collect this discussion in the following proposition, that will be proven in detail in section 5.

Proposition 4. *Assume $\mu>0$, $0<\rho<\sqrt{2}$, $p>2$ and let the functions $h_j's$ be as in* (2.2)-(2.4). *Then, for every $\alpha>0$ sufficiently small and for m fixed functions $\phi_1,\dots,\phi_m$, satisfying that*

$$\|\phi_j\|_{2,p,\mu,\rho}\le 1,\quad j=1,\dots,m$$

equation (2.9) *has a unique solution* $\psi = \Psi(\phi_1, \ldots, \phi_m)$. *Even more, the operator* $\psi = \Psi(\phi_1, \ldots, \phi_m)$ *turns out to be lipschitz in every* ϕ_j. *More precisely,* $\psi = \Psi(\phi_1, \ldots, \phi_m)$ *satisfies that*

$$\begin{aligned} \|\psi\|_X &:= \|D^2\psi\|_{p,\mu,\sim} + \|(1 + R^\mu_\alpha(x))\, D\psi\|_{L^\infty(\mathbb{R}^3)} + \|(1 + R^\mu_\alpha(x))\, \psi\|_{L^\infty(\mathbb{R}^3)} \\ &\leq C \left(\alpha^{2+\tau} + \alpha^\tau \sum_{j=1}^m \|\phi_j\|_{2,p,\mu,\rho} \right) \end{aligned} \tag{2.16}$$

and

$$\|\Psi(\phi_j) - \Psi(\hat{\phi}_j)\|_X \leq C\alpha^\tau \|\phi_j - \hat{\phi}_j\|_{2,p,\mu,\rho}. \tag{2.17}$$

Hence, using Proposition 4, we solve equation (2.10) with $\psi = \Psi(\phi_1, \ldots, \phi_m)$. Let us set

$$\begin{aligned} \mathtt{N}_l(\phi_1, \ldots, \phi_l, \ldots, \phi_m) \;:=\; & B_l(\phi_l) \;+\; [F'(\zeta_{l,2}\mathrm{w}) - F'(w(t))]\phi_l \\ & +\; \zeta_{l,2}(F'(\mathrm{w}) + 2)\Psi(\phi_1, \ldots, \phi_m) \;+\; \zeta_{l,2} N[\phi_l + \Psi(\phi_1, \ldots, \phi_m)]. \end{aligned}$$

So, setting $\Phi = (\phi_1, \ldots, \phi_m)$, we only need to solve

$$\partial_{tt}\phi_l \;+\; \Delta_{M_\alpha}\phi_l \;+\; F'(w_l(t))\,\phi_l \;=\; -\,S_l(\mathrm{w}) - \mathtt{N}_l(\Phi), \quad \text{in } M_\alpha \times \mathbb{R} \tag{2.18}$$

for every $l = 1, \ldots, m$.

To solve problem (2.18), we solve a nonlinear problem in ϕ_l, in such a way that we eliminate the parts of the error that do not contribute to the projections onto $w'(t)$. Using the fact that the error $S(U_0)$ has the size

$$\|S_l(\mathrm{w}) + \alpha^2(\Delta_M h_l + |A_M|^2 h_l)\|_{p,2,\rho} \leq \alpha^{2\,-\,\tau}, \quad \tau \in (0, 1) \tag{2.19}$$

and as we will see in section 2.7, $\mathtt{N}_l(\phi)$ satisfies that

$$\|\mathtt{N}_l(\Phi)\|_{p,4,\rho} \leq C\alpha^{3-\tau}$$

and is Lipschitz with small Lipschitz constant, a direct application of the contraction mapping principle in a ball of radius $\mathcal{O}(\alpha^{2-\tau})$ in the norm $\|\phi_l\|_{2,p,2,\rho}$, allows us to solve the projected system

$$\partial_{tt}\phi_l + \Delta_{M_\alpha}\phi_l + F'(w_l(t))\phi_l = \,-\,S_l(\mathrm{w}) - \mathtt{N}_l(\Phi) \;+\; c_l(\mathrm{y})w'(t), \quad \text{in } M_\alpha \times \mathbb{R}. \tag{2.20}$$

$$\int_{\mathbb{R}} \phi_l\,(\mathrm{y}, t)w'(t)dt \;=\; 0, \quad l = 1, \ldots, m. \tag{2.21}$$

where

$$c_l(\mathrm{y}) \;=\; \int_{\mathbb{R}} [S_l(\mathrm{w}) + \mathtt{N}_l(\Phi)]\, w'(t)dt, \quad \forall\, l = 1, \ldots, m.$$

This solution ϕ_l, defines a Lipschitz operator $\phi_l = \Phi_l(\mathrm{v}_1, \ldots, \mathrm{v}_m)$ for the product norm

$$\|(\mathrm{v}_1, \ldots, \mathrm{v}_m)\|_* := \sum_{j=1}^m \|\mathrm{v}_j\|_{*,\delta}.$$

This information is collected in the following proposition.

Proposition 5. *Assume $0 < \mu \le 2$, $0 < \rho < \sqrt{2}$ and $p > 2$. For every $\alpha > 0$ small enough, there exists an universal constant $C > 0$, such that system (2.20)-(2.21) has a unique solution $(\phi_1, \dots, \phi_m) = \Phi(\mathrm{v}_1, \dots, \mathrm{v}_m)$, satisfying*

$$\|\Phi\|_{2,p,2,\rho} \le C\alpha^{2-\tau}$$

and

$$\|\Phi(\mathrm{v}_1, \dots, \mathrm{v}_m) - \Phi(\hat{\mathrm{v}}_1, \dots, \hat{\mathrm{v}}_m)\|_{2,p,2,\rho} \le C\,\alpha^{2-\tau}\|(\mathrm{v}_1, \dots, \mathrm{v}_m) - (\hat{\mathrm{v}}_1, \dots, \hat{\mathrm{v}}_m)\|_{*,\delta}.$$

To conclude the proof of Theorem 2.1.1, we adjust the vector function $\mathrm{v} = (\mathrm{v}_1, \dots, \mathrm{v}_m)$ in such a way that

$$c_l(\mathrm{y}) = \int_{\mathbb{R}} [S_l(\mathrm{w}) + \mathtt{N}_l(\Phi)]\, w'(t)dt = 0, \quad \forall\, l = 1, \dots, m.$$

We find that making this projection zero is equivalent to solve the nonlinear and nonlocal equation

$$\alpha^2\left(\Delta_M h_l + |A_M|^2 h_l\right) - a_0[e^{-\sqrt{2}(h_l - h_{l-1})} \;-\; e^{-\sqrt{2}(h_{l+1} - h_l)}]$$
$$= \alpha^2 G_{l,1}(\mathrm{v}) \;+\; \alpha^2 G_{l,2}(\mathrm{v}) \tag{2.22}$$

where

$$\alpha^2 G_{l,1}(\mathrm{v}) \;:=\; \int_{\mathbb{R}} \zeta_{j2}\,\big[-\alpha^3(t+h_j)\,a_1(\alpha y, \alpha(t+h_j))\,h_j'' w'(t) -$$
$$-\;\alpha^2\,\left(\Delta_M h_j - \alpha\,(t+h_l)\,a_1(\alpha y, \alpha(t+h_l))h_j''\right)\,w_j'(t+h_l-h_j) \;+\; \mathrm{R}_l\big]\,w'(t)dt$$
$$\alpha^2 G_{l,2}(\mathrm{v}) \;:=\; \int_{\mathbb{R}} \mathtt{N}_l(\Phi) w'(t)dt.$$

where we set $\Phi = (\Phi_1, \dots, \Phi_m)$ and the constant a_0 is given by

$$\left(\int_{\mathbb{R}} (w'(t))^2 dt\right) a_0 = \int_{\mathbb{R}} 6(1 - w^2(t))\, e^{-\sqrt{2}\,t} w'(t)dt.$$

From lemma 2.4.4 we have that for any $p > 2$ and $0 < \mu < 4 - \frac{2}{p}$

$$\left\|\int_{\mathbb{R}} \mathtt{N}_l(\Phi) w'(t)dt\right\|_{p,\mu} \le C\alpha^{3-\tau}$$

so that

$$\|G_{l,2}(\mathrm{v})\|_{p,\mu} \le C\alpha^{1-\tau}.$$

On the other hand,

$$\|G_{l,2}(\mathrm{v}) - G_{l,2}(\hat{\mathrm{v}})\|_{p,\mu} \le C\alpha^{-1}\|\Phi(\mathrm{v}) - \Phi(\hat{\mathrm{v}})\|_{p,2,\rho} \le C\alpha^{1-\tau}\|\mathrm{v} - \hat{\mathrm{v}}\|_{*,\delta}$$

Finally, using

$$\mathrm{Q}(\mathrm{v}) := G_{l,1}(\mathrm{v}) + G_{l,2}(\mathrm{v})$$

and lemmas 1 and 3, we see that a direct application of contraction mapping principle for a vector function v in the ball

$$\|(\mathrm{v}_1,\dots,\mathrm{v}_m)\|_* := \sum_{j=1}^{m} \|\mathrm{v}_j\|_{*,\delta} \le K\sigma^{-\frac{5}{4}}$$

for some $K>0$ chosen large and independent of $\alpha>0$, yields the existence of functions $\mathrm{v}_1,\dots,\mathrm{v}_m$ satisfying (2.4), so that

$$c_l(\mathrm{y}) = \int_{\mathbb{R}} \left[S_l(\mathrm{w}) + \mathtt{N}_l(\Phi)\right] w'(t)dt = 0, \quad \forall\, l=1,\dots,m$$

and this completes the proof of the theorem. In subsequent section we present the proofs of the auxiliary results mentioned this section.

2.7 Gluing reduction and solution to the projected problem.

In this section, we prove propositions 4 and 5. The notations we use in this section have been set up in sections 5 and 6.

We begin by fixing functions $\phi_1,\dots,\phi_m$ such that $\|\phi_l\|_{2,p,\mu,\rho} \le 1$ for $l=1,\dots,m$, we solve problem (2.9). Observe that there exist constants $a<b$, independent of α, such that

$$0 < a \le Q_\alpha(x) \le b, \quad \text{for every } x \in \mathbb{R}^3$$

where we set

$$Q_\alpha(x) \;=\; 2 - \left(1 - \sum_{j=1}^{m} \zeta_{j2}\right)\left[F'(\mathrm{w}) + 2\right].$$

Using this remark we study the problem

$$\Delta\psi - Q_\alpha(x)\psi = g(x), \quad x \in \mathbb{R}^3 \tag{2.1}$$

for a given $g=g(x)$ such that

$$\|g\|_{p,\mu,\sim} := \sup_{x\in\mathbb{R}^3} (1 + R^\mu(\alpha x))\|g\|_{L^p(B_1(x))}.$$

Solvability theory for equation (2.1) is collected in the following lemma whose proof follows the same lines as in lemma 7.1 in [14] and lemma 5.1 in [16].

Lemma 2.7.1. *Assume $p>2$ and $\mu\ge 0$. There exists a constant $C>0$ and $\alpha_0>0$ small enough such that for $0<\alpha<\alpha_0$ and any given $g=g(x)$ with $\|g\|_{p,\mu,\sim}<\infty$, equation (2.1) has a unique solution $\psi=\psi(g)$, satisfying the a-priori estimate*

$$\|\psi\|_X \le C\|g\|_{p,\mu,\sim}$$

where

$$\|\psi\|_X := \|D^2\psi\|_{p,\mu,\sim} + \|(1 + R^\mu_\alpha(x))\, D\psi\|_{L^\infty(\mathbb{R}^3)} + \|(1 + R^\mu_\alpha(x))\,\psi\|_{L^\infty(\mathbb{R}^3)}.$$

Now we prove Proposition 4. Denote by X, the space of functions $\psi \in W^{2,p}_{loc}(\mathbb{R}^3)$ such that $\|\psi\|_X < \infty$ and let us denote by $\Gamma(g) = \psi$ the solution to the equation (2.1) from the previous lemma. We see that the linear map Γ is continuous i.e

$$\|\Gamma(g)\|_X \leq C\|g\|_{p,\mu,\sim}.$$

Using this we can recast (2.9) as a fixed point problem, in the following manner

$$\psi = -\Gamma\left(\left(1-\sum_{j=1}^{m}\zeta_{j2}\right)S(\mathrm{w}) \,+\, g_1 \,+\, \left(1-\sum_{j=1}^{m}\zeta_{j2}\right)N\left[\sum_{i=1}^{m}\zeta_{i3}\phi+\psi\right]\right) \tag{2.2}$$

where

$$g_1 \;=\; \sum_{j=1}^{m} 2\,\nabla\zeta_{j2}\cdot\nabla_{\mathcal{N}_\alpha}\varphi_j \;+\; \varphi_j\Delta\zeta_{j2}.$$

Under conditions (2.3)-(2.4) and $\max_{1\leq l\leq m}\|\phi_l\|_{2,p,\mu,\rho} \leq 1$, we estimate the size of the right-hand side in (2.2).

Recall that $S(\mathrm{w}) = \beta_\alpha(x)S(U_0) + E$, where

$$|D_y\,E| \,+\, |E| \leq Ce^{-\frac{\eta}{\alpha}}r^{-4}(\alpha y).$$

So, we estimate directly using (2.15), to get

$$\begin{aligned}\left|\left(1-\sum_{j=1}^{m}\zeta_{j2}\right)S(\mathrm{w})\right| \;&\leq\; C\sum_{j=1}^{m}\alpha^2(1+r(\alpha y))^{-2}e^{-\rho|t|}\,(1-\zeta_{j2})\\ &\leq\; C\alpha^{2+\tau}(1+r(\alpha y))^{-4+\tau},\end{aligned}$$

this means that

$$\left|\left(1-\sum_{j=1}^{m}\zeta_{j2}(x)\right)S(\mathrm{w})\right| \leq C\alpha^{2+\tau}(1+R_\alpha(x))^{-4+\tau}.$$

Consequently we get, for $\mu \leq 4$ that

$$\left\|\left(1-\sum_{j=1}^{m}\zeta_{j,2}\right)S(\mathrm{w})\right\|_{p,4-\tau,\sim} \;\leq\; C\alpha^{2+\tau}$$

.

As for the second term in the right-hand side of (2.2), the following holds true

$$\begin{aligned}|2\nabla\zeta_{j,2}\cdot\nabla\phi_j + \phi_j\Delta\zeta_{j,2}| \;&\leq\; C(1-\zeta_{j2})(1+r^\mu(\alpha y))^{-1}e^{-\rho|t|}\|\phi_j\|_{2,p,\mu,\rho}\\ &\leq\; C\alpha^\tau(1+r^{\mu+2-\tau}(\alpha y))^{-1}\|\phi_j\|_{2,p,\mu,\rho}.\end{aligned}$$

for some $0<\tau<1$, provided $0<\rho<\sqrt{2}$. This implies that

$$\|2\nabla\zeta_{j,2}\cdot\nabla\phi_j + \phi_j\Delta\zeta_{j,2}\|_{p,\mu+2-\tau,\sim} \leq C\alpha^\tau\sum_{j=1}^{m}\|\phi_j\|_{2,p,\mu,\rho}.$$

Finally we must check the lipschitz character of $(1-\sum_{j=1}^m \zeta_{j2})N[\sum_{i=1}^m \zeta_{i2}\phi_i+\psi]$. Take $\psi_1,\psi_2\in X$. Then

$$\left(1-\sum_{j=1}^m \zeta_{j2}\right)\left|N\left[\sum_{i=1}^m \zeta_{i2}\phi_i+\psi_1\right]-N\left[\sum_{i=1}^m \zeta_{i2}\phi_i+\psi_2\right]\right|\le$$

$$\le\left(1-\sum_{j=1}^m \zeta_{j2}\right)\left|F(\mathrm{w}+\sum_{i=1}^m \zeta_{j1}\phi_i+\psi_1)-F(\mathrm{w}+\sum_{i=1}^m \zeta_{i1}\phi_i+\psi_2)-F'(\mathrm{w})(\psi_1-\psi_2)\right|$$

$$\begin{aligned}&\le\ C\left(1-\sum_{j=1}^m \zeta_{j2}\right)\sup_{s\in[0,1]}\left|\sum_{i=1}^m \zeta_{i1}\phi_i+s\psi_1+(1-s)\psi_2\right||\psi_1-\psi_2|\\&\le\ C\alpha^\tau\left(\sum_{i=1}^m\|\phi_i\|_{\infty,\mu,\rho}+\|\psi_1\|_X+\|\psi_2\|_X\right)|\psi_1-\psi_2|\end{aligned}$$

So, we see that

$$\left\|\left(1-\sum_{j=1}^m \zeta_{j2}\right)N\left[\sum_{i=1}^m \zeta_{i2}\phi_i+\psi_1\right]-\left(1-\sum_{j=1}^m \zeta_{j2}\right)N\left[\sum_{i=1}^m \zeta_{i2}\phi_i+\psi_2\right]\right\|_{p,2\mu,\sim}$$
$$\le C\alpha^\tau\|\psi_1-\psi_2\|_{\infty,2\mu}.$$

In particular, we take advantage of the fact that $N(\varphi)\sim\varphi^2$, to find that

$$\left\|\left(1-\sum_{j=1}^m \zeta_{j2}\right)N\left(\sum_{i=1}^m \zeta_{i2}\phi_i\right)\right\|_{p,2\mu,\sim}\le C\alpha^{2\tau}\sum_{j=1}^m\|\phi_j\|^2_{2,p,\mu,\rho}\,.$$

Consider $\widetilde{\Gamma}:X\to X$, $\widetilde{\Gamma}=\widetilde{\Gamma}(\psi)$ the operator given by the right-hand side of (2.2). From the previous remarks we have that $\widetilde{\Gamma}$ is a contraction provided α is small enough and so we have found $\psi=\widetilde{\Gamma}(\psi)$ the solution to (2.9) with

$$\|\psi\|_X\le C\left(\alpha^{2+\tau}+\alpha^\tau\sum_{j=1}^m\|\phi_j\|_{2,p,\mu,\rho}\right)$$

We can check directly that $\Psi(\Phi)=\psi$ is Lipschitz in $\Phi=(\phi_1,\ldots,\phi_m)$, i.e

$$\|\Psi(\Phi_1)-\Psi(\Phi_2)\|_X\le$$

$$C\left\|\left(1-\sum_{j=1}^m \zeta_{j2}\right)\left[N\left(\sum_{i=1}^m \zeta_{i2}\phi_{i\,1}+\Psi(\Phi_1)\right)-N\left(\sum_{i=1}^m \zeta_{i2}\phi_{i\,2}+\Psi(\Phi_2)\right)\right]\right\|_{p,2\mu,\sim}$$
$$\le C\alpha^\tau\left(\|\Psi(\Phi_1)-\Psi(\Phi_2)\|_X+\|\Phi_1-\Phi_2\|_{2,p,\mu,\rho}\right)$$

Hence for α small, we conclude

$$\|\Psi(\Phi_1) - \Psi(\Phi_2)\|_X \leq C\alpha^{\tau}\|\Phi_1 - \Phi_2\|_{2,p,\mu,\rho}.$$

Now we solve system

$$\partial_{tt}\phi_l + \Delta_{M_\alpha}\phi_l + F'(w_l(t))\phi_l = \,-S_l(\mathrm{w}) - \mathtt{N}_l(\phi_l) \,+\, c_l(\mathrm{y})w'(t), \quad \text{in } M_\alpha \times \mathbb{R}.$$

$$\int_{\mathbb{R}} \phi_l(\mathrm{y},t)w'(t)dt \,=\, 0.$$

To do so, we need to study solvability for the linear equation

$$\partial_{tt}\phi + \Delta_{M_\alpha}\phi + F'(w(t))\phi = g(\mathrm{y},t) + c(\mathrm{y})\,w'(t), \quad \text{in } M_\alpha \times \mathbb{R} \tag{2.3}$$

$$\int_{\mathbb{R}} \phi(\mathrm{y},t)w'(t)dt = 0. \tag{2.4}$$

Solvability of (2.3)-(2.4) is based upon the fact that the heteroclinic solution $w(t)$ is nondegenerate in the sense, that the following property holds true.

Lemma 2.7.2. *Assume that $\phi \in L^\infty(\mathbb{R}^3)$ and assume $\phi = \phi(x_1,x_2,t)$ satisfies*

$$L(\phi) := \partial_{tt}\phi + \Delta_{\mathbb{R}^2}\phi + F'(w(t))\phi = 0, \qquad \text{in } \mathbb{R}^2 \times \mathbb{R}. \tag{2.5}$$

Then $\phi(x_1,x_2,t) \,=\, C\,w'(t)$, for some constant $C \in \mathbb{R}$.

For the detailed proof of this lemma we refer the reader, for instance, to lemma 5.1 in [14],lemma 6.1 in [16] and references therein. The linear theory we need to solve system (2.21), is collected in the following proposition, whose proof is again contained in essence in lemma 5.2 in [14] and lemma 6.2 in [16].

Proposition 6. *Assume $p > 2$, $0 < \rho < \sqrt{2}$ and $\mu \geq 0$. There exist $C > 0$, an universal constant, and $\alpha_0 > 0$ small such that, for every $\alpha \in (0,\alpha_0)$ and any given g with $\|g\|_{p,\mu,\rho} < \infty$, problem (2.3)-(2.4) has a unique solution (ϕ,c) with $\|\phi\|_{p,\mu,\rho} < \infty$, satisfying the apriori estimate*

$$\|D^2\phi\|_{p,\mu,\rho} + \|D\phi\|_{\infty,\mu,\rho} + \|\phi\|_{\infty,\mu,\rho} \leq C\|g\|_{p,\mu,\rho}.$$

Using Proposition 6, we are ready to solve system (2.20)-(2.21). First, recall that as stated in (2.14)

$$\|S_l(\mathrm{w}) + \alpha^2(\Delta_M h_l + |A_M|^2 h_l)\|_{p,2,\rho} \leq C\alpha^{2-\tau} \tag{2.6}$$

From the discussion in 6.2, we have a nonlocal operator $\psi = \Psi(\phi_1,\ldots,\phi_m)$. Recall that for $\Phi = (\phi_1,\ldots,\phi_m)$,

$$\mathtt{N}_l(\Phi) := \,B_l(\phi_l) + \left[F'(\zeta_{l2}\mathrm{w}) - F'(w_l(t))\right]\phi_l \,+\, \zeta_{l2}[F'(\mathrm{w}) + 2]\Psi(\Phi) + \zeta_{l2}N(\phi_l + \Psi(\Phi)).$$

Let us denote

$$N_1(\Phi) := \,B_l(\phi_l) + \left[F'(\zeta_{l2}\mathrm{w}) - F'(w_l(t))\right]\phi_l$$

$$N_2(\Phi) := \ \zeta_{l2}\,[F'(\mathrm{w})+2]\,\Psi(\Phi), \quad N_3(\Phi) := \zeta_{l2} N(\phi_l + \Psi(\Phi)).$$

We need to investigate the Lipschitz character of N_i, $i = 1, 2, 3$. We begin with N_3. Observe that

$$|N_3(\Phi_1) - N_3(\Phi_2)| = \zeta_{l2}|N(\phi_{l1} + \Psi(\Phi_1)) - N(\phi_{l2} + \Psi(\Phi_2))|$$

$$\leq C\zeta_{l2} \sup_{\tau\in[0,1]} |\tau(\phi_{l1} + \Psi(\Phi_1)) + (1-\tau)(\phi_{l2} + \Psi(\phi_{l2}))| \cdot |\phi_{l1} - \phi_{l2} + \Psi(\Phi_1) - \Psi(\Phi_2)|$$

$$\leq C\left[|\Psi(\Phi_2)| + |\phi_{l1} - \phi_{l2}| + |\Psi(\Phi_1) - \Psi(\Phi_2)| + |\phi_{l2}|\right] \cdot \left[|\phi_{l1} - \phi_{l2}| + |\Psi(\Phi_1) - \Psi(\Phi_2)|\right].$$

This implies that

$$\|N_3(\Phi_1) - N_3(\Phi_2)\|_{p,2\mu,\rho} \leq$$

$$\leq C\left[\alpha^{2+\tau} + \sum_{j=1}^{m} \|\phi_{j1}\|_{p,\mu,\rho} + \sum_{j=1}^{m} \|\phi_{j2}\|_{p,\mu,\rho}\right] \cdot \sum_{j=1}^{m} \|\phi_{j1} - \phi_{j2}\|_{p,\mu,\rho}.$$

Now we check on $N_1(\Phi)$. Clearly, we just have to pay attention to $B_l(\phi_l)$. But notice that $B_l(\phi_l)$ is linear on ϕ_l and

$$B_l(\phi_l) = -\alpha^2\left\{h_l''(\alpha y) + \frac{\alpha y}{1+(\alpha y)^2}h_l'(\alpha y) + \frac{2(t+h_l)}{(1+(\alpha y)^2)^2}\right\}\partial_t\phi_l$$

$$-2\alpha h_l'(\alpha y)\partial_{ty}\phi_l + \alpha^2[h_l'(\alpha y)]^2\partial_{tt}\phi_l + D_{\alpha,h_l}(\phi_l),$$

where the differential operator D_{α,h_l} is given in (2.4). Hence, from the assumptions (2.2)-(2.7) made on the functions $\mathrm{v}_j's$, we have that

$$\|N_1(\Phi_1) - N_1(\Phi_2)\|_{p,2+\mu,\rho} \leq C\alpha\|\Phi_1 - \Phi_2\|_{p,\mu,\rho}.$$

Then, assuming that $\max_{1\leq j\leq m}\|\phi_j\|_{2,p,\mu,\rho} \leq A\alpha^{2-\tau}$, we have that

$$\|\mathtt{N}_l(\Phi)\|_{p,2+\mu,\rho} \leq C\alpha^{3-\tau}$$

Setting $T(g) = \phi$ the linear operator given by the Lemma 6, we recast problem (2.20) as the fixed point problem

$$\phi_l = T(-S_l(\mathrm{w}) - \mathtt{N}_l(\Phi)) =: \mathcal{T}_l(\Phi), \quad l = 1, \ldots, m.$$

in the ball

$$B_\alpha := \left\{(\phi_1, \ldots, \phi_m) : \|\phi_j\|_{2,p,2,\rho} \leq A\,\alpha^{2-\tau}, \quad j = 1, \ldots, m\right\}$$

where, clearly we are working in the space of function $\Phi \in W^{2,p}_{loc}(M_\alpha \times \mathbb{R})$, endowed with the norm

$$\|\Phi\|_{**} := \sum_{j=1}^{m} \|\phi_j\|_{2,p,2,\rho}.$$

Observe that

$$\|\mathcal{T}_l(\Phi_1) - \mathcal{T}_l(\Phi_2)\|_{**} \leq C\,\|\mathtt{N}_l(\Phi_1) - \mathtt{N}_l(\Phi_2)\|_{p,4,\rho} \leq C\,\alpha\,\|\Phi_1 - \Phi_2\|_{**}, \quad \Phi_1, \Phi_2 \in B_\alpha.$$

On the other hand, because C and K are universal constants and taking A large enough independent of $\alpha > 0$, we have that

$$\|\mathcal{T}_l(\Phi)\|_{**} \leq C\ (\|S_l(\mathrm{w})\|_{p,2,\rho} + \|N_l(\Phi)\|_{p,4,\rho}) \leq A\,\alpha^{2-\tau}, \quad \phi \in B_\alpha.$$

Hence, the mapping $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_m)$ is a contraction from the ball B_α onto itself. From the contraction mapping principle we get a unique solution

$$\Phi = \Phi(\mathrm{v}_1, \dots, \mathrm{v}_m)$$

as required. As for the Lipschitz character of $\Phi(\mathrm{v}_1, \dots, \mathrm{v}_m)$ it comes from a lengthy by direct computation from the fact that

$$\|\Phi(\mathrm{v}_1, \dots, \mathrm{v}_m) - \Phi(\tilde{\mathrm{v}}_1, \dots, \tilde{\mathrm{v}}_m)\|_{2,p,2,\rho} \leq C \sum_{j=1}^{m} \|S_j(\mathrm{w}, \mathrm{v}_1, \dots, \mathrm{v}_m) - S_j(\mathrm{w}, \tilde{\mathrm{v}}_1, \dots, \tilde{\mathrm{v}}_m)\|_{p,2,\rho} +$$

$$+ \sum_{j=1}^{m} \|N_j(\,\Phi(\mathrm{v}_1, \dots, \mathrm{v}_m)\,) - Nj(\,\Phi(\tilde{\mathrm{v}}_1, \dots, \tilde{\mathrm{v}}_m)\,)\|_{p,4,\rho}.$$

We left to the reader to check on the details of the proof of the following estimate

$$\|\Phi(\mathrm{v}_1, \dots, \mathrm{v}_m) - \Phi(\tilde{\mathrm{v}}_1, \dots, \tilde{\mathrm{v}}_m)\|_{2,p,2,\rho} \leq C\alpha^{2-\tau} \sum_{j=1}^{m} \|\mathrm{v}_j - \tilde{\mathrm{v}}_j\|_{*,\delta}.$$

for $(\mathrm{v}_1, \dots, \mathrm{v}_m)$ and $(\tilde{\mathrm{v}}_1, \dots, \tilde{\mathrm{v}}_m)$ satisfying (2.3) and (2.4). This completes the proof of proposition 5 and consequently the proof of Theorem 2.1.1.

Chapter 3

Two Logarithmical Ended Solutions to the Allen-Cahn Equation in $\mathbb{R}^3$

In this section, we sketch the construction of a family of bounded axially symmetric solutions to

$$\Delta u + u(1-u^2) = 0, \quad \text{in } \mathbb{R}^3. \tag{3.1}$$

whose zero level set is close to a radially symmetric solution of the Toda system of PDEs in $\mathbb{R}^2$ and so that, outside of a compact set, has four logarithmical connected components. This is another step in the program towards the study and the classification of finite Morse index solutions to (3.1), since these solutions are expected to have Morse index 2. Much of the developments done in this chapter are similar to those done in previous sections.

3.1 Statement of the main result

In order to state our main result, we consider a smooth radially symmetric solution (q_1, q_2) of the *Toda System*

$$\begin{aligned} \Delta\, q_1 + a_0 e^{-\sqrt{2}\,(q_2-q_1)} &= 0, \quad \text{in } \mathbb{R}^2 & (3.1)\\ \Delta\, q_2 - a_0 e^{-\sqrt{2}\,(q_2-q_1)} &= 0, \quad \text{in } \mathbb{R}^2. & (3.2) \end{aligned}$$

where $a_0 > 0$ is a positive constant to be determined. To be more precise, we assume that $-q_1 = q_2 = q$, and the function q is a solution to the Liouville equation

$$\Delta\, q - a_0 e^{-2\sqrt{2}\,q} = 0, \quad \text{in } \mathbb{R}^2 \tag{3.3}$$

$$q(0) = a > 0, \quad \partial_r\, q(0) = 0 \tag{3.4}$$

given explicitly by

$$q(r,\rho) = \frac{1}{2\sqrt{2}} \log\left(\frac{\sqrt{2}\, a_0}{4\,\rho} \left(1 + \rho\, r^2\right)^2 \right). \tag{3.5}$$

From the fact that $q(0) = a > 0$, we obtain

$$\log\left(\frac{\sqrt{2}\,a_0}{4\,\rho}\right) = \frac{2\,a}{\sqrt{2}}.$$

We remark that, for every $\alpha > 0$, the functions

$$-\frac{\sqrt{2}}{2}\log\left(\frac{1}{\alpha}\right) + q_1(\alpha r), \quad \frac{\sqrt{2}}{2}\log\left(\frac{1}{\alpha}\right) + q_2(\alpha r), \quad r > 0.$$

are also smooth radially symmetric solutions to (3.4)-(3.5). Now, for $\alpha > 0$ small, consider two smooth parameter functions v_1, v_2 satisfying that for $j = 1, 2$

$$\|v_j\|_* := \|D^2 v_j\|_{\infty,2,\sigma} + \|(1+|x'|)Dv_j\|_{L^\infty(\mathbb{R}^2)} + \|\log(2+|x'|)v_j\|_{L^\infty(\mathbb{R}^2)} = \mathcal{O}(\alpha^{\tau_1}) \tag{3.6}$$

for some $\tau_1 \in (0, 2)$, and let us define the functions

$$\mathrm{f}_{1\alpha}(r) := -\frac{\sqrt{2}}{2}\log\left(\frac{1}{\alpha}\right) + q_1(\alpha r) + v_1(\alpha r)$$

$$\mathrm{f}_{2\alpha}(r) := \frac{\sqrt{2}}{2}\log\left(\frac{1}{\alpha}\right) + q_2(\alpha r) + v_2(\alpha r).$$

and denote $M_{j\alpha}$ the graph of the function $\mathrm{f}_{j\alpha}$. The main result we present in this section is the following.

