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ALGEBRAIC MODELS OF CONCEPTUAL METAPHOR: CONTRIBUTIONS TO THE UNDERSTANDING OF MATHEMATICS LEARNING PROCESSES.

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Resumen

Esta tesis estudia un fenómeno cognitivo humano llamado *Metáfora Conceptual* en el contexto del aprendizaje y razonamiento de la Matemática. La metáfora permite el entendimiento de un concepto abstracto llamado *destino*, e.g. números, en términos de un concepto mas concreto llamado *fuente*, e.g. un montón de caramelos. A menudo, algunos razonamientos son llevados desde la fuente al destino permitiendo inferir propiedades de este último. Esto es conocido como *razonamiento por analogía*.

Por un lado, evidencia empírica indica que la metáfora es capaz de potenciar el aprendizaje. Evidencia convergente son los reportes de científicos que admiten el uso de analogías mientras están desarrollando sus teorías. Por otro lado, algunas personas son contrarias al uso de analogías en educación. Ellos argumentan que los políticos y comunicadores usan analogías de manera que la audiencia llegue a conclusiones erroneas; sugiriendo que las analogías afectan el razonamiento objetivo. Esta discusión pone de relieve la necesidad de investigar el fenómeno para entender su alcance y sus limitaciones.

Este trabajo presenta un modelo formal para la metáfora que puede ser usado como un marco para estudiar el aprendizaje por analogías. Debido a que el modelo es abstracto, el capítulo 1 es utilizado para concretar las ideas: utilizamos nuestro formalismo para analizar en profundidad un ejemplo bien conocido. Así mismo, el capítulo 5 presenta formalizaciones de otras metáforas que son encontradas frecuentemente en la enseñanza de la matemática.

El modelo formal es construido en el capítulo 4 donde la fuente y el destino de una metáfora son formalizados por un concepto clave llamado *dominio*. Algunos resultados de este capítulo son acompañados por interpretaciones cognitivas, como por ejemplo, los teoremas 40, 41, 42, y la proposición 26, que pueden ser vistos como una descripción de cómo una analogía llevaría razonamientos de la fuente al destino. Además, los teoremas 30 y 31 sugieren mecanismos para el proceso de aprendizaje por analogía. Finalmente, el capítulo 4 presenta algunas construcciones teóricas como productos y coproductos de dominios.

Nuestro modelo de metáfora relaciona dos dominios, cada uno definido como una mezcla de lenguaje y semántica. La mayoría de los resultados del capítulo 4 necesitan la premisa de que los dos lenguajes involucrados son "compatibles". Matemáticamente, se necesita una función capaz de preservar la estructura determinada por una operación sintáctica llamada *sustitución*. Aplicando teoría de grafos y teoría de la unificación, esta noción de compatibilidad es caracterizada para el caso de los términos del lenguage en el capítulo 2. En el capítulo 3, esta noción de compatibilidad es caracterizada para el caso de las fórmulas de un lenguage mediante la adaptación de los métodos del capítulo anterior.

Finalmente, un apéndice (*Relational Spaces*) presenta otro enfoque para estudiar la metáfora. Allí, los dominios son definidos con la semántica solamente, ignorando el lenguage. La mayoría de los resultados enfatizados arriba se pierden o se debilitan sugiriendo que la información "abstracta" provista por los símbolos y la recursión provista por la gramática del lenguaje son necesarios para imitar el comportamiento de la metáfora. Esta observación, junto con otros resultados, podría indicar la existencia de una relación entre la propiedad de recursión de los lenguages humanos [1] y la habilidad de aprender por analogías.

[1]Noam Chomsky sostiene que la recursión es el componente unicamente humano de la facultad del lenguage [49].

Summary

This thesis studies a human cognitive phenomenon called *Conceptual Metaphor* in the context of mathematics learning and reasoning. Metaphor enables the understanding of an abstract concept called *target*, e.g. numbers, in terms of a more concrete concept called *source*, e.g. piles of candies. Often, inferences from the source are carried to the target and applied there yielding some conclusions about the target. This is known as *reasoning by analogy*.

On the one hand, empirical evidence indicates that metaphor enhances learning and converging evidence is provided by working scientists who report the use of analogies while developing their theories. On the other hand, some people advise against its usage in education. They argue that politicians and communicators often lead people into erroneous conclusions by using metaphor, and then, analogies undermine objective reasoning. This discussion highlights the need for research to shed light into the learning mechanics underlying metaphor in order to understand its scope and limitations.

This work presents a formal model of metaphor which can be used as a framework to study learning by analogy. Since the model is abstract, we use Chapter 1 to make ideas more concrete: we use our formalism for analize deeply a well known example. Along these lines, Chapter 5 presents formalizations of other metaphors frequently encountered in mathematics teaching.

The model is built in Chapter 4 where the source and the target of a metaphor are formalized by a key concept named *domain*. Some results of this chapter are accompanied by cognitive interpretations, as for example, Theorems 40, 41, 42, and Proposition 26 can be seen as descriptions of how an analogy carries reasonings from its source to its target. Also, Theorems 30 and 31 suggest models for the process of learning by analogy. Finally, Chapter 4 presents some theoretical constructions such as products and coproducts of domains.

Our metaphor model relates two domains, each one defined as a mixture of language and semantics. Most results of Chapter 4 need the premise that the two involved languages are "compatible". Mathematically, they need a map able to preserve the structure determined by a syntactical operation called *substitution*. This compatibility notion is characterized for the case of language terms in Chapter 2 by applying unification theory and graph theory. And in Chapter 3, this compatibility notion is characterized for the case of the language formulas by adapting the methods of the previous chapter.

Finally, one Appendix (*Relational Spaces*) presents another approach to study metaphor. There, domains are defined with semantics only, leaving language aside. Most of the results emphasized above are lost or at least weakened, suggesting that the "abstract" information provided by symbols and the "recursion" provided by the grammar of the language are necessary to mimic metaphor's behavior. This observation, together with other results of this thesis, might point to a relation between the recursion property of human languages[2] and the ability of learning by analogy.

[2]The linguist Noam Chomsky claims that recursion is the only human component of the faculty of language [49].

A mi abuela, por su cariño inmenso.
A mi madre y a mi tía Carmen por su constante apoyo.
A Miguel, mi primo.

Agradecimientos

La presentación de este trabajo finaliza el recorrido de un pequeño tramo del sendero que, con algo de suerte y bastante esfuerzo, permitirá que nuestras futuras generaciones alcancen un desarrrollo pleno. Recorrer dicho sendero es construir una educación de calidad. Aunque el presente trabajo es un aporte pequeñísimo, apenas cuantificable, dentro de ese proyecto inmenso; aun así reune los aportes de una cantidad inmensa de personas e instituciones que han sembrado tiempo y confianza en el autor de esta tesis. Talvez éste es el lugar adecuado para expresarles reconocimiento y agradecimiento por dicho apoyo.

Empezaré por contar como inicia esta historia. En el año 2006 decido estudiar en la Universidad de Chile motivado por uno de sus profesores, Pablo Dartnell, quien me comentó acerca de un nuevo proyecto que estaban iniciando dentro del Departamento de Ingeniería Matemática (DIM). La idea principal era realizar investigación en el campo educativo pero, a diferencia de otros enfoques, querían hacerlo utilizando el método científico y mediante el uso de modelos matemáticos para representar fenómenos de aprendizaje. Siempre quise contribuir al campo educativo, así que el entusiasmo de Pablo me fué contagiado rápidamente. Avanzamos a la siguiente etapa: Pablo me presenta a Roberto Araya, un investigador que ha dedicando gran parte de sus esfuerzos a comprender y mejorar el sistema educativo chileno. En una conversación casual Roberto se veía interesado en explorar algo llamado “metáforas” y me propuso estudiarlo. Mi mente empezó a rondar obsesivamente la idea de Roberto. Acepté la propuesta.

Aquí empieza el proyecto cuyo resultado tiene el lector entre sus manos: lo hermoso del descubrimiento, la emoción del aprendizaje, lo arriesgado de la empresa, la frustración de los intentos fallidos, y ... continúa la lista. Sin duda alguna, las discusiones con Pablo Dartnell, Roberto Araya, David Gomez y Renato Lewin pusieron sobre la mesa las intuiciones e ideas iniciales que dieron forma a este trabajo. Otro aporte fueron los Seminarios de Educación del Centro de Modelamiento Matemático (CMM) organizado por Leonor Varas, la cercanía con proyectos de educaci ón gestionados por el Centro de Investigación Avanzada en Educación de la Universidad de Chile (CIAE) y mi inclusión en el proyecto “Videojuegos Educativos” en donde trabajé con Cristián Reyes.

Por otro lado, talvez debí pensar con mas detenimiento antes de empezar a investigar los misterios que encierran las metáforas. El hecho de que no hubiesen modelos matemáticos creados para este fenomeno implicó la difícil tarea (al menos para mí) de crear un modelo matemático desde cero. Esto trajo como consecuencia varios momentos complicados donde el apoyo y la buena vibra de los amigos jugó un rol fundamental. En este aspecto, la oficina del DIM fué el baluarte donde se celebraron aquellas juntas con David, Alvaro, Marcelo y Andrea que estuvieron llenas de carcajadas y momentos distendidos (extremadamente necesarios). En otras ocasiones, Pablo dejaba alegremente su rol de profesor guía para transformarse en un amigo locuaz y divertido. Así mismo quiero agradecer a Omar, Luciano, Cristian, Selene y Aurelie por apoyarme con alegría durante los años que duró este proyecto. Un agradecimiento especial a toda mi familia por el apoyo brindado a la distancia. En particular, a mi primo Miguel por acordarse y venir a visitarme cuando fué necesario.

Además agradezco mucho a Rafael Nuñez y Rafael Correa por tomarse el tiempo para leer y mejorar este trabajo por medio de sus comentarios y observaciones. Mi gratitud a las instituciones que financiaron esta tesis: al DIM, que mediante un proyecto MECESUP me otorgó una beca para realizar mis estudios de doctorado. Al CMM que mediante una beca complementaria me ayudó a financiar la parte final de esta investigación. Al CIAE (Centro de Investigación Avanzada en Educación) por la invitación a varios cursos y discusiones realizados en sus instalaciones. Al Departamento de Ciencias Básicas de la Universidad del Bío-Bío por permitirme utilizar una pequeña porción del tiempo de trabajo para completar esta tarea.

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Part I

Algebraic Models of Conceptual Metaphor: Contributions to the Understanding of Mathematics Learning Processes.

Chapter 1

Introduction and Motivation

> And I cherish more than anything else the Analogies, my most trustworthy masters. They know all the secrets of nature, and they ought to be least neglected in Geometry.
>
> Kepler

A growing body of evidence indicates that, for a country, high rates of education are essential for achieving high levels of economic growth[46]. There is no debate on such claims, an effective educational system make pupils become agents of positive change. Nonetheless, economics aside, we believe it is important to make an analogous reflection at an individual level: people with access to quality education are empowered for pursuing their own goals and achieve a good quality of life.

This thesis is aimed to contributing to the improvement of educational systems through a better understanding of the human's learning phenomenon. At least three crucial tasks could be enlightened by such understanding: identification of good patterns for *communicating knowledge*, discovery of principles for *designing effective tools* for teaching and learning, and pointing out strategies for *maintaining the learner engaged*. Undoubtedly, the amount of work required to address any of these issues puts them out of the scope of this thesis. However, problems of this magnitude might be approached by institutions such as the CIAE (Center for the Advanced Reasearch in Education of the University of Chile) whose creation has gathered a multidisciplinary team of researchers to work around education themes.

This thesis studies a phenomenon that has been named *Conceptual Metaphor*[1] by cognitive scientists. We call it only *metaphor* or sometimes *analogy*. The referred subject is a vast domain of knowledge and, to perform the present study, we have made some strong assumptions that shall be signaled into the relevant contexts. But first, let us comment why we selected this subject of study in preference to others: there are serious claims that postulate metaphor as the natural way in which people learn to reason about concepts that are new or abstract. If these claims are right, there are deep implications for education. And to this respect, the influential psychologist Steven Pinker says [96]

> I think that metaphor really is a key to explaining thought and language. The human mind comes equipped with an ability to penetrate the cladding of sensory appearance and discern the abstract construction underneath—not always on demand, and not infallibly, but often enough and insightfully enough to shape the human condition. Our powers of analogy allow us to apply ancient neural structures to newfound subject matter, to discover hidden laws and systems in nature, and not least, to amplify the expressive power of language itself.

In brief, an analogy is a cognitive process that uses knowledge of an structure, named *source* domain, as a guide to reason about another structure called *target* domain. This process is pervasive in human thought for reasoning and in human language for comunication. And this might be the cause that analogy has been claimed to be the core of human cognition (see for example [31, 71]). Even if someone disagrees with such a high level of importance for analogy in human thought, it is undeniable that the phenomenon has received plenty of attention from many researchers in diverse fields such as poetics, linguistics, psichology, education [3, 98, 106, 53], cognitive science [30, 51, 71] and artificial intelligence [92, 119, 81, 74, 42] among others.

This reveals a deep need, across related areas, for the creation of models that shed some light into the understanding of the analogy phenomenon. As a manifestation of this need, one of the 33 basic research problems proposed for automated reasoning is exclusively devoted to analogy [123, 124]. In this context, the main contribution of this thesis is a formalization of this cognitive process together with some arguments for supporting the idea that such model might open up a way

[1]Literary metaphors like *juliet is the sun* or *the world is a stage* are different in nature and not relevant for this work.

for applying mathematical analysis on the study of the phenomenon (some previous attempts in the path of formalizing the phenomenon can be found in [41, 40, 55]).

1.1 Metaphors and the Learning of Mathematics

A metaphor is a particular relation between a well known idea –such as three candies– and a new or abstract idea —the number 3–. Experimental evidence shows that such mechanism can be used succesfully to enhance methods for teaching and learning mathematics.

For instance, the chilean researcher Roberto Araya and his collaborators used videos to teach how to solve algebraic equations to 236 students divided in two groups[3]. The first group was taught with a classical symbolic strategy and the second one was taught by using balances in equilibrium to introduce the concept of an equation. In a post-test, the second group performed significantly better.

Furthermore, this study obtains a striking result: students with a below average mathematics performance who watched the analogies video performed about the same as students with an above average performance who watched the classical strategy. A study with similar characteristics is reported in [98]. Complementary to this line of research, the psychologist Rafael Nuñez has identified some conceptual metaphors that seem to be at the heart of some mathematical concepts and might be useful to teach (or learn) complex ideas such as continuous functions and infinity[89, 87].

More evidence comes from a classical study performed by the psychologist Robert Siegler and his collaborators. In a serie of studies[101, 105], they show how children's understanding about walking along a path can be used to improve their number understanding. First, let us give some context. In [104] Siegler shows that children represent numbers in a kind of "mental line of numbers" which, over time, develops from a logarithmic representation to a linear one [2]. Then, he observes that arithmetic performance depends crucially on such representation: the more linear the representation is, the better the performance becomes (see also [44]).

Without instruction, this change of representation is very limited and takes place over long

[2] A huge amount of evidence indicates that our understanding of numbers depends on two core systems that our brain uses to represent numbers [23]. The above discussion refers to one called Aproximate Number System (ANS).

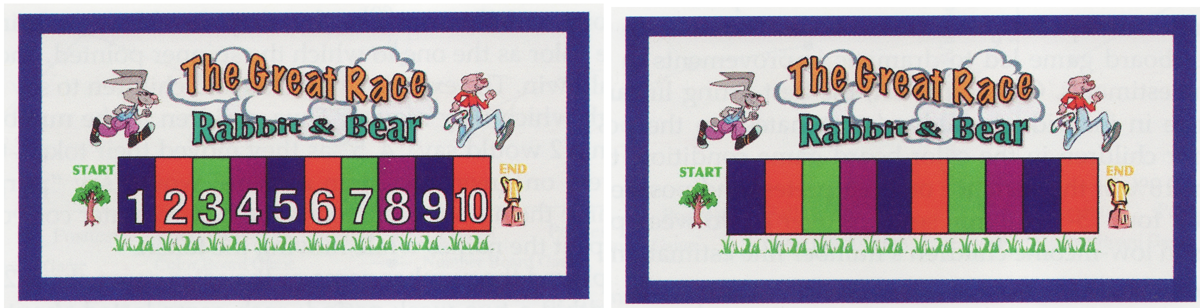

Figure 1.1: Differences between the boards of the Siegler's experiment.

periods of time. However, in [101, 105] Siegler triggers such change by means of a metaphor. He designed two versions of a game called "The Great Race" (TGR). Both versions of the game had many aspects in common: the boards were horizontal arrangements of ten squares with "start" and "end" written before the left-most square and after the right-most square respectively, they were designed to be played between the child and the experimentalist. The winner was the one who arrived to the end square first, in each turn one player could move 1 or 2 squares towards the right.

The differences between the two versions were only that the first board had the numbers 1 - 10 listed from left to right, and when the players moved they had to recite the numbers they were steping, for example, "three" and "four." Whereas the second version of the game (thought as to be used by a control group) had only colors in the squares, and when the players moved they had to recite such colors, saying for example "blue" and "red" (See Figure 1.1).

The first graders who played the first version of the game, changed their internal logarithmic representation for a linear one which led them to improve their performance in diverse numerical tasks. The control group did not. The gains remained 9 weeks later. On another condition [105], a third version of the game is based on a circular board divided in ten parts with numbers 1-10 listed on them. The group which played this version of the game did not improve its representation or its performance in the numerical tasks.

Because of the results, it is clear that the first version of the game helps learning. It is equally clear that the second and third versions do not. By using the predictions of our model, the results of these studies can be formally explained: in few words, the first version pinpoints the meaningful matchups between the path and the numbers while the second version does not suggest such matching. The third version makes such suggestion but supports misleading inferences when working with numbers: in a circular board might be possible that the number 9 is at the left side of the number 2 suggesting [3] that 9 would be lesser than 2, or that there are only two numbers[4] between 9 and 2. Throughout this chapter we shall study the TGR metaphor (TGRM), i.e., the metaphor underlying the first version of the TGR game. This running example shall enable us to present our work.

1.2 Our Algebraic Approach to Formalize Metaphor

In this work we develop a formal model of metaphor in order to analyze the phenomenon. We adopt this approach for several reasons; in particular because the authors are mathematicians. We do not believe that this is the only approach, it might just be the more accesible to us. However, we believe that a formal model can shed a good deal of light into the understanding of the phenomenon.

We have found almost no formal models of the phenomenon in the literature (however see [41, 40, 55] and [84]). In particular, we have not found formal models that account for the properties of metaphor that are related to human learning. For the sake of completeness of this work, some computational models shall be analized in the first part of Chapter 2. Our aim in what follows is to show the benefits of having formal models for phenomena by considering an education related case.

The *Flynn Paradox* appears in the following manner. The psychologist Arthur Jensen argues that widely accepted estimates of the heritability of IQ[5] —the fraction of the variance of IQ in a population caused by differences in genetic endowment— render environmental explanations of large IQ differences between groups implausible [57]. On the other hand, James Flynn discovered a substantial and long-sustained increase[25] in test scores measuring IQ. In simpler words, people within a given generation are smarter than their parents. This effect, observed in many parts of the world, has been named *The Flynn Effect* after his discoverer. This raises the Flynn Paradox: the

[3] Many studies [17] have provided evidence that people associate their left hand and left visual field with small numbers and their right hand and right visual field with large numbers.

[4] In a circular board you can walk from the square labeled by 9 through two squares labeled by 10 and 1 to arrive at the square labeled by 2.

[5] An intelligence quotient, or IQ, is a score derived from one of several standardized tests designed to assess intelligence.

fact that IQ gains are mainly environmental is pointed out by the Flynn Effect; Jensen's analysis suggests that such gains should not exist.

One of the most popular explanations of the rise in IQ, thus far, comes from a formal model proposed in [19] by Dickens and Flynn. Such work challenges Jensen's classical analysis by creating a formal model which assumes a strong reciprocal causation between phenotypic IQ and environment. This reciprocal causation produces a environment/gene correlation which leads ultimately to solve the paradox by showing how enviromental causes of the rise in IQ are compatible with large estimates of heritability.

Our point here is that such correlation had been observed by other researchers (see [103]) previous to the introduction of the Flynn's model. However, such correlation was ignored or badly interpreted. The clue which leads Dickens and Flynn to draw conclusions essentially different from the traditional ones is the analysis performed with the help of their formal model. This example clarifies our argument: *given certain assumptions*, to draw logical conclusions is easier when a formal model is available.

The model built in this thesis has an *algebraic* nature which describes a metaphor as a mapping $f : A \to B$. Let us consider our formalization of the TGRM as an example: the elements of A are the board squares without symbols on them, the elements of B are the numbers $1, 2, 3, ..., 10$, and f is the map which matches each square of A with its respective numerical symbol (see Figure 1.2). Because the sets A and B are not necessarily numerical sets, the analysis of our model shall require methods borrowed from universal algebra, formal logic and category theory. Next paragraph shows why.

Recall that the TGR game was designed to facilitate the learning of numbers. This is the main reason why the set B has the standard structure of numbers including, but not limited to, (1) the ordering, (2) the distance and (3) the successor operation. Notice how the design of the TGR board gives to A some similar structure: firstly, there exists an order relation since the squares are ordered from left to right. Secondly, there exist an operation which gives the notion of distance: the effort of walking from one square to another square. Thirdly, a player move from one square to the next one at the right because of the game's rules. These structures can be studied by considering A and B as *algebras* (in Universal Algebra) or by understanding them as *models* (in Formal Logic).

The core of our model is described as follows. Observe that, depending on the metaphor that is being analized, two elements of A might be related in a very specific way. The concern of our model is to reveal *if* such relation is *preserved* through the map $f : A \to B$. For example, in the formalization of the TGRM, the binary relation describing that square a_1 is at the left of square a_2 is preserved into a binary relation between numbers: the number printed on a_1 is *lesser than* the number printed on a_2. Another example is that the operation which makes the player move one square to the right is preserved into the successor operation of numbers.

The model presented here is able to represent a variety of metaphors since it has some "parameters" whose values can be "adjusted". Such values are not numerical but strings of symbols[6]taken from a *first order language* which plays the rol of a set of feasible values. For instance, when formalizing the TGRM, we associate two first order languages L_A and L_B to the sets A and B respectively. Our model's construction ensures[7] that every string of symbols from L_A references[8] one relation defined on A. Figure 1.2 depicts a possible formalization of the TGRM. There, the languages L_A and L_B have been selected in an intuitive way. The elements of Π and Ψ are the values of the "parameters" associated to such formalization.

Our next goal is to argue for the validity and usefulness of the model presented here. Developing formal models is full of challenges and the first one is to find a set of assumptions which command the essential behavior of the phenomenon. For highlighting the importance of this, notice that if the premises are erroneous, the conclusions might be misleading. For example, Jensen's argument is based on the assumption that there is no correlation between IQ heritability and environment. As we saw, such assumption yield the Flynn Paradox when contrasted to the collected data.

Thus, when the model's predictions conflict with the phenomenon's data, it is worth reviewing the model's premises. This points to a way of validating our formal model. The test is passed if there is consistency between the phenomenon's collected data and the model's predictions[9]. In what follows, we enumerate the assumptions that our model makes together with its predictions and results as a way to give our readers information that permits them evaluate the model.

[6]Those are terms and formulas from formal logic.

[7]It shall be formalized later, but it is the standard way in which mathematical logic relates a language with a model of the language.

[8]Notice the analogous way in which a spoken word or a image points to a concept in our mind.

[9]As an illustration of this, when Dickens and Flynn presented their model they wrote, "it predicts that even in adults, radical environmental change should produce significant changes in IQ" and afterwards they suggested a way to validate their model: "If we could test IQ before and after periods of incarceration, or before and after joining religious cults that significantly restrict people's control over their lifestyles, we might observe large changes in IQ surprising from the perspective of the standard model."

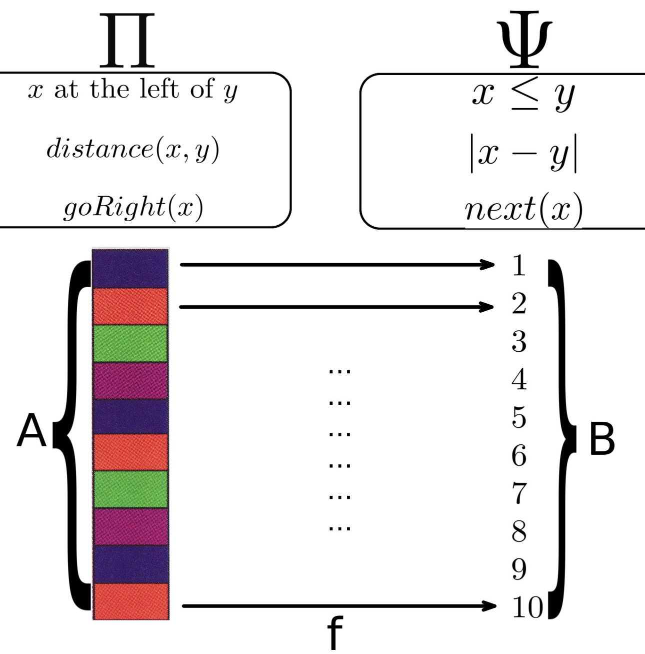

Figure 1.2: The set Π contains elements from the language L_A associated to A. In the same way, the set Ψ contains elements from the language L_B associated to B.

1.3 Hypotheses and Results of our Model

The aim of this section is to clarify our formalization and to signal the path for validating our model. In first place we state the assumptions that our model makes and thereafter we provide evidence supporting these scenarios. This process must be handled with care since it is not easy to distinguish between assumptions that have been *made on the phenomenon* in order to make it simple and susceptible of formalization and assumptions that have been *imposed by the mathematical tools* that were used in the formulation of the model.

In second place, but more important for validation, we present some of the results of our model together with (arguably) cognitive interpretations suggesting that some of the predictions of our model are consistent with the observed behavior of metaphor. In what follows, in order to differentiate between the two different conceptual systems involved in each metaphor, we shall call metaphor's *source* the (usually) more concrete one and metaphor's *target* the (generally) new or more abstract one.

1.3.1 Hypothesis: A cognitive view of Metaphor

In the cognitive field there are two dominant views of metaphor. One of them is based on the idea that metaphor is a comparison based on the similarity of two concepts. This idea is known as the *categorization theory of metaphor* and proposses that metaphors are class-inclusion statements in which the target is said to be a member of a category represented by the source. Since Amos Tversky has shown [117] that people's similarity judgments are assimetrical, this theory of metaphor captures the non-symetrical behavior[10] of metaphor.

However, our model assumes the second view of metaphor which is characterized by the *structure mapping theory* [30] introduced in 1983 by the psychologist Dedre Gentner as a framework for studying analogy[11]. In what follows, we introduce Gentner's theory by considering three spoken expressions that might be used by the players of the TGR game:

1. Now we are *tied* since we are standing on *the same number*.
2. I am *losing* because the number I am standing on is *at the left* of the number you are on.
3. *Go* one number to the *right*, from square 3 to the *next* square 4.

These three expressions really allude to a single, unstated metaphor: ARITHMETIC IS A COMPETITION ALONG A PATH. Since it is literally impossible for a person to be standing up on a number, the classical theory of metaphor would have said that the above statements are just ornamented versions of other unstated assertions which really describe the situation. However, the linguist George Lakoff and the philosopher Mark Johnson observed that these sort of expressions are pervasive in everyday language. Based on this, they claimed that metaphor is a fundamental mechanism for thinking and reasoning. As supporting evidence, Lakoff presents a vast collection of *conceptual metaphors* such as LOVE IS A JOURNEY, ARGUMENT IS WAR, or CAUSES OF BEHAVIOR ARE FORCES and its associated metaphorical expressions[70] (See Table 1.1).

Along these lines, but at the heart of learning mathematics, George Lakoff and Rafael Nuñez suggest a way in which metaphor is seen as involved in the construction of mathematical knowledge [71]. By using their ideas, the TGRM (or ARITHMETIC IS A COMPETITION ALONG A PATH)

[10] Metaphors are not fully reversible. Notice that "My lawyer is a shark" has definitely a meaning while "This shark is a lawyer" (arguably) does not.

[11] It should be clear that there is certainly not a consensus among researchers about which of the two theories is more convincing.

Our relationship has hit a dead-end street. It's stalled; we can't keep going the way we've been going. It's been a long, bumpy road. We're spinning our wheels. Our relationship is off the track. Look how far we've come.

Table 1.1: LOVE IS A JOURNEY's metaphorical expressions taken from [72].

might be explained in the following way:

> The two competitors are two different variables. The values taken by these two variables are the positions on the board occupied by the two competitors during the game. The feeling of getting ahead or falling behind while playing the game yields the comparison between two numbers. The equality of two variables is seen as both competitors standing on the same position. The successor of a number is seen as the square spatially adjacent at the right. Adding two numbers can be represented by the distance gained by two consecutive walks. A substraction might be seen as the distance left until the game's goal.

It is clear that a metaphor links two different conceptual systems, as for example, the TGR gaming and the numbers. Less clear is how such link is characterized. The structure mapping theory of Gentner postulates that the core of such relation is determined *systematically* by a mapping between the entities of the different domains. For example, the map depicted in Table 1.2 could determine the TGRM. Furthermore, in a serie of studies, Gentner and her collaborators have shown that a focus on relationships is the key to the power of analogy as a tool of reasoning [30, 33, 35] (see also [52]).

Our model *assumes* that metaphor's behaviour is governed by principles resembling[12] those postulated by Gentner. Additionally, the examples presented in Chapter 5 are based on constructions given in [71] by Nuñez and Lakoff.

[12]Those principles are formalized in Chapter 4, Definition 28 and Definition 37.

SOURCE		TARGET
Child	$\rightarrow$	Variable x
Opponent	$\rightarrow$	Variable y
Child is loosing	$\rightarrow$	$x < y$
Tied competitors	$\rightarrow$	$x = y$
Child's advance of one square to the right	$\rightarrow$	$x + 1$ (successor of x)
First square at the left of the board	$\rightarrow$	Number 1
Seconth square at the left of the board	$\rightarrow$	Number 2
....	$\rightarrow$	...
n-th square of the board	$\rightarrow$	Number n

Table 1.2: ARITHMETIC IS A COMPETITION ALONG A PATH's ontological map.

1.3.2 Hypothesis: Representation of the Information

In this section we introduce some of the strongest assumptions of our model. First of all, notice that conceptual metaphor is a highly complex process whose input is multimodal; this means that a person which is processing a conceptual metaphor might be sending/receiving information through multiple channels such as speaking, gesturing, hearing, watching or feeling and this is done by means of words, sounds, touches, gestures, pictures, diagrams, etc. (see [88]). Observe how this information is processed in parallel and thus it is hard to imagine how these different channels of information are composed, integrated and codified. It is even harder to build a formal model of this reality. Our formal model can not deal with such reality and just make the assumption that all the information was previously integrated and flows through a single, serial channel that codifies the information in the form of a language which, is non necessarily verbal but, has clear rules for composing its elements.

Also, our model makes an assumption about the representation of information in the human mind. The attentive reader might have noticed, from the brief descripition of our model in the previous section, that the information is represented[13] by a combination of (1) sets and (2) strings of symbols where the first would be interpreted as the "semantic content" and the second as the "symbolic referent"[14]. This assumption of our model can be grounded in the cognitive theory of dual coding developed by Paivio [93] who claims that mental representations are coded in an analog

[13]See the formal definition of "domains" in Chapter 4, Definition 27 and Definition 35.
[14]See Appendix C for a model without a formal language.

system and in a verbal-symbolic system. In order to formalize a metaphor, our model mimics these two systems. The first by selecting a suitable set (together with the appropriate structure) and the second by choosing a first order language suitable for the particular context.

Additionally, a very similar structure for the representation of information has been proppposed in the context of language. The psychologist Steven Pinker asserts in his book "Word and Rules" that

> ... The human language system also appears to be built out of two kinds of mental tissue. It has a lexicon of words, which refer to common things such as people, places, objects, and actions, and which are handled by a mechanism for storing and retrieving items in memory. And it has a grammar of rules, which refer to novel relationships among things, and which is handled by a mechanism for combining and analyzing sequences of symbols.

Along the same lines, but in the specific context of spatial representations, Kosslyn *et al.* [65, 66] have shown that humans distinguish between categorical and coordinate spatial representations. The coordinate system codes for spatial position in a quantitatively precise way (a coordinates set sort) and the categorical system defines space in more qualitative and conceptually distinct classes (for instance, left *vs* right, or above *vs* below).

1.3.3 Hypothesis: Independent Representations of Target and Source

Two opposite views of Conceptual Metaphor are widely spread in cognitive science. The first one postulates that most metaphors in our language are *dead*. Simply put, it means that when someone uses any of the metaphorical expressions in Table 1.1, as for example, "it has been a long, bumpy road," he is not thinking in a journey (not even unconsciously). He is thinking directly in terms of love.

The second view was postulated by Lakoff and claims that the human mind can only think in concrete experiences as objects, forces, sounds, images and that all other ideas are metaphorical references to these basic scenarios. Under this theory we can not think of a relationship of love without calling into mind some kind of journey. This implies that when someone says or hears a metaphorical expression from Table 1.1, she must be (might be in an unconscious manner) thinking in a journey.

Some of the defenders of this last position have claimed that this is because, in our brain, there is a single representation underlying these two concepts. For example, Walsh introduced the ATOM theory (A Theory Of Magnitude) or the concept that a single underlying magnitude representation in the brain links the three domains of space, time and number. The alternative proposed by Dehaene S., Cohen L., and Pinel P. ([18] p. 120) posits that distinct representations exist in the brain for the domains of space, time and number although some mechanisms are shared between them.

In this context, recall that our proposal models a metaphor as a map from the metaphor's source to the metaphor's target. The fact that, in our model, we can change the source's structure (the target's structure) without affecting the target's structure (the source's structure) suggests that the assumptions that our model makes are closer to Dehaene's proposal than to Walsh's view. This might imply that our model would be consistent with assertions like (1) the representations of the source and the target of a metaphor might be coded in different brain circuitry, (2) that they might be accesed independently, and (3) that metaphor exists as a wealth of associations and inter-relations between them.

We are aware that considerable controversy surrounds these assumptions. However, these assumptions might be supported by the Glucksberg's experiment [15] and the findings reported by

[15]The psychologists Boaz Keysar, Samuel Glucksberg and their collaborators showed chains of dead metaphorical expressions built around a Conceptual Metaphor [60]:

> "Love is a patient," said Lisa. "I feel that this relationship is on its last legs. How can we have a strong marriage if you keep admiring other women?" "It's your jealously," said Tom.

If Lakoff's view is right, these expressions should be understood through metaphorical mappings and then, the subjects would understand a new example, like "You're infected with this disease", more quickly than if it came out after a bland lead-in that made no mention of the metaphor:

> "Love is a challenge," said Lisa. "I feel that this relationship is in trouble. How can we have a strong marriage if you keep admiring other women?" "It's yout jealously," said Tom.

Contrary to the expected by Lakoff's Theory, there were no difference between the understanding times. This suggests that people ignored the source of the metaphor (patient) and was able to think directly on terms of the target: love. However, in a third condition, the subjects were forced to activate the underlying conceptual metaphor because the lead-in sentences had *unconventional* metaphorical expressions:

> "Love is a patient," said Lisa. "I feel that this relationship is about to flatline. How can we have a strong marriage if you keep admiring other women?" "It's your jealously," said Tom.

David Kemmerer who, along the lines of the TIME IS SPACE[16] metaphor, has shown that some patients with brain damage can lose their ability to understand prepositions for space, as in *She's at the corner* and *She ran through the forest,* while retaining their ability to understand the same prepositions for time as in *She arrived at 1:30* and *She worked through the evening* [60]. Other pacients show the opposite pattern. Also, further evidence suggests that, in the brain, the number's representation can be disassociated from the representation of space (see [24]).

Additionally, in order to clarify this matter, we can consider evidence reported by Bowdle and Gentner which suggests that metaphors, when used often enough, shift from being interpreted through metaphorical mappings to being dead metaphors [32, 9]. They call this behaviour *the career of metaphor*.

1.3.4 Two Problems related to Metaphor

We think that, in the context of learning, metaphor should be addressed with two different perspectives. Next, we propose two problems that highlight the subtle differences between those perspectives. To exemplify them, we shall consider the TGR metaphor that was introduced previously.

The first analysis considers the point of view of a teacher and it might be necessary for understanding the design of the TGR game. Here, we want to think about how a metaphor is *created* inside of the teacher's mind in order to help his student to learn. In the teacher's brain, the most likely scenario is that the target's metaphor is well coded and represented. Such target becomes the input for a twofold problem: (i) choosing the metaphor's source from a variety of well coded candidates by (ii) pinpointing the structural similarities between each candidate and the target. Name this *the teacher's metaphor problem*. A *solution* for this problem, given a fixed learner, is a metaphor able to help the learner to acquire knowledge of the target. For example, since playing the TGR game led children to gain numerical knowledge, the board—and the rules—of the TGR game is a solution for this problem.

This time readers understood the probe sentence about being infected with a disease more quickly, as quick as when the story was about being literally infected with a real disease. The conclusions of the study are that people can read through metaphorical expressions to the source and target of the underlying Conceptual Metaphor, but only when the metaphor is fresh.

[16] It has been claimed that the abstract domain of time gets its relational structure from the more concrete domain of space by the psychologist Lera Borditsky and her collaborators [77].

For the second case, the emphasis is set on the learning process and the analysis seeks to elaborate an hypotesis that might explain the learner's cognitive gains. Let us consider a learner and a given metaphor together with its source and its target. Inside the learner's mind, the most likely scenario is that the metaphor's source is well coded while the metaphor's target has a faulty/incomplete representation. In this case, we are interested in understand the process performed by the learner's brain that, by using the information provided by the metaphor, modifies its internal structures in a way to obtain an "improved" representation of the metaphor's target. We call it *the learner's metaphor problem* for later reference.

A *solution* for this problem might take the form of an algorithm whose inputs are: a source, a target, and a metaphor between them. This algorithm has to change the encoding of some particular structures of the target in order to mimic the associated structures of the source. Notice that the TGR game together with those children showing gains of knowledge after playing it, show that this problem actually has a solution: while the children were playing the game, their brains must have made use of a solution of this problem in order to learn.

Also, it is worth observing that a combination of the two stated problems yields a third problem: *the researcher's metaphor problem*. A researcher, aiming to structure a fuzzy target in his mind, creates his own[17] metaphors by evaluating[18] different metaphor's sources.

1.3.5 Result: The Learner's Metaphor Problem

The first step leading to a satisfactory solution for the learner's metaphor problem should be a proposal of the structures supporting it. That is to say, we must define precisely what is the source,

[17]Donald Schön recounts an engineering problem faced by the designers of the first paintbrushes made with synthetic bristles[82]. Compared to brushes with natural bristles, the new ones glopped paint onto a surface unevenly, and none of the improvements the engineers could think up (varying the diameter, splitting the ends like hairs) fixed the problem. Then someone said, "You know, a paintbrush is a kind of pump!" When a painter bends the brush against a surface, he forces paint through the spaces between the bristles, which act like channels or pipes. A natural brush, it turned out, forms a gradual curve when it is bent against a wall, whereas the synthetic one formed a sharp angle, blocking the channels like a crimp in a garden hose. By varying the density of the bristles along their lengths, the engineers got the synthetic brush to curve more gently and thus to deliver paint more evenly. The crucial first step was the pump metaphor, which got them to reconstruct a brush from one big paddle that whipes paint against the surface to a set of channels that punt it out to their ends.

[18]It is clear that different sources might draw different conclussions. The researcher must experiment in order to validate the source and identify the adequate conclussions.

the target and the metaphor. In this aspect, we believe that our model makes a contribution in the understanding of the learner's metaphor problem. Furthermore, the proofs of theorems 30 and 31 in Chapter 4 can be seen as describing processes which might be consistent with solutions for such problem. To support this affirmation, we present Theorem 31 in the context of the TGR game. We warn the reader that such presentation might make Theorem 31 appear trivial but, such risk is taken in order to aid the presentation.

Let us set the ideal scenario for this problem: On one hand, an ideal child knows everything about walking along a path. Also, he knows the rules to play the TGR game. This means that he knows that some squares are closer than others and—because of the game's rules—that he must walk only toward the right side of the board, etc. On the other hand, though he might name numbers and even recognize their numerals, he might have no clue about their structure. That is to say, even when he might recognize the symbol "3", he does not know that the successor of 3 is 4, or that 2+3 =5, or that 5 is lesser than 8[19]. For this discussion, let us assume that his numerical knowledge can be depicted as the upper box of Figure 1.3.

$$\boxed{1 \to 2 \quad 3 \quad 4 \to 5 \quad 8}$$
$$\Downarrow$$
$$\boxed{1 \to 2 \to 3 \to \ldots \to 8 \to 9 \to 10}$$

Figure 1.3: An interpretation of Theorem 31 as a solution of the learner's metaphor problem in the context of the TGR game. Before playing, the children only have some vague concept of some numbers and possibly a vague concept of "next" in numbers. Then, TGRM comes and the target becomes like the box below. The arrows depict the action of the operation "*next*".

Before presenting Theorem 31, let us introduce its two main hypotheses. In the context of the TGR game, these would be that (i) the board of the TGR game can be "generated", which means that the player can visit all the squares of the board by walking one step at a time. The second hypothesis would be that (ii) the child's brain supports[20] certain recursive symbolic operations. This last hypothesis can be better understood as follows.

[19]We are describing an hypothetical student, real students might usually know at least some of these facts, depending on their age and other factors.

[20]It is important to observe that such hypothesis is paraphrased "in the context of the TGR game" because we do not assert or assume that a person is able to support infinite recursion. Actually, the mentioned hypothesis is imposed by the mathematical tools that were used in the construction of the model. Since a formal language is embedded into our model, its grammatical properties, such as the recursive generation of an infinite set of words, are reflected in the results. However, we point out that (1) there is at least some evidence pointing that the brain has a mild capacity for carrying on recursive processes until certain level and that (2) the linguist Noam Chomsky together with his collaborators claim

$$\begin{array}{rcl} x & \to & x \\ goRight(x) & \to & next(x) \\ goRight(goRight(x)) & \to & next(next(x)) \\ goRight(goRight(goRight(x))) & \to & next(next(next(x))) \\ ... & \to & ... \\ goRight(goRight(...(goRight(x)))) & \to & next(next(...(next(x)))) \end{array}$$

Table 1.3: The recursivity hypothesis of Theorem 31 contextualized to the TGRM.

In a formalization of the TGR metaphor, the operation that makes the child move (from one square to another) is represented through a symbolic string[21] such as $goRight(x)$. This way, when x is refering the first position from the left, $goRight(x)$ refers the second position, $goRight(goRight(x))$ refers the third position and so on. Also, recall—from the TGR game—that when the child moves from 3 to 4 he must recite "three" and then "four", and that numbers 3 and 4 are printed in the third and fourth squares respectively. These rules and perceptions might be creating (perhaps unconsciously) an operation which for 1 returns 2, for 2 returns 3, for 3 returns 4 and so on. Thus, when the child moves, the design of the TGR game highlights a correspondence between the "movement operation" and an analogous operation in the realm of numbers. Our formalization would represent such numerical operation by a string, let us say, $next(x)$.

In this scenario, the recursivity hypothesis of Theorem 31 might be interpreted as that the child's brain supports the generation of the strings in Table 1.3 together with the depicted matches. Theorem 31 assumes the two hypotheses above described and asserts that a more accurate representation of the target can be built (see lower box of Figure 1.3). The proof of the theorem is based on the idea that (1) for the game domain, the variable x is referencing the first square, (2) for the numbers domain, the same string x is referencing the number 1, and (3) that strings of symbols can be used to represent the missing numbers (if 3 were not present, it would be represented by $next(next(x))$; if 4 was missing, it would be represented by $next(next(next(x)))$, and so on) and then—sort of—"copying" the source's structure into the target. To work out this idea, the Theorem's proof makes use of some concepts from Category Theory, as for example, the building of a quotient object.

that recursion is the only uniquely human component of the faculty of language [49]. These two points together with the fact that our model, through recursion, represents accurately some instances of the phenomenon, might serve to argue for a link between the capacity of analogize and certain recursive processes of the mind. A detailed discussion of this subject is presented in Appendix B.

[21] This would be a *term* in formal logic.

Let us make some comments about this process of "learning" described here. Observe that it is based on syntactical operations which are independent of the context, and then, it is a general process in the sense that it can be applied to any metaphor that, like the TGRM, is susceptible of formalization. Also, consider that for the sake of clarity, the previous example is extremely simple: it has a source and a target with ten elements each and the only operations involved are *goRight* and *next*. However, Theorem 31 is general enough to handle any finite number of operations defined on sources or targets with a countable number of elements on them.

On other aspect, many models of nature are obtained as solutions of optimization problems. In a way, the model of "learning" presented here is not an exception. Theorem 31 ensures that the new target satisfies a universal property[22]: the new metaphor's target is characterized by its ability of representing *all* the information concerning the source's relevant structure while having the minimum number of elements. It can be thought of as an efficient storage of new information. This might be not true for human learning and thus this shows a weakness of our model. This weakness might be introduced by the simplification of the phenomenon, necessary to make metaphor susceptible of analysis.

Along the same lines, the proof of Theorem 31 describes a static, one-step process. In Figure 1.3 the upper box represents the numbers's understanding of the child. This process acts when the child plays the TGR game by transforming the target in the lower box of Figure 1.3. In human learning, the real process has some dynamics: the gains of knowledge are obtained over time and after repeated applications of the metaphor. However, we believe that the proof of Theorem 31 might be a basis to generate models that include such dynamics.

$$\boxed{1 \to 2 \quad 3 \quad 4 \to 8 \to 5}$$

Figure 1.4: Let us think about what happens when a metaphor tries to give the adequate structure to this set of "numbers".

This section ends by posing a question for future research. Suppose that the child of our example thinks that the number 8 is less than the number 5 (see Figure 1.4). It would be interesting to investigate what our model predicts (if it can) about what happens when the TGRM tries to re-structure this faulty target. We believe that the answer might be related or interpreted as the psychological term "cognitive dissonance."

[22]Observe that universal properties in the language of category theory are viewed as some kind of the "most efficient solution" for a problem.

1.3.6 Limitation: The Teacher's Metaphor Problem

The problem that we refer as the teacher's metaphor problem is the most frequently issue discussed in metaphor research. Our model can not account for this problem since we consider the source and the target of metaphor as given. This way, we avoid dealing with the problem of how to select an adequate source between many candidates. In our running example, the TGRM, the metaphor's source is not chosen between some candidates, is given beforehand.

In the same way, we consider the structural matches between source and target as given. This means that, in our example of the TGRM, the model does not address the problem of finding out if the operation *goRight* is matched with the relation *next*; such correspondence is given beforehand. Despite that, our model makes a contribution by providing a definition which, given that *goRight* is matched to *next*, enables us to identify a metaphor in the mapping that associates a number to each square of the board.

However, some particular issues related to this problem can be enlightened by our model. For example, the identification of some inference tools that metaphor can carry from the source to the target. This particular contribution of this work is extense and possibly important and thus we have devoted the entire next section to present the associated results. We continue this section by posing two observations that are related to the teacher's metaphor problem.

The first observation is that seems like that certain kind of information is never carried by the metaphor from the source to the target. As far as we know, the reason is unknown. Let us exemplify this conundrum with the TGR game. The metaphor's game associates (or maps) the position of each square with the numeral imprinted on the square. This is reflected in the fact that children in the game group improve on the estimate the position of numbers on a numerical line. However, it seems like that such positions are not associated with the square's color, that is to say, children would be unable to arrange colors on a line[23] (and consider that the numeral is just a variation of color). Somehow, the design of the game and its dynamics marks the imprinted numerals as salient target's features whose structure should be carried by the metaphor, and at the same time, leaves the square's colors as irrelevant features.

[23]However, see a neurological condition called synesthesia where letters or numbers are perceived as inherently colored.

The second observation refers to loose and overlapping analogies that might be even harmful for teaching. This might happen when the selected source has some structure which, when transported through the metaphor, suggests that false statements in the target are true (or viceversa) and thus misleads the reasoning of the learner. This feature of metaphor is real and, in a negative sense, very powerful: some politicians, marketers and media make abuse of this characteristic of metaphor in order to manipulate public opinion[24]. However, in the realm that interest us, this issue is nicely approached in "The Stuff of Thought" written by Steven Pinker. The relevant paragraphs are reproduced next.

> Loose and overlapping analogies are also a mark of bad science writing and teaching. The immune system is like a sentinel, except when it's a lock and key; no, wait, it's a garbage collector! The best science writers, in contrast, pinpoint the meaningful matchups in an analogy and intercept the misleading ones. In *The Blind Watchmaker*, Richard Dawkins explains how sexual selection can produce flamboyant displays like the outsize tail of a widowbird. Traits in males that are attractive to females can vary wildly over the course of evolution, Dawkins notes, because there are many stable combinatios of a tail length preferred by females and an actual tail length in the population (which is itself a compromise between the length preferred by previous generations of choosy females and the length that is optimal for flight). Mathematicians call this "a line of equilibria," and to establish the conditions that produce it they require abstruse equations. But Dawkins explains the idea as follows:
>
> Suppose that a room has both a heating device and a cooling device, each with its own thermostat. Both thermostats are set to keep the room at the same fixed temperature, 70 degrees F. If the temperature drops below 70, the heater turns itself on and the refrigerator turns itself off. If the temperature rises above 70, the refrigerator turns itself on and the heater turns itself off. The analogue of the widow bird's tail length is not the temperature (which remains constant at 70°) but the total rate of consumption of electricity. The point is that there are lots of different ways in which the desired temperature can be achieved. It can be achieved by both devices working very hard, the heater belting out hot air and the refrigerator working flat out to neutralize the heat. Or it can be achieved by the heater putting out a bit less heat, and the cooler working correspondingly less hard to neutralize it. Or it can be achieved by both devices working scarcely at all. Obviously, the latter is the more desirable solution from the point of view of the electricity bill but, as far as the object of maintaining the fixed temperature is

[24] See the book "Manufacturing Consent" by Noam Chomsky.

> concerned, every one of a large series of working rates is equally satisfactory. We have a *line* of equilibrium points, rather than a single point.
>
> In the passages I have underlined,—says Pinker—Dawkins anticipates how his readers might misconnect the entities in the world to the entitites in the analogy, and redirects their gaze to the intended points of correspondence.

1.3.7 Result: Arguments to justify "Reasoning by Analogy"

Before entering into the discussion, let us get insight on the nature of metaphor by considering examples of bad ones. Here are three outstanding examples which were taken from the widely circulated list of the World's Worst Analogies:

- John and Mary had never met. They were like two hummingbirds who had also never met.
- Her eyes were like two brown circles with big black dots in the center.
- The red brick wall was the color of a brick-red Crayola crayon.

The psychologist Steven Pinker suggests that the above expressions should not be considered as metaphorical since they fail the test of supporting an inference. In these metaphors, knowing about the source adds nothing to our knowledge of the target.

Along these lines, linking metaphors and reasoning, Dedre Gentner has noticed that many scientific theories were first stated as analogies, and often are still best explained that way: heat is like a fluid, evolution is like selective breeding, the atom is like a solar system, genes are like coded messages [28]. Also, in [29] Gentner presents empirical evidence suggesting that inference patterns are carried from the source to the target when using analogies. In general, the scientific community recognizes that analogies play a key role in problem solving and science.

However, since metaphors can change people's decisions or opinions by framing adequately the subject,[25] their powers have been used by politicians and marketers with controversial pur-

[25]The best way, Pinker suggests, to show the effects of framing is describing the Tversky and Kahneman's 1981 study: identical events, different metaphors, flipped decision. They posed the following problem to a group of doctors [115]: "A new strain of flu is expected to kill 600 people. Two programs to combat the disease have been proposed." One subgroup of them were then presented with the following decision:

> If program A is adopted, 200 people will be saved, If program B is adopted, there is a one-third prob-

poses. This fact has been used by detractors of the educational and explanatory usage of analogy in education[43]. The scientist Bipin Indurkhya , who has studied metaphor in deep from a computational perspective, makes this observation by pointing out the usual argument[56]:

> ... certain similarities are pointed out between the source and the target as a "justification" for reaching an erroneous conclusion about the target.... an inference based only on some existing similarity between the source and the target —and nothing else— is about as justified as a random inference.

also, he suggests that the key of this controversy is the *problem of justification of analogy*. This problem asks for a justification to trust in the inferences yielded by an analogy. It has been largely discussed by many scholars but it has never been given a satisfactory answer (see discussion in [56]). Our model of metaphor might help to end this discussion by showing that some inferences given by an analogy *can be* justified.

Let us set ourselves in the context of the metaphor's teacher problem, that is to say, in the teacher's head where the source and the target are totally structured. In this work, the first model of metaphor is given through a concept named *homomorphism* (Definition 37, Chapter 4). This is a map between the source and the target which ensures that the relevant facts of the source shall be reflected in the relevant facts of the target.

> ability that 600 people will be saved and two-thirds probability that no people will be saved. Which of the two programs would you favor?

Most doctors in this group chose program A, the sure option, rather than program B, the risky one. The other subgroup of doctors was presented with the following dilemma:

> If program C is adopted, 400 people will die, If program D is adopted, there is a one-third probability that nobody will die and a two-thirds probability that 600 people will die. Which of the two programs would you favor?

If you are like most of the doctors who faced *this* choice, you become a gambler selecting program D, and avoid program C, the sure option. Notice that both dilemmas above are exactly the same, but the below version is paraphrased in a negative way (or as a loss) rather than the positive way (or a gain), saving people, which is used in the first version. The difference in wording points to a difference in metaphors. Independently, has been show that people has an aversion to loss, which means, for example, that people do not mind paying for something with a credit card even when they are told that there is a discount for cash, but they do mind paying the same amount if they are told there is a surcharge for using credit. The combination of people's loss aversion with the effects of framing explains the paradoxical result: the "gain" metaphor made the doctors risk-averse; the "loss" metaphor made them gamblers. Crucially, observe that the decision that doctors made was not an everyday or random decision. It was a decision where human lives were at stake.

Let us exemplify this concept by the TGR metaphor. Select the binary relations (i) *at the left of*, (ii) the *equality of squares* and (iii) the unary operation *goRight* as the three relevant relations in the source. On the other hand, let us select the following relations in the target: (i) less than , (ii) the equality of numbers, and (iii) the unary operation next. In our formalism, the map f which associates the board's square a with the number n_a printed on it, is an homomorphism *because* it satisfies three conditions: (1) whenever square a is at the left of square b, n_a is lesser than n_b, (2) if a and b are the same square, then their printed numbers are also the same, and (3) if the square b is right-adjacent to the square a, then $n_b = n_a + 1$. Nothing else has to be checked to ensure that f is a homomorphism between the game's board and the numbers.

Significantly, our model predicts that along with these facts, other facts shall be brought from the source to the target. We use the above example to illustrate the results of our model. To make ideas concrete, we imagine that a child is playing the TGR game against an opponent.

An interpretation of Theorem 40 shall raise the following example. Assume that the child is at square a and his opponent is in the right-adjacent square b. If the child advances one square and reaches his opponent, this fact will be reflected (in the target) as a combination of the equality of numbers and the successor operation: $n_a + 1 = n_b$.

Theorem 41 ensures that logical operators used in the source can be transported to the target. For example, assume the child's position is left-adjacent to the opponent's position. Once the child makes his next move, only two alternatives are possible:[26] the opponent and the child share the same position, or the opponent is at the left of the child. Let us denote: the child's next position as a and the opponent's next position as b. These two exclusive possibilities raised in the source are reflected in two exclusive alternatives in the target: $n_a = n_b$, or $n_b < n_a$.

Let us introduce Proposition 26 by the following example. Observe that in any state of the TGR game, exactly one of the three following facts is true: the child is at the left of the opponent, the opponent is at the left of the child, or both share the same position. Such fact shall be reflected in the target as the trichotomy law: $x < y$, or $y < x$, or $x = y$ holds for any pair (x, y) of numbers.

In order to exemplify another result, we want to observe that f is stronger than a homomorphism: it is what we call a *formal metaphor* (see Definition 38). Basically, this means that the

[26]In the game, each player can advance only one or two squares in each turn.

relevant facts in the target are reflected at least once in the source. In more precise words, f satisfies two additional conditions: (4) whenever $n_a < n_b$, there exists squares a and b such that $f(a) = n_a$, $f(b) = n_b$ and a is at the left of b. Finally, f satifies that (5) if $n_a = n_b$, then there exists squares a and b such that $f(a) = n_a$, $f(b) = n_b$ and $a = b$. Because of (4) and (5), we know that f is a (formal) metaphor between the game's board and the numbers[27].

In this context, Theorem 42 ensures that a formal metaphor is able to carry *tools of inference* from the source to the target. For example, assume that the child masters the following board's rule of inference: in the previous turn, my position a was at the left of my oponnent's position b. In this turn, we both moved one square to the right. Then, I am still at the left of my opponent. An interpretation of the second literal of Theorem 42 suggests that metaphor carries this inference tool to obtain the analogous (if-then) implication in the target: if $n_a < n_b$, then $n_a + 1 < n_b + 1$.

The above examples were given in order to exemplify some of our results. Since this is an introductory discussion, we have not explicitated the hypotheses of these results. However, each one of them needs some kind of recursivity hypothesis similar to the one discussed and exemplified in the section devoted to the learner's metaphor problem (see Table 1.3). Such hypotheses are deeply analized in Chapter 2 and Chapter 3 by using formal methods. The results of these chapters are not presented here because they, even when they do have a mathematical interest, have barely a cognitive interpretation. The introduction of Chapter 2 provides a description of the mathematical work done there.

1.3.8 Result: Tools for Designing Learning Materials

Observe how, in this introduction, we have analyzed in deep the TGR game together with its underlying metaphor. We believe that this kind of analysis can be applied into the design of learning and teaching materials. We have taken a first step in this direction: by using the definitions and results presented in this work, we designed a game aimed to teach a mathematically complex subject: exponential functions. An exploratory study was performed with teenager students in order to estimate the effects of such game in its academic performance. Encouragingly, the results are

[27] It might be worth pointing out that a formal metaphor is not equivalent to the algebraic notion of an "isomorphism". Formally, every isomorphism is a metaphor and every metaphor is a homomorphism, but the reversed implications are not true.

consistent with our proposal. We provide further details in Appendix A of this thesis for those readers which are curious about the game, its metaphor-based design and the methodology of the study.

Chapter 2

Term Morphisms: Structure preserving mappings for analogy models based on first order languages

2.1 Abstract

Analogy models use knowledge about one structure (named *source*) as a guide to reason about an unknown structure called *target*. When both structures are represented by descriptions, each one written in its own formal language, the analogical reasoning process is based on a correspondence relating strings of the source description with strings of the target description. In this work we assume that such correspondence is a mapping $F : S \to T$, where S is the set of terms used in the description of the source and T is the set of terms used in the description of the target. We generate a larger set of terms S^* by recursively combining the elements of S through substitution of terms. Then, by using graph-based representations of terms, we find conditions that characterize those mappings $F : S \to T$ that can be extended to mappings $F^* : S^* \to T^*$ which preserve the structure given by the substitution operation. Since these conditions might be hard to check by a computer, a simple assumption on the set S enables us to use Unification Theory to formulate these conditions in a more applicable way. We also provide an algebraic interpretation of analogies together with examples in that context, in order to clarify the nature of the structure preserved by the mapping

$F^* : S^* \to T^*$.

2.2 Introduction

For the purposes of this work, we use the terms analogy and metaphor indistinctly. Analogy has received plenty of attention from many researchers in various fields of science. In fields related to artificial intelligence, analogy models are used as tools for giving "reasoning habilities" to machines (see [92, 119, 81, 74, 112]) giving rise to applications such as some expert systems and theorem provers. Furthermore, one of the 33 basic research problems proposed for automated reasoning in [123] is devoted to analogy (details in [124]). In fields like psichology and cognitive science, some computational models for analogy have been developed in order to understand the human mind, as for example in [34, 73, 113]. A review of computational models for analogy can be found in [64].

One of the most common approaches to implement those computational models and applications is to conceive analogy as involving a *mapping* between two objects S and T, called the *source* and the *target* respectively. These objects can be (depending on the model) domains of knowledge, theorems, formal knowledge, reasoning methods, proofs, etc. and the whole idea is to use some knowledge about the source object S to represent, understand and process the (generally not well-known) target object T. Notice that the mapping $F : S \to T$ is a key ingredient of this kind of model because it allows for some structure embedded into the source object to be projected into the target object in order to facilitate its representation and processing.

Some ideas have been proposed to capture the laws which characterize the mapping $F : S \to T$. Psychological principles governing the analogical process are proposed in [30] and used in [34] as a basis to build a heuristic algorithm to obtain F; [121] implements a corpus based algorithm that would serve to build F; [42] gives a heuristic algorithm based on a generalization of *anti-unification* to determine such mapping; mathematical models for $F : S \to T$ are presented in [55] and [54] where S and T are represented by formal languages in the former and algebras in the latter; in [116, 118] the authors formalize similarity (wich can be seen as a particular case of metaphor) with mathematical tools and then support their conclusions with experimental evidence. The interest on the mapping $F : S \to T$ shown across different fields of science, suggests that the study of its properties is an important subject in the path to understanding the phenomenon of analogy.

In the present work, we have choosen a specific context to study some theoretical properties of the mapping $F : S \to T$. Most analogy models represent each object by means of formal languages ([34, 7, 119, 81, 42]) i.e. they need a description of the source object in a *source language* and a description of the target object in a *target language*. In those models, each "object" becomes a collection of symbol strings formed according to the rules of its own (formal) language and $F : S \to T$ associates symbol strings of the source object description with symbol strings of the target object description.

As an example of such context, [81] proposes an analogy based theorem prover. In that study, a theorem and its proof are considered the source object. The target object is a theorem (similar to the source theorem but slightly different) and the idea is to use the proof of the source theorem as a "proof plan" which leads to the creation of a proof for the target theorem. In this case, the mapping F relates syntactic elements (definitions, axioms, formulas) of the source object to syntactic elements of the target theorem. Thus, the theorem prover uses the structure embedded in the source theorem proof, to create a proof for the target theorem.

In our research, we assume that S and T are sets of terms[1] belonging to different *First Order Languages*. As any other language, a first order language is recursive by definition, which implies that terms belonging to a finite set S can be combined (by applying the language rules) to obtain a countable set of terms S^*. The research question that we adress is whether there exists a mapping $F^* : S^* \to T^*$ wich preserves the combination rules and extends $F : S \to T$. In this study, we use tree-based representations of terms to express conditions characterizing the extensibility of mappings $F : S \to T$. Since these conditions turn out to be hard to check for a computer, we make an assumption on the set S which enables us to use *Unification Theory* (see [75]) to express such conditions in a way that, in some cases, a machine would find more useful.

We believe that the subject adressed in the above paragraph is important for three reasons. First, if an analogy model uses a mapping $F : S \to T$ which satisfies the assumptions of this work, then the conditions we found could be checked in order to extend the model's applicability by using the mapping $F^* : S^* \to T^*$ instead of just the mapping $F : S \to T$. Second, the analysis made in this study together with the proposed model for analogy could be used to shed some light on the principles which rule the analogical process. For example, from our study it is clear that $F : S \to T$ is not extensible in case that a term $s \in S^*$ has two different tree-based representations a_1, a_2 wich

[1]This study can be extended to the case when S and T are sets of formulae.

are related by F (in a specific sense) to two diferent terms $t_1, t_2 \in T^*$. This theoretical property could be compared with observed facts from real world analogies to investigate if they can be related and interpreted. The third reason is an algebraic interpretation of this work which shall be discussed in section 2.8.1.

2.3 First Order Languages and the Extensibility Problem

2.3.1 First Order Languages

A First Order Language $\mathscr{L}^S$ is determined in a standard way (see [20, 58]) by giving a symbol set S. The symbol set S is the disjoint union of a countable set of variables, a set of function symbols, a set of relation symbols and a set of logical connectives. Throughout this work we assume that all such languages are built over the same set $V = \{x_1, x_2, x_3, ...\}$ of variables. The set of terms of $\mathscr{L}^S$ is defined as the minimal set $Term(\mathscr{L}^S)$ which satisfies:

1. if $x \in V$, then $x \in Term(\mathscr{L}^S)$, and
2. for any $n \in \mathbb{N}$, if $f \in S$ is an n-ary function symbol and $t_1, ..., t_n \in Term(\mathscr{L}^S)$, then $f(t_1, ..., t_n) \in Term(\mathscr{L}^S)$.

Since we are not going to deal with formulae here, it is not necessary to take into account the set of relation symbols and the set of logical connectives. A mapping $\rho : V \to Term(\mathscr{L}^S)$ is called a *substitution* if the set $\{v \in V \mid v \neq \rho(v)\}$ is finite. We denote a substitution by $\rho = \{v_1 \leftarrow t_1, v_2 \leftarrow t_2, ..., v_n \leftarrow t_n\}$ where $\rho(v_i) = t_i \neq v_i$ for $i = 1, ..., n$ and $\rho(v) = v$ otherwise. Observe that ρ can always be extended in a unique way to a mapping $\overline{\rho} : Term(\mathscr{L}^S) \to Term(\mathscr{L}^S)$ by defining $\overline{\rho}(t)$ as the term obtained by simoultaneously replacing in t each ocurrence of v_i for t_i. Sometimes, we shall write $t\rho$ instead of writing $\overline{\rho}(t)$, or $t\dfrac{t_1 t_2 ... t_n}{v_1 v_2 ... v_n}$ if we wish to make the substitution explicit. Let $\Pi \subseteq Term(\mathscr{L}^S)$ be a set of terms, we shall say "ρ in Π" when $(Im(\rho) \setminus V) \subseteq \Pi$, where $Im(\rho)$ is the image of ρ.

Definition 1. Given a non empty set $\Pi \subseteq Term(\mathscr{L}^S)$, the set of terms *generated by* Π is defined as the minimal set Π^* which satisfies the two following conditions:

1. If $x \in V$, then $x \in \Pi^*$.

2. For any substitution ρ in Π^* , if $t \in \Pi$ then $t\rho \in \Pi^*$.

It is worth observing that for any Π, $V \subseteq \Pi^*$. Additionally, observe that if $\Pi = V_0$ where V_0 is any subset of V, then $\Pi^* = V$. In what follows, let $\mathscr{L}^S$ and $\mathscr{L}^T$ be two first order languages which should be interpreted as the "source language" and the "target language" respectively.

2.3.2 Statement of the Extensibility Problem

From now on, consider $\Pi \subseteq Term(\mathscr{L}^S)$ and $\Psi \subseteq Term(\mathscr{L}^T)$ two sets that do not contain variables (i.e. $\Pi \cap V = \Psi \cap V = \emptyset$).

Definition 2. A mapping $F : \Pi \to \Psi$ which preserves variables is called a *term translation*, i.e. for all $t \in \Pi$

$$V(t) = V(F(t)) \tag{2.1}$$

where $V(t)$ denotes the set of variables which occur in the term t.

Example 1. Let $S = \{+, \cdot, 0, 1\}$ and $T = \{\cup, \times, \emptyset, a\}$ be two symbol sets. Consider $\Pi = \{0, 1, x_1 + x_2, x_3 \cdot x_4, x_1 + (x_3 \cdot x_4)\}$ and $\Psi = \{a \times \emptyset, a, x_1 \cup x_2, (x_3 \times x_4) \cup x_3, x_1 \cup (x_3 \cup x_4)\}$. The bijective mapping $F : \Pi \to \Psi$ which assigns the i-th element of Ψ to the i-th element of Π (considering the order in which the elements of sets were listed) satisfies the conditions to be called a term translation.

Definition 3. We say that a term translation $F^* : \Pi^* \to \Psi^*$ is a *term morphism from* Π^* *to* Ψ^* when it satisfies the following two conditions:

1. If $x \in V$, then $F^*(x) = x$.

2. If $t' \in \Pi$ and ρ is a substitution in Π^*, then

$$F^*(t'\rho) = F^*(t')(F^* \circ \rho). \tag{2.2}$$

Observe that the last definition is not a classic notion, we define it in order to state our research problem. In this work we intend to characterize the existence of a term morphism $F^* : \Pi^* \to \Psi^*$ which extends a given term translation $F : \Pi \to \Psi$. For such morphism, equation (2.2) can be rewritten as:

$$F^*(t'\rho) = F(t')(F^* \circ \rho) \tag{2.3}$$

It is easy to find examples where a term translation $F : \Pi \to \Psi$ can not be extended by any term morphism (see example 6 in the appendix of this chapter). The following proposition, which can be proved by straightforward computations, shows that a term morphism F^* which extends a term translation F is characterized by its effect on all those substitutions ρ such that $Im(\rho) \subseteq \Pi^*$.

Proposition 1. *A term morphism $F^* : \Pi^* \to \Psi^*$ extends a term translation $F : \Pi \to \Psi$ if and only if for any substitution ρ in Π^*, $\gamma = F^* \circ \rho$ is a substitution in Ψ^* and makes following diagram commute:*

$$\begin{tikzcd} \Pi \arrow[r, "F"] \arrow[d, "\overline{\rho}"] & \Psi \arrow[d, "\overline{\gamma}"] \\ \Pi^* \arrow[r, "F^*"] & \Psi^* \end{tikzcd}$$

2.4 Tree Representations for Terms

Next, we introduce a representation of terms which enables us to handle different ways to build a term $t \in \Pi^*$ by using recursive substitution. We denote a *directed graph* by a pair (N, E) where N is the set of *nodes* and E is the set of *arcs*. A *rooted directed tree* is a directed graph $G = (N, E)$ which is connected, has no cycles and there exists a special node r called *root* from which all *paths* of the graph are directed. By vu we shall mean the arc which begins at node v and ends at node u. The set $\delta(v) = \{vu \in E \mid u \in N\}$ will be the set of arcs which begin at node v.

Definition 4. Let $\Pi \subseteq Term(\mathscr{L}^S) \setminus V$ be a non empty set. A *tree over* Π is a 3-tuple (G, ν, e) where $G = (N, E)$ is a rooted directed tree, $\nu : N \to \Pi \cup V$ and $e : E \to V$ are two mappings such that for any node $v \in N$:

- if $\nu(v) \in V$ then $\delta(v) = \emptyset$, and
- $\nu(v) \in \Pi$ if and only if $|\delta(v)| = |V(\nu(v))|$ and $e(\delta(v)) = V(\nu(v))$.

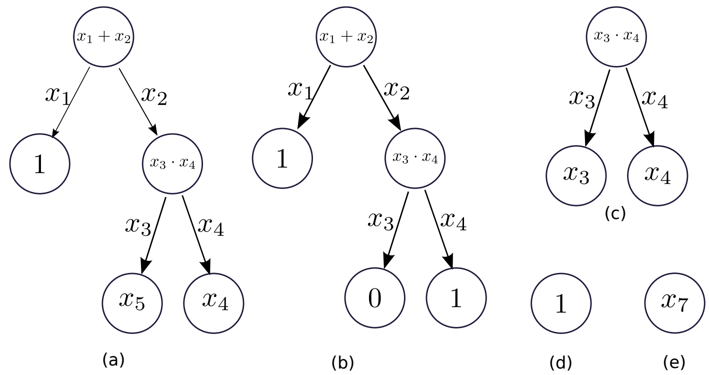

Figure 2.1: (a), (b) and (c) are proper trees, (c) is a fundamental tree, (d) and (e) are atomic trees, for $\Pi = \{0, 1, x_1 + x_2, x_3 \cdot x_4\}$.

Given (G, ν, e) a tree over Π, we shall often refer to $\nu(v)$ as the "label" of the node v. Analogously, we shall speak of the "label" on the arc uv, meaning $e(uv)$. With this terminology, a tree over Π is a rooted directed tree (N, E) with labels, such that every node $v \in N$ has only two alternatives (see Fig. 2.1):

1. if the label of v is a variable, it must be a leaf.
2. if the label of v is a non variable term $t \in \Pi$, then (and only then) the arcs steming from v are in 1-1 correspondence to the variables of t, via their labels.

When Π is clear from the context, we shall say "a tree" instead of "a tree over Π". A tree which has only one node (the root node) is called *atomic*, otherwise it is called *proper*. Notice that for any atomic tree, the label of its only node is a variable or it is a term without variables. In the case of a proper tree, the same condition is true for any leaf:

- v is a leaf if and only if its label is a variable or it is a term without variables.

A proper tree is *fundamental* if every arc rv (which begins at the root node) satisfies $e(rv) = \nu(v)$. Observe that fundamental trees have only two levels. The root r is at first level, labeled by a term $t \in \Pi$, and each variable of t giving rise to a leaf in the second level (see Fig. 2.1 (c)).

Next, we give some definitions relating trees with terms in Π^*. Let a be a tree, we shall say that $x_1, ..., x_n$ are variables of a, for some $x_1, \ldots, x_n \in V$, if and only if $x_1, \ldots, x_n$ are labels of some leaves of a. Let a' be a tree with variables $x_1,...,x_n$ and let $a_1,\dots,a_n$ be trees. We denote by $a'\frac{a_1...a_n}{x_1...x_n}$ the tree which results from replacing every leaf with label x_i of a' by the tree a_i.

Define the set Π° as the set of all trees over Π. Observe that every proper tree $a \in \Pi^\circ$ can be written as $a = a'\frac{a_1...a_n}{x_1...x_n}$ where a' is a fundamental tree and $a_1,...,a_n$ are trees. To see this, take a' as the fundamental tree determined by the node r (the root of a) and the edge set $\delta(r)$, while $a_1,...,a_n$ are the trees determined by the $n = |\delta(r)|$ children of r. We say that $a', a_1,...,a_n$ is the *decomposition* of a. We have just sketched the proof of the following proposition:

Proposition 2. *Every $a \in \Pi^\circ$ is atomic or can be written as $a = a'\frac{a_1...a_n}{x_1...x_n}$, where a' is a fundamental tree and $a_1,...,a_n$ are trees.*

This proposition will enable us to use induction to prove statements about elements of Π° by defining the *inductor* of $a \in \Pi^\circ$:

$$inductor(a) = \begin{cases} 0 & \text{if } a \text{ is fundamental,} \\ length(a) & \text{otherwise .} \end{cases}$$

where $length(a)$ is the length of the longest path in a. We can apply induction by showing first that a statement holds for any tree with inductor 0 (i.e. the atomic and fundamental trees). And then the inductive step can be performed by observing that in any tree $a = a'\frac{a_1...a_n}{x_1...x_n}$, the inductor of a' is zero and $inductor(a_i) < inductor(a)$ for $i \in \{1,...,n\}$.

Let us link the above paragraphs to terms. First, we are going to define a "projection" map $\pi : \Pi^\circ \to \Pi^*$ in the following inductive way:

$$\pi(a) = \begin{cases} \nu(r), & \text{if } a \text{ is atomic or fundamental,} \\ \pi(a')\dfrac{\pi(a_1)...\pi(a_n)}{x_1...x_n}, & \text{otherwise, where } a', a_1,..,a_n \text{ is the decomposition of } a. \end{cases}$$

It is straightforward that this mapping is well defined and surjective.

Let us now introduce some concepts for trees which are analogous to concepts introduced for terms. A *tree-substitution* in Π° is a function $\lambda : V \to \Pi^\circ$ such that the set $\{\lambda(x) \mid x \in V,\ \pi(\lambda(x)) \neq x\}$ is finite. In analogy with term substitutions we are going to denote λ as $\{x_1 \leftarrow a_1,...,x_n \leftarrow a_n\}$ where $a_i = \lambda(x_i)$ for $i \in \{1,...,n\}$ and if $x \neq x_i$ for all $i \in \{1,...,n\}$, then $\pi(\lambda(x)) = x$. A tree-substitution $\lambda = \{x_1 \leftarrow a_1, \ldots, x_n \leftarrow a_n\}$ can be applied to a tree a, meaning that any leaf of a labeled by x is replaced by the tree $\lambda(x)$. The tree obtained in this way shall be denoted by $a\lambda$, or $a\frac{a_1 \ldots a_n}{x_1 \ldots x_n}$ if we wish to make the tree-substitution explicit. The following property is straightforward.

Lemma 1. *Let $a' \in \Pi^\circ$ be a tree and $\lambda = \{x_1 \leftarrow a_1, \ldots, x_n \leftarrow a_n\}$ be a tree-substitution in Π°, then*

$$\pi(a'\frac{a_1 \ldots a_n}{x_1 \ldots x_n}) = \pi(a')\frac{\pi(a_1) \ldots \pi(a_n)}{x_1 \ldots x_n}.$$

From this lemma, it is clear that if F^* is a term morphism which extends a term translation F, and $a', a_1, \ldots, a_n$ is the decomposition of a, then:

$$F^*(\pi(a'\frac{a_1 \ldots a_n}{x_1 \ldots x_n})) = F(\pi(a'))\frac{F^*(\pi(a_1)) \ldots F^*(\pi(a_n))}{x_1 \ldots x_n}. \tag{2.4}$$

2.5 First Characterization of Extensibility

Definition 5. Let $F : \Pi \to \Psi$ be a term translation. The *tree map associated to F* is the mapping denoted by $F^\circ : \Pi^\circ \to \Psi^\circ$ which takes $a = (G, \nu, e) \in \Pi^\circ$ and changes the node labels of the tree by:

$$F^\circ(a) = (G, F \circ \nu, e) \tag{2.5}$$

where $F : \Pi \cup V \to \Psi \cup V$ is the extension that acts as the identity on variables.

Thus, F° takes a tree $a \in \Pi^\circ$ and changes the label $t \in \Pi \cup V$ of each node to the label $F(t) \in \Psi \cup V$, transforming a in a tree belonging to Ψ° (see Fig. 2.2). From the above definition, next lemma follows in a straightforward way.

Lemma 2. *Let $a \in \Pi^\circ$. If $a = a'\frac{a_1 \ldots a_n}{x_1 \ldots x_n}$, then $F^\circ(a) = F^\circ(a')\frac{F^\circ(a_1) \ldots F^\circ(a_n)}{x_1 \ldots x_n}$.*

It is worth noticing that $F^\circ(a)$ is a well defined tree over Ψ for every $a \in \Pi^\circ$. Therefore, (unlike F^*) the tree map F° always exists. When F^* does exist, next result describes its relation with F°.

Lemma 3. *Let $F : \Pi \to \Psi$ be a term translation. A map $F^* : \Pi^* \to \Psi^*$ is a term morphism which*

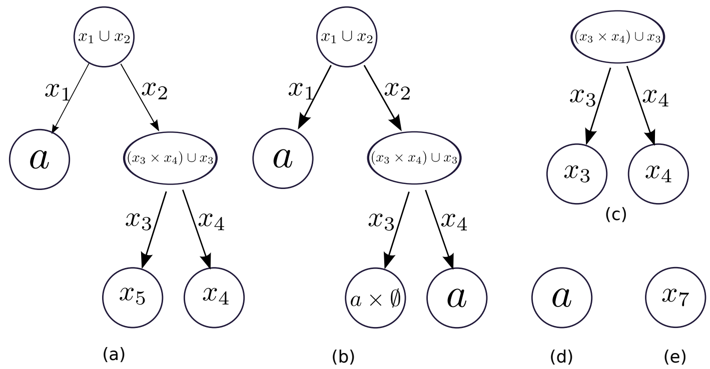

Figure 2.2: Outcomes of the tree map F° associated to F (from example 1) when it is applied on trees depicted in figure 2.1.

extends F if and only if $F^* \circ \pi = \pi \circ F^\circ$ *i.e. the following diagram commutes:*

$$\begin{tikzcd} \Pi^\circ \arrow[r, "F^\circ"] \arrow[d, "\pi"'] & \Psi^\circ \arrow[d, "\pi"] \\ \Pi^* \arrow[r, "F^*"'] & \Psi^* \end{tikzcd} \tag{2.6}$$

Proof. We use induction to prove that diagram 2.6 commutes necessarily when F^* is a term morphism. Let us take $a \in \Pi^\circ$. If $inductor(a) = 0$, a is an atomic or a fundamental tree. In either case

$$F^*(\pi(a)) = F^*(\nu(r)) = F(\nu(r)) = \pi(F^\circ(a)).$$

To perform the inductive step, let us assume that $inductor(a) = k > 0$ and that the condition follows necessarily for every tree of inductor less than k. Let $a', a_1, ..., a_n$ be the decomposition of a. Then:

$$F^*(\pi(a)) = F^*(\pi(a')\frac{\pi(a_1)...\pi(a_n)}{x_1...x_n}) = F(\pi(a'))\frac{F^*(\pi(a_1))...F^*(\pi(a_n))}{x_1...x_n}$$

since F^* is a term morphism extending F. Using the induction hypotheses, the last expression is equal to

$$\pi(F^\circ(a'))\frac{\pi(F^\circ(a_1))...\pi(F^\circ(a_n))}{x_1...x_n} = \pi(F^\circ(a'\frac{a_1...a_n}{x_1...x_n})) = \pi(F^\circ(a)).$$

Let us prove that the commutativity of diagram 2.6 is enough to ensure that F^* is a term morphism extending F. That F^* is a term translation extending F follows directly from the definition of F° and the commutation of diagram 2.6. Let us show that F^* is a term morphism. Take $t \in \Pi^*$ such that $t = t'\alpha$ for some $t' \in \Pi$ and $\alpha = \{x_1 \leftarrow t_1, ..., x_n \leftarrow t_n\}$ in Π^*. Take $a', a_1, ..., a_n \in \Pi^\circ$ where a' is fundamental and such that $\pi(a') = t'$ and for $i \in \{1, .., n\}$, $\pi(a_i) = t_i$. Consider the following computations:

$$F^*(t) = F^*(t'\alpha) = F^*(\pi(a'\frac{a_1, \dots, a_n}{x_1, \dots, x_n})) = \pi(F^\circ(a'\frac{a_1, \dots, a_n}{x_1, \dots, x_n}))$$

$$= \pi(F^\circ(a'))\frac{\pi(F^\circ(a_1)), \dots, \pi(F^\circ(a_n))}{x_1, \dots, x_n} = F(\pi(a'))\frac{F^*(\pi(a_1)), \dots, F^*(\pi(a_n))}{x_1, \dots, x_n}$$

$$= F(t')\frac{F^*(t_1), \dots, F^*(t_n)}{x_1, \dots, x_n} = F(t')(F^* \circ \alpha).$$

Therefore, F^* is a term morphism which extends F. $\square$ $\square$

The uniqueness of F^*, when it exists, comes as a straightforward corollary of this lemma, given that the map π is surjective.

Proposition 3. *Let $F : \Pi \to \Psi$ be a term translation. If there exists a term morphism $F^* : \Pi^* \to \Psi^*$ which extends F, then F^* is unique.*

Let us now define an equivalence relation "$\sim$" in Π° by "$a \sim b$ if and only if $\pi(a) = \pi(b)$". We denote the equivalence class of a by $\overline{a}$. Let us consider the quotient set $\Pi^\circ / \sim$ and its surjective projection map $p : \Pi^\circ \longrightarrow \Pi^\circ / \sim$. It is clear that there exists a bijection $i : \Pi^\circ / \sim \longrightarrow \Pi^*$ which satisfies $i \circ p = \pi$.

The following observation shall be used in next theorem. If for all $a, b \in \Pi^\circ$, $a \sim b$ implies $F^\circ(a) \sim F^\circ(b)$, then (and only then) we can define the mapping

$$F' : \Pi^\circ / \sim \longrightarrow \Psi^\circ / \sim \text{ given by } F'(\overline{a}) = \overline{F^\circ(a)}.$$

Theorem 4. *Let $F : \Pi \to \Psi$ be a term translation and $F^\circ : \Pi^\circ \to \Psi^\circ$ its associated tree map. The following sentences are equivalent:*

1. *For all $a, b \in \Pi^\circ$, $a \sim b$ implies $F^\circ(a) \sim F^\circ(b)$.*

2. *There exists a term morphism* $F^* : \Pi^* \to \Psi^*$ *which extends* $F : \Pi \to \Psi$.

$$
\begin{tikzcd}
 & \Pi^\circ \arrow[r, "F^\circ"] \arrow[dl, "\pi"'] \arrow[d, "p"'] & \Psi^\circ \arrow[r, "\pi"] \arrow[d, "p"'] & \Psi^* \\
\Pi^* \arrow[r, "i^{-1}"'] & \Pi^\circ/\sim \arrow[r, "F'"'] & \Psi^\circ/\sim \arrow[ur, "i"'] &
\end{tikzcd}
\tag{2.7}
$$

Proof. $1 \implies 2$. Assume condition 1. Therefore, $F' : \Pi^\circ/\sim \to \Psi^\circ/\sim$ can be defined satisfying $F' \circ p = p \circ F^\circ$. Since $i \circ p = \pi$, diagram (2.7) commutes. Define $F^* = i \circ F' \circ i^{-1}$. Then, applying Lemma 3, F^* is a term morphism which extends F.

$2 \implies 1$. Let us take $a, b \in \Pi^\circ$ and assume that $a \sim b$. Therefore, $\pi(a) = \pi(b)$ which implies that

$$F^*(\pi(a)) = F^*(\pi(b)). \tag{2.8}$$

If F^* is a term morphism[2] which extends F, the diagram in Lemma 3 commutes and then, equation 2.8 is equivalent to $\pi(F^\circ(a)) = \pi(F^\circ(b))$. Thus, condition 1 holds. □ □

2.6 Unification Theory and Tree Representation of Terms

Theorem 4 give us a condition which characterizes the existence of a term morphism extending a given term translation. In general, checking such conditions requires going over an infinite set A of trees. Section 2.7 will give us a way to characterize the existence of such morphism by checking such conditions in a set A' (potentially) smaller than A. In this section, we develop the necessary tools for the results in section 2.7.

Let $\alpha_1 = \{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$ be a substitution. The set $dom(\alpha_1) = \{x_1, \ldots, x_n\}$ is called *domain* of α_1. Additionally, we say that the set of variables $ran(\alpha_1) = \bigcup_{i=1}^n V(t_i)$ is the *range* of α_1. Consider $\alpha_2 = \{y_1 \leftarrow s_1, \ldots, y_m \leftarrow s_m\}$ another substitution. The substitution given by $\overline{\alpha_2} \circ \alpha_1$ is called the *composition* of substitutions α_1 and α_2. We shall denote it by $\alpha_1 \bullet \alpha_2$. Notice that $\alpha_1 \bullet \alpha_2$ can be represented by the set $\{x_1 \leftarrow t_1\alpha_2, \ldots, x_n \leftarrow t_n\alpha_2, y_1 \leftarrow s_1, \ldots, y_m \leftarrow s_m\}$ after removing from

[2] Straightforward computations show that such term morphism satisfies $F^* = i \circ F' \circ i^{-1}$.

it all elements of the form $x_i \leftarrow x_i$, as well as $y_k \leftarrow s_k$ if $y_k = x_i$ for some $i \in \{1,...,n\}$. A substitution α is called *idempotent* if $\alpha \bullet \alpha = \alpha$. Some well known properties of substitutions are the following:

1. For any term t, $t(\alpha_1 \bullet \alpha_2) \equiv (t\alpha_1)\alpha_2$.
2. $\alpha_1 \bullet (\alpha_2 \bullet \alpha_3) = (\alpha_1 \bullet \alpha_2) \bullet \alpha_3$.
3. α is idempotent if and only if $dom(\alpha) \cap ran(\alpha) = \emptyset$.

From now on we differentiate the mathematical equality denoted by "$=$" from the syntactic equality between terms denoted by "$\equiv$". An *equation of terms* is an expression of the form "$t_1 = t_2$", where t_1 and t_2 are terms. A *unifier* of such equation is a substitution α which satisfies $t_1\alpha \equiv t_2\alpha$. Let $E = \{t_1 = s_1, ..., t_n = s_n\}$ be a set of equations. The substitution α is a *unifier* of E when $t_i\alpha \equiv s_i\alpha$ for every $i \in \{1,..,n\}$. We denote the set of unifiers of E by $Unif(E)$. In case that E has only one element, we shall omit parentheses from notation to write "$t = s$" instead of $\{t = s\}$. A unifier $\theta \in Unif(E)$ is called a *mgu (most general unifier)* of E if it satisfies: $\gamma \in Unif(E)$ if and only if there exists a substitution α such that $\gamma = \theta \bullet \alpha$. If $Unif(E)$ is a nonempty set, then E has an idempotent mgu also, which can be obtained by applying the *solved form algorithm* (see [75]). Clearly, if θ is an idempotent mgu of E, then

$$\gamma \in Unif(E) \text{ if and only if } \gamma = \theta \bullet \gamma \tag{2.9}$$

A substitution α is a *variable substitution* when its image is a set of variables. Otherwise, we call α a *proper* substitution. A variable substitution $\alpha : V \to V$ is called a *permutation of variables* if it is bijective.

Proposition 4. *If θ_1 is an idempotent mgu of E, then θ_2 is an idempotent mgu of E iff there exists a permutation of variables $\phi = \{x_1 \leftarrow y_1, ..., x_k \leftarrow y_k, y_1 \leftarrow x_1, ..., y_k \leftarrow x_k\}$ such that $\{x_1 \leftarrow y_1, ..., x_k \leftarrow y_k\} \subseteq \theta_1$ and $\theta_2 = \theta_1 \bullet \phi$.*

From this proposition, it follows that all idempotent mgu of E are similar. For details of unification theory and proofs of the above propositions see [75].

Let us now link the above concepts with the tree-based representation developed for terms. A tree-substitution $\alpha = \{y_1 \leftarrow b_1, ..., y_m \leftarrow b_m\}$ can be extended in a unique way to a mapping

$\overline{\alpha} : \Pi^\circ \to \Pi^\circ$ by defining $\overline{\alpha}(a) = a\alpha$ for any $a \in \Pi$. Consider $\alpha_1 = \{x_1 \leftarrow a_1, ..., x_n \leftarrow a_n\}$ another tree-substitution. The tree-substitution $\overline{\alpha} \circ \alpha_1$ shall be denoted by $\alpha_1 \star \alpha$. Notice that $\alpha_1 \star \alpha$ can be represented by the set $\{x_1 \leftarrow a_1\alpha, ..., x_n \leftarrow a_n\alpha, y_1 \leftarrow b_1, ..., y_m \leftarrow b_m\}$ after removing from it all elements of the form $x_i \leftarrow a_i\alpha$ where $\pi(a_i\alpha) = x_i$, as well as $y_k \leftarrow b_k$ if $y_k = x_i$ for some $i \in \{1, ..., n\}$. Given a set of terms Π, some straightforward properties of tree-substitutions are:

1. For any tree $a \in \Pi^\circ$, $a(\alpha_1 \star \alpha_2) = (a\alpha_1)\alpha_2$.
2. $\alpha_1 \star (\alpha_2 \star \alpha_3) = (\alpha_1 \star \alpha_2) \star \alpha_3$.
3. Let α_1, α_2 be tree-substitutions in Π°, then $\pi \circ (\alpha_1 \star \alpha_2) = (\pi \circ \alpha_1) \bullet (\pi \circ \alpha_2)$.

Definition 6. Let $F : \Pi \to \Psi$ be a term translation, α a substitution in Π^* and β a substitution in Ψ^*.

1. A tree-substitution $\alpha^\circ : V \to \Pi^\circ$ is called a *lifting* of α if and only if $\pi \circ \alpha^\circ = \alpha$.
2. Two substitutions α in Π^* and β in Ψ^* are *compatible (through F)* iff for every lifting α° of α, the tree-substitution $F^\circ \circ \alpha^\circ$ is a lifting of β.

Proposition 5. *Let $F : \Pi \to \Psi$ be a term translation, $s, t \in \Pi$ and θ_1 in Π^* an idempotent mgu of $s = t$. If there exists $\hat{\theta}_1 \in Unif(F(s) = F(t))$ compatible with θ_1, then for any idempotent mgu θ_2 of $s = t$, there exists $\hat{\theta}_2 \in Unif(F(s) = F(t))$ compatible with θ_2.*

Proof. By Proposition 4, there exists a substitution ϕ such that $\theta_2 = \theta_1 \bullet \phi$. Let us show that $\hat{\theta}_2 = \hat{\theta}_1 \bullet \phi$ is compatible with θ_2. Let $\theta_2^\circ : V \to \Pi^\circ$ be a lifting of θ_2. Since ϕ is a permutation of variables, it is straightforward that $\theta_2^\circ = \theta_1^\circ \star \phi^\circ$ for some suitable liftings θ_1° of θ_1 and ϕ° of ϕ. Now, the compatibility of $\hat{\theta}_2$ and θ_2 follows from:

$$\pi \circ F^\circ \circ \theta_2^\circ = \pi \circ F^\circ \circ (\theta_1^\circ \star \phi^\circ) = (\pi \circ F^\circ \circ \theta_1^\circ) \bullet (\pi \circ F^\circ \circ \phi^\circ) = \hat{\theta}_1 \bullet \phi = \hat{\theta}_2.$$

On the other hand, $F(s)\hat{\theta}_1 \equiv F(t)\hat{\theta}_1$ implies that $F(s)\hat{\theta}_1 \bullet \phi \equiv F(t)\hat{\theta}_1 \bullet \phi$. Therefore, $\hat{\theta}_2 \in Unif(F(s) = F(t))$. □ □

Similar techniques are used to prove the next result. (see Appendix).

Proposition 6. *Let θ_1 be an idempotent mgu of $s = t$ which is* not *in Π^*. The following statements are true:*

1. *Any idempotent mgu θ_2 of $s = t$ is* not *in Π^*.*
2. *If for any substitution α, $\theta_1 \bullet \alpha$ is not in Π^*, then for any substitution α' and for any idempotent mgu θ_2 of $s = t$, the substitution $\theta_2 \bullet \alpha'$ is not in Π^*.*

Definition 7. Let $t, s \in \Pi$ be a pair of terms and θ be an idempotent mgu of the equation $t = s$. A substitution λ is called *minimal respect to* θ whenever $\theta \bullet \lambda$ is in Π^* and there are no substitutions α, β such that β in Π^* is proper, $dom(\beta) \subseteq ran(\alpha)$, $\theta \bullet \alpha$ in Π^* and $\lambda = \alpha \bullet \beta$.

Proposition 7 below, justifies the name "minimal" given above to some substitutions. For its proof, we shall need the straightforward Lemma 5 stated below. The *length* of a substitution $\alpha = \{x_1 \leftarrow t_1, ..., x_n \leftarrow t_n\}$ is defined by $length(\alpha) = length(\alpha(t_1)) + ... + length(\alpha(t_n))$.

Lemma 5. *Let $\gamma_0, \gamma_1, \alpha$ be substitutions such that $\gamma_0 = \gamma_1 \bullet \alpha$. If α is proper and $dom(\alpha) \subseteq ran(\gamma_1)$, then $length(\gamma_1) < length(\gamma_0)$.*

Proposition 7. *Let θ be an idempotent mgu of $t = s$. If γ is a substitution such that $\theta \bullet \gamma$ is in Π^*, then there exists a minimal substitution λ (respect to θ) and a substitution α in Π^* such that $\gamma = \lambda \bullet \alpha$.*

Proof. If γ is minimal respect to θ, we consider $\lambda = \gamma$ and α the identity substitution. Otherwise, put $\gamma_0 = \gamma$. Since $\theta \bullet \gamma_0$ is in Π^* and γ_0 is not minimal, there exists a substitution γ_1 and a proper substitution α_1 in Π^* such that $dom(\alpha_1) \subseteq ran(\gamma_1)$, $\theta \bullet \gamma_1$ is in Π^* and $\gamma_0 = \gamma_1 \bullet \alpha_1$. The above argument can be repeated a finite number of times only since by Lemma 5, $length(\gamma_k) < length(\gamma_{k-1})$. Therefore, at some k_0 we obtain a substitution γ_{k_0} which is minimal respect to θ. Take $\lambda = \gamma_{k_0}$ and $\alpha = \alpha_{k_0} \bullet ... \bullet \alpha_1$. It is easy to see that $\alpha \in \Pi^*$ since α_i is in Π^* for all $i \in \{1, 2, ..., k_0\}$. □ □

Lemma 6. *Let $s, t \in \Pi$. Let θ_1, θ_2 be two idempotent mgu of $\{s = t\}$ and let ϕ be the invertible substitution such that $\theta_2 = \theta_1 \bullet \phi$. A substitution λ is minimal respect to θ_2 if and only if the substitution $\phi \bullet \lambda$ is minimal respect to θ_1.*

The proof of Lemma 6 is straightforward. The proof of Lemma 6 can be found in the appendix.