Theorem 3.1.1. *For all sufficiently small $\alpha > 0$ there exists an smooth axially symmetric bounded solution u_α to problem* (3.1) *such that*

$$u_\alpha(x', z) = w\left(z - f_{1\alpha}(\alpha x')\right) - w\left(z - f_{2\alpha}(\alpha x')\right) - 1 + o(1), \quad \text{as } \alpha \to 0$$

for $x = (r\cos\theta, r\sin\theta, z)$, $|z| < \frac{\eta}{\alpha} + \frac{1}{2\alpha}\log(1 + (\alpha r)^2)$. These solutions have the properties that they are axially symmetric and they converge to ± 1 away from the graphs of the functions $f_{j\alpha}$, i.e

$$u_\alpha(x', z) = u_\alpha(|x'|, z) = u_\alpha(|x'|, -z), \quad u_\alpha^2(x) \to 1, \quad \text{as } dist(x, M_{j\alpha}) \to \infty, \quad \text{for } x = (x', z).$$

In addition, the location of the interfaces $f_{j\alpha}'s$ is governed at main order by the Toda system on $\mathbb{R}^2$, i.e

$$\Delta_{\mathbb{R}^2} q_1 + a_0\, e^{-\sqrt{2}(q_2 - q_1)} = 0, \quad \Delta_{\mathbb{R}^2} q_2 - a_0\, e^{-\sqrt{2}(q_2 - q_1)} = 0$$

and

$$f_{2\alpha} - f_{1\alpha} \geq \sqrt{2}\log\left(\frac{1}{\alpha}\right) + \log\left(1 + |\alpha x'|\right).$$

3.2 Toda system in $\mathbb{R}^2$ and its linearization

In this part we describe the Toda System of PDEs, that will govern the location of the interfaces of the solution, namely

$$\Delta f_1 + a_0 e^{-\sqrt{2}(f_2 - f_1)} = g_1, \quad \text{in } \mathbb{R}^2 \tag{3.1}$$
$$\Delta f_2 - a_0 e^{-\sqrt{2}(f_2 - f_1)} = g_2, \quad \text{in } \mathbb{R}^2. \tag{3.2}$$

where $a_0 > 0$ is a positive constant. Let us look for an radially symmetric smooth solution to (3.1)-(3.2) having the form

$$f_1(x') \;=\; q_1(x') + v_1(x'), \quad f_2(x') \;=\; q_2(x') + v_2(x'), \quad x' \in \mathbb{R}^2 \tag{3.3}$$

where

$$\Delta\, q_1 + a_0 e^{-\sqrt{2}\,(q_2 - q_1)} \;=\; 0, \quad \text{in } \mathbb{R}^2 \tag{3.4}$$

$$\Delta\, q_2 - a_0 e^{-\sqrt{2}\,(q_2 - q_1)} \;=\; 0, \quad \text{in } \mathbb{R}^2. \tag{3.5}$$

As mentioned before, let us assume further that $-q_1 = q_2 = q$, so that the function q must be a solution to the Liouville equation

$$\Delta\, q - a_0 e^{-2\sqrt{2}\, q} = 0, \quad \text{in } \mathbb{R}^2 \tag{3.6}$$

$$q(0) = a > 0, \quad \partial_r\, q(0) = 0 \tag{3.7}$$

It is known that every radially symmetric solution to (3.1)-(3.7) is given by

$$q(r, \rho, \gamma) = \frac{1}{2\sqrt{2}} \log\left(\frac{\sqrt{2}\, a_0}{4\, \rho\, \gamma^2} \left(1 + \rho\, r^{2\gamma}\right)^2 \right) + \frac{(2\gamma - 2)}{2\sqrt{2}} \log{(r)} \tag{3.8}$$

Since we are looking for smooth solutions, this forces $\gamma = 1$, so that

$$q(r, \rho) = \frac{1}{2\sqrt{2}} \log\left(\frac{\sqrt{2}\, a_0}{4\, \rho} \left(1 + \rho\, r^2\right)^2 \right). \tag{3.9}$$

From the fact that $q(0) = a > 0$, we obtain

$$\log\left(\frac{\sqrt{2}\, a_0}{4\, \rho} \right) = \frac{2\, a}{\sqrt{2}}.$$

Observe that ρ is a free parameter depending on the initial condition (3.7).

Linearizing (3.1)-(3.2) around the exact solution (q_1, q_2), and decoupling this equation as we did in section 2.2, we obtain the nonlinear system

$$\Delta\, v_1 + \sqrt{2}\, a_0 e^{-\sqrt{2}\,(q_2 - q_1)} v_1 + \mathrm{N}(v_1) \;=\; \widetilde{g}_1, \quad \text{in } \mathbb{R}^2 \tag{3.10}$$

$$\Delta\, v_2 = \widetilde{g}_2, \quad \text{in } \mathbb{R}^2 \tag{3.11}$$

where we consider right-hand side functions g_j satisfying that

$$\|\widetilde{g}_j\|_{\infty,\beta,\sigma} : \|(1 + |x|^\beta)\widetilde{g}_j\|_{C^{0,\sigma}(\mathbb{R}^2)} < \infty, \quad j = 1, 2$$

for some $\sigma \in (0, 1)$ and $\beta \geq 0$ and where we have denoted

$$\mathrm{N}(v_1) = -e^{-2\sqrt{2}\, q} \left[e^{-\sqrt{2}\, v_1} - 1 + \sqrt{2}\, v_1 \right]. \tag{3.12}$$

Let us consider first the linear system associated to (3.10)-(3.11), namely

$$\Delta v_1 + \sqrt{2}\, a_0 e^{-\sqrt{2}\,(q_2-q_1)} v_1 \,=\, \widetilde{g}_1, \quad \text{in } \mathbb{R}^2 \tag{3.13}$$

$$\Delta v_2 = \widetilde{g}_2, \quad \text{in } \mathbb{R}^2 \tag{3.14}$$

Since our setting is radially symmetric, we deal with this system using variations of parameters formula. Setting $r = |x'|$, we observe that (3.14) has a radially symmetric smooth solution given by

$$v_2(r) := \int_r^\infty \xi\, \log(\xi)\, \widetilde{g}_2(\xi) d\xi + \log(r) \int_0^r \xi\, \widetilde{g}_2(\xi)\, d\xi$$

Clearly v_2 is smooth and taking $\beta > 2$, we see directly from this formula that

$$\|v_2\|_* \le C \|q_2\|_{\infty,\beta,\sigma}$$

where

$$\|v_2\|_* := \|D^2 v_2\|_{\infty,2,\sigma} \,+\, \|(1+|x'|) D v_2\|_{L^\infty(\mathbb{R}^2)} + \|\log(2+|x'|) v_2\|_{L^\infty(\mathbb{R}^2)}. \tag{3.15}$$

Next we solve equation (3.13). Taking derivatives respect to ρ and γ in (3.8), we find that the functions

$$\psi_1(r) = \frac{1-\rho\, r}{1+\rho\, r^2}, \quad \psi_2(r) = \frac{\log(r)\,(\rho\, r^2 - 1)}{1+\rho\, r^2} - 1 \tag{3.16}$$

span the kernel of (3.13). Observe that ψ_2 is clearly singular at the origin. Observe also that

$$\partial_r\, \psi_1(r) = \frac{4\rho\, r}{(1+\rho\, r^2)^2}, \quad \partial_r\, \psi_2(r) = \frac{-1+\rho^2\, r^4 + 4\rho\, r^2\, \log(r)}{r(1+\rho\, r^2)^2} \tag{3.17}$$

From (3.17) we find that

$$|\partial_r\, \psi_1(r)| \le \frac{C\, r}{1+r^4}, \quad |\partial_r\, \psi_2(r)| \le \frac{C}{r}, \quad r > 0. \tag{3.18}$$

We set as right inverse for (3.13) the function

$$v_1(r) = \psi_1(r) \int_r^\infty \xi\, \psi_2(\xi)\, q_1(\xi)\, d\xi + \psi_2(r) \int_0^r \xi\, \psi_1(\xi)\, q_1(\xi)\, d\xi$$

Using (3.18), we directly observe that $\partial_r v_2(0) = 0$ and it can be checked that

$$\|v_1\|_* \le C \|q_1\|_{\infty,\beta,\sigma}$$

provided that $\beta > 2$.

With this discussion, we are now in position to invert the linear system (3.10)-(3.11). We collect this information in the following lemma

Lemma 3.2.1. *Assume $\sigma \in (0,1)$ and $\beta > 2$ and consider a vector function $(\widetilde{g}_1, \widetilde{g}_2)$ satisfying that*

$$\|\widetilde{g}_j\|_{\infty,\beta,\sigma} \le C\alpha^{\kappa_1}$$

for some small parameter $\alpha > 0$ and some $\kappa_1 > 0$. Then system (3.10)-(3.11) *has a unique solution* (v_1, v_2) *satisfying that*

$$\|v_j\|_* \le C \max_{k=1,2} \|\widetilde{g}_k\|_{\infty,\beta,\sigma}, \quad j = 1, 2.$$

Even more this solution turns out to be Lipschitz in the vector function $(\widetilde{g}_1, \widetilde{g}_2)$*, namely*

$$\|v_j - \hat{v}_j\|_* \le C \max_{k=1,2} \|\widetilde{g}_k - \hat{g}_k\|_{\infty,\beta,\sigma}, \quad j = 1, 2.$$

The proof of this lemma is straightforward from the previous comments and proceeding as in section 2.4. Let us remark that in the case where $\widetilde{g}_j$, $j = 1, 2$, are nonlocal operator in (v_1, v_2) having small Lipschitz character a direct application of Banach fixed point theorem will also lead to the existence of a unique solution to (3.10)-(3.11).

3.3 Approximate solution to the projected problem

Now that we have described the location of the nodal set of our solution, we proceed to set up our approximation. Consider a radially symmetric solution (q_1, q_2) to the system

$$\Delta\, q_1 + a_0 e^{-\sqrt{2}\,(q_2 - q_1)} = 0, \quad \Delta\, q_2 - a_0 e^{-\sqrt{2}\,(q_2 - q_1)} = 0, \quad \text{in } \mathbb{R}^2.$$

where $a_0 > 0$ is a positive constant to be determined. Recall from the previous section that $-q_1 = q_2 = q$, and the function q is a solution to the Liouville equation

$$\Delta\, q - a_0 e^{-2\sqrt{2}\, q} = 0, \quad \text{in } \mathbb{R}^2, \quad q(0) = a > 0, \quad \partial_r\, q(0) = 0 \tag{3.1}$$

given explicitly by

$$q(r, \rho) = \frac{1}{2\sqrt{2}} \log\left(\frac{\sqrt{2}\, a_0}{4\,\rho} \left(1 + \rho\, r^2\right)^2 \right). \tag{3.2}$$

We remark again that, for every $\alpha > 0$, the functions

$$-\frac{\sqrt{2}}{2} \log\left(\frac{1}{\alpha}\right) + q_1(\alpha x'), \quad = \frac{\sqrt{2}}{2} \log\left(\frac{1}{\alpha}\right) + q_2(\alpha x'), \quad r > 0.$$

are also smooth radially symmetric solutions to (3.1)-(3.2).

Now, for $\alpha > 0$ small, consider two smooth parameter functions v_1, v_2 satisfying that for $j = 1, 2$

$$\|v_j\|_* := \|D^2 v_j\|_{\infty,2,\sigma} + \|(1 + |x'|) D v_j\|_{L^\infty(\mathbb{R}^2)} + \|\log(2 + |x'|) v_j\|_{L^\infty(\mathbb{R}^2)} \le K \alpha^{\tau_1} \tag{3.3}$$

for some $\tau_1 \in (0, 2)$, where $K > 0$ and τ_1 will be chosen later and independent of $\alpha > 0$ small. Let us consider again the functions

$$\mathrm{f}_{1\alpha}(x') = -\frac{\sqrt{2}}{2} \log\left(\frac{1}{\alpha}\right) + q_1(\alpha x') + v_1(\alpha x')$$

$$\mathrm{f}_{2\alpha}(x') = \frac{\sqrt{2}}{2}\log\left(\frac{1}{\alpha}\right) + q_2(\alpha x') + v_2(\alpha x').$$

Proceeding as we in the proof of Theorem 2.1.1. we write

$$w_j(x) := (-1)^{j-1} w(z - \mathrm{f}_{j\alpha}(x')), \quad x = (x', z) \in \mathbb{R}^3$$

and we consider as local approximation the function

$$U_0(x) \;=\; w_1(x) + w_2(x) - 1, \quad x \in \mathbb{R}^3. \tag{3.4}$$

Proceeding as in section 2.5, let us next consider sets

$$A_j := \left\{ x = (x', z) : |z - \mathrm{f}_{j\alpha}(x')| \le \frac{\mathrm{f}_{2\alpha}(x') - \mathrm{f}_{1\alpha}(x')}{2} \right\}, \quad j = 1, 2.$$

Writing $z = t + \mathrm{f}_{j\alpha}(x')$, we notice that A_j can be described as

$$A_j := \left\{ x = (x', t) : |t| \le \frac{\mathrm{f}_{2\alpha}(x') - \mathrm{f}_{1\alpha}(x')}{2} \right\}, \quad j = 1, 2.$$

Hence we can estate the following lemma regarding the error of this approximation in the set A_j.

Lemma 3.3.1. *For $l = 1, 2$ and every $x \in A_l$, $x = (x', t)$, we have that*

$$\begin{aligned} (-1)^{l-1}\, S(U_0) \;=\; &-\Delta_{\mathbb{R}^2} f_{l\alpha}(x') w'(t) + \; 6\left(1 - w^2(t)\right)\, e^{-\sqrt{2}t}\, e^{-\sqrt{2}(f_{2\alpha} - f_{1\alpha})} \\ &+ |\nabla f_{l\alpha}|^2 w''(t) - \; \Delta_{\mathbb{R}^2} f_{j\alpha}\, w_j'(t + f_{l\alpha} - f_{j\alpha}) \;-\; |\nabla (f_{j\alpha} - f_{l\alpha})|^2 w_j''(t + f_{l\alpha} - f_{j\alpha}) \\ &- \; |\nabla f_{l\alpha}|^2 w''(t + f_{l\alpha} - f_{j\alpha}) \;+\; \mathrm{R}_l(\alpha x', t, v_1, v_2, Dv_1, Dv_2) \end{aligned} \tag{3.5}$$

where $\mathrm{R}_l = \mathrm{R}_l(\alpha y, t, p, q)$ *is smooth on its arguments and*

$$|D_p \mathrm{R}_l(\alpha y, t, p, q)| \;+\; |D_q \mathrm{R}_l(\alpha y, t, p, q)| \;+\; |\mathrm{R}_l(\alpha y, t, p, q)| \le C\, \alpha^{2+\tau}(1 + |\alpha y|)^{-4} e^{-\rho\,|t|} \tag{3.6}$$

for some $\tau > 0$ small and some $0 < \rho < \sqrt{2}$ and where

$$p = (v_1, v_2), \quad q = (Dv_1, Dv_2).$$

The proof of this lemma follows the same lines of lemma (3.3.1), with no significant changes and actually with easier computations of the laplacian, so we leave its verification to the reader.

Next step, consists on defining the global approximation to the solution. We simply consider a smooth cut-off function $\beta \in C_c^{\infty}(\mathbb{R})$, such that $\beta(t) = 1$, for $|t| \le 1/2$ and $\beta(t) = 0$, for $|t| \le 1$. Now, for $\alpha > 0$ small we define the cut-off function and consider the cut-off function defined by

$$\beta_\alpha(x) \;=\; \beta(|z| - \frac{\eta}{\alpha} - 6\sqrt{2}\; \log(|\alpha x'| + 3), \quad x = (x', z) \in \mathbb{R}^3.$$

We see that β_α is supported in a region that expands logarithmically in $|\alpha x'|$ and we consider as global approximation the function

$$\mathrm{w}(x) \;:=\; \beta_\alpha(x) U_0(x) \;+\; (1 - \beta_\alpha(x))(-1). \tag{3.7}$$

We compute the new error as follows

$$S(\mathrm{w}) \;=\; \Delta \mathrm{w} + F(\mathrm{w}) \;=\; \beta_\alpha(x)\, S(U_0) \;+\; E$$

where

$$E \;=\; 2\nabla\beta_\alpha \nabla U_0 \;+\; \Delta\beta_\alpha (U_0+1) \;+\; F(\beta_\alpha U_0 - (1-\beta_\alpha)) - \beta_\alpha F(U_0).$$

Due to the choice of $\beta_\alpha(x)$ and the explicit form the term E has, the error created only takes into account values of β_α for $x \in \mathbb{R}^3$ in the region

$$|z| \geq \frac{\eta}{\alpha} + 4\ln(|\alpha x'|) - 2, \quad x = (x', z) \in \mathbb{R}^3$$

and so, we get the following estimate for the term E

$$|D_{x'}\, E| \;+\; |E| \leq C e^{-\frac{\eta}{\alpha}} (1 + |\alpha x'|)^{-4}.$$

We observe that the error E decays rapidly and is exponentially small in $\alpha > 0$, so that its contribution is negligible.

3.4 Outline of the Lyapunov-Schmidt Reduction

Since we want to measure how far is approximation (3.7) from being a genuine solution to our problem, we need to find an appropriate functional setting to work with. In order to do so, let us consider a weight function $W_{\mu,c} := \sum_{j=1}^{2} W_{\mu,c,j}$ designed in the following way. First

$$W_{\mu,c,j}(x', z) := (1 + |\alpha x'|^{\mu})\, e^{c|z - \mathrm{f}_{\mathrm{j}\alpha}(x')|}, \quad (x', z) \in A_j$$

while in the lower part of $\mathbb{R}^3 - (A_1 \cup A_2)$, we consider $W_{\mu,c}$ satisfying that

$$c\,(1 + |\alpha x'|^{\mu+2}) \leq W_{\mu,c}(x' z) \leq C\,(1 + |\alpha x'|^{\mu+2}).$$

Observe in this case that $|z| = dist(M_{1\alpha}, x)$. We ask for the same behavior in the upper part of $\mathbb{R}^3 - (A_1 \cup A_2)$ with the corresponding changes.

Hence, for ϕ an axially symmetric function, respect to the z-axis, and abusing notation already introduce in the previous section we define the weighted norms

$$\|\phi\|_{C^{0,\sigma}_{\mu,c}(\mathbb{R}^3)} := \sup_{x \in \mathbb{R}^3} W_{\mu,c}(x)\, \|\phi\|_{C^{0,\sigma}(B_1(x))}. \tag{3.1}$$

$$\|\phi\|_{\infty,\mu,c} := \|W_{\mu,c}(x)\, \phi\|_{L^\infty(\mathbb{R}^3)}. \tag{3.2}$$

Some remarks are in order. Observe that the weight function $W_{\mu,x}(x)$, basically measures polynomial decay in $|\alpha x'|$ and exponential decay along the z direction associated to each one of the surfaces $M_{j\alpha}$. Hence, it is clear that the norms defined in (3.1)-(3.2) also depend on $\alpha > 0$, though this dependence is not explicit in the definition.

Next, let us recall that our goal is to find an axially symmetric solution to the problem

$$S(U) = \Delta U + U(1-U^2) = 0, \quad \text{in } \mathbb{R}^2 \times \mathbb{R}$$

which is close to the function w defined in (3.7). A crucial observation we make is that, under assumptions (3.5), directly from lemma (3.3.1) and the choice of the functional setting, the error created by the function w has the size

$$\|S(\mathrm{w})\|_{C^{0,\sigma}_{\mu,c}(\mathbb{R}^3)} \le C\alpha^{2(1-\tau_0)} \tag{3.3}$$

where $\mu \in (0,2]$, $0 < c < \sqrt{2}$ and $\tau_0 = \frac{c}{2\sqrt{2}}$

We proceed as we did in section 2.6, without no further changes. In fact most of the developments here, are a repetition of what has already done, with basically the same estimations. Hence, we rather prefer to give an outline of the scheme.

Define for $l = 1,2$ and $n \in \mathbb{N}$, the cut off function for $x = (x', t + \mathrm{f}_{l\alpha}) \in \mathbb{R}^2$

$$\zeta_{l,n}(x) \;=\; \beta\left(|t| - \frac{1}{2}\left[\mathrm{f}_{2\alpha}(x') - \mathrm{f}_{1\alpha(x')}\right] + n\right).$$

As before, we look for a solution to (3.1) of the form

$$U = \zeta_{1,3}(x)\varphi_1 + \zeta_{2,3}(x)\varphi_2 + \psi$$

so that we fall into a system of elliptic PDES for φ_1, φ_1 and ψ similar to (2.9)-(2.10).

The linear theory needed to solve this problem is a copy of the one sketched in section 2.7, for the system

$$\partial_{tt}\phi_l(x',t) + \Delta_{\mathbb{R}^2}\phi_l(x',t) + F'(w(t))\phi_l(x',t) \;=\; g(x',t) + c_l(x')w'(t), \quad \in \mathbb{R}^2 \times \mathbb{R} \tag{3.4}$$

$$\Delta\psi(x) - 2\psi(x) \;=\; h(x), \quad x \in \mathbb{R}^3 \tag{3.5}$$

in the class of axially symmetric functions and in the topology induced by the norms defined in (3.1)-(3.2). As we already saw, the Lyapunov-Schmidt reduction scheme is based upon the fact that we can find right hand-sides to (3.4)-(3.5) so that the functions $c_l(x') = 0$, $l = 1,2$.

3.5 Solving the reduced problem

Let us recall that

$$\mathrm{w}(x) \;:=\; \beta_\alpha(x)U_0(x) \;+\; (1-\beta_\alpha(x))(-1) \tag{3.1}$$

where

$$U_0(x) \;=\; w_1(x) + w_2(x) - 1, \quad w_j(x) := (-1)^{j-1}w(z_j - \mathrm{f}_{j\alpha}(x')), \quad x \in \mathbb{R}^3 \tag{3.2}$$

and

$$\beta_\alpha(x) \;=\; \beta(|z| - \frac{\eta}{\alpha} - 6\sqrt{2}\;\log(|\alpha x'| + 3), \quad x = (x', z) \in \mathbb{R}^3.$$

for a cut-off function $\beta \in C^\infty_c(\mathbb{R})$, such that $\beta(t) = 1$, for $|t| \le 1/2$ and $\beta(t) = 0$, for $|t| \le 1$.

We estate it in the following proposition, in order to collect estimates regarding (3.3).

Proposition 7. *Assume $c \in (0, \sqrt{2})$ and $\tau_2 > 0$ are fixed and satisfying that*

$$\tau_2 \in \left(\frac{c}{2\sqrt{2}}, 1 \right).$$

Assume that the functions $f_{j\alpha}$ satisfy condition (3.5)*. The there exists a constant $C > 0$ independent of $\alpha > 0$ such that*

$$\|S(\mathrm{w})\|_{C^{0,\sigma}_{2,c}(\mathbb{R}^3)} \leq C\alpha^{2(1-\tau_2)}. \tag{3.3}$$

and

$$\|S(\mathrm{w}, \mathrm{v}) - S(\mathrm{w}, \widetilde{\mathrm{v}})\|_{C^{0,\sigma}_{2,c}(\mathbb{R}^3)} \leq C\alpha^{2(1-\tau_2)}\|\mathrm{v} - \widetilde{\mathrm{v}}\|_*. \tag{3.4}$$

where

$$\|\mathrm{v}\|_* := \sum_{j=1}^{2} \|D^2 v_j\|_{\infty,2,\lambda} + \|(1 + |x|)Dv_j\|_{L^\infty(\mathbb{R}^2)} + \|\log(2 + |x|)v_j\|_{L^\infty(\mathbb{R}^2)} \tag{3.5}$$

Remark: This estimate does not use the fact that the vector function (f_1, f_2) is an exact solution of the Toda system (3.1)-(3.2).

In what comes next, we derive the system that governs the location of the interfaces, namely a system of PDE's that will guarantee that

$$c_j(x') = 0, \quad j = 1, 2.$$

In order to determine this functions, let us multiply the equation (3.5) by $\zeta_{l,2}(x)w_j$ and we integrate in t to get that at main order

$$-\int_{\mathbb{R}} S(\mathrm{w})\zeta_{j,2}w(t)dt - \mathcal{O}(\alpha^{2+\tau}(1 + |\alpha x'|)^{-3}) = c_j(x') \int_{\mathbb{R}} w_j^2 \zeta_{j,2} dt.$$

Hence using lemma 3.3.1, and setting

$$c^* := \int_{\mathbb{R}} |w'(t)|^2 dt, \quad a_0 = \int_{\mathbb{R}} (1 - w^2(t))w'(t)e^{-\sqrt{2}\,t}dt$$

we find that

$$c_j(x') = -\alpha^2 c^* \Delta_{\mathbb{R}^2} \mathrm{f}_{j\alpha}(x') + (-1)^j \, a_0 \, e^{-\sqrt{2}(\mathrm{f}_{2\alpha} - \mathrm{f}_{1\alpha})} + \mathcal{O}(\alpha^{2+\tau}(1 + |\alpha x'|)^{-3}).$$

It is not hard to check from 3.3.1 that the error involved in this system has Lipschitz constant of order $\mathcal{O}(\alpha^{2+\tau})$. Hence we see that making $(c_1(x'), c_2(x')) = (0, 0)$ is equivalent to a system of the form

$$\Delta f_1 + a_0 e^{-\sqrt{2}\,(f_2 - f_1)} = G_1(v_1, v_2), \quad \text{in } \mathbb{R}^2 \tag{3.6}$$

$$\Delta f_2 - a_0 e^{-\sqrt{2}\,(f_2 - f_1)} = G_2(v_1, v_2), \quad \text{in } \mathbb{R}^2. \tag{3.7}$$

With similar estimations to those in proposition (5) we find that the functions G_j satisfy the following estimates

$$\|G_j(v_1, v_2)\|_{\infty,\sigma,3} \leq K\alpha^{\kappa_1}$$

$$\|G_j(v_1, v_2) - G_j(\widetilde{v}_1, \widetilde{v}_2)\|_{\infty,\sigma,3} \leq C\alpha^{\kappa_1}\|(v_1, v_2) - (\widetilde{v}_1, \widetilde{v}_2)\|_*$$

As mentioned before a direct application of Banach Fixed point theorem finishes the proof.

Chapter 4

Two ended solution for the Inhomogeneous Allen-Cahn equation in $\mathbb{R}^2$

In this chapter we consider the equation

$$\alpha^2 \operatorname{div}(a(x)\nabla u(x)) + a(x)f(u) = 0, \quad \text{in } \mathbb{R}^2, \quad f(u) = u(1-u^2) \tag{4.1}$$

where the function $a(x)$ is a smooth positive function, and we construct a new family of solutions, whose zero level set has, outside any large ball, two asymptotically half lines as connected components. As far as our knowledge goes, little is known about entire solutions to (4.1) in the case that $a(x)$ is not identically constant, having a single transition close to a noncompact curve.

4.1 Statement of the main result

In this part, we will consider a smooth noncompact simple curve $\Gamma = \gamma(\mathbb{R})$, where $\gamma : \mathbb{R} \to \Gamma \subset \mathbb{R}^2$ is parameterized by arc-length. We denote by $\nu : \Gamma \to \mathbb{R}^2$ the choice of the normal vector to Γ, so that the curve is positively oriented. Points $x \in \mathbb{R}^2$ that are $\delta-$close to this curve, with δ small, can be represented using Fermi coordinates as follows

$$x = \gamma(s) + z \cdot \nu(s) =: X(s,z), \quad |z| < \delta, \quad s \in \mathbb{R}.$$

so that, the map $x \mapsto (s,z)$ defines a local diffeomorphism. Any smooth curve $\delta-$close to Γ in a C^m-topology can be parameterized by

$$\gamma_h(s) = \gamma(s) + h(s)\nu(s)$$

where h is a small C^m-function. The weighted length of Γ_h is given by

$$l_\Gamma(h) := \int_{\Gamma_h} a(x)d\vec{r} = \int_{-\infty}^{+\infty} a\left(\gamma_h(s)\right)|\dot{\gamma}_h(s)|ds$$
$$= \int_{-\infty}^{+\infty} a(s,h(s))|\dot{\gamma} + h\dot{\nu} + h'\nu|ds.$$

Since $|\dot{\gamma}| = 1$ and $\dot{\nu}(s) = k(s)\dot{\gamma}(s)$, where k is the signed curvature of Γ, we find that

$$l_h(h) = \int_{-\infty}^{+\infty} a(s,h(s))[(1+kh)^2 + |h'|^2]ds.$$

We say that Γ is a stationary curve respect to the function $a(x)$, if and only if,

$$l'_\Gamma[h] = \int_{\Gamma_h} (\partial_z a(s,0) + a(s,0)k(s))h(s)ds = 0, \quad \forall h \in C_c^\infty(\mathbb{R}).$$

This is equivalent to say that the curve Γ satisfies

$$\partial_z a(s,0) = k(s)a(s,0), \quad s \in \mathbb{R}. \tag{4.1}$$

Regarding the stability properties of the stationary curve Γ and the second variation of the length functional l_Γ,

$$l''_\Gamma(h,h) = \int_{-\infty}^{+\infty} \left(a(s,0)|h'(s)|^2 + [\partial_{zz}a(s,0) - 2k^2(s)]h^2(s) \right)ds$$

we have the Jacobi operator of Γ, corresponding to the linear differential operator

$$\mathcal{J}_{a,\Gamma}(h) = h''(s) + \frac{\partial_s a(s,0)}{a(s,0)}h'(s) - \left[\partial_{zz}a(s,0) - 2k^2(s)\right]h(s). \tag{4.2}$$

We say that the stationary curve Γ is also nondegenerate respect to the potential $a(x)$, if and only if, the bounded kernel of $\mathcal{J}_{a,\Gamma}$ is the trivial one. The nondegeneracy condition basically implies that $\mathcal{J}_{a,\Gamma}$ has an appropriate right inverse, so that the curve is isolated in a properly chosen topology.

In order to state the main result of this chapter, let us first list our set of assumptions on the function $a(x)$ and the curve Γ. As for the function $a : \mathbb{R}^2 \to \mathbb{R}$, we assume that

$$a \in C^5(\mathbb{R}^2), \quad 0 < m < a(x) \leq M \tag{4.3}$$

for some positive constants m, M. Assume also that in the local Fermi coordinates

$$|k_\Gamma(s)| + |k'_\Gamma(s)| + |k''_\Gamma(s)| \leq \frac{C}{(1+|s|)^{1+\alpha}} \tag{4.4}$$

for some $\alpha > 0$. In particular, condition (4.4) implies that

$$\dot{\gamma}_\pm := \lim_{s\to\pm\infty} \dot{\gamma}(s)$$

are well defined directions in $\mathbb{R}^2$. We must assume further the non-parallelism condition

$$-1 \leq \langle \dot{\gamma}_+, \dot{\gamma}_- \rangle < 1. \tag{4.5}$$

From condition (4.5), we have that the mapping $x = X(s,z)$ provides local coordinates in a region of the form

$$\mathcal{N}_\delta := \{x = X(s,z) : |z| < \delta + c_0|s|\}$$

where $c_0 > 0$ is a small constant. Finally, abusing notation, by setting $a(s,z) = a(X(s,z))$, we assume that

$$|\nabla_{s,z} a(s,z)| \leq \frac{C}{(1+|s|)^{1+\alpha}}, \quad |D^2_{s,z}\, a(s,z)| \leq \frac{C}{(1+|s|)^{2+\alpha}} \tag{4.6}$$

where $\alpha > 0$ is as in (4.4). Next, we proceed to state the main result.

Theorem 4.1.1. *Assume that $a(x)$ is a smooth potential and let Γ be a smooth stationary and nondegenerate simple curve respect to the length functional $\int_\Gamma a(x)$. Assume also that conditions (4.3)-(4.6) are satisfied. Then for any $\alpha > 0$ small enough, there exists a smooth bounded solution u_α to the inhomogeneous Allen-Cahn equation (4.1), such that*

$$u_\alpha(x) = w\left(\frac{z - h(s)}{\alpha}\right) + O(\alpha^2), \quad \text{for} \quad x = X(s,z), \quad |z| < \delta \tag{4.7}$$

where the function h satisfies

$$\|h\|_{C^1(\mathbb{R})} \leq C\alpha.$$

This solution converges to ± 1, away from Γ, namely

$$u^2_\alpha(x) \to \pm 1, \quad dist(\Gamma, x) \to \infty.$$

Remark 4.1.1: Throughout the proof of this theorem, we obtain an explicit description for u_α and its derivatives. We apply infinite dimensional reduction method in the spirit of the pioneering work due to Floer and Weinstein [20]. The presentation follows the same structure as in Chapter 2. Section 4.2 deals with the geometrical setting need to set up the proof of theorem 4.1.1. Section 4.3 is devoted to find a good approximation of a solution to (4.1). Next, we sketch the proof of theorem 4.1.1 in section 4.4, while leaving complete details for subsequent sections. In section 4.5 we present the invertibility theory for the Jacobi operator of the curve Γ and we give some examples for the function $a(x)$ and the curve Γ, where our result applies.

4.2 Geometrical background

In this section we write,in some appropriate coordinate system, the differential operator

$$\alpha^2 \Delta_{\bar{x}} u + \alpha^2 \frac{\nabla_{\bar{x}} a}{a} \cdot \nabla_{\bar{x}} u \tag{4.1}$$

involved in equation (4.1). First, observe that the obvious scaling $\bar{x} = \alpha x$ and setting $v(x) := u(\alpha x)$, transforms (4.1) into

$$\Delta_x v + \alpha \frac{\nabla_{\bar{x}} a}{a} \cdot \nabla_x v \tag{4.2}$$

Let us consider a large dilation of the curve Γ, namely $\Gamma_\alpha := \alpha^{-1}\Gamma$, for $\alpha > 0$ small. Next, we introduce local translated Fermi coordinates near Γ_α

$$\begin{aligned} X_{\alpha,h}(s,t) &= X_\alpha(s, t + h(\alpha s)) \\ &= \frac{1}{\alpha}\gamma(\alpha s) + (t + h(\alpha s))\,\nu(\alpha s) \end{aligned} \tag{4.3}$$

where $h \in C^2(\mathbb{R})$ is a bounded function. From assumption (4.5), we see that the mapping $X_{\alpha,h}(s,t)$ gives local coordinates in the region

$$\mathcal{N}_{\alpha,h} = \left\{ x = X_{\alpha,h}(s,t) \in \mathbb{R}^2 : |t + h(\alpha s)| < \frac{\delta}{\alpha} + c_0|s| \right\}$$

which is a dilated tubular neighborhood around Γ_α translated in h.

Now, we give an expression for the euclidean laplacian in terms of the coordinates $X_{\alpha,h}$. The proof goes as in section 2.2, so we leave details for the reader.

Lemma 4.2.1. *On the open neighborhood $\mathcal{N}_{\alpha,h}$ of Γ_α, the euclidean laplacian has the following expression when is computed in the coordinate $x = X_{\alpha,h}(s,t)$ reads as*

$$\begin{aligned} \Delta_{X_{\alpha,h}} &= \partial_{tt} + \partial_{ss} - 2\alpha h'(\alpha s)\partial_{st} - \alpha^2 h''(\alpha s)\partial_t + \alpha^2 |h'(\alpha s)|^2 \partial_{tt} \\ &\quad - \alpha[k(\alpha s) + \alpha(t + h(\alpha s))k^2(\alpha s)] \cdot \partial_t + D_{\alpha,h}(s,t) \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} D_{\alpha,h}(s,t) &:= \alpha(t+h)A_0(\alpha s, \alpha(t+h))[\partial_{ss} - 2\alpha h'(\alpha s)\partial_{ts} - \alpha^2 h''(\alpha s)\partial_t + \alpha^2|h'(\alpha s)|^2\partial_{tt}] \\ &\quad + \alpha^2(t+h)B_0(\alpha s, \alpha(t+h))[\partial_s - \alpha h'(\alpha s)\partial_t] \\ &\quad + \alpha^3(t+h)^2 C_0(\alpha s, \alpha(t+h))\partial_t \end{aligned} \tag{4.5}$$

for which

$$A_0(\alpha s, \alpha[t + h(\alpha s)]) = 2k(\alpha s) + \alpha O(|[t + h(\alpha s)]k^2(\alpha s)|) \tag{4.6}$$

$$B_0(\alpha s, \alpha[t + h(\alpha s)]) = \dot{k}(\alpha s) + \alpha O(|(t + h(\alpha s))\dot{k}(\alpha s) \cdot k^2(\alpha s)|) \tag{4.7}$$

$$C_0(\alpha s, \alpha[t + h(\alpha s)]) = k^3(\alpha s) + \alpha O(|(t + h(\alpha s))k^4(\alpha s)|) \tag{4.8}$$

are smooth functions and these relations can be derived.

Next, we derive an expression for the second term in equation (4.2), in terms of the Fermi coordinates. We collect the computations in the following lemma, whose proof is also left to the reader since it is a straight forward calculation using the chain rule.

Lemma 4.2.2. *In the open neighborhood $\mathcal{N}_{\alpha,h}$ of Γ_α, we have the validity of the following expression*

$$\alpha\frac{\nabla_X a}{a} \cdot \nabla_{X_{\alpha,h}} = \alpha\left[\frac{\partial_t a}{a}(\alpha s, 0) + \alpha(t + h(\alpha s))\left(\frac{\partial_{tt} a}{a}(\alpha s, 0) - \left|\frac{\partial_t a}{a}(\alpha s, 0)\right|^2\right)\right]\partial_t$$

$$+\ \alpha\frac{\partial_{\boldsymbol{s}} a}{a}(\alpha s,0)[\partial_s-\alpha h'(\alpha s)\cdot\partial_t]+E_{\alpha,h}(s,t)$$

where

$$E_{\alpha,h}(s,t):=\alpha^2(t+h(\alpha s))D_0(\alpha s,\alpha(t+h))[\partial_s-\alpha h'(\alpha s)\cdot\partial_t]$$
$$+\alpha^3(t+h(\alpha s))^2F_0(\alpha s,\alpha(t+h))\partial_t \tag{4.9}$$

and for which the next functions are smooth

$$D_0(\alpha s,\alpha(t+h))=\partial_{\boldsymbol{t}}\left[\frac{\partial_{\boldsymbol{s}} a}{a}\right](\alpha s,0)+\alpha O\left((t+h(\alpha s))\partial_{\boldsymbol{tt}}\left[\frac{\partial_t a}{a}\right]\right)$$
$$+A_0(\alpha s,\alpha(t+h))\cdot\frac{\partial_{\boldsymbol{s}} a}{a}(\alpha s,\alpha(t+h)) \tag{4.10}$$
$$F_0(\alpha s,\alpha(t+h))=\frac{1}{2}\partial_{\boldsymbol{tt}}\left[\frac{\partial_t a}{a}\right](\alpha s,0)+\alpha O\left((t+h(\alpha s))\partial_{\boldsymbol{ttt}}\left[\frac{\partial_t a}{a}\right]\right) \tag{4.11}$$

and where $A_0(\alpha s,\alpha(t+h))$ given by (4.6). Further, these relations can be differentiated.

Using lemmas 4.2.1 and 4.2.2, the fact that the curve Γ satisfies condition (4.1), we can write expression (4.2) in coordinates $x=X_{\alpha,h}(s,t)$ as

$$\Delta_{X_{\alpha,h}}+\alpha\frac{\nabla_X a(\alpha x)}{a(\alpha x)}\cdot\nabla_{X_{\alpha,h}}\ =$$
$$\partial_{tt}\ +\ \partial_{ss}\ +\ \alpha\frac{\partial_{\boldsymbol{s}} a}{a}(\alpha s,0)\partial_s$$
$$-\,\alpha^2\left\{h''(\alpha s)+\frac{\partial_{\boldsymbol{s}} a}{a}(\alpha s,0)\,h'(\alpha s)+\left[2\,k^2(\alpha s)-\frac{\partial_{\boldsymbol{tt}} a}{a}(\alpha s,0)\right]h(\alpha s)\right\}\partial_t$$
$$-\alpha^2\left[k^2(\alpha s)-\frac{\partial_{\boldsymbol{tt}} a}{a}(\alpha s,0)+\left|\frac{\partial_{\boldsymbol{t}} a}{a}(\alpha s,0)\right|^2\right]t\,\partial_t-2\alpha h'(\alpha s)\partial_{st}+\alpha^2|h'(\alpha s)|^2\partial_{tt}$$
$$+\alpha(t+h(\alpha s))A_0(\alpha s,\alpha(t+h))[\partial_{ss}-2\alpha h'(\alpha s)\partial_{ts}-\alpha^2h''(\alpha s)\partial_t+\alpha^2|h'(\alpha s)|^2\partial_{tt}$$
$$+\ \alpha^2\,(t+h(\alpha s))\tilde{B}_0(\alpha s,\alpha(t+h))[\partial_s-\alpha h'(\alpha s)\partial_t]\ +\ \alpha^3(t+h(\alpha s))^2\tilde{C}_0(\alpha s,\alpha(t+h))\partial_t \tag{4.12}$$

where

$$\tilde{B}_0(\alpha s,\alpha(t+h)):=B_0(\alpha s,\alpha(t+h))+D_0(\alpha s,\alpha(t+h)) \tag{4.13}$$
$$\tilde{C}_0(\alpha s,\alpha(t+h)):=C_0(\alpha s,\alpha(t+h))+F_0(\alpha s,\alpha(t+h)). \tag{4.14}$$

4.3 Approximation of the solution and preliminary discussion

To begin with, let us consider the parameter function $h\in C^2(\mathbb{R})$, for which we assume further $h=h(\boldsymbol{s})$ the apriori estimate

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})}:=\|h\|_{L^\infty(\mathbb{R})}+\|(1+|\,\boldsymbol{s}\,|)^{1+\alpha}h'\|_{L^\infty(\mathbb{R})}+\sup_{\boldsymbol{s}\in\mathbb{R}}(1+|\,\boldsymbol{s}\,|)^{2+\alpha}\|h''\|_{C^{0,\lambda}(\boldsymbol{s}-1,\boldsymbol{s}+1)}\leq\mathcal{K}\alpha \tag{4.1}$$

for a certain constant $\mathcal{K} > 0$ that will be chosen later, but independent of $\alpha > 0$.

Let us consider $w(t)$, the solution to the ODE

$$w''(t) + w(t)(1 - w^2(t)) = 0, \quad w'(t) > 0, \quad w(\pm\infty) = \pm 1.$$

So, in the region $\mathcal{N}_{\alpha,h}$, we choose as first approximate for a solution to (4.1), the function

$$u_0(x) := w(z - h(\alpha s)) = w(t), \quad x = X_{\alpha,h}(s,t) \in \mathcal{N}_{\alpha,h} \tag{4.2}$$

where $z = t + h(\alpha s)$ designates the normal coordinate to Γ. Setting

$$S(v) := \Delta_x v + \alpha \frac{\nabla_{\bar{x}} a}{a} \nabla_x v + v(1 - v^2) = 0, \quad \text{in } \mathbb{R}^2 \tag{4.3}$$

and using expression (4.12), we compute the error $S(u_0)$ in the region $\mathcal{N}_{\alpha,h}$ to find that

$$\begin{aligned} S(u_0) \;=\; & -\alpha^2 \mathcal{J}_a[h](\alpha s) w'(t) - \alpha^2 \left[2k^2(\alpha s) \;-\; \frac{\partial_{\boldsymbol{tt}} a}{a}(\alpha s, 0) \right] t w'(t) \;+\; \alpha^2 |h'(\alpha s)|^2 w''(t) \\ & + \alpha(t + h(\alpha s)) A_0(\alpha s, \alpha(t+h))[-\alpha^2 h''(\alpha s) w'(t) + \alpha^2 |h'(\alpha s)|^2 w''(t)] \\ & + \alpha^2 (t + h(\alpha s)) \tilde{B}_0(\alpha s, \alpha(t+h)) \left(-\alpha h'(\alpha s) w'(t)\right) \\ & + \alpha^3 (t + h(\alpha s))^2 \tilde{C}_0(\alpha s, \alpha(t+h)) w'(t) \end{aligned} \tag{4.4}$$

with $A_0, B_0, \tilde{C}_0$ are given in (4.6)-(4.13)- (4.14), respectively.

Now, for every fixed $s \in \mathbb{R}$, let us consider the L^2-projection, given by

$$\Pi(s) := \int_{-\infty}^{+\infty} S(u_0)(s,t) w'(t) dt$$

where for simplicity we are assuming that coordinates are defined for all t, since the difference with the integration taken in all the actual domain for t produces only exponentially small terms. From (4.4), we observe that

$$\begin{aligned} \Pi(\alpha s) = & -\alpha^2 \mathcal{J}_a[h](\alpha s) \int_{\mathbb{R}} |w'(t)|^2 dt \\ & - \alpha^3 \int_{\mathbb{R}} (t+h) A_0(\alpha s, \alpha(t+h)) \left[h''(\alpha s) \, |w'(t)|^2 - |h'(\alpha s)|^2 \, w''(t) w'(t) \right] dt \\ & - \alpha^3 \int_{\mathbb{R}} \left[(t+h) \tilde{B}_0(\alpha s, \alpha(t+h)) h'(\alpha s) \;-\; (t+h)^2 \tilde{C}_0(\alpha s, \alpha(t+h)) \right] |w'(t)|^2 dt \end{aligned} \tag{4.5}$$

where we have used that $\int_{-\infty}^{+\infty} t |w'(t)|^2 dt = 0$, $\int_{-\infty}^{+\infty} w''(t) w'(t) dt = 0$, to get rid of the terms of order α^2.

Making this projection equal to zero is equivalent to the nonlinear differential equation for h

$$\mathcal{J}_{a,\Gamma}[h] = h''(\boldsymbol{s}) + \frac{\partial_s a(\boldsymbol{s}, 0)}{a(\boldsymbol{s}, 0)} h'(\boldsymbol{s}) - Q(\boldsymbol{s}) h(\boldsymbol{s}) = G_0[h] \;, \; \forall \boldsymbol{s} \in \mathbb{R} \tag{4.6}$$

where we have set
$$Q(\boldsymbol{s}) = \frac{\partial_{tt} a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)} - 2k^2(\boldsymbol{s})$$
and G_0 consists in the remaining terms of (4.5). G_0 is easily checked to be a Lipschitz in h, with small Lipschitz constant. Here is where the *nondegeneracy condition* on the curve Γ makes its entrance, since we need to invert the operator $\mathcal{J}_{a,\Gamma}$, in such way that equation (4.6) can be set as a fixed problem for a contraction mapping of a ball of the form (4.1).

It will be necessary to pay attention to the terms
$$\alpha^2 \left[\frac{\partial_{tt} a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)} - 2k^2(\boldsymbol{s})\right] tw'(t), \quad \alpha^3 C_0(\alpha s, \alpha[t + h(\alpha s)])w'(t) \tag{4.7}$$
since they are involved in the size of $S(u_0)$ up to $O(\alpha^2)$ and because the solvability of the nonlinear Jacobi equation (4.6) depends strongly on the fact that the error created by our choice of the approximation, is sufficiently small in $\alpha > 0$. Let us mention that the second term in (4.7) must be improve only due to technical reason concerning our choice of the functional analytical scheme. We improve our choice of the approximation throughout the following argument. Let us consider the ODE
$$\psi_0''(t) + f'(w(t))\psi_0(t) = tw'(t)$$
which has a unique bounded solution with $\psi_0(0) = 0$, given explicitly by the variation of parameters formula
$$\psi_0(t) = w'(t) \int_0^t |w'(s)|^{-2} ds \cdot \int_{-\infty}^t s|w'(s)|^2 ds.$$
Since $\int_{\mathbb{R}} s|w'(s)|^2 ds = 0$, the function $\psi_0(s)$ satisfies that $\psi_0(t) \sim e^{-\sqrt{2}|t|}$ as $|t| \to \infty$.

Analogously, consider $g(t) := t^2 w'(t)$ and note that we can write
$$g = C_g w' + g_\perp$$
where $g_\perp$ denotes the orthogonal projection of g onto w' in $L^2(\mathbb{R})$, given by
$$g_\perp(t) := t^2 w'(t) - \left(\int_{\mathbb{R}} \tau^2 |w'(\tau)|^2 d\tau / \int_{\mathbb{R}} |w'(\tau)|^2 d\tau\right) w'(t).$$
Thus by setting
$$\psi_1(t) = w'(t) \int_0^t |w'(s)|^{-2} ds \cdot \int_{-\infty}^t g_\perp(t) \cdot w'(s) ds$$
this formula not only provides a bounded solution of $\psi_1''(t) + f'(w(t))\psi_1(t) = g_\perp(t)$, since $\int_{\mathbb{R}} g_\perp(t)w'(t)dt = 0$, but also provides a solution with exponential decay $\psi_1(t) \sim e^{-\sqrt{2}|t|}$ as $|t| \to +\infty$, given that $g_\perp$ exhibits this exponential decay. Hence, we choose as new approximation, the function
$$u_1(s,t) := u_0(s,t) + \varphi_1(s,t) = w(t) + \varphi_1(s,t) \tag{4.8}$$
where
$$\begin{aligned} \varphi_1(s,t) \quad := \quad & \alpha^2 \left[2k^2(\alpha s) - \frac{\partial_{\boldsymbol{tt}} a(\alpha s,0)}{a(\alpha s,0)}\right] \psi_0(t) \\ & -\alpha^3 \left[k^3(\alpha s) + \frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\alpha s, 0)\right] \psi_1(t) \end{aligned} \tag{4.9}$$

which can be easily seen to behave like $\varphi_1(s,t) = O(\alpha^2(1+|\alpha s|)^{-2-\alpha}e^{-\sqrt{2}|t|})$, thanks to assumptions (4.4)-(4.6) we have made on the curve Γ and the potential a, and to the previous observation on $\psi_0(t),\ \psi_1(t)$.

Now, to analyze the error terms created by the Allen-Cahn equation (4.3) on the second approximation $u_1(s,t)$, note that

$$S(u_0+\varphi_1) = S(u_0) \ + \ \Delta_x\varphi_1 + \alpha\frac{\nabla_{\bar{x}}a}{a}\nabla_x\varphi_1 + f'(u_0)\varphi_1 + +N_0(\varphi_1) \tag{4.10}$$

where

$$N_0(\varphi_1) = f(u_0+\varphi_1) - f(u_0) - f'(u_0)\varphi_1 \tag{4.11}$$

From the definition of φ_1, we find that

$$\begin{aligned} S(u_1) \ &= \ S(u_0) \ + \ \alpha^2\left[2k^2(\alpha s) - \frac{\partial_{tt}a(\alpha s,0)}{a(\alpha s,0)}\right] tw'(t) \\ &\quad - \alpha^3\left[k^3(\alpha s) + \frac{1}{2}\partial_{tt}\left(\frac{\partial_t a}{a}\right)(\alpha s,0)\right] g_\perp(t) \\ &\quad + \left[\Delta_x + \alpha\frac{\nabla_{\bar{x}}a}{a}\nabla_x - \partial_{tt}\right]\varphi_1 \ + \ N_0(\varphi_1). \end{aligned} \tag{4.12}$$

Analyzing the new error created by φ_1, we readily check using the expansions for the differential operators (4.4)-(4.9) and the definition (4.11), that

$$\begin{aligned} \left[\Delta_x + \alpha\frac{\nabla_{\bar{x}}a}{a}\nabla_x - \partial_{tt}\right]\varphi_1 \ + \ N_0(\varphi_1) \ &= \ -\alpha^4\, Q''(\alpha s)\psi_0) + \alpha^4\big[\mathcal{J}_a[h](\alpha s) - tQ(\alpha s)\big]Q(\alpha s)\psi_0' \\ &\quad - \alpha^4\left(\frac{\partial_{\boldsymbol{s}}a(\alpha s,0)}{a(\alpha s,0)}Q'(\alpha s)\psi_0 + 2h_1'(-Q'(\alpha s)\psi_0') + |h_1'|^2 Q(\alpha s)\psi_0''\right) \\ &\quad + O(\alpha^4)w(t)(-Q(\alpha s)\psi_0)^2 + O(\alpha^5(1+|\alpha s|)^{-4-\alpha}e^{-\sqrt{2}|t|}) \end{aligned} \tag{4.13}$$

where we recall the convention

$$Q(\boldsymbol{s}) = \frac{\partial_{\boldsymbol{tt}}a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)} - 2k^2(\boldsymbol{s}), \quad \boldsymbol{s}\in\mathbb{R}. \tag{4.14}$$

and have used that the error terms in the differential operator evaluated in φ_1, associated to $A_0(\alpha s,\alpha(t+h)), \tilde{B}_0(\alpha s,\alpha(t+h)), \tilde{C}_0(\alpha s,\alpha(t+h))$ behave like $O(\alpha^5(1+|\alpha s|)^{-4-2\alpha}e^{-\sqrt{2}|t|})$, given that h has a bounded size is αs by (4.1), and since $\varphi_1(s,t)$ has smooth dependence in αs with size $O(\alpha^2(1+|\alpha s|)^{-2-\alpha}e^{-\sqrt{2}|t|})$.

Therefore, the error (4.13) is can be written as

$$\left[\Delta_x + \alpha\frac{\nabla_{\bar{x}}a}{a}\nabla_x - \partial_{tt}\right]\varphi_1 + N_0(\varphi_1) = \alpha^4 Q(\alpha s)\psi_0'(t)h''(\alpha s) + R_0(\alpha s,t,h) \tag{4.15}$$

where the function $R_0 = R_0(\alpha s,t,h(\alpha s),h'(\alpha s))$ has Lipschitz dependence in variables h, h' on the ball

$$\|h\|_{L^\infty(\mathbb{R})} + \|(1+|\boldsymbol{s}|^{1+\alpha})h'\|_{L^\infty(\mathbb{R})} \le \mathcal{K}\alpha.$$

Moreover, under our set of assumptions and the observation made on ψ_0, it turns out that for any $\lambda \in (0,1)$:

$$\|R_0(\alpha s, t, h)\|_{C^{0,\lambda}(B_1(s,t))} \le C\alpha^4(1+|\alpha s|)^{-4-\alpha}e^{-\sqrt{2}|t|}.$$

With this remarks, we can write the error of u_1, in (4.12) as

$$S(u_1) = -\alpha^2 \mathcal{J}_a[h](\alpha s)w'(t) + \alpha^4 Q(\alpha s)\psi_0'(t)h''(\alpha s)$$
$$- \alpha^3(t+h)A_0(\alpha s, \alpha(t+h))h''(\alpha s)w'(t) \;+\; R_1(\alpha s, t, h(\alpha s), h'(\alpha s)) \tag{4.16}$$

where

$$R_1 = \alpha^2|h'|^2 w''(t) + R_0(\alpha s, t) + \alpha^3(t+h)A_0(\alpha s, \alpha(t+h))|h'|^2 w''(t)$$
$$- \alpha^3(t+h)\tilde{B}_0(\alpha s, \alpha(t+h))h'w'(t) + \alpha^4(t+h)\,O\left(\partial_{\boldsymbol{ttt}}\left(\frac{\partial_{\boldsymbol{t}} a}{a}\right) + k^4\right)t^2 w'(t). \tag{4.17}$$

Furthermore, $R_1 = R_1(\alpha s, t, h(\alpha s), h'(\alpha s))$ satisfies that

$$|\partial_\imath R_1(\alpha s, t, \imath, \jmath)| + |\partial_\jmath R_1(\alpha s, t, \imath, \jmath)| + |R_1(\alpha s, t, \imath, \jmath)| \le C\alpha^4(1+|\alpha s|)^{-2-2\alpha}e^{-\sqrt{2}|t|}$$

with the constant C above depending on the number $\mathcal{K}$ of condition (4.1), but independent of $\alpha > 0$.

We can summarize this discussion by saying that

$$S(u_1) \;+\; \alpha^2\mathcal{J}_a[h](\alpha s)w'(t) \;=\; O(\alpha^4(1+|\alpha s|)^{-2-\alpha}e^{-\sqrt{2}|t|}), \quad x = X_{\alpha,h}(s,t) \in \mathcal{N}_{\alpha,h}. \tag{4.18}$$

The approximation $u_1(x)$ in (4.8) will be sufficient for our purposes. However, it is defined only in the region

$$\mathcal{N}_{\alpha,h} = \left\{x = X_{\alpha,h}(s,t) \in \mathbb{R}^2 / \; |t + h(\alpha s)| < \frac{\delta}{\alpha} + c_0|s| =: \rho_\alpha(s)\right\} \tag{4.19}$$

Since we are assuming that Γ is a connected and simple and that it also possesses two ends departing from each other, it follows that $\mathbb{R}^2 \setminus \Gamma_\alpha$ consists of precisely two components S_+ and S_-. Let us use the convention that ν_α points towards S_+. The previous comments allow us to define in $\mathbb{R}^2 \setminus \Gamma_\alpha$ the function

$$\mathbb{H}(x) := \begin{cases} +1 & \text{if } x \in S_+ \\ -1 & \text{if } x \in S_- \end{cases} \tag{4.20}$$

Let us consider $\eta(s)$ a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$, and define

$$\zeta_3(x) := \begin{cases} \eta(|t + h(\alpha s)| - \rho_\alpha(s) + 3) & \text{if } x \in \mathcal{N}_{\alpha,h} \\ 0 & \text{if } x \notin \mathcal{N}_{\alpha,h} \end{cases} \tag{4.21}$$

where ρ_α is defined in (4.19).

Next, we consider as global approximation the function $\mathtt{w}(x)$ defined as

$$\mathtt{w} := \zeta_3 \cdot u_1 + (1 - \zeta_3) \cdot \mathbb{H} \tag{4.22}$$

where $u_1(x)$ is given by (4.8). Using that $\mathbb{H}(\alpha^{-1}\,\bar{x})$ is an exact solution to (4.1) in $\mathbb{R}^2 \setminus \Gamma$, the error of global approximation can be computed as

$$S(\mathtt{w}) \;=\; \Delta_x \mathtt{w} + \alpha \frac{\nabla_{\bar{x}} a}{a} \nabla_x \mathtt{w} + f(\mathtt{w}) \;=\; \zeta_3 S(u_1) + E \tag{4.23}$$

where $S(u_1)$ is computed in (4.16) and the term E is given by

$$E = \Delta_x \zeta_3 (u_1 - \mathbb{H}) + 2\nabla_x \zeta_3 \nabla_x (u_1 - \mathbb{H}) + (u_1 - \mathbb{H}) \frac{\nabla_{\bar{x}} a}{a} \nabla_x \zeta_3 \\ + f\big(\zeta_3 u_1 + (1-\zeta_3)\mathbb{H}\big) - \zeta_3 f(u_1) \tag{4.24}$$

It is worth to mention that the from the form of the neighborhood $\mathcal{N}_{\alpha,h}$ in (4.19), and from the choice of u_1, one can readily check that for every $x = X_{\alpha,h} \in \mathcal{N}_{\alpha,h}$

$$|u_1(x) - \mathbb{H}(x)| \le e^{-\sqrt{2}|t+h(\alpha s)|}, \quad \rho_\alpha - 2 < |t + h(\alpha s)| < \rho_\alpha - 1$$

and therefore

$$|E| \le C\; e^{-\sqrt{2}|t+h(\alpha s)|} \le C e^{-\sqrt{2}\delta/\alpha} \cdot e^{-c|s|} e^{-\sigma|t|}$$

for some $0 < \sigma < \sqrt{2}$ and $c > 0$ small.

4.4 The proof of Theorem 4.1.1

In this section we sketch the proof of theorem 4.1.1 leaving the detailed proofs of every proposition mentioned here for subsequent sections.