Proposition 8. *Let $F : \Pi \to \Psi$ be a term translation and $s, t \in \Pi$. If the set of unifers of $s = t$ is nonempty, then the next two statements are equivalent:*

1. *For any θ_2 idempotent mgu of $t = s$ and any λ_2 minimal respect to θ_2, there exists an unifier $\widehat{\lambda_2}$ of $F(t) = F(s)$ compatible with $\theta_2 \bullet \lambda_2$.*
2. *There exists θ_1 an idempotent mgu of $t = s$ such that for any substitution λ_1 minimal respect to θ_1, there exists a unifier $\widehat{\lambda_1}$ of $F(t) = F(s)$ compatible with $\theta_1 \bullet \lambda_1$.*

Proof. We only have to show that $2 \implies 1$. Assume that 2 holds. Let us take θ_2 an arbitrary idempotent mgu of $t = s$ and λ_2 any substitution minimal respect to θ_2. By Proposition 4, $\theta_2 = \theta_1 \bullet \phi$ for some invertible substitution ϕ. By Lemma 6, the substitution $\phi \bullet \lambda_2$ is minimal respect to θ_1. Therefore, hypotheses 2 implies that there exists $\widehat{\lambda_1}$ a unifier of $F(t) = F(s)$ compatible with $\theta_1 \bullet (\phi \bullet \lambda_2) = (\theta_2 \bullet \phi^{-1}) \bullet (\phi \bullet \lambda_2) = \theta_2 \bullet \lambda_2$. Since θ_2 is arbitrary, the result follows. □ □

2.7 Conditions for Extensibility

2.7.1 Characterization of Extensibility

Given a term translation $F : \Pi \to \Psi$, Theorem 4 tells us that F can not be extended to a term morphism if there are trees $a, b \in \Pi^\circ$ such that $\pi(a) \equiv \pi(b)$ and $\pi(F^\circ(a)) \not\equiv \pi(F^\circ(b))$. The first part of that assertion: "there exist trees $a, b \in \Pi^\circ$ such that $\pi(a) \equiv \pi(b)$", is equivalent to the condition "there exist terms $s, t \in \Pi$ and substitutions α_1, α_2 in Π^* satisfying $t\alpha_1 \equiv s\alpha_2$". Clearly, when $V(s) \cap V(t) = \emptyset$, we can rewrite this last condition in terms of unifiers as "there exist terms $s, t \in \Pi$ and a substitution α in Π^* belonging to $Unif(s = t)$". We shall take advantage of such observation by saying that a set Π of terms is *disjoint*, when it does not contain variables and for any pair of terms $s \neq t$ in Π, $V(t) \cap V(s) = \emptyset$.

Definition 8. A term translation $F : \Pi \to \Psi$ is called *unifiable* if for every pair $t, s \in \Pi$, every θ idempotent mgu of $t = s$ and every minimal substitution λ respect to θ, there exists a unifier $\widehat{\lambda}$ in Ψ^* of the equation $F(t) = F(s)$ compatible with $\theta \bullet \lambda$.

From Proposition 8 and Lemma 6, the following characterization of a unifiable term translation follows.

Corollary 7. *A term translation $F : \Pi \to \Psi$ is unifiable if and only if every pair of terms $t, s \in \Pi$ satisfies:*

1. *$Unif(t = s)$ is empty, or*
2. *There exists θ an idempotent mgu of $t = s$ such that for every minimal substitution λ respect to θ there exists a unifier $\widehat{\lambda}$ in Ψ^* of the equation $F(t) = F(s)$ compatible with $\theta \bullet \lambda$.*

Next theorem characterizes the existence of a term morphism $F^* : \Pi^* \to \Psi^*$ extending a given term translation $F : \Pi \to \Psi$, when Π is disjoint. For the proof of the following result, we define the *length* of a term t as the number of function symbols which occur in it.

Theorem 2.A (Characterization of Extensibility)**.** Let Π be disjoint. Given a term translation $F : \Pi \to \Psi$, the following statements are equivalent:

1. $F : \Pi \to \Psi$ is unifiable.
2. There exists a term morphism $F^* : \Pi^* \to \Psi^*$ which extends $F : \Pi \to \Psi$.

Proof. By Theorem 4, we shall prove that 1 is equivalent to

2'. For every pair of trees $a, b \in \Pi^\circ$, $a \sim b$ implies $F^\circ(a) \sim F^\circ(b)$.

$1 \implies$ 2'. Let us define $\kappa : \Pi^\circ \times \Pi^\circ \to \mathbb{N}$ by:

$$\kappa(a,b) = \begin{cases} length(\pi(a)) & \text{if } a \sim b, \\ 0 & \text{otherwise.} \end{cases}$$

We are going to apply induction on $\kappa(a,b)$. The base step is proved by observing that if $\kappa(a,b) = 0$, then $a \nsim b$ (and the result is trivially true) or $a \sim b$ with a and b atomic trees whose only label is a variable. Therefore, $\pi(a) \equiv \pi(b) \equiv x$ for some $x \in V$ and then $F^\circ(a) \sim F^\circ(b)$ follows. To prove the induction step, we will show that 2 is true for any pair a,b of trees such that $k = \kappa(a,b) > 0$ by first assuming that 2 holds for any pair of trees c,d with $\kappa(c,d) < k$.

Since $k > 0$, it follows that $a \sim b$ and $length(\pi(a)) = length(\pi(b)) = k$. Observe that, or a is an atomic tree whose only label is not a variable, or a has a decomposition. The same observation is valid for the tree b. In the case that a and b are atomic, the result follows directly since $\pi(a) \equiv \pi(b)$ implies $a = b$ and then $F(a) = F(b)$. The case where a is atomic and b has a decomposition (or simmetrically the case where b is atomic and a has a decomposition) can be proved in a similar way to the case where both trees have a decomposition[3]. Therefore, let us assume that $a = a' \frac{a_1,\dots,a_n}{x_1,\dots,x_n}$ and $b = b' \frac{b_1,\dots,b_m}{y_1,\dots,y_m}$.

There are two alternatives:

Case 1: $\{x_1,\dots,x_n\} \cap \{y_1,\dots,y_m\} \neq \emptyset$.

Since Π is disjoint, this implies that $\{x_1,\dots,x_n\} = \{y_1,\dots,y_m\}$ and $\pi(a') \equiv \pi(b')$. Since a' and b' are fundamental, $a' = b'$ and then $F^\circ(a') = F^\circ(b')$. Besides, without loss of generality we can assume that $x_i = y_i$ for $i \in \{1,..,n\}$. It is easy to see that $\pi(a) \equiv \pi(b)$ and $\pi(a') \equiv \pi(b')$ imply that $\pi(a_i) \equiv \pi(b_i)$ for $i \in \{1,...n\}$. Now, we can use the induction hypotheses since $\kappa(a_i,b_i) < k$ and conclude that $\pi(F^\circ(a_i)) \equiv \pi(F^\circ(b_i))$. The computation below shows that the result follows in this case:

$$\begin{aligned}
\pi(F^\circ(a)) &= \pi(F^\circ(a' \frac{a_1,\dots,a_n}{x_1,\dots,x_n})) \\
&= \pi(F^\circ(a')) \frac{\pi(F^\circ(a_1)),\dots,\pi(F^\circ(a_n))}{x_1,\dots,x_n} \\
&= \pi(F^\circ(b')) \frac{\pi(F^\circ(b_1)),\dots,\pi(F^\circ(b_n))}{x_1,\dots,x_n} \\
&= \pi(F^\circ(b' \frac{b_1,\dots,b_n}{x_1,\dots,x_n})) \\
&= \pi(F^\circ(b)).
\end{aligned}$$

Case 2: $\{x_1,\dots,x_n\} \cap \{y_1,\dots,y_m\} = \emptyset$.

[3] Considering the tree-substitution $\gamma^\circ = \{y_1 \leftarrow b_1,\dots,y_m \leftarrow b_m\}$ instead of $\gamma^\circ = \{x_1 \leftarrow a_1,\dots,x_n \leftarrow a_n, y_1 \leftarrow b_1,\dots,y_m \leftarrow b_m\}$ below.

Let us consider the tree-substitution $\gamma^\circ = \{x_1 \leftarrow a_1, ..., x_n \leftarrow a_n, y_1 \leftarrow b_1, ..., y_m \leftarrow b_m\}$. Since $a \sim b$, the substitution $\gamma = \pi \circ \gamma^\circ$ is a unifier of the equation $\pi(a') = \pi(b')$. Therefore, there exists an idempotent mgu θ of such equation. Because $\gamma = \theta \bullet \gamma$ is in Π^*, there exists a substitution λ minimal respect to θ and a substitution α in Π^*, such that $\gamma = \lambda \bullet \alpha$ and $\theta \bullet \lambda$ in Π^*. Let us take arbitrary liftings ϕ° of $\theta \bullet \lambda$ and α° of α and observe that $\gamma = \theta \bullet \gamma = (\theta \bullet \lambda) \bullet \alpha = \pi \circ (\phi^\circ \star \alpha^\circ)$. That is, $\pi \circ \gamma^\circ = \pi \circ (\phi^\circ \star \alpha^\circ)$, which implies that for $z \in \{x_1, .., x_n, y_1, ..., y_m\}$,

$$\gamma^\circ(z) \sim \phi^\circ(z)(\alpha^\circ).$$

Notice that $\gamma^\circ(z) \in \{a_1, ..., a_n, b_1, ..., b_m\}$. Since $\kappa(\gamma^\circ(z), \phi^\circ(z)(\alpha^\circ)) < k$, we can apply our induction hypotheses concluding that $F^\circ(\gamma^\circ(z)) \sim F^\circ(\phi^\circ(z)(\alpha^\circ))$ which in turn implies that $\pi \circ F^\circ \circ \gamma = \pi \circ F^\circ \circ (\phi^\circ \star \alpha^\circ)$. From this, both computations below follow:

$$\pi \circ F^\circ(a) \equiv \pi \circ F^\circ(a'\gamma^\circ) \equiv \pi(F^\circ(a'))(\pi \circ F^\circ \circ \gamma^\circ) \tag{2.10}$$
$$\equiv \pi(F^\circ(a'))(\pi \circ F^\circ \circ (\phi^\circ \star \alpha^\circ)) \tag{2.11}$$
$$\equiv \pi \circ F^\circ(a'(\phi^\circ \star \alpha^\circ)), \tag{2.12}$$

$$\pi \circ F^\circ(b) \equiv \pi \circ F^\circ(b'\gamma^\circ) \equiv \pi(F^\circ(b'))(\pi \circ F^\circ \circ \gamma^\circ) \tag{2.13}$$
$$\equiv \pi(F^\circ(b'))(\pi \circ F^\circ \circ (\phi^\circ \star \alpha^\circ)) \tag{2.14}$$
$$\equiv \pi \circ F^\circ(b'(\phi^\circ \star \alpha^\circ)). \tag{2.15}$$

The facts that θ is an idempotent mgu of $\pi(a') = \pi(b')$, λ is minimal respect to θ and ϕ° is a lifting of $\theta \bullet \lambda$, imply that $\widehat{\phi} = \pi \circ F^\circ \circ \phi^\circ$ is a unifier of equation $\pi \circ F^\circ(a') = \pi \circ F^\circ(b')$, since F is unifiable. Thus:

$$\begin{gathered}\pi \circ F^\circ(a'(\phi^\circ \star \alpha^\circ)) \equiv \pi \circ F^\circ(a')(\pi \circ F^\circ \circ \phi^\circ) \bullet (\pi \circ F^\circ \circ \alpha^\circ) \\ \equiv (\pi \circ F^\circ(a')\widehat{\phi})(\pi \circ F^\circ \circ \alpha^\circ) \equiv (\pi \circ F^\circ(b')\widehat{\phi})(\pi \circ F^\circ \circ \alpha^\circ) \\ \equiv \pi \circ F^\circ(b')(\pi \circ F^\circ \circ \phi^\circ) \bullet (\pi \circ F^\circ \circ \alpha^\circ) \equiv \pi \circ F^\circ(b'(\phi^\circ \star \alpha^\circ)).\end{gathered} \tag{2.16}$$

From equations 2.10, 2.13 and 2.16, $F^\circ(a) \sim F^\circ(b)$ follows.

2' $\implies$ 1.

Let us show that $F : \Pi \to \Psi$ is unifiable. Let us take a pair of terms $s, t \in \Pi$ and consider the equation $t = s$. If $Unif(t = s) = \emptyset$, there is nothing to prove. Otherwise, take θ an idempotent mgu of $t = s$. If there are no minimal substitutions respect to θ, the result follows. If there are, let us take a substitution λ, minimal respect to θ. Let us show that there exists $\widehat{\phi}$ in Ψ^* a unifier of $F(s) = F(t)$ compatible with $\theta \bullet \lambda$. To find $\widehat{\phi}$ and show the compatibility assertion, let us take two arbitrary liftings ϕ° and ϕ_1° of $\theta \bullet \lambda$ and notice that for every $x \in V$, $\phi^\circ(x) \sim \phi_1^\circ(x)$. Thus, by hypotheses 2, $F^\circ(\phi^\circ(x)) \sim F^\circ(\phi_1^\circ(x))$. Since ϕ° and ϕ_1° are arbitrary, $\widehat{\phi} = \pi \circ F \circ \phi^\circ$ is compatible with $\theta \bullet \lambda$. Let us show that $\widehat{\phi}$ is a unifier of $F(s) = F(t)$. Let us choose $a', b' \in \Pi^\circ$ such that $\pi(a') \equiv t$ and $\pi(b') \equiv s$. Since $s(\theta \bullet \lambda) \equiv t(\theta \bullet \lambda)$, it follows that $(a'\phi^\circ) \sim (b'\phi^\circ)$ and then $F^\circ(a'\phi^\circ) \sim F^\circ(b'\phi^\circ)$, which implies that $F(t)\widehat{\phi} \equiv F(s)\widehat{\phi}$. Therefore, $\widehat{\phi}$ is a unifier of $F(t) = F(s)$. □ □

The following discussion is aimed to drop the assumption "Π is disjoint" in Theorem 2.A. To this aim, let us consider a fixed order in the set of variables $V = \{z_1, z_2, ...\}$. We shall be interested in the case where Π is finite[4]. Let Π be ordered in the following way: $\Pi = \{t_1, ..., t_m\}$. Given a term translation $F : \Pi \to \Psi$, we construct inductively a finite family of invertible substitutions $\{\phi_t\}_{t \in \Pi}$ as follows: For each $i = 1, ..., m$ write $V(t_i) = \{x_1, ..., x_n\}$ and consider the invertible substitution $\phi_{t_i} = \{x_1 \leftarrow w_1, ..., x_n \leftarrow w_n\}$ where $w_1, ..., w_n$ are the *first n* variables in V which satisfy: for any $k \in \{1, ..., n\}$, the variable w_k does not occur in any term $t \in \Pi$ neither it occurs in any term $t_j \phi_{t_j}$ with $j < i$.

Definition 9. Let $F : \Pi \to \Psi$ be a term translation with Π a finite set. We define the *disjoint term translation* associated to F as the term translation $F' : \Pi' \to \Psi'$ where $\Pi' = \{t\phi_t\}_{t \in \Pi}$, $\Psi' = \{F(t)\phi_t\}_{t \in \Pi} \cup (\Psi \setminus Im(F))$ and F' given by $F'(t\phi_t) = F(t)\phi_t$.

Notice that Π' is disjoint, $(\Pi')^* = \Pi^*$ and $(\Psi')^* = \Psi^*$. Now we are ready to prove another version of Theorem 2.A.

Theorem 2.B (Characterization of Extensibility)**.** Let $F : \Pi \to \Psi$ be a term translation with Π finite. The following statements are equivalent:

1. The disjoint term translation $F' : \Pi' \to \Psi'$ associated to F is unifiable.

[4] This finiteness hypothesis can be weakened to Π countable and the set $V \setminus (\bigcup_{t \in \Pi} V(t))$ infinite.

2. There exists a term morphism $F^* : \Pi^* \to \Psi^*$ which extends $F : \Pi \to \Psi$.

Proof. Let us assume that the disjoint term translation $F' : \Pi' \to \Psi'$ associated to F is unifiable. By applying Theorem 2.A to $F' : \Pi' \to \Psi'$, we obtain a term morphism $(F')^* : \Pi^* \to \Psi^*$ which extends F'. Let us take $t \in \Pi$ and consider the following computation:

$$(F')^*(t) = (F')^*(t(\phi_t)(\phi_t)^{-1}) = F'(t\phi_t)\phi_t{}^{-1} = F(t)\phi_t\phi_t{}^{-1} = F(t).$$

Therefore, $(F')^*$ extends $F : \Pi \to \Psi$.

On the other hand, let us assume that the term morphism $F^* : \Pi^* \to \Psi^*$ extends $F : \Pi \to \Psi$. Let us take a term $t\phi_t \in \Pi'$ and consider the following computation:

$$F^*(t\phi_t) = F(t)\phi_t = F'(t\phi_t).$$

Thus F^* extends $F' : \Pi' \to \Psi'$. Since Π' is disjoint, by Theorem 2.A we conclude that $F' : \Pi' \to \Psi'$ is unifiable. □ □

2.7.2 Sufficient Conditions for Extensibility

Let $F : \Pi \to \Psi$ be a term translation. The existence of a term morphism extending F for certain classes of sets Π, is characterized by Theorem 2 (versions A and B) in terms of minimal substitutions. However, notice that minimal substitutions are theoretical devices which, in computational models or applications might be hard to work with. This section presents another version of Theorem 2 which makes use of a restricted family of minimal substitutions.

Definition 10. A term translation $F : \Pi \to \Psi$ will be called *strongly unifiable* if for every pair of terms $s, t \in \Pi$, every idempotent mgu θ of equation $t = s$ satisfies:

1. If θ is not in Π^*, then $\theta \bullet \alpha$ is not in Π^* for every substitution α.
2. If θ is in Π^*, then there exists a unifier $\widehat{\theta}$ in Ψ^* of $F(t) = F(s)$ compatible with θ.

According to Definition 10, to verify that $F : \Pi \rightarrow \Psi$ is strongly unifiable we must go over all the pairs $s,t \in \Pi$ and check that every idempotent mgu of $t = s$ satisfies conditions 1 and 2 above. However, the following result shows that for each equation $t = s$, it is enough to find just one idempotent mgu satisfying conditions 1 and 2. Next corollary follows directly from Propositions 5 and 6.

Corollary 8. *Let $F : \Pi \rightarrow \Psi$ be a term translation. $F : \Pi \rightarrow \Psi$ is strongly unifiable iff every pair of terms $s,t \in \Pi$ satisfies one of the two following conditions:*

- *There are no unifiers of equation $t = s$.*
- *There exists an idempotent mgu θ of equation $t = s$ which satisfies:*
  1. *If θ is not in Π^*, then $\theta \bullet \alpha$ is not in Π^* for every substitution α.*
  2. *If θ is in Π^*, then there exists a unifier $\widehat{\theta}$ in Ψ^* of $F(t) = F(s)$ compatible with θ.*

Next two theorems give us conditions to ensure the extensibility of a term translation.

Theorem 3.A (Sufficient Conditions for Extensibility)**.** Let $F : \Pi \rightarrow \Psi$ be a term translation with Π disjoint. If $F : \Pi \rightarrow \Psi$ is strongly unifiable, then there exists a term morphism $F^* : \Pi^* \rightarrow \Psi^*$ which extends $F : \Pi \rightarrow \Psi$.

Proof. If $F : \Pi \rightarrow \Psi$ is strongly unifiable, then it is unifiable. Therefore, this theorem is a direct consequence of Theorem 2.A. □ □

Theorem 3.B. Let $F : \Pi \rightarrow \Psi$ be a term translation with Π finite. If the disjoint term translation $F' : \Pi' \rightarrow \Psi'$ associated to F is strongly unifiable, then there exists a term morphism $F^* : \Pi^* \rightarrow \Psi^*$ which extends $F : \Pi \rightarrow \Psi$.

Proof. If $F' : \Pi' \rightarrow \Psi'$ is strongly unifiable, then it is unifiable. Therefore, this theorem is a direct consequence of Theorem 2.B. □ □

2.8 Discussion and Conclusions

The results of this study give us conditions which characterize the existence of a term morphism extending a given term translation. We obtained such conditions by using graph-based representations of terms and applying unification theory. In this concluding section we shall discuss two points that we consider relevant for the understanding of the scope of this work. In the first point we shall give algebraic interpretations for some of the concepts introduced in this work. These interpretations enables us to argue the second point: the usefulness of this work as a framework for some analogy models.

2.8.1 An algebraic interpretation of this study

Suppose $\mathscr{L}^S$ and $\mathscr{L}^T$ are two different first order languages. Let $F : \Pi \to \Psi$ be a term translation where $\Pi \subseteq Term(\mathscr{L}^S) \setminus V$ and $\Psi \subseteq Term(\mathscr{L}^T) \setminus V$. The goal of the following discussion is to clarify the algebraic relation determined by F between the set $Term(\mathscr{L}^S)$ and the set $Term(\mathscr{L}^T)$. We shall need some definitions following [4].

A *signature* is a set $\mathscr{F}$ of *function symbols* such that a nonnegative integer n is assigned to each member f of $\mathscr{F}$. This integer is called the *arity* of f and f is said to be an *n-ary function symbol*. Given a signature $\mathscr{F}$, an $\mathscr{F}$-algebra is an ordered pair (A, Σ) where A is a nonempty set and Σ is a family of finitary operations on A indexed by $\mathscr{F}$ such that corresponding to each n-ary function symbol $f \in \mathscr{F}$ there is an n-ary operation f^A on A. When Σ is clear from the context, we shall refer to the $\mathscr{F}$-algebra A instead of (A, Σ). Let A and B be two $\mathscr{F}$-algebras. A mapping $\alpha : A \to B$ is called a *homomorphism* from A to B if for each n-ary function symbol $f \in \mathscr{F}$ and every $a_1, ..., a_n \in A$,

$$\alpha(f^A(a_1, ..., a_n)) = f^B(\alpha(a_1), ..., \alpha(a_n)).$$

Since our study deals only with the set of terms of a language, we shall assume that the symbol sets S and T are signatures i.e. they have only function symbols. The set of terms $Term(\mathscr{L}^S)$ could be considered as an S-algebra by determining the family of operations Σ_S in the standard way. However, we are interested in endowing the sets of terms $Term(\mathscr{L}^S)$ and $Term(\mathscr{L}^T)$ with another family of operations as follows: Let us take the set Π as a signature by considering each term $t \in \Pi$ with $V(t) = \{x_1, ..., x_n\}$ as a n-ary function symbol. Let f_t be the n-ary operation on $Term(\mathscr{L}^S)$ defined by:

$$f_t : (t_1, ..., t_n) \to t\frac{t_1, \ldots, t_n}{x_1, \ldots, x_n}.$$

We denote this family of operations by Σ_Π. Analogously, we can consider Ψ as a signature which determines a family $\{f_s\}_{s\in\Psi} = \Sigma_\Psi$ of operations on $Term(\mathscr{L}^T)$. Consider the Π-algebra (Π^*, Σ_Π) and the Ψ-algebra (Ψ^*, Σ_Ψ). Observe that when the term morphism $F^* : \Pi^* \to \Psi^*$ exists, it relates these algebras in the following way:

$$F^*(f_t(t_1, ..., t_n)) = f_{F(t)}(F^*(t_1), ..., F^*(t_n)). \tag{2.17}$$

The above equation (2.17) suggests endowing the set Ψ^* with the structure of a Π-algebra. For each $t \in \Pi$ with $V(t) = \{x_1, ..., x_n\}$ define an n-ary operation $f_{F(t)}$ on $Term(\mathscr{L}^T)$ by:

$$f_{F(t)} : (s_1, ..., s_n) \to F(t)\frac{s_1, \ldots, s_n}{x_1, \ldots, x_n}.$$

Let us call this family of operations $\Sigma_{F(\Pi)}$. Equation 2.17 says that when $F^* : \Pi^* \to \Psi^*$ exists, it is a homomorphism from the Π-algebra (Π^*, Σ_Π) to the Π-algebra $(\Psi^*, \Sigma_{F(\Pi)})$ (notice that $\Sigma_{F(\Pi)} \subseteq \Sigma_\Psi$). Therefore, the results of this work give us conditions on a term translation $F : \Pi \to \Psi$ to be extended by a homomorphism of Π-algebras.

2.8.2 A model for analogy

In this section we show how this work can be related to the modelling of analogies. To this aim, we introduce a theoretical model for analogy. The basic idea underlying this model is that the source and the target domains of an analogy can be formalized as algebras. A first approach for this idea can be found in [54] where the "analogy mapping" was represented by another algebra. Based on such work, a computational model for solving proportional analogy problems in string domains was developed in [15].

Let us introduce our theoretical model for analogy (TMA). Let $\mathscr{A}$ be an S-algebra and $\mathscr{B}$ a T-algebra where S and T are signatures. A *formal analogy problem* with *source* $\mathscr{A}$ and *target* $\mathscr{B}$ is a pair (F, h) where $F : \Pi \to \Psi$ is a term translation with $\Pi \subseteq Term(\mathscr{L}^S) \setminus V$, $\Psi \subseteq Term(\mathscr{L}^T) \setminus V$, and $h: A' \rightarrow B'$ is a mapping with $A' \subseteq \mathscr{A}$ and $B' \subseteq \mathscr{B}$.

Definition 11. An *analogy solution* of the formal analogy problem (F,h) will be a homomorphism of Π-algebras $h_F : \mathscr{U}(A') \rightarrow \mathscr{U}(B')$ extending h, where $\mathscr{U}(A')$ is the subalgebra of $(\mathscr{A}, \{f_t^{\mathscr{A}}\}_{t\in\Pi})$ generated by A' and $\mathscr{U}(B')$ is the subalgebra of $(\mathscr{B}, \{f_{F(t)}^{\mathscr{B}}\}_{t\in\Pi})$ generated by B'.

Notice that since h_F restricted to A' is h, and since A' generates $\mathscr{U}(A')$ as a Π-algebra, then if h_F exists, it is unique[5]. That is to say, a formal analogy problem has at most one solution.

Let us show how our model works by using examples taken from [15]. A proportional analogy has the form " A is to B as C is to D", abbreviated as $A : B :: C : D$. The associated analogy problem has the form $A : B :: C : X$ where A, B, C are given and X is unknown. Consider the following examples in string domains:

1.	abba	:	abab	::	pqrrqp	:	?(pqrpqr)
2.	abba	:	abbbbba	::	pqrrpq	:	?(pqrrrrrpq)

In order to model these analogy problems, let us consider the set P of all non-null strings composed from letters $a, b, ..., z$. Let us endow P with two operations: "Append" and "Symmetry". The operation append ($\alpha^P : P^2 \rightarrow P$) takes two strings s_1 and s_2 and returns the appended string $s_1 s_2$. The operation symmetry inverts the order of letters in a string i.e. ($\sigma^P : P \rightarrow P$) takes a string $l_1 l_2 .... l_{n-1} l_n$ and returns the string $l_n l_{n-1} .. l_2 l_1$. Observe that P is an $\mathscr{S}$-algebra by considering the signature $\mathscr{S} = \{\alpha, \sigma\}$.

Example 2. Let us model the analogy problem *abba* : *abab* :: *pqrrqp* :? and propose a solution through the model.

Let us consider the set of terms $\Pi = \{\alpha(X, \sigma(Y)), \alpha(X,Y)\}$ and let (I,h) be the analogy problem with source P and target P where $I : \Pi \rightarrow \Pi$ is the identity term translation and $h : \{ab\} \rightarrow \{pqr\}$ is determined by $h(ab) = pqr$. Let $\mathscr{U}(\{ab\})$ be the subalgebra of $(P, \{f_t\}_{t\in\Pi})$ generated by

[5]It is well known that for two algebra morphisms $h, f : A \rightarrow B$, if $h|_{A'} = f|_{A'}$ (where A' generates the algebra A), then $h = f$.

$\{ab\}$ and $\mathscr{U}(\{pqr\})$ the subalgebra of $(P, \{f_t\}_{t\in\Pi})$ generated by $\{pqr\}$. It can be shown [6] that there exists a homomorphism of Π-algebras $h_I : \mathscr{U}(\{ab\}) \to \mathscr{U}(\{pqr\}))$ which extends h. Therefore, h_I is an analogy solution of (I,h). Clearly, $abba, abab \in \mathscr{U}(\{ab\})$ $pqrrqp, pqrpqr \in \mathscr{U}(\{pqr\})$ and notice that because h_I is a homomorphism of Π-algebras extending h, it satisfies necessarily that $h_I(abba) = pqrrqp$ and $h_I(abab) = pqrpqr$, which is the answer to the original problem that this model provides.

Example 3. Let us model the proportional analogy *abba* : *abbbbba* :: *pqrrpq* :? and propose a solution through the model. To this aim, set

$$\Pi = \{\alpha(\alpha(X,Y),\alpha(Y,X)), \alpha(\alpha(\alpha(\alpha(\alpha(X,Y),Y),Y),Y),\alpha(Y,X))\}$$

and let (I,h) be the analogy problem with source P and target P, where $I : \Pi \to \Pi$ is the identity term translation and $h : \{a,b\} \to \{pq,r\})$ is the mapping determined by $h(a) = pq$, $h(b) = r$. Let $\mathscr{U}(\{a,b\})$ be the subalgebra of $(P, \{f_t\}_{t\in\Pi})$ generated by $\{a,b\}$ and $\mathscr{U}(\{pq,r\})$ the subalgebra of $(P, \{f_t\}_{t\in\Pi})$ generated by $\{pq,r\}$. By using the semigroup structure of the set P with the operation α, it is easy to show the existence of a homomorphism of Π-algebras $h_I : \mathscr{U}(\{a,b\}) \to \mathscr{U}(\{pq,r\}))$ which extends h. Therefore, h_I is an analogy solution of (I,h). Clearly, *abba*, $abbbbba \in \mathscr{U}(\{a,b\})$, $pqrrpq, pqrrrrrpq \in \mathscr{U}(\{pq,r\})$ and observe that because h_I is a homomorphism of Π-algebras extending h, it satisfies necessarily that $h_I(abba) = pqrrpq$ and $h_I(abbbbba) = pqrrrrrpq$, which is the answer to the proposed problem that this model provides.

Observe that when presenting a *formal analogy problem* (F,h) with source $\mathscr{A}$ and target $\mathscr{B}$, associated to the analogy problem $A : B :: C : X$, we have already selected an important number of parameters for the model. First, the sets of operations of the source and the target algebras together with the relations between them are determined by the selection of the mapping $F : \Pi \to \Psi$. Second, the sets of generators of the subalgebras involved in the analogy solution are determined by the selection of the mapping $h : A' \to B'$. Furthermore, examples 1 and 2 illustrate what was remarked after definition 11, namely that the choice of h guarantees the uniqueness of the solution h_F, when it exists. Somehow, the mapping h forces the way h_F must be defined. On the other hand, notice that even though the choice of (F,h) is not unique, neither it is completely arbitrary. For example,

[6] It is easy to show that there exists a homomorphism h' between the semigroups $(\mathscr{U}(\{ab,ba\}),\alpha)$ and $(\mathscr{U}(\{pqr,rqp\}),\alpha)$ which extends h and satisfies $h' \circ \sigma = \sigma \circ h'$. Since the symmetry operation satisfies $\sigma(\alpha(X,Y)) = \alpha(\sigma(Y),\sigma(X))$, it can be seen that h' restricted to $\mathscr{U}(\{ab\})$ (the subalgebra of $(P, \{f_t\}_{t\in\Pi})$ generated by $\{ab\}$) is the homomorphism h_I we are looking for (see appendix for details).

in the first two examples of proportional analogy problems $A : B :: C : X$, the mappings F and h were chosen in a way that A, B belong to the source subalgebra, C belongs to the target subalgebra and the possible analogy solution h_I satisfies $h_I(A) = C$. In those examples, $X = h_I(B)$ is the solution of the problem compatible with our choices.

Next example highlights the fact that although our model shares some common features with the model presented in [15], the two models are essentially different in nature. To illustrate this, we cite a paragraph from [15], where the authors say: "there is an assumption built into our formalism: namely that the same object occurring at different places in an algebraic term is the same". Such assumption implies that analogies like "*ababa* : *abbaa* :: *cdcdg* :?" can not be handled. This is because in the first term of the analogy problem, the model can not distinguish between the first and the last occurrence of the letter "*a*". Our model can still state a formal analogy problem in this case, however, the fact that it ends up having no solution, suggests the further need for tools which would help us in the path of building a clearer description of the conditions under which an analogy has one or more solutions.

Example 4. Let us model the following proportional analogy problem

$$ababa : abbaa :: cdcdg :? \tag{2.18}$$

Let us set $\Pi = \{t_1, t_2\}$ where $t_1 = \alpha(\alpha(X,X),Y)$ and $t_2 = \alpha(\alpha(X,\sigma(X)),Y)$. Consider the analogy problem (I,h) with source P and target P where $I : \Pi \to \Pi$ is the identity term translation and $h : \{ab,a\} \to \{cd,g\})$ is determined by $h(ab) = cd$ and $h(a) = g$. Let $\mathscr{U}(\{ab,a\})$ be the subalgebra of $(P, \{f_t\}_{t\in\Pi})$ generated by $\{ab,a\}$ and $\mathscr{U}(\{cd,g\})$ the subalgebra of $(P, \{f_t\}_{t\in\Pi})$ generated by $\{cd,g\}$. Clearly, $ababa$, $abbaa \in \mathscr{U}(\{ab,a\})$ and $cdcdg, cddcg \in \mathscr{U}(\{cd,g\})$. Let us assume for a while that there exists a homomorphism $h_I : \mathscr{U}(\{ab,a\}) \to \mathscr{U}(\{cd,g\}))$ of Π-algebras which extends h. Such homomorphism should satisfy $h_I(ababa) = cdcdg$ and $h_I(abbaa) = cddcg$, and then it would arguably provide a plausible behavior of the phenomenon. However, observe that the assumption of the existence of h_I leads us into contradiction: $f_{t_1}(ababa,a) = f_{t_2}(ababa,a) = ababaababaa$, and then

$$h_I(ababaababaa) = f_{t_1}(cdcdg,g) = cdcdgcdcdgg,$$

$$h_I(ababaababaa) = f_{t_2}(cdcdg,g) = cdcdggdcdcg.$$

That is, $cdcdgcdcdgg = cdcdggdcdcg$. Therefore, there is no such homomorphism extending h, and then, there is no analogy solution for the formal analogy problem (I,h).

In this last example, our choices were motivated by the fact that a plausible solution for the problem 2.18 is *cddcg*. However, notice that another reasonable solution candidate for the problem would be *cddgg*. Thus, the aim of the above example is not to show that the problem 2.18 has no solutions. The example shows that for the particular choice of (F,h) made in the formalization of the problem, the model does not provide solutions. However, it is possible that the TMA model gives solutions for other selections of (F,h) which formalize the problem. This observation suggests some topics for future research. For example, in the context of proportional analogy problems of the kind $A : B :: C : X$, how to characterize solutions of all adequate choices of (F,h) (all the associated formal analogy problems with source $\mathscr{A}$ and target $\mathscr{B}$). Then, for instance, we could say that $X = D$ is the unique solution for a given proportional analogy problem, when the analogy solution h_F for every choice of F and h satisfies $h_F(B) = D$.

The following example is different in nature. It shows that our model, in addition to proportional analogies in string domains, can be applied to more general analogies where the domains can be represented by algebras of terms. Specifically, it sketches how the results obtained in section 2.7 can be considered in the context of the TMA model.

Example 5. Let S and T be two signatures and let $F : \Pi \to \Psi$ be a term translation, where $\Pi \subseteq Term(\mathscr{L}^S)$ and $\Psi \subseteq Term(\mathscr{L}^T)$. Let us consider the analogy problem (F,Id) with source $Term(\mathscr{L}^S)$ and target $Term(\mathscr{L}^T)$ where $Id : V \to V$ is the identity mapping between variables. The results of section 2.7 give us conditions to be checked in $F : \Pi \to \Psi$ to verify that $F^* : \Pi^* \to \Psi^*$ exists. And when it does exist, F^* is an analogy solution of the analogy problem (F,Id) above considered.

2.9 Appendix

This section contains some examples and proofs which have not been included in the chapter either because its easyness of because it deviates the attention from the goal of the chapter.

Example 6. Let $S = \{f, e_1, e_2\}$ and $T = \{h_1, h_2, h_3, h_4\}$ be two symbol sets where e_1, h_1 are unary function symbols, e_2, h_2 are binary function symbols, f, h_3 are ternary function symbols and h_4 is a 4-ary function symbol. Consider the following terms belonging to $Term(\mathscr{L}^S)$:

- $Z_1 = e_1(u)$
- $Z_2 = e_2(v_1, v_2)$
- $Z_3 = f(e_1(x_1), x_2, x_3)$
- $Z_4 = f(y_1, y_2, e_2(y_3, y_4))$

It is easy to see that every term $t \in \Pi^*$ can be built in one unique way from Π, when $\Pi = \{Z_3, Z_4\}$ or $\Pi = \{Z_1, Z_3, Z_4\}$ or $\Pi = \{Z_2, Z_3, Z_4\}$. However, let us consider $\Pi = \{Z_1, Z_2, Z_3, Z_4\}$ and $t = f(e_1(z_1), z_2, e_2(z_3, z_4)) \in \Pi^*$. Observe that

$$t = f(e_1(z_1), z_2, e_2(z_3, z_4)) = Z_3 \frac{z_1\ z_2\ e_2(z_3, z_4)}{x_1\ x_2\ x_3} = Z_4 \frac{e_1(z_1)\ z_2\ z_3\ z_4}{y_1\ y_2\ y_3\ y_4}$$

On the other hand, let $\Psi = \{h_1(u), h_2(v_1, v_2), h_3(x_1, x_2, x_3), h_4(y_1, y_2, y_3, y_4)\}$. Let $F : \Pi \to \Psi$ be a term translation which associates the term $Z_i \in \Pi$ to the i-th term in Ψ (considering the order in which the elements of Ψ were listed). If we assume that there exists a term morphism F^* which extends F, it should satisfy:

$$F^*(t) = F^*(Z_3 \frac{z_1\ z_2\ e_2(z_3, z_4)}{x_1\ x_2\ x_3}) = F^*(Z_4 \frac{e_1(z_1)\ z_2\ z_3\ z_4}{y_1\ y_2\ y_3\ y_4})$$

which means that,

$$F(Z_3) \frac{F^*(z_1)\ F^*(z_2)\ F^*(e_2(z_3, z_4))}{x_1\ x_2\ x_3} = F(Z_4) \frac{F^*(e_1(z_1))\ F^*(z_2)\ F^*(z_3)\ F^*(z_4)}{y_1\ y_2\ y_3\ y_4}$$

$$h_3(x_1, x_2, x_3) \frac{z_1\ z_2\ h_2(z_3, z_4)}{x_1\ x_2\ x_3} = h_4(y_1, y_2, y_3, y_4) \frac{h_1(z_1)\ z_2\ z_3\ z_4}{y_1\ y_2\ y_3\ y_4}$$

which yields the following contradiction:

$$h_3(z_1, z_2, h_2(z_3, z_4)) = h_4(h_1(z_1), z_2, z_3, z_4)$$

Proposition 1 can be proved in the following way:

Proposition. *A term morphism $F^* : \Pi^* \to \Psi^*$ extends a term translation $F : \Pi \to \Psi$ if and only if for any substitution ρ in Π^*, $\gamma = F^* \circ \rho$ is a substitution in Ψ^* and makes following diagram commute:*

$$\begin{tikzcd} \Pi \arrow[r, "F"] \arrow[d, "\overline{\rho}"] & \Psi \arrow[d, "\overline{\gamma}"] \\ \Pi^* \arrow[r, "F^*"] & \Psi^* \end{tikzcd}$$

Proof. Let F^* be a term morphism which extends F. Let us show the necessity of the condition. If ρ is a substitution in Π^*, then $F^* \circ \rho$ is a substitution in Ψ^*. Let us take any $t' \in \Pi$ and thus $t'\rho \in \Pi^*$. Consider the following computation:

$$(F^* \circ \overline{\rho})(t') = F^*(t'\rho) = F(t')(F^* \circ \rho) = F(t')\gamma = (\overline{\gamma} \circ F)(t').$$

Since $t' \in \Pi$ is arbitrary, it follows that $F^* \circ \overline{\rho} = \overline{\gamma} \circ F$.

On the other hand, assume that for any substitution ρ in Π^*, $\gamma = F^* \circ \rho$ is a substitution in Ψ^* such that $F^* \circ \overline{\rho} = \overline{\gamma} \circ F$. For any $t' \in \Pi$:

$$F^*(t'\rho) = F^*(\overline{\rho}(t')) = (F^* \circ \overline{\rho})(t') = (\overline{\gamma} \circ F)(t') = (\overline{(F^* \circ \rho)} \circ F)(t') = F(t')(F^* \circ \rho)$$

and thus F^* extends F. $\square$

Proposition. *The term morphism $F^* : \Pi^* \to \Psi^*$ of Theorem 4, satisfies:*

$$F^* = i \circ F' \circ i^{-1}.$$

Proof. If there exists a term morphism $F^* : \Pi^* \to \Psi^*$ which extends $F : \Pi \to \Psi$, condition 1 holds. Therefore, we can define F' by $F'(\overline{a}) = \overline{F^\circ(a)}$ and the diagram of the theorem commutes. Let us take any $t \in \Pi^*$, some $a \in \Pi^\circ$ such that $\pi(a) = t$ and consider the following computations:

$$F^*(t) = F^*(\pi(a)) = \pi(F^\circ(a)) = i(p(F^\circ(a)))$$

$$= i(F'(p(a))) = i(F'(i^{-1}(\pi(a)))) = i(F'(i^{-1}(t)))$$

since t is arbitrary, $F^* = i \circ F' \circ i^{-1}$ holds. $\square$

Proposition. *Let θ_1 be an idempotent mgu of $s = t$ which is* not *in Π^*. The following statements are true:*

1. *Any idempotent mgu θ_2 of $s = t$ is* not *in Π^*.*
2. *If for any substitution α, $\theta_1 \bullet \alpha$ is not in Π^*, then for any substitution α' and for any idempotent mgu θ_2 of $s = t$, the substitution $\theta_2 \bullet \alpha'$ is not in Π^*.*

Proof. By contradiction both statements are clear:

The first statement: If we assume that $\theta_2 = \theta_1 \bullet \phi$ is in Π^*, it follows that $\theta_2 \bullet \phi^{-1} = \theta_1$ is in Π^*. This is a contradiction with the hypotheses.

The second statement: If we assume that $\theta_2 \bullet \alpha'$ in Π^* it means that $\theta_1 \bullet (\phi \bullet \alpha')$ in Π^*. It contradicts the hypotheses.

□

Proposition. *Let $F : \Pi \to \Psi$ be a term translation and $s, t \in \Pi$. If the set of unifers of $s = t$ is nonempty, then the next two statements are equivalent:*

1. *For any θ_2 idempotent mgu of $t = s$ and any λ_2 minimal respect to θ_2, there exists an unifier $\widehat{\lambda_2}$ of $F(t) = F(s)$ compatible with $\theta_2 \bullet \lambda_2$.*
2. *There exists θ_1 an idempotent mgu of $t = s$ such that for any substitution λ_1 minimal respect to θ_1, there exists a unifier $\widehat{\lambda_1}$ of $F(t) = F(s)$ compatible with $\theta_1 \bullet \lambda_1$.*

Proof. This proof is done by reduction to absurd. Assume that λ is minimal respect to θ_2 and that $\phi \bullet \lambda$ is *not* minimal respect to θ_1. Thus, there are substitutions α and β such that β in Π^* is proper, $dom(\beta) \subseteq ran(\alpha)$, $\theta_1 \bullet \alpha$ in Π^* and $\phi \bullet \lambda = \alpha \bullet \beta$. Since ϕ is invertible, it means that substitution $(\phi^{-1} \bullet \alpha)$ and β satisfy the following: β in Π^* is a proper substitution, $dom(\beta) \subseteq ran(\phi^{-1} \bullet \alpha)$, $\theta_2 \bullet (\phi^{-1} \bullet \alpha) = \theta_1 \bullet \alpha$ in Π^* and $\lambda = (\phi^{-1} \bullet \alpha) \bullet \beta$. It contradicts the fact that λ is minimal respect to θ_2. Therefore, substitution $\phi \bullet \lambda$ is minimal respect to θ_1. □

Proposition. *Let t be a term with $V(t) = \{x_1, ..., x_n\}$. If $t\frac{t_1, \ldots, t_n}{x_1, \ldots, x_n} \equiv t\frac{s_1, \ldots, s_n}{x_1, \ldots, x_n}$, then $(t_j \equiv s_j)$, for every $j \in \{1, ..., n\}$.*

Proof. Let us show that by induction on $k = length(t\frac{t_1, \ldots, t_n}{x_1, \ldots, x_n})$. The base step is trivial. If $k = 0$ we have that

$$x\frac{y}{x} \equiv x\frac{z}{x}$$

and then $z \equiv y$ follows.

Let us show the induction step for $k = length(t\frac{t_1, \ldots, t_n}{x_1, \ldots, x_n})$ by assuming that the proposition holds for any $m < k$. Notice that $t = f(u_1, ..., u_l)$ for some l-ary function symbol f and terms $u_1, ..., u_l$. Therefore, $t\frac{t_1, \ldots, t_n}{x_1, \ldots, x_n} \equiv t\frac{s_1, \ldots, s_n}{x_1, \ldots, x_n}$ is equivalent to $f(u_1, ..., u_l)\frac{t_1, \ldots, t_n}{x_1, \ldots, x_n} \equiv f(u_1, ..., u_l)\frac{s_1, \ldots, s_n}{x_1, \ldots, x_n}$. Thus, it follows that for every $i \in \{1, .., l\}$,

$$u_i\frac{t_1, \ldots, t_n}{x_1, \ldots, x_n} \equiv u_i\frac{s_1, \ldots, s_n}{x_1, \ldots, x_n}.$$

Now, since $m_i = length(u_i\frac{t_1, \ldots, t_n}{x_1, \ldots, x_n}) < k$, we apply the induction hypothesis on every m_i. Since every x_j occurs in at least some u_i, we conclude that for every $j \in \{1, .., n\}$, $t_j \equiv s_j$. $\square$ $\square$

Proposition. *There exists $h' : (\mathscr{U}(\{ab, ba\}), \alpha) \to (\mathscr{U}(\{pqr, rqp\}), \alpha)$*

Proof. Let $P^* = P \cup \{\varnothing\}$ where $\varnothing$ is the empty string. Notice that (P^*, α) is a semigroup.

Let us denote by $\mathscr{U}\{ab, ba\}$ to the subsemigroup of (P^*, α) generated by $\{ab, ba\}$. Let us observe that for $s \in \mathscr{U}\{ab, ba\}$ there is only two options. s begins with "a" or (exclusive) s begins with "b".

- In the first case: $s = \alpha(ab, x)$ for some $x \in \mathscr{U}\{ab, ba\}$.
- In the second case: $s = \alpha(ba, x)$ for some $x \in \mathscr{U}\{ab, ba\}$.