We look for a solution u of the inhomogeneous Allen-Cahn equation (4.2) in the form

$$u = \mathtt{w} + \varphi \tag{4.1}$$

where $\mathtt{w}$ is the global approximation defined in (4.22) and φ is small in some suitable sense. We find that φ must solve the following nonlinear equation

$$\Delta_x \varphi + \alpha \frac{\nabla_{\bar{x}} a}{a} \nabla_x \varphi + f'(\mathtt{w})\varphi \;=\; -S(\mathtt{w}) \;-\; N_1(\varphi) \tag{4.2}$$

where

$$S(\mathtt{w}) := \Delta_x \mathtt{w} + \alpha \frac{\nabla_{\bar{x}} a}{a} \nabla_x \mathtt{w} + f(\mathtt{w}) \tag{4.3}$$

$$N_1(\varphi) := f(\mathtt{w} + \varphi) - f(\mathtt{w}) - f'(\mathtt{w})\varphi \tag{4.4}$$

We introduce several norms that will allow us to set up an appropriate functional scheme to solve (4.2). Let us consider $\eta(s)$, a cut-off function with $\eta(s) = 1$ for $s < 1$ and $\eta = 0$ for $s > 2$, we define

$$\zeta_n(x) := \begin{cases} \eta\left(|t + h(\alpha s)| - \rho_\alpha(s) + n\right) & \text{if } x \in \mathcal{N}_{\alpha,h} \\ 0 & \text{if } x \notin \mathcal{N}_{\alpha,h} \end{cases} \tag{4.5}$$

where ρ_α and $\mathcal{N}_{\alpha,h}$ are set in (4.19). Let us consider $\lambda \in (0,1)$, $b_1, b_2 > 0$ fixed and satisfying that $b_1^2 + b_2^2 < (\sqrt{2} - \tau)/2$ for $\tau > 0$. Define the weight function $K(x)$, for $x = (x_1, x_2)$, as follows

$$K(x) := \zeta_2(x) \left[e^{\sigma|t|/2}(1 + |\alpha s|)^\mu \right] + (1 - \zeta_2(x))e^{b_1|x_1| + b_2|x_2|}. \tag{4.6}$$

For a function $g(x)$ defined in $\mathbb{R}^2$, we set the norms

$$\|g\|_{L^\infty_K(\mathbb{R}^2)} := \sup_{x \in \mathbb{R}^2} K(x) \|g\|_{L^\infty(B_1(x))} \tag{4.7}$$

$$\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} := \sup_{x \in \mathbb{R}^2} K(x) \|g\|_{C^{0,\lambda}(B_1(x))} \tag{4.8}$$

On the other hand, consider $\alpha > 0$ and certain $\mu \geq 0$, $0 < \sigma < \sqrt{2}$. For functions $g(s,t)$ and $\phi(s,t)$ defined in whole $\mathbb{R} \times \mathbb{R}$, we set

$$\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} := \sup_{(s,t) \in \mathbb{R} \times \mathbb{R}} (1 + |\alpha s|)^\mu e^{\sigma|t|} \|g\|_{C^{0,\lambda}(B_1(s,t))} \tag{4.9}$$

$$\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} := \|D^2\phi\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|D\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} + \|\phi\|_{L^\infty_{\mu,\sigma}(\mathbb{R}^2)} \tag{4.10}$$

Finally, given $\alpha > 0$ and $\lambda \in (0,1)$, consider,for a function f defined in $\mathbb{R}$, the norm

$$\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} := \sup_{\boldsymbol{s} \in \mathbb{R}} (1 + |\boldsymbol{s}|)^{2+\alpha} \|f\|_{C^{0,\lambda}(\boldsymbol{s}-1, \boldsymbol{s}+1)}. \tag{4.11}$$

Recall also that the parameter function $h(\boldsymbol{s})$ satisfies for some $\lambda \in (0,1)$

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq \mathcal{K}\alpha$$

where

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} := \|h\|_{L^\infty(\mathbb{R})} + \|(1 + |\boldsymbol{s}|)^{1+\alpha} h'\|_{L^\infty(\mathbb{R})} + \|h''\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})}. \tag{4.12}$$

In order to solve (4.2), let us look for a solution φ of problem, having the form

$$\varphi(x) = \zeta_3(x)\phi(s,t) + \psi(x), \quad \text{for } x \in \mathbb{R}^2 \tag{4.13}$$

where ϕ is defined in $\Gamma_\alpha \times \mathbb{R}$ and ψ is defined in entire $\mathbb{R}^2$. Using that $\zeta_3 \cdot \zeta_4 = \zeta_4$, we get that (4.2) reads as

$$\begin{aligned} S(\mathtt{w} + \varphi) = \zeta_3 \left[\Delta_x \phi + \alpha \frac{\nabla_{\bar{x}} a}{a} \nabla_x \phi + f'(u_1)\phi \right] \\ + \zeta_4 \left[\, [f'(u_1) - f'(H(t))]\psi + N_1(\psi + \phi) + S(u_1) \right] \\ + \Delta_x \psi + \alpha \frac{\nabla_{\bar{x}} a}{a} \nabla_x \psi + [(1 - \zeta_4) f'(u_1) + \zeta_4 f'(H(t))]\psi + (1 - \zeta_3) S(\mathtt{w}) \\ + (1 - \zeta_4) N_1(\psi + \zeta_3 \phi) + 2\nabla_x \zeta_3 \nabla_x \phi + \phi \Delta_x \zeta_3 + \alpha \phi \frac{\nabla_{\bar{x}} a}{a} \nabla_x \zeta_3 \end{aligned}$$

where $H(t)$ is some increasing smooth function satisfying

$$H(t) = \begin{cases} +1 & \text{if } t > 1 \\ -1 & \text{if } t < -1. \end{cases} \tag{4.14}$$

In this way, we will have constructed a solution $\varphi = \zeta_3\phi + \psi$ to problem (4.2) if we require that the pair (ϕ, ψ) satisfies the coupled system below

$$\Delta_x \phi + \alpha \frac{\nabla_{\bar{x}} a}{a} \nabla_x \phi + f'(u_1)\phi$$
$$+ \zeta_4[f'(u_1) - f'(H(t))]\psi + \zeta_4 N_1(\psi + \phi) + S(u_1) = 0 \quad \text{for } |t| < \frac{\delta}{\alpha} \tag{4.15}$$

$$\Delta_x \psi + \alpha \frac{\nabla_{\bar{x}} a}{a} \nabla_x \psi + [(1-\zeta_4)f'(u_1) + \zeta_4 f'(H(t))]\psi + (1-\zeta_3)S(\mathtt{w})$$
$$+ (1-\zeta_4)N_1(\psi + \zeta_3\phi) \ + \ 2\nabla_x\zeta_3 \nabla_x \phi + \phi\Delta_x\zeta_3 + \alpha\phi\frac{\nabla_{\bar{x}} a}{a}\nabla_x \zeta_3 = 0, \quad \text{in} \quad \mathbb{R}^2 \tag{4.16}$$

Next, we will extend equation (4.15) to entire $\mathbb{R} \times \mathbb{R}$. To do so, let us set

$$\mathtt{B}(\phi) = \zeta_0 \tilde{\mathtt{B}}_0(\phi) := \zeta_0[\Delta_x - \partial_{tt} - \partial_{ss}]\phi \tag{4.17}$$

where Δ_x is expressed in local coordinates, using formula (4.4), and $\mathtt{B}(\phi)$ is understood to be zero for $|t + h(\alpha s)| > \rho_\alpha(s,t) - 2$. Thus equation (4.15) is extended as

$$\partial_{tt}\phi + \partial_{ss}\phi + \alpha\frac{\nabla_{\bar{x}} a}{a}\nabla_x\phi + \mathtt{B}(\phi) + f'(w(t))\phi = -\tilde{S}(u_1)$$
$$- \left\{[f'(u_1) - f'(w)]\phi + \zeta_4[f'(u_1) - f'(H(t))]\psi + \zeta_4 N_1(\psi + \phi)\right\}, \ \text{ in } \ \mathbb{R} \times \mathbb{R} \tag{4.18}$$

where we have denoted

$$\tilde{S}(u_1)(s,t) = -\alpha^2 \mathcal{J}_a[h](\alpha s) w'(t) + \alpha^4 Q(\alpha s)\psi_0'(t) \cdot h''(\alpha s)$$
$$+ \zeta_0 \left\{ \alpha^3 (t+h) A_0(\alpha s, \alpha(t+h)) h''(\alpha s) \cdot w''(t) + R_1 \right\}. \tag{4.19}$$

Recall from (4.17) that

$$R_1 = R_1(\alpha s, t, h(\alpha s), h'(\alpha s))$$

satisfies

$$|\partial_\imath R_1(\alpha s, t, \imath, \jmath)| + |\partial_\jmath R_1(\alpha s, t, \imath, \jmath)| + |R_1(\alpha s, t, \imath, \jmath)| \le C\alpha^4(1 + |\alpha s|)^{-2-2\alpha} e^{-\sqrt{2}|t|}. \tag{4.20}$$

We notice that$\tilde{S}(u_1)$ coincides with $S(u_1)$ in the region where $\zeta_0 \equiv 1$, while outside the support of ζ_0 the terms of $S(u_1)$ which are not defined for all t, are cut off.

To solve the resulting system (4.16)-(4.18), we focus first on solving equation (4.16) in ψ for a fixed and small ϕ. We make ue of the important observation that the term $[(1-\zeta_4)f'(u_1) + \zeta_4 f'(H)]$, is uniformly negative and so the operator in (4.16) is qualitatively similar to $\Delta_x + \alpha\nabla_{\bar{x}} a/a \cdot \nabla_x - 2$. A direct application of the contraction mapping principle lead us to the existence of a solution $\psi = \Psi(\phi)$, according to the next proposition whose detailed proof is carried out in Section 4.5.

Proposition 8. *Let $\lambda \in (0,1)$, $\sigma \in (0,\sqrt{2})$, $\mu \in (0, 2+\alpha)$. There is $\alpha_0 > 0$, such that for any small $\alpha \in (0,\alpha_0)$ the following holds. Given ϕ with $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le 1$, there is a unique solution $\psi = \Psi(\phi)$ to equation (4.16) with*

$$\|\psi\|_X := \|D^2\psi\|_{C^{0,\lambda}_K(\mathbb{R}^2)} + \|D\psi\|_{L^\infty_K(\mathbb{R}^2)} + \|\psi\|_{L^\infty_K(\mathbb{R}^2)} \le Ce^{-\sigma\delta/2\alpha} \tag{4.21}$$

Besides, Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_X \le Ce^{-\sigma\delta/2\alpha}\|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{4.22}$$

where the norms $L^\infty_K, C^{0,\lambda}_K, C^{2,\lambda}_{\mu,\sigma}$ are defined in (4.7)-(4.8)-(4.10).

Using proposition 8, we solve (4.18) replacing ψ with the nonlocal operator $\psi = \Psi(\phi)$. Setting

$$\begin{aligned}\mathtt{N}(\phi) := \mathtt{B}(\phi) + \alpha\frac{\nabla_{\bar{x}}a}{a}\nabla_x\phi + [f'(u_1) - f'(w)]\phi \\ + \zeta_4[f'(u_1) - f'(H(t))]\Psi(\phi) + \zeta_4 N_1(\Psi(\phi) + \phi)\end{aligned} \tag{4.23}$$

our problem is reduced to find a solution ϕ to the following nonlinear, nonlocal problem

$$\partial_{tt}\phi + \partial_{ss}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathtt{N}(\phi) \quad \text{in} \quad \mathbb{R}\times\mathbb{R}. \tag{4.24}$$

Before solving (4.24), we consider the problem of finding a (ϕ, c) a solution to the following nonlinear projected problem

$$\begin{cases} \partial_{tt}\phi + \partial_{ss}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathtt{N}(\phi) + c(s)w'(t) \quad \text{in} \quad \mathbb{R}\times\mathbb{R} \\ \displaystyle\int_{\mathbb{R}} \phi(s,t)w'(t)dt = 0, \quad \forall s \in \mathbb{R}. \end{cases} \tag{4.25}$$

Solving problem (4.25) amounts to eliminate the part of the right hand side in (4.24), that do not contribute to the projections onto $w'(t)$, namely $\int_{\mathbb{R}}[\tilde{S}(u_1) + N(\phi)]w'(t)dt$. Since, we have that

$$\|\tilde{S}(u_1) + \alpha^2\mathcal{J}_a[h](\alpha s)\cdot w'(t)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le C\alpha^4 \tag{4.26}$$

and due to the fact that $\mathtt{N}(\phi)$ defines a contraction within a ball centered at zero with radius $O(\alpha^4)$ in norm C^1, we conclude the existence of a unique small solution of problem (4.25) whose size is $O(\alpha^4)$ in this norm. This solution ϕ turns out to define an operator in h, namely $\phi = \Phi(h)$, which exhibits a Lipschitz character in norms $\|\cdot\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$. We collect the discussion in the following proposition.

Proposition 9. *Given $\lambda \in (0,1)$, $\mu \in (0, 2+\alpha]$ and $\sigma \in (0,\sqrt{2})$, there exists a constant $K > 0$ such that the nonlinear projected problem (4.25) has a unique solution $\phi = \Phi(h)$ with*

$$\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le K\alpha^4. \tag{4.27}$$

Besides Φ has small a Lipschitz dependence on h satisfying condition (4.1), in the sense

$$\|\Phi(h_1) - \Phi(h_2)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le C\alpha^3\|h_1 - h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})} \tag{4.28}$$

for any $h_1, h_2 \in C^{2,\lambda}_{loc}(\mathbb{R})$ with $\|h_i\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})} \le \mathcal{K}\alpha$.

The proof of this proposition is left to section 4.5, where a complete study of the linear theory needed to solve is discussed.

In order to conclude the proof of Theorem 4.1.1, we have to adjust the parameter function h so that the nonlocal term

$$c(s)\int_{\mathbb{R}}|w'(t)|^2dt = \int_{\mathbb{R}}\tilde{S}(u_1)(\alpha s,t)w'(t)dt + \int_{\mathbb{R}}\mathtt{N}(\Phi(h))(s,t)w'(t)dt \tag{4.29}$$

becomes identically zero, and consequently we obtain a genuine solution to equation (4.1). Setting $c_* := \int_{\mathbb{R}}|w'(t)|^2dt$, using expression (4.19), and carrying out the same computation we did in (4.5), we obtain that

$$\int_{\mathbb{R}}\tilde{S}(u_1)(\alpha s,t)w'(t)dt = -c_*\alpha^2\mathcal{J}_a[h](\alpha s) + c_*\alpha^2 G_1(h)(\alpha s) \tag{4.30}$$

where

$$\begin{aligned} c_*G_1(h)(\alpha s) &:= \alpha h''(\alpha s)\int_{\mathbb{R}}\zeta_0(t+h)A_0(\alpha s,\alpha(t+h))w''(t)w'(t)dt \\ &+ \alpha^2 Q(\alpha s)h''(\alpha s)\int_{\mathbb{R}}\psi_0'(t)w'(t)dt + \alpha^{-2}\int_{\mathbb{R}}\zeta_0\ R_1(\alpha s,t,h,h')w'(t)dt \end{aligned} \tag{4.31}$$

and we recall that R_1 is of size $O(\alpha^4)$ in the sense of (4.20). Thus setting

$$c_*G_2(h)(\alpha s) := \alpha^{-2}\int_{\mathbb{R}}\mathtt{N}(\Phi(h))(s,t)w'(t)dt, \quad \mathtt{G}(h)(\alpha s) := G_1(h)(\alpha s) + G_2(h)(\alpha s) \tag{4.32}$$

it turns out that equation (4.29) is equivalent to

$$c(s)\cdot c_* = -c_*\alpha^2\mathcal{J}_{a,\Gamma}[h](\alpha s) + c_*\alpha^2 G_1(h)(\alpha s) + c_*\alpha^2 G_2(h)(\alpha s)$$

Therefore the condition $c(s) = 0$ is equivalent to the following nonlinear problem on h

$$\mathcal{J}_{a,\Gamma}[h](\alpha s) = h''(\alpha s) + \frac{\partial_{\boldsymbol{s}} a(\alpha s,0)}{a(\alpha s,0)}h'(\alpha s) - Q(\alpha s)h(\alpha s) = \mathtt{G}[h](\alpha s), \quad \text{in } \mathbb{R} \tag{4.33}$$

Consequently, we will have proved Theorem 4.1.1, if we find a function h, solving equation (4.33).

Hence, we need to devise a corresponding solvability theory for the linear problem

$$\mathcal{J}_a[h](\boldsymbol{s}) = f(\boldsymbol{s}), \quad \text{in } \mathbb{R} \tag{4.34}$$

and we look for suitable conditions on the curve and on the potential a, that guarantees the property already stated. The next result addresses this matter.

Proposition 10. *Given $\alpha > 0$, $\lambda \in (0,1)$, and a function f with $\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} < \infty$, assume that Γ is a smooth curve satisfying* (4.1). *If, Γ is nondegenerate respect to the potential a and conditions 4.3- hold, then there exists a unique bounded solution h of problem* (4.34)*, and exists a positive constant $C = C(a,\Gamma,\alpha)$ such that*

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \le C\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} \tag{4.35}$$

with the norms defined in (4.11)-(4.12).

In section 4.4, we study in detail the proof of this proposition. For the time being, let us note that, G is a small operator of size $O(\alpha)$ uniformly on functions h satisfying (4.1). Hence Proposition 10 plus the contraction mapping principle yield the next result, which ensures the solvability of the nonlinear Jacobi equation. Its detailed proof can be found in section 7.

Proposition 11. *Given $\alpha > 0$ and $\lambda \in (0,1)$, there exist a positive constant $\mathcal{K} > 0$ such that for any $\alpha > 0$ small enough the following holds. There is a unique solution h of (4.33) on the region (4.12), namely $\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq \mathcal{K}\alpha$.*

and this concludes the proof of Theorem 4.1.1. The rest of the chapter is devoted to give fairly detailed proofs of every result mentioned here.

4.5 The Jacobi Operator $\mathcal{J}_{a,\Gamma}$

This section is meant to provide a complete proof of proposition 11. Recall that the Jacobi operator of the curve Γ associated to the potential a, corresponds to the linear operator

$$\mathcal{J}_{a,\Gamma}[h](\boldsymbol{s}) = h''(\boldsymbol{s}) + \frac{\partial_{\boldsymbol{s}} a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)} h'(\boldsymbol{s}) - Q(\boldsymbol{s})h(\boldsymbol{s}) \tag{4.1}$$

where we recall that

$$Q(\boldsymbol{s}) := \frac{\partial_{\boldsymbol{tt}} a(\boldsymbol{s},0)}{a(\boldsymbol{s},0)} - 2k^2(\boldsymbol{s}) \tag{4.2}$$

Recall also that we are assuming the curve Γ to be nondegenerate, which means that the only bounded solution to

$$\mathcal{J}_{a,\Gamma}[h](\boldsymbol{s}) = 0, \quad \forall \boldsymbol{s} \in \mathbb{R}$$

is the trivial one.

In order to find accurate information on the kernel of (4.1), we consider the auxiliary equation

$$\frac{d}{d\boldsymbol{s}}\left(p(\boldsymbol{s})\frac{d}{d\boldsymbol{s}}h\right) - q(\boldsymbol{s})h = 0, \text{ in } \mathbb{R} \tag{4.3}$$

where we assume that $p, q : \mathbb{R} \to \mathbb{R}$ satisfy the following

$$p \in C^1[0,+\infty) \cap L^\infty[0,+\infty), \; q \in C^1[0,+\infty) \tag{4.4}$$

$$p(\boldsymbol{s}) \geq p_0 > 0, \quad \forall \boldsymbol{s} \geq 0 \tag{4.5}$$

$$\lim_{\boldsymbol{s}\to\pm\infty} p(\boldsymbol{s}) =: p(\pm\infty) > 0 \tag{4.6}$$

$$|p(\boldsymbol{s})| + (1+|\boldsymbol{s}|)^{2+\alpha}|p'(\boldsymbol{s})| \leq C, \quad \forall \boldsymbol{s} \geq 0 \tag{4.7}$$

$$|q(\boldsymbol{s})| + |q'(\boldsymbol{s})| \leq \frac{C}{1+|\boldsymbol{s}|^{2+\alpha}}, \quad \forall \boldsymbol{s} \geq 0 \tag{4.8}$$

for some constants $\alpha > -1$, $\beta_0 > 0$ and $C > 0$.

The first result concerns the decay for the derivative of a solution to the auxiliary equation, provided that p and q decay sufficiently fast.

Lemma 4.5.1. *Suppose $\alpha > -1$, and consider a one-sided bounded solution $h \in L^\infty[0,\infty)$ of (4.3), for which functions p and q fulfill (4.4) to (4.8). Then there is a constant $C = C(p,q,\alpha,h) > 0$ such that*

$$|h'(\boldsymbol{s})| \leq \frac{C}{|\boldsymbol{s}|^{1+\alpha}}, \quad \forall \boldsymbol{s} > 0$$

where $C(p,q,\alpha,h) = \|p^{-1}\|_{L^\infty[0,\infty)}\|h\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|)^{2+\alpha}q\|_{L^\infty[0,\infty)}$.

Proof. Observe first that thanks to assumptions (4.4)-(4.6), it holds

$$p(\boldsymbol{s}) = p(+\infty) - \int_{\boldsymbol{s}}^{+\infty} p'(\xi)d\xi \tag{4.9}$$

Now, since h solves the equation, then for $\boldsymbol{s}_1 > \boldsymbol{s}_2 > 0$ we have

$$\begin{aligned}|p(\boldsymbol{s}_1)h'(\boldsymbol{s}_1) - p(\boldsymbol{s}_2)h'(\boldsymbol{s}_2)| &\leq \int_{\boldsymbol{s}_1}^{\boldsymbol{s}_2} |q(\boldsymbol{s})h(\boldsymbol{s})| \\ &\leq \|h\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|)^{2+\alpha}q\|_{L^\infty[0,\infty)} \left|\int_{\boldsymbol{s}_1}^{\boldsymbol{s}_2} \frac{1}{1+|\xi|^{2+\alpha}}d\xi\right| \\ &\leq C(q,h)\left|\frac{1}{|\boldsymbol{s}_1|^{1+\alpha}} - \frac{1}{|\boldsymbol{s}_2|^{1+\alpha}}\right|\end{aligned}$$

where $C(q,h) := C\cdot\|h\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|)^{2+\alpha}q\|_{L^\infty[0,\infty)} \in \mathbb{R}$ is fixed. In particular using that $1+\alpha > 0$, it follows that

$$\lim_{\boldsymbol{s}_1\to+\infty} |p(\boldsymbol{s}_1)h'(\boldsymbol{s}_1)| \leq |p(\boldsymbol{s}_2)h'(\boldsymbol{s}_2)| + C(q,h)\frac{1}{|\boldsymbol{s}_2|^{1+\alpha}} < +\infty$$

which implies that $p(+\infty)h'(\infty) \in \mathbb{R}$. From this, we can rewrite equation (4.3) in its integral form

$$p(\boldsymbol{s})h'(\boldsymbol{s}) = p(+\infty)h'(+\infty) - \int_{\boldsymbol{s}}^{+\infty} q(\xi)h(\xi)d\xi. \tag{4.10}$$

but using (4.9), we find that

$$p(+\infty)h'(\boldsymbol{s}) - h'(\boldsymbol{s})\int_{\boldsymbol{s}}^{+\infty} p'(\xi)d\xi = p(+\infty)h'(+\infty) - \int_{\boldsymbol{s}}^{+\infty} q(\xi)h(\xi)d\xi$$

and so

$$p(+\infty)h'(s) = p(+\infty)h'(+\infty) + h'(s)\int_s^{+\infty} p'(\xi)d\xi - \int_s^{+\infty} q(\xi)h(\xi)d\xi.$$

Integrating again between 0 and $\boldsymbol{s}$, we obtain an expression for the solution h of (4.3)

$$p(+\infty)h(s) = p(+\infty)h(0) + p(\infty)h'(+\infty)s$$
$$+ \underbrace{\int_0^s h'(\xi)\int_\xi^{+\infty} p'(\tau)d\tau d\xi}_{I} - \underbrace{\int_0^s \int_\xi^{+\infty} q(\tau)h(\tau)d\tau d\xi}_{II} \tag{4.11}$$

Let us estimate these integrals. We first estimate integral I

$$\begin{aligned}|I| &\leq \int_0^s |h'(\xi)| \int_\xi^{+\infty} |p'(\tau)|d\tau \\ &\leq C\|h'\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|^{2+\alpha})p'\|_{L^\infty[0,\infty)} \int_0^{\boldsymbol{s}}\int_\xi^{+\infty} \frac{1}{1+|\tau|^{2+\alpha}}d\tau d\xi \\ &\leq C_{h',p',\alpha}\int_0^{\boldsymbol{s}} \frac{1}{1+|\xi|^{1+\alpha}}d\xi = O(1+|\boldsymbol{s}|^{-\alpha})\end{aligned}$$

where $C_{h',p',\alpha} := C\|h'\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|^{2+\alpha})p'\|_{L^\infty[0,\infty)}$. In the same way, we estimate II

$$\begin{aligned}|II| &\leq \int_0^{\boldsymbol{s}}\int_\xi^{+\infty} |q(\tau)|\ |h(\tau)|d\tau d\xi \\ &\leq C\|h\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|^{2+\alpha})q\|_{L^\infty[0,\infty)} \int_0^{\boldsymbol{s}}\int_\xi^{+\infty} \frac{d\tau d\xi}{1+|\tau|^{1+\alpha}} \\ &\leq C_{h,q,\alpha}(1+|\boldsymbol{s}|)^{-\alpha}\end{aligned}$$

with $C_{h,q,\alpha} := C\|h\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|)^{2+\alpha}q\|_{L^\infty[0,\infty)}$.

Since h is bounded, we deduce from (4.11) that

$$O(1) = p(+\infty)h(0) + p(+\infty)h'(+\infty)\boldsymbol{s} + O(1+|\boldsymbol{s}|^{-\alpha}). \tag{4.12}$$

Dividing (4.12) by $\mathtt{b}s > 0$ and taking $\boldsymbol{s} \to +\infty$, we get that

$$0 = p(+\infty)h'(+\infty)$$

provided that $\alpha > -1$. From (4.6), it follows that $h'(+\infty) = 0$. In particular, the latter fact together with formula (4.10), imply that

$$p(\boldsymbol{s})h'(\boldsymbol{s}) = \int_{\boldsymbol{s}}^\infty q(\xi)h(\xi)d\xi$$

and consequently

$$|h'(\boldsymbol{s})| \leq C\|p^{-1}\|_{L^\infty[0,\infty)}\|h\|_{L^\infty[0,\infty)}\|(1+|\boldsymbol{s}|^{2+\alpha})q\|_{L^\infty[0,\infty)}\frac{1}{1+|\boldsymbol{s}|^{1+\alpha}}$$

which completes the proof of the estimate. □

The core of this section is reflected in the next result.

Lemma 4.5.2. *Let $\alpha > 0$, and suppose function q satisfies* (4.4)-(4.8). *Then the equation*

$$u''(\boldsymbol{s}) - q(\boldsymbol{s})u(\boldsymbol{s}) = 0, \qquad in \quad \mathbb{R} \tag{4.13}$$

has two linearly independent smooth solutions $u, \tilde{u}$, so that as $s \to +\infty$

$$u(\boldsymbol{s}) = \boldsymbol{s} + O(1) + O(|\boldsymbol{s}|^{-1-\alpha}), \qquad \tilde{u}(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}) \tag{4.14}$$

$$u'(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}), \qquad \tilde{u}'(\boldsymbol{s}) = O(|\boldsymbol{s}|^{-1} + |v|^{-1-\alpha}) \tag{4.15}$$

Proof. To begin with, we look for a solution $u(\boldsymbol{s}) = \boldsymbol{s}v(\boldsymbol{s})$, so that, multiplying equation (4.13) by $\boldsymbol{s}$, we find that v satisfies

$$\frac{d}{d\boldsymbol{s}}\left(\boldsymbol{s}^2 v'(\boldsymbol{s})\right) - q(\boldsymbol{s})\boldsymbol{s}^2 v(\boldsymbol{s}) = 0 \tag{4.16}$$

Now, consider the functions

$$x(\boldsymbol{s}) := \boldsymbol{s}^2 v'(\boldsymbol{s}), \quad y(\boldsymbol{s}) := v(\boldsymbol{s}) \tag{4.17}$$

so that, equation (4.16) becomes the linear system of differential equations

$$\begin{cases} x'(\boldsymbol{s}) = q(\boldsymbol{s})\boldsymbol{s}^2 y(\boldsymbol{s}) \\ y'(\boldsymbol{s}) = \dfrac{1}{\boldsymbol{s}^2}x(\boldsymbol{s}) \end{cases}, \quad \forall \boldsymbol{s} \in [\boldsymbol{s}_0, +\infty) \tag{4.18}$$

Integrating this system between $\boldsymbol{s}_0$ and $\boldsymbol{s}$ we obtain the identities

$$y(\boldsymbol{s}) = y(\boldsymbol{s}_0) + \int_{\boldsymbol{s}_0}^{s} \frac{1}{\xi^2} x(\xi) d\xi$$

$$x(\boldsymbol{s}) = x(\boldsymbol{s}_0) + \int_{\boldsymbol{s}_0}^{\boldsymbol{s}} q(\xi)\xi^2 y(\xi) d\xi \tag{4.19}$$

In particular, we deduce an explicit formula for $y(\boldsymbol{s})$, given by

$$y(\boldsymbol{s}) = y(\boldsymbol{s}_0) + x(\boldsymbol{s}_0)\left(\frac{1}{\boldsymbol{s}_0} - \frac{1}{\boldsymbol{s}}\right) + \int_{\boldsymbol{s}_0}^{\boldsymbol{s}} y(\tau)q(\tau)\tau^2 \left(\frac{1}{\tau} - \frac{1}{\boldsymbol{s}}\right) d\tau \tag{4.20}$$

In this way, we can estimate $y(\boldsymbol{s})$ for $\boldsymbol{s} \geq \boldsymbol{s}_0$ as

$$|y(\boldsymbol{s})| \leq |y(\boldsymbol{s}_0)| + |x(\boldsymbol{s}_0)| \left(\frac{1}{\boldsymbol{s}_0} - \frac{1}{\boldsymbol{s}}\right) + \int_{\boldsymbol{s}_0}^{\boldsymbol{s}} |y(\tau)| \; |q(\tau)|\tau \left(1 - \frac{\tau}{\boldsymbol{s}}\right) d\tau.$$

From Gronwall's inequality we find the estimate

$$|y(\boldsymbol{s})| \leq \left(|y(\boldsymbol{s}_0)| + \frac{2|x(\boldsymbol{s}_0)|}{\boldsymbol{s}_0}\right) \exp\left(\int_{\boldsymbol{s}_0}^{\boldsymbol{s}} |q(\tau)|\tau \left(1 - \frac{\tau}{\boldsymbol{s}}\right) d\tau\right) \tag{4.21}$$

Notice that, for any $\boldsymbol{s} \geq \tau > \boldsymbol{s}_0 :\ \left|\tau\left(1-\frac{\tau}{\boldsymbol{s}}\right)\right| \leq 2\tau = O(\tau)$. This fact combined with the decay of $q(\boldsymbol{s})$, leads to

$$|y(\boldsymbol{s})| \leq C_{q,\alpha}(|y(\boldsymbol{s}_0)| + \frac{2}{\boldsymbol{s}_0}|x(\boldsymbol{s}_0)|)$$

where $C_{q,\alpha} := C\|(1+|\boldsymbol{s}|)^{2+\alpha}q\|_{L^\infty[\boldsymbol{s}_0,+\infty)} \int_{\boldsymbol{s}_0}^{\infty} |\tau|^{-1-\alpha}d\tau$. From (4.20) it follows that for any $\boldsymbol{s}_1 > \boldsymbol{s}_2 \geq \boldsymbol{s}_0 > 0$:

$$|y(\boldsymbol{s}_1) - y(\boldsymbol{s}_2)| \leq |x(\boldsymbol{s}_0)|\left(\frac{1}{\boldsymbol{s}_2} - \frac{1}{\boldsymbol{s}_1}\right) + C\int_{\boldsymbol{s}_2}^{\boldsymbol{s}_1} |q(\tau)|\tau d\tau$$

implying that $y(+\infty) \in \mathbb{R}$. Moreover, same formula (4.20) yields

$$y(+\infty) = y(\boldsymbol{s}_0) + \frac{x(\boldsymbol{s}_0)}{\boldsymbol{s}_0} + \int_{\boldsymbol{s}_0}^{+\infty} y(\tau)q(\tau)\tau d\tau$$

which allows us to write

$$y(\boldsymbol{s}) - y(+\infty) = -\frac{x(\boldsymbol{s}_0)}{\boldsymbol{s}_0} - \int_{\boldsymbol{s}_0}^{\boldsymbol{s}} y(\tau)q(\tau)\frac{\tau^2}{\boldsymbol{s}}d\tau - \int_{\boldsymbol{s}}^{+\infty} y(\tau)q(\tau)\tau d\tau$$

In particular, by choosing the constants to be $y(+\infty) = 1,\ x(\boldsymbol{s}_0) = 0$, we finally deduce

$$y(\boldsymbol{s}) = 1 - \int_{\boldsymbol{s}_0}^{\boldsymbol{s}} y(\tau)q(\tau)\frac{\tau^2}{\boldsymbol{s}}d\tau - \int_{\boldsymbol{s}}^{+\infty} y(\tau)q(\tau)\tau d\tau \tag{4.22}$$

Additionally, the derivative $y'(\boldsymbol{s}) = v'(\boldsymbol{s})$ can be obtained from $x(\boldsymbol{s})$ using relation (4.19), as

$$v'(\boldsymbol{s}) = \frac{x(\boldsymbol{s})}{\boldsymbol{s}^2} = \frac{0}{\boldsymbol{s}^2} + \frac{1}{\boldsymbol{s}^2}\int_{\boldsymbol{s}_0}^{\boldsymbol{s}} q(\xi)\xi^2 y(\xi)d\xi. \tag{4.23}$$

Now that $y(\boldsymbol{s})$ is bounded in $[\boldsymbol{s}_0, +\infty)$, similar arguments as in section 2.4 imply the same estimates for the integrals in (4.22)-(4.23), since

$$\left|\int_{\boldsymbol{s}}^{+\infty} y(\tau)q(\tau)\tau d\tau\right| = O(|\boldsymbol{s}|^{\alpha}), \quad \left|\int_{\boldsymbol{s}_0}^{\boldsymbol{s}} y(\tau)q(\tau)\frac{\tau^2}{\boldsymbol{s}}d\tau\right| = O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha})$$

From these estimates, we conclude that

$$\begin{aligned} v(\boldsymbol{s}) = y(\boldsymbol{s}) &= 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}) \\ v'(\boldsymbol{s}) &= O(|\boldsymbol{s}|^{-2} + |\boldsymbol{s}|^{-1-\alpha}) \end{aligned}$$

So the asymptotic behavior of the first solution follows, as $\alpha > 0$ and by definition of u:

$$\left.\begin{aligned} u(\boldsymbol{s}) &= \boldsymbol{s}\left(1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha})\right) = \boldsymbol{s} + O(1 + |\boldsymbol{s}|^{1-\alpha}) \\ u'(\boldsymbol{s}) &= v(\boldsymbol{s}) + \boldsymbol{s}v'(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}) \end{aligned}\right. , \quad \boldsymbol{s} \geq \boldsymbol{s}_0$$

which finishes the analysis of the profile of the first solution found to equation (4.13).

To conclude, we find $\tilde{u}$ using reduction of order formula, to find that

$$\tilde{u}(\boldsymbol{s}) = \left(\int_{\boldsymbol{s}}^{\infty} u^{-2}(\xi)d\xi\right) \cdot u(\boldsymbol{s})$$

and directly from this, one gets

$$\begin{aligned} \tilde{u}(\boldsymbol{s}) &= C(\boldsymbol{s})u(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}) \\ \tilde{u}'(\boldsymbol{s}) &= C'(\boldsymbol{s})u(\boldsymbol{s}) + C(\boldsymbol{s})u'(\boldsymbol{s}) = O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-2} + |\boldsymbol{s}|^{-1-\alpha}) \end{aligned}, \quad \text{for } \boldsymbol{s} >> \boldsymbol{s}_0$$

which concludes the proof of Lemma 4.5.2. □

Now proceed to state the main result of this section, which characterize the profile of the kernel of the Jacobi operator.

Proposition 12. *Let $\Gamma \subset \mathbb{R}^2$ be a stationary non-degenerate curve as in respect to a. Assume also that conditions (4.3)-(10) are satisfied, for $\alpha > 0$ and additionally the potential stabilizes on the curve at infinity, namely*

$$a(\pm\infty, 0) := \lim_{\boldsymbol{s}\to\pm\infty} a(\boldsymbol{s}, 0) > 0 \in \mathbb{R}. \tag{4.24}$$

Then, there are two linearly independent elements in the kernel of h_1, h_2 of (4.1) satisfying that

$$\begin{aligned} h_i(\boldsymbol{s}) &= |\boldsymbol{s}| + O(1) + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}) \\ h_i'(\boldsymbol{s}) &= O(1) + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-1-\alpha}) \end{aligned}, \quad \text{as} \quad (-1)^i \boldsymbol{s} \to +\infty \tag{4.25}$$

and they are bounded functions as $(-1)^{i+1}\boldsymbol{s} \to \infty$. Furthermore, in the region where the latter happens, it holds

$$|h_i(\boldsymbol{s})| + (1 + |\boldsymbol{s}|^{1+\alpha})|h_i'(\boldsymbol{s})| \leq C, \quad \text{as} \quad (-1)^{i+1}\boldsymbol{s} \to +\infty \tag{4.26}$$

Proof. We look for solutions $h(s) = a(s,0)^{-1/2} \cdot u(s)$ to (4.1), which means that u solves the auxiliary equation

$$u''(\boldsymbol{s}) - \tilde{q}(\boldsymbol{s})u(\boldsymbol{s}) = 0, \quad \text{in } \mathbb{R}$$

where

$$\tilde{q}(\boldsymbol{s}) := \frac{\partial_{\boldsymbol{tt}} a(\boldsymbol{s}, 0)}{a(\boldsymbol{s}, 0)} - 2k^2(\boldsymbol{s}) + \frac{1}{2}\frac{\partial_{\boldsymbol{ss}} a(\boldsymbol{s}, 0)}{a(\boldsymbol{s}, 0)} - \frac{1}{4}\left|\frac{\partial_{\boldsymbol{s}} a(\boldsymbol{s}, 0)}{a(\boldsymbol{s}, 0)}\right|^2.$$

Now, thanks to the assumptions we have made on $a(\boldsymbol{s}, \boldsymbol{t})$ and Γ, it follows that

$$(1 + |\boldsymbol{s}|)^{2+\alpha}|\tilde{q}(\boldsymbol{s})| \leq C.$$

Therefore, applying lemma 4.5.2 on the region $[0, +\infty)$, we deduce the existence of two solutions linearly independent of equation (4.5) in $\mathbb{R}$, denoted by $u(\boldsymbol{s})$ and $\tilde{u}(\boldsymbol{s})$, which satisfies the right-sided asymptotic behavior as $s \to +\infty$

$$\begin{aligned} u(\boldsymbol{s}) &= s + O(1) + O(|\boldsymbol{s}|^{1-\alpha}), \qquad \tilde{u}(\boldsymbol{s}) = 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}) \\ u'(\boldsymbol{s}) &= 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}), \qquad \tilde{u}'(\boldsymbol{s}) = O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-1-\alpha}). \end{aligned} \tag{4.27}$$

Applying Lemma 4.5.2 again, but this time on the region $(-\infty, 0]$, we obtain two other solutions $v(s)$ and $\tilde{v}(s)$ linearly independent of equation (4.5) in $\mathbb{R}$, that now satisfy the left-sided asymptotic behavior as $s \to -\infty$

$$\begin{aligned} v(\boldsymbol{s}) &= |\boldsymbol{s}| + O(1) + O(|\boldsymbol{s}|^{1-\alpha}), \qquad & \tilde{v}(\boldsymbol{s}) &= 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}) \\ v'(\boldsymbol{s}) &= 1 + O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-\alpha}), & \tilde{v}'(\boldsymbol{s}) &= O(|\boldsymbol{s}|^{-1} + |\boldsymbol{s}|^{-1-\alpha}). \end{aligned} \tag{4.28}$$

We remark that the non-degeneracy of curve Γ, implies that $\tilde{u}(\boldsymbol{s})$ cannot be bounded on $(-\infty, 0]$. Recall also that $\{u, \tilde{u}\}$ and $\{v, \tilde{v}\}$ represent two different basis of the vector space of solutions to the equation (4.5). So that, for some constats $\alpha_i 4$, for $i = 1, \cdot, 4$, we have that

$$\forall \boldsymbol{s} \in \mathbb{R}: \quad u(\boldsymbol{s}) = \alpha_1 v(\boldsymbol{s}) + \alpha_2 \tilde{v}(\boldsymbol{s}), \quad \tilde{u}(\boldsymbol{s}) = \alpha_3 v(\boldsymbol{s}) + \alpha_4 \tilde{v}(\boldsymbol{s}). \tag{4.29}$$

From the previous discussion about $\tilde{u}$, we observe that not only that $\tilde{u}$ grows at most at a linear rate on $(-\infty, 0]$, but also that the non-degeneracy property implies $\alpha_3 \neq 0$. Hence, the function $h_1(\boldsymbol{s}) := \alpha_3^{-1} a(\boldsymbol{s}, 0)^{-1/2} \tilde{u}(\boldsymbol{s})$ belongs in the kernel of $\mathcal{J}_{a,\Gamma}$, satisfying (4.25)-(4.26) for $i = 1$.

The same argument can be applied to $\tilde{v}(\boldsymbol{s})$ to find the function $h_2(\boldsymbol{s}) := a(\boldsymbol{s}, 0)^{-1/2} u(\boldsymbol{s})$ behaving as predicted and clearly, being linear independent with h_1. This completes the proof of the proposition. ☐

Once we have described the kernel of (4.1), it is straightforward to check the following proposition, whose proof is left to the readers.

Proposition 13. *Under the same set of assumptions as in proposition 12 and given $\alpha > 0$, $\lambda \in (0, 1)$ and a function f with $\|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} < +\infty$, then the equation*

$$\mathcal{J}_a[h](\boldsymbol{s}) = f(\boldsymbol{s}), \quad \boldsymbol{s} \in \mathbb{R}$$

has a unique bounded solution, given by the variation of parameters formula

$$h(\boldsymbol{s}) = -h_1(\boldsymbol{s}) \int_{-\infty}^{\boldsymbol{s}} a(\xi, 0) h_2(\xi) f(\xi) d\xi - h_2(\boldsymbol{s}) \int_{\boldsymbol{s}}^{+\infty} a(\xi, 0) h_1(\xi) f(\xi) d\xi \tag{4.30}$$

In addition, there is some positive constant $C = C(a, \Gamma, \alpha)$ such that

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq C \|f\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} \tag{4.31}$$

where

$$\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} := \|h\|_{L^\infty(\mathbb{R})} + \|(1 + |s|)^{1+\alpha} h'\|_{L^\infty(\mathbb{R})} + \sup_{s \in \mathbb{R}} (1 + |\boldsymbol{s}|)^{2+\alpha} \|h''\|_{C^{0,\lambda}(\boldsymbol{s}-1, \boldsymbol{s}+1)}$$

4.6 Gluing reduction and solution to the projected problem

This section is devoted to give fairly detailed proves of propositions 8 and 9. In what follows, we refer to the notation and to the objects introduced in sections 3 and 4.

In this part we prove proposition 8. To do so, let us first consider the linear problem

$$\Delta_x \psi + \alpha \frac{\nabla_{\bar{x}} a}{a} \nabla_x \psi - W_\alpha(x)\psi = g(x), \quad \text{in} \quad \mathbb{R}^2 \tag{4.1}$$

where

$$-W_\alpha(x) := [(1-\zeta_4) f'(u_1) + \zeta_4 f'(H(t))].$$

Observe that the dependence in α is implicit on the cut-off function ζ_4, defined in (4.5).

Let us observe that for any $\alpha > 0$ small enough, the term W_α satisfies the global estimate $0 < \beta_1 < W_\alpha(x) < \beta_2$ for a certain positive constants β_1, β_2. In fact, we can chose $\beta_1 := \sqrt{2} - \tau$ for any arbitrary small $\tau > 0$. To address the study of this equation, recall the definition of the weighted norms:

$$\|g\|_{L^\infty_K(\mathbb{R}^2)} := \sup_{x \in \mathbb{R}^2} K(x) \|g\|_{L^\infty(B_1(x))}, \quad \|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} := \sup_{x \in \mathbb{R}^2} K(x) \|g\|_{C^{0,\lambda}(B_1(x))}$$

with K is given by (4.6).

Lemma 4.6.1. *For any $\lambda \in (0,1)$, there are numbers $C > 0$, and $\alpha_0 > 0$ small enough, such that for $0 < \alpha < \alpha_0$ and any given continuous function $g = g(x)$ with $\|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} < +\infty$, the equation* (4.1) *has a unique solution $\psi = \Psi(\phi)$ satisfying the a priori estimate:*

$$\|\psi\|_X := \|D^2\psi\|_{C^{0,\lambda}_K(\mathbb{R}^2)} + \|D\psi\|_{L^\infty_K(\mathbb{R}^2)} + \|\psi\|_{L^\infty_K(\mathbb{R}^2)} \le C \|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \tag{4.2}$$

The proof of this lemma follows the same lines of lemma 4.1 in [16] with no significant changes. We leave details to the reader, but we do comment on the estimate

$$\|\psi\|_{L^\infty_K(\mathbb{R}^2)} \le C \|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \tag{4.3}$$

for $\alpha > 0$ small enough and any bounded solution ψ of (4.1). It follows directly from a sub-supersolution scheme, using that $b_1^2 + b_2^2 < (\sqrt{2} - \tau)/2$ and the fact that the function

$$\psi_0(x) := e^{R_0} \|\psi\|_\infty \cdot \left\{ \zeta_3(x) [e^{-\sigma|t|/2} (1 + |\alpha s|)^{-\mu}] + (1 - \zeta_3(x)) e^{-b_1|x_1| - b_2|x_2|} \right\}$$

ca be readily checked to be a positive supersolution of (4.1), provided that $R_0 > 0$ sufficiently large.

Hence, we can use the maximum principle within the annulus $B_{R_1}(\vec{0}) \setminus B_{R_0}(\vec{0})$ with a barrier function of the form $\psi_0 + \theta e^{\sqrt{\beta_1/2}(|x_1| + |x_2|)}$ for $\theta > 0$ small, to find that

$$K(x)|\psi(x)| \le M \|\psi\|_{L^\infty(\mathbb{R}^2)} \le \tilde{M} \|g\|_{C^{0,\lambda}_K(\mathbb{R}^2)}, x \in \mathbb{R}^2.$$

Now we have all the ingredients need for the proof of proposition 8. Let us set $\psi := \Upsilon(g)$ the solution of equation (4.1) predicted by lemma 4.6.1. We can write problem (4.16) as a fixed point problem in the space X of functions $\psi \in C^{2,\lambda}_{loc}(\mathbb{R}^2)$ with $\|\psi\|_X < \infty$, as

$$\psi = \Upsilon(g_1 + G(\psi)), \quad \psi \in X \tag{4.4}$$

where

$$g_1 := (1-\zeta_3)S(\mathtt{w}) + 2\nabla_x\zeta_3\nabla_x\phi + \phi\Delta_x\zeta_3 + \alpha\phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3,$$
$$G(\psi) := (1-\zeta_4)N_1(\psi+\zeta_3\phi).$$

Consider $\mu \in (0, 2+\alpha)$, $\sigma \in (0, \sqrt{2})$ and $\alpha > 0$ fixed and a function h satisfying (4.12). Consider also a function $\phi = \phi(s,t)$, satisfying $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq 1$.

Note that the derivatives of ζ_3 are nontrivial only within the region $\rho_\alpha - 2 < |t + h(\alpha s)| < \rho_\alpha - 1$, with ρ_α defined in (4.19). Taking into account the weight $K(x)$ (4.6), we find that

$$\begin{aligned} K(x)\left|2\nabla_x\zeta_3\nabla_x\phi + \phi\Delta_x\zeta_3 + \alpha\phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3\right| &\leq C_a K(x) e^{-\sigma|t|}(1+|\alpha s|)^{-\mu}\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\ &\leq C_a\ e^{-\sigma\delta/2\alpha} e^{\sigma/2(-c_0|s|+2+|h|)}\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \end{aligned}$$

provided that

$$c_0 < \frac{b_2\delta}{a_2}, \qquad \frac{c_0\,\theta}{1-\theta} \leq b_2$$

conditions that holds, since we can take $c_0 > 0$ small enough,independent of $\alpha > 0$ and θ small depending maybe on c_0. At the end, there are some constants $\tilde{c}_0$ and $\tilde{\delta} > 0$, depending on Γ and $a(x,y)$, such that the right hand side satisfies for $x \in \mathbb{R}^2$

$$\left\|2\nabla_x\zeta_3\nabla_x\phi + \phi\Delta_x\zeta_3 + \phi\frac{\nabla_{\bar{x}}a}{a}\nabla_x\zeta_3\right\|_{C^{0,\lambda}B(x,1)} \leq C_{a,\Gamma}\ e^{-\sigma\tilde{\delta}/\alpha}e^{-\tilde{c}_0|x|}\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

where these constants are explicitly $\tilde{\delta} := \delta - c_0 a_2/b_2$, $\tilde{c}_0 := \sigma\theta c_0/b_2$, and where we emphasize that $C_{a,\Gamma}$ does not depend on α.

Expressions (4.23)-(4.24) for $S(\mathtt{w})$, imply that $\|S(\mathtt{w})\|_{C^{0,\lambda}_{\mu,\sqrt{2}}(\mathbb{R}^2)} \leq C\alpha^3$. In particular, the exponential decay exhibited by w', w'', ψ_0, ψ_1 in $t-$variable imply

$$|(1-\zeta_3)S(\mathtt{w})| = |(1-\zeta_3)\zeta_3 S(u_1) + (1-\zeta_3)E| \leq C_a\ e^{-\sqrt{2}|t|}(1+|\alpha s|)^{-2-\alpha}$$

Now since this error term is vanishing everywhere but on the region $\rho_\alpha - 2 < |t + h(\alpha s)| < \rho_\alpha - 1$, we can use the definition (4.6) of the weight function $K(x)$ to prove that

$$\begin{aligned} K(x)|(1-\zeta_3)S(\mathtt{w})(x)| &\leq e^{\sigma|t|/2}(1+|\alpha s|)^{\mu-2-\alpha}\ C_a e^{-\sigma|t|/2}e^{-(\sqrt{2}-\sigma/2)|t|} \\ &\leq C_a e^{-(\sqrt{2}-\sigma/2)(\delta/\alpha + c_0|s| - |h| - 2)} \leq Ce^{-\sigma\tilde{\delta}/\alpha} \end{aligned}$$

where we have used the expression (4.19) for ρ_α, and we set $\tilde{\delta} := (\sqrt{2}/\sigma - 1/2)\delta >> \delta/2$. Further, the regularity in the $s-$variable of the functions involved in g_1, imply that

$$\|g_1\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \leq Ce^{-\sigma\delta/2\alpha}$$

On the other hand, consider the set for $A > 0$ large

$$\Lambda = \{\psi \in X :\ \|\psi\|_X \leq A \cdot e^{-\sigma\delta/2\alpha}\} \tag{4.5}$$

The definitions of N_1 in (4.4) and G in (4.6), lead us to the following computations

$$(1-\zeta_4)|N_1(\Psi(\phi_1)+\zeta_3\phi_1)-N_1(\Psi(\phi_2)+\zeta_3\phi_2)| \le$$
$$C_{\mathtt{w}}(1-\zeta_4)\sup_{t\in(0,1)}|t\Psi(\psi_1)+(1-t)\Psi(\psi_2)+\zeta_3(t\phi_1+(1-t)\phi_2)|\cdot|\Psi(\psi_1)-\Psi(\psi_2)|$$

together with

$$\begin{aligned}|G(\psi_1)-G(\psi_2)| &\le (1-\zeta_4)\sup_{\xi\in(0,1)}|DN_1(\xi\psi_1+(1-\xi)\psi_2+\zeta_3\phi)[\psi_1-\psi_2]| \\ &\le C\|f''(\mathtt{w})\|_\infty(1-\zeta_4)\sup_{\xi\in(0,1)}|\xi\psi_1+(1-\xi)\psi_2+\zeta_3\phi|\cdot|\psi_1-\psi_2|\end{aligned}$$

The latter, plus the regularity in the $s-$variable leads the Lipschitz character of G:

$$\|G(\psi_1)-G(\psi_2)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \le C_A e^{-\sigma\delta/\alpha}\|\psi_1-\psi_2\|_{C^{0,\lambda}_K(\mathbb{R}^2)}$$

while

$$\|G(0)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \le C_{\mathtt{w}}\|(1-\zeta_4)\zeta_3^2\phi^2\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \le Ce^{-\sigma\delta/\alpha}$$

In order to use the fixed point theorem, we need to estimate the size of the nonlinear operator

$$\begin{aligned}\|\Upsilon(g_1+G(\psi))\|_X &\le \|\Upsilon(g_1+G(\psi)-G(0))\|_X+\|\Upsilon(G(0))\|_X \\ &\le C(\|g_1\|_{C^{0,\lambda}_K(\mathbb{R}^2)}+\|G(\psi)-G(0)\|_{C^{0,\lambda}_K(\mathbb{R}^2)}+\|G(0)\|_{C^{0,\lambda}_K(\mathbb{R}^2)}) \\ &\le C(C_a\ e^{-\sigma\delta/2\alpha}+e^{-\sigma\delta/\alpha}\|\psi\|_{C^{0,\lambda}_K(\mathbb{R}^2)}) \\ &\le Ce^{-\sigma\delta/2\alpha}(1+\|\psi\|_X)\end{aligned}$$

additionally, we also have

$$\begin{aligned}\|\Upsilon(g_1+G(\psi_1))-\Upsilon(g_1+G(\psi_2))\|_X &\le C\|G(\psi_1)-G(\psi_2)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} \\ &\le Ce^{-\sigma\delta/\alpha}\|\psi_1-\psi_2\|_X\end{aligned}$$

where in both inequalities we used that Υ is a linear and bounded operator.

This means that the right hand side of equation (4.4) defines a contraction mapping on Λ into itself, provided that the number A in definition (4.5) is taken large enough and $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}} \le 1$. Hence applying Banach fixed point theorem follows the existence of a unique solution $\psi=\Psi(\phi)\in\Lambda$.

In addition, it is direct to check that

$$\|\Psi(\phi_1)-\Psi(\phi_2)\|_X \le C_a\ e^{-\sigma\delta/2\alpha}\|\phi_1-\phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R})}+C\ e^{-\sigma\delta/\alpha}\|\Psi(\phi_1)-\Psi(\phi_2)\|_X \tag{4.6}$$

from where the Lipschitz dependence (4.22) of $\Psi(\phi)$ follows and this concludes the proof of Lemma 8

The purpose of the whole section is to give a proof of proposition 9, which deals with the solvability of the nonlinear projected problem (4.25) for ϕ.

At the core of the proof of proposition 9, is the fact that the heteroclinic solution $w(t)$ of the ODE

$$w''(t) + w(t)(1 - w^2(t)) = 0, \quad w'(t) > 0, \quad w(\pm\infty) = \pm 1$$

is $L^\infty-$nondegenerate in the sense of the following lemma.

Lemma 4.6.2. *Let ϕ be a bounded and smooth solution of the problem*

$$L(\phi) = 0 \quad in \quad \mathbb{R}^2 \tag{4.7}$$

Then necessarily $\phi(s,t) = Cw'(t)$, with $C \in \mathbb{R}$.

For a detailed proof of this lemma we refer the reader to lemma 6.1 in [16] and references there in.

Next, let us consider the linear projected problem

$$\begin{cases} \partial_{tt}\phi + \partial_{ss}\phi + f'(w)\phi = g(s,t) + c(s)w'(t) \text{ in } \mathbb{R} \times \mathbb{R} \\ \displaystyle\int_{\mathbb{R}} \phi(s,t)w'(t)dt = 0, \quad \forall s \in \mathbb{R} \end{cases} \tag{4.8}$$

Assuming that the corresponding operations can be carried out, for every fixed s, we can multiply the equation by $w'(t)$ and integrate by parts, to find that

$$c(s) = -\frac{\int_{\mathbb{R}} g(s,t)w'(t)dt}{\int_{\mathbb{R}} |w'(t)|^2 dt} \tag{4.9}$$

Hence, if ϕ solves problem (4.8), then ϕ eliminates the part of g which does not contribute to the projection onto $w'(t)$. This means, that ϕ solves the same equation, but with g replaced by $\tilde{g}$, where

$$\tilde{g}(s,t) = g(s,t) - \frac{\int_{\mathbb{R}} g(s,\tau)w'(\tau)d\tau}{\int_{\mathbb{R}} |w'(\tau)|^2 d\tau} w'(t). \tag{4.10}$$

Observe that the term $c(s)$ in problem (4.25) has a similar role, except that we cannot find it so explicitly, since this time the PDE in ϕ is nonlinear and nonlocal.

Now, we show that the linear problem (4.8) has a unique solution ϕ, which respects the size of g in norm (4.9), up to its second derivatives. We collect the discussion in the following proposition, whose proof is basically that contained in proposition 6.1 in [16] and proposition 4.1 in [14].

Proposition 14. *Given $\mu \geq 0$ and $0 < \sigma < \sqrt{2}$, there is a constant $C > 0$ such that for all sufficiently small $\alpha > 0$ the following holds. For any g with $\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} < \infty$, the problem (4.8) with $c(s)$ defined in (4.9), has a unique solution ϕ with $\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} < \infty$. Furthermore, this solution satisfies the estimate*

$$\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\|g\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}. \tag{4.11}$$

Now, we are in a position to proof proposition 9. Recall from section 4, that proposition 9 refers to the solvability of the projected problem

$$\begin{aligned} &\partial_{tt}\phi + \partial_{ss}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathtt{N}(\phi) + c(s)w'(t) \quad \text{in} \quad \mathbb{R}\times\mathbb{R} \\ &\int_{\mathbb{R}} \phi(s,t)w'(t)dt = 0, \quad \text{for all } s\in\mathbb{R}. \end{aligned} \tag{4.12}$$

where we recall that

$$\mathtt{N}(\phi) := \underbrace{\mathtt{B}(\phi) + [f'(u_1) - f'(w)]\phi + \alpha\nabla_{\bar{x}}a/a\cdot\nabla_x\phi}_{\mathtt{N}_1(\phi)} + \underbrace{\zeta_4[f'(u_1) - f'(H(t))]\Psi(\phi)}_{\mathtt{N}_2(\phi)} + \underbrace{\zeta_4 N_1(\Psi(\phi)+\phi)}_{\mathtt{N}_3(\phi)} \tag{4.13}$$

considering that the operators N_1 and $\mathtt{B}$ are given in (4.4)-(4.17).

Let us define $\phi := T(g)$ as the operator providing the solution predicted in proposition 14. Then (4.12) can be recast as the fixed point problem

$$\phi = T(-\tilde{S}(u_1) - \alpha^2\mathcal{J}_{a,\Gamma}[h]\,w'(t) - \mathtt{N}(\phi)) =: \mathcal{T}(\phi), \quad \|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le K\alpha^4 \tag{4.14}$$

Claim 1. *Given $\alpha>0$, $0<\mu<2+\alpha$ and $0<\sigma<\sqrt{2}$, there is some constant $C>0$, possibly depending on the constant $\mathcal{K}$ of* (4.12) *but independent of α, such that for $M>0$ and ϕ_1,ϕ_2 satisfying*

$$\|\phi_i\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le M\alpha^4, \quad i=1,2.$$

then the nonlinearity $\mathtt{N}$ *behaves locally Lipschitz, as*

$$\|\mathtt{N}(\phi_1) - \mathtt{N}(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le C\alpha\,\|\phi_1-\phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{4.15}$$

where the operator $\mathtt{N}$ *is given in* (4.13)*.*

To prove this claim, we analyze each of its components $\mathtt{N}_i$ from (4.13). Let us start with $\mathtt{N}_1$. Note that its first term corresponds to a second order linear operator with coefficients of order α plus a decay of order at least $O((1+|\alpha s|)^{-1-\alpha})$. In particular, recall from (4.17) that $\mathtt{B} = \zeta_0\tilde{\mathtt{B}}_0$, where in coordinates $[\Delta_x - \partial_{tt} - \partial_{ss}]$ amounts

$$\begin{aligned} \tilde{B}_0 =& -2\alpha h\partial_{st} - \alpha[k(\alpha s) + \alpha(t+h)k^2(\alpha s)]\partial_t + \alpha(t+h)A_0(\alpha s, \alpha(t+h)) \\ &\cdot[\partial_{ss} - 2h'\partial_t + \alpha^2|h'|^2\partial_{tt}] + \alpha^2(t+h)B_0(\alpha s,\alpha(t+h))[\partial_s - \alpha h'\partial_t] \qquad (4.16)\\ &- \alpha^2 h''\partial_t + \alpha^2|h'|^2\partial_{tt} \\ &+ \alpha^3(t+h)^2C_0(\alpha s,\alpha(t+h))\partial_t \qquad (4.17) \end{aligned}$$

Analyzing each term, leads to

$$\|\mathtt{B}(\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \le C\alpha\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}.$$

Thus, $\mathtt{N}_1$ satisfies

$$\|\mathtt{N}_1(\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\alpha\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}. \tag{4.18}$$

On the other hand, consider functions ϕ_i, with

$$\|\phi_i\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq M\alpha^3, \quad i = 1, 2$$

Now, let us analyze $\mathtt{N}_2$, by noting that for any $(s,t) \in \mathbb{R}^2$ the definition (4.8) implies that

$$\begin{aligned}
&\|\mathtt{N}_3(\phi_1) - \mathtt{N}_3(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\qquad \leq C \sup_{(s,t)\in\mathbb{R}^2} e^{(\sigma/2-\sqrt{2})|t|} \sup_{x\in\mathbb{R}^2} K(x)\|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))} \\
&\qquad \leq C\|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}_K(\mathbb{R}^2)} = Ce^{-\sigma\delta/2\alpha}\|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}.
\end{aligned} \tag{4.19}$$

In order to analyze $\mathtt{N}_4$, note that the definition (4.4) of N_1 also implies

$$\begin{aligned}
&|\mathtt{N}_4(\phi_1) - \mathtt{N}_4(\phi_2)| \leq |\zeta_4 N_1(\Psi(\phi_1) + \phi_1) - \zeta_4 N_1(\Psi(\phi_2) + \phi_2)| \\
&\quad \leq C\zeta_4 \sup_{\xi\in(0,1)} |\xi(\Psi(\phi_1) + \phi_1) + (1-\xi)(\Psi(\phi_2) + \phi_2)| \cdot (|\phi_1 - \phi_2| + |\Psi(\psi_1) - \Psi(\psi_2)|)
\end{aligned}$$

taking into account the region of $\mathbb{R}^2$ we are considering, it is possible to make appear de weight $K(x)$ in (4.6). Therefore thanks to the hypothesis on ϕ_i and Lemma 8, we obtain

$$\begin{aligned}
&\|\mathtt{N}_4(\phi_1) - \mathtt{N}_4(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\quad \leq C \sup_{(s,t)\in\mathbb{R}^2} \Big\{ e^{\sigma|t|/2}[\|\phi_1\|_{C^{0,\lambda}(B_1(s,t))} + \|\phi_2\|_{C^{0,\lambda}(B_1(s,t))} + \|\Psi(\phi_1)\|_{C^{0,\lambda}(B_1(x))} + \|\Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))}] \\
&\qquad\qquad \cdot e^{\sigma|t|/2}(1+|\alpha s|)^{\mu}[\|\phi_1 - \phi_2\|_{C^{0,\lambda}(B_1(s,t))} + \|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))}] \Big\} \\
&\leq C \sup_{(s,t)\in\mathbb{R}^2} \Big\{ \Big[\|\phi_1\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + K(x)(\|\Psi(\phi_1)\|_{C^{0,\lambda}(B_1(x))} + \|\Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))}) \Big] \\
&\qquad\qquad \cdot (e^{-\sigma|t|/2}\|\phi_1 - \phi_2\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + K(x)\|\Psi(\phi_1) - \Psi(\phi_2)\|_{C^{0,\lambda}(B_1(x))}) \Big\} \\
&\leq C(\|\phi_1\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\Psi(\phi_1)\|_X + \|\Psi(\phi_2)\|_X)[\|\phi_1 - \phi_2\|_{C^{0,\lambda}_{\mu,\sigma}} + \|\Psi(\phi_1) - \Psi(\phi_1)\|_X] \\
&\leq 2C(\alpha^3 + e^{-\sigma\delta/2\alpha}) \Big\{ \|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + e^{-\sigma\delta/2\alpha}\|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \Big\}
\end{aligned} \tag{4.20}$$

To reach a conclusion, we note from (4.18)-(4.19) and (4.20) that choosing $\alpha > 0$ small enough we obtain the validity of inequality (4.15). The proof of Claim 1 is concluded. □

To conclude the proof of proposition 9,we make the observation that the formula (4.19) and estimate (4.20) ensure that, for any $0 < \mu \leq 2 + \alpha$, $\sigma \in (0, \sqrt{2})$ and $\lambda \in (0,1)$ it holds

$$\|\tilde{S}(u_1) + \alpha^2 \mathcal{J}_a[h] \cdot w'\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\alpha^4 \tag{4.21}$$

Let us assume now that $\phi_1, \phi_2 \in B_\alpha$, where

$$B_\alpha := \{\phi \in C^{2,\lambda}_{loc}(\mathbb{R}^2) \ / \ \|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq K\alpha^4\}$$

for a constant K to be chosen. Note that using Claim 1, we are able to estimate the size of $\mathtt{N}(\phi)$ for any $\alpha > 0$ sufficiently small, as follows

$$\begin{aligned}
\|\mathtt{N}(\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} &\leq C\|\mathtt{N}(0)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + C\alpha\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&= C\|\zeta_4[f'(u_1) - f'(H)]\Psi(0) + \zeta_4 N_1(\Psi(0))\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + C\alpha\|\phi\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\leq C \sup_{t\in\mathbb{R}} e^{-\sigma\delta/2\alpha} \cdot \|\Psi(0)\|_X + \|\Psi(0)\|_X^2 + CK\alpha^5 \\
&\leq \tilde{C}\alpha^5 \quad \forall\, \phi \in B_\alpha
\end{aligned} \tag{4.22}$$

for some constant $\tilde{C}$, independent of K.