Therefore, s can be built in a unique way (except for associativity). From this, it is easy to define $h' : (\mathscr{U}\{ab, ba\}, \alpha) \to (\mathscr{U}\{pqr, rqp\}, \alpha)$ inductively:

$$h'(s) = \begin{cases} pqr & \text{if } s = ab, \\ rqp & \text{if } s = ba, \\ \alpha(pqr, h'(x)) & \text{if } s = \alpha(ab, x), \\ \alpha(rqp, h'(x)) & \text{if } s = \alpha(ba, x), \end{cases}$$

Observe that h' is well defined since α is an associative operation. Additionally, we can define in $\mathscr{U}\{ab, ba\}$ and $\mathscr{U}(\{pqr, rqp\})$ the operation σ("Symmetry") by $\sigma(ab) = ba$, $\sigma(ba) = ab$ and $\sigma(\alpha(x,y)) = \alpha(\sigma(y), \sigma(x))$ and then h' satisfies $h' \circ \sigma = \sigma \circ h'$.

We just have to restrict h' to the subalgebra of $(P, \{f_t\}_{t\in\Pi})$ generated by $\{ab\}$, that is $h_I = h'|_{(P,\{f_t\}_{t\in\Pi})}$. Notice that h_I is a morphism of Π-algebras since h' preserves the operation α and the operation σ. $\square$

Chapter 3

Formula Morphisms

The subject addressed on chapter 2, namely the existence of a term morphism which extends a given term translation, is the result of formalizing metaphor based in a key assumption: that a set of terms is able to accurately describe a domain of knowledge. Often enough, such assumption is not true. The aim of this chapter is to analyze the case when the assumption made is that a set of terms *together* with a set of formulas describe accurately a domain of knowledge. The present chapter is a necessary complement to the analysis already done in the previous chapter since the usage of formulas is a well known way to describe relations in a domain of knowledge. This chapter strongly depends on Chapter 2.

We assume that every first order language $\mathscr{L}^S$ mentioned here is built on the same fixed numerable set of variables $V = \{x, y, z, ...\}$. The present discussion will assume that a language is determined by a set S of relation and function symbols in the standard way (see[20, 58]). However, we shall restrict ourselves to work with formulas with no quantifiers, i.e. we shall work with the set of logical connectives $\{\neg, \wedge, \vee\}$. We shall write $Form(\mathscr{L}^S)$ refering to the set of formulas of such language.

3.1 Extending Formula Translations by Substituting Terms

3.1.1 Formula Translations and Substitutions of Terms

Let S and Q be symbol sets which determine the first order languages $\mathscr{L}^S$ and $\mathscr{L}^Q$ respectively, and let $\Sigma \subseteq Form(\mathscr{L}^S), \Omega \subseteq Form(\mathscr{L}^Q)$ be two non empty sets.

Definition 12. A mapping $F : \Sigma \to \Omega$ is called a *formula translation*, if it preserves the *free variables* of formulas, i.e. for all $\varphi \in \Sigma$,

$$V(\varphi) = V(F(\varphi))$$

where $V(\varphi)$ denotes the set of free variables of the formula φ.

Example 7. Let $S = \{\subset, \in, \equiv\}$ and $Q = \{\leq, =\}$ be two symbol sets. The former is usually associated to set theory while the latter is used to work with arithmetic. Given $\Sigma = \{(x \subset y) \vee (x \equiv y)\}$ and $\Omega = \{x \leq y\}$, the rule below, define a formula translation $F : \Sigma \to \Omega$.

$$F((x \subset y) \vee (x \equiv y)) = x \leq y$$

We use the same meaning of the expression "a variable x occurs free in a formula φ" as the usual meaning in mathematical logic. We need now define how to substitute a term t for a variable x in a formula φ where x occurs free, thus obtaining a formula ψ. Observe that because our language has no quantifiers, next definition of substitution do not need the usual amount of care which is necessary[1] to define the substitution in a way that the formula φ expresses the same about x as ψ does about t. In chapter 2 we defined a substitution $\rho : V \to Term(\mathscr{L}^S)$ together with the way to apply it to a term t. In this section, we define the procedure wich is used to apply a substitution ρ to a formula φ.

[1] Next example (taken from [20]) illustrates why a certain amount of care in such definition is necessary. Let $\varphi := \exists z\, z + z = x$, if we interpret this formula in the natural numbers, the formula is true iff the interpretation of x is an even number. If we replace the variable x by y in φ we obtain the formula $\exists z\, z + z = y$ wich states that y is even. But if we replace the variable x by z, we obtain the formula $\exists z\, z + z = z$, wich no longer says that z is even; in fact, this formula is true in the natural numbers regardless of the assignment for z (because $0 + 0 = 0$). We can see that the meaning of the formula is altered by applying the substitution without enough care. The problem is that at the place where x occurred free, the variable z gets bound. The definition of substitution addresses this kind of problems.

Definition 13. Let S be a fixed symbol set, $\rho = \{x_1 \leftarrow t_1, ..., x_n \leftarrow t_n\}$ a substitution where $t_i \in Term(\mathscr{L}^S)$ and let $\varphi \in Form(\mathscr{L}^S)$. We define the formula $\varphi\rho$ (or $\varphi\frac{t_1...t_n}{x_1...x_n}$) wich is said to be obtained from φ by *simultaneous substitution* of $t_1, ..., t_n$ for $x_1, ..., x_n$ in the following inductive way:

1. $[t = s]\rho \ := \ t\rho = s\rho$.
2. $R(s_1, .., s_r)\rho := R(s_1\rho, .., s_r\rho)$.
3. $[\neg\varphi]\rho := \neg[\varphi\rho]$
4. $[\varphi \vee \psi]\rho := \varphi\rho \vee \psi\rho$.
5. $[\varphi \wedge \psi]\rho := \varphi\rho \wedge \psi\rho$.

Unification Theory adapted for Formulas

This section is devoted to develop the theory that shall help us to approach next section (section 3.1.2). We shall borrow some ideas from Unification Theory in order to work with the formulas of the language. For a given pair of formulas $\varphi, \psi \in Form(\mathscr{L}^S)$, we say that the string "$\varphi = \psi$" is a *formula equation* (or simply, an equation when "formula" is understood from context). For the sake of avoiding confusion, we shall use the symbol "$\equiv$" to denote the syntactic equality of formulas and terms. An *unifier* of the equation $\varphi = \psi$ is a substitution $\alpha : V \to Term(\mathscr{L}^S)$ such that $\varphi\alpha \equiv \psi\alpha$. Let $D = \{\varphi_1 = \psi_1, ..., \varphi_n = \psi_n\}$ be a set of equations, an *unifier* of D is a substitution α wich is a unifier of every equation in D. We shall denote the set of unifiers of D by $Unif(D)$. The following discussion is aimed to show that if $Unif(D) \neq \emptyset$, there exists a set $T_D = \{t_1 = s_1, ..., t_m = s_m\}$ of term equations such that α is a unifier of D if and only if α is a unifier of T_D. This is a key result enabling us to use definitions given (in Section 2.6) for a set of term equations, in the context of formula equations. Let us consider the following algorithm:

Algorithm for Unification of Formulas

Input: A set of formula equations $\{\chi_1 = \xi_1, ..., \chi_n = \xi_n\}$.
Initialize the set of formula equations $D = \{\chi_1 = \xi_1, ..., \chi_n = \xi_n\}$.

Initialize the set of term equations $T_D = \emptyset$.
Select any equation $\chi = \xi$ from D for which one of the following procedures (depending on the form of the equation) can be applied:

1.	$\varphi_1 \wedge \varphi_2 = \psi_1 \wedge \psi_2$	Replace the equation in D by the pair of equations $\varphi_1 = \psi_1, \varphi_2 = \psi_2$.
2.	$\varphi_1 \vee \varphi_2 = \psi_1 \vee \psi_2$	Replace the equation in D by the pair of equations $\varphi_1 = \psi_1, \varphi_2 = \psi_2$.
3.	$\neg\varphi_1 = \neg\psi_1$	Replace the equation in D by the equation $\varphi_1 = \psi_1$.
4.	$R(s_1, ..., s_k) = R(t_1, ..., t_k)$, where R is a k-ary relation symbol	Delete the equation from D, and add to T_D the k equations $s_i = t_i$ for $i \in \{1, ..., k\}$.
5.	Any other kind of equation	Halt the algorithm with failure.

Repeat until failure or $D = \emptyset$.
Output the set of equations of terms T_D.

The above algorithm implements the idea of inductively analyzing the formulas to check if the pair of formulas of each equation can be (via a substitution) syntactically equal. It is straightforward that when the algorithm fails, $Unif(D) = \emptyset$. Besides, it is clear that if the algorithm does not fail and returns an empty set $T_D = \emptyset$ if and only if the input is empty.

Proposition 9. *The Algorithm for Unification of Formulas finishes in a finite number of steps.*

Proof. It is straightforward by noticing that applying steps 1, 2 or 3, the number of logical connectors occurring in the set of equations is reduced by two. By applying step 4, one equation is eliminated from D. □

For a given input $E = \{\chi_1 = \xi_1, ..., \chi_n = \xi_n\}$, let us define $\mathscr{E}$ as the set which contains all the equations of formulas that were put in the set D at least once during the execution of the algorithm.

Lemma 9. *If the Algorithm for Unification of Formulas does not fail, then $Unif(E) \subseteq Unif(\mathscr{E})$.*

Proof. Let us assume that the algorithm does not fail and let α be a substitution. Let us show this lemma by proving that $\alpha \notin Unif(\mathscr{E})$ implies $\alpha \notin Unif(E)$. If $\alpha \notin Unif(\mathscr{E})$, there exists an equation $\varphi_0 = \psi_0$ belonging to $\mathscr{E}$ such that $\varphi_0\alpha \not\equiv \psi_0\alpha$. The lemma follows, if $\varphi_0 = \psi_0$ belongs to E. Otherwise, $\varphi_0 = \psi_0$ must be the result of the application of step 1, 2, 3 or 4 of the algorithm to an equation $\varphi_1 = \psi_1$ such that $\varphi_1\alpha \not\equiv \psi_1\alpha$. Repeating recursively this procedure, we shall find an equation $\varphi_k = \psi_k$ belonging to E such that $\varphi_k\alpha \not\equiv \psi_k\alpha$ which means that $\alpha \notin Unif(E)$. We have shown that $\alpha \in Unif(E)$ implies $\alpha \in Unif(\mathscr{E})$. $\square$

On the other hand, since $E \subseteq \mathscr{E}$, $Unif(\mathscr{E}) \subseteq Unif(E)$. Therefore, applying last lemma, we obtain that $Unif(\mathscr{E}) = Unif(E)$. We shall use this fact in the proof of the following property:

Proposition 10. *If for an input $E = \{\chi_1 = \xi_1, ..., \chi_n = \xi_n\}$ the Algorithm for Unification of Formulas does not fail and returns a nonempty set $T_E = \{t_1 = s_1, ..., t_m = s_m\}$, then $Unif(T_E) = Unif(E)$.*

Proof. First, we shall show that $Unif(T_E) \subseteq Unif(\mathscr{E})$. Let α be a unifier belonging to $Unif(T_E)$. We shall prove by induction on n that $\chi\alpha \equiv \xi\alpha$ for all equations $\chi = \xi$ in $\mathscr{E}$ such that n is the number of logical connectives that occur in such equation. The base case $n = 0$ is shown easily since $\chi = R(s_1, .., s_k)$ and $\xi = R(t_1, ..., t_k)$, where the equations $s_i = t_i$ belongs to T_E for $i \in \{1, .., k\}$. Since $\alpha \in Unif(T_E)$, $R(s_1\alpha, .., s_k\alpha) \equiv R(t_1\alpha, ..., t_k\alpha)$, implying that $\chi\alpha \equiv \xi\alpha$. The induction step. Let us take $n > 0$ and assume that the result holds for all equations $\eta = \vartheta$ with a number of logical connectives less that n. We are going to show that $\chi\alpha \equiv \xi\alpha$ holds for $\chi = \xi$ in $\mathscr{E}$. Since the Algorithm does not fail and $n > 0$, only steps 1, 2, 3 or 4 can be applied to the equation $\chi = \xi$. Each applicable step is handled in a separate case. We only address cases 1 and 3 because the rest of the cases are analogous:

- Case 1: $\chi = \varphi_1 \wedge \varphi_2$, $\xi = \psi_1 \wedge \psi_2$. By using the induction hypothesis we conclude that $\varphi_1\alpha \equiv \psi_1\alpha$ and $\varphi_2\alpha \equiv \psi_2\alpha$ and then $\chi\alpha \equiv \xi\alpha$.
- Case 2: $\chi \equiv \neg\varphi$ and $\xi \equiv \neg\psi$. By using the induction hypothesis we conclude that $\varphi\alpha \equiv \psi\alpha$ and then $\chi\alpha \equiv \xi\alpha$.

Since $\chi = \xi$ was an arbitrary equation from $\mathscr{E}$, we have shown that $Unif(T_E) \subseteq Unif(\mathscr{E})$.

Let us show that $Unif(E) \subseteq Unif(T_E)$. To this aim, let us show that $\alpha \notin Unif(T_E)$ implies $\alpha \notin Unif(E)$. Let us assume that $\alpha \notin Unif(T_E)$ and then there exists a term equation $t = s \in T_E$ such that $t\alpha \not\equiv s\alpha$. Since step (4) of the algorithm is the only step which adds equations to the set T_E, there exists a formula equation $R(t_1, .., t, ..t_k) = R(s_1, ..., s, ...s_k) \in \mathscr{E}$ such that $R(t_1, .., t, ..t_k)\alpha \not\equiv R(s_1, ..., s, ...s_k)\alpha$. Therefore, $\alpha \notin Unif(\mathscr{E})$ and, because of Lemma 9, it means that $\alpha \notin Unif(E)$. $\square$

Therefore, for each set D of formula equations, there is an associated set T_D of term equations (given by the Algorithm for Unification of Formulas) such that $Unif(D) = Unif(T_D)$. In the following, we shall say that a unifier μ of a set of formula equations D is an *mgu (most general unifier)* of D if and only if μ is an mgu of the associated set of term equations T_D.

Minimal Substitutions and Unifiable Formula Translations

Next definition is a generalization of Definition 7 in Chapter 2, which shall be needed in this section to characterize formula translations that can be extended.

Definition 14. Let $\Pi \subseteq Term(\mathscr{L}^S)$ and let θ be an idempotent mgu of a set of formula equations. A substitution $\lambda : V \to Term(\mathscr{L}^S)$ is called *minimal (for* Π*) respect to* θ whenever $\theta \bullet \lambda$ is in Π^* and there are no substitutions α, β such that β in Π^* is proper, $dom(\beta) \subseteq ran(\alpha)$, $\theta \bullet \alpha$ in Π^* and $\lambda = \alpha \bullet \beta$.

Here we do not present the proofs of Proposition 11, Lemma 10 and Proposition 12 below, because they are just slight paraphrasings of the proofs of Proposition 7, Lemma 6 and Proposition 8 in Chapter 2.

Proposition 11. *Let* $\Pi \subseteq Term(\mathscr{L}^S)$*. Let* θ *be an idempotent mgu of a set of formula equations. If* γ *is a substitution such that* $\theta \bullet \gamma$ *is in* Π^**, then there exists a substitution* λ *minimal (for* Π *respect to* θ*) and a substitution* α *in* Π^* *such that* $\gamma = \lambda \bullet \alpha$*.*

Lemma 10. *Let* $\Pi \subseteq Term(\mathscr{L}^S)$*. If* θ_1, θ_2 *are two idempotent mgu's of a set of formula equations and* ϕ *is the invertible substitution such that* $\theta_2 = \theta_1 \bullet \phi$*, then a substitution* λ *is minimal (for* Π*) respect to* θ_2 *if and only if the substitution* $\phi \bullet \lambda$ *is minimal (for* Π*) respect to* θ_1*.*

Proposition 12. *Let* $\Pi \subseteq Term(\mathscr{L}^S) \setminus V$*,* $\Psi \subseteq Term(\mathscr{L}^Q) \setminus V$*,* $\Sigma \subseteq Form(\mathscr{L}^S)$ *and* $\Omega \subseteq Form(\mathscr{L}^Q)$*. Let* $F : \Sigma \to \Omega$ *be a formula translation and let* $T : \Pi \to \Psi$ *be a term translation. If* $\varphi, \psi \in \Sigma$ *and* $Unif(\varphi = \psi)$ *is nonempty, then the next two statements are equivalent:*

1. *For any* θ_2 *idempotent mgu of* $\varphi = \psi$ *and any* λ_2 *minimal (for* Π*) respect to* θ_2*, there exists an unifier* $\widehat{\lambda_2}$ *of* $F(\varphi) = F(\psi)$ *compatible with* $\theta_2 \bullet \lambda_2$*.*

2. *There exists* θ_1 *an idempotent mgu of* $\varphi = \psi$ *such that for any substitution* λ_1 *minimal (for* Π*) respect to* θ_1*, there exists a unifier* $\widehat{\lambda_1}$ *of* $F(\varphi) = F(\psi)$ *compatible with* $\theta_1 \bullet \lambda_1$*.*

Definition 15. Let $T : \Pi \to \Psi$ be a term translation where $\Pi \subseteq Term(\mathscr{L}^S)$ and $\Psi \subseteq Term(\mathscr{L}^Q)$ and let $\Sigma \subseteq Form(\mathscr{L}^S), \Omega \subseteq Form(\mathscr{L}^Q)$. A formula translation $F : \Sigma \to \Omega$ is called *unifiable (respect to* T*)* if for every pair $\varphi, \psi \in \Sigma$, every idempotent mgu θ of $\varphi = \psi$ and every minimal substitution λ respect to θ, there exists a unifier $\widehat{\lambda}$ in Ψ^* of the equation $F(\varphi) = F(\psi)$ compatible (through T) with $\theta \bullet \lambda$.

From Lemma 10 and Proposition 12 follows the next characterization of a unifiable formula translation.

Corollary 11 (Characterization)**.** *Let* $T : \Pi \to \Psi$ *be a term translation. A formula translation* $F : \Sigma \to \Omega$ *is unifiable respect to* T *if and only if every pair of formulas* $\varphi, \psi \in \Sigma$ *satisfies:*

1. $Unif(\varphi = \psi)$ *is empty, or*

2. *There exists* θ *an idempotent mgu of* $\varphi = \psi$ *such that for every* λ *minimal substitution (for* Π*) respect to* θ *there exists a unifier* $\widehat{\lambda}$ *in* Ψ^* *of the equation* $F(\varphi) = F(\psi)$ *compatible (through* T*) with* $\theta \bullet \lambda$*.*

3.1.2 Statement of the Problem and Characterization Theorems

Given a set of formulas $\Sigma \subseteq Form(\mathscr{L}^S)$ and a set of terms $\Pi \subseteq Term(\mathscr{L}^S)$, we define the set of formulas

$$\Sigma \otimes \Pi = \{\varphi\rho \mid \varphi \in \Sigma, \rho \text{ in } \Pi^* \}.$$

(where $\Omega \subseteq Form(\mathscr{L}^Q)$ and $\Psi \subseteq Term(\mathscr{L}^Q) \setminus V$),

The purpose of the previous chapter was the study of conditions that characterize those term translation $T : \Pi \to \Psi$ that can be extended to term morphisms $T^* : \Pi^* \to \Psi^*$. When such extension exists, we say that T is an *extensible* term translation. Observe that T^* is determined because when it exists, it is unique. This section is aimed to present conditions which guarantee that a formula translation $F : \Sigma \to \Omega$ can be mixed up with a term translation $T : \Pi \to \Psi$ into a single formula translation $L_{F,T} : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ which preserves the substitution operation, in more precise words, we look for a formula translation $L_{F,T}$ satisfying:

$$L_{F,T}(\varphi\rho) = F(\varphi)(T^* \circ \rho). \tag{3.1}$$

Clearly, a necessary condition is the existence of the term morphism $T^* : \Pi^* \to \Psi^*$ extending $T : \Pi \to \Psi$. However, notice that even when T is extensible, it is possible [2] that for $\varphi, \psi \in \Sigma$ and ρ, γ in Π^*, $\varphi\rho \equiv \psi\gamma$ and $F(\varphi)(T^* \circ \rho) \not\equiv F(\psi)(T^* \circ \gamma)$ meaning that the mapping $L_{F,T} : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ given by equation 3.1 above, is not well defined. In what follows, we discuss some conditions which ensure us that L is well defined by equation 3.1.

A set of formulas $\Sigma \subseteq Form(\mathscr{L}^S)$ is *disjoint* when for any $\varphi, \psi \in \Sigma$, if $\varphi \not\equiv \psi$, then $V(\varphi) \cap V(\psi) = \emptyset$. For a given extensible term translation $T : \Pi \to \Psi$, next theorem characterizes those formula translations $F : \Sigma \to \Omega$ with Σ disjoint, which make $L_{F,T}$ a well defined mapping. In what follows, when F and T are understood from context, we shall write L instead of $L_{F,T}$.

Theorem 12 (First Extension Characterization)**.** *Let $F : \Sigma \to \Omega$ be a formula translation where Σ is disjoint and let $T : \Pi \to \Psi$ be an extensible term translation. The following statements are equivalent:*

1. *$F : \Sigma \to \Omega$ is unifiable respect to T.*
2. *The mapping $L : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ given by*

$$L(\varphi\rho) = F(\varphi)(T^* \circ \rho),$$

is well defined.

Proof. Let us show that 1 implies 2. To this aim, we must show that if $\varphi\rho \equiv \psi\gamma$, then $F(\varphi)T^* \circ \rho \equiv F(\psi)T^* \circ \gamma$. Let us assume that $\varphi\rho \equiv \psi\gamma$. Since F preserves variables, without loss of

[2] An example: Take two signatures, namely $L_1 = \{a, b, R\}$ and $L_2 = \{A, B, S\}$ where a, b, A, B are constants, R is a binary relation symbol and S is a unary relation symbol. Set $\Sigma = \{R(x, b), R(a, y)\}$, $\Omega = \{S(x), S(y)\}$ and let $F : \Sigma \to \Omega$ be the only formula translation that can be defined between those sets. Let $\Pi = \{a, b\}$, $\Psi = \{A, B\}$ and let $T : \Pi \to \Psi$ be the term translation defined by $T(a) = A, T(b) = B$. Observe that $\Pi^* = V \cup \{a, b\}$, $\Psi^* = V \cup \{A, B\}$ and that T^* it is the map which extends T and acts as the identity on variables. In this context take $\varphi\rho = R(x, b)\frac{a}{x}$ and $\psi\gamma = R(a, y)\frac{b}{y}$ and notice that $\varphi\rho \equiv \psi\gamma \equiv R(a, b)$. However, $F(\varphi)T \circ \rho \equiv S(A)$ differs from $F(\psi)T \circ \gamma \equiv S(B)$.

generality we can assume that $\rho = \{x_1 \leftarrow t_1, ...x_n \leftarrow t_n\}$ and that $\gamma = \{y_1 \leftarrow s_1, ..., y_m \leftarrow s_m\}$ where $V(\varphi) = \{x_1, ..., x_n\}$ and $V(\psi) = \{y_1, ..., y_m\}$. There are two alternatives:

Case 1:

$\{x_1, ..., x_n\} \cap \{y_1, ..., y_m\} \neq \emptyset$. Because Σ is disjoint, $\varphi \equiv \psi$ and $\{x_1, ..., x_n\} = \{y_1, ..., y_m\}$. Thus, $\varphi\rho \equiv \varphi\gamma$ implies that $\rho = \gamma$ and then, $F(\varphi)(T^* \circ \rho) \equiv F(\psi)(T^* \circ \gamma)$.

Case 2:

$\{x_1, ..., x_n\} \cap \{y_1, ..., y_m\} = \emptyset$. Consider the substitution $\alpha = \{x_1 \leftarrow t_1, ..., x_n \leftarrow t_n, y_1 \leftarrow s_1, ..., y_m \leftarrow s_m\}$. Clearly, $\varphi\rho \equiv \psi\gamma$ if and only if $\varphi\alpha \equiv \psi\alpha$. Thus, α is an unifier of equation $\varphi = \psi$ and there exists an idempotent mgu θ of such equation. Because $\alpha = \theta \bullet \alpha$ is in Π^*, there exists a substitution λ minimal (for Π) respect to θ and a substitution α' in Π^* such that $\alpha = \lambda \bullet \alpha'$ and $\theta \bullet \lambda$ in Π^*. Since F is unifiable respect to T, it follows that there exists a substitution $\widehat{\lambda}$ compatible with $\theta \bullet \lambda$ which is a unifier of $F(\varphi) = F(\psi)$. The following computations give us the sought result:

$$\begin{gathered}
F(\varphi)\widehat{\lambda} \equiv F(\psi)\widehat{\lambda} \\
(F(\varphi)\widehat{\lambda})(T^* \circ \alpha') \equiv (F(\psi)\widehat{\lambda})(T^* \circ \alpha') \\
F(\varphi)(T^* \circ (\theta \bullet \lambda)) \bullet (T^* \circ \alpha') \equiv F(\psi)(T^* \circ (\theta \bullet \lambda)) \bullet (T^* \circ \alpha') \\
F(\varphi)(T^* \circ (\theta \bullet \lambda \bullet \alpha')) \equiv F(\psi)(T^* \circ (\theta \bullet \lambda \bullet \alpha')) \\
F(\varphi)(T^* \circ (\theta \bullet \alpha)) \equiv F(\psi)(T^* \circ (\theta \bullet \alpha)) \\
F(\varphi)(T^* \circ \alpha) \equiv F(\psi)(T^* \circ \alpha) \\
F(\varphi)(T^* \circ \rho) \equiv F(\psi)(T^* \circ \gamma).
\end{gathered}$$

Let us show, by using contradiction, that 2. implies 1. Let us assume that L is well defined, but $F : \Sigma \to \Omega$ is not unifiable respect to T. Therefore, there exist an idempotent mgu θ of $\varphi = \psi$ (where $\varphi, \psi \in \Sigma$), and a minimal sustitution λ (for Π respect to θ) such that every unifier $\widehat{\lambda}$ of $F(\varphi) = F(\psi)$ is not compatible with $\theta \bullet \lambda$. In case that $F(\varphi) = F(\psi)$ has no unifiers, the contradiction follows immediately: since L is well defined, $\varphi(\theta \bullet \lambda) \equiv \psi(\theta \bullet \lambda)$ implies $F(\varphi)T^* \circ (\theta \bullet \lambda) \equiv F(\psi)T^* \circ (\theta \bullet \lambda)$, and then $\widehat{\lambda} = T^* \circ (\theta \bullet \lambda)$ is a unifier of $F(\varphi) = F(\psi)$. Hence, let us analize the case in which $F(\varphi) = F(\psi)$ has unifiers and $\widehat{\lambda} = T^* \circ (\theta \bullet \lambda)$ is one of them. Since

$F:\Sigma\rightarrow\Omega$ is not unifiable respect to T, $\widehat{\lambda}$ is not compatible with $\theta\bullet\lambda$ and then, there exists a lifting λ° of $\theta\bullet\lambda$ which satisfies $\pi\circ\lambda^{\circ}=\theta\bullet\lambda$ and $\pi\circ T^{\circ}\circ\lambda^{\circ}\neq\widehat{\lambda}$. From the first expression, $T^{*}\circ(\theta\bullet\lambda)=T^{*}\circ(\pi\circ\lambda^{\circ})=\widehat{\lambda}$ and since $(\pi\circ T^{\circ})\circ\lambda^{\circ}\neq\widehat{\lambda}$, we get $T^{*}\circ\pi\neq\pi\circ T^{\circ}$. From Lemma 3 in Chapter 2, T is not extensible, which contradicts the hypotheses of the theorem. □

The goal of the following discussion is to drop the assumption "Σ is disjoint" in the earlier theorem. To this aim, let us consider a fixed order in the set of variables $V=\{z_1,z_2,\ldots\}$. We shall be interested in the case where Σ is finite. Let us consider a particular ordering for $\Sigma=\{\varphi_1,\ldots,\varphi_m\}$. Given a formula translation $F:\Sigma\rightarrow\Omega$, we construct inductively a finite family of invertible substitutions $\{\phi_{\varphi_i}\}_{i=1}^{m}$ as follows: For each $i=1,\ldots,m$, write $V(\varphi_i)=\{x_1,\ldots,x_n\}$ and consider the invertible substitution $\phi_{\varphi_i}=\{x_1\leftarrow w_1,\ldots,x_n\leftarrow w_n\}$ where $w_1,\ldots,w_n$ are the *first* n variables in V which satisfy: for any $k\in\{1,\ldots,n\}$, the variable w_k does not occur in any formula $\varphi\in\Sigma$ neither it occurs in any term $\varphi_j\phi_{\varphi_j}$ with $j<i$.

Definition 16. Let $F:\Sigma\rightarrow\Omega$ be a formula translation where $\Sigma=\{\varphi_1,\ldots,\varphi_n\}$ is a finite set. We define the *disjoint formula translation* associated to F as the formula translation $F':\Sigma'\rightarrow\Omega'$ where $\Sigma'=\{\varphi\phi_{\varphi}\}_{\varphi\in\Sigma}$, $\Omega'=\{F(\varphi)\phi_{\varphi}\}_{\varphi\in\Sigma}\cup(\Omega\setminus Im(F))$ and F' is given by $F'(\varphi\phi_{\varphi})=F(\varphi)\phi_{\varphi}$.

Observe that Σ' is disjoint. Additionally, notice that $\Sigma'\otimes\Pi=\Sigma\otimes\Pi$ when we consider any set Π of terms. Now we are ready to prove another version of Theorem 12.

Theorem 13 (Second Extension Characterization)**.** *Let $F:\Sigma\rightarrow\Omega$ be a formula translation where Σ is finite and let $T:\Pi\rightarrow\Psi$ be an extensible term translation. The following statements are equivalent:*

1. *The disjoint formula translation $F':\Sigma'\rightarrow\Omega'$ associated to F is unifiable respect to T.*
2. *The mapping $L:\Sigma\otimes\Pi\rightarrow\Omega\otimes\Psi$ given by*

$$L(\varphi\rho)=F(\varphi)(T^{*}\circ\rho),$$

is well defined.

Proof. To show that 1 implies 2, let us assume that the disjoint formula translation $F':\Sigma'\rightarrow\Omega'$ associated to F is unifiable respect to T. By Theorem 12 above, the mapping $L':\Sigma'\otimes\Pi\rightarrow\Omega'\otimes\Psi$

given by $L'((\varphi\phi_\varphi)\rho) = F'(\varphi\phi_\varphi)(T^* \circ \rho)$ is well defined. It is easy to see that the substitution $\phi_\varphi^{-1} \bullet \rho$ is in Π^* and then, there exists a well defined mapping $L_0 : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ given by:

$$\begin{aligned} L_0(\varphi\rho) &= F'(\varphi\phi_\varphi)T^* \circ (\phi_\varphi^{-1} \bullet \rho) = F(\varphi)\phi_\varphi \bullet (T^* \circ (\phi_\varphi^{-1} \bullet \rho)) \\ &= F(\varphi)T^* \circ (\phi_\varphi \bullet \phi_\varphi^{-1} \bullet \rho) = F(\varphi)(T^* \circ \rho). \end{aligned}$$

Therefore, the equality $\varphi\rho \equiv \psi\gamma$ implies that $L_0(\varphi\rho) \equiv L_0(\psi\gamma)$. The above computations show that $\varphi\rho \equiv \psi\gamma$ only if $F(\varphi)(T^* \circ \rho) \equiv F(\psi)(T^* \circ \gamma)$. Hence, the mapping $L : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ of 2 is well defined.

Let us show that 2 implies 1. Since $L(\varphi\rho) = F(\varphi)(T^* \circ \rho)$ is well defined, there is a well defined mapping $L_1 : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ given by:

$$\begin{aligned} L_1(\varphi\rho) &= L(\varphi\ (\phi_\varphi \bullet \rho)) = F(\varphi)T^* \circ (\phi_\varphi \bullet \rho) = F(\varphi)(T^* \circ \phi_\varphi) \bullet (T^* \circ \rho) \\ &= F(\varphi)\phi_\varphi \bullet (T^* \circ \rho) = F'(\varphi\phi_\varphi)(T^* \circ \rho). \end{aligned}$$

The above computations show that the disjoint formula translation $F' : \Sigma' \to \Omega'$ associated to F is extended by L_1 which is well defined. Since Σ' is disjoint, Theorem 12 ensures that the mapping F' is unifiable respect to T. $\square$

3.1.3 Sufficient Conditions

For a formula translation $F : \Sigma \to \Omega$ and a term translation $T : \Pi \to \Psi$, Theorem 12 and Theorem 13 make use of minimal substitutions in order to characterize when the mapping $L : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ is well defined by $L(\varphi\rho) = F(\varphi)(T^* \circ \rho)$. However, minimal substitutions are theoretical devices which, in computational models or applications might be hard to work with. This section presents other versions of these theorems which make use of a restricted family of minimal substitutions and, hopefully, for applications they might be more useful than the general Theorems.

Definition 17. Let $T : \Pi \to \Psi$ be an extensible term translation. A formula translation $F : \Sigma \to \Omega$ will be called *strongly unifiable respect to* T if for every pair of formulas $\varphi, \psi \in \Sigma$, every idempotent mgu θ of equation $\varphi = \psi$ satisfies:

1. If θ is not in Π^*, then $\theta \bullet \alpha$ is not in Π^* for every substitution α.

2. If θ is in Π^*, then there exists a unifier $\widehat{\theta}$ of $F(\varphi) = F(\psi)$ in Ψ^* compatible (through T) with θ.

According to Definition 17, to verify that $F : \Sigma \to \Omega$ is strongly unifiable respect to T, we must go over all the pairs of formulas $\varphi, \psi \in \Sigma$ checking that every idempotent mgu of the equation $\varphi = \psi$ satisfies conditions 1 and 2 above. However, the following result, which follows directly from Lemma 10 and Proposition 12, shows that for each equation $\varphi = \psi$, it is enough to find just one idempotent mgu satisfying conditions 1 and 2.

Corollary 14. *Let $T : \Pi \to \Psi$ be an extensible term translation. A formula translation $F : \Sigma \to \Omega$ is strongly unifiable (respect to T) iff every pair of formulas $\varphi, \psi \in \Sigma$ satisfies one of the two following conditions:*

- *There are no unifiers of equation $\varphi = \psi$.*
- *There exists an idempotent mgu θ of equation $\varphi = \psi$ which satisfies:*
  1. *If θ is not in Π^*, then $\theta \bullet \alpha$ is not in Π^* for every substitution α.*
  2. *If θ is in Π^*, then there exists a unifier $\widehat{\theta}$ in Ψ^* of $F(\varphi) = F(\psi)$ compatible (through T) with θ.*

Let $F : \Sigma \to \Omega$ be a formula translation which is strongly unifiable respect to a term translation T. Let $\varphi, \psi \in \Sigma$ such that $Unif(\varphi = \psi) \neq \emptyset$ and let θ be an mgu of such set. Observe that there are two cases:

- If θ is not in Π^*: Since every substitution α makes the substitution $\theta \bullet \alpha$ being not in Π^*, there are no minimal substitutions (for Π) respect to θ.
- If θ is in Π^*: From the definition of minimal substitutions, the only minimal substitutions (for Π) respect to θ are substitutions of variables $\phi : V \to V$.

From this observation follows that any formula translation $F : \Sigma \to \Omega$ which is strongly unifiable respect to a term translation T, is unifiable respect to T. We can then state two new theorems, which are versions of Theorem 12 and Theorem 13.

Theorem 15. *Let $T : \Pi \to \Psi$ be an extensible term translation. Let $F : \Sigma \to \Omega$ be a formula translation with Σ disjoint. If $F : \Sigma \to \Omega$ is strongly unifiable respect to T, then the mapping $L : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ given by*

$$L(\varphi\rho) = F(\varphi)(T^* \circ \rho),$$

is well defined.

Proof. If $F : \Sigma \to \Omega$ is strongly unifiable respect to T, then it is unifiable respect to T. Therefore, this theorem is a direct consequence of Theorem 12. □

Theorem 16. *Let $T : \Pi \to \Psi$ be an extensible term translation. Let $F : \Sigma \to \Omega$ be a formula translation with Σ finite. If the disjoint formula translation $F' : \Sigma' \to \Omega'$ is strongly unifiable respect to T, then the mapping $L : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ given by*

$$L(\varphi\rho) = F(\varphi)(T^* \circ \rho),$$

is well defined.

Proof. If $F' : \Pi' \to \Psi'$ is strongly unifiable respect to T, then it is unifiable respect to T. Therefore, this theorem is a direct consequence of Theorem 13. □

3.2 Extending Formula Translations by Formula Morphisms

In this section we look for conditions which enable us to extend formula translations $F : \Sigma \to \Omega$ by mappings wich preserve the structure of the logical relations ocurring in formulas. First, observe that $Form(\mathscr{L}^S)$ can be seen as an algebra whose signature is

$$\mathscr{F} = \{\vee, \wedge, \neg\} \tag{3.2}$$

where $\wedge$ and $\vee$ are binary function symbols and $\neg$ is a unary function symbol. Since formulas are elements of this algebra, we shall represent formulas by using an adaptation of the graph-based technique to represent terms, which was introduced in section 2.4.

3.2.1 Formula Morphisms

We begin this section by presenting some basic definitions which are necessary to express the problem addressed in this section:

Definition 18. Let $\Sigma \subseteq Form(\mathscr{L}^S)$ be a nonempty set of formulas, the set $\Sigma^+ \subseteq Form(\mathscr{L}^S)$ is defined as the minimal set which satisfies:

1. if $\varphi \in \Sigma$, then $\varphi \in \Sigma^+$.
2. if $\varphi \in \Sigma^+$, then $\neg\varphi \in \Sigma^+$.
3. if $\varphi, \psi \in \Sigma^+$, then $\varphi \wedge \psi \in \Sigma^+$.
4. if $\varphi, \psi \in \Sigma^+$, then $\varphi \vee \psi \in \Sigma^+$.

Definition 19. A formula translation $F^+ : \Sigma^+ \to \Omega^+$ is called a *formula morphism* when satisfies the following conditions:

- If $\varphi, \psi \in \Sigma^+$, then $F^+(\varphi \wedge \psi) = F^+(\varphi) \wedge F^+(\psi)$.
- If $\varphi, \psi \in \Sigma^+$, then $F^+(\varphi \vee \psi) = F^+(\varphi) \vee F^+(\psi)$.
- If $\varphi \in \Sigma^+$, then $F^+(\neg\varphi) = \neg F^+(\varphi)$.

Let $F : \Sigma \to \Omega$ be a formula translation. We want to study if it is possible to extend F to a formula morphism $F^+ : \Sigma^+ \to \Omega^+$. Next example shows that such extension can not always be achieved.

Example 8. Let us consider two signatures, namely $L_1 = \{R, G\}$ and $L_2 = \{\mathscr{R}, \mathscr{G}\}$ where all symbols are unary relation symbols. Set $\Sigma = \{G(x), R(y), G(x) \vee R(y)\}$, $\Omega = \{\mathscr{G}(x), \mathscr{R}(y), \mathscr{G}(x) \wedge \mathscr{R}(y)\}$ and let $F : \Sigma \to \Omega$ be the only definable formula translation between Σ and Ω. Assume that there were a formula morphism $F^+ : \Sigma^+ \to \Omega^+$ extending F. Such F^+ must satisfy $F^+(G(x) \vee R(y)) \equiv \mathscr{G}(x) \vee \mathscr{R}(y)$ since it is a formula morphism and $F^+(G(x) \vee R(y)) \equiv \mathscr{G}(x) \wedge \mathscr{R}(y)$ because F^+ extends F. This two conditions are impossible to satisfy and then F^+ does not exists.

The above example emphasizes the issue: F does not take care of the fact that Σ^+ allows two different ways of building the formula $G(x) \vee R(y)$. Next section introduces a method which enables us to characterize those formula translation that can be extended to formula morphisms.

3.2.2 Trees Over a Formula Set and Characterization Lemma.

In order to handle different ways of building a formula $\varphi \in \Sigma^+$, we introduce a graph-based representation of formulas, analogous to the representation introduced for terms in Chapter 2, Section 2.4. Recall that for a directed graph $G = (N,E)$ we denote the arc which begins at node v and ends at node u by vu, the set $\delta(v) = \{vu \in E \mid u \in N\}$ is the set of arcs which begin at node v and the definition of the signature

$$\mathscr{F} = \{\vee, \wedge, \neg\}$$

Definition 20. Let $\Sigma \subseteq Form(\mathscr{L}^S)$ be a non empty set. A *tree over* Σ is a 3-tuple (G, ν, e) where $G = (N,E)$ is a rooted directed tree and $\nu : N \to \Sigma \cup \mathscr{F}$, $e : E \to \{1,2\}$ are two mappings such that for any node $v \in N$:

- v is a leaf if and only if $\nu(v)$ is a formula $\varphi \in \Sigma$.
- If v is not a leaf, then there are two exclusive alternatives for $\nu(v)$:
  1. if $\nu(v) = \neg$, then there is exactly one arc vu and $e(vu) = 1$.
  2. if $\nu(v) = \wedge$ or $\nu(v) = \vee$, then there are exactly two arcs steaming from v, namely vu_1 and vu_2 such that $e(vu_1) = 1$ and $e(vu_2) = 2$.

Let (G, ν, e) be a formula tree over Σ. We call the *label* of the node v to $\nu(v)$ and the *order* of the arc uv to $e(uv)$ (see Figure 3.1). With this new terminology, a tree over Σ is a directed tree (N,E) with root r, labeled nodes and ordered arcs such that every node satisfies: if the node is a leaf, then its label is a formula in Σ. Otherwise, the number of childrens of the node is equal to the arity of its label (which is a logical operation). Observe that the set of arcs steming from each node v is an ordered set and then, two trees wich differ only in the ordering of $\delta(v)$ for some v, are different. Figure 3.1 shows an example: the tree depicted in (c) is different from the tree depicted in (d).

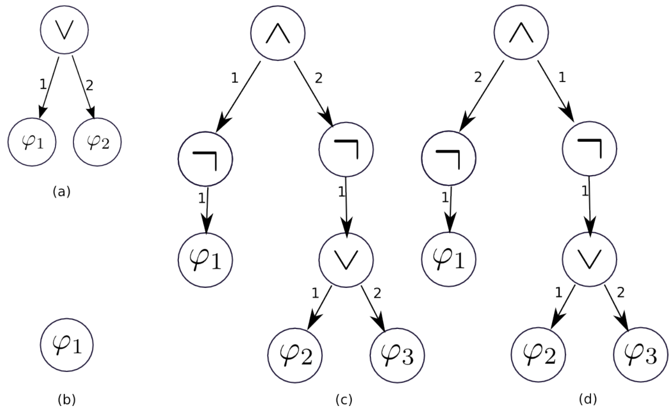

Figure 3.1: Different formula trees, where the tree depicted in (b) is atomic.

When Σ is clear from the context, we shall say "a tree" instead of "a tree over Σ". Because we shall use induction for some proofs, we introduce the *length* of a tree as the cardinality of the set of arcs of its longest directed path. A tree which has only one node (the root node) is called *atomic*, otherwise it is called *proper*. When working with proper trees we shall adopt a self exaplanatory notation that resembles the algebra of formulas: Let a be a proper tree whose root node r is labeled by a connective $c \in \mathscr{F}$. We shall write a_i to denote the subtree of a whose root is the i-th children of r. Then, depending on the logical connective c we shall write $\neg a_1$ or $a_1 \wedge a_2$ or $a_1 \vee a_2$ instead of a.

We define the set Σ° as the set of all trees over Σ. For a given a formula translation $F : \Sigma \to \Omega$, we define the mapping $F^\circ : \Sigma^\circ \to \Omega^\circ$ which acts in a tree $a = (G, \nu, e) \in \Sigma^\circ$ as follows:

$$F^\circ(a) = (G, F \circ \nu, e) \in \Omega^\circ$$

where we have considered a mapping $F : \Sigma \cup \mathscr{F} \to \Omega \cup \mathscr{F}$ extended to act as the identity on the symbols belonging to $\mathscr{F}$. This definition means that F° acts on a tree a by simply looking at its leaves and reeplacing their labels $\varphi \in \Sigma$ by $F(\varphi) \in \Omega$. For example, consider $\Sigma = \{\varphi_1, \varphi_2, \varphi_3\}$, $\Omega = \{\psi_1, \psi_2, \psi_3\}$ and let $F : \Sigma \to \Omega$ be the formula translation determined by $F(\varphi_i) = \psi_i$ for $i \in \{1, 2, 3\}$. The values of F° applied on the trees depicted in Figure 3.1 are drawn below in Figure 3.2.

Some straightforward properties of the mapping F° are described in the following proposition:

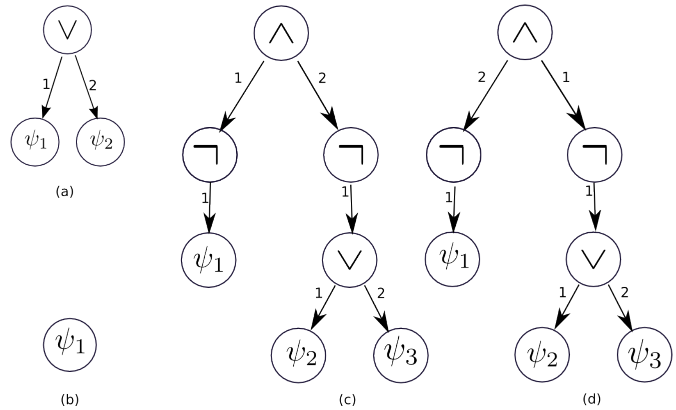

Figure 3.2: The mapping F° applied to trees depicted in Figure 3.1.

Proposition 13. *Let $F : \Sigma \to \Omega$ be a formula translation. The mapping $F^\circ : \Sigma^\circ \to \Omega^\circ$ satisfies the following properties:*

- $F^\circ(\neg a_1) = \neg F^\circ(a_1)$.
- $F^\circ(a_1 \wedge a_2) = F^\circ(a_1) \wedge F^\circ(a_2)$.
- $F^\circ(a_1 \vee a_2) = F^\circ(a_1) \vee F^\circ(a_2)$.

Now we introduce a way to associate to every tree $a \in \Sigma^\circ$, a formula $\mu(a) \in \Sigma^+$. Let us consider the mapping $\mu : \Sigma^\circ \to \Sigma^+$, defined in the following inductive way:

- If a is atomic, then $\mu(a) = \nu(r)$.
- If $a = \neg a_1$, then $\mu(a) = \neg\mu(a_1)$.
- If $a = a_1 \wedge a_2$, then $\mu(a) = \mu(a_1) \wedge \mu(a_2)$.
- If $a = a_1 \vee a_2$, then $\mu(a) = \mu(a_1) \vee \mu(a_2)$.

Take for example the trees in Figure (3.1). The mapping μ associates the following formulas to these trees: $\varphi_1 \vee \varphi_2$ for (a), φ_1 for (b), $\neg\varphi_1 \wedge \neg(\varphi_2 \vee \varphi_3)$ for (c) and $\neg(\varphi_2 \vee \varphi_3) \wedge \neg\varphi_1$ for (d). It can be shown by using an inductive argument that $\mu : \Sigma^\circ \to \Sigma^+$ is a surjective mapping and then, for any formula $\varphi \in \Sigma^+$ there exists at least one tree $a \in \Sigma^\circ$ such that $\mu(a) \equiv \varphi$. In such case, we say that a is a *lifting* of φ.

Lemma 17 (Characterization Lemma)**.** *Let $F : \Sigma \to \Omega$ be a formula translation. A formula translation $F^+ : \Sigma^+ \to \Omega^+$ is a formula morphism extending F if and only if $F^+ \circ \mu = \mu \circ F^\circ$ i.e., the following diagram commutes:*

$$\begin{tikzcd} \Sigma^\circ \arrow[r, "F^\circ"] \arrow[d, "\mu"] & \Omega^\circ \arrow[d, "\mu"] \\ \Sigma^+ \arrow[r, "F^+"] & \Omega^+ \end{tikzcd}$$

Proof. Let us assume that F^+ is a formula morphism which extends F. We must show that for any $a \in \Sigma^\circ$, $F^+ \circ \mu(a) \equiv \mu \circ F^\circ(a)$. To this aim, we will use induction on the length of a. The base step is proved for $n = length(a) = 0$. In this case a and $F^\circ(a)$ are atomic and then $\mu(a) \in \Sigma$ and $\mu(F^\circ(a)) \in \Omega$. As F^+ extends F, $\mu(F^\circ(a)) \equiv F(\mu(a)) \equiv F^+(\mu(a))$. The induction step is done by showing that for any tree a with $n = length(a) > 0$, $F^+ \circ \mu(a) = \mu \circ F^\circ(a)$, assuming that last equation holds for any tree b such that $length(b) < n$. Consider the three possible alternatives:

1. $a = \neg a_1$.
2. $a = a_1 \wedge a_2$.
3. $a = a_1 \vee a_2$.

We shall analize only the first case, since the other alternatives are handled in an analogous way. Let us assume that $a = \neg a_1$. Since $length(a_1) < length(a)$, by applying the induction hypothesis and since F^+ is a formula morphism, we obtain that :

$$\begin{aligned} F^+(\mu(a)) &\equiv F^+(\neg\mu(a_1)) \equiv \neg F^+(\mu(a_1)) \equiv \neg\mu(F^\circ(a_1)) \equiv \mu(\neg F^\circ(a_1)) \equiv \mu(F^\circ(\neg a_1)) \\ &\equiv \mu(F^\circ(a)). \end{aligned}$$

From the proof of 1 and the analog proofs of 2 and 3, we conclude that for every $a \in \Sigma^\circ$, $\mu(F^\circ(a)) \equiv F^+(\mu(a))$ holds.

Conversely, let us assume now that the term translation $F^+ : \Sigma^+ \to \Omega^+$ makes the above diagram commute. That F^+ extends F follows from the equalities $F^+ \circ \mu(a) = \mu \circ F^\circ(a)$ for every atomic tree a. Let us show that F^+ is a formula morphism. To this aim, let us take $\psi, \xi \in \Sigma^+$ with liftings a_ψ and a_ξ respectively. The following computation (together with identical ones for the rest of the connectives) shows that F^+ is a formula morphism:

$$\begin{aligned} F^+(\psi \wedge \xi) &= F^+(\mu(a_\psi \wedge a_\xi)) = \mu(F^\circ(a_\psi \wedge a_\xi)) = \mu(F^\circ(a_\psi)) \wedge \mu(F^\circ(a_\xi)) \\ &= F^+(\psi) \wedge F^+(\xi). \end{aligned}$$

□

Last lemma (Lemma 17) together with the fact that μ is surjective, show that when the formula morphism $F^+ : \Sigma^+ \to \Omega^+$ extending F exists, it must be unique.

3.2.3 Characterization Theorems

Let "$\sim$" be the equivalence relation on Σ° determined by the fibers of the mapping μ. That is, for $a, b \in \Sigma^\circ$, $a \sim b$ if and only if $\mu(a) \equiv \mu(b)$. Let us denote by $\overline{a}$ the equivalence class of a and by $\Sigma^\circ/\sim$, $\pi : \Sigma^\circ \to \Sigma^\circ/\sim$ the quotient set of Σ° and its associated projection mapping respectively. Note that since μ is a surjective mapping, there is a natural bijection $i : \Sigma^\circ/\sim \to \Sigma^+$.

Theorem 18. *Let $F : \Sigma \to \Omega$ be a formula translation. Next two conditions are equivalent:*

- *For every $a, b \in \Sigma^\circ$, $a \sim b$ implies $F^\circ(a) \sim F^\circ(b)$.*
- *There exists a formula morphism $F^+ : \Sigma^+ \to \Omega^+$ extending F.*

Proof. Let us assume that for every $a, b \in \Sigma^\circ$, $a \sim b$ implies $F^\circ(a) \sim F^\circ(b)$. Then, $F' : \Sigma^\circ/\sim \to \Omega^\circ/\sim$, defined by:

$$F'(\overline{a}) = \overline{F^\circ(a)} \tag{3.3}$$

is a well defined mapping which makes following diagram commute:

$$\begin{tikzcd}
\Sigma^+ & \Sigma^\circ \arrow[l, "\mu"'] \arrow[r, "F^\circ"] \arrow[d, "\pi"] & \Omega^\circ \arrow[d, "\pi"] \arrow[dr, "\mu"] & \\
& \Sigma^\circ/\sim \arrow[ul, "i"] \arrow[r, "F'"] & \Omega^\circ/\sim \arrow[r, "i"] & \Omega^+
\end{tikzcd}$$

Let us take $F^+ = i \circ F' \circ i^{-1}$. Then, by Lemma 17, F^+ is a formula morphism which extends F.

Conversely, assume that there exists a formula morphism $F^+ : \Sigma^+ \to \Omega^+$ extending F. Let us take $a, b \in \Sigma^\circ$ such that $a \sim b$. Therefore $\mu(a) \equiv \mu(b)$, which implies that $F^+(\mu(a)) \equiv F^+(\mu(b))$. Applying Lemma 17, we obtain that $\mu(F^\circ(a)) \equiv \mu(F^\circ(b))$. Therefore, $F^\circ(a) \sim F^\circ(b)$. □

The conditions given in theorem 18 are hard to verify because they must be checked in every pair of trees of the infinite set Σ°. For the case of terms in chapter 2, we used unification theory in order to work out the respective conditions and express them in a way easier to verify. Unfortunately, we know of no analogous theory to apply in the case of formulas. Nevertheless, we can borrow some ideas from unification theory and apply them here. Let us consider an algorithm which is an adaptation of the well known Unification Algorithm for terms. From now on, we shall denote by a_φ a tree $a \in \Sigma^\circ$ which is atomic and whose only label is $\varphi \in \Sigma$. Notice that $b_\varphi, p_\varphi, q_\varphi$ will have the same meaning.

Adapted Unification Algorithm for Formula Trees

The input for this algorithm is a pair of trees $a, b \in \Sigma^\circ$. During its execution, the algorithm keeps a set of pairs of trees E, initially set $E = \{(a,b)\}$. The algorithm chooses any pair of trees from the set E to wich a numbered step from below applies (the action taken by the algorithm is determined by the form of the pair of trees selected):

1.a	$(c(a_1,a_2), c(b_1,b_2))$	Replace the pair in E by the pairs $(a_1,b_1),(a_2,b_2)$.
1.b	$(c(a_1), c(b_1))$	Replace the pair in E by the pair (a_1,b_1).
2.	$(c_1(a), c_2(b))$	where $c_1 \neq c_2$. Halt with failure.
3.	(a, b_φ)	where a is non atomic. Replace by the pair (b_φ, a).
4.	(a_φ, b_ψ)	where $\varphi \not\equiv \psi$. Halt with failure.

where $c, c_1, c_2 \in \mathscr{F}$ are logical connectives. Notice that the first step has options (a) and (b) for cases where the connectives are unary and binary respectively. The algorithm ends when failure has been returned or when no step can be applied. In the latter case, the algorithm returns the set E as output.

It is easy to see that the algorithm ends: An application of the step 1 diminishes strictly the total number of nodes present in the trees in E. Additionally, step 3 can be applied consecutively just to a finite number of pairs of nodes in E before applying a step of another kind. Therefore, for any input, after a finite number of times of applying (1) or (3), the algorithm either fails or returns the set E as output.

We say that a set of pairs of trees $E = \{(a_1,b_1),...,(a_n,b_n)\}$ is *adequate* when $a_i \sim b_i$, for all $i \in \{1,..,n\}$. In order to prove Lemma 19 below, we shall need the following straightforward proposition.

Proposition 14. *The next assertions hold:*

1. $a_1 \wedge a_2 \sim b_1 \wedge b_2$ *if and only if* $a_1 \sim b_1$ *and* $a_2 \sim b_2$.
2. $a_1 \vee a_2 \sim b_1 \vee b_2$ *if and only if* $a_1 \sim b_1$ *and* $a_2 \sim b_2$.
3. $\neg a_1 \sim \neg b_1$ *if and only if* $a_1 \sim b_1$.

Lemma 19. *Let $a,b \in \Sigma^\circ$ be the input for the Adapted Unification Algorithm. If the algorithm fails, then $a \nsim b$. Otherwise, the output D is adequate if and only if $a \sim b$.*

Proof. It is clear that the algorithm fails, only if $a \nsim b$. Let us assume that the adapted unification algorithm does not fail and outputs a set D of pairs of trees. At each application of steps 1 or 3, the algorithm replaces its set E of pairs of trees by a new set E' of pairs of trees. From proposition 14 follows directly that E is adequate if and only if E' is adequate. Since initially $E = \{(a,b)\}$, we conclude that $a \sim b$ if and only if the output of the algorithm is adequate. □

Next theorem yields a characterization which is one of the goals for this section. Before presenting the theorem, let us consider the set all liftings of formulas in Σ given by

$$\mu^{-1}(\Sigma) = \{a \in \Sigma^\circ \mid \mu(a) \in \Sigma\}. \tag{3.4}$$

Theorem 20 (First Characterization Theorem)**.** *Let $F : \Sigma \to \Omega$ be a formula translation. The following two assertions are equivalent:*

1. *For every pair of trees $a,b \in \mu^{-1}(\Sigma)$, $a \sim b$ implies $F^\circ(a) \sim F^\circ(b)$.*

2. *There exists a formula morphism $F^+ : \Sigma^+ \to \Omega^+$ extending F.*

Proof. Because of Theorem 18, we can ensure that assertion 2 is equivalent to

3. For every $c, d \in \Sigma^\circ$, $c \sim d$ implies $F^\circ(c) \sim F^\circ(d)$.

Since $\mu^{-1}(\Sigma) \subseteq \Sigma^\circ$, it is clear that 2 implies 1.

Instead of proving $1 \Rightarrow 2$, we are going to prove that $1 \Rightarrow 3$. Assume that assertion 1 holds. Let us take $c, d \in \Sigma^\circ$ such that $c \sim d$. We must show that $F^\circ(c) \sim F^\circ(d)$. Since $c \sim d$, if we apply the Adapted Unification Algorithm to (c, d), the algorithm must return an adequate set of pairs of trees E with form $E = \{(p_{\varphi_1}, p_1), ..., (p_{\varphi_k}, p_k)\}$ (otherwise, the steps 1 or 3 of the algorithm could be further applied). Crucially, observe that the output of the Adapted Unification Algorithm depends exclusively on the graph structure and the labels of the non-leaf nodes of the input. Additionally, observe that the trees c and $F^\circ(c)$ are isomorphic as directed trees and they have the same labels in the non-leaf nodes (and the same is true for trees d and $F^\circ(d)$). Therefore, if we apply the Adapted Unification Algorithm to the pair $(F^\circ(c), F^\circ(d))$, the algorithm must not fail and it must return a set of pairs of trees $E' = \{(q_{F(\varphi_1)}, F^\circ(p_1)), ..., (q_{F(\varphi_k)}, F^\circ(p_k))\}$. Since E is adequate, we can use assertion 1 to conclude that $q_{F(\varphi_i)} \sim F^\circ(p_i)$ for every $i \in \{1, .., k\}$. Therefore, E' is adequate, and then $F^\circ(c) \sim F^\circ(d)$. $\square$

Formula Morphisms and Substitutions of Terms

Let $T : \Pi \to \Psi$ be an extensible term translation. We shall say that a formula translation $F : \Sigma \to \Omega$ is *extensible* respect to T when the formula translation $L : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ given by $L(\varphi\rho) = F(\varphi)T^* \circ \rho$ is well defined. In other words, formula translations $F : \Sigma \to \Omega$ which are extensible respect to T are those characterized by Theorem 12 and Theorem 13. Additionally, Theorem 15 and Theorem 16 have provided conditions that guarantee that $F : \Sigma \to \Omega$ is extensible respect to T.

Let $T : \Pi \to \Psi$ be an extensible term translation and $F : \Sigma \to \Omega$ a formula translation which is extensible respect to T. Let $L : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ be the formula translation defined by

$$L(\varphi\rho) = F(\varphi)T^* \circ \rho. \tag{3.5}$$

The aim of this section is to find conditions ensuring that L can be extended to a formula morphism $L^+ : (\Sigma\otimes\Pi)^+ \to (\Omega\otimes\Psi)^+$. Observe that Theorem 20 gives one of such conditions: For every pair of trees $a,b \in \mu^{-1}(\Sigma\otimes\Pi)$, if $a \sim b$, then $L^\circ(a) \sim L^\circ(b)$. Notice that such condition seems hard to verify because the definition of $\Sigma\otimes\Pi$ makes $\mu^{-1}(\Sigma\otimes\Pi)$ an infinite set even when Σ is finite. At the end of this section, we shall present Theorem 24 which gives conditions ensuring that L can be extended to a formula morphism $L^+ : (\Sigma\otimes\Pi)^+ \to (\Omega\otimes\Psi)^+$ and those conditions are described in terms of the set Σ°. Additionally, conditions in Theorem 24 require checking a finite number (exponential though) of conditions in the case when Σ is finite.

From now on, we give a rather arbitrary but fixed order to the leaves of any formula tree. Given any formula tree, consider its leaves with the order determined by the visit of its nodes in a Depth First Traversal fashion ruled by the pattern: visit the root, then follow the arc with order 1 and finally follow the arc with order 2, if it exists. This is just the way we have chosen to identify each leaf of a given tree with a number. Additionally, we assume that the set V of variables is an ordered set.

Definition 21. Let $b \in (\Sigma\otimes\Pi)^\circ$ be a formula tree with length n and let β be a substitution in Π^*. The formula tree denoted by $b\beta$ is the formula tree determined by the next rule which is inductive on the length of b.

- If b is atomic and φ is its only label, then $b\beta$ is the atomic tree whose only label is $\varphi\beta$.
- If $b = \neg a$, then $b\beta = \neg(a\beta)$.
- If $b = a_1 \wedge a_2$, then $b\beta = (a_1\beta) \wedge (a_2\beta)$.
- If $b = a_1 \vee a_2$, then $b\beta = (a_1\beta) \vee (a_2\beta)$.

Next straightforward lemma follows directly from the definition of $b\beta$.

Lemma 21. *Let $b \in (\Sigma\otimes\Pi)^\circ$ and let β be a substitution in Π^*. Then,*

$$\mu(b\beta) \equiv \mu(b)\beta$$

Proof. We are going to proof this lemma by induction on n which is the length of b. The base step is done by showing the result for $n = 0$. If $n = 0$, b is an atomic three and it has only one label, namely φ. It is clear that $\mu(b)\beta \equiv \varphi\beta \equiv \mu(b\beta)$.

The induction step is done by showing that our result holds for a tree of length n by assuming that the lemma is true for any tree of lesser length. There are three cases, but we handle just two of them because the other one is totally analogous.

First case: Assume that $b = \neg a$. Notice that $\mu(b)\beta \equiv \mu(\neg a)\beta \equiv \neg\mu(a)\beta$. Because the length of a is less than the length of b, the induction hypotheses ensures us that $\mu(a)\beta \equiv \mu(a\beta)$. Therefore, the result follows from the next computations: $\mu(b)\beta \equiv \neg\mu(a)\beta \equiv \neg\mu(a\beta) \equiv \mu(\neg(a\beta)) \equiv \mu(b\beta)$.

Second case: $b = a_1 \wedge a_2$. Notice that $\mu(b)\beta \equiv \mu(a_1 \wedge a_2)\beta = \mu(a_1)\beta \wedge \mu(a_2)\beta$. The induction hypotheses applied on a_1 and a_2 ensures us that $\mu(a_1)\beta = \mu(a_1\beta)$ and $\mu(a_2)\beta = \mu(a_2\beta)$. Then, the following computations hold: $\mu(b)\beta \equiv \mu(a_1)\beta \wedge \mu(a_2)\beta \equiv \mu(a_1\beta) \wedge \mu(a_2\beta) \equiv \mu(a_1\beta \wedge a_2\beta) = \mu(b\beta)$. The case when $b = a_1 \vee a_2$ is handled in the exact same way. □

For any tree $b \in \Sigma^\circ$ with leaves $\{1,2,...,n\}$, we are going to associate a tree $b^* \in (\Sigma \otimes \Pi)^\circ$ and a family of n substitutions $\{\phi_i\}_{i=1}^n$ in the following way: Let $\varphi_1,...,\varphi_n \in \Sigma$ be the n labels on the n leaves of b. Let us pick $k \in \{1,2,...,n\}$ and let let $x_1,...,x_m$ be the exact m free variables of the label φ_k. It is easy to show that there is exactly one substitution ϕ_k which satisfies the following three conditions: (1) it makes $\phi_k(x_i) = z_i$, where $\{z_1,...,z_m\}$ are the first m variables in V which satisfy: for any $i \in \{1,...,m\}$, the variable z_i does not occur free in any formula in Σ neither it occurs free in any formula $\varphi_j\phi_j$ with $j < k$, (2) it is invertible (and thus $\phi_k^{-1} = \phi_k$), and (3) its domain has minimal cardinality. We write b^* to denote the tree which results of reeplacing the label φ_i of the tree b by the label $\varphi_i\phi_i$, for $i = 1,2,....,n$. We say that b^* is the *twin* of b and the substitutions in $\{\phi_j\}_{j=1}^n$ are the *substitutions of the twin* b^*.

Notice that b and b^* are isomorphic as directed trees and that they both have the exact same labels in the non-leaf nodes. Crucially, observe that the labels $\varphi_1\phi_1, \varphi_2\phi_2, ..., \varphi_n\phi_n$ on the leaves of b^* satisfy $V(\varphi_i\phi_i) \cap V(\varphi_j\phi_j) = \emptyset$, when $i \neq j$. That is, the leaves of b^* have labels on them whose sets of variables are mutually disjoint.

Now, let us associate to the formula tree b the substitution $\beta : V \to V$ defined by

$$\beta(x) = \begin{cases} \phi_1^{-1}(x), & \text{if } x \in V(\varphi_1\phi_1), \\ \phi_2^{-1}(x), & \text{if } x \in V(\varphi_2\phi_2), \\ \ldots & \ldots \\ \phi_n^{-1}(x), & \text{if } x \in V(\varphi_n\phi_n), \\ x, & \text{otherwise.} \end{cases} \tag{3.6}$$

and observe that the substitution β is well defined since $V(\varphi_i\phi_i) \cap V(\varphi_j\phi_j) = \emptyset$ when $i \neq j$. Next proposition follows directly from the definitions in the preceding paragraphs.

Proposition 15. *Let $b \in \Sigma^\circ$ be a tree with labels $\varphi_1, \varphi_2, \ldots, \varphi_n$. Let $\{\phi_i\}_{i=1}^n$ be the substitutions of the twin b^* and let β be the substitution defined by 3.6. Then, $b^*\beta = b$ and $\mu(b^*)\beta \equiv \mu(b)$.*

Proof. Let us show that $b^*\beta = b$ by using an inductive argument. Notice that b and b^* are isomorphic as directed trees and that there exists only one of such isomorphisms. By using this isomorphism, any subtree a of b determines uniquely a subtree $a^\star$ of b^*. We are going to prove that for any subtree a of b, $a^\star\beta = a$. In particular, it will show that $b^*\beta = b$.

We are going to use induction on the length of a. The base step is done for the case when a is a leaf of b. In such case, $a^\star$ is a leaf of b^* and let φ be the only label on a. It means that $a^\star$ has the label $\varphi\phi_1$, therefore $a^\star\beta$ has the label $\varphi\phi_1\beta$. From the definition of β it follows that $\varphi\phi_1\beta \equiv \varphi$. This shows that $a^\star\beta = a$.

The induction step is done when is shown that $a^\star\beta = a$ holds for a subtree a whose length is $n > 0$ by assuming that the proposition holds for any subtree of b with lesser length. There are three cases, but we shall address just two of them because the other one is totally analogous.

First case: $a = \neg a_1$. In this case, it is clear that $a^\star = \neg a_1^\star$ where a_1 is a subtree of a. Thus, the length of a_1 is lesser than n. It is clear that $a^\star\beta = (\neg a_1^\star)\beta = \neg(a_1^\star\beta)$ and then by induction hypotheses, we obtain that $\neg(a_1^\star\beta) = \neg(a_1) = a$. This means that $a^\star\beta = a$.

Second case: $a = a_1 \wedge a_2$. In this case, it is clear that $a^\star = a_1^\star \wedge a_2^\star$. Notice that a_1, a_2 are subtrees of a and then both of them have a length lesser than n. Observe that $a^\star\beta = (a_1^\star \wedge a_2^\star)\beta =$

$(a_1^\star\beta) \wedge (a_2^\star\beta)$ and then by induction hypotheses, we conclude that $(a_1^\star\beta) \wedge (a_2^\star\beta) = a_1 \wedge a_2 = a$. We have shown that $a^\star\beta = a$. The case $a = a_1 \vee a_2$ is handled in the exact same way. In particular when $a = b$, $a^\star = b^*$. This observation shows that $b^*\beta = b$ and an application of Lemma 21 shows that $\mu(b^*)\beta \equiv \mu(b)$ holds. □

Because of this last proposition, which ensures that $b = b^*\beta$ and $\mu(b) = \mu(b^*)\beta$, we say that the pair (b^*, β) is the *representation of* b.

Proposition 16. *Let $T : \Pi \to \Psi$ be an extensible term translation and let $F : \Sigma \to \Omega$ be a formula translation. Assume that $L : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ is well defined by $L(\varphi\rho) \equiv F(\varphi)T^* \circ \rho$. Let $b \in \Sigma^\circ$ and (b^*, β) its representation. For any substitution γ in Π^*, $L^\circ(b\gamma) = (L^\circ(b^*)\beta)\ T^* \circ \gamma$.*

Proof. Let us assume that b has m leafs and that $\{\phi_i\}_{i=1}^m$ are the substitutions of the twin b^*. Let us show this lemma by using an inductive argument. Notice that the trees $L^\circ(b)$ and $L^\circ(b^*)$ are isomorphic as trees and there exists only one isomorphism between these trees. We associate to every subtree a of b the subtree $a^\star$ of b^* which is determined by such isomorphism. We are going to show that for every subtree a of b and the associated subtree $a^\star$ of b^* it holds that $L^\circ(a\gamma) = (L^\circ(a^\star)\beta)\ T^* \circ \gamma$. In particular, this will show that $L^\circ(b\gamma) = (L^\circ(b^*)\beta)\ T^* \circ \gamma$.

We are going to use induction on the length of the subtree a. When the length of a is zero, it is a leaf. Let us assume that a is a leaf whose label is a formula $\varphi_1 \in \Sigma$. Then, the only label of $L^\circ(a\gamma)$ is $F(\varphi_1)T^* \circ \gamma$. On the other hand, the only label of $a^\star$ is the formula $\varphi_1\phi_1$ and then, the unique label of $L^\circ(a^\star)$ is $L(\varphi_1\phi_1) \equiv F(\varphi_1)T^* \circ \phi_1 \equiv F(\varphi_1)\phi_1$ because T^* acts as the identity on variables. Since F preserves variables, it is clear that $(L^\circ(a^\star)\beta)\ T^* \circ \gamma$ has the formula $(F(\varphi_1)\phi_1\beta)\ T^* \circ \gamma \equiv F(\varphi_1)\ T^* \circ \gamma$ as its only label. We have shown that $L^\circ(a\gamma) = (L^\circ(a^\star)\beta)\ T^* \circ \gamma$.

The induction step is done by showing that $L^\circ(a\gamma) = (L^\circ(a^\star)\beta)\ T^* \circ \gamma$ holds when a is a subtree of b with length $n > 0$ and we assume that the equality holds for any subtree of b whose length is lesser than n. There are three cases depending on the root of a.

First case: Let $a = \neg a_1$ and consequently $a^\star = \neg a_1^\star$. The following computations use the fact that L° has the properties described in Proposition 13: $(L^\circ(a^\star)\beta)\ T^* \circ \gamma \equiv (L^\circ(\neg a_1^\star)\beta)\ T^* \circ \gamma \equiv \neg(L^\circ(a_1^\star)\beta)\ T^* \circ \gamma$. In this last expression, an application of the induction hypotheses yields $\neg(L^\circ(a_1^\star)\beta)\ T^* \circ \gamma \equiv \neg(L^\circ(a_1\gamma)) \equiv L^\circ(\neg a_1\gamma) \equiv L^\circ(a\gamma)$. We have shown that $L^\circ(a\gamma) \equiv (L^\circ(a^\star)\beta)\ T^* \circ \gamma$.

Second case: Let $a = a_1 \wedge a_2$ and consequently $a^\star = a_1^\star \wedge a_2^\star$. Proposition 13 enables us to perform the following computations: $(L^\circ(a^\star)\beta)\ T^* \circ \gamma = (L^\circ(a_1^\star \wedge a_2^\star)\beta)\ T^* \circ \gamma = (L^\circ(a_1^\star)\beta \wedge L^\circ(a_2^\star)\beta)\ T^* \circ \gamma = (L^\circ(a_1^\star)\beta)\ T^* \circ \gamma \wedge (L^\circ(a_2^\star)\beta)\ T^* \circ \gamma$. Applying the induction hypothesis, we obtain that last expression is equal to $L^\circ(a_1\gamma) \wedge L^\circ(a_2\gamma) = L^\circ(a_1\gamma \wedge a_2\gamma) = L^\circ(a\gamma)$. We have shown that $L^\circ(a\gamma) \equiv (L^\circ(a^\star)\beta)\ T^* \circ \gamma$.

Third case: This case assumes that $a = a_1 \vee a_2$ which is handled in the exact same way that the second case. □

Next, we define the set in which we shall be looking for conditions that Theorem 24 uses to characterize those formula translation L that can be extended to a formula morphism L^+. To this aim, let us consider the trees $b \in \Sigma^\circ$ whose twins b^* are such that there exists unifiers of the equation $\mu(b^*) = \varphi$ for some $\varphi \in \Sigma$. In other words, formally speaking, we define the next two sets:

$$\Delta_\Sigma = \{b \in \Sigma^\circ \mid \text{ exists } \varphi \in \Sigma \text{ such that the equation } \mu(b^*) = \varphi \text{ has unifiers }\},$$

$$\Delta_\Sigma^* = \{b^* \in (\Sigma \otimes \Pi)^\circ \mid b \in \Delta_\Sigma\}$$

and observe that there is a one to one correspondence between Δ_Σ and Δ_Σ^*. Additionally, when Σ is finite, these two sets are finite.

The next step to perform is to find a way to relate trees in Δ_Σ^* with some trees in $(\Omega \otimes \Psi)^\circ$. The ideal way to do that would be by using map $L^\circ : (\Sigma \otimes \Pi)^\circ \to (\Omega \otimes \Psi)$. However, under the assumption that $T : \Pi \to \Psi$ is a term translation and $F : \Sigma \to \Omega$ a formula translation, the mapping L might be not well defined and consequently the map L° would not be either. To avoid extra hypotheses in Theorem 24 (requesting L being well defined), in what follows we define a map $L^\star$ which is always well defined and shall enable us to achieve the sought result.

Let $T : \Pi \to \Psi$ be a term translation and $F : \Sigma \to \Omega$ a formula translation. Let $b \in \Delta_\Sigma$ with k leaves whose labels are $\varphi_1, \varphi_2, ..., \varphi_k$, let b^* be its twin and $\{\phi_j\}_{j=1}^k$ the *substitutions of the twin* b^*. Define the mapping $L^\star : \Delta_\Sigma^* \to (\Omega \otimes \Psi)^\circ$ in the following way: the tree $L^\star(b^*)$ is the result of reeplacing in each leaf of b the label φ_i by the label $F(\varphi_i)\phi_i$. Observe that b, b^* and $L^\star(b^*)$ are isomorphic as directed trees. Because F preserves variables, for every $i \in \{1, 2, ..., k\}$ the label $F(\varphi_i)\phi_i$ on the tree $L^\star(b^*)$ has the exact same set of variables as the label $\varphi_i\phi_i$ of the twin b^*. Finally, observe that sets of variables of the labels of $L^\star(b^*)$ are mutually disjoint.

Definition 22. Let $T : \Pi \to \Psi$ be a term translation and let $F : \Sigma \to \Omega$ be a formula translation. We say that F and T are *coupled* when for every $\varphi \in \Sigma$ and every $b \in \Delta_\Sigma$, if θ is an idempotent mgu of the equation $\varphi = \mu(b^*)$ and λ is a minimal substitution (for Π) respect to θ, then there exists an unifier $\widehat{\lambda}$ in Ψ^* of the equation $F(\varphi) = \mu(L^\star(b^*))$ which is compatible with $\theta \bullet \lambda$.

Lemma 22. *Let $T : \Pi \to \Psi$ be a term translation and $F : \Sigma \to \Omega$ a formula translation. If F and T are coupled, then F is unifiable respect to T.*

Proof. Let us assume that F and T are coupled. To show that F is unifiable respect to T, take $\varphi, \psi \in \Sigma$ and let θ be an mgu of $\varphi = \psi$ and let λ be a minimal substitution respect to θ. We must show that there exists a unifier $\widehat{\lambda}$ in Ψ^* of the equation $F(\varphi) = F(\psi)$ compatible (through T) with $\theta \bullet \lambda$. Let us take the atomic tree $b_\psi \in \Sigma^\circ$ which satisfies $\mu(b_\psi) \equiv \psi$ and let (b^*, β) be its representation. Then $b_\psi = b^*\beta$ and $\psi \equiv \mu(b_\psi) \equiv \mu(b^*)\beta$. Because θ is a unifier of $(\varphi = \psi)$, it follows that $\varphi\theta \equiv \psi\theta \equiv \mu(b^*)\beta \bullet \theta$. From the definition of the representation (b^*, β) it is clear that β does not affect those variables in $V(\varphi)$ and then $\varphi\theta \equiv \varphi(\beta \bullet \theta) \equiv \mu(b^*)(\beta \bullet \theta)$ which means that $\beta \bullet \theta$ is a unifier of the equation $\varphi = \mu(b^*)$. If $\beta \bullet \theta$ were not an mgu of $\varphi = \mu(b^*)$, then θ would not be an mgu of $\varphi = \psi$. In other words, $\beta \bullet \theta$ must be an mgu of $\varphi = \mu(b^*)$. Therefore, since F and T are coupled, there must exist a unifier $\widehat{\lambda}$ of $F(\varphi) = \mu(L^\star(b^*))$ which is compatible with $\beta \bullet \theta \bullet \lambda$ and then

$$F(\varphi)\widehat{\lambda} \equiv \mu(L^\star(b^*))\widehat{\lambda}.$$

Let us define a substitution γ in Ψ^* by $\gamma(x) = \widehat{\lambda}(x)$ if $x \in V(\varphi) \cup V(\psi)$ and $\gamma(x) = x$ otherwise. Observe that the intersection of the sets of free variables $V(F(\varphi))$ and $V(\mu(L^\star(b^*)))$ must be empty because of the definition of b^* and the fact that F preserves variables. Therefore,

$$F(\varphi)\widehat{\lambda} \equiv F(\varphi)\gamma. \tag{3.7}$$

On the other hand, because $\widehat{\lambda}$ is compatible (through T) with $\beta \bullet \theta \bullet \lambda$, it is not difficult to see that $\mu(L^\star(b^*))\widehat{\lambda} \equiv \mu(L^\star(b^*))\beta\ \gamma$. Additionally, observe that $\mu(L^\star(b^*))\beta \equiv F(\psi)$ and then $\mu(L^\star(b^*))\widehat{\lambda} \equiv F(\psi)\gamma$. From this and equality 3.7, we conclude that $F(\varphi)\gamma \equiv F(\psi)\gamma$ which means that γ is a unifier of the equation $F(\varphi) = F(\psi)$. Because $\widehat{\lambda}$ is compatible with $\beta \bullet \theta \bullet \lambda$, γ is compatible with $\theta \bullet \lambda$. We have shown that F is unifiable respect to T. $\square$

Notice that from last lemma together and an application of Theorem 12, it follows that when F and T are coupled, F is extensible respect to T and consequently the map $L : (\Sigma \otimes \Pi) \to (\Omega \otimes \Psi)$

is well defined by $L(\varphi\rho) \equiv F(\varphi)T^* \circ \rho$. In such case, it is easy to see that L° extends $L^\star$. We need one more tool before presenting Theorem 24 which is the last theorem of this section:

Second Adapted Unification Algorithm for Formula Trees

The input for this algorithm is a pair of trees $a, b \in (\Sigma \otimes \Pi)^\circ$. During its execution, the algorithm keeps a set of pairs of trees E, initially set $E = \{(a,b)\}$. The algorithm chooses any pair of trees from the set E to wich a numbered step from below applies (the action taken by the algorithm is determined by the form of the pair of trees selected):

1.a	$(a_1 \wedge a_2, b_1 \wedge b_2)$	Replace the pair in E by the pairs $(a_1,b_1),(a_2,b_2)$.
1.b	$(a_1 \vee a_2, b_1 \vee b_2)$	Replace the pair in E by the pairs $(a_1,b_1),(a_2,b_2)$.
2	$(\neg a_1, \neg b_1)$	Replace the pair in E by the pair (a_1,b_1).
3.	(a_1, b_φ)	where a_1 is non atomic. Replace by the pair (b_φ, a_1).
4.	(a_1, b_1)	if both a_1, b_1 are non-atomic and the labels on its roots are different connectives, then halt with failure.

where b_φ denotes an atomic tree whose only label is the formula φ. The algorithm ends when failure has been returned or when no step can be applied. In the latter case, the algorithm returns the set E as output.

We can conclude easily that the algorithm ends: Any application of the steps 1 or 2 diminishes strictly the total number of nodes in the trees belonging to E. Additionally, step 3 can be applied consecutively just a finite number of times to different pairs of nodes in E before applying another different step. Therefore, for any input, after a finite number of times of applying 1,2 or 3, the algorithm either fails or returns the set E as output. It is clear that if the algorithm fails, $\mu(a) \not\equiv \mu(b)$. On the other hand, the goal of the above algorithm is to give the next straightforward characterization for two trees $a, b \in (\Sigma \otimes \Pi)^\circ$ which satisfy $\mu(a) \equiv \mu(b)$.

Lemma 23. *Let $a, b \in \Sigma^\circ$ be the input for the Second Adapted Unification Algorithm. Assume that the algoritm does not fails and outputs the set of trees $E = \{(a_{\varphi_1}, b_1), (a_{\varphi_2}, b_2), ..., (a_{\varphi_m}, b_m)\}$. Then, $\mu(a) \equiv \mu(b)$ if and only if for every $i \in \{1,2,...,m\}, \mu(a_{\varphi_i}) \equiv \varphi_i \equiv \mu(b_i)$.*

Proof. If $\mu(a) \equiv \mu(b)$, it is clear that for every $i \in \{1,2,...,m\}, \mu(a_{\varphi_i}) \equiv \varphi_i \equiv \mu(b_i)$. The other direction follows directly from the definition of μ and a simple analysis made to each one of the steps defined in the Second Adapted Unification Algorithm. $\square$

Next, the main Theorem of this section.

Theorem 24 (Second Characterization Theorem)**.** *Let $T : \Pi \to \Psi$ be an extensible term translation and let $F : \Sigma \to \Omega$ be a formula translation. The two following sentences are equivalent.*

1. *The formula translation F and the term translation T are coupled.*
2. *There exists a formula morphism $L^+ : (\Sigma \otimes \Pi)^+ \to (\Omega \otimes \Psi)^+$ which extends the formula translation $L : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ defined by equation 3.5.*

Proof. Let us show $1 \Rightarrow 2$. Assume that the formula translation F and the term translation T are coupled. Then, Lemma 22 ensures us that F is unifiable respect to T and the map L exists and is well defined by $L(\varphi\rho) \equiv F(\varphi)T^* \circ \rho$. To show that there exists a formula morphism $L^+ : (\Sigma \otimes \Pi)^+ \to (\Omega \otimes \Psi)^+$ extending L, we are going to use Theorem 20. That is to say that we will show for every pair of trees $a, b \in \mu^{-1}(\Sigma \otimes \Pi)$, if $\mu(a) \equiv \mu(b)$, then $\mu(L^\circ(a)) \equiv \mu(L^\circ(b))$. To this aim, let us take $a, b \in \mu^{-1}(\Sigma \otimes \Pi)$ and assumme $\mu(a) \equiv \mu(b)$. We must show that $\mu(L^\circ(a)) \equiv \mu(L^\circ(b))$. Let us give to the Second Adapted Unification Algorithm the pair (a, b) as input and notice that because $\mu(a) \equiv \mu(b)$, the algorithm does not fail and (wihout loss of generality) let us assume that it outputs a set of pairs of trees:

$$D = \{(a_{\chi_1}, d'_1), (a_{\chi_2}, d'_2), ..., (a_{\chi_n}, d'_n)\}.$$

Since $\mu(a) \equiv \mu(b)$, an application of Lemma 23 shows that $\chi_i \equiv \mu(d'_i)$ for $i \in \{1, 2, ..., n\}$.

Notice that a and $L^\circ(a)$ are isomorphic as trees and the same is true for b and $L^\circ(b)$. Then, if the input for the Second Adapted Unification Algorithm is the pair $(L^\circ(a), L^\circ(b))$, the algorithm does not fail and outputs a set of pairs of trees

$$E = \{(c_{L(\chi_1)}, L^\circ(d'_1)), (c_{L(\chi_2)}, L^\circ(d'_2)), ..., (c_{L(\chi_n)}, L^\circ(d'_n))\}.$$

Our strategy to prove this theorem is to show that $L(\chi_i) \equiv \mu(L^\circ(d'_i))$ holds, for $i \in \{1, 2, .., n\}$. Once done, an application of Lemma 23 yields $\mu(L^\circ(a)) \equiv \mu(L^\circ(b))$ and the theorem has been shown. Let us execute this strategy by picking an arbitrary index k from $\{1, 2, ..., n\}$. Such index determines the pair (a_{χ_k}, d'_k) and the pair $(c_{L(\chi_k)}, L^\circ(d'_k))$ from D and E respectively. In order to avoid a cumbersome notation, let us drop the subindex k and let us denote these pairs as (a_χ, d') and $(c_{L(\chi)}, L^\circ(d'))$ respectively. If we can show that $L(\chi) \equiv \mu(L^\circ(d'))$ by assuming that $\chi \equiv \mu(d')$, the proof is complete since the forgotten index was chosen arbitrarily.

Observe that d' is a formula tree over $(\Sigma \otimes \Pi)^+$ and that χ belongs to $(\Sigma \otimes \Pi)$. Let us assume without loss of generality that the leaves of d' are $\psi_1, \psi_2, ..., \psi_m$. Because these labels belong to $\Sigma \otimes \Pi$, there exists $\varphi_1, \varphi_2, ..., \varphi_m \in \Sigma$ and substitutions $\rho_1, \rho_2, ..., \rho_m$ in Π^* such that $\psi_i = \varphi_i \rho_i$ for $i \in \{1, 2, ..., m\}$. Set a new tree denoted by d as the one obtained by reeplacing the ψ-labels on d' by φ-labels. Notice that d and d' are isomorphic as trees. Observe that $d \in \Sigma^\circ$ and then it has a twin d^* and let $\{\phi_j\}_{j=1}^m$ be the substitutions of the twin d^*. Let us define a substitution ρ given by $\rho(x) = x(\phi_i^{-1} \bullet \rho_i)$, if $x \in V(\varphi_i \phi_i)$ for $i \in \{1, 2, ..., m\}$ and $\rho(x) = x$ otherwise. Observe that $\varphi_i \phi_i$ is the label on the leaf i of the twin d^* and then, it is clear that ρ is well defined because the sets of variables $V(\varphi_i \phi_i)$ are mutually disjoint. From the definition of ρ, it is clear that $d^* \rho = d'$ holds and then, an application of Lemma 21 yields

$$\chi \equiv \mu(d') \equiv \mu(d^*)\rho \tag{3.8}$$

Additionally, because $\chi \in \Sigma \otimes \Pi$, there exists $\varphi \in \Sigma$ and a substitution γ in Π^* such that $\chi \equiv \varphi\gamma$ and then, equation 3.8 above gives

$$\varphi\gamma \equiv \mu(d^*)\rho.$$

Because of the definition of the twin d^*, the sets $V(\varphi)$ and $V(\mu(d^*))$ are disjoint. Such property enables us to ensure that there exists a unifier α of equation $\varphi = \mu(d^*)$, because we can define $\alpha(x) = \gamma(x)$ if $x \in V(\varphi)$ and $\alpha(x) = \rho(x)$ otherwise. Let θ be a mgu of $\varphi = \mu(d^*)$ and then, since α in Π^*, there exists a substitution λ minimal respect to θ and a substitution α' in Π^* such that $\theta \bullet \lambda \bullet \alpha' = \alpha$. It is clear that $d \in \Delta_\Sigma$ because $\varphi \in \Sigma$ and $d \in \Sigma^\circ$ and then, since F and T are coupled, there exists a unifier $\widehat{\lambda}$ of equation $F(\varphi) = \mu(L^\circ(d^*))$ which is compatible with $\theta \bullet \lambda$. Next computations shall help us to prove the sought result.

$$
\begin{gathered}
F(\varphi)\widehat{\lambda} \equiv \mu(L^\circ(d^*))\widehat{\lambda} \\
(F(\varphi)\widehat{\lambda})\ T^* \circ \alpha' \equiv (\mu(L^\circ(d^*))\widehat{\lambda})\ T^* \circ \alpha' \\
(F(\varphi)T^* \circ \theta \bullet \lambda)\ T^* \circ \alpha' \equiv (\mu(L^\circ(d^*))T^* \circ \theta \bullet \lambda)\ T^* \circ \alpha' \\
F(\varphi)T^* \circ (\theta \bullet \lambda \bullet \alpha') \equiv \mu(L^\circ(d^*))T^* \circ (\theta \bullet \lambda \bullet \alpha') \\
F(\varphi)T^* \circ \alpha \equiv \mu(L^\circ(d^*))T^* \circ \alpha
\end{gathered}
$$

From the last expression above together with the fact that F preserves variables and $V(\varphi) \cap V(\mu(L^\circ(d^*))) =$

$\emptyset$, it is clear that $F(\varphi)\ T^* \circ \gamma \equiv \mu(L^\circ(d^*))T^* \circ \rho$ and then:

$$L(\chi) \equiv \mu(L^\circ(d^*))T^* \circ \rho \tag{3.9}$$

Finally, we conclude by making the observation that the tree $L^\circ(d^*)\ T^* \circ \rho$ is equal to the tree $L^\circ(d')$. Such observation follows from two facts: (1) the trees d' and d^* are isomorphic as directed trees implying that $L^\circ(d')$ and $L^\circ(d^*)$ are isomorphic as directed trees, and (2) the fact that the i-th leaf of $L^\circ(d^*)\ T^* \circ \rho$ has the label $F(\varphi_i \phi_i)T^* \circ \rho \equiv (F(\varphi_i)\phi_i \phi_i^{-1})\ T^* \circ \rho_i \equiv F(\varphi)T^* \circ \rho_i$ which is in fact the label on the i-th leaf of $L^\circ(d')$. Since $L^\circ(d^*)\ T^* \circ \rho = L^\circ(d')$, an application of Lemma 21 yields $\mu(L^\circ(d^*)T^* \circ \rho) \equiv \mu(L^\circ(d^*))\ T^* \circ \rho \equiv \mu(L^\circ(d'))$. From the last equality and equation 3.9, we conclude that $L(\chi) \equiv \mu(L^\circ(d'))$. The proof of $1 \Rightarrow 2$ is ended by this equality. However, it is worth noticing that because L is well defined, such equality does not depend on the selected formulas $\varphi, \varphi_1, \varphi_2, ..., \varphi_m \in \Sigma$.

To show that $2 \Rightarrow 1$, we present an argument which uses contradiction. Assume that there exists a formula morphism $L^+ : (\Sigma \otimes \Pi)^+ \to (\Omega \otimes \Psi)^+$ which extends $L : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ which is well defined by equation 3.5 and assume that the formula translation F and the term translation T are *not* coupled. The last assumption implies the existence of $\varphi \in \Sigma$, $b \in \Delta_\Sigma$, θ mgu of equation $\varphi = \mu(b^*)$ and a minimal substitution λ respect to θ such that there is no unifier of the equation $F(\varphi) = \mu(L^\circ(b^*))$ compatible with $\alpha = \theta \bullet \lambda$. In such case, notice that α is in Π^* and $\varphi\ \alpha \equiv \mu(b^*)\ \alpha$. Because L^+ exists and it is well defined, the last equality implies that

$$L^+(\varphi\ \alpha) \equiv L^+(\mu(b^*)\ \alpha) \tag{3.10}$$

On the other hand, observe that the trees $L^\circ(b^*\alpha)$ and $L^\circ(b^*)T^* \circ \alpha$ are isomorphic as trees. Additionally, it is not dificult to see that they both have the exact same labels on each of its leaves. These observations can be used to show the following equality of trees:

$$L^\circ(b^*\alpha) = L^\circ(b^*)T^* \circ \alpha \tag{3.11}$$

this equality of trees, together with Lemma 21 and the commutation of L^+ and L° given by the

Characterization Lemma, can be used to perform the following computations:

$$\begin{aligned} L^{+}(\mu(b^{*})\alpha) &\equiv L^{+}(\mu(b^{*}\alpha)) \\ &\equiv \mu(L^{\circ}(b^{*}\alpha)) \\ &\equiv \mu(L^{\circ}(b^{*})\ T^{*}\circ\alpha) \\ &\equiv \mu(L^{\circ}(b^{*}))\ T^{*}\circ\alpha \end{aligned}$$

the last line above together with the observation that $L^{+}(\varphi\ \alpha) = F(\varphi)T^{*}\circ\alpha$ (which holds because $\varphi \in \Sigma$), enables us to rewrite equality 3.10 as:

$$F(\varphi)T^{*}\circ\alpha \equiv \mu(L^{\circ}(b^{*}))\ T^{*}\circ\alpha.$$

Therefore, $T^{*}\circ\alpha$ is an unifier of equation $F(\varphi) = \mu(L^{\circ}(b^{*}))$ and since $\alpha = \theta\bullet\lambda$, α is compatible with $\theta\bullet\lambda$. This contradicts our initial assumption. We have shown that the existence of L^{+} implies that F and T are coupled. □

We close this chapter making some comments regarding the implementation of last theorem in a computer algorithm. Let $T : \Pi \to \Psi$ be an extensible term translation and $F : \Sigma \to \Omega$ a formula translation. If Σ is finite, we could implement an algorithm to verify the conditions of last theorem in order to ensure that L is well defined and that L^{+} exists. Such algorithm should verify that F and T are coupled. It implies to find the set Δ_{Σ}, which is finite in case Σ is finite. Because there are some well known efficient algorithms for dealing with the problem of unification (see for example [80]), it is clear the design of an easy algorithm (exponential but finite in the case of looking for all the combinations) that can be used to locate Δ_{Σ} and then build $L^{\star}(\Delta_{\Sigma})$. This can be done easily. Now, the algorithm must pick all the pairs (φ, b) where $\varphi \in \Sigma$ and $b \in \Delta_{\Sigma}$ and check if they satisfy the conditions of the Definition 22 (in order to verify that F and T are coupled). For checking these conditions, there are three degrees of dificulty that we describe next. For an arbitrary pair (φ, b) let us consider the equation $\varphi = \mu(b^{*})$ and notice that there are three different cases:

1. There are no unifiers of $\varphi = \mu(b^{*})$: In this case the condition is trivially satisfied.
2. The mgu θ of $\varphi = \mu(b^{*})$ is a substitution in Π^{*}: In such case, every minimal substitution λ change variables by variables. Therefore, if the algorithm verifies that the substitution $T^{*}\circ\theta$ is an unifier of the equation $F(\varphi) = L^{\star}(b^{*})$, the condition is checked. With this strategy, the verification of the condition is easy for this case.

3. The mgu θ of $\varphi = \mu(b^*)$ is a substitution which is not in Π^*: In this case, because Σ is finite, there must be a finite number of minimal substitutions λ. However, finding them all might be challenging. If it can be done, the algorithm must verify that all the substitutions $T^* \circ (\theta \bullet \lambda)$ are unifiers of the equation $F(\varphi) = L^\star(b^*)$. If this is true, the condition is checked. More research is needed in order to get a clear sense of the difficulty of this case.