Then from the estimates (4.21)-(4.22) follows that the right hand side of the projected problem (4.12) defines an operator $\mathcal{T}$ applying the ball B_α into itself, provided K is fixed sufficiently large and independent of $\alpha > 0$. Indeed using the alternative definition of $\mathcal{T}$, and Proposition 14, we can easily find an estimate for the size of ϕ, through

$$\begin{aligned}
\|\mathcal{T}(\phi)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} &= \|T(-\tilde{S}(u_1) - \alpha^2\mathcal{J}_a[h]w' - \mathtt{N}(\phi))\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\leq \|T\|(\|\tilde{S}(u_1) + \alpha^2\mathcal{J}_a[h]w'\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\mathtt{N}(\phi)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}) \leq C\alpha^4
\end{aligned}$$

Further, $\mathcal{T}$ is also a contraction mapping of B_α in norm $C^{2,\lambda}_{\mu,\sigma}$ provided that $\mu \leq 2 + \alpha$, since Claim (1) asserts that $\mathtt{N}$ has Lipschitz dependence in ϕ:

$$\begin{aligned}
\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} &= \| - T(\mathtt{N}(\phi_1) - \mathtt{N}(\phi_2))\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \\
&\leq C\|\mathtt{N}(\phi_1) - \mathtt{N}(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\alpha\ \|\phi_1 - \phi_2\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}
\end{aligned}$$

So by taking $\alpha > 0$ small, we can use the contraction mapping principle to deduce the existence of a unique fixed point ϕ to equation (4.14), and thus ϕ turns out to be the only solution of problem (4.12). This justify the existence of ϕ, as required.

On the other hand, the Lipschitz dependence (4.28) of Φ in h, follows from the fact that

$$\|\mathcal{T}(\Phi(h_1)) - \mathcal{T}(\Phi(h_2))\|_{C^{2,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C(\|\tilde{S}(u_1,h_1) - \tilde{S}(u_1,h_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} + \|\mathtt{N}(\Phi_1) - \mathtt{N}(\Phi_1)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)})$$

A series of lengthy but straightforward computations, leads to (4.28) and so the proof is complete.

4.7 The proof of proposition 11

In this section, we will finish the proof of Theorem 4.1.1 by proving proposition 11. Recall that the reduced problem (4.33) reads as

$$\mathcal{J}_a[h](\alpha s) := h''(\alpha s) + \frac{\partial_s a(\alpha s, 0)}{a(\alpha s, 0)} h'(\alpha s) - Q(\alpha s)h(\alpha s) \ = \ \mathtt{G}(h)(\alpha s) \quad \text{in } \mathbb{R} \tag{4.1}$$

where $Q(\boldsymbol{s})$ was defined in (4.2), and the operator $\mathtt{G} = G_1 + G_2$ was given in (4.31)-(4.32).

We will make use of the following technical lemma, whose proof is left to the reaader.

Lemma 4.7.1. *Let $\Theta = \Theta(s,t)$ be a function defined in $\mathbb{R} \times \mathbb{R}$, such that, for any $\lambda \in (0,1)$, $\mu \in (1, 2+\alpha]$ and $\sigma \in (0, \sqrt{2})$*

$$\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} := \sup_{(s,t)\in\mathbb{R}\times\mathbb{R}} e^{\sigma|t|}(1+|\alpha s|)^{\mu}\|\Theta\|_{C^{0,\lambda}(B_1(s,t))} < +\infty$$

Then the function defined in $\mathbb{R}$ as

$$Z(\alpha s) := \int_{\mathbb{R}} \Theta(s,t) w'(t) dt$$

satisfies for some constant $C = C(w, \mu, \sigma) > 0$ the following estimate:

$$\|Z\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\alpha^{-1}\|\Theta\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \tag{4.2}$$

Let us apply Lemma (4.7.1) to the function $\Theta(s,t) := \mathtt{N}(\Phi(h))(s,t)$, to estimate the size of the operator G_2 in (4.32), where we recall that

$$G_2(h)(\alpha s) := c_*^{-1}\alpha^{-2}\int_{\mathbb{R}} \mathtt{N}(\Phi(h))(s,t)w'(t)dt.$$

We can estimate the size of the projection of $\mathtt{N}$ using the previous estimate (4.2), and the bound (4.22) for the size of $\mathtt{N}$:

$$\|G_2(h)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\alpha^{-3}\|\mathtt{N}(\Phi(h))\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\alpha^2 \tag{4.3}$$

Likewise, for $\phi_i = \Phi(h_i)$, $i = 1, 2$ it holds similarly that

$$\|G_2(h_1) - G_2(h_2)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\alpha^{-3}\|\mathtt{N}(\phi_1) - \mathtt{N}(\phi_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)}$$

Nonetheless, using (4.15) and proposition 9, it follows that

$$\|\mathtt{N}(\phi_1) - \mathtt{N}(\phi_1)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\alpha^4\|h_1 - h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}.$$

The previous estimates allow us to deduce

$$\|G_2(h_1) - G_2(h_2)\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R}^2)} \leq C\alpha\|h_1 - h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}.$$

Furthermore, from (4.3) we also have that

$$\|G_2(0)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\alpha^2 \tag{4.4}$$

for some $C > 0$ possibly depending on $\mathcal{K}$.

Next, we consider

$$c_*G_1(h_1) = \alpha h_1''(\boldsymbol{s})\int_{\mathbb{R}} \zeta_0(t+h_1)A_0(\boldsymbol{s},\alpha(t+h_1))w''(t)w'(t)dt$$
$$+\alpha^2 Q(\boldsymbol{s})h_1''(\boldsymbol{s})\int_{\mathbb{R}}\psi_0'(t)w'(t)dt + \alpha^{-2}\int_{\mathbb{R}}\zeta_0\ R_1(\boldsymbol{s},t,h_1,h_1')w'(t)dt$$

It is direct to check, from (4.6) and (4.17) the following estimate on the Lipschitz character for $G_1(h)$

$$\|G_1(h_1)-G_1(h_2)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\alpha\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}.$$

Now, a simple but crucial observation we make is that

$$c_*G_1(0) = \alpha^{-2}\int_{\mathbb{R}}\zeta_0\ R_1(\alpha s,t,0,0)w'(t)dt$$

has the size

$$\|G_1(0)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\alpha^{-2}\|R_1\|_{C^{0,\lambda}_{\mu,\sigma}(\mathbb{R})} \leq C_2\alpha \tag{4.5}$$

for some constant C_2 independent of $\mathcal{K}$ in (4.1). Therefore, the entire operator $\mathtt{G}(h)$ inherits a Lipschitz character in h, from those of G_1, G_2:

$$\|\mathtt{G}(h_1)-\mathtt{G}(h_2)\|_{C^{0,\lambda}_{\mu,*}(\mathbb{R})} \leq C\alpha\|h_1-h_2\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})}. \tag{4.6}$$

Further, estimates (4.4)-(4.5) imply that $\mathtt{G}$ is such

$$\|\mathtt{G}(0)\|_{C^{2,\lambda}_{\mu,*}(\mathbb{R})} \leq 2C_2\alpha \tag{4.7}$$

Now let $h = T(f)$ be the linear operator defined in Proposition 10, and let $\mathtt{G}$ be the nonlinear operator given in (4.32). Consider the Jacobi nonlinear equation (4.1), but this time written as a fixed point problem: Find some h such that

$$h = T(\mathtt{G}(h)), \quad \|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq \mathcal{K}\alpha \tag{4.8}$$

Observe that

$$\|T(\mathtt{G}(h))\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq C\left(\|\mathtt{G}(h)-\mathtt{G}(0)\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} + \|\mathtt{G}(0)\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})}\right)$$
$$\leq C\,\alpha\left(1+\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})}\right)$$

where we made use of (4.6)-(4.7). Observe also that

$$\|T(\mathtt{G}(h_1))-T(\mathtt{G}(h_2))\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq C\|\mathtt{G}(h_1)-\mathtt{G}(h_2)\|_{C^{0,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq C\alpha\|h_1-h_2\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})}$$

Hence choosing $\mathcal{K}>0$, large enough but independent of $\alpha>0$, we find that if α is small, the operator $T\circ\mathtt{G}$ is a contraction on the ball $\|h\|_{C^{2,\lambda}_{2+\alpha,*}(\mathbb{R})} \leq \mathcal{K}\alpha$. As a consequence of the Banach's fixed point theorem, obtain the existence of a unique fixed point of the problem (4.8). This finishes the proof of Proposition 11 and consequently, the proof of our theorem.

4.8 Examples

To get a better understanding of the geometrical settings of this chapter, we present some examples that portray the nature of the curves and of the potentials we are thinking of, and how they interact in a way that they meet all the hypotheses of Theorem 4.1.1.

In what follows, we will admit curves that can be represented as the graph of some function. Let us consider a smooth function $f : \mathbb{R} \to \mathbb{R}$, $f = f(\boldsymbol{x})$, and a parameterized curve $\Gamma := \{\gamma(\boldsymbol{x}) / \ \boldsymbol{x} \in \mathbb{R}\} \subset \mathbb{R}^2$ such that

$$\gamma(\boldsymbol{x}) = (\boldsymbol{x}, f(\boldsymbol{x})), \quad \dot{\gamma}(\boldsymbol{x}) = (1, f'(\boldsymbol{x})) \tag{4.1}$$

In addition we choose the normal ν of Γ oriented negatively, meaning that the wedge product $\dot{\gamma}(\boldsymbol{x}) \times \nu(\boldsymbol{x})$ points in the opposite direction than e_3, the generator of the $z-$axis in $\mathbb{R}^3$. This forces

$$\nu(\boldsymbol{x}) = \frac{1}{\sqrt{1 + |f'(\boldsymbol{x})|^2}}(f'(\boldsymbol{x}), -1)$$

Let us also consider a potential defined in Euclidean coordinates $a = a(\boldsymbol{x}, \boldsymbol{y})$, adopting the convention where $(\boldsymbol{x}, \boldsymbol{y}) := (\bar{x}, \bar{y})$, which satisfies all the hypothesis (4.3)-(4.6) supposed for this Chapter.

Recall from the criticality condition (4.1), that in order for Γ to be a stationary curve with respect to the weighted arc-length $l_{a,\Gamma}$, is necessary that the potential a and the curvature k satisfy the equation

$$\partial_{\boldsymbol{t}} a(\boldsymbol{s}, 0) = k(\boldsymbol{s}) \cdot a(\boldsymbol{s}, 0) \ , \quad \text{a.e. } \boldsymbol{s} \in \mathbb{R} \tag{4.2}$$

Denoting $X(\boldsymbol{x}, \boldsymbol{t}) := \gamma(\boldsymbol{x}) + \boldsymbol{t}\nu(\boldsymbol{x})$, we can now set the potential written in this coordinates as

$$\tilde{a}(\boldsymbol{x}, \boldsymbol{t}) := a \circ X(\boldsymbol{x}, \boldsymbol{t}) = a\left(\boldsymbol{x} + \frac{\boldsymbol{t} f'(\boldsymbol{x})}{\sqrt{1 + |f'(\boldsymbol{x})|^2}}, \ f(\boldsymbol{x}) - \frac{\boldsymbol{t}}{\sqrt{1 + |f'(\boldsymbol{x})|^2}}\right) \tag{4.3}$$

Accordingly, relation (4.3) implies that the criticality condition (4.2) amounts to the following equation in Euclidean coordinates

$$\frac{\partial_{\boldsymbol{x}} a(\boldsymbol{x}, f(\boldsymbol{x})) f'(\boldsymbol{x})}{\sqrt{1 + |f'(\boldsymbol{x})|^2}} - \frac{\partial_{\boldsymbol{y}} a(\boldsymbol{x}, f(\boldsymbol{x}))}{\sqrt{1 + |f'(\boldsymbol{x})|^2}} = \frac{f''(\boldsymbol{x})}{(1 + |f'(\boldsymbol{x})|^2)^{3/2}} \cdot a(\boldsymbol{x}, f(\boldsymbol{x})) \tag{4.4}$$

where it has been used the classical formula for the curvature of Γ as given in (4.1),

$$k(\boldsymbol{x}) = f''(\boldsymbol{x})(1 + |f'(\boldsymbol{x})|^2)^{-3/2}$$

Example 1: The x-axis

For the sake of simplicity, let us find a some particular kind of stationary curve. We will be interested in finding $\Gamma \subset \mathbb{R}^2$ as a straight line on the Euclidean plane, further, we want this line to be the

x-axis. Nonetheless, the stationarity of this line must be with respect to some nontrivial potential $a(\boldsymbol{x},\boldsymbol{y}) \not\equiv 1$ that does not represent the classic Euclidean metric in $\mathbb{R}^2$, case in which all straight lines are trivially known as stationary curves.

With this purpose, let us set the function $f(\boldsymbol{x}) \equiv 0$ in (4.1), implying that $\Gamma = \overrightarrow{0X}$. In particular, adopting the convention $e_i := (\delta_{i1}, \delta_{i2})$ with δ_{ij} denoting the Kronecker delta, we have on the curve that $\nu(\boldsymbol{x}) \equiv e_2$, thus the Fermi coordinates are reduced simply to the Euclidean coordinates, namely $X(\boldsymbol{x},\boldsymbol{t}) = \boldsymbol{x}e_1 + \boldsymbol{t}e_2 = (\boldsymbol{x},\boldsymbol{t})$.

In this simplified context, it turns out that the criticality condition (4.4) is reduced to

$$-\partial_{\boldsymbol{y}} a(\boldsymbol{x},0) = 0, \quad \forall \boldsymbol{x} \in \mathbb{R} \tag{4.5}$$

Therefore, we only need to find a nontrivial potential $\tilde{a}(\boldsymbol{x},\boldsymbol{t}) = a(\boldsymbol{x},\boldsymbol{y})$ in such way the x-axis becomes a stationary curve, and also a nondegenerate curve.

Claim 2. *Given any $\alpha > 0$, the following potential*

$$a(\boldsymbol{x},\boldsymbol{y}) := \frac{1}{(1+|\boldsymbol{x}|)^{2+\alpha}} \cdot \left(\frac{\boldsymbol{y}^2}{\cosh(\boldsymbol{y})} \right) + 1 \tag{4.6}$$

satisfies all the requirements previously indicated, in relation with the curve $\Gamma = \overrightarrow{0X}$.

Proof.-
Let us note that $a(\boldsymbol{x},\boldsymbol{y})$ is smooth, globally bounded, and bounded below far away from zero. Further, it is direct that $\overrightarrow{0X}$ is a stationary curve relative to $l_{a,\Gamma}$ since solves equation (4.5)

$$\partial_{\boldsymbol{y}} a(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{(1+|\boldsymbol{x}|)^{2+\alpha}} \left(\frac{2\boldsymbol{y} - \boldsymbol{y}^2 \sinh(\boldsymbol{y})}{\cosh^2(\boldsymbol{y})} \right) \quad \Rightarrow \quad \partial_{\boldsymbol{y}} a(\boldsymbol{x},0) = 0, \quad \forall \boldsymbol{x} \in \mathbb{R}$$

Now to see that $\overrightarrow{0X}$ is a nondegenerate curve, just note that the potential achieves it minimum exactly on the region defined by the $\boldsymbol{x}$-axis, and moreover, around this curve the potential is strictly convex in the $\boldsymbol{y}$-direction. The latter translates in the fact that $\partial_{\boldsymbol{yy}} a(\boldsymbol{x},0) > 0$, given

$$\partial_{\boldsymbol{yy}} a(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{(1+|\boldsymbol{x}|)^{2+\alpha}} \left(\frac{2 - 2\boldsymbol{y}\sinh(\boldsymbol{y}) - \boldsymbol{y}^2\cosh(\boldsymbol{y})}{\cosh^2(\boldsymbol{y})} - \frac{2(2\boldsymbol{y} - \boldsymbol{y}^2\sinh(\boldsymbol{y}))\sinh(\boldsymbol{y})}{\cosh^3(\boldsymbol{y})} \right)$$

$$\Rightarrow \quad \partial_{\boldsymbol{yy}} a(\boldsymbol{x},0) = \frac{2}{(1+|\boldsymbol{x}|)^{2+\alpha}} > 0, \quad \forall \boldsymbol{x} \in \mathbb{R}$$

Taking this into account, note that $a(\boldsymbol{x},\boldsymbol{y})$ and $k(\boldsymbol{x}) \equiv 0$ are such that term

$$Q(\boldsymbol{x}) := \frac{\partial_{\boldsymbol{yy}} a(\boldsymbol{x},0)}{a(\boldsymbol{x},0)} - 2k^2(\boldsymbol{x})$$

fulfills the following conditions

$$Q(\boldsymbol{x}) > 0, \quad \text{and} \quad |Q(\boldsymbol{x})| \leq \frac{2}{(1+|\boldsymbol{x}|)^{2+\alpha}}, \quad \forall \boldsymbol{x} \in \mathbb{R}$$

Hence we deduce that $\Gamma = \overrightarrow{0X}$ is a nondegenerate curve with respect to the potential $a(\boldsymbol{x}, \boldsymbol{y})$ given in (4.6), finishing the proof of Claim 2. $\square$

Using the software MATLAB v2010, we plot the potential on the square $[-10, 10] \times [-10, 10]$, and we illustrate in color red the respective stationary curve Γ.

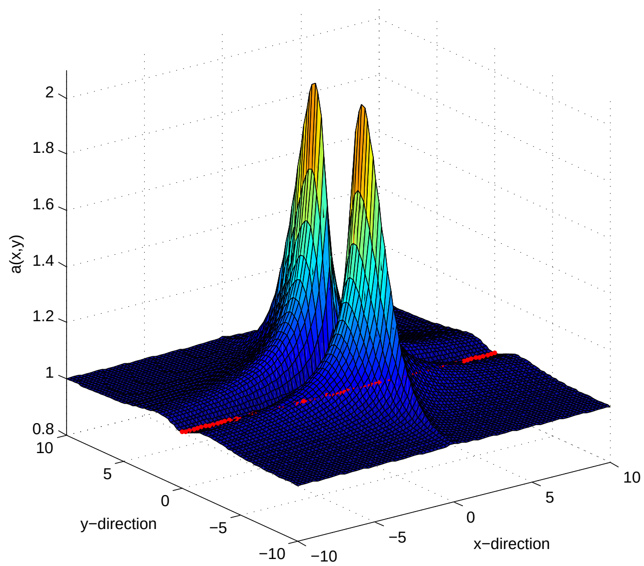

Figure 1: Potential $a(\boldsymbol{x}, \boldsymbol{y})$ (4.6) with geodesic $\Gamma = 0\vec{X}$, for $\alpha = 10^{-2}$.

Example 2: Asymptotic straight line

This time we consider a different type of curve $\Gamma \subset \mathbb{R}^2$. For $\omega \neq 0$, let us set function $f(\boldsymbol{x}) := \sqrt{1 + \omega^2 x^2}$, so that Γ converges asymptotically to straight lines as $|\boldsymbol{x}| \to \infty$. We have to exhibit some nontrivial potential $a(\boldsymbol{x}, \boldsymbol{y})$ for which Γ be nondegenerate geodesic relative to the arclength $\int_\Gamma a(\vec{x})$. Since this curve is not exactly a straight line, we don't get any simplification of the Fermi coordinates $X(\boldsymbol{x}, \boldsymbol{t})$. Therefore, we will assume a weaker dependence of the potential in Euclidean variables, namely $a = a(\boldsymbol{y})$. Note that

$$f'(\boldsymbol{x}) = \frac{\omega^2 \boldsymbol{x}}{\sqrt{1 + \omega^2 \boldsymbol{x}^2}}, \; f''(\boldsymbol{x}) = \frac{\omega^2}{(\sqrt{1 + \omega^2 \boldsymbol{x}^2})^{3/2}}, \; \frac{f''(\boldsymbol{x})}{1 + |f'(\boldsymbol{x})|^2} = \frac{\omega^2}{f^3(\boldsymbol{x}) + \omega^2 f(\boldsymbol{x})(f^2(\boldsymbol{x}) - 1)} \tag{4.7}$$

So, given the dependence of a only on $\boldsymbol{y}-$variable, criticality condition (4.4) amounts to

$$\frac{a'(f(\boldsymbol{x}))}{a(f(\boldsymbol{x}))} = \frac{-f''(\boldsymbol{x})}{1 + |f'(\boldsymbol{x})|^2} = g(f(x)) \tag{4.8}$$

with $g(\boldsymbol{y}) := -\omega^2[(1 + \omega^2)\boldsymbol{y}^3 - \omega^2 \boldsymbol{y}]^{-1}$.
We can solve directly this ordinary differential equation (4.8), for a in $\boldsymbol{y}-$variable.

$$\log(a(\boldsymbol{y})) = \int g(\boldsymbol{y}) d\boldsymbol{y} + M \quad \Leftrightarrow \quad a(\boldsymbol{y}) = M \exp\left(\int \frac{-\omega^2 d\boldsymbol{y}}{(1 + \omega^2)\boldsymbol{y}^3 - \omega^2 \boldsymbol{y}} \right)$$

This integral can be computed using partial fraction decomposition, noting the factorization $\boldsymbol{y}^3 - \omega^2/(1+\omega^2)\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{y}-y_+)(\boldsymbol{y}-y_-)$ in which $y_\pm := \pm\omega(\sqrt{1+\omega^2})^{-1}$. Then,

$$\frac{A}{\boldsymbol{y}} + \frac{B}{\boldsymbol{y}-y_+} + \frac{C}{\boldsymbol{y}-y_-} = \frac{(A+B+C)\boldsymbol{y}^2 + (-y_-B - y_+C)\boldsymbol{y} - \omega^2/(1+\omega^2)A}{\boldsymbol{y}^3 - \omega^2/(1+\omega^2)\boldsymbol{y}}$$

which leads to a linear system, solved by $A = 1,\ B = -\frac{1}{2},\ C = -\frac{1}{2}$. Hence we obtain

$$a(\boldsymbol{y}) = M \exp\left(\int \frac{d\boldsymbol{y}}{\boldsymbol{y}} - \int \frac{d\boldsymbol{y}}{2(\boldsymbol{y}-y_+)} - \int \frac{d\boldsymbol{y}}{2(\boldsymbol{y}-y_-)}\right) = \frac{M\boldsymbol{y}}{\sqrt{(\boldsymbol{y}-y_+)(\boldsymbol{y}-y_-)}}$$

For this construction, we will need to consider a slight modification of function a as follows. We say that the potential $\hat{a} : \mathbb{R}^2 \to \mathbb{R}$ is an *admissible left-extension* of function $a(x,y)$, provided that *I)* $\hat{a}$ be smooth bounded function, of at least $C^2(\mathbb{R}^2)$ class. *II)* $\hat{a}(x,y) = a(x,y)$ for points with $y \geq \omega^2/(1+\omega^2)$. *III)* $\hat{a}$ is uniformly positive, bounded below away from zero. We state the following

Claim 3. *Given $|\omega| \leq 1/\sqrt{2}$, any admissible left-extension of the potential given below*

$$a(\boldsymbol{x},\boldsymbol{y}) := \frac{\sqrt{1+\omega^2}\boldsymbol{y}}{\sqrt{(1+\omega^2)\boldsymbol{y}^2 - \omega^2}} \tag{4.9}$$

induces a metric in $\mathbb{R}^2$ for which $\Gamma = \left\{(\boldsymbol{x}, \sqrt{1+\omega^2\boldsymbol{x}^2})\right\}_{\boldsymbol{x}\in\mathbb{R}}$ is a nondegenerate geodesic.

Proof.-

Regardless the value of the parameter $\omega \neq 0$, it can be readily checked that within the region $y \geq 2\omega/\sqrt{1+\omega^2}$, function (4.9) is smooth, bounded, and uniformly positive. Moreover, this potential satisfies the asymptotic stability on the curve Γ, since $f(\boldsymbol{x}) \to +\infty$ as $|\boldsymbol{x}| \to +\infty$ and additionally $\lim_{\boldsymbol{y}\to+\infty} a(\boldsymbol{x},\boldsymbol{y}) = 1,\ \forall \boldsymbol{x} \in \mathbb{R}$. The previous construction of $a(\boldsymbol{x},\boldsymbol{y})$ was intended to build a potential satisfying the criticality condition (4.8) for the curve generated by $f(\boldsymbol{x}) = \sqrt{1+\omega^2x^2}$. Thus Γ is a geodesic for the arclength $\int_\Gamma a(\vec{x})$. All these features of a ensure that any *admissible left-extension* will provide a potential with the desired properties to induce a smooth metric in $\mathbb{R}^2$, fulfilling hypothesis (4.1) of Theorem 4.1.1. Notwithstanding, in order to estimate the derivatives of $\tilde{a}$ we need to compute first

$$\partial_{\boldsymbol{x}}\tilde{a}(\boldsymbol{x},0) = a'(f(\boldsymbol{x}))f'(\boldsymbol{x}), \quad \partial_t\tilde{a}(\boldsymbol{x},0) = -a'(f(\boldsymbol{x}))[1+|f'(\boldsymbol{x})|^2]^{-1}$$
$$\partial_{\boldsymbol{xx}}\tilde{a}(\boldsymbol{x},t) = a''(f(\boldsymbol{x}))|f'(\boldsymbol{x})|^2 + a'(f(\boldsymbol{x}))f''(\boldsymbol{x}), \quad \partial_{\boldsymbol{tt}}\tilde{a}(\boldsymbol{x},t) = (-1)^2a''(f(\boldsymbol{x}))[1+|f'(\boldsymbol{x})|^2]^{-1}$$
$$\partial_{\boldsymbol{xt}}\tilde{a}(\boldsymbol{x},t) = -a''(f(\boldsymbol{x}))f'(\boldsymbol{x})/[1+|f'(\boldsymbol{x})|^2] - 2a'(f(\boldsymbol{x}))f'(\boldsymbol{x})f''(\boldsymbol{x})/[1+|f'(\boldsymbol{x})|^2]^2$$

Moreover a tedious but simple calculation shows that

$$a'(\boldsymbol{y}) = \frac{-\omega^2\sqrt{1+\omega^2}}{(\boldsymbol{y}^2+\omega^2(\boldsymbol{y}^2-1))^{3/2}}, \quad a''(\boldsymbol{y}) = \frac{3\omega^2(1+\omega^2)^{3/2}\boldsymbol{y}}{[(1+\omega^2)\boldsymbol{y}^2-\omega^2]^{5/2}}$$

Therefore, taking into account the decay (4.7) of $f(\boldsymbol{x})$ and its derivatives, follows that this potential satisfies condition (10) of Theorem 4.1.1, for $\alpha = 2 > 0$. It only remains to prove the nondegeneracy property of the curve Γ. It can be checked the positiveness of the term $Q(\boldsymbol{x})$, in fact

$$2k^2(\boldsymbol{x}) = \frac{2|f''(\boldsymbol{x})|^2}{(1+|f'(\boldsymbol{x})|^2)^{3/2}} = \frac{2\omega^2}{(1+(\omega^2+\omega^4)\boldsymbol{x}^2)^3}, \quad \partial_{\boldsymbol{tt}}\tilde{a}(\boldsymbol{x},0) = a''(f(\boldsymbol{x}))\frac{1+\omega^2\boldsymbol{x}^2}{1+(\omega^2+\omega^4)\boldsymbol{x}^2}$$

so by the definition $Q(\boldsymbol{x}) = \partial_{tt}\tilde{a}(\boldsymbol{x},0)/\tilde{a}(\boldsymbol{x},0) - 2k^2(\boldsymbol{x})$ we obtain

$$Q(\boldsymbol{x}) \geq \min\{1, \|a\|_\infty^{-1}\} \left(\frac{3\omega^2(1+\omega^2)^{3/2}\sqrt{1+\omega^2\boldsymbol{x}^2}}{[(1+\omega^2)(1+\omega^2\boldsymbol{x}^2)-\omega^2]^{5/2}} \cdot \frac{1+\omega^2\boldsymbol{x}^2}{1+(\omega^2+\omega^4)\boldsymbol{x}^2} - \frac{2\omega^2}{(1+(\omega^2+\omega^4)\boldsymbol{x}^2)^3} \right)$$

$$> C_a \left(\frac{3\omega^2(1+\omega^2)^{3/2}(1+\omega^2\boldsymbol{x}^2)^{1/2}}{(1+\omega^2)^{5/2}(1+\omega^2\boldsymbol{x}^2)^{5/2}} - \frac{2\omega^2}{(1+(\omega^2+\omega^4)\boldsymbol{x}^2)^3} \right)$$

$$\geq C_a \left(\frac{3\omega^2}{(1+\omega^2)(1+\omega^2\boldsymbol{x}^2)^2} - \frac{2\omega^2}{(1+\omega^2\boldsymbol{x}^2)^3} \right) = \frac{C_a\omega^2}{(1+\omega^2\boldsymbol{x}^2)^2} \left(\frac{3}{1+\omega^2} - \frac{2}{1+\omega^2\boldsymbol{x}^2} \right)$$

Hence choosing $\omega \in \mathbb{R}\setminus\{0\}$ with $|\omega| \leq 1/\sqrt{2}$, we get that $Q(\boldsymbol{x}) > 0$ in the entire domain $\mathbb{R}$. Finally the term $Q(\boldsymbol{x})$ decays polynomially at a rate $O((1+|\boldsymbol{x}|)^{-4})$ as a consequence of the decay of the potential and the squared curvature, which finishes the proof of Claim 3. □

Remark 1. *We emphasize the fact that the criticality condition for* Γ *and the nondegeneracy property are tested only within the semi-space* $\boldsymbol{y} \geq 1$*, which involve only the part (4.9) of the admissible left-extension, since* $\hat{a}(\boldsymbol{x},\boldsymbol{y}) = a(\boldsymbol{y})$ *in this region and the curve complies* $|f(\boldsymbol{x})| \geq 1$.