Chapter 4

Metaphor Model

4.1 Introduction

The mainstream of cognitive science research related to the study of metaphor has traditionally been dominated by approaches which assume that a domain of knowledge can be represented by means of a symbolic language together with its grammar (a set of rules to manipulate such symbolic representations). For some examples of these approaches we recommend the reader to review some of the models that are described under the label "symbolic models" in [64]. However, in the last 20 years this perspective has been challenged from several directions within the cognitive science field. One of such directions is a new trend on cognitive science called *embodied cognition* which postulates the idea that the motor system strongly influences cognition, and that some of the elements of the mind are represented in a distributed fashion and embedded into the entire human body. The mathematical model for metaphor that we present in this chapter is based in some of the ideas presented in the work of various researchers in such area. In particular, the model presented in this chapter is influenced by the principles for metaphor postulated in [71] and [34].

In what follows we present a model for metaphor which is consistent with the idea that symbolic and embodied approaches for understanding cognition are complementary. One of our key hypothesis is that a domain of knowledge can be represented by two components. The first component is a symbolic language associated to the domain and the second component is the semantics (or meaning) of such domain. Under this perspective, a metaphor between two domains of knowledge shall emerge as the result of an interaction between four components: the languages and the semantics associated to these domains. We are going to formalize these concepts by using some basic concepts of *model theory*.

Let us provide now an outline of this chapter. The first section is called “Term Domains” where our first model for knowledge domains is introduced. A little more complete model for knowledge domains shall be obtained by adding some more structure to term domains. This is done in the second section which is called “Formula Domains”.

The section 4.2 is devoted to term domains. In the first part, this section introduce some basic concepts of model theory as structure, signature and homomorphism between structures. Next section defines the set of symbolic constructions called *terms* which shall be the language associated to term domains. The semantic counterpart of terms shall be a set of operations defined on a set of objects belonging to the domain. Next, the model for metaphor is presented under the concept of a *homomorphism* between term domains which is just a map that preserves the structure of term domains. We end this section by presenting the key theorem (described in the next paragraph) for term domains. In order to pursue its proof, we need to introduce some concepts such as *quotient domains* and *term morphisms* and show some intermediate results.

The key result of the section devoted to term domains is presented in Theorem 31 and may be summarized as follows. Let A be a domain with language L_1 and let L_2 be another language. Theorem 31 describes conditions in terms of the languages L_1 and L_2 which enable us to build a domain B which is the largest domain with language L_2 that admits a homomorphism $A \to B$. The proof given for this theorem uses words of language L_2 as elemental blocks to construct the new domain B. Then, the model presented here has a kind of ”structure creation power”. We would like to interpret this result as the property of creativeness which is attributed to metaphor. That is consistent with findings presented in the cognitive literature, for example, in [77] it is suggested that the concept of time is indeed created in our minds by means of this property of metaphors. We transcribe here a paragraph of such work. “It appears that abstract domains such as time are indeed shaped by metaphorical mappings from more concrete and experiential domains such as space”. Similar findings are reported for numbers and other abstract concepts.

We conclude the part of this chapter devoted to term domains by presenting the *product domain* and the *coproduct domain* for a given family of term domains. These domains have the expected properties for a product and a coproduct of mathematical objects [1]. Such properties are shown in Theorem 34 and Theorem 33. We study them in this chapter because we believe that products and coproducts of domains can be linked to other cognitive phenomeenumeratena. In particular, we think that the product domain can be related to a cognitive phenomenon called Blending (See [21] for a brief introduction to Blending). For some research linking coproducts and cognitive development see [94].

The section 4.3 is devoted to formula domains. In this section is defined the set of symbolic constructions called *formulas* which shall be part of the language associated to formula domains. The semantic counterpart of formulas shall be relations defined on the set of objects belonging to the domain. In this context, we present two models for metaphor under the names of *homomorphism* and *metaphor*. The first is a map which preserves the structure of a formula domain and the later is just a stronger version of the earlier. The name "metaphor" given to the second kind of maps is justified by Theorem 42 which shows that a metaphor between two domains preserves the consequence between two formulas (consequence might be taken as a formal model of human inference). Overall, the results of this section show that a homomorphism preserves the truthness of some well characterized families of formulas. Such results are presented in Theorem 40 and Theorem 41.

We conclude the section devoted to formula domains by presenting the *product domain* and the *coproduct domain* for a given family of formula domains. These domains have the expected universal properties for a product and a coproduct of mathematical objects [2]. Such properties are shown in Theorem 46 and Theorem 45. We have already given the reason why we studied those kind of constructions here.

4.2 Term Domains

This section is aimed to provide a formalization of domains of knowlege which is necessary for our modeling of the phenomenon of metaphor. As we said, such formalization shall have two compo-

[1] From a point of view of Cathegories Theory.
[2] From a point of view of Cathegories Theory.

nents: a syntactic component determined by a language and a semantic component determined by a set of operations over a set of elements of a given structure. Let us begin by sketching the basic concepts provided to us by Model Theory.

4.2.1 Basics of Model Theory

This section is adapted from the book "A shorter model Theory" [50]. Because this section is just a brief (and then necessarily incomplete) review of the subject, we recommend that readers unfamiliar with this topic review the mentioned book in order to have examples and details of the concepts presented here. We begin with the definition of a structure.

Definition 23. A *structure* Q is an object with the following four ingredients:

1. A set called the *domain* of Q, written $dom(Q)$ or $dom\ Q$. The elements of $dom(Q)$ are called the *elements* of the structure Q.
2. A set of elements of Q called *constant elements*, each of which is named by one or more *constants*. If c is a constant, we write c^Q for the constant element named by c.
3. For each positive integer n, a set of n-ary operations on $dom(Q)$ (i.e. maps from $(dom\ Q)^n$ to $dom\ Q$), each of which is named by one or more n-ary *function symbols*. If F is a function symbol, we denote by F^Q the function that F names.
4. For each positive integer n, a set of n-ary relations on $dom(Q)$ (i.e. subsets of $(dom\ Q)^n$), each of which is named by one or more n-ary *relation symbols*. If R is a relation symbol, we denote by R^Q the relation that R names.

Except where we say otherwise, any of the sets (1- 4) above may be empty. The constant, relation and function symbols can be any mathematical objects, not necessarily written symbols; but for peace of mind we shall assume that, for instance, a 3-ary relation symbol doesn't also appear as a 3-ary function symbol or a 2-ary relation symbol.

We would like to emphasize that Definition 23 accounts for an important amount of mathematical objects. Some important examples of Definition 23 are graphs, linear orderings, groups and vector spaces. In other studies, a structure is defined as a pair $(dom(A), \alpha)$ where α is a mapping taking each symbol s to the corresponding item s^A. As an example consider the structure whose domain is the set of real numbers and it is defined by $(\mathbb{R}, \alpha)$ where the domain of α is the set $\{+, -, \cdot, 0, 1, \leq\}$ and it makes: constants 0 and 1 naming the numbers 0 and 1, a 2-ary relation symbol $\leq$ naming the relation $\leq$, 2-ary function symbols $+$ and $\cdot$ naming addition and multiplication respectively, and a 1-ary function symbol naming minus.

Definition 24. The *signature* of a structure Q is specified by giving the set of constants of Q, and for each separate $n > 0$, the set of n-ary relation symbols of Q and the set of n-ary function symbols of Q.

We assume that the signature of a structure can be read off directly from the structure. The symbol L will be used to stand for signatures. In case that a structure Q has signature L, we shall say that Q is an L-structure. The sequences of elements of an structure shall be denoted by $\overline{q}$, $\overline{a}$ etc. A *tuple in* Q (or from Q) is a finite sequence of elements of Q; it is an n-tuple if it has length n. When it doesn't lead into confusion, the length of a sequence or tuple shall be determined by the context.

Definition 25. Let L be a signature and let A and B be L-structures. By a *homomorphism* f from A to B, in symbols $f : A \to B$, we shall mean a function f from $dom(A)$ to $dom(B)$ with the following three properties.

1. For each constant c of L, $f(c^A) = c^B$.
2. For each $n > 0$ and each n-ary function symbol F of L and n-tuple $\overline{a}$ from A,
$$f(F^A(\overline{a})) = F^B(f\overline{a}).$$
3. For each $n > 0$ and each n-ary relation symbol R of L and n-tuple $\overline{a}$ from A,
$$\text{if } \overline{a} \in R^A, \text{ then } f\overline{a} \in R^B.$$

where the above notation $f\overline{a}$ means that when $\overline{a}$ is $(a_0, ..., a_{n-1})$, then $f\overline{a}$ means $(fa_0, ..., fa_{n-1})$.

An *isomorphism* is a bijective homomorphism whose inverse is a homomorphism. An isomorphism $f : A \to A$ is called an *automorphism* of A. Definition 25 accounts for many structure preserving maps between mathematical objects such as graph homomorphisms, group homomorphisms, linear maps, etc. We recommend the reader to review [50] for detailed examples. We presented Definition 25 here because it shall serve us as an introductory idea to the key concept of a *homomorphism between domains* that we introduce in next section.

4.2.2 Term Domains

Let us start by defining the formal meaning of the word "language" that we have been loosely using until now. It shall be a set of symbol strings in the precise sense of Definition 26 below. Every language has a numerable set of *variables*. These shall be denoted by symbols $x, y, z, \dots$, etc., and one of their purposes is to serve as temporary labels for elements of a structure. Any symbol can be used as a variable, provided it is not already being used for something else. We assume that all languages mentioned in this chapter share the same set of variables which shall be denoted by V.

Definition 26. The set of *terms* of a signature L is the smallest set of strings of symbols which satisfies:

1. Every variable $x \in V$ is a term.
2. Every constant of L is a term.
3. If $n > 0$, F is a n-ary function symbol of L and $t_1, \dots, t_n$ are terms, then the expression $F(t_1, \dots, t_n)$ is a term.

If t is a term of L, we shall say that t is an L-term. The *complexity* of a term is the number of function (including constant) symbols occurring in it, counting each occurrence separately. We shall use some inductive arguments on the complexity of terms, which is possible because if t occurs as part of s, then s has a greater complexity than t.

If we introduce a term t as $t(\overline{x})$, this will always mean that $\overline{x}$ is a sequence $(x_0, x_1, \dots)$, possibly infinite, of distinct variables, and every variable which occurs in t is among the variables in $\overline{x}$. In the same context we may write $t(\overline{s})$, where $\overline{s}$ is a sequence of terms $(s_0, s_1, \dots.)$; then $t(\overline{s})$ means the term obtained from t by reeplacing s_0 in place of x_0, s_1 in place of x_1, etc., throughout t. It is well known that every term t of L can be constructed in only one way. Such is easily seen from two facts: (i) if t is a constant then t is not of the form $F(\overline{s})$, and (ii) if $t = F(s_1, s_2, \dots, s_m) = G(r_1, r_2, \dots, r_n)$ then $F = G$, $m = n$ and for each $i \leq n$, $s_i = r_i$.

Next, we describe how terms stand for elements of a structure. Let Q be an L structure, $t(\overline{x})$ an L- term where $\overline{x} = (x_0, x_1, \dots)$ and $\overline{q} = (q_0, q_1, \dots)$ be a sequence of elements of Q. Then the element of Q denoted by $t^Q(\overline{q})$ (or $t^Q\overline{q}$) is defined by induction on the complexity of t in the following manner

1. if t is the variable x_i then $t^Q(\overline{q})$ is q_i.
2. if t is a constant c then $t^Q(\overline{q})$ is the element c^Q,
3. if t is of the form $F(s_1,...,s_n)$ where F is a n-ary function symbol and each s_i is a term $s_i(\overline{x})$, then $t^Q(\overline{q})$ is the element $F^Q(s_1^Q(\overline{q}),...,s_n^Q(\overline{q}))$.

Notice that from the three items above, the way in which each n-ary term t determines an n-ary operation t^Q on the set of elements of Q is standard and clear.

Next definition describes a first formalization of a "domain of knowledge". Our model assumes that an L-structure can be seen as a context [3] and that a domain of knowledge can be represented by a collection of operations which can be defined in terms of such context. Next definition makes it precisely.

Definition 27. Let Q be an structure with signature L. A *term domain* (in the context of Q) is a pair (A,Π) where Π is a finite set of L-terms and A is a nonempty set of elements of Q such that for every term $t \in \Pi$ and every tuple $\overline{a}$ of elements of A, the element $t^Q(\overline{a})$ belongs to A. The elements of A are called the *objects* of the domain.

We shall omit systematically the context Q of a term domain (A,Π) in the following straightforward manner. Because the definition above requires that for every $t \in \Pi$, the set A to be closed with respect to the operation t^Q, we shall denote by t^A the operation t^Q restricted to A. Additionally, when specifying the signature of the context becomes necessary, we shall say that (A,Π) is an "L-domain" meaning that L is the signature of the context Q. When it does not lead into confusion, we shall say "domain" instead of "term domain".

Let Q be an L-structure, Π a set of L-terms and A' be a set of elements of Q. It follows from Definition 27 that there is a unique smallest term domain (A,Π) such that $A' \subseteq A$; we call (A,Π) the *term domain generated by* A'. We call A' a *set of generators* for (A,Π). A term domain is said to be *finitely generated* if it is generated by some finite set of elements.

[3] The philosopher Josef Stern postulates in his book "Metaphor in Context" that there is a systematic context-dependence of metaphorical interpretation. There are various studies whose conclusions support such theory (see for example [37]).

4.2.3 Homomorphisms and Quotient Domains

The model for metaphor that is proposed in this section relates two domains when both share some common structure. A first approximation to this idea was given in the definition of a homomorphism between structures (Definition 25) which is a map which preserves the structure between two mathematical objects sharing the same signature (or language).

However, we are looking for a model that is able to address the case when there is some common structure between two mathematical objects with different signatures (different languages). In order to achieve that goal, in what follows, we are going to adapt the concept of a homomorphism between two L-structures.

Recall that every signature uses the same numerable set V of variables to build its associated set of terms. Let Π be a set of L_1-terms and Ψ a set of L_2-terms. A map $F : \Pi \to \Psi$ is called a *term translation* when it preserves the set of variables for each term $t \in \Pi$. In other words, t and $F(t)$ have the exact same set of variables.

Definition 28. Let (A,Π) be an L_1-domain, (B,Ψ) an L_2-domain and $F : \Pi \to \Psi$ a term translation. A mapping $f : A \to B$ is called an *F-homomorphism* from the domain (A,Π) to the domain (B,Ψ), if for every $t \in \Pi$ and every tuple $\overline{a}$ in A,

$$f(t^A\overline{a}) = F(t)^B(f\overline{a}). \tag{4.1}$$

Notice that condition (2) of the definition of a homomorphism between structures (Definition 25) is very similar to the condition imposed by equation 4.1 above. The only difference is that equation 4.1 relates two signatures by means of a term translation F.

We shall denote term translations with capital letters and homomorphisms between domains with lower case letters. When possible, we shall use the lower case and upper case version of the same letter to denote a homomorphism and its related term translation. For example, we are going to denote an F-homomorphism by f and a P-homomorphism by p.

When F and f are surjective (injective), we say that f is a surjective (injective) F-homomorphism. An F-homomorphism which is surjective and injective is called an F-isomorphism. It is straightforward that if f is an F-isomorphism, then f^{-1} is an F^{-1}-isomorphism. If there exists an F-isomorphism $f:(A,\Pi)\to(B,\Psi)$, we say that (A,Π) and (B,Ψ) are *F-isomorphic* (or just isomorphic if F is clear from the context). It is also straightforwad that if $f:(A,\Pi)\to(B,\Psi)$ is an F-homomorphism and $g:(B,\Psi)\to(C,\Omega)$ is an G-homomorphism, then the map $(g\circ f):A\to C$ is an $(G\circ F)$-homomorphism.

Lemma 25. *Let (A,Π) be an L_1-domain generated by A' and let (B,Ψ) an L_2-domain. If $f:A\to B$ and $g:A\to B$ are two F-homomorphisms such that $f_{|A'}=g_{|A'}$, then f and g are equal.*

Proof. We are going to show that the set A can be obtained as a numerable union of sets. Let us define

$$\begin{aligned} E_0 &= A', \\ E_1 &= \{t^A\overline{a} \mid t\in\Pi, \overline{a} \text{ is a tuple in } E_0\}, \\ E_{n+1} &= \{t^A\overline{a} \mid t\in\Pi, \overline{a} \text{ is a tuple in } E_n\}. \end{aligned}$$

Since (A,Π) is generated by A', an straightforward argument[4] shows that

$$A=\bigcup_{n\geq 0} E_n. \tag{4.2}$$

Now, it is easy to show that $f_{|E_n}=g_{|E_n}$ for every $n\in\mathbb{N}$ by using induction on n and the fact that f and g are F-homomorphisms. Therefore, from equation 4.2, we conclude that $f=g$. □

Let us denote equivalence relations by caligraphic letters. Let $\mathscr{A}$ be an equivalence relation on a set A, we shall denote the equivalence class of some element a by $[a]_{\mathscr{A}}$, the associated quotient set by $A/\mathscr{A}$ and its projection mapping $a\to[a]_{\mathscr{A}}$ by $p_{\mathscr{A}}$. When the subindexes of $[a]_{\mathscr{A}}$ and $p_{\mathscr{A}}$ are clear from context, we shall omit them.

Most of the equivalence relations encountered in this chapter are induced by the fibers of certain maps. For example, the map $f:A\to B$ determines an equivalence relation on A by letting $a\sim b$ exactly when $f(a)=f(b)$. When f is an F-homomorphism, we call such equivalence relation the

[4]It is clear that $\cup E_n\subseteq A$, since (A,Π) is a domain. On the other hand, $A'\subseteq\cup E_n$ and notice that $(\cup E_n,\Pi)$ is a domain. Therefore, $A\subseteq\cup E_n$.

kernel of f and denote it by $ker(f)$. In such case, the equivalence class of a is denoted by $[a]_f$, the associated quotient set by A/f and the projection map by p_f.

Let (A,Π) be an L_1-domain, (B,Ψ) an L_2-domain and $f : A \to B$ an F-homomorphism. Our next discussion is aimed to introduce a domain called the *quotient domain of* (A,Π) *by* f. For reasons that shall be aparent later, we want this new domain to be an L_1-domain. To this aim, we need to determine a set of terms $\Pi_F \subseteq \Pi$ which has exactly one member of each equivalence class of the set Π/F (where F is the equivalence relation determined by the fibers of $F : \Pi \to \Psi$). We are going to determine uniquely such set Π_F by giving a fixed order to Π, as for example $\Pi = \{t_1, t_2, t_3, ...t_n\}$, and selecting the subset of terms Π_F wich satisfies our requirement and minimizes the sum of subindexes. Observe that the set of terms Π_F is selected in a way that there exists a bijection with the quotient set Π/F. Then, we can determine uniquely the map $P : \Pi \to \Pi_F$ as the map which for $t \in \Pi$ and $s \in \Pi_F$ satisfies $P(t) = s$ exactly when $F(t) = F(s)$. It is clear that P is a surjective term translation.

Let us define some operations in A/f such that the pair $(A/f, \Pi_F)$ becomes an L_1-domain. Let $s \in \Pi_F$ and let $\overline{[a]}$ be a tuple in A/f. Since $\Pi_F \subseteq \Pi$, we can define the operation $s^{A/f}$ by

$$s^{A/f}\overline{[a]} = [s^A \overline{a}]. \tag{4.3}$$

We must show that 4.3 is well defined. Let $\overline{a}$ and $\overline{b}$ two tuples of A such that $\overline{[a]} = \overline{[b]}$. Let us show that $[s^A \overline{a}] = [s^A \overline{b}]$, through the following computations

$$\begin{aligned} [s^A \overline{a}] &= f^{-1}(f(s^A \overline{a})) \\ &= f^{-1}(F(s)^A f \overline{a}) \\ &= f^{-1}(F(s)^A f \overline{b}) \\ &= f^{-1}(f(s^A \overline{b})) \\ &= [s^A \overline{b}]. \end{aligned}$$

The domain $(A/f, \Pi_F)$ with the above set of operations is called the *quotient domain of* (A,Π) *by* f and denoted by $(A,\Pi)/f$.

Proposition 17. *Let* (A,Π) *be an* L_1*-domain,* (B,Ψ) *an* L_2*-domain. If* $f : A \to B$ *is an* F*-homomorphism, then the projection map* $p : A \to A/f$ *is a surjective* P*-homomorphism from the domain* (A,Π) *to* *the quotient domain* $(A,\Pi)/f$.

Proof. It is clear that P and p are surjective maps. To show that p is a P-homomorphism let us take $t \in \Pi$ and a tuple $\overline{a}$ in A. We must show that $p(t^A\overline{a}) = P(t)^{A/f}p\overline{a}$. Let $P(t) = s$ and consider next computations

$$\begin{aligned}
p(t^A\overline{a}) &= [t^A\overline{a}]_f \\
&= f^{-1}(f(t^A\overline{a})) \\
&= f^{-1}(F(t)^A f\overline{a}) \\
&= f^{-1}(F(s)^A f\overline{a}) \\
&= f^{-1}(f(s^A\overline{a})) \\
&= [s^A\overline{a}] \\
&= s^{A/f}\overline{[a]} \\
&= P(t)^{A/f}p\overline{a}.
\end{aligned}$$

where the fourth and last equality were obtained because $P(t) = s$ is equivalent to $F(t) = F(s)$. Therefore, the projection map p is a P-homomorphism. $\square$

Theorem 26. *Let* (A,Π) *be an* L_1*-domain and* (B,Ψ) *an* L_2*-domain. If* $f : A \to B$ *is an* F*-homomorphism, then there exists an injective* F'*-homomorphism* $f' : A/f \to B$*, uniquely determined, wich satisfies* $F = F' \circ P$ *and makes next diagram commute*

$$\begin{tikzcd}
(A,\Pi) \arrow[r, "f"] \arrow[d, "p"'] & (B,\Psi) \\
(A,\Pi)/f \arrow[ur, dashed, "f'"'] &
\end{tikzcd} \tag{4.4}$$

Proof. The term translation F' is uniquely determined by $F'(t) = F(t)$ for $t \in \Pi_F$. That is because we want F' satisfying $F' \circ P = F$. Observe that F' is injective. For analog reasons, f' must be the mapping determined by $f'[a] = f(a)$. We shall show that f' is an F'-homomorphism by showing that $f'(t^{A/f}\overline{[a]}) = F'(t)^B f'\overline{[a]}$. Let us take $t \in \Pi_F$ and a tuple $\overline{[a]}$ in A/f. Because $F'(t) = F(t)$, next computations follow

$$\begin{aligned} f'(t^{A/f}\overline{[a]}) &= f'([t^A\overline{a}]) \\ &= f(t^A\overline{a}) \\ &= F(t)^B f\overline{a} \\ &= F'(t)^B f'\overline{[a]}. \end{aligned}$$

Then, f' is an F'-homomorphism from the L_1-domain $(A,\Pi)/f$ to the L_2-domain (B,Ψ). ◻

Corollary 27. *Let (A,Π) be an L_1-domain and (B,Ψ) an L_2-domain. If $f : A \to B$ is a surjective F-homomorphism, then the quotient L_1-domain $(A,\Pi)/f$ is isomorphic to the L_2-domain (B,Ψ).*

Proof. It follows directly from the proof of Theorem 26. Since in this case F and f are surjective, the mappings F' and f' are bijective. Therefore, domains $(A,\Pi)/f$ and (B,Ψ) are F-isomorphic. ◻

Lemma 28. *Let (A,Π), (B,Ψ) and (C,Ω) be three domains. Let $f : A \to B$ be a surjective F-homomorphism and $g : A \to C$ a G-homomorphism. If for every $t \in \Pi$, $[t]_F \subseteq [t]_G$ and $ker(f) \subseteq ker(g)$*[5] *then there exists a unique term translation H which satisfies $H \circ F = G$ and a unique H-homomorphism $h : B \to C$ which makes the below diagram commute. The map h is surjective exactly when g is surjective and h is injective exactly when $ker(f) = ker(g)$.*

$$\begin{tikzcd} (A,\Pi) \arrow[r, "f"] \arrow[dr, "g"'] & (B,\Psi) \arrow[d, dashed, "h"] \\ & (C,\Omega) \end{tikzcd}$$

Proof. The mappings f and g determine the quotient domains $(A,\Pi)/f$ and $(A,\Pi)/g$ respectively. Consider the next two diagrams below (the left-sided for term translations and the right-sided for homomorphisms between domains).

[5] Condition $ker(f) \subseteq ker(g)$ is equivalent to say that for every $a \in A$, $[a]_f \subseteq [a]_g$.

$$\begin{tikzcd}
(A,\Pi)/f \arrow[rr, "F'"] \arrow[dd, "K"'] & & (B,\Psi) \\
& (A,\Pi) \arrow[ul, "P_f"'] \arrow[ur, "F"] \arrow[dl, "P_g"] \arrow[dr, "G"'] & \\
(A,\Pi)/g \arrow[rr, "G'"'] & & (C,\Omega)
\end{tikzcd}
\qquad
\begin{tikzcd}
(A,\Pi)/f \arrow[rr, "f'"] \arrow[dd, "k"'] & & (B,\Psi) \\
& (A,\Pi) \arrow[ul, "p_f"'] \arrow[ur, "f"] \arrow[dl, "p_g"] \arrow[dr, "g"'] & \\
(A,\Pi)/g \arrow[rr, "g'"'] & & (C,\Omega)
\end{tikzcd}$$

Our strategy to prove this theorem shall be executed in three steps: (i) we shall define a term translation $K : \Pi_F \to \Pi_G$ satisfying $K \circ P_f = P_g$. Once done, Theorem 26 ensures that the above (left-sided) diagram is commutative. (ii) we shall define a map $k : A/f \to A/g$ satisfying $k \circ p_f = p_g$ and prove that such map is a K-homomorphism. In such case, another application of Theorem 26 shows that the above (right-sided) diagram is commutative, and (iii) because f' is an F'-isomorphism, we can define $h = g' \circ k \circ f'^{-1}$ and $H = G' \circ K \circ F'^{-1}$. We shall conclude this proof by observing that because of the commutation of the above diagrams, h is an H-homomorphism.

Step (i): Notice that for every $t \in \Pi$, $[t]_F \subseteq [t]_G$, and then we can define K by $K([t]_F) = [t]_G$. Notice that this is the only definition of K which satisfies $K \circ P_f = P_g$. Therefore, the above (left-sided) diagram is commutative.

Step (ii): Condition $ker(f) \subseteq ker(g)$ is equivalent to the statement: equivalence classes determined by f are included in the equivalence classes determined by g. Therefore, we can define $k : A/f \to A/g$ by $k([a]_f) = [a]_g$. Such definition of k is the only one satisfying $k \circ p_f = p_g$. We prove now that k is a K-homomorphism. Let $\overline{[a]}_f$ be a tuple in A/f and $s \in \Pi_F$. Since $\Pi_F \subseteq \Pi$, $s \in \Pi$ and $P_f(s) = s$. Therefore, $K(s) = K(P_f(s)) = P_g(s)$. This observation shall be used in the below computations which show that k is a K-homomorphism.

$$\begin{aligned}
k(s^{A/f}\overline{[a]}_f) &= k([s^A\overline{a}]_f) \\
&= [s^A\overline{a}]_g \\
&= P_g(s)^{A/g}\overline{[a]}_g \\
&= K(s)^{A/g}\, k(\overline{[a]}_f).
\end{aligned}$$

An application of Theorem 26 makes the above (right-sided) diagram commute.

Step (iii): Because f is a surjective F-homomorphism, f' and F' are invertible. We define $H =$

$G' \circ K \circ F'^{-1}$ and $h = g' \circ k \circ f'^{-1}$. Since the composition of homomorphisms is a homomorphism, the commutation of the two diagrams above imply that h is an H-homomorphism. To conclude this proof, notice that k was uniquely determined, g' and f' are unique, and then h is uniquely determined. In the same way K was uniquely determined, F' and G' are unique. Therefore, H is uniquely determined. Moreover, observe that f' is bijective, k is always a surjective map and then h is a surjective map exactly when g is a surjective map. On the other hand, g' and f' are always injective and k is an injective map exactly when $ker(f) = ker(g)$. Therefore, h is injective exactly when $ker(f) = ker(g)$. □

Proposition 18. *Let (A,Π), (B,Ψ) and (C,Ω) be three domains. Let $f : A \to B$ be a surjective F-homomorphism and $g : A \to C$ a G-homomorphism such that for every $t \in \Pi$, $[t]_F \subseteq [t]_G$ and $ker(f) \subseteq ker(g)$. Let h be the H-homomorphism determined by Lemma 28 above. The following three assertions hold:*

(1) If g is a surjective G-homomorphism, then h is a surjective H-homomorphism.
(2) If $ker(f) = ker(g)$ and for every for every $t \in \Pi$, $F^{-1}(F(t)) = G^{-1}(G(t))$, then h is an injective H-homomorphism.
(3) If g is a surjective G-homomorphism, $ker(f) = ker(g)$ and for every $t \in \Pi$, $F^{-1}(F(t)) = G^{-1}(G(t))$, then h is an H-isomorphism.

Proof. To show (1), observe that condition for every $t \in \Pi$, $F^{-1}(F(t)) \subseteq G^{-1}(G(t))$, means that every equivalence class determined by the fibers of F is contained in exactly one equivalence class determined by the fibers of G. Such condition implies that the term translation K defined in the proof of Lemma 28 is a surjective term translation. Therefore, since F' and G' are bijective and $H = G' \circ K \circ F'^{-1}$, we conclude that H is surjective. Because g is a surjective G-homomorphism, Lemma 28 ensures that h is surjective and then h is a surjective H-homomorphism.

To show (2), observe that the condition for every $t \in \Pi$, $F^{-1}(F(t)) = G^{-1}(G(t))$ means that the equivalence relation determined by the fibers of F is equal to the equivalence relation determined by the fibers of G. In such case, the term translation K defined in the proof of Lemma 28 is a bijective term translation. Since F' is bijective, G' is injective and $H = G' \circ K \circ F'^{-1}$, we conclude that H is injective. Because $ker(f) = ker(g)$, Lemma 28 ensures that h is injective. Therefore, h is an injective H-homomorphism. Finally, it is clear that statement (3) follows directly from statements (1) and (2). □

Let (A,Π) be an L-domain, $I:\Pi \to \Pi$ the identity term translation and $\{c\}$ a singleton. Observe that the pair $(\{c\},\Pi)$ is an L-domain [6] and that the map $e: A \to \{c\}$ is an I-homomorphism with the largest kernel $ker(e) = A \times A$. On the other hand, the identity map $i: A \to A$ is an I-homomorphism with the smallest kernel $ker(i) = \{(a,a) \mid a \in A\}$.

Let $\mathscr{K}$ be the set of kernels of all I-homomorphisms which are defined on the domain (A,Π) and let R be a binary relation defined on A. Let us consider the set defined by $\mathscr{K}(R) = \{K \in \mathscr{K} | R \subseteq K\}$. Next proposition (Proposition 19) shows that the intersection of an arbitrary collection of kernels from $\mathscr{K}(R)$ belongs to $\mathscr{K}(R)$. The *hull* of R is the intersection of all kernels belonging to $\mathscr{K}(R)$ and then, by Proposition 19 below, it is the smallest equivalence relation which contains R and is the kernel of some I-homomorphism. Next proposition ensures that the hull of any binary relation exists, is well defined and it is uniquely determined.

Proposition 19. *Let (A,Π) be an L-domain and let R be a binary relation on A. Assume that $\{K_\alpha\}_{\alpha \in D}$ is a collection of kernels from $\mathscr{K}(R)$. Then, the intersection $\bigcap_{\alpha \in D} K_\alpha$ belongs to $\mathscr{K}(R)$.*

Proof. Because $\{K_\alpha\}_{\alpha \in D}$ is a collection of kernels from $\mathscr{K}(R)$, it is easy to see that

$$K = \bigcap_{\alpha \in D} K_\alpha = \{(a,b) \in A \times A \mid (a,b) \in K_\alpha \text{ for every } \alpha \in D\}$$

is an equivalence relation which contains R. We want to show that K is the kernel of some I-homomorphism $k: (A,\Pi) \to (B,\Pi)$. In more detail, we shall show that $B = A/K$ i.e. the quotient set of A by K. In order to determine the domain $(A/K,\Pi)$ we define its operations. For every $t \in \Pi$, define $t^{A/K}$ in A/K by

$$t^{A/K}\overline{[a]}_K = [t^A\overline{a}]_K \tag{4.5}$$

where $\overline{[a]}_K$ is a tuple of elements of A/K. We should check that equation 4.5 defines well the operation $t^{A/K}$. To this aim, let us take two tuples $\overline{[a]}_K$ and $\overline{[b]}_K$ in A/K such that $\overline{[a]}_K = \overline{[b]}_K$ and let us show that $[t^A\overline{a}]_K = [t^A\overline{b}]_K$. Because K is the intersection of the K_α's, the affirmation $\overline{[a]}_K = \overline{[b]}_K$ implies that $\overline{[a]}_{K_\alpha} = \overline{[b]}_{K_\alpha}$ for every $\alpha \in D$. Because every K_α is a kernel, their associated quotient domains are well defined and then, $\overline{[a]}_{K_\alpha} = \overline{[b]}_{K_\alpha}$ implies $[t^A\overline{a}]_{K_\alpha} = [t^A\overline{b}]_{K_\alpha}$ for every $\alpha \in D$. Since K is the intersection of all the K_α's, we conclude $[t^A\overline{a}]_K = [t^A\overline{b}]_K$. Therefore, it has been shown that

[6] Such observation is true because for each $t \in \Pi$ we can define the operation $t^{\{c\}}$ by $t^{\{c\}}(\overline{c}) = c$ where $\overline{c}$ is any tuple in $\{c\}$

that $(A/K,\Pi)$ is a domain. Additionally, it is easy to see that $k : A \to A/K$ given by $k(a) = [a]_K$ is an I-homomorphism by performing straightforward computations[7]. Since K contains R and K is the kernel of k, we conclude that K belongs to $\mathscr{K}(R)$. $\square$

Let (A,Π) be an L domain and R be a binary relation defined on A. In what follows, we associate to R an I-homomorphism ρ_R whose kernel coincides with the hull of R. It is done in the following way: Let $r : (A,\Pi) \to (C,\Pi)$ be an I-homomorphism whose kernel is equal to the hull of R. Without loss of generality we can assume that r is surjective and then, by Corollary 27, (C,Π) is isomorphic to the quotient $(A,\Pi)/r$. The kernel of its projection map is equal to the hull of R because it is equal to the kernel of r. We shall denote such projection map by ρ_R (or just ρ when R is clear from the context) and we shall refer to it as the *I-homomorphism associated to the hull of R*. Next lemma shows how it gives a way to factorize homomorphisms between domains.

Lemma 29. *Let $g : (A,\Pi) \to (B,\Psi)$ be an G-homomorphism and let R be a binary relation defined on A. If $R \subseteq ker(g)$, then there exists a unique G-homomorphism $g' : A/\rho_R \to B$ such that the below diagram commutes. The map g' is surjective exactly when g is surjective and g' is injective exactly when $ker(g)$ is equal to the hull of R.*

$$\begin{tikzcd}
(A,\Pi) \arrow[r, "\rho"] \arrow[dr, "g"'] & (A,\Pi)/\rho \arrow[d, dashed, "g'"] \\
& (B,\Psi)
\end{tikzcd}$$

Proof. Let ρ be the I-homomorphism associated to the hull of R. We want to apply Lemma 28 (with $\rho = f$ and $g = g$). Let us check the hypotheses. Observe that ρ is a surjective I-homomorphism whose kernel is the hull of R. Because $R \subseteq ker(g)$, it is clear that $ker(\rho) \subseteq ker(g)$. Because ρ is an I-homomorphism and I is the identity term translation, the condition $[t]_I \subseteq [t]_G$ is trivially satisfied for every $t \in \Pi$. Then, an application of Lemma 28 yields a unique G-homomorphism $g' : A/\rho \to B$ with the properties stated above. $\square$

[7] $k(t^A\overline{a}) = [t^A\overline{a}]_K = t^{A/K}\overline{[a]}_K = I(t)^{A/K}\overline{[a]}_K$

4.2.4 Term Morphisms and Term Domains

The aim of this section is to stablish Corollary 30 and Theorem 31 below. But before that, we need to introduce a key concept which enable us to express the hypotheses of these two results. Recall that we have fixed V as the set of variables. A mapping α which assigns a term to every variable is called a *substitution* when the set $\{v \in V \mid v \neq \alpha(v)\}$ is finite. Such finiteness together with a fixed order in the set of variables enable us to denote a substitution α as an n-tuple of terms $\bar{s} = (s_1, s_2, ..., s_n)$ meaning that $\alpha(v_i) = s_i \neq v_i$ for $i = 1, ..., n$ and $\alpha(v) = v$ otherwise.

A substitution $\bar{s}$ can be applied to a term t, obtaining the term $t\bar{s}$ which, we recall, is the term obtained by simoultaneously replacing in the term t, each ocurrence of v_i by s_i. Given a set of terms Π, we say that $\bar{s}$ *is in* Π, when $\bar{s}$ is a tuple of elements in Π.

Definition 29. Let L be a signature and Π a set of L-terms. The set of terms *generated by* Π, which is denoted by Π^*, is defined as the minimal set which satisfies the two following conditions:

1. Π^* contains all the variables.
2. if $t \in \Pi$ and $\bar{s}$ is in Π^* , then $t\bar{s} \in \Pi^*$.

Observe that the pair (Π^*, Π) can be taken as an L-domain. To see that, define its set of operations in the following way. Let $t \in \Pi$ and $\bar{s}$ a sustitution in Π^*, the operation t^{Π^*} on the set Π^* is given by

$$t^{\Pi^*}(\bar{s}) = t\,\bar{s}. \tag{4.6}$$

The following discussion is aimed to analyze the relation of the domain (Π^*, Π) with a domain of the form (A, Π). An *assignment* is a mapping $\sigma : V \to A$. It can be easily proved[8] that σ can be extended in a unique way to a map $\widehat{\sigma} : \Pi^* \to A$ which satisfies $\widehat{\sigma}(t\bar{s}) = t^A(\widehat{\sigma}\bar{s})$ for every $t \in \Pi$ and $\bar{s}$ in Π^*. Observe that the set of operations defined by (4.6) above makes V a generator of the L-domain (Π^*, Π). This last observation would be interpreted in a context of universal algebra by saying that the domain (Π^*, Π) has the *universal mapping property*[9] for the class of domains of the form (A, Π).

[8] It is standard that an assignment can be extended in a unique way to a mapping from the set of L-terms to A. We can restrict such mapping to Π^* to obtain the desired extension.

[9] To say that the domain (Π^*, Π) has the universal mapping property for the class of domains of the form (A, Π) means that for every assignment $\alpha : V \to A$ there is a I-homomorphism $\widehat{\alpha} : (\Pi^*, \Pi) \to (A, \Pi)$ which extends α. In such case V would be a set of free generators of (Π^*, Π) and (Π^*, Π) would be freely generated by V.

Proposition 20. *Let* (A,Π) *be an L-domain. If* $\sigma : V \to A$ *is an assignment, then there exists a unique I-homomorphism* $\widehat{\sigma} : (\Pi^*,\Pi) \to (A,\Pi)$ *which extends* σ.

Proof. We have already discussed that σ has a unique extension $\widehat{\sigma} : \Pi^* \to A$ which satisfies $\widehat{\sigma}(t\overline{s}) = t^A(\widehat{\sigma}\overline{s})$ for every $t \in \Pi$ and $\overline{s}$ in Π^*. From formula 4.6, it is clear that $\widehat{\sigma}(t^{\Pi^*}(\overline{s})) = \widehat{\sigma}(t\overline{s}) = t^A(\widehat{\sigma}\overline{s})$. Therefore, $\widehat{\sigma}$ is an I-homomorphism. □

For a given term translation $F : \Pi \to \Psi$, we are going to analyze an F-homomorphism in particular which is important because of its nature. The existence of a particular F-homomorphism $F^* : (\Pi^*,\Pi) \to (\Psi^*,\Psi)$ is addressed in Chapter 2. Three observations are in order for appreciating its nature: (i) the map F^* is defined on a set of symbolic strings, (ii) the map F^* takes values on a set of symbolic strings, and (iii) the map F^* preserves a syntactic operation called substitution. Those observations suggest that F^* can be analyzed by using syntactic methods. The precise definition of F^* is given next.

Definition 30. Let Π be a set of L_1-terms and Ψ be a set of L_2-terms. A term translation $F^* : \Pi^* \to \Psi^*$ is a *term morphism from* Π^* *to* Ψ^* when it satisfies two conditions:

1. If v is a variable, then $F^*(v) = v$.
2. If $t \in \Pi$ and $\overline{s}$ is a substitution in Π^*, then

$$F^*(t\overline{s}) = F^*(t)\; F^*\overline{s}, \tag{4.7}$$

where if $\overline{s} = (s_1, s_2, ..., s_n)$, then $F^*\overline{s}$ means $(F^*(s_1), F^*(s_2), ..., F^*(s_n))$.

Notice that Definition 30 represents the idea of a sustitution-preserving mapping between a set of L_1-terms and a set of L_2-terms. However, the next straightforward proposition says that a term morphism is just a particular case of a homomorphism between domains.

Proposition 21. *Let* $F^* : \Pi^* \to \Psi^*$ *be a term morphism and let* F *be* F^* *restricted to* Π. *Then,* F^* *is an* F*-homomorphism from the domain* (Π^*,Π) *to the domain* (Ψ^*,Ψ).

One of the key hypothesis in Corollary 30 and Theorem 31 shall be the existence of a term morphism $F^* : \Pi^* \to \Psi^*$ extending a term translation F. Some conditions ensuring the existence of F^* are provided in Chapter 2.

Proposition 22. *Let (A,Π) be an L_1-domain, (B,Ψ) an L_2-domain and let $f : A \to B$ be an F-homomorphism. Assume that there exists a term morphism $F^* : \Pi^* \to \Psi^*$ which extends F. If $\alpha : V \to A$ is an assignment, then the I-homomorphism $\widehat{\beta}$ which extends $\beta = f \circ \alpha$ makes the following diagram commute.*

$$\begin{tikzcd}
(\Pi^*,\Pi) \arrow[r, "F^*"'] \arrow[d, "\widehat{\alpha}"] & (\Psi^*,\Psi) \arrow[d, "\widehat{\beta}"] \\
(A,\Pi) \arrow[r, "f"'] & (B,\Psi)
\end{tikzcd}$$

Proof. Let us consider the two F-homomorphisms from (Π^*,Π) to (B,Ψ) given by $f \circ \widehat{\alpha}$ and $\widehat{\beta} \circ F^*$. Notice that for each variable $v \in V$, $f \circ \widehat{\alpha}(v) = \beta(v) = \beta \circ F(v) = \beta \circ F^*(v)$. Therefore, by observing that the set of variables generates the domain (Π^*,Π), an application of Lemma 25 leads us to conclude that $f \circ \widehat{\alpha} = \widehat{\beta} \circ F^*$. That is to say, the above diagram commutes. □

Theorem 30 (Extension)**.** *Let (A,Π) be an L_1-domain generated by A', (B,Ψ) an L_2-domain and let $f' : A' \to B$ be a map. Assume that there exists a term morphism $F^* : \Pi^* \to \Psi^*$ which extends a given term translation $F : \Pi \to \Psi$. Let $\alpha : V \to A'$ be a surjective assignment and set $\beta = f' \circ \alpha$. If $ker(\widehat{\alpha}) \subseteq ker(\widehat{\beta} \circ F^*)$, then f' can be extended to a F-homomorphism $f : A \to B$ which makes next diagram commute. Additionally, f is uniquely determined.*

$$\begin{tikzcd}
(\Pi^*,\Pi) \arrow[r, "\widehat{\alpha}"] \arrow[d, "F^*"] & (A,\Pi) \arrow[d, dashed, "f"] \\
(\Psi^*,\Psi) \arrow[r, "\widehat{\beta}"] & (B,\Psi)
\end{tikzcd}$$

Proof. To show this result, we shall apply Lemma 28 (with $f = \widehat{\alpha}$ and $g = \widehat{\beta} \circ F^*$). We check now the hypotheses. Because the assignment α is surjective, the I-homomorphism $\widehat{\alpha}$ which extends α is surjective. Observe that for every $t \in \Pi$, the condition $[t]_I \subseteq [t]_F$ is trivially satisfied. Additionally, $ker(\widehat{\alpha}) \subseteq ker(\widehat{\beta} \circ F^*)$ is one of the hypotheses of this corollary. An application of Lemma 28 yields a uniquely determined F-homomorphism f which makes the above diagram commute. Such commutation is used in straightforward computations[10] which show that f extends f'. □

Theorem 31. *Let (A,Π) be an L_1-domain generated by a countable set A' , Ψ a set of L_2-terms and $F : \Pi \to \Psi$ a surjective term translation. If there exists a term morphism $F^* : \Pi^* \to \Psi^*$ extending*

[10]Let $a \in A'$. Notice that for some $x \in V$, $f(a) = f(\alpha(x)) = \beta(F^*(x)) = \beta(x) = f' \circ \alpha(x) = f'(a)$.

F, then there exists an L_2-domain (B,Ψ) together with a surjective F-homomorphism $e:(A,\Pi)\to(B,\Psi)$ which satisfy: for every F-homomorphism $h:(A,\Pi)\to(C,\Psi)$, there exists a unique I-homomorphism $h':(B,\Psi)\to(C,\Psi)$ which makes the below diagram commute. Additionally, h' is surjective exactly when h is surjective and h' is injective exactly when $ker(e)=ker(h)$.

$$\begin{tikzcd}
(A,\Pi) \arrow[r, "h"] \arrow[dr, "e"'] & (C,\Psi) \\
 & (B,\Psi) \arrow[u, dashed, "h'"']
\end{tikzcd}$$

Proof. Let us choose and fix a surjective assignment $\alpha:V\to A'$. This is always possible because A' is numerable. Since A' generates (A,Π), the assignment α can be extended in a unique way to a surjective I-homomorphism $\widehat{\alpha}:(\Pi^*,\Pi)\to(A,\Pi)$. Let us consider the binary relation R on Ψ^* given by

$$R=\{(F^*t,F^*s)\mid(t,s)\in ker(\widehat{\alpha})\}. \tag{4.8}$$

Let ρ be the I-homomorphism associated to the hull of R and set $(B,\Psi)=(\Psi^*,\Psi)/\rho$. We want to apply Lemma 28 (with $f=\widehat{\alpha}$ and $g=\rho\circ F^*$), to obtain a uniquely determined F-homomorphism $e:(A,\Pi)\to(B,\Psi)$ which makes the following diagram commute.

$$\begin{tikzcd}
(\Pi^*,\Pi) \arrow[r, "\widehat{\alpha}"'] \arrow[d, "F^*"'] & (A,\Pi) \arrow[d, dashed, "e"'] \\
(\Psi^*,\Psi) \arrow[r, "\rho"] & (B,\Psi)
\end{tikzcd} \tag{4.9}$$

Let us check the hypotheses of Lemma 28: For every $t\in\Pi$, the condition $[t]_I\subseteq[t]_F$ is trivially satisfied. For the other condition, we need to show that $ker(\widehat{\alpha})\subseteq ker(\rho\circ F^*)$. To this aim, let us take a pair $(t,s)\in ker(\widehat{\alpha})$. Then $(F^*t,F^*s)\in R$ and since $R\subseteq ker(\rho)$, it is clear that $(F^*t,F^*s)\in ker(\rho)$. It means that $\rho(F^*(t))=\rho(F^*(s))$ and then $(t,s)\in ker(\rho\circ F^*)$. We have shown that $ker\widehat{\alpha}\subset ker(\rho\circ F^*)$. Additionally, notice that ρ is surjective by definition, and since F is surjective, F^* must be surjective[11]. Therefore, $\rho\circ F^*$ is surjective. The above checks enable us to apply Lemma 28 and obtain a surjective F-homomorphism $e:(A,\Pi)\to(B,\Psi)$ which is uniquely determined and makes diagram 4.9 commute.

Let us show that for every F-homomorphism $h:(A,\Pi)\to(C,\Pi)$, there exists a unique I-

[11] It shall be clear in Chapter 2, where term morphisms are studied in detail.

homomorphism $h' : (B,\Psi) \to (C,\Psi)$ such that $h = h' \circ e$. Let $h : (A,\Pi) \to (C,\Pi)$ be an F-homomorphism. Let us define the assignment β by $\beta = h \circ \alpha$ and let $\widehat{\beta}$ be the I-homomorphism extending β. Let us consider the following diagram.

$$\begin{tikzcd}
 & (\Pi^*,\Pi) \arrow[dl, "F^*"'] \arrow[dr, "\widehat{\alpha}"] & \\
(\Psi^*,\Psi) \arrow[r, "\rho"] \arrow[dr, "\widehat{\beta}"'] & (B,\Psi) \arrow[d, dashed, "h'"] & (A,\Pi) \arrow[l, "e"'] \arrow[dl, "h"] \\
 & (C,\Psi) &
\end{tikzcd} \tag{4.10}$$

We want to apply Lemma 29 with the binary relation R defined by 4.8 and $g = \widehat{\beta}$. Notice that such application will give us an I-homomorphism $h' : (B,\Psi) \to (C,\Psi)$ which is uniquely determined and satisfies $h' \circ \rho = \widehat{\beta}$. This formula, together with the commutation of Diagram 4.9 and Proposition 22 will show that Diagram 4.10 commutes. Therefore, $h = h' \circ e$. Notice that Lemma 29 ensures that h' is surjective exactly when $\widehat{\beta}$ is surjective and that h' is injective exactly when $ker(\widehat{\beta}) = ker(\rho)$. To conclude this proof we must: (i) check the hypotheses of Lemma 29, (ii) prove that if h is surjective, then $\widehat{\beta}$ is surjective. (iii) prove that the statement $ker(e) = ker(h)$ implies the statement $ker(\widehat{\beta}) = ker(\rho)$.

For step (i), we must show that $R \subseteq ker(\beta)$. To this aim us take $(t', s') \in R$ and let us show that $(t', s') \in ker(\beta)$. Since $(t', s') \in R$ and F^* is surjective, there exists $(t, s) \in \Pi^*$ such that $F^* t = t'$, $F^* s = s'$ and $\widehat{\alpha}(s) = \widehat{\alpha}(t)$. Then, $h(\widehat{\alpha}(s)) = h(\widehat{\alpha}(t))$. Proposition 22 shows that this last equality is equivalent to the equality $\beta \circ F^*(t) = \beta \circ F^*(s)$. It means that $(t', s') \in ker(\beta)$. We have shown that $R \subseteq ker(\beta)$. Therefore, Diagram 4.10 commutes.

For step (ii), let us assume that h is surjective and then the composition $h \circ \widehat{\alpha}$ is surjective since $\widehat{\alpha}$ is surjective. From the commutation of Diagram 4.10 it is clear that $\widehat{\beta} \circ F^* = h \circ \widehat{\alpha}$ and then $\widehat{\beta} \circ F^*$ is surjective. Therefore, $\widehat{\beta}$ must be surjective.

For step (iii), let us assume that $ker(e) = ker(h)$ and observe that such condition is equivalent to say that for every $a, b \in A$, $e(a) = e(b) \Leftrightarrow h(a) = h(b)$. This statement implies that for every $s, t \in \Pi^*$, $e(\widehat{\alpha}(s)) = e(\widehat{\alpha}(t)) \Leftrightarrow h(\widehat{\alpha}(s)) = h(\widehat{\alpha}(t))$. We can rewrite this last equivalence by using the fact that Diagram 4.10 commutes. It produces that for every $s, t \in \Pi^*$, $\rho(F^*(s)) = \rho(F^*(t)) \Leftrightarrow \widehat{\beta}(F^*(s)) = \widehat{\beta}(F^*(t))$ holds. Since F^* is surjective, we conclude that for every $s', t' \in \Psi^*$, $\rho(s') =$

$\rho(t') \Leftrightarrow \widehat{\beta}(s') = \widehat{\beta}(t')$ holds. This last equivalence shows that $ker(\rho) = \ker(\widehat{\beta})$. $\square$

Let us make an observation that would be useful to give some interpretations of the last theorem. Notice that the domain (B,Ψ) is the quotient domain $(\Psi^*,\Psi)/\rho$ and then its elements are equivalence classes of the terms in Ψ^*. That means that the elements of this domain are strings on the language determined by Ψ^*. The observation is as follows: these elements can be seen also as (equivalence classes of) strings on the language determined by Π^*. Such observation follows directly from the diagram 4.9, above, and the fact that $\rho \circ F^*$ is a surjective F-homomorphism; an application of Corollary 27 ensures that domains $(\Pi^*,\Pi)/(\rho \circ F^*)$ and (B,Ψ) are F'-isomorphic for some term translation F'.

Corollary 32. *Let (A,Π) be an L_1-domain generated by A', Ψ a set of L_2-terms and $F : \Pi \to \Psi$ a bijective term translation. Assume that there exists a term morphism $F^* : \Pi^* \to \Psi^*$ which extends F. If there exists a surjective assignment $\alpha : V \to A'$ whose extension $\widehat{\alpha}$ satisfies*

$$(\forall s,t \in \Pi^*) \;\; F^*(s) = F^*(t) \Rightarrow \widehat{\alpha}(s) = \widehat{\alpha}(t), \tag{4.11}$$

then there exists an L_2-domain (B,Ψ) which is F-isomorphic to (A,Π).

Proof. Because we assume that F^* exists and extends F, we can apply Theorem 31 to obtain the domain (B,Ψ) and the F-homomorphism $e : (A,\Pi) \to (B,\Psi)$ which makes Diagram 4.9 commute. Additionally, observe that $\widehat{\alpha}$ satisfies condition 4.11 which is equivalent to $ker(F^*) \subseteq ker(\widehat{\alpha})$. This observation and the fact that F is bijective, enable us to apply Lemma 28 (with $f = F^*$ and $g = \widehat{\alpha}$) to obtain an F^{-1}-homomorphism $u : (\Psi^*,\Psi) \to (A,\Pi)$ which is uniquely determined, surjective and makes next diagram commute

$$\begin{tikzcd} (\Pi^*,\Pi) \arrow[r, "\widehat{\alpha}"] \arrow[d, "F^*"'] & (A,\Pi) \arrow[d, "e"] \\ (\Psi^*,\Psi) \arrow[r, "\rho"] \arrow[ur, dashed, "u"] & (B,\Psi) \end{tikzcd} \tag{4.12}$$

We want to use Lemma 29 to factorize u. To this aim, recall that $ker(\rho)$ is the hull of $R = \{(F^*t, F^*s) \mid (t,s) \in ker(\widehat{\alpha})\}$. If we are able to prove that $R \subseteq ker(u)$, we can use Lemma 29 to obtain an uniquely determined F^{-1}-homomorphism $k : (B,\Psi) \to (A,\Pi)$ which satisfies $u = k \circ \rho$.

It makes next diagram commute

$$\begin{tikzcd}
(\Pi^*,\Pi) \arrow[r, "\widehat{\alpha}"] \arrow[d, "F^*"'] & (A,\Pi) \arrow[d, bend left, "e"] \\
(\Psi^*,\Psi) \arrow[r, "\rho"] \arrow[ur, "u"] & (B,\Psi) \arrow[u, bend left, "k"]
\end{tikzcd}$$

and then $k \circ e \circ \widehat{\alpha} = \widehat{\alpha}$ and $e \circ k \circ \rho = \rho$. Because ρ and $\widehat{\alpha}$ are surjective maps, they are right cancelleable and then $k \circ e = id_A$ and $e \circ k = id_B$. It follows that $e^{-1} = k$ and that the domains (A,Π) and (B,Ψ) are F-isomorphic.

To end this proof, it is only left to show that $R \subseteq ker(u)$. To this aim, let us take a pair $(t,s) \in \Pi^*$. Assume that $(F^*t, F^*s) \in R$, and then $\widehat{\alpha}(t) = \widehat{\alpha}(s)$ by definition of R. Last equality can be rewritten by using Diagram 4.12 which yields $u(F^*(t)) = u(F^*(s))$. This equality proves that (F^*t, F^*s) belongs to $ker(u)$. Therefore, $R \subseteq ker(u)$ and the corollary follows. □

4.2.5 Products for Term Domains

Coproducts for Term Domains

This section is devoted to building a domain which acts as the coproduct of an indexed collection $\{(B_\alpha, \Psi_\alpha)\}_{\alpha \in D}$ of L_α-domains. For the purposes of this section we shall need to use the *disjoint union* of sets (denoted by $\uplus$) which is a modified union operation that indexes the elements according to which set they originated in. Let us observe that when $\{X_\alpha\}_{\alpha \in D}$ is an indexed collection of sets, every element x of the set $\biguplus_{\alpha \in D} X_\alpha$ corresponds to exactly one element $y \in X_\lambda$ and exactly one index $\lambda \in D$. Without loss of generality, we shall identify x and y in order to avoid a cumbersome notation. Such identification can be thought as if the X_α's were mutually disjoint [12].

Given a collection $\{(B_\alpha, \Psi_\alpha)\}_{\alpha \in D}$ of L_α-domains, we need a signature to determine the language of the coproduct domain. Let us define the set of function symbols L as the set of terms

$$L = \biguplus_{\alpha \in D} \Psi_\alpha,$$

[12] It can be done without loss of generality because, if there is a common element between two X_α we can index one of them making them mutually disjoint.

and observe that for some $\lambda \in D$, each element s of L is an L_λ-term. However, we emphasize that s shall be treated as a function symbol of the new signature L. The arity of each function symbol $s \in L$ is given by the cardinality of the set of variables of the term s. For each function symbol s in L we determine exactly one L-term in the following way. Denote by $\bar{x}_s$ the set of variables occurring in s ordered in the way in which they occur in s. Let us define the following set of L-terms

$$\oplus_{\alpha\in D}\Psi_\alpha = \{s(x_1,x_2,....,x_n) \mid s \in L \text{ and } \bar{x}_s = (x_1,x_2,...,x_n)\} \tag{4.13}$$

and notice that for $\lambda \in D$, the map $I_\lambda : \Psi_\lambda \to \oplus_{\alpha\in D}\Psi_\alpha$ defined by $I_\lambda(s) = s(x_1,x_2,....,x_n)$ is an injective term translation. In what follows, we build the semantics of the coproduct domain.

Definition 31. For a given collection $\{(B_\alpha,\Psi_\alpha)\}_{\alpha\in D}$ of term domains, its set of *words* is given by $B = \biguplus_{\alpha\in D} B_\alpha$. The set of *phrases* over the family $\{(B_\alpha,\Psi_\alpha)\}_{\alpha\in D}$ is denoted by $\oplus_{\alpha\in D}B_\alpha$ (or $\oplus B_\alpha$) and inductively defined as the minimal set satisfying two conditions

1. every word is a phrase, i.e. $B \subseteq \oplus B_\alpha$.
2. if $s \in L$ is a n-ary function symbol and $\overline{p} = (p_1,...,p_n)$ is a n-tuple of phrases such that $p_k \notin B_\lambda$ for some k, then $s(p_1,...,p_n)$ is a phrase.

It is clear that any phrase p can be uniquely written as $p = s\overline{q}$ where $s \in L$ and $\overline{q}$ is a tuple of phrases. Such assertion follows from Definition 31 and the following observation. We can define a new signature L' by considering all elements of B as new constant symbols of L. Clearly, the set of phrases $\oplus_{\alpha\in D}B_\alpha$ can be embedded into the set of terms of this new signature L'. It is straightforward that every L'- term can be uniquely written as $t = s\overline{w}$ where s is a function symbol of L' and $\overline{w}$ is a tuple of L'-terms and then, each phrase $p \in \oplus_{\alpha\in D}B_\alpha$ is uniquely written as $p = s\overline{q}$.

Le us observe that $\oplus_{\alpha\in D}B_\alpha$ is an inductive set. The *complexity* of $p \in \oplus_{\alpha\in D}B_\alpha$ is the number of L-function symbols which occur on p (counting repeated ocurrences). Observe that if $p = s\overline{q}$, then each phrase in the tuple $\overline{q}$ has a complexity smaller than the complexity of p. Additionally, words are the unique phrases with zero complexity.

By $\oplus_{\alpha\in D}(B_\alpha,\Psi_\alpha)$ we denote the L-domain $(\oplus B_\alpha, \oplus_{\alpha\in D}\Psi_\alpha)$ whose set of operations is defined as follows. Let $s(x_1,x_2,...,x_n) \in \oplus_{\alpha\in D}\Psi_\alpha$ and let $\lambda \in D$ such that $s \in \Psi_\lambda$. Let $\overline{p} = (p_1,p_2,...,p_n)$

be a n-tuple of phrases. The operation $s^{\oplus B_\alpha}$ is defined in the set of phrases by

$$s^{\oplus B_\alpha}(\overline{p}) = \begin{cases} s^{B_\lambda}(\overline{p}), & \text{if } p_i \in B_\lambda \text{ for every } i \in \{1,2,...,n\}. \\ s(p_1,p_2,....,p_n), & \text{otherwise.} \end{cases}$$

From this definition, the following proposition is straighforward.

Proposition 23. *Let λ be an arbitrary element of D. The injection $i_\lambda : B_\lambda \to \oplus_{\alpha\in D}B_\alpha$ defined by $i_\lambda(b) = b$, is a I_λ-homomorphism from the term domain $(B_\lambda, \Psi_\lambda)$ to the term domain $\oplus_{\alpha\in D}(B_\alpha, \Psi_\alpha)$.*

Definition 32. Let $\{(B_\alpha, \Psi_\alpha)\}_{\alpha\in D}$ be an indexed collection of term domains. The *coproduct domain* of $\{(B_\alpha, \Psi_\alpha)\}_{\alpha\in D}$ is the term domain $\oplus_{\alpha\in D}(B_\alpha, \Psi_\alpha)$ together with the family of I_λ-homomorphisms $i_\lambda : B_\lambda \to \oplus B_\alpha$.

Theorem 33. *Let (A,Π) be a L'-domain and let $\{(B_\alpha, \Psi_\alpha)\}_{\alpha\in D}$ be a collection of L_α-domains. For every $\lambda \in D$, let $F_\lambda : \Psi_\lambda \to \Pi$ be a term translation. If for every $\lambda \in D$ there exists a F_λ-homomorphism $f_\lambda : B_\lambda \to A$, then there exist a unique term translation $F : \oplus_{\alpha\in D}\Psi_\alpha \to \Pi$ which satisfies $F \circ I_\lambda = F_\lambda$ and a unique F-homomorphism $f : \oplus_{\alpha\in D}B_\alpha \to A$ which makes the following diagram commute*

$$\begin{tikzcd} (A,\Pi) & \\ \oplus_{\alpha\in D}(B_\alpha,\Psi_\alpha) \arrow[u, dashed, "f"] & (B_\lambda,\Psi_\lambda) \arrow[l, "i_\lambda"] \arrow[ul, "f_\lambda"'] \end{tikzcd}$$

Proof. Let us define the term translation $F : \oplus_{\alpha\in D}\Psi_\alpha \to \Pi$ in the unique way which satisfies $F \circ I_\lambda = F_\lambda$ for every $\lambda \in D$, that is to say that, if $s(x_1,x_2,...,x_n) \in \oplus_{\alpha\in D}\Psi_\alpha$ with $s \in \Psi_\lambda$, then $F(s(x_1,x_2,...,x_n)) = F_\lambda(s)$. Let us use induction on the complexity of phrases to define $f : \oplus_{\alpha\in D}B_\alpha \to A$. Let $p \in \oplus B_\alpha$ be a phrase and set

$$f(p) = \begin{cases} f_\lambda(p) & \text{if } p \in B_\lambda, \\ F_\lambda(s)^A f\overline{w} & \text{if } p = s\overline{w} \text{ with } s \in \Psi_\lambda \text{ and } \overline{w} \text{ a tuple of phrases.} \end{cases} \tag{4.14}$$

where if $\overline{w} = (w_1,w_2,...,w_n)$, then $f\overline{w} = (f(w_1), f(w_2),..., f(w_n))$.

Since $f \circ i_\lambda = f_\lambda$ for every $\lambda \in D$, it is clear that the definition of f makes the above diagram commute. Let us show that f is an F-homomorphism. Let us take a n-ary term $s(x_1,x_2,...,x_n) \in$

$\oplus_{\alpha \in D}\Psi_\alpha$ with $s \in \Psi_\lambda$ and let $\overline{p}$ be a n-tuple of phrases. We must show that $f(s^{\oplus B_\alpha}\overline{p}) = F(s)^A f\overline{p}$. There are two alternatives: The first alternative is when every component of the tuple $\overline{p}$ is a word belonging to B_λ. Then:

$$\begin{aligned} f(s^{\oplus B_\alpha}\overline{p}) &= f(s^{B_\lambda}\overline{p}) \\ &= f_\lambda(s^{B_\lambda}\overline{p}) \\ &= F_\lambda(s)^A f_\lambda(\overline{p}) \\ &= F(s)^A f\overline{p}. \end{aligned}$$

Otherwise, (second alternative) there exists at least one component of the tuple $\overline{p}$ which does not belong to B_λ. Then, from the definition of f, it is clear that:

$$\begin{aligned} f(s^B\overline{p}) &= F_\lambda(s)^A f\overline{p} \\ &= F(s)^A f\overline{p} \end{aligned}$$

To complete this proof we are going to show that f is unique. Let us assume that there exist another F-homomorphism $f' : \oplus_{\alpha\in D}B_\alpha \to A$ which makes the diagram in the statement of this theorem commute. Let us make two observations. First, $f' \circ i_\lambda = f_\lambda$ for every $\lambda \in D$, which implies that on words with zero complexity $f = f'$. Second, f' must satisfy the equation of a F-homomorphism

$$f'(t^B\overline{w}) = F(t)^A f'\overline{w}, \tag{4.15}$$

for every word (in particular for those words with complexity larger than zero). These two observations mean that equation 4.14 is satisfied when f is replaced by f'. Since equation 4.14 defines f, we conclude that f and f' are equal. □

Complementing last theorem, notice that if there exists a F-homomorphism $f : \oplus_{\alpha\in D}B_\alpha \to A$, it is clear that for every $\lambda \in D$, the map $f_\lambda = f \circ i_\lambda$ is an $(F \circ I_\lambda)$-homomorphism.

Products for Term Domains

This section is devoted to building the product domain of an indexed collection $\{(B_\alpha, \Psi_\alpha)\}_{\alpha\in D}$ of term domains. Because of a technical requirement we shall assume that the index set D has a distinguished element 0. To define the product domain, we must first determine the signature associated to this domain. We warn the reader that the election of the function symbols for this new signature shall be very particular and non usual. However, such election simplifies the proofs for this section. We define the set L of function symbols as the set of sequences of terms given by:

$$L = \{(t_\alpha)_{\alpha \in D} \in \prod_{\alpha \in D} \Psi_\alpha \mid \text{ for all } \beta, \gamma \in D,\ V(t_\beta) = V(t_\gamma)\}.$$

In what follows, we determine exactly one L-term for every function symbol s of L. Observe that each function symbol s of L is an indexed sequence of terms whose components share the exact same set x_s of variables. Let us assume that $\bar{x}_s = (x_1, x_2, ..., x_n)$ is the tuple obtained by ordering the set x_s in the way these variables appear in t_0. This enables us to determine uniquely the following set of L-terms

$$\otimes_{\alpha \in D} \Psi_\alpha = \{s(x_1, x_2, ..., x_n) \mid s \in L \text{ and } \bar{x}_s = (x_1, x_2, ..., x_n)\}, \tag{4.16}$$

and observe that for any $\lambda \in D$, the map $P_\lambda : \otimes_{\alpha \in D} \Psi_\alpha \to \Psi_\lambda$ which applied to $s(x_1, x_2, ..., x_n)$ with $s = (t_\alpha)_{\alpha \in D}$ returns the term t_λ (i.e. $P_\lambda(s(x_1, x_2, ..., x_n)) = t_\lambda$) is a term translation.

On the other hand, we need to address the semantics of the product domain. Let $B = \prod_{\alpha \in D} B_\alpha$ be the cartesian product of the family of sets $\{B_\alpha\}_{\alpha \in D}$ and $p_\lambda : \prod_{\alpha \in D} B_\alpha \to B_\lambda$ its coordinate maps. We denote by $\otimes_{\alpha \in D}(B_\alpha, \Psi_\alpha)$ the term domain $(B, \otimes_{\alpha \in D} \Psi_\alpha)$ whose family of operations is determined as follows. Let $s \in \otimes_{\alpha \in D} \Psi_\alpha$ be a n-ary term and $\bar{b}$ a n-tuple in B. The operation $s^B : B^n \to B$ is defined componentwise by

$$(s^B \bar{b})_\lambda = P_\lambda(s)^{B_\lambda} p_\lambda \bar{b}, \tag{4.17}$$

where, if $\bar{b} = (b_1, b_2, ..., b_n)$ with $b_i = (b_{i\alpha})_{\alpha \in D}$, then $p_\lambda \bar{b} = (b_{1\lambda}, b_{2\lambda}, ..., b_{n\lambda})$. From formula 4.17 and the observation that $p_\lambda(s^B \bar{b}) = (s^B \bar{b})_\lambda$, next proposition is straighforward.

Proposition 24. *Let λ be an arbitrary element of D. The coordinate map $p_\lambda : \prod_{\alpha \in D} B_\alpha \to B_\lambda$ is a P_λ-homomorphism from the term domain $\otimes_{\alpha \in D}(B_\alpha, \Psi_\alpha)$ to the term domain $(B_\lambda, \Psi_\lambda)$.*

Definition 33. Let $\{(B_\alpha, \Psi_\alpha)\}_{\alpha \in D}$ be an indexed collection of term domains. Its *product domain* is the term domain $\otimes_{\alpha \in D}(B_\alpha, \Psi_\alpha)$ together with the family of P_λ-homomorphisms $p_\lambda : \prod_{\alpha \in D} B_\alpha \to B_\lambda$.

Next theorem shows that the domain $\otimes_{\alpha\in D}(B_\alpha,\Psi_\alpha)$ satisfies the expected properties of a product object. The above definitions make next theorem straightforward. However, for the sake of clarity, details are given.

Theorem 34. *Let (A,Π) be an L'-domain and $\{(B_\alpha,\Psi_\alpha)\}_{\alpha\in D}$ a family of L_α-domains. For every $\lambda\in D$ let $F_\lambda:\Pi\to\Psi_\lambda$ be a term translation. If for every $\lambda\in D$ there exists a F_λ-homomorphism $f_\lambda:A\to B_\lambda$, then there exists a unique term translation $F:\Pi\to\oplus_{\alpha\in D}\Psi_\alpha$ which satisfies $P_\lambda\circ F=F_\lambda$ and a unique F-homomorphism $f:A\to\prod_{\alpha\in D}B_\alpha$ which makes next diagram commute:*

$$\begin{tikzcd}
(A,\Pi) \arrow[d, dashed, "f"] \arrow[dr, "f_\lambda"] & \\
\otimes_\alpha(B_\alpha,\Psi_\alpha) \arrow[r, "p_\lambda"'] & (B_\lambda,\Psi_\lambda)
\end{tikzcd}$$

Proof. Let $F:\Pi\to\otimes_{\alpha\in D}\Psi_\alpha$ be the term translation defined componentwise by $F(s)_\lambda=F_\lambda(s)$. Notice that it is the unique definition which satisfies $P_\lambda\circ F=F_\lambda$ for every $\lambda\in D$. Let $B=\prod_{\alpha\in D}B_\alpha$ and let us define $f:A\to B$ componentwise by

$$f(a)_\lambda=f_\lambda(a).$$

Notice that $p_\lambda\circ f=f_\lambda$ and then, f is the unique mapping which makes the above diagram commute. In order to show that f is a F-homomorphism, let us take a n-ary term $s\in\Pi$, and a n-tuple $\overline{a}\in A$. Notice that for each index $\lambda\in D$, next computations follow

$$\begin{aligned}
f(s^A\overline{a})\ _\lambda &= f_\lambda(s^A\overline{a})\\
&= F_\lambda(s)^{B_\lambda}f_\lambda(\overline{a})\\
&= (F(s)_\lambda)^{B_\lambda}\ f_\lambda(\overline{a})\\
&= (P_\lambda\circ F(s))^{B_\lambda}\ p_\lambda\circ f(\overline{a})\\
&= p_\lambda(F(s)^Bf(\overline{a}))\\
&= [F(s)^Bf(\overline{a})]_\lambda
\end{aligned}$$

and then, $f(s^A\overline{a})=F(s)^Bf(\overline{a})$. This proves that f is an F-homomorphism. $\square$

Complementary to the above theorem, let (A,Π) be an L'-domain and let $F:\Pi\to\otimes_{\alpha\in D}\Psi_\alpha$ be a term translation. Notice that if there exists a F-homomorphism f from (A,Π) to the product domain $\otimes_{\alpha\in D}(B_\alpha,\Psi_\alpha)$, the composition $f_\lambda = p_\lambda\circ f$ is an $(P_\lambda\circ F)$-homomorphism uniquely determined for each $\lambda\in D$.

4.3 Formula Domains

The goal of this section is to enhace the metaphor model presented in Section 4.2. This new model will be able to handle some cases where term domains are not useful. This cases occur when it is necessary to represent some relations between the objects of the domain but those relations can not be modeled through operations defined in the domain. Another goal of this section is to provide a set of formulas which can be associated to a meaning in term domains. This section is highly dependent on Chapter 2 and Chapter 3 of this work. Additionally, we recommend the reader to review Section 4.2.1 before reading next section.

4.3.1 A Little More of Model Theory

Let L be a signature. The *atomic formulas* of L are the strings of symbols determined inductively by the two following conditions

1. If s and t are terms of L, then the string $s=t$ is an atomic formula of L.
2. If $n>0$, R is an n-ary relation symbol of L and $t_1,...,t_n$ are terms of L, then the expression $R(t_1,t_2,..,t_n)$ is an atomic formula of L.

Let Q be an L-structure. An *assignment* is a map $\sigma:V\to dom(Q)$ from the set of variables V to the set of elements of Q. We shall denote them by $\sigma:V\to Q$. It is well known that σ can be extended in a unique way to a map $\widehat{\sigma}:Term(L)\to Q$ which for every n-ary function symbol $f\in L$ and every tuple $t_1,...,t_n$ of L-terms satisfies

$$\widehat{\sigma}(f(t_1,...,t_n)) = f^Q(\widehat{\sigma}(t_1),...,\widehat{\sigma}(t_n)).$$

Additionally, let us consider an n-ary relation symbol $R \in L$ and let R^Q be its associated n-ary relation defined on the set of elements of Q. Given an assignment σ, we can associate the atomic formula $R(t_1, t_2, .., t_n)$ to the truth value $R^Q(\widehat{\sigma}(t_1), ..., \widehat{\sigma}(t_n))$. In case that $R^Q(\widehat{\sigma}(t_1), ..., \widehat{\sigma}(t_n))$ holds, we write $\sigma \models_Q R(t_1, ..., t_n)$ and say that σ *satisfies* the atomic formula $R(t_1, ..., t_n)$. The below theorem of model theory describes a tight relation between homomorphisms of L-structures (see Definition 25 in Section 4.2) and atomic formulas of L.

Theorem 35. *Let A and B be L-structures and let $f : dom(A) \to dom(B)$ be a map.*

1. *If f is a homomorphism, then for every assignment $\sigma : V \to A$,*

$$f \circ \widehat{\sigma} = \widehat{f \circ \sigma}.$$

2. *The map f is a homomorphism if and only if every atomic formula ϕ of L and every assignment $\sigma : V \to A$ satisfy*

$$\text{if } \sigma \models_A \phi \text{ , then } f \circ \sigma \models_B \phi.$$

The symbols $\neg$, $\wedge$, $\vee$, $\forall$ and $\exists$ stand for the meanings “not”, “and”, “or”, “for all elements” and “there is an element” respectively. These symbols and meanings are standard and well known in mathematical logic. The class of the *first order formulas* for the signature L is denoted by $Form(L)$ and is defined as the smallest class of formulas determined by the following conditions:

1. all atomic formulas of L are formulas of L.
2. if ϕ is a formula, then the expression $\neg\phi$ is a formula.
3. if ϕ and ψ are formulas, then the expressions $\phi \wedge \psi$ and $\phi \vee \psi$ are formulas.
4. if ϕ is a formula and x is a variable, then the expressions $\exists x\phi$ and $\forall x\phi$ are formulas.

The formulas which go into the making of a formula ϕ are called the subformulas of ϕ. The quantifiers $\forall y$ (“for all y”) and $\exists y$ (“there is y”) bind variables in the way that is standard in logic. In the same standard way, we distinguish between free and bound occurrences of variables in a formula. The *free variables* of a formula ϕ are those variables which have free occurrences in ϕ. The concept of *satisfaction* is extended to the class of all L-formulas in the following standard way.

Definition 34. Let Q be an L-structure and let $\sigma : V \to Q$ be an assignment. We say that σ *satisfies* a formula ϕ in Q (in symbols $\sigma \models_Q \phi$), when the inductive construction of ϕ satisfies one of the following conditions:

1. if ϕ is atomic, then $\sigma \models \phi$ holds if σ satisfies ϕ.
2. if $\phi = \neg\varphi$, then $\sigma \models \phi$ if and only if $\sigma \models \varphi$ does not hold.
3. if $\phi = \varphi \wedge \psi$, then $\sigma \models \phi$ if and only if $\sigma \models \varphi$ and $\sigma \models \psi$.
4. if $\phi = \varphi \vee \psi$, then $\sigma \models \phi$ if and only if $\sigma \models \varphi$ or $\sigma \models \psi$.
5. if $\phi = \exists x\varphi$, then $\sigma \models \phi$ if and only if exists $q \in Q$ such that $\sigma\frac{q}{x} \models \varphi$.
6. if $\phi = \forall x\varphi$, then $\sigma \models \phi$ if and only if for every $q \in Q$, $\sigma\frac{q}{x} \models \varphi$.

where $\sigma\frac{q}{x}$ denotes the assignment whose values are all equal to the values of σ, except for the variable x where $\sigma\frac{q}{x}(x) = q$.

A *substitution* is a map $\rho : V \to Term(L)$ which associates an L-term to every variable. A substitution ρ can be applied to an an L-term s by replacing every variable x occurring in s by $\rho(x)$. The new term obtained from such process is denoted by $s\rho$. Moreover, it can be shown that such process determines uniquely an extension $\widehat{\rho} : Term(L) \to Term(L)$ which for every n-ary function symbol $f \in L$ and every tuple $t_1,...,t_n$ of L-terms satisfies

$$\widehat{\rho}(f(t_1,...,t_n)) = f(\widehat{\rho}(t_1),...,\widehat{\rho}(t_n)).$$

Additionally, a substitution ρ can be applied to an L-formula φ. However, a careful process is needed (see Definition 13 in Chapter 3). The formula which results from such process is denoted by $\varphi\rho$. A well known result of Mathematical Logic is the next one which is provided here without proof.

Lemma 36 (Substitution Lemma). *Supposse that Q is an L-structure, s is an L-term and φ is an L-formula. Let $\sigma : V \to Q$ be an assignment and let $\rho : V \to Term(L)$ be a substitution. Next two assertions hold.*

1. $\widehat{\sigma}(s\rho) = \widehat{\sigma} \circ \widehat{\rho}(s) = \widehat{\widehat{\sigma} \circ \rho}(s)$.
2. $\sigma \models \varphi\rho$ *if and only if* $\widehat{\sigma} \circ \rho \models \varphi$

4.3.2 Formula Domains and its Homomorphisms

In this section we are going to formalize knowledge domains. The formalization presented below has the ability of representing general relations between objects of the domain. However, we have restricted ourselves to work with no quantified formulas. The analysis of the case when quantified formulas are included is beyond the scope of this work. Such analysis will require new techniques and more research. From now on, every formula mentioned in this chapter shall have no quantifiers. Let us begin with the definition of a formula domain.

Definition 35. Assume that Q is an L-structure. Let (A,Π) be a term domain in the context of Q and Σ a finite non-empty set of non quantified L-formulas. The triple (A,Σ,Π) is called a *formula domain* (in the context of Q) when for every $\varphi \in \Sigma$, every t occurring in φ and every assignment $\sigma : V \to A$, the element $\widehat{\sigma}(t)$ belongs to A.

Observe that the definition of a formula domain (A,Σ,Π) requires A closed for the family of operations determined by the terms occurring in the formulas of Σ. In what follows, we shall need some basic facts about term domains. Most of them were previously introduced in Section 4.2. However, we rewrite them below in a way well suited for the goals of this section.

Let (A,Π) be a term domain in the context of Q and let Π^* (Definition 29) be the set of L-terms generated by Π. When the image of a substitution ρ is contained in Π^* (i.e. $\rho : V \to \Pi^*$), it can be extended in a unique way to a map $\widehat{\rho} : \Pi^* \to \Pi^*$ which is an I-homomorphism of term domains. When considering another substitution $\gamma : V \to \Pi^*$, follows from Lemma 25 that the substitution $\widehat{\rho} \circ \gamma : V \to \Pi^*$ can be uniquely extended to the I-homomorphism $\widehat{\rho} \circ \widehat{\gamma} : \Pi^* \to \Pi^*$. By an analogous reasoning, when $\sigma : V \to A$ is an assignment, the composition $\widehat{\sigma} \circ \widehat{\rho} : \Pi^* \to A$ is the uniquely determined I-homomorphism which extends the assignment $\widehat{\sigma} \circ \rho$. Last assertion follow from the fact that each assignment $\sigma : V \to A$ can be extended in a unique way to an I-homomorphism $\widehat{\sigma} : \Pi^* \to A$. These observations suggest to think about Π^* as a collection of terms that can be interpreted in the domain (A,Π). [13]

In the next discussion we seek to convince our readers that the formulas of the collection $(\Sigma \otimes \Pi)^+$ (Definition 18) can be interpreted in the domain (A,Σ,Π) where the meaning of the word "interpretation" is formalized in Definition 36 below. Recall that formulas in $(\Sigma \otimes \Pi)^+$ are obtained as combinations of formulas from $\Sigma \otimes \Pi = \{\varphi\rho \mid \varphi \in \Sigma, \rho \text{ in } \Pi^*\}$ by means of the syntactic operators $\wedge, \vee$ and $\neg$.

Definition 36. Assume that Q is an L-structure and let (A,Σ,Π) be a formula domain with context Q. We say that the extension $\widehat{\sigma}$ of an assignment $\sigma : V \to A$ *satisfies* a formula $\varphi \in (\Sigma \otimes \Pi)^+$ in the

[13] To reinforce this suggestion, it is easy to create examples where $\widehat{\sigma}(t) \notin A$, if $t \notin \Pi^*$.

domain (A,Σ,Π) exactly when φ is satisfied by σ in Q (as in Definition 34). In symbols,

$$\widehat{\sigma} \models_A \varphi \text{ if and only if } \sigma \models_Q \varphi. \tag{4.18}$$

Observe that given an assignment σ, the definition of satisfaction in an L-structure Q (Definition 34) gives an inductive procedure which uses the elements of Q to assign a truth value to L-formulas. It might seem like $\widehat{\sigma} \models_A \varphi$ in (4.18) above depends not only on the elements of A but on the elements of Q. Such impression is false. It turns out that when $\sigma : V \to A$ and $\varphi \in (\Sigma \otimes \Pi)^+$, we just need the elements of A to apply the procedure of Definition 34 to determine if $\sigma \models_A \varphi$ holds. Next two straightforward results prove this last affirmation.

Proposition 25. *Assume that (A,Σ,Π) is a formula domain. Let s be a term which occurs in $\varphi \in \Sigma$ and let ρ be a substitution in Π^*. If $\sigma : V \to A$ is an assignment, then $\widehat{\sigma}(s\rho) \in A$.*

Proof. Let $\sigma : V \to A$ be an assignment. Because ρ is a substitution whose image is a subset of Π^*, the composition $\widehat{\sigma} \circ \rho$ is an assignment whose image is a subset of A. The Substitution Lemma ensures that the extension of $\widehat{\sigma} \circ \rho$ is $\widehat{\sigma} \circ \widehat{\rho}$. Since (A,Σ,Π) is a formula domain and s is a term ocurring in a formula of Σ, the element $\widehat{\sigma} \circ \widehat{\rho}(s) = \widehat{\sigma}(s\rho)$ belongs to A. □

Next corollary follows directly from the above proposition and the definition of $(\Sigma \otimes \Pi)^+$.

Corollary 37. *Let (A,Σ,Π) be a formula domain and $\sigma : V \to A$ an assignment. If s is any term occurring in any formula $\varphi \in (\Sigma \otimes \Pi)^+$, then $\widehat{\sigma}(s)$ belongs to A.*

Last corollary together with Proposition 25 show that $\sigma \models_A \varphi$ in Definition 36 depends only on elements of A. In other words, formula domains have enough information to determine the truth value of any formula in $(\Sigma \otimes \Pi)^+$. Next straightforward lemma shall be useful to perform some computations later on. by using graph theory and unification theory.

Lemma 38 (Substitution Lemma for Domains)**.** *Let (A,Σ,Π) be a formula domain (in a context Q). Let $\varphi \in (\Sigma \otimes \Pi)^+$ and $\rho : V \to \Pi^*$ a substitution. Then, for any assignment $\sigma : V \to A$,*

$$\widehat{\sigma} \models_A \varphi\rho \quad \textit{if and only if} \quad \widehat{\sigma} \circ \widehat{\rho} \models_A \varphi$$

Proof. Let us assume that $\widehat{\sigma} \models_A \varphi\rho$ and then, by definition $\sigma \models_Q \varphi\rho$. An application of the Substitution Lemma (Lemma 36) yields $\widehat{\sigma} \circ \rho \models_Q \varphi$. Since $\widehat{\sigma} \circ \rho$ has form $V \to A$, it means that $\widehat{\sigma} \circ \widehat{\rho} \models_A \varphi$ by definition again. On the other hand, if we assume that $\widehat{\sigma} \circ \widehat{\rho} \models_A \varphi$, by definition $\widehat{\sigma} \circ \rho \models_Q \varphi$. An application of Lemma 36 produces $\sigma \models_Q \varphi\rho$. It means that $\widehat{\sigma} \models_A \varphi\rho$. ◻

This study continues by determining the meaning of a *formula homomorphism* between two formula domains (Definition 37 below). To this aim, recall that a map between two sets of formulas (each one of them having possibly a different signature) is called a *formula translation*, if it preserves the free variables of formulas. In the same manner, a mapping between two sets of terms (each one of them having possibly a different signature) is called a *term translation* when it preserves the variables of terms.

Let (A,Σ,Π) and (B,Ω,Ψ) be two formula domains. Assume that $T : \Pi \to \Psi$ is a term translation and that $f : A \to B$ is an T-homomorphism from the term domain (A,Π) to the term domain (B,Ψ). Let $\varphi \in (\Sigma \otimes \Pi)^+$ and $\phi \in (\Omega \otimes \Psi)^+$. We say that f *preserves* φ in ϕ when for every assignment $\sigma : V \to A$,

$$\text{if } \widehat{\sigma} \models_A \varphi, \text{ then } \widehat{f \circ \sigma} \models_B \phi. \tag{4.19}$$

Definition 37. Let (A,Σ,Π) and (B,Ω,Ψ) be two formula domains. Let $f : A \to B$ be a T-homomorphism from (A,Π) to (B,Ψ) where $T : \Pi \to \Psi$ is a term translation. Assume that $F : \Sigma \to \Omega$ is a formula translation. We say that $f : A \to B$ is an *F-homomorphism respect to T* if for every $\varphi \in \Sigma$, f preserves φ in $F(\varphi)$.

Next, we present some properties of formula homomorphisms. Those results show that a formula homomorphism is able to preserve other formulas than those in Σ. Assume that φ in $(\Sigma \otimes \Pi)^+$. In order to say that a formula homomorphim preserves φ in ϕ, we must have a formula $\phi \in (\Omega \otimes \Psi)^+$ in first place. The problem of how to associate a formula ϕ to φ was analized in Chapter 2 and Chapter 3 of this work. For the sake of completeness of this discussion, next paragraph presents a brief (and necessarily incomplete) summary of the concepts introduced in these chapters. These concepts will enable us to present the results for this chapter.

Let L_1 and L_2 be two signatures. Let $T : \Pi \to \Psi$ be a term translation and let $F : \Sigma \to \Omega$ be a formula translation where Σ is a set of L_1-formulas, Ω is a set of L_2-formulas, Π is a set of L_1-terms and Ψ is a set of L_2-terms. If $T^* : \Pi^* \to \Psi^*$ is a T-homomorphism from (Π^*,Π) to (Ψ^*,Ψ) which extends T and acts as the identity on variables, we say that T^* is the *term morphism* extending T. Such T^* does not always exists, but when it exists, it must be unique. Assume that T^* is a term morphism extending T and consider the problem of extending F to a formula translation $L : (\Sigma \otimes \Pi) \to (\Omega \otimes \Psi)$ which for every $\varphi \in \Sigma$ and every substitution $\rho : V \to \Pi^*$ satisfies

$$L(\varphi\rho) = F(\varphi)\ (T^* \circ \rho). \tag{4.20}$$

Again, L does not always exists. When it exists, it must be unique. Finally, suppose that L exists and consider the problem of extending $L : \Sigma \otimes \Pi \to \Omega \otimes \Psi$ to a formula translation $L^+ : (\Sigma \otimes \Pi)^+ \to (\Omega \otimes \Psi)^+$ which preserves logical connectors at a syntactic level. If it exists, we say that L^+ is a *formula morphism* extending L. We recommend the reader to review Chapter 3, Definition 19 where a formula morphism is defined. Later on, we shall need to perform some computations. Next lemma will help us with that.

Lemma 39. *Let (A, Π) be an L_1-domain, (B, Ψ) an L_2-domain, $T : \Pi \to \Psi$ a term translation and $f : A \to B$ an T-homomorphism. If there exists a term morphism T^* which extends T, then for any substitution $\rho : V \to \Pi^*$ and any assignment $\sigma : V \to A$, the following diagram commutes.*

$$\begin{tikzcd}
\Pi^* \arrow[r, "T^*"] \arrow[d, "\widehat{\rho}"'] & \Psi^* \arrow[d, "\widehat{T^* \circ \rho}"] \\
\Pi^* \arrow[r, "T^*"'] \arrow[d, "\widehat{\sigma}"'] & \Psi^* \arrow[d, "\widehat{f \circ \sigma}"] \\
A \arrow[r, "f"'] & B
\end{tikzcd} \tag{4.21}$$

Proof. Observe that every arrow in the diagram is an homomorphism of term domains: every vertical arrow in the diagram is an I-homomorphism and every horizontal arrow in the diagram is an T-homomorphism. Therefore, any composition of arrows which goes from left to right is an T-homomorphism. Consequently, we can show the result by using the following reasoning. For every variable x, it is clear that $(T^* \circ \rho) \circ T^*(x) = (T^* \circ \rho) \circ I(x) = T^* \circ \rho(x)$. Since (Π^*, Π) is a term domain generated by the set of variables, Lemma 25 ensures us that the respective extensions are equal i.e. $T^* \circ \widehat{\rho} = (\widehat{T^* \circ \rho}) \circ T^*$. That means that the upside part of the diagram commutes. The same reasoning applied to the downside part of the diagram enables us to conclude that $f \circ \widehat{\sigma} = \widehat{f \circ \sigma} \circ T^*$. It means that the below square of the diagram commutes. Therefore, the whole diagram commutes and our result follows. □

Theorem 40. *Let (A,Σ,Π) and (B,Ω,Ψ) be two formula domains, $T:\Pi\to\Psi$ a term translation and $F:\Sigma\to\Omega$ a formula translation. Assume that $L:\Sigma\otimes\Pi\to\Omega\otimes\Psi$ is well defined by equation 4.20. If $f:A\to B$ is an F-homomorphism (respect to T), then for every formula $\psi\in\Sigma\otimes\Pi$, f preserves ψ in $L(\psi)$.*

Proof. We must show that for every $\psi\in\Sigma\otimes\Pi$ and every assignment $\sigma:V\to A$,

$$\widehat{\sigma}\models_A \ \psi \ \text{ implies } \ \widehat{f\circ\sigma}\models_B \ L(\psi). \tag{4.22}$$

Because $\psi\in\Sigma\otimes\Pi$, we can set $\psi=\varphi\rho$ with $\varphi\in\Sigma$ and ρ in Π^*. Assume that $\widehat{\sigma}\models_A\varphi\rho$ and then by Lemma 38 we obtain that $\widehat{\sigma}\circ\widehat{\rho}\models_A\varphi$ which means that $\widehat{\widehat{\sigma}\circ\rho}\models_A\varphi$. Because f is an F-homomorphism, we conclude that $f\circ\widehat{\widehat{\sigma}\circ\rho}\models_B F(\varphi)$. By using Lemma 39 it is easy to see that $f\circ\widehat{\widehat{\sigma}\circ\rho}=\widehat{f\circ\sigma}\circ(\widehat{T^*\circ\rho})$. Therefore, $\widehat{f\circ\sigma}\circ(\widehat{T^*\circ\rho})\models_B F(\varphi)$ holds. An application of the Substitution Lemma (Lemma 38) yields $\widehat{f\circ\sigma}\models_B F(\varphi)(T^*\circ\rho)$. The result follows because $F(\varphi)(T^*\circ\rho)=L(\varphi\rho)=L(\psi)$. □

Our next result concerns a subset of formulas of $(\Sigma\otimes\Pi)^+$. To prove it, we will need to perform induction. To this aim, we are going to associate a *size* to every formula of $(\Sigma\otimes\Pi)^+$. The size of a formula shall be directly related to the concept of a tree over a set of formulas introduced in Chapter 3. Let us consider the set of trees over $\Sigma\otimes\Pi$ denoted by $(\Sigma\otimes\Pi)^\circ$. A tree $a\in(\Sigma\otimes\Pi)^\circ$ is called a *lifting* of $\varphi\in(\Sigma\otimes\Pi)^+$ when its projection $\mu(a)$ is equal to φ (see Chapter 3, Section 3.2.2). Each tree has a *length* which is just the cardinality of the set of edges of its maximal directed path. The *size* of a formula $\varphi\in(\Sigma\otimes\Pi)^+$ is the minimum of the set of lengths of all its liftings. Since every $\varphi\in\Sigma\otimes\Pi$ has a lifting which has only one node and no edges, it is clear that a formula φ has zero size if and only if $\varphi\in\Sigma\otimes\Pi$. To express our next result, define the set Δ as the minimal subset of $(\Sigma\otimes\Pi)^+$ which contains $\Sigma\otimes\Pi$ and satisfies: if $\varphi,\phi\in\Delta$, then $\varphi\wedge\phi\in\Delta$ and $\varphi\vee\phi\in\Delta$. Observe that Δ is the set of all the combinations of formulas from $\Sigma\otimes\Pi$ by means of the operators $\wedge$ and $\vee$.

Theorem 41. *Let (A,Σ,Π) and (B,Ω,Ψ) be two formula domains, $T:\Pi\to\Psi$ a term translation, $F:\Sigma\to\Omega$ a formula translation and $f:A\to B$ an F-homomorphism respect to T. Assume that $L:\Sigma\otimes\Pi\to\Omega\otimes\Psi$ is well defined by equation 4.20 and exists. If there exists a formula morphism $L^+:(\Sigma\otimes\Pi)^+\to(\Omega\otimes\Psi)^+$ extending L, then for every $\psi\in\Delta$, f preserves ψ in $L^+(\psi)$.*

Proof. Let us show that for every assignment $\sigma : V \to A$,

$$\text{if} \quad \widehat{\sigma} \models_A \psi, \quad \text{then} \quad \widehat{f \circ \sigma} \models_B L^+(\psi). \tag{4.23}$$

We use induction on the size of ψ. The base step is provided by Theorem 40 since ψ having zero size is equivalent to $\psi \in \Sigma \otimes \Pi$. Let us prove the inductive step. Let us show that the theorem holds for a formula $\psi \in \Delta$ whose size is $n > 1$. Our induction hypothesis is that the theorem is true for every formula in Δ with size $n-1$ or less. There are two different cases. First, assume that $\psi = \varphi \wedge \phi$ where $\varphi \in \Delta$ and $\phi \in \Delta$. It is clear that φ and ϕ have sizes $n-1$ (or less) because if not, the size of ψ would be greater than n. Thus, we can apply the inductive hypothesis on both formulas. If $\widehat{\sigma} \models_A \varphi \wedge \phi$, then $\widehat{\sigma} \models_A \varphi$ and $\widehat{\sigma} \models_A \phi$. By applying the induction hypotheses in φ and ϕ we obtain that $\widehat{f \circ \sigma} \models_B L^+(\varphi)$ and $\widehat{f \circ \sigma} \models_B L^+(\phi)$. Since L^+ is a formula morphism, we conclude that $\widehat{f \circ \sigma} \models_B L^+(\varphi \wedge \phi)$ and the result holds for this case. The case $\psi = \varphi \vee \phi$ is totally analogous to the first one. The analysis of these two cases, shows that the theorem holds. □

Theorem 40 and Theorem 41 gives us conditions which guarantee that f preserves a formula φ in a formula ψ. This preservation is a notion which depends on assignments. However, there is a notion of preservation of formulas which does not depends on assignments. We describe such notion next. Let (A, Σ, Π) be a formula domain and $\varphi \in (\Sigma \otimes \Pi)^+$ a formula. We say that A is a *model* of φ (in symbols $A \models \varphi$) when for every assignment $\sigma : V \to A$, $\widehat{\sigma} \models_A \varphi$ holds. From this definition, next proposition is straighforward.

Proposition 26. *Let (A, Σ, Π), (B, Ω, Ψ) be two formula domains, $\varphi \in (\Sigma \otimes \Pi)^+$, $\phi \in (\Omega \otimes \Psi)^+$ and $T : \Pi \to \Psi$ a term translation. Let $f : A \to B$ be a surjective T-homomorphism from (A, Π) to (B, Ψ) and assume that f preserves φ in ϕ. If A is a model for φ, then B is a model for ϕ, i.e. if $A \models \varphi$, then $B \models \phi$.*

Proof. Since f is surjective, for every assignment $\lambda : V \to B$, there exists an assignment $\sigma : V \to A$ such that $\lambda = f \circ \sigma$. Additionally, $A \models \varphi$ means that $\widehat{\sigma} \models_A \varphi$ holds for every assignment $\sigma : V \to A$. It implies that for every $\lambda : V \to B$, $\widehat{\lambda} \models_B \phi$ because f preserves φ in ϕ. Therefore, $B \models \phi$. □

We finalize this section by introducing a map which is a stronger version of a homomorphism between formula domains.

Definition 38. Assume that (A, Σ, Π) and (B, Ω, Ψ) are two formula domains with possibly distinct contexts. Let $F : \Sigma \to \Omega$ be a formula translation and let $T : \Pi \to \Psi$ be a term translation. An F-homomorphism $f : A \to B$ respect to T is called a *metaphor respect to F and T* when next two conditions are satisfied

1. F is injective, and
2. for every formula ψ in the image of F and every assignment $\lambda : V \to B$, if $\widehat{\lambda} \models_B \psi$, then there exists $\sigma : V \to A$ such that $f \circ \sigma = \lambda$ and $\widehat{\sigma} \models_A F^{-1}(\psi)$.

The last theorem of this section aims to summarize two properties of the mapping above defined as metaphor. In order to express these two properties we need to formalize "consequence". Let (A, Σ, Π) be a formula domain and let $\varphi, \psi \in (\Sigma \otimes \Pi)^+$. By $\psi \models_A \varphi$ we denote the fact that φ is a *consequence* of ψ which means that for every assignment $\sigma : V \to A$,

$$\text{if} \quad \widehat{\sigma} \models_A \psi, \quad \text{then} \quad \widehat{\sigma} \models_A \varphi. \tag{4.24}$$

Theorem 42. *Let (A, Σ, Π) and (B, Ω, Ψ) be two formula domains. Let $F : \Sigma \to \Omega$ be a formula translation and let $T : \Pi \to \Psi$ be a term translation. Let $\phi, \psi_1, \psi_2, \ldots, \psi_m, \varphi_1, \varphi_2, \ldots, \varphi_n \in \Sigma$ and set $\psi = \bigvee_{i=1}^{m} \psi_i$, $F(\psi) = \bigvee_{i=1}^{m} F(\psi_i)$, $\varphi = \bigvee_{i=1}^{n} \varphi_i$ and $F(\varphi) = \bigvee_{i=1}^{n} F(\varphi_i)$. Suppose that $f : A \to B$ is a metaphor respect to F and T. The following two sentences hold.*

1. *if $A \models \neg\phi$, then $B \models \neg F(\phi)$.*

2. *if $\psi \models_A \varphi$, then $F(\psi) \models_B F(\varphi)$.*

Proof. Let us show 1. We use *Reductio ad absurdum* to prove this statement. Let us assume that (i) $A \models \neg\phi$ holds, and (ii) $B \models \neg F(\phi)$ does not hold. From assumption (ii), there exists an assignment $\lambda : V \to B$ such that $\widehat{\lambda} \models_B F(\phi)$. Because f is a metaphor, there must exist an assignment $\sigma : V \to A$ such that $f \circ \sigma = \lambda$ and $\widehat{\sigma} \models_A \phi$. This contradicts our assumption (i). We have shown that $A \models \neg\phi$ implies $B \models \neg F(\phi)$.

Let us show 2. For this statement we present a proof by contradiction. Let us assume that (a) $\psi \models_A \varphi$ holds, and (b) $F(\psi) \models_B F(\varphi)$ does not hold. From assumption (b), there exists an assignment $\lambda : V \to B$ such that

$$\widehat{\lambda} \models_B F(\psi) \text{ holds, and } \widehat{\lambda} \models_B F(\varphi) \text{ does not hold.} \tag{4.25}$$

Since $\widehat{\lambda} \models_B F(\psi)$, there exists some $i \in \{1,..,m\}$ such that $\widehat{\lambda} \models_B F(\psi_i)$. Then, and because f is a metaphor, there exists some some assignment $\sigma : V \to A$ such that $f \circ \sigma = \lambda$ and $\widehat{\sigma} \models_A \psi_i$. It means that $\widehat{\sigma} \models_A \psi$ and because of assumption (a), $\widehat{\sigma} \models_A \varphi$. Therefore, for some $j \in \{1,...,n\}$, $\widehat{\sigma} \models_A \varphi_j$. Using the fact that f is an F-homomorphism and that $f \circ \sigma = \lambda$ we obtain that $\widehat{\lambda} \models_B F(\varphi_j)$ which means that $\widehat{\lambda} \models_A F(\varphi)$. This contradicts assumption 4.25 above. Therefore, $\psi \models_A \varphi$ implies $F(\psi) \models_B F(\varphi)$. □

Next, we provide plausible interpretations for the properties of metaphor described in last theorem. The first property says that when a formula ϕ can not be satisfied in the base domain, the metaphor induces the belief that the analogous formula $F(\phi)$ can not be satisfied in the target domain either. For the interpretation of the second property we take consequence as a model for human inference. In such case, metaphors are inference-preserving maps. Notice that such property endows this model of the humand mind with the capacity of performing inferences in unknown domains.

4.3.3 Quotients and Products for Formula Domains

This section is aimed to present three constructions on formula domains which act as quotient, product and coproduct objects respectively. It shall be shown in Theorem43, Theorem 45 and Theorem 46 that such constructions satisfy the expected universal properties.

Aimed to improve the readability of the next discussion we stick to the following convention. Let (A,Σ,Π) and (B,Ω,Ψ) be two formula domains with possibly different contexts. We shall use the lower case, the upper case and the upper case marked with a tilde versions of the same letter to denote a formula homomorphism, its formula translation and its term translation. For example, by f we shall denote an F-homomorphism respect to $\tilde{F}$, where $F : \Sigma \to \Omega$ is a formula translation and $\tilde{F} : \Pi \to \Psi$ is a term translation. Additionally, when we say that $f : A \to B$ *is an F-homomorphism respect to $\tilde{F}$*, we imply the fact that the term translation $\tilde{F} : \Pi \to \Psi$ and the formula translation

$F : \Sigma \to \Omega$ are given. In cases where the term translation $\tilde{F}$ associated to f is clear from the context, we just say that f is an F-homomorphism.

Quotient Domains

Let (A,Σ,Π) and (B,Ω,Ψ) be two formula domains with possibly distinct contexts. Let $f : A \to B$ be an F-homomorphism respect to $\tilde{F}$. Our next discussion is aimed to introduce a domain called the *quotient domain of* (A,Σ,Π) *by* f which is a formula domain built on the quotient term domain $(A,\Pi)/f$. As in the case of term domains, we are interested in that this new domain has the same signature which is associated to the domain (A,Σ,Π). To this aim, we need to determine a set of formulas $\Sigma_F \subseteq \Sigma$ which has exactly one representant for each equivalence class of the quotient of Σ by the fibers of $F : \Sigma \to \Omega$ i.e. Σ/F. We determine uniquely Σ_F by giving a fixed order to $\Sigma = \{\phi_1,\phi_2,\phi_3,...\phi_n\}$ and selecting the subset of terms Σ_F wich satisfies the requirement and minimizes the sum of the subindexes of its elements. Observe that there is a bijection between the set of formulas Σ_F and the quotient set Σ/F. Therefore, we determine uniquely the surjective formula translation $P : \Sigma \to \Sigma_F$ which for $\phi \in \Sigma$ and $\varphi \in \Sigma_F$ satisfies $P(\phi) = \varphi$ exactly when $F(\phi) = F(\varphi)$.

Recall that $(A,\Pi)/f$ is the quotient domain of the term domain (A,Π) by the $\tilde{F}$-homomorphism $f : A \to B$ which was defined [14] by the pair $(A/f,\Pi_{\tilde{F}})$. Additionally, recall the fact that $p : A \to A/f$ is the projection map which is an $\tilde{P}$-homomorphism from (A,Π) to $(A,\Pi)/f$. We want to show that the triple $(A/f,\Sigma_F,\Pi_{\tilde{F}})$ is a formula domain. To this aim, we determine an n-ary relation R_ϕ defined on the set A/f for each formula $\phi \in \Sigma_F$ with free variables $x_1,x_2,...,x_n$. Consider $\sigma : V \to A/f$ an assignment and let $R_\phi^{A/f}(\sigma(x_1),\sigma(x_2),....,\sigma(x_n))$ holds if and only if

$$\text{there exists an assignment } \gamma : V \to A \text{ such that } p \circ \gamma = \sigma \text{ and } \widehat{\gamma} \models_A \phi. \tag{4.26}$$

From this definition, it is clear that the projection map $p : A \to A/f$ becomes a surjective P-homomorphism from the formula domain (A,Σ,Π) to the formula domain $(A/f,\Sigma_F,\Pi_{\tilde{F}})$.

Definition 39. Let (A,Σ,Π), (B,Ω,Ψ) be two formula domains with possibly distinct contexts. Let $f : A \to B$ be an F-homomorphism respect to $\tilde{F}$. We denote by $(A,\Sigma,\Pi)/f$ to the triple

[14] See definition 33 for details.

$(A/f, \Sigma_F, \Pi_F)$ which is the *quotient domain* of (A, Σ, Π) respect to f when the projection map $p : A \to A/f$ is considered.

Despite the fact that the above definition makes next theorem straightforward, we include the details in the proof.

Theorem 43. *Let (A, Σ, Π) and (B, Ω, Ψ) be two formula domains with possibly distinct contexts. If $f : A \to B$ is an F-homomorphism respect to $\tilde{F}$, then there exists a unique map $f' : A/f \to B$ which is an injective F'-homomorphism respect to $\tilde{F}'$ and makes below diagram commute. The maps F' and $\tilde{F}'$ are injective, uniquely determined and satisfy $F = F' \circ P$, $\tilde{F} = \tilde{F}' \circ \tilde{P}$.*

$$\begin{tikzcd} (A, \Sigma, \Pi) \arrow[r, "f"] \arrow[d, "p"'] & (B, \Omega, \Psi) \\ (A, \Sigma, \Pi)/f \arrow[ur, dashed, "f'"'] & \end{tikzcd} \tag{4.27}$$

Proof. The formula translation F' is injective and uniquely determined by $F'(\varphi) = F(\varphi)$ for $\varphi \in \Sigma_F$. Notice that such is the only definition which makes F' satisfy $F' \circ P = F$. In the same way, the term translation $\tilde{F}'$ is injective and uniquely determined by $\tilde{F}'(t) = \tilde{F}(t)$ for $t \in \Pi_F$. Additionally, f' is the mapping uniquely determined by $f'([a]) = f(a)$ where $[a] \in A/f$ is the class of $a \in A$. Theorem 26 shows that f' is an $\tilde{F}'$-homomorphism.

We want to show that f' is an F'-homomorphism respect to $\tilde{F}'$. To this aim, let $\sigma : V \to A/f$ be an assignment and let $\phi \in \Sigma_F$ be a formula. We must show that if $\sigma \models_{A/f} \phi$, then $\widehat{f' \circ \sigma} \models_B F'(\phi)$. Assume that $\sigma \models_{A/f} \phi$ and then, by definition of the quotient domain, there must exists some assignment $\gamma : V \to A$ such that $\widehat{\gamma} \models_A \phi$ and $p \circ \gamma = \sigma$. Because f is an F-homomorphism, $\widehat{f \circ \gamma} \models_B F(\phi)$ holds. It means that $\widehat{f' \circ p \circ \gamma} \models_B F(\phi)$ since $f' \circ p = f$. Thus, $\widehat{f' \circ \sigma} \models_B F'(\phi)$ holds because $p \circ \gamma = \sigma$ and $\phi \in \Sigma_F$. Therefore, $f' : A/f \to B$ is an injective F'-homomorphism respect to $\tilde{F}'$. □

Let (A, Σ, Π) and (B, Ω, Ψ) be two formula domains. We say that $f : A \to B$ is an *F-isomomorphism respect to* $\tilde{F}$, when the three maps $f, F, \tilde{F}$ are invertible and $f : A \to B$ is an F-homomorphism respect to $\tilde{F}$ whose inverse is an F^{-1}-homomorphism respect to $\tilde{F}^{-1}$. The formula domains (A, Σ, Π) and (B, Ω, Ψ) are *F-isomorphic* when there exists an F-isomomorphism between them. It is clear that an F-isomorphism between formula domains preserves a formula φ in $F(\varphi)$ if and only if it preserves its negation $\neg\varphi$ in $\neg F(\varphi)$.

Observe that Theorem 43 is a classical proof of the universal property of a quotient object. It is interesting that in this case, such result does *not* imply that when f is a surjective homomorphism, the quotient domain $(A,\Sigma,\Pi)/f$ is F-isomorphic[15] to (B,Ω,Ψ). Next corollary shows that such result is true, when f is a metaphor.

Corollary 44. *Let* (A,Σ,Π) *and* (B,Ω,Ψ) *be two formula domains with possibly distinct contexts. Let* $f : A \to B$ *be an* F*-homomorphism respect to* $\tilde{F}$*, where* f, F *and* $\tilde{F}$ *are surjective maps. If* f *is a metaphor respect to* F *and* $\tilde{F}$*, then the quotient domain* $(A,\Sigma,\Pi)/f$ *is isomorphic to the formula domain* (B,Ω,Ψ)*.*

Proof. Observe that maps f', F' and $\tilde{F}'$ defined in the proof of the above theorem (Theorem 43) are all bijective because f, F and $\tilde{F}$ are all surjective. Corollary 27 shows that f'^{-1} is an $\tilde{F}^{-1}$-homomorphism. We must show that the map $f'^{-1} : B \to A/f$ is an F'^{-1}-homomorphism. To this aim, let $\lambda : V \to B$ be an assignment and let $\varphi \in \Omega$ be a formula, we will show that if $\widehat{\lambda} \models_B \varphi$, then $\widehat{f'^{-1} \circ \lambda} \models_{A/f} F'^{-1}(\varphi)$. Let us assume that $\widehat{\lambda} \models_B \varphi$. Because f is a metaphor, there must exists an assignment $\sigma : V \to A$ such that $f \circ \sigma = \lambda$ and $\widehat{\sigma} \models_A F^{-1}(\varphi)$. Therefore, since p is an P-homomorphism, $\widehat{p \circ \sigma} \models_{A/f} P \circ F^{-1}(\varphi)$. We can rewrite this last expression as $\widehat{f'^{-1} \circ f} \circ \sigma \models_{A/f} F'^{-1}(\varphi)$ since $p = f'^{-1} \circ f$ and $F'^{-1} = P \circ F^{-1}$. Therefore, $\widehat{f'^{-1} \circ \lambda} \models_{A/f} F'^{-1}(\varphi)$ holds and the corollary follows. □

Coproducts for Formula Domains

This section is devoted to build a formula domain which acts as the coproduct of an indexed collection $\{(B_\alpha, \Omega_\alpha, \Psi_\alpha)\}_{\alpha \in D}$ of formula domains where each domain has an associated signature L_α. This new formula domain shall be based on the coproduct $\oplus_{\alpha \in D}(B_\alpha, \Psi_\alpha)$ of term domains. We recommed our reader to review Section 4.2.5 of this chapter where the coproduct of term domains was introduced.

We need a signature L which can be associated to the new coproduct of formula domains. Let

[15] To see that, consider a huge relation in the target domain as for example $B \times B$.

us make this in the following way. Consider each formula of $\biguplus_{\alpha\in D}\Omega_\alpha$ as a *relation symbol* whose arity is the cardinality of its set of free variables. The new signature is given by

$$L = L' \cup \biguplus_{\alpha\in D} \Omega_\alpha$$

where L' is the signature associated to the coproduct of term domains $\oplus_{\alpha\in D}(B_\alpha, \Psi_\alpha)$.

In order to define the coproduct of formula domains, we need to build a set of L-formulas. Let R be an n-ary relation symbol of L. Observe that R is an L_λ-formula belonging to some Ω_λ, and assume that $x_1, x_2, ..., x_n$ are its free variables ordered in the way which they occur in the formula. We build exactly one L-formula for each relation simbol $R \in L$ by defining the following set of atomic L-formulas:

$$\oplus_\alpha \Omega_\alpha = \{R(x_1, x_2, ..., x_n) \mid R \in \biguplus_{\alpha\in D} \Omega_\alpha\}.$$

For each $\lambda \in D$, define the injection $I_\lambda : \Omega_\lambda \to \oplus_\alpha \Omega_\alpha$ by $I_\lambda(R) = R(x_1, x_2, ..., x_n)$ and notice that such map is a formula translation.

Recall that the coproduct of term domains is defined by $\oplus_{\alpha\in D}(B_\alpha, \Psi_\alpha) = (\oplus_\alpha B_\alpha, \oplus_{\alpha\in D}\Psi_\alpha)$ where $\oplus_\alpha B_\alpha$ is the set of phrases [16]. Recall that the injection maps $i_\lambda : B_\lambda \to B$ are defined by $i_\lambda(b) = b$. We shall denote by $\oplus_\alpha(B_\alpha, \Omega_\alpha, \Psi_\alpha)$ the formula domain $(\oplus_\alpha B_\alpha, \oplus_\alpha \Omega_\alpha, \oplus_\alpha \Psi_\alpha)$ whose set of relations is determined in the following manner. Let $R(x_1, x_2, ..., x_n) \in \oplus_\alpha \Omega_\alpha$ with $R \in \Omega_\lambda$. The relation $R^{\oplus_\alpha B_\alpha}$ on $\oplus_\alpha B_\alpha$ is defined by:

$$R^{\oplus_\alpha B_\alpha}(p_1, p_2, ..., p_n) \text{ holds if and only if } p_1, p_2, ..., p_n \in B_\lambda \text{ and } \sigma \models_{B_\lambda} R, \tag{4.28}$$

251 where $\sigma : V \to B_\lambda$ is determined by $\sigma(x_i) = p_i$ for $i = 1, 2, ..., n$ and $\sigma(x) = p_1$ otherwise. The above definition makes next proposition straightforward. However, we give a proof in case the reader want to follow the details.

Proposition 27. *Let λ be an arbitrary element of D. The injection $i_\lambda : B_\lambda \to \oplus_{\alpha\in D} B_\alpha$ is an I_λ-homomorphism respect to $\tilde{I}_\lambda$.*

Proof. Proposition 23 shows that the injection $i_\lambda : B_\lambda \to \oplus_{\alpha\in D} B_\alpha$ defined by $i_\lambda(b) = b$ is an $\tilde{I}_\lambda$-homomorphism. We must show that i_λ is an I_λ-homomorphism respect to $\tilde{I}_\lambda$. To this aim, let

[16] see Definition 31.

$R \in \Omega_\lambda$ be a formula, $\gamma : V \to B_\lambda$ be an assignment. Assume that $\widehat{\gamma} \models_{B_\lambda} R$. From the above equivalence 4.28 follows directly that $\widehat{i_\lambda \circ \gamma} \models_B I_\lambda(R)$. □

Definition 40. $\{(B_\alpha, \Omega_\alpha, \Psi_\alpha)\}_{\alpha \in D}$ be an indexed collection of formula domains. The *coproduct domain* of such collection is the formula domain $\oplus_\alpha(B_\alpha, \Omega_\alpha, \Psi_\alpha)$ together with the family $i_\lambda : B_\lambda \to \oplus_{\alpha \in D} B_\alpha$ of I_λ-homomorphisms respect to $\tilde{I}_\lambda$.

Theorem 45. *Let (A, Σ, Π) be a formula domain and let $\{(B_\alpha, \Omega_\alpha, \Psi_\alpha)\}_{\alpha \in D}$ be a family of formula domains with possibly distinct contexts. Assume that for every $\lambda \in D$, $f_\lambda : B_\lambda \to A$ is an F_λ-homomorphism respect to $\tilde{F}_\lambda$. Then, there exists a unique map $f : \oplus_{\alpha \in D} B_\alpha \to A$ which is an F-homomorphism respect to $\tilde{F}$ and makes diagram 4.29 below commute. The maps F and $\tilde{F}$ are uniquely determined and satisfy $F \circ I_\lambda = F_\lambda$ and $\tilde{F} \circ \tilde{I}_\lambda = \tilde{F}_\lambda$.*

$$\begin{tikzcd}
(A, \Sigma, \Pi) & \\
\oplus_{\alpha \in D}(B_\alpha, \Omega_\alpha, \Psi_\alpha) \arrow[u, dashed, "f"] & (B_\lambda, \Omega_\lambda, \Psi_\lambda) \arrow[l, "i_\lambda"] \arrow[ul, "f_\lambda"']
\end{tikzcd} \tag{4.29}$$

Proof. The term translation $\tilde{F} : \oplus_{\alpha \in D} \Psi_\alpha \to \Pi$ and the mapping $f : \oplus_\alpha B_\alpha \to A$ are defined exactly as in the proof of Theorem 33. Observe that these are the unique ways in which $\tilde{F}$ satisfies $\tilde{F} \circ \tilde{I}_\lambda = \tilde{F}_\lambda$ for every $\lambda \in D$ and f makes the above diagram commute. Theorem 33 shows that f is an $\tilde{F}$-homomorphism and that it is unique.

Let us define the formula translation F in the only way which satisfies $F \circ I_\lambda = F_\lambda$. For a formula $R(x_1, x_2, ..., x_n) \in \oplus_\alpha \Omega_\alpha$ with $R \in \Omega_\lambda$, define $F(R(x_1, x_2, ..., x_n)) = F_\lambda(R)$. Clearly, F is a formula translation since F_λ is a formula translation. We must show that f is an F-homomorphism. Let $\sigma : V \to A$ be an assignment and let $R(x_1, x_2, ..., x_n) \in \oplus_\alpha \Omega_\alpha$ with $R \in \Omega_\lambda$. We must show that if $\widehat{\sigma} \models_{\oplus_\alpha B_\alpha} R(x_1, x_2, ..., x_n)$, then $\widehat{f \circ \sigma} \models_A F(R(x_1, x_2, ..., x_n))$. Let us assume that $\widehat{\sigma} \models_{\oplus_\alpha B_\alpha} R(x_1, x_2, ..., x_n)$ and then equivalence 4.28 ensures that an assignment $\alpha : V \to B_\lambda$ satisfying $\alpha(x_i) = \sigma(x_i)$ for $i = 1, 2, ..., n$ makes $\widehat{\alpha} \models_{B_\lambda} R$ hold. Since f_λ is an F_λ-homomorphism, it follows that $\widehat{f_\lambda \circ \alpha} \models_A F_\lambda(R)$. Therefore, $\widehat{f \circ \sigma} \models_A F(R(x_1, x_2, ..., x_n))$ follows because $f_\lambda = f \circ i_\lambda$, $\sigma(x_i) = i_\lambda \circ \alpha(x_i)$ for $i = 1, 2, ..., n$ and $F_\lambda = F \circ I_\lambda$. □

Products for Formula Domains

This section is devoted to build a formula domain which acts as the product of an indexed collection $\{(B_\alpha, \Omega_\alpha, \Psi_\alpha)\}_{\alpha \in D}$ of formula domains where each domain has an associated signature L_α. Because of a technical requirement we shall assume that the index set D has a distinguished element 0. We recommed our reader to review Section 4.2.5 of this chapter where the product of term domains is introduced, because this new formula domain shall be based on the product of term domains $\otimes_{\alpha \in D}(B_\alpha, \Psi_\alpha)$.

We need a signature L which can be associated to this new domain, namely the product of formula domains. To this aim, let us consider the following set of sequences of formulas

$$K = \{(\phi_\alpha)_{\alpha \in D} \in \prod_{\alpha \in D} \Omega_\alpha \mid \text{ for all } \alpha, \beta \in D,\ V(\phi_\alpha) = V(\phi_\beta)\}. \tag{4.30}$$

where $V(\phi)$ denotes the set of free variables of ϕ. Observe that each element of K is a sequence (possibly infinite) of formulas which share the exact same set of free variables. This is the sequence's *associated* set of variables. We will consider each sequence of K as a *relation symbol* of L whose arity is determined by the cardinality of its associated set of variables. The new signature L is defined by $L = L' \cup K$ where L' is the signature of the product of term domains $\otimes_{\alpha \in D}(B_\alpha, \Psi_\alpha)$.

In order to determine the product of formula domains we need to build a set of L-formulas. Let $R \in K$ be an n-ary relation symbol of L. Because R is a tuple $(\phi_\alpha)_{\alpha \in D}$, we can assume that $x_1, x_2, ..., x_n$ is its associated set of variables ordered in the way which they occur in the formula ϕ_0. We build exactly one L-formula for each relation simbol $R \in L$ by defining the following set of atomic L-formulas:

$$\otimes_\alpha \Omega_\alpha = \{R(x_1, x_2, ..., x_n) \mid R \in K\}.$$

Let $R(x_1, x_2, ..., x_n) \in \otimes_\alpha \Omega_\alpha$ with $R = (\phi_\alpha)_{\alpha \in D}$ and observe that for any $\lambda \in D$, the mapping $P_\lambda : \otimes_\alpha \Omega_\alpha \to \Omega_\lambda$ defined by $P_\lambda(R(x_1, x_2, ..., x_n)) = \phi_\lambda$ is a formula translation.

Set $B = \prod_{\alpha \in D} B_\alpha$ and recall that the product of term domains is defined by $\otimes_{\alpha \in D}(B_\alpha, \Psi_\alpha) = (B, \otimes_\alpha \Psi_\alpha)$ together with the $\tilde{P}$-homomorphisms $p_\lambda : B \to B_\lambda$. By $\otimes_\alpha(B_\alpha, \Omega_\alpha, \Psi_\alpha)$ we denote the formula domain $(B, \otimes_\alpha \Omega_\alpha, \otimes_\alpha \Psi_\alpha)$ whose set of relations is determined in the following way. Let $R(x_1, x_2, ..., x_n) \in \otimes_\alpha \Omega_\alpha$ with $R = (\phi_\alpha)_{\alpha \in D}$ and let $((b_{1\alpha})_{\alpha \in D}, (b_{2\alpha})_{\alpha \in D}, ..., (b_{n\alpha})_{\alpha \in D})$ be a n-tuple in B. The n-ary relation R^B on B is defined by

$$R^B((b_{1\alpha})_{\alpha\in D},(b_{2\alpha})_{\alpha\in D},\dots,(b_{n\alpha})_{\alpha\in D}) \text{ if and only if } \widehat{\alpha}_\lambda \models_{B_\lambda} \phi_\lambda, \text{ for every } \lambda \in D, \tag{4.31}$$

where $\alpha_\lambda(x_i) = b_{i\lambda}$ for $i = 1,2,\dots,n$. The above equivalence 4.31 makes next proposition straightforward.

Proposition 28. *Let $\lambda \in D$. The coordinate map $p_\lambda : B \to B_\lambda$ is a P_λ-homomorphism respect to $\tilde{P}_\lambda$ from the formula domain $\otimes_{\alpha\in D}(B_\alpha,\Omega_\alpha,\Psi_\alpha)$ to the formula domain $(B_\lambda,\Omega_\lambda,\Psi_\lambda)$.*

Proof. Proposition 24 shows that p_λ is a $\tilde{P}_\lambda$-homomorphism from $\otimes_{\alpha\in D}(B_\alpha,\Psi_\alpha)$ to (B_λ,Ψ_λ). Let us show that p_λ is a P_λ-homomorphism respect to $\tilde{P}_\lambda$. To this aim, let $\sigma : V \to B$ be an assignment and $R(x_1,x_2,\dots,x_n)$ a formula in $\otimes_\alpha\Omega_\alpha$ with $R = (\phi_\alpha)_{\alpha\in D}$.We must show that if $\sigma \models_B R(x_1,x_2,\dots,x_n)$, then $\widehat{p_\lambda \circ \sigma} \models_{B_\lambda} P_\lambda(R(x_1,x_2,\dots,x_n))$. Assume that $\sigma \models_B R(x_1,x_2,\dots,x_n)$, then equivalence 4.31 implies that $\widehat{\alpha}_\lambda \models_{B_\lambda} \phi_\lambda$ where $\alpha_\lambda(x_i)$ is equal to the λ-th component of $\sigma(x_i)$ for $i = 1,2,\dots,n$. Therefore, $\widehat{p_\lambda \circ \sigma} \models_{B_\lambda} P_\lambda(R(x_1,x_2,\dots,x_n))$ since $\alpha_\lambda(x_i) = p_\lambda(\sigma(x_i))$ for $i = 1,2,\dots,n$ and $P_\lambda(R(x_1,x_2,\dots,x_n)) = \phi_\lambda$. It shows that p_λ is an P_λ-homomorphism. □

Definition 41. Let $\{(B_\alpha,\Omega_\alpha,\Psi_\alpha)\}_{\alpha\in D}$ be an indexed collection of formula domains. The *product domain* of $\{(B_\alpha,\Omega_\alpha,\Psi_\alpha)\}_{\alpha\in D}$ is the formula domain $\otimes_{\alpha\in D}(B_\alpha,\Omega_\alpha,\Psi_\alpha)$ together with the collection $p_\lambda : \prod_{\alpha\in D} B_\alpha \to B_\lambda$ of P_λ-homomorphisms respect to $\tilde{P}_\lambda$.

Theorem 46. *Let (A,Σ,Π) be a formula domain and let $\{(B_\alpha,\Omega_\alpha,\Psi_\alpha)\}_{\alpha\in D}$ be a family of formula domains with possibly distinct contexts. Assume that for every $\lambda \in D$, $f_\lambda : A \to B_\lambda$ is an F_λ-homomorphism respect to $\tilde{F}_\lambda$. Then there exists a unique map $f : A \to \prod_{\alpha\in D} B_\alpha$ which is an F-homomorphism respect to $\tilde{F}$ and makes diagram 4.32 below commute. The maps F and $\tilde{F}$ are uniquely determined and satisfy $P_\lambda \circ F = F_\lambda$ and $\tilde{P}_\lambda \circ \tilde{F} = \tilde{F}_\lambda$.*

$$\begin{tikzcd}
(A,\Sigma,\Pi) \arrow[d, dashed, "f"] \arrow[dr, "f_\lambda"] & \\
\otimes_\alpha(B_\alpha,\Omega_\alpha,\Psi_\alpha) \arrow[r, "p_\lambda"'] & (B_\lambda,\Omega_\lambda,\Psi_\lambda)
\end{tikzcd} \tag{4.32}$$

Proof. The term translation $\tilde{F} : \otimes_{\alpha\in D}\Psi_\alpha \to \Pi$ and the mapping $f : \oplus_\alpha B_\alpha \to A$ are defined exactly as in the proof of Theorem 34. Observe that these are the only ways in which $\tilde{F}$ satisfies $\tilde{F} \circ P_\lambda = F_\lambda$

for every $\lambda \in D$ and that f makes the above diagram commute. Theorem 34 shows that f is an uniquely determined $\tilde{F}$-homomorphism.

Let $\varphi \in \Sigma$ be a formula whose free variables are $\{x_1, x_2, ..., x_n\}$. Let us define the formula translation $F : \Sigma \to \otimes_\alpha \Omega_\alpha$ by $F(\varphi) = R(x_1, x_2, ..., x_n)$ with $R = (F_\lambda(\varphi))_{\lambda \in D}$ and notice that such definition is the only one satisfying $P_\lambda \circ F = F_\lambda$. Let us show that $f : A \to B$ is an F-homomorphism respect to $\tilde{F}$. To this aim, let $\varphi \in \Sigma$ be an n-ary formula with variables $x_1, x_2, ..., x_n$, and let $\sigma : V \to A$ be an assignment. We must show that if $\widehat{\sigma} \models_A \varphi$, then $\widehat{f \circ \sigma} \models_B F(\varphi)$. Let us assume that $\widehat{\sigma} \models_A \varphi$. Since for every $\lambda \in D$, f_λ is an F_λ-homomorphism, $\widehat{f_\lambda \circ \sigma} \models_{B_\lambda} F_\lambda(\varphi)$ holds for any $\lambda \in D$. Set $R = (F_\lambda(\varphi))_{\lambda \in D}$ and then equivalence 4.31 shows that the associated relation $R^B((f_\lambda \circ \sigma(x_1))_{\lambda \in D}, (f_\lambda \circ \sigma(x_2))_{\lambda \in D}, ..., (f_\lambda \circ \sigma(x_n))_{\lambda \in D})$ holds in B. From here, it is clear that $\widehat{f \circ \sigma} \models_B R(x_1, x_2, ..., x_n)$ and then $\widehat{f \circ \sigma} \models_B F(\varphi)$ holds. □

Complementing last theorem, observe that if $f : A \to B$ is an F-homomorphism respect to $\tilde{F}$ from (A, Σ, Π) to the product domain $\otimes_{\alpha \in D}(B_\alpha, \Omega_\alpha, \Psi_\alpha)$, then for each $\lambda \in D$ the composition $f_\lambda = p_\lambda \circ f$ is an uniquely determined $(P_\lambda \circ F)$-homomorphism respect ot $\tilde{P} \circ \tilde{F}$.

Chapter 5

Applications

The aim of this chapter is to give a detailed presentation of some applications of metaphor models developed in previous chapters and provide some arguments that support our approach.

First we will give a detailed and intuitive introduction to three well known metaphors used world wide in mathematical training, followed by the construction of their respective formal models. Through this presentation we will give examples of the concepts and constructs of the models, making observations about the properties usually attributed to metaphors in cognitive literature that our models succesfully captures after appropriate interpretations.

In the first section we will introduce the *object collection metaphor* for natural numbers, providing some references to previous studies concerning it. After that, we will build the formal models for this metaphor and -as a result- we will give examples of domains, metaphor, homomorphisms and some comments about the plausibility of the model.

In the second section we introduce the *two-pan balance metaphor* for linear equations and comment on some research that has been done on teaching taking advantage of this metaphor and its impact on learning. Again, we will construct the formal models for this metaphor using our approach and, in the process, we will be able to give some examples of the notions of domain, product domain, quotient domain, metaphor and homomorphism.

In the third section we will speak about the *motion along a path metaphor* for integer numbers and explicitate the formal models for this metaphor. We will stress the resemblance of the construction of this metaphor with the standard (formal) construction of the integer numbers suggesting that the standard mathematical construction of the integer numbers might be strongly linked to some metaphor of motion.

5.1 Object Collection Metaphor for the Natural Numbers

This metaphor is special in our discussion because it is one of the fundamental motivations for this entire work. This metaphor is nicely described from the cognitive point of view in the book “Where mathematics comes from” [71] under the name of “Arithmetic as Object Collection”. For related work about its use in mathematical training see [6].

For the sake of the self containment of the present work we present a description of this metaphor. Nevertheless, we warn the reader that we do not intend to give a description as complete and accurate as those that might be found in a cognitive science treaty dealing with this phenomenon.

Imagine a normal child who has not yet built in her mind a strong enough concept of number, yet she has some skills like the ability to group objects and the capacity of assigning cardinal numbers to groups of objects among others. How do you teach the first basics of arithmetic to a child like this? The first thing that you would *not* do, of course, is write down the Peano axioms and tell her -this is the whole number arithmetic-, unless you want to scare her for good. The experienced teacher knows that a better strategy to introduce her to the number concept is to take advantage of knowledge that is already structured in her mind. She knows the meaning of the words “grouping” or “joining” (wich we shall understand as joining two piles of objects into one), she can relate groups of objects with their cardinal (number of objects) and she can compare the cardinality of those piles of objects doing some kind of counting with her hands. At this point it is clear that there is a conceptual domain in her mind composed by piles of objects and its manipulative operations (pile, join, compare) and that somehow the child knows its structure by experience. The whole idea is to take advantage of that domain (name it the *pile domain*) to teach arithmetic. One way to do this is by building a metaphor wich explains the relations between numbers based in the existing relations in this domain.

In terms of [71] and [30], the metaphor that we are talking about takes advantage of the similarity between the structure of the pile domain (whatever its structure in the child's mind would be) and the structure of the natural numbers (formally given by the Peano axioms, for instance) to relate these two "domains of knowledge" and produce into the child mind her first representation of natural numbers and their structure. Of course, this is just an idealistic view wich simplifies the phenomenon in order to build a manageable model because it is known that even babies at age of months can count an add [17]. There are many ways to implement this metaphor for mathematical training, look for example in [102] and in [6] where different teaching methodologies based on this metaphor are commented.

In the present work we focus on the traditional approach to this metaphor, that is to say, we are going to assume that the source domain (namely *Piles*) is a *real* conceptual domain, structured in our minds by means of sensorial information and physical experience among other influences. We assume that the main objects of this domain are piles and that our minds attributes some basic structure to them as - for example - that two piles A, B can be joined in a new pile denoted $A \uplus B$, two piles can be compared by counting their elements, that some propositions about piles can be formulated in the language used for this domain and they can be perceived as false or true, etc.

In the next section we will develop a formal model of this domain (in definition (42)) that will be called a pile system, but we have to warn the reader that the formal model of this domain might not catch all the intuitive properties that the real domain has. Notice for example that in the real domain when piles are going to be joined, they are always assumed pairwise disjoint, while on the other hand, since the developed formalization of this domain is based on set theory, the definition of a pile system gives us sets to be interpreted as piles and they are not necessarily disjoint whenever they are to be joined. Despite these inconvenients, the model we shall introduce is good enough for our purposes i.e. explore the structural similarity with $\mathbb{N}$ and show the usefulness of the model of metaphor developed in previous chapters. We emphasize that we do not claim that the model given by definition (42) is necessarily the best model that can be built for this domain nor that is the only way to achieve a formalization for it. For those readers who prefer to work with a "closer to reality" model, the appendix of this chapter develops a model wich catch those intuitive properties.

The fundamental idea underlying the modelling of this metaphor is the assumption that the metaphor can be viewed as a mapping $h : Piles \to \mathbb{N}$ wich associates to every pile of objects $A \in Piles$ its cardinality $|A| \in \mathbb{N}$. The structural similarity determined by that mapping can be described now by listing the adequate relations and operations in the base domain *Piles* wich are good to explain (or preserve) the selected relations and operations in the target domain $\mathbb{N}$:

1. Notice that we can explain the sum operation in the target domain $\mathbb{N}$ by means of the join operation defined in the source. In other words, we can infer that $4+3=7$ in $\mathbb{N}$ because in the pile domain we can take a pile with four objects and a pile with tree objects and verify that the join of these piles is a pile with seven objects (as in the figure 5.1). The observation here is that the cardinality of the join of two piles is equal to the sum of the cardinalities of the two piles, formally this property is expressed by the equation

$$h(A \uplus B) = h(A) + h(B) = |A| + |B|.$$

   where A, B are piles of objects. These kinds of models are formally stated in the examples 9, 40 and 41 by associating the operation of "join" ($\uplus$) in the pile domain with the addition ($+$) in the natural numbers.

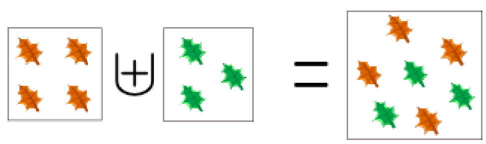

Figure 5.1: Join operation in the piles domain

2. The order relation ($\leq$) of natural numbers is another fundamental part of their structure, the metaphor is able to explain that structure based on the assumption that our mind can compare two piles in order to find some commonalities between them. For example, in the picture (5.2-a) we would conclude that $3 \leq 7$ because we see that the left pile is part of the right pile wich is bigger. We should reach the same conclusion in the picture (5.2-b) because we can make an association between piles such that to each leaf in the left pile corresponds one and only one leaf in the right pile.

   Our model is able to formalize the approachs previously described. Furthermore, our model let us categorize those alternatives because each one of them has different properties. The first approach can be modelled by selecting $\subseteq$ as the appropriate relation in the pile system (the formal model) (this is formalized in the examples 9, 37, 38 below) and notice that

$$\text{If } A \subseteq B, \text{ then } h(A) \leq h(B)$$

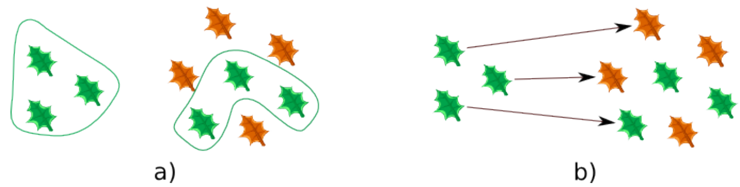

Figure 5.2: Options to explain the relation $\leq$ in $\mathbb{N}$. a) Use of $\subseteq$. b) Use of $\hookrightarrow$.

For the second alternative, consider the relation between two piles A and B denoted by $A \hookrightarrow B$ wich is given by the posibility of associating every object of the pile A with some (different) object of the pile B (this relationship is expressed in the formalized example (10) as the existence of an injective function $f : A \to B$). Observe that this relation is tightly related to $\leq$, since it is characterized by:

$$A \hookrightarrow B \text{ if and only if } h(A) \leq h(B).$$

3. Another relation in $\mathbb{N}$ of capital importance is the identity relationship, this is an important part of the structure of $\mathbb{N}$ wich helps us to build and understand other concepts, for example: the result of an operation, the associativity property of the operations, the commutativity property of operations and so on. Although in the pile domain we have the identity relationship $(A = A)$, it seems that it is not the best relationship to explain the structure of the identity relationship in $\mathbb{N}$, this is because there could be two different piles that have the same number of objects i.e. $A \neq B$ but $h(A) = h(B)$. In its place let us consider the relationship denoted by $A \equiv B$ wich holds if and only if you can associate each object from A with one object in B in a 1 to 1 way (i.e. there exist a bijective function $f : A \to B$ as it is formalized in the example (16)).This relationship seems more related to the concept of number because:

$$A \equiv B \text{ iff } h(A) = h(B)$$

wich could be interpreted in applications as: “a number is the collection of piles wich have the same cardinality”.

4. The multiplication is another emblematic structure of $\mathbb{N}$. The metaphor can use some kind of product operation defined in the Piles domain to introduce it. It is plausible to argue that this operation raises from experience and might be understood as a sort of repetition of piles. For example, imagine that you need to transport trees, and assume that you have two carss

to acomplish this task. Each car has the capacity to transport three trees each time. Figure (5.3) depicts the situation where you have made one trip and therefore the total amount of transported trees is represented by a pile M of trees built by repeating one pile of trees as many times as cars you have at your disposal.

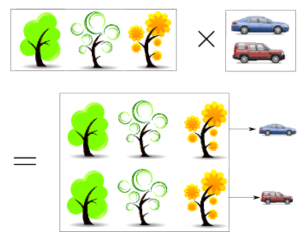

Figure 5.3: Product of piles.

The formal model lets us define a metaphor wich is able to introduce the multiplication based on this domain of piles (the pile system), noticing that in the case of the pile system we can use the product of sets and observe that

$$h(A \times B) = h(A) \cdot h(B),$$

and furthermore

$$(A \times B) \equiv C \text{ if and only if } h(A) \cdot h(B) = h(C).$$

This discussion is formalized in the examples 18 and 43.

5. Notice that we can now explore some of the properties of the operations in $\mathbb{N}$ by means of the different structures that we have been introducing, for example the associativity of the sum can be explained by the fact:

$$(A \uplus B) \uplus C \equiv A \uplus (B \uplus C) \text{ if and only if } (h(A) + h(B)) + h(C) = h(A) + (h(B) + h(C))$$

the commutativity of the multiplication by the fact that:

$$A \times B \equiv B \times A \;\text{ if and only if }\; h(A) \cdot h(B) = h(B) \cdot h(A)$$

6. Notice that there are other properties, relations and operations in $\mathbb{N}$ that can be explored with this model in the spirit of looking for structural similarity, but by now, we want to focus the

reader's attention in the fact that our model of metaphor predicts that some of the inferences (or entailments or consequences) can be borrowed from the domain of piles by the domain of natural numbers, and in the developed theory there exists a theorem wich guarantees it (theorem 42 from chapter 4). An example of such a thing is the following. Observe that in the domain of piles it holds that $A \uplus C \hookrightarrow B \uplus C$ is a consequence of $A \hookrightarrow B$, in other words:

$$\text{for any pile } C, \;\; A \hookrightarrow B \text{ implies } A \uplus C \hookrightarrow B \uplus C.$$

Our claim is that, if we are using h as metaphor we should have that:

$$\text{for any natural } c, \;\; a \leq b \;\text{ implies }\; a + c \leq b + c.$$

And this is exactly what the theorem (42) from chapter 4 guarantees. This is formalized and analized in the examples 19 and 44 of the following section.

It is worth to observe that concepts like "$\hookrightarrow$", "$\equiv$" and "$\times$" have been accuratelly formalized by set theory (as for example in ZFC, Zermelo-Fraenkel set theory with the axiom of choice) and cardinal numbers theory. Our contribution is just to order these ideas into a cognitive context and in an appropriate way in order to build up a model for metaphor.

We have so far given an overview of the resemblance between the structures *Piles* or collections of objects and the set of natural numbers $\mathbb{N}$ determined by the mapping h, and furthermore, we have sketched how the preservation of inferences between those domains is modelled.

Formal Model Based on Domains

To begin with the formal study of this metaphor, our first step will be to explicitate formally the "knowledge domains" between wich the metaphor is defined. In order to achieve this, we have to explicitate a symbol set S and some related S-structure for each domain of knowledge. The symbol set S determines the language for the domain.

We begin by specifying the semantics of the formal model, in other words, we are first going to introduce the S-structure used to model this domain. A pile system is just an attempt to represent formally the human-like perception of the "object collections" domain. This formalization shall only partially resemble the "object collections" domain but it is enough to fulfill our purposes. Furthermore, observe that the following formalization is just one possible approach and there would be many ways to accomplish this task.

Definition 42. A collection P of finite sets is a *pile system* if:

1. $\emptyset \in P$.
2. There exists distinguished elements