Using the software MATLAB v2010, we plot an admissible left-extension of the potential on the square $[-10,10]\times[-10,10]$, and we illustrate in color red the respective stationary curve $\Gamma_\omega := \left\{(\boldsymbol{x}, \sqrt{1+\omega^2\boldsymbol{x}^2}) : \ \boldsymbol{x} \in \mathbb{R}\right\}$.

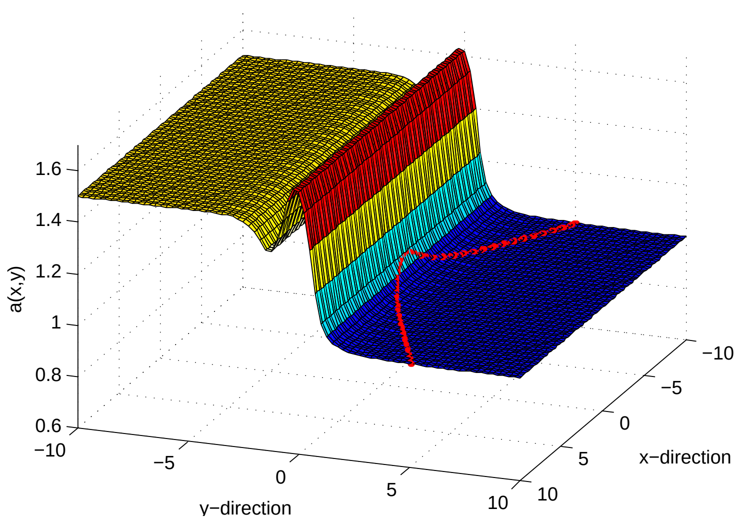

Figure 2: Potential $\hat{a}(\boldsymbol{x},\boldsymbol{y})$ (4.9) with Γ_ω as nondegenerate geodesic, for $\omega = 1/2$.

Chapter 5

Conclusions

General Conclusions

The Lyapunov-Schmidt reduction method has proven to be a very versatile and useful tool to prove existence of solutions to semilinear elliptic equations. In this thesis work we took advantage of this method in different scenarios, where the method granted either qualitative properties or asymptotic behavior of the solutions we have found.

In this regard, the families of solutions found in chapters 2 and 3 do enjoy axial symmetry, though the linear theory, associated to the projected problems there, did not make use of this special property. This is due to the invariance of the laplacian under rotations and the invariances of the surfaces from where the whole scheme is built.

An important aspect of the existence results we present in chapters 2,3 and 4, is the lack of compactness in the domain. This of course makes the analysis of linear operators more involved. We took advantage of the fact that the reduced problems, in the three chapters, were in essence odes and in this setting explicit right inverses for linear operators can be found from variations of parameters formula.

Another remark is in order. In the introduction we paid special attention to the relation between Allen-Cahn equation and minimal surfaces. We remark that, even for solutions with multiple transitions layers, minimal surfaces are the candidates to be nodal sets, or to be close to nodal sets of solutions to the Allen-Cahn equation, though we have not stressed this out explicitly.

Conclusions from chapter 2

In the introduction we mentioned that the next step in generalizing De Giorgi's conjecture consists on understanding and classifying entire solutions to

$$\Delta u + u(1-u^2) = 0, \quad \text{in } \mathbb{R}^N$$

having finite Morse index. The geometry of level sets for this type of solutions, as we have seen, is very rich and nontrivial. In chapter 2, we constructed a new family of solutions to the Allen-Cahn equation with any arbitrary number of transition layers diverging logarithmically from a catenoid. From the fact that the linear operator associated to the reduced problem enjoys the form

$$\delta \Delta_M h + |A_M|^2 h = q, \quad \delta \to 0.$$

and as we saw the inverse of this operator may blow up as $\delta \to 0$, it is expected that the Morse index of this family of solutions also blows up as the catenoid becomes more dilated.

A natural question that arises from this chapter is whether the construction can be taken into the setting of more complicated minimal surfaces, where no axially symmetry is present. A first step on this issue would be to investigate for instance the Costa-Hoffman-Meeks minimal surface, which is invariant under a the discrete group of dihedral symmetries and reflections.

Conclusions for chapter 3

The Lyapunov-Schmidt reduction scheme done for the construction of the family of solutions from chapter 3, follows basically the same lines of the construction in chapter 2. The main difference is of course the reduced problem. While in chapter 2, we needed to find an explicit solution of the Jacobi- Toda system through an approximation scheme, in chapter 3, a smooth radially symmetric solution to Liouville equation is available. Another important difference in both reduced problems comes from the linear Jacobi-Toda operator from chapter 2 and the Liouville linearized operator in chapter 3. Though the topologies where we invert are very much alike, the nature of both operator is very different.

Hence, together with the family of solutions from chapter 2, with two catenoidal transitions, at a first glance would seem to be very much alike since they both have logarithmical ends, but they are expected to differ strongly on its Morse index. Verifying this last statements should be of course the next step on the program presented in chapters 2 and 3.

Conclusions for chapter 4

In the introduction we mentioned the relation between the solutions of the inhomogeneous Allen-Cahn equation (1.8), and the properties of the potential $a(x)$ involved in this PDE. This opened a question about the existence of a smooth bounded solution u with transition near a given noncompact curve $\Gamma \subset \mathbb{R}^2$. More specifically, in determining sufficient conditions on $a(x)$ and Γ in order to build such solutions. In this direction, Theorem 4.1.1 provides some specific conditions on both, the potential $a(x, y)$ and the curve Γ, of which we point out the following:

- The smoothness and the uniform positiveness of the potential $a(x, y)$,
- The polynomial decay along the curve of the potential, and the decay of curvature k_Γ,
- The stationarity of Γ relative to $l_{a,\Gamma}$ plus a nondegeneracy, in relation to the existence of bounded kernel of the Jacobi operator $\mathcal{J}_a[h]$.

Furthermore as expected, it turns out that the solution u depends strongly on the potential $a(x)$. Indeed, the construction method forces the solution u to depend on some perturbation h, that is ultimately determined by $a(x)$; h needs to solve the nonlinear Jacobi equation (4.33).

A precise qualitative description is presented in Proposition 12, for the asymptotic behavior of a solution to $\mathcal{J}_{a,\Gamma}[h] = 0$, provided some conditions on the coefficients of the equation. The study of the kernel of the Jacobi operator is the key aspect from which we obtain the desired invertibility. Proposition 13 assures the sufficiency for the variation of parameters formula, to provide a smooth bounded solution of $\mathcal{J}_{a,\Gamma}[h] = f$, for a locally Hölder right-hand side decaying polynomially. Further, the polynomial decay is inherited to h' and h'' as stated in (4.31). In addition, the last Proposition also shows a high regularity for the solution here provided, unlike what presented on classical contexts of invertibility, where the inverse of $\mathcal{J}_M$ is usually defined in functional spaces of weaker regularity.

The nondegeneracy condition of Γ supposed in Theorem 4.1.1, basically implies that the curve Γ is isolated in some proper topology, so we do not have bounded kernel.This assumptions is a simplification the study of the invertibility theory of the Jacobi operator in our case. It would be interesting to study if the same results holds when removing this condition.

In another topic, it is worth mentioning a previous stage of this study, where we dealt with a slight simplification of the context in this thesis work. It was studied the existence of a solution u to the inhomogeneous Allen-Cahn equation (4.1), in the case where the potential $a : \mathbb{R}^2 \to \mathbb{R}$ has the form $a(x) = 1 + \chi(x)$, where function χ has compact support. An interesting result arose from this analysis, characterizes the *nondegeneracy condition* of the unbounded curve Γ in terms of the solvability of an related ODE in a compact domain. More explicitly, we proved

Proposition 15. *Let Γ be an unbounded curve, intersecting the set $\Omega := \operatorname{supp}(\chi) \subset \mathbb{R}^2$. Assume that the portion of Γ contained in Ω is parametrized as $\Gamma \cap \Omega := \gamma([s_1, s_2])$. Then Γ is a nondegenerate curve with respect to the arclength $\int_\Gamma a(x)$, if and only if, the following Neumann boundary value problem*

$$\begin{cases} \mathcal{J}_{a,\Gamma}[h](s) = 0, & \textit{in} \quad (s_1, s_2) \\ h'(s_1) = h'(s_2) = 0 \end{cases} \tag{5.1}$$

does not have the eigenvalue $\lambda = 0$.

From this fact we can easily describe the kernel of $\mathcal{J}_{a,\Gamma}$, since the nontrivial behavior of a bounded basis h_1, h_2 it only could arise on the compact portion Ω, depending on the existence of the eigenfunction associated to $\lambda = 0$. Another appealing geometrical property arisen in this context is related to the stationarity for geodesics. It can be shown, using a similar argument than carried out in [12] for the analysis of the nondegeneracy in $\mathbb{R}^2$ on a bounded domain, that a necessary condition for a curve Γ to be geodesic related to the length $\int_\Gamma a(x)$, is that Γ must *cross perpendicularly the boundary* $\partial\Omega$, which requires that on each point of intersection $P \in \Gamma \cap \partial\Omega$ the tangent vector $\hat{t}$ of the curve must be perpendicular to the normal of the boundary ν_Ω.

On the other hand, we must say that the two examples 4.8-4.8 exhibited in Chapter 4.2 constitutes a major contribution to the understanding of Differential Geometry in relation to Partial Differential Equations. There are only a few examples of this kind in the literature, because of the difficulty in finding nontrivial geometrical configurations in which geodesic curves which are *non-degenerate* with respect to some arclength $\int_\Gamma a(x)$.

There are natural extensions of this work, that can lead to future works. One open problem consists in a variant of this work, on the existence of smooth bounded solutions u to the inhomogeneous Allen-Cahn equation (4.1) with multiple transitions near an noncompact curve Γ, whose positions are expected to be governed by a Toda-type system. Some other cases consist in the study of the same equation in a variety of settings, where the potential $a(x)$ is less smooth or has some singularities, or where the uniform positiveness does not hold.

Concluding remarks

We finish this chapter first, refereing the reader to the discussion presented in the Appendix A, where multiplicity results are given for a nonlinear system of PDE's with symmetric coupling as applications of a variational form of the Lyapunov-Schmidt reduction method. At the core of this variational reduction is the structure of the equation

$$-\Delta u = f(u), \quad \in \Omega$$

for $\Omega \subset \mathbb{R}^N$ bounded and smooth, and f an even super linear and subcritical nonlinearity. This fact is striking since it comes directly from the structure of the system rather than the nonlinearity.

One of the interesting cases that are yet to be solved, is the case of asymptotically linear f, having slope λ that do not cross any eigenvalues of the $-\Delta$.

References

[1] S. Allen and J. W. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, Acta. Metall. 27 (1979), 1084-1095.

[2] L. Ambrosio, X. Cabré, Entire solutions of semilinear elliptic equations in R3 and a conjecture of De Giorgi, Journal Amer. Math. Soc. 13 (2000), 725-739.

[3] E. Bombieri, E. De Giorgi, E. Giusti, Minimal cones and the Bernstein problem. Invent. Math. 7 1969 243-268.

[4] L. Caffarelli, A. Cordoba, Uniform convergence of a singular perturbation problem. Comm. Pure and Appl. Math. XLVI pp 1-12 1995.

[5] L. Caffarelli, A. Cordoba, Phase transitions: uniform regularity of the intermediate layers. J. Reine Angew. Math. 593 (2006), 209-235.

[6] L. Caffarelli, A. Cordoba, Phase transition: uniform regularity of the intermediate layers, J. Reine Angew. Math. 593, pp 209-235, 2006.

[7] A. Castro, A. Lazer, Critical point theory and the number of solutions of a nonlinear Dirichlet problem, Ann. Mat. Pura Appl., (4) 120 (1979), 113-137.

[8] A. Castro, Reduction methods via minimax, Lecture Notes in Mathematics 957, Differential Equations, Springer Berlin-New York, 1982, pp.1-20.

[9] J. Cossio and C. Vélez, Soluciones no triviales para un problema de Dirichlet asintóticamente lineal, Rev. Colombiana Mat. Vol 37, 2003, 25-36.

[10] C.J. Costa, Example of a complete minimal immersions in R3 of genus one and three embedded ends, Bol. Soc. Bras. Mat. 15(1-2)(1984), 47-54.

[11] E. N. Dancer, Stable and finite Morse index solutions on Rn or on bounded domains with small diffusion. Trans. Amer. Math. Soc. 357 (2005), no. 3, 1225-1243.

[12] M. del Pino, M. Kowalczyk, F. Pacard, J. Wei, Multiple end solutions to the Allen-Cahn equation in $\mathbb{R}^2$, 2008

[13] M. del Pino, M. Kowalczyk, J. Wei, The Toda system and clustering interfaces in the Allen-Cahn equation, Arch. Rat. Mech. Anal. 190 (2008), 141-187.

[14] M. del Pino, Manuel,M. Kowalczyk, J, Wei, Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces. Rev. Un. Mat. Argentina 50 (2009), no. 2, 95-107, 35-40 (49-05)

[15] M. del Pino, M. Kowalczyk, J.C. Wei, J. Yang, Interface foliations of a positively curved manifold near a closed geodesic. Geom. Funct. Anal. 20 (2010), no. 4, 918-957.

[16] M. del Pino, M. Kowalczyk, J. Wei, On De Giorgi Conjecture in Dimensions $N \geq 9$ (final version) Annals of Mathematics 174 (2011), no.3, 1485-1569.

[17] , Z.Du and C.Gui, Interior layers for an inhomogeneous Allen-Cahn equation,J. Differential Equations, 249, pp 215–239 2010.

[18] Z.Du and L.Wang, Interface foliation for an inhomogeneous Allen-Cahn equation in Riemmanian Manifolds, Calc. Var, DOI 10.1007, March s00526-012-0521-4, 2012

[19] Z.Du and B.Lai, Transition layers for an inhomogeneus Allen-Cahn equation in Riemannian Manifolds, preprint.

[20] A. Floer, A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. J.Funct. Anal. 69 (1986), no. 3, 397-408.

[21] N. Ghoussoub and C. Gui, On a conjecture of De Giorgi and some related problems, Math. Ann. 311 (1998),481-491.

[22] L. Hauswirth and F. Pacard, Higher genus Riemann minimal surfaces, Invent. Math.169(2007), 569-620.

[23] D. Hoffman and W.H. Meeks III, A complete embedded minimal surface in R3 with genus one and three ends, J. Diff. Geom. 21 (1985), 109-127.

[24] D. Hoffman and W. H. Meeks III. The strong halfspace theorem for minimal surfaces. Inventiones Math., 101, 373-377, 1990.

[25] D. Hoffman and W.H. Meeks III, Embedded minimal surfaces of finite topology, Ann. Math. 131 (1990), 1-34.

[26] D. Hoffman and H. Karcher, Complete embedded minimal surfaces of finite total curvature. In Geometry V, Encyclopaedia Math. Sci.vol. 90, pp.5-93, 262-272. Springer Berlin (1997).

[27] J. E. Hutchinson, Y.Tonegawa, Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory, Calc.Var. 10, pp 49–84, 2000.

[28] N. Kapuleas, Complete embedded minimal surfaces of finite total curvature. J. Differential Geom. 45 (1997), 95-169.

[29] M. Kowalczyk, On the existence and Morse index of solutions to the Allen-Cahn equation in two dimensions, Ann. Mat. Pura Appl. 184 4 (2005), no. 1, 17-52.

[30] F. J. López, A. Ros, On embedded complete minimal surfaces of genus zero, J. Differential Geom. 33 (1991), no. 1, 293-300.

[31] J. Pérez, A. Ros, The space of properly embedded minimal surfaces with finite total curvature. Indiana Univ. Math. J. 45 (1996), no. 1, 177-204.

[32] L. Modica, Convergence to minimal surfaces problem and global solutions of $\Delta u = 2(u3 - u)$. Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), pp. 223-244, Pitagora, Bologna,(1979).

[33] S. Nayatani, Morse index of complete minimal surfaces. In: Rassis, T.M. (ed.) The Problem of Plateau, pp. 181-189(1992)

[34] S. Nayatani, Morse index and Gauss maps of complete minimal surfaces in Euclidean 3-space, Comm. Math. Helv. 68(4)(1993), 511-537.

[35] R. Osserman, A survey of minimal surfaces, Math. Studies 25, Van Nostrand, New York 1969.

[36] F. Pacard and M. Ritoré, From the constant mean curvature hypersurfaces to the gradient theory of phase transitions, J. Differential Geom. 64 (2003), no. 3, 359-423.

[37] P. Padilla and Y. Tonegawa, On the convergence of stable phase transitions, Comm. Pure Appl. Math. 51 (1998), 551-579.

[38] O. Savin, Regularity of at level sets in phase transitions, Annals of Mathematics, 169 (2009), 41-78.

[39] R. Schoen, Uniqueness, symmetry, and embeddedness of minimal surfaces, J. Differential Geom. 18 (1983), 791-809.

[40] J. Smoller, Shock waves and reaction-dffusion equations, Springer Verlag, Berlin 1992.

[41] Y. Tonegawa, Phase Field model with a variable chemical potential, Proc. Roy. Soc. Edinburgh Sect. A 132 (2002), no. 4, 993-1019.

[42] M.Traizet, An embedded minimal surface with no symmetries. J. Differential Geom. 60 (2002) 103-153.

Apenddix A

On a nonlinear elliptic system with symmetric Copuling

ON A NONLINEAR ELLIPTIC SYSTEM WITH SYMMETRIC COUPLING [1]

Oscar Agudelo

Departamento de Ingeniería Matemática, Universidad de Chile
Casilla 170 Correo 3, Santiago de Chile
oagudelo@dim.uchile.cl

Bernhard Ruf

Dipartimento di Matematica, Universit degli Studi di Milano
20122 Milano, Italia
bernhard.ruf@unimi.it

Carlos Vélez

Escuela de Matemáticas, Universidad Nacional de Colombia, Sede Medellín
Apartado Aéreo 3840, Medellín, Colombia
cauvelez@unal.edu.co

Abstract

Multiplicity results are proved for the nonlinear elliptic system

$$\begin{cases} -\Delta u + g(v) &= 0 \\ -\Delta v + g(u) &= 0 \qquad \text{in } \Omega, \\ \quad u = v &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear C^1-function which satisfies additional conditions. No assumption of symmetry on g is imposed.

[1]This work was supported by Progetto UniALA, offered by Fondazione Cariplo and Universit degli Studi di Milano in 2010. The first author was also partly funded by the chilean grant MECESUP, proyect UCH0607.

Extensive use is made of a global version of the Lyapunov-Schmidt reduction method due to Castro and Lazer and of symmetric versions of the Mountain Pass Theorem.

Keywords: Elliptic system, Lyapunov-Schmidt reduction method, Mountain Pass Theorem.

---

1. Introduction

It is well-known that a symmetry in a differential equation often generates the existence of multiple solutions. Consider e.g. the superlinear and subcritical equation

$$-\Delta u = f(u) \ , \quad \text{in} \ \ \Omega \ , \ \ u|_{\partial\Omega} = 0 \ , \tag{1.1}$$

where $f \in C(\mathbb{R})$ is a superlinear and subcritical nonlinearity. If $f(u)$ is an odd function, then the equation has the symmetry $u \mapsto -u$. Using the concept of index theories (e.g. the Krasnoselskii genus), one shows that this symmetry implies that the equation has infinitely many solutions.

In this article we consider a semilinear elliptic system in which the symmetry is not given by an odd nonlinearity, but by a *symmetric coupling.* We consider systems of the following form

$$\begin{cases} -\Delta u + g(v) &= 0 \\ -\Delta v + g(u) &= 0 \qquad \text{in} \ \ \Omega, \\ \qquad u = v &= 0 \qquad \text{on} \ \ \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1-function satisfying some assumptions to be specified later, but is not required to be odd. Note that this system allows the following symmetry:

$$T_1 : (u, v) \mapsto (v, u).$$

Indeed, looking at the associated functional (supposing it is well-defined)

$$J(u,v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega G(u) + \int_\Omega G(v) \ , \tag{1.3}$$

where $G(s) = \int_0^s g(t)dt$ is the primitive of g, we see that this functional is invariant under the group action $T = \{id, T_1\}$.

Thus, one may try to proceed similarly as for equation (1.1) by defining a suitable index. However, one encounters two major problems. First, the functional is strongly indefinite due to the first term in the functional. Second, the group T has an infinite-dimensional fixed point space, given by the pairs of functions of the form $\{(u,u)\}$. We overcome these difficulties by performing an infinite dimensional Lyapunov-Schmidt reduction (following Castro-Lazer [5]). Surprisingly, the resulting reduced functional $\widetilde{J}$ has the classical $\mathbb{Z}_2$-symmetry $\{id, -id\}$ (although, as we emphasize, no oddness assumption is taken for the nonlinearity), and so classical variational methods for the existence of multiple solutions can be employed.

We will denote by $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ the sequence of eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω. Also, $\{\varphi_j\}_j$ will denote an orthonormal basis, in $H_0^1(\Omega)$, of eigenfunctions of $-\Delta$ in Ω with Dirichlet boundary condition. We will study the existence of multiple solutions for problem (1.2) under three different sets of conditions. For the first two sets, we assume g satisfies

(g_0) $g(0) = 0$ and

(g_1) $\inf_{t\in\mathbb{R}} g'(t) > -\lambda_1$.

First, we consider the *superlinear setting*, in which we assume

(g_2) There exists a positive constant C such that

$|g(t)| \leq C(1 + |t|^p)$, where $p \in (1, \frac{N+2}{N-2})$ for all $t \in \mathbb{R}$, and

(g_3) There exists $R > 0$ such that $0 < \mu G(t) \le t g(t)$, for $|t| > R$, where $\mu > 2$.

Secondly, we also consider the *asymptotically linear setting*, in which g is assumed to satisfy

(g_4) $g'(\infty) := \lim_{|t|\to\infty} \frac{g(t)}{t} \in (\lambda_k, \lambda_{k+1})$ for some $k \ge 1$.

Our main results read as follows.

Theorem A. *(superlinear case) If g satisfies $(g_0) - (g_3)$, problem (1.2) has infinitely many solutions.*

We observe that conditions (g_2) and (g_3) include the "classical" nonlinearity $g(t) = t|t|^{p-1}$. But we emphazise that Theorem A holds true for a more general kind of nonlinearities, e.g. $g(t) = (t^+)^p - (t^-)^q$, for $t \in \mathbb{R}$ and $1 < p, q < (N+2)/(N-2)$, without any further restriction on p and q.

In the asymptotically linear framework we have the following analogue of Theorem A.

Theorem B. *(asymptotically linear case) Assume g satisfies $(g_0) - (g_1)$ and (g_4). If, in addition, $g'(0) < \lambda_j$ for $j \le k$, then problem (1.2) has (at least) $2(k - j + 1)$ nontrivial solutions.*

On the other hand, we consider a third setting, in which we only assume

(g_5) $\sup_{t\in\mathbb{R}} g'(t) < \lambda_1$.

We observe that under condition (g_5), system (1.2) is equivalent to the system

$$\begin{cases} -\Delta u &= h(v) \\ -\Delta v &= h(u) \qquad \text{in} \quad \Omega, \\ \quad u &= v = 0 \qquad \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

where $h = -g$ satisfies $\inf h' > -\lambda_1$. We point out that (1.4) is the very analogue in systems of the single-equation problem (1.1). In this direction we prove the following result which shows that system (1.2) (or, equivalently, system (1.4)) has a *strong hidden symmetry.*

Theorem C. *Assume g satisfies (g_5). Then (u, v) is a solution of* (1.2) *if and only if $u \equiv v$ and*

$$-\Delta u + g(u) = 0 \ , \quad in \ \ \Omega \ , \ u|_{\partial\Omega} = 0 \ . \tag{1.5}$$

In other words, under condition (g_5), solving system (1.2) *is equivalent to solving the single-equation problem* (1.5).

System (1.2) is Hamiltonian and our approach to it is variational, i.e. we define an energy functional $J : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$ by

$$J(u, v) = \int_\Omega (\nabla u \cdot \nabla v + G(u) + G(v))\, d\zeta,$$

where $G(t) := \int_0^t g(s)ds$. Assuming either (g_2) or (g_4), this functional is of class C^1 (see [11]) and

$$\partial_u J(u, v)\varphi = \int_\Omega (\nabla\varphi \cdot \nabla v + g(u)\varphi)\, d\zeta, \qquad \forall u, v, \varphi \in H_0^1(\Omega), \tag{1.6}$$

and

$$\partial_v J(u, v)\psi = \int_\Omega (\nabla u \cdot \nabla\psi + g(v)\psi)\, d\zeta, \qquad \forall u, v, \psi \in H_0^1(\Omega). \tag{1.7}$$

Thus, because of classical regularity theory (see [9]), critical points of J agree with classical solutions of problem (1.2). We then prove Theorem A and B showing the existence of critical points of J. Because of the form of the system

$$(u, v) \text{ is a solution of (1.2) if and only if } (v, u) \text{ is a solution of (1.2),} \tag{1.8}$$

as can be easily verified. This fact provides some symmetry on the functional J when it is written in appropriate coordinates.

The paper is organized as follows: in Section 2 we recall the Castro-Lazer version of the Lyapunov-Schmidt reduction method in an abstract setting. We then show that our functional J satisfies the conditions of such setting. In Section 3 we prove Theorem A and in Section 4 we prove Theorem B. In proving them, we recall and use appropriate symmetric versions of the Mountain Pass Theorem of Ambrosetti and Rabinowitz. Finally, in Section 5 we prove Theorem C.

2. Preliminaries

We begin by stating a global version of the Lyapunov-Schmidt method (see [4] and [5]).

Lemma 2.1. *Let H be a real separable Hilbert space. Let Z and W be closed subspaces of H such that $H = Z \oplus W$. Let $J : H \longrightarrow \mathbb{R}$ a function of class C^1. If there exist $m > 0$ and $\sigma > 1$ such that*

$$\langle \nabla J(\boldsymbol{z} + \boldsymbol{w}) - \nabla J(\boldsymbol{z} + \boldsymbol{w}_1), \boldsymbol{w} - \boldsymbol{w}_1 \rangle \geq m \|\boldsymbol{w} - \boldsymbol{w}_1\|_H^\sigma \quad \forall \boldsymbol{z} \in Z \quad \forall \boldsymbol{w}, \boldsymbol{w}_1 \in W \tag{2.1}$$

then:

(i) There exists a continuous function $\phi : Z \to W$ such that

$$J(\boldsymbol{z} + \phi(\boldsymbol{z})) = \min_{\boldsymbol{w} \in W} J(\boldsymbol{z} + \boldsymbol{w}).$$

Moreover, given $\boldsymbol{z} \in Z$, $\phi(\boldsymbol{z})$ is the unique element of W such that

$$\langle \nabla J(\boldsymbol{z} + \phi(\boldsymbol{z})), \boldsymbol{w} \rangle = 0 \qquad \forall \boldsymbol{w} \in W. \tag{2.2}$$

(ii) The functional $\widetilde{J} : Z \to \mathbb{R}$, defined by $\widetilde{J}(\boldsymbol{z}) := J(\boldsymbol{z} + \phi(\boldsymbol{z}))$ for $\boldsymbol{z} \in Z$, is of class C^1. Moreover,

$$D\widetilde{J}(\boldsymbol{z})\boldsymbol{h} = \langle \nabla \widetilde{J}(\boldsymbol{z}), \boldsymbol{h} \rangle = \langle \nabla J(\boldsymbol{z} + \phi(\boldsymbol{z})), \boldsymbol{h} \rangle \qquad \forall \boldsymbol{z}, \boldsymbol{h} \in Z. \tag{2.3}$$

(iii) Given $\boldsymbol{z} \in Z$, $\boldsymbol{z}$ is a critical point of $\widetilde{J}$ if and only if $\boldsymbol{z} + \phi(\boldsymbol{z})$ is a critical point of J.

Assuming (g_1) and either (g_2) or (g_4), we intend to apply Lemma 2.1 to the functional $J : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$ defined as

$$J(u, v) = \int_\Omega (\nabla u \cdot \nabla v + G(u) + G(v)) \, d\zeta,$$

where $G(t) := \int_0^t g(s)ds$. First, it is well-known that assuming either (g_2) or (g_4), this functional is of the class C^1 (see [11]) and

$$\partial_u J(u,v)\varphi = \int_\Omega (\nabla\varphi \cdot \nabla v + g(u)\varphi)\, d\zeta, \qquad \forall u, v, \varphi \in H_0^1(\Omega), \tag{2.4}$$

and

$$\partial_v J(u,v)\psi = \int_\Omega (\nabla u \cdot \nabla \psi + g(v)\psi)\, d\zeta, \qquad \forall u, v, \psi \in H_0^1(\Omega). \tag{2.5}$$

Let us take $H = H_0^1(\Omega) \times H_0^1(\Omega)$ equipped with the inner product $\langle (u_1, v_1), (u_2, v_2)\rangle = \langle u_1, u_2\rangle_{H_0^1} + \langle v_1, v_2\rangle_{H_0^1}$. Here, $\langle f_1, f_2\rangle_{H_0^1} = \int_\Omega \nabla f_1 \cdot \nabla f_2$. Let us define $W := \{\mathbf{w} = (w, w) : w \in H_0^1(\Omega)\}$ and $Z := \{\mathbf{z} = (z, -z) : z \in H_0^1(\Omega)\}$. Then $H_0^1(\Omega) \times H_0^1(\Omega) = Z \oplus W$. Let us verify (2.1). Let $\mathbf{z} \in Z$ and $\mathbf{w}, \mathbf{w}_1 \in W$. Then

$$\begin{aligned}
&\langle \nabla J(\mathbf{z} + \mathbf{w}) - \nabla J(\mathbf{z} + \mathbf{w}_1), \mathbf{w} - \mathbf{w}_1\rangle \\
&= \langle \nabla J(z + w, -z + w) - \nabla J(z + w_1, -z + w_1), (w - w_1, w - w_1)\rangle \\
&= [\partial_u J(z + w, -z + w) - \partial_u J(z + w_1, -z + w_1)](w - w_1) \\
&\quad + [\partial_v J(z + w, -z + w) - \partial_v J(z + w_1, -z + w_1)](w - w_1) \\
&= 2\int_\Omega |\nabla(w - w_1)|^2 + \int_\Omega [g(z + w) - g(z + w_1)](w - w_1) \\
&\quad + \int_\Omega [g(-z + w) - g(-z + w_1)](w - w_1).
\end{aligned}$$

Because of (g_1), there exists $\epsilon \in (0, \lambda_1)$ such that $g'(t) \geq -\lambda_1 + \epsilon$ for all $t \in \mathbb{R}$. Thus, the Mean Value Theorem, the previous identities, and Poincare Inequality give us

$$\begin{aligned}
&\langle \nabla J(\mathbf{z} + \mathbf{w}) - \nabla J(\mathbf{z} + \mathbf{w}_1), \mathbf{w} - \mathbf{w}_1\rangle \\
&\geq 2\int_\Omega |\nabla(w - w_1)|^2 + 2(-\lambda_1 + \epsilon)\int_\Omega (w - w_1)^2 \\
&\geq 2\int_\Omega |\nabla(w - w_1)|^2 + 2\frac{(-\lambda_1 + \epsilon)}{\lambda_1}\int_\Omega |\nabla(w - w_1)|^2 \\
&= 2\frac{\epsilon}{\lambda_1}\int_\Omega |\nabla(w - w_1)|^2 = \frac{\epsilon}{\lambda_1}\|\mathbf{w} - \mathbf{w}_1\|_H^2.
\end{aligned}$$

We have then verified the hypotheses of Lemma 2.1. Thus, there exist a continuous function $\mathbf{w} \equiv \phi : Z \longrightarrow W$ and a functional $\widetilde{J} : Z \longrightarrow \mathbb{R}$ which satisfy (i), (ii) and (iii). Because of (iii), our concern becomes the existence of critical points of the functional $\widetilde{J}$.

Observe that, given $\mathbf{z} = (z, -z) \in Z$, $\mathbf{w}(\mathbf{z}) = (w(z), w(z))$ and

$$\begin{aligned} \widetilde{J}(\mathbf{z}) &= J(z + w(z), -z + w(z)) \\ &= \textstyle\int_\Omega [|\nabla w(z)|^2 - |\nabla z|^2 + G(z + w(z)) + G(-z + w(z))]\, d\zeta. \end{aligned} \tag{2.6}$$

The symmetry of problem (1.2) expressed by condition (1.8) is translated into the following lemma.

Lemma 2.2. *If g satisfies (g_1) and either (g_2) or (g_4), then the function $\boldsymbol{w} \equiv \phi$ and the functional $\widetilde{J}$ are even.*

Proof. Let $\mathbf{z} = (z, -z) \in Z$. First, let us verify that

$$\langle \nabla J(-z + w(z), z + w(z)), (\varphi, \varphi) \rangle = 0 \ , \qquad \forall\ \varphi \in H_0^1(\Omega)$$

which, by uniqueness in (i) of Lemma 2.1, implies that $\mathbf{w}(\mathbf{z}) = \mathbf{w}(-\mathbf{z})$. Indeed, observe that

$$\begin{aligned} &\langle \nabla J(-z + w(z), z + w(z)), (\varphi, \varphi) \rangle \\ &= \partial_u J(-z + w(z), z + w(z))\varphi + \partial_v J(-z + w(z), z + w(z))\varphi \\ &= \int_\Omega \nabla\varphi \cdot \nabla(z + w(z)) + g(-z + w(z))\varphi \, d\zeta + \int_\Omega \nabla(-z + w(z)) \cdot \nabla\varphi + g(z + w(z))\varphi \, d\zeta \\ &= \int_\Omega \nabla\varphi \cdot \nabla(-z + w(z)) + g(z + w(z))\varphi \, d\zeta + \int_\Omega \nabla(z + w(z)) \cdot \nabla\varphi + g(-z + w(z))\varphi \, d\zeta \\ &= \partial_u J(z + w(z), -z + w(z))\varphi + \partial_v J(z + w(z), -z + w(z))\varphi \\ &= \langle \nabla J(z + w(z), -z + w(z)), (\varphi, \varphi) \rangle = 0 \ , \qquad \forall\ \varphi \in H_0^1(\Omega). \end{aligned}$$

Hence, given $z \in H_0^1(\Omega)$,

$$\begin{aligned} \widetilde{J}(-\mathbf{z}) &= J(-z + w(-z), z + w(-z)) \\ &= J(-z + w(z), z + w(z)) \\ &= \textstyle\int_\Omega (|\nabla w(z)|^2 - |\nabla(-z)|^2 + G(-z + w(z)) + G(z + w(z)))d\zeta \\ &= J(z + w(z), -z + w(z)) \\ &= \widetilde{J}(\mathbf{z}). \end{aligned}$$

$\square$

Remark 1: Observe that from condition (g_1) and Lemma 2.1, we conclude that the set of candidates to be solutions of (1.2) is contained in the graph $\{\mathbf{z} + \mathbf{w}(\mathbf{z}) : \mathbf{z} \in Z\}$. From condition (g_0) we have $\mathbf{w}(\mathbf{0}) = \mathbf{0}$. Hence, combining these two facts, we observe that under $(g_0) - (g_1)$ the unique solution (u, v) of (1.2) with $u \equiv v$, i.e living in the set of fixed points of the action group, is the trivial one. Compare this with Theorem C.

3. Proof of Theorem A

Throughout this section we assume g satisfies (g_0), (g_1), (g_2) and (g_3). To prove Theorem A we make use of the following version of the Symmetric Mountain Pass Theorem (see e. g. [11]). We recall that if E is a Banach space and $I \in C^1(E, \mathbb{R})$, a sequence $\{e_n\}$ in E is a (PS)-sequence for the functional I, provided that

$$\forall n \in \mathbb{N}, \; |I(e_n)| \leq C \quad \text{and} \quad DI(e_n) \longrightarrow 0, \; n \to \infty. \tag{3.1}$$

The functional I is said to satisfy the (PS)-condition on E if every (PS)-sequence in E has a convergent subsequence.

Theorem 3.1. *Let $E = E_1 \oplus E_2$ be an infinite dimensional Banach space, where E_1 is a finite dimensional subspace. Let us assume $I \in C^1(E, \mathbb{R})$ is even, satisfies the Palais-Smale condition and $I(0) = 0$. Assume, in addition, I satisfies:*

- (I_1) *There exist positive constants α and ρ such that $I|_{\partial B_\rho \cap E_2} \geq \alpha$.*
- (I_2) *For each finite dimensional subspace $X \subset E$ there exists an $R = R(X) > 0$ such that $I|_{X \setminus B_R(0)} \leq 0$.*

Then I possesses an unbounded sequence of critical values.

We apply Theorem 3.1 to the functional $-\widetilde{J}$. To this end, let $j \in \mathbb{N}$ such that $g'(0) < \lambda_j$. We take $E_1 := \langle (\varphi_1, -\varphi_1) \ldots, (\varphi_{j-1}, -\varphi_{j-1}) \rangle \subset Z$ and $E_2 = E_1^{\perp} \subset Z$.

Claim 1: Under assumptions (g_0)-(g_3) functional $-\widetilde{J}$ satisfies (I_1).

Proof. Let us consider the functional $F : H_0^1(\Omega) \longrightarrow \mathbb{R}$ defined as

$$\begin{aligned} F(z) &= -J(z,-z) = \int_\Omega (|\nabla z|^2 - G(z) - G(-z))\, d\zeta \\ &= \int_\Omega (\frac{1}{2}|\nabla z|^2 - G(z))\, d\zeta + \int_\Omega (\frac{1}{2}|\nabla(-z)|^2 - G(-z))\, d\zeta. \end{aligned}$$

Because of hypothesis (g_0) and the variational characterization of λ_j (see [11] or [6]), $F|_{\langle\varphi_1\dots,\varphi_{j-1}\rangle^\perp}$ has a strict local minimum at zero and there exist positive constants α and ρ such that

$$F(z) \geq \alpha \qquad \forall z \in \partial B_\rho \cap \langle\varphi_1 \dots, \varphi_{j-1}\rangle^\perp \subset H_0^1(\Omega).$$

Hence, for each $\mathbf{z} = (z, -z) \in \partial B_{\sqrt{2}\rho} \cap E_2 \subset Z$,

$$-\widetilde{J}(\mathbf{z}) = -\min_{w\in H_0^1(\Omega)} J(z+w, -z+w) \geq -J(z,-z) = F(z) \geq \alpha. \qquad \square$$

Claim 2: Under assumptions (g_0)-(g_3) the functional $-\widetilde{J}$ satisfies (I_2).

Proof. Let X be a finite dimensional subspace of Z. Then, there exists a constant $\gamma_X > 0$ such that $\|z\|^2 \leq \gamma_X \|z\|_{L^2}^2$ for all $\mathbf{z} = (z,-z) \in X$. Using hypothesis (g_3) and integrating,

$$G(t) \geq a|t|^\mu - b$$

where $a > 0$ and $b > 0$ are constants. Since $\mu > 2$, given any $\alpha > 0$, there exists a constant C_α such that

$$a|t|^\mu - b \geq \frac{\alpha}{2} t^2 + C_\alpha$$

(for this, simply consider $h(t) := a|t|^\mu - \frac{\alpha}{2}t^2 - b$, which is bounded below and continuous). Thus,

$$G(t) \geq \frac{\alpha}{2}t^2 + C_\alpha \qquad \forall t \in \mathbb{R}.$$

Therefore, given $\mathbf{z} = (z,-z) \in X$, $\mathbf{w}(\mathbf{z}) = (w(z), w(z))$,

$$G(z + w(z)) + G(-z + w(z)) \geq \frac{\alpha}{2}(z + w(z))^2 + \frac{\alpha}{2}(-z + w(z))^2 + 2C_\alpha.$$

We then have

$$
\begin{aligned}
-\widetilde{J}(\mathbf{z}) &= \int_\Omega [|\nabla z|^2 - |\nabla w(z)|^2 - G(z+w(z)) - G(-z+w(z))]\, d\zeta \\
&\le \gamma_X \int_\Omega z^2 d\zeta - \alpha \int_\Omega z^2 d\zeta - \alpha \int_\Omega (w(z))^2 d\zeta - 2\widehat{C}_\alpha \\
&\le (\gamma_X - \alpha) \int_\Omega z^2 - 2\widehat{C}_\alpha \ .
\end{aligned}
$$

Thus, taking $\alpha > \gamma_X$, we have that

$$
-\widetilde{J}(\mathbf{z}) \longrightarrow -\infty\,, \quad \text{as } \ \|\mathbf{z}\| \to \infty,\ \mathbf{z} \in X.
$$

Since, X is arbitrary we have verified (I_2). ■

It remains to show that $\widetilde{J}$ satisfies the Palais-Smale condition.

Lemma 3.1. *Under the assumptions* (g_0)*-*(g_3) *the functional* $\widetilde{J}$ *satisfies the (PS)-condition.*

Proof. Observe that from (2.2) and (2.3), it suffices to verify that J satisfies the Palais-Smale condition. Let $\{(u_n, v_n)\}_n \subset H_0^1(\Omega) \times H_0^1(\Omega)$ be a (PS)-sequence. We want to extract a strongly convergent subsequence. Due to the form of DJ, the compactness on the Sobolev Embeddings and Vainberg's Lemma (see e.g. [10]), we just have to prove that $\{u_n\}_n$ and $\{v_n\}_n$ are bounded sequences in $H_0^1(\Omega)$.

Condition (3.1) implies that there exists a sequence $\{\varepsilon_n\}_n$, $\varepsilon_n > 0$ and $\varepsilon_n \to 0^+$ so that

$$
|DJ(u_n, v_n)[\phi, \psi]| \le \varepsilon_n(\|\phi\| + \|\psi\|), \quad \forall \phi, \psi \in H_0^1(\Omega). \tag{3.2}
$$

We take as test functions $\phi = \frac{1}{2}u_n$ and $\psi = \frac{1}{2}v_n$ to get

$$
\begin{aligned}
C \ &+ \frac{\varepsilon_n}{2}(\|u_n\| + \|v_n\|) \\
&\ge \tfrac{1}{2} DJ(u_n, v_n)[u_n, v_n] - J(u_n, v_n) \\
&= \int_\Omega \{-G(v_n) - G(u_n)\} + \frac{1}{2}\int_\Omega \{g(u_n)u_n + g(v_n)v_n\} \\
&\ge \frac{1}{2}\int_\Omega \{g(v_n)v_n - \mu G(v_n)\} + \frac{1}{2}\int_\Omega \{g(u_n)(u_n) - \mu G(u_n)\} \\
&\qquad + \Big(\frac{\mu}{2} - 1\Big)\int_\Omega \{G(v_n) + G(u_n)\}.
\end{aligned}
$$

So, changing the constant C if necessary, we find by (g_3) that

$$\int_\Omega G(u_n) + G(v_n) \leq C\,[1 + \varepsilon_n(\|u_n\| + \|v_n\|)]. \tag{3.3}$$

Since $\{J(u_n, v_n)\}_n$ is bounded, we can choose a large positive constant C such that

$$\left| \int_\Omega \nabla u_n \cdot \nabla v_n + \int_\Omega G(u_n) + G(v_n) \right| \leq C. \tag{3.4}$$

Because of hypothesis (g_3), $|G(t)| - G(t) = 0$, for every $|t| \geq R$, so it is a bounded function. Thus, we get from (3.3) and (3.4) that

$$\begin{aligned} \left|\int_\Omega \nabla u_n \cdot \nabla v_n\right| &\leq \int_\Omega |G(u_n)| + |G(v_n)| + C \\ &\leq \int_\Omega G(u_n) + G(v_n) + C \\ &\leq C[1 + \varepsilon_n(\|u_n\| + \|u_n\|)]. \end{aligned} \tag{3.5}$$

From (3.2), testing against $[\phi, \psi] = [u_n, v_n]$, we obtain

$$\left| 2\int_\Omega \nabla u_n \cdot \nabla v_n + \int_\Omega g(u_n)u_n + g(v_n)v_n \right| \leq \varepsilon_n(\|u_n\| + \|u_n\|).$$

So, by (3.5) we obtain

$$\int_\Omega g(u_n)u_n + g(v_n)v_n \leq C[1 + \varepsilon_n(\|u_n\| + \|u_n\|)]. \tag{3.6}$$

On the other hand, using again (3.2) and testing against $[\phi, \psi] = [0, u_n]$, we have

$$\left| \int_\Omega |\nabla u_n|^2 + g(v_n)u_n \right| \leq \varepsilon_n \|u_n\|. \tag{3.7}$$

Now let us estimate the second term in left-hand side of inequality (3.7). Using Hölder inequality we have

$$\left| \int_\Omega g(v_n)u_n \right| \leq \left(\int_\Omega |g(v_n)|^{1+\frac{1}{p}} \right)^{\frac{p}{1+p}} \left(\int_\Omega |u_n|^{1+p} \right)^{\frac{1}{1+p}} \tag{3.8}$$

Now note that for suitable positive constants c, d_1, d_2,

$$|g(t)|^{1+\frac{1}{p}} \leq c\,|g(t)||t| + d_1 \leq c\,g(t) + d_2. \tag{3.9}$$

Indeed, the first inequality in (3.9) follows from hypothesis (g_2), since

$$|g(t)|^{\frac{1}{p}} \leq C\,|t| + d:$$

- for $|t| \geq 1$

$$\begin{aligned} |g(t)|^{1+\frac{1}{p}} &\leq C\,|g(t)|\,|t| + d\,|g(t)| \\ &\leq C\,|g(t)|\,|t| + d\,|g(t)|\,|t|. \end{aligned}$$

- for $|t| \leq 1$ we see that $|g(t)|$ is simply bounded. So the first inequality in (3.9) holds. As for the second inequality in (3.9), we write

$$|g(t)|\,|t| = g(t)\cdot t + |g(t)|\,|t| - g(t)\cdot t,$$

and observe that, because of (g_3), $|g(t)|\,|t| - g(t)\cdot t = 0$, for $|t| \geq R$. So this difference remains bounded in $\mathbb{R}$ and the inequality holds.

From (3.6), (3.8) and (3.9), we get that

$$\begin{aligned} |\int_\Omega g(v_n)u_n| &\leq \big(c\int_\Omega g(v_n)v_n + d_2\big)^{\frac{p}{1+p}}\|u_n\|_{L^{1+p}} \\ &\leq \big(C[1+\varepsilon_n(\|u_n\|+\|v_n\|)]\big)^{\frac{p}{1+p}}\|u_n\|. \end{aligned}$$

Then, by (3.7),

$$\int_\Omega |\nabla u_n|^2 \leq \varepsilon_n\|u_n\| + (C[1+\varepsilon_n(\|u_n\|+\|v_n\|)])^{\frac{p}{1+p}}\,\|u_n\|.$$

In a similar fashion, taking $[\phi,\psi] = [v_n, 0]$ in (3.2), we get the analogous estimate

$$\int_\Omega |\nabla v_n|^2 \leq \varepsilon_n\|v_n\| + (C[1+\varepsilon_n(\|u_n\|+\|v_n\|)])^{\frac{p}{1+p}}\,\|v_n\|.$$

Joining these two estimates we obtain

$$\|u_n\|^2 + \|v_n\|^2 \leq \varepsilon_n(\|u_n\|+\|v_n\|) + C\,(\|u_n\|+\|v_n\|)^{\frac{2p+1}{1+p}} + K.$$

Since $\frac{2p+1}{1+p} < 2$, the sequence $\{(u_n,v_n)\}_n$ is bounded in H and the proof of the lemma is complete. □

4. Proof of Theorem B

Throughout this section we assume that g satisfies (g_0), (g_1) and (g_4). To prove Theorem B we make use of the following version of the Symmetric Mountain Pass Theorem (see e.g. [2], [3], and [12]).

Theorem 4.1. *Let $E = E_1 \oplus E_2$ be a real Banach space, where E_1 is a finite dimensional subspace. Let $X \subset E$ be a finite dimensional subspace of E such that $\dim E_1 < \dim X$. Suppose that $I \in C^1(E, \mathbb{R})$ is an even functional, satisfying $I(\boldsymbol{0}) = 0$ and*

- (I_1') *There exists a positive constant ρ such that $I|_{\partial B_\rho \cap E_2} \geq 0$.*
- (I_2') *There exists $M > 0$ such that $\max_{\boldsymbol{z} \in X} I(\boldsymbol{z}) < M$.*

If I satisfies the Palais-Smale condition at level c, for every $c \in [0, M]$, then I possesses (at least) $\dim X - \dim E_1$ pairs of nontrivial critical points.

As in Section 3, we take $E_1 := \langle(\varphi_1, -\varphi_1) \ldots, (\varphi_{j-1}, -\varphi_{j-1})\rangle$ and $E_2 = E_1^\perp$. As we proved in the previous section, the fact that $-\widetilde{J}$ satisfies (I_1') comes from hypothesis (g_0) and the variational characterization of the eigenvalues, i.e. the local structure of the functional around zero in this case is similar to that of the superlinear setting.

Claim: Under hypotheses (g_0), (g_1) and (g_4), the functional $-\widetilde{J}$ satisfies (I_2').

Proof. Let us take $X = \langle(\varphi_1, -\varphi_1) \ldots, (\varphi_k, -\varphi_k)\rangle$. Since $g'(\infty) > \lambda_k$, taking a number $\alpha \in (\lambda_k, g'(\infty))$ it follows that

$$G(t) > \frac{\alpha}{2} t^2 + C_\alpha \qquad \forall t \in \mathbb{R}.$$

The remaining of this proof is very similar to the proof of Claim 2 in Section 3 by simply using the inequality

$$\|x\|^2 \leq \lambda_k \int_\Omega x^2 \qquad \forall x \in \langle \varphi_1, ..., \varphi_k \rangle.$$

From this, given $\mathbf{z} = (z, -z) \in X$,

$$-\widetilde{J}(\mathbf{z}) \leq (\lambda_k - \alpha)\|z\|_{L^2}^2 + \widetilde{C}_\alpha \longrightarrow -\infty \text{ as } \|\mathbf{z}\| \to \infty, \mathbf{z} \in X . \qquad \square$$

It remains to show that $\widetilde{J}$ satisfies the Palais-Smale condition. In this case, we follow the ideas of the corresponding proof for the problem with one equation and asymptotic (nonresonant) nonlinearities, although our proof requires a bit more of technicalities.

Lemma 4.1. *Under assumptions* (g_0), (g_1) *and* (g_4) *the functional* $\widetilde{J}$ *satisfies the (PS)-condition.*

Proof. As before, from (2.2) and (2.3), it suffices to verify that J satisfies the Palais-Smale condition. We take a (PS)-sequence $\{(u_n, v_n)\}_n$ in $H_0^1(\Omega) \times H_0^1(\Omega)$ and again it is sufficient to prove that this sequence is bounded. In this case, we argue by contradiction. Let us assume that $\{\|(u_n, v_n)\|\}_n$ is not bounded. Passing to a subsequence, denoted the same for simplicity of notation, we can say that either $\|u_n\| \to \infty$ or $\|v_n\| \to \infty$. We claim that

(I) if $\|u_n\| \to \infty$, then there exists a subsequence $\|v_{n_k}\| \to \infty$, and

(II) if $\|v_n\| \to \infty$, then there exists a subsequence $\|u_{n_k}\| \to \infty$.

Indeed, let us prove (I) arguing by contradiction. If $\|u_n\| \to \infty$ and $\|v_n\| \leq C$, then, passing to a subsequence we have that

$$v_n \rightharpoonup v, \quad \text{in} \quad H_0^1(\Omega) \qquad \tfrac{u_n}{\|u_n\|} \rightharpoonup \bar{u}, \quad \text{in} \quad H_0^1(\Omega)$$

$$v_n \to v, \quad \text{in} \quad L^r(\Omega) \qquad \tfrac{u_n}{\|u_n\|} \to \bar{u}, \quad \text{in} \quad L^r(\Omega), \quad \text{for} \quad r \in [1, \tfrac{2N}{N-2}).$$

There exists a sequence $\{\varepsilon_n\}_n$, $\varepsilon_n > 0$ and $\varepsilon_n \to 0^+$ so that

$$|DJ(u_n, v_n)[\phi, \psi]| \leq \varepsilon_n(\|\phi\| + \|\psi\|), \quad \forall \phi, \psi \in H_0^1(\Omega). \tag{4.1}$$

Testing $\partial_v J(u_n, v_n)$ against $\frac{u_n}{\|u_n\|}$ and using (4.1) we get that

$$\left| \|u_n\| + \int_\Omega g(v_n) \frac{u_n}{\|u_n\|} \right| \leq \varepsilon_n.$$

From (g_4), $|g(t)| \leq C(1 + |t|)$ for all $t \in \mathbb{R}$. Using Vainberg's Lemma (see [10]) we have that

$$\int_\Omega g(v_n) \frac{u_n}{\|u_n\|} \longrightarrow \int_\Omega g(v)\, \bar{u}$$

and so we get

$$\|u_n\| \longrightarrow -\int_\Omega g(v)\, \bar{u}, \text{ as } n \to \infty.$$

This contradicts our initial assumption. We proceed in an analogue way to prove (II) and therefore the claim is proved.

Now, using the claim, and passing to a subsequence, we can assume without loss of generality that:

$$\|u_n\| \to \infty \quad \text{and} \quad \|v_n\| \to \infty.$$

Hence, there exist $u, v \in H_0^1(\Omega)$ such that

$$\tfrac{u_n}{\|u_n\|} \rightharpoonup \bar{u}, \quad \text{in} \quad H_0^1(\Omega) \qquad \tfrac{v_n}{\|v_n\|} \rightharpoonup \bar{v}, \quad \text{in} \quad H_0^1(\Omega)$$

$$\tfrac{u_n}{\|u_n\|} \to \bar{u}, \quad \text{in} \quad L^r(\Omega) \qquad \tfrac{v_n}{\|v_n\|} \to \bar{v}, \quad \text{in} \quad L^r(\Omega), \quad \text{for} \quad r \in [1, \tfrac{2N}{N-2}).$$

We claim that $\{\|u_n\|\}_n$ and $\{\|v_n\|\}_n$ go to infinity at the same rate. More precisely, we claim that

$$\lim_{n\to\infty} \frac{\|u_n\|}{\|v_n\|} = 1. \tag{4.2}$$

To prove this claim, we first test $\partial_u J(u_n, v_n)$ against $\frac{v_n}{\|v_n\|}$ and then divide by $\|u_n\|$ to get

$$\left| \frac{\|v_n\|}{\|u_n\|} + \int_\Omega \frac{g(u_n)}{\|u_n\|} \cdot \frac{v_n}{\|v_n\|} \right| \le \frac{\varepsilon_n}{\|u_n\|}. \tag{4.3}$$

Assumption (g_4) implies that $g(t) = g'(\infty)t + \gamma(t)$, where $\gamma(t) = o(t)$, as $|t| \to \infty$. Then,

$$\int_\Omega \frac{g(u_n)}{\|u_n\|} \frac{v_n}{\|v_n\|} = g'(\infty) \int_\Omega \frac{v_n}{\|v_n\|} \frac{u_n}{\|u_n\|} + \int_\Omega \gamma(u_n) \frac{v_n}{\|v_n\|\|u_n\|}. \tag{4.4}$$

Now we show that

$$\int_\Omega \gamma(u_n) \frac{v_n}{\|v_n\|\|u_n\|} \longrightarrow 0.$$

Indeed, just observe that given $\varepsilon > 0$ arbitrary, there exists $T \ge 0$ such that

$$\left| \frac{\gamma(t)}{t} \right| < \varepsilon, \;\; \text{for} \;\; |t| \ge T.$$

On the other hand, $\gamma(t) = g(t) - g'(\infty)t$ is continuous in $[-T, T]$ and so it is bounded in $[-T, T]$. Thus, it follows that

$$\begin{aligned}
\int_\Omega \left| \gamma(u_n) \frac{v_n}{\|v_n\|\|u_n\|} \right| &\le \int_{\{|u_n|>T\}} + \int_{\{|u_n|\le T\}} \\
&\le \varepsilon \int_\Omega \left| \frac{u_n}{\|u_n\|} \frac{v_n}{\|v_n\|} \right| + \frac{C_T}{\|u_n\|} \int_\Omega \left| \frac{v_n}{\|v_n\|} \right| \\
&\le C\varepsilon + \frac{C_T}{\|u_n\|} C \\
&\le 2C\varepsilon, \quad \text{for } n \text{ large enough.}
\end{aligned}$$

Hence, we can take the limit in (4.4) to get

$$\int_\Omega \frac{g(u_n)}{\|u_n\|} \frac{v_n}{\|v_n\|} \longrightarrow \int_\Omega g'(\infty)\, \bar{u}\, \bar{v}.$$

This and (4.3) give

$$\frac{\|v_n\|}{\|u_n\|} \longrightarrow -\int_\Omega g'(\infty)\,\bar{u}\,\bar{v}. \tag{4.5}$$

Arguing in a similar fashion, but now testing $\partial_v J(u_n, v_n)$ against $\frac{u_n}{\|u_n\|}$, we also obtain

$$\frac{\|u_n\|}{\|v_n\|} \longrightarrow -\int_\Omega g'(\infty)\,\bar{u}\,\bar{v}, \tag{4.6}$$

which together with (4.5) implies that actually $\int_\Omega g'(\infty)\,\bar{u}\,\bar{v} = -1$ and therefore the claim is proved.

Let us now take $\phi \in H_0^1(\Omega)$. Using (4.1) we have that

$$\left|\int_\Omega \nabla\phi \cdot \nabla\left(\frac{v_n}{\|v_n\|}\right) + \frac{g(u_n)}{\|v_n\|}\phi\right| \longrightarrow 0. \tag{4.7}$$

Due to the weak convergence of $\frac{v_n}{\|v_n\|}$ to $\bar{v}$, we know that

$$\int_\Omega \nabla\phi \cdot \nabla\left(\frac{v_n}{\|v_n\|}\right) \longrightarrow \int_\Omega \nabla\phi \cdot \nabla\bar{v}. \tag{4.8}$$

On the other hand, (4.2) implies that

$$\int_\Omega \frac{g(u_n)}{\|v_n\|}\,\phi \longrightarrow \int_\Omega g'(\infty)\,\bar{u}\,\phi. \tag{4.9}$$

To see why this is true, it is enough to notice that

$$\int_\Omega \frac{g(u_n)}{\|v_n\|}\,\phi = \int_\Omega \frac{g(u_n)}{\|u_n\|} \cdot \frac{\|u_n\|}{\|v_n\|}\,\phi = \frac{\|u_n\|}{\|v_n\|}\int_\Omega \frac{g'(\infty)u_n + \gamma(u_n)}{\|u_n\|}\,\phi$$

and arguing as above, it can be proved that $\int_\Omega \frac{\gamma(u_n)}{\|u_n\|}\,\phi \longrightarrow 0$.

From (4.7), (4.8) and (4.9), we have proven that

$$\forall\phi \in H_0^1(\Omega) : \int_\Omega \nabla\bar{v} \cdot \nabla\phi + g'(\infty)\,\bar{u}\,\phi = 0. \tag{4.10}$$

Using (4.5) and reasoning analogously, we also get that

$$\forall\phi \in H_0^1(\Omega) : \int_\Omega \nabla\bar{u} \cdot \nabla\phi + g'(\infty)\,\bar{v}\,\phi = 0. \tag{4.11}$$

From relations (4.10) and (4.11), testing both integrals against $\phi = \bar{v} + \bar{u}$ we obtain

$$\int_\Omega |\nabla(\bar{u} + \bar{v})|^2 = -g'(\infty)\int_\Omega (\bar{v} + \bar{u})^2.$$

Since $g'(\infty) > 0$, $\bar{v} = -\bar{u}$. Replacing this in any of the relations (4.10) or (4.11) we get that $\bar{u} = -\bar{v} \in H_0^1(\Omega)$ is a weak solution, and actually a classical one, to the problem

$$\begin{cases} -\Delta u = g'(\infty)\, u \;\; \text{in } \Omega \\ \qquad u = 0 \;\; \text{on } \partial\Omega. \end{cases}$$

This, as well as (4.5) and (4.6), imply that $g'(\infty) = \lambda_j$ for some $j \in \mathbb{N}$. This contradicts hypothesis (g_4). Hence, a contradiction is reached assuming that $\{\|(u_n, v_n)\|\}_n$ is unbounded, and the conclusion of the lemma follows. □

5. Proof of Theorem C

Assume condition (g_5). Let us assume (u, v) is a solution of (1.2). Multiply the first equation in (1.2) by $u - v$, and then multiply the second equation by $u - v$. Taking the difference of both results, we get

$$\int_\Omega |\nabla(u-v)|^2 + (g(v) - g(u))(u - v) = 0$$

or, equivalently,

$$\int_\Omega |\nabla(u-v)|^2 = \int_\Omega (g(u) - g(v))(u - v).$$

Because of Mean Value Theorem and (g_5), we have that

$$\int_\Omega |\nabla(u-v)|^2 \leq (\lambda_1 - \epsilon) \int_\Omega (u-v)^2,$$

for some small $\epsilon > 0$. From Poincar's Inequality we conclude that $u \equiv v$.

[1] R. A. Adams, *Sobolev Spaces*, New York, Academic Press, 1975.

[2] A. Ambrosetti, P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973) 349-381.

[3] P. Bartolo, V. Benci, D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity*, Nonlinear Anal. TMA 7 (9) (1983) 981-1012.

[4] A. Castro, *Reduction methods via minimax*, Lecture Notes in Mathematics 957, Differential Equations, Springer Berlin-New York, 1982, pp.1-20.

[5] A. Castro, A. Lazer, *Critical point theory and the number of solutions of a nonlinear Dirichlet problem*, Ann. Mat. Pura Appl., (4) 120 (1979), 113-137.

[6] J. Cossio and C. Vélez, *Soluciones no triviales para un problema de Dirichlet asintóticamente lineal*, Rev. Colombiana Mat. Vol 37, 2003, 25-36.

[7] D. G. De Figueiredo and P. L. Felmer, *On Supercuadratic Elliptic Systems*, Trans. of AMS, Volume 343, Number 1, 1994, pp. 99-116.

[8] M. Furtado, E. Silva, M. Xavier, *Multiplicity and Concentration of Solutions for Elliptic Systems with Vanishing Potentials*, J. Diff. Eq. 249 (2010), pp. 2377-2396.

[9] D. Gilbart, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer Verlag. Berlin 1977.

[10] D. Mitrovic and D. Zubrinic, *Fundamentals of applied functional analysis*, Pitman Monographs and surveys in pure and applied Mathematics, 91. Addison Wesley. Longman Inc., 1998

[11] P. H. Rabinowitz, *Minimax methods in Critical Point theory with applications to Differential Equations*, Regional Conference Series in Mathematics, number 65. AMS, Providence, R.I., 1986.

[12] E.A.B. Silva, *Critical point theorems and applications to differential equations*, Ph.D. Thesis, University of Wisconsin-Madison, 1988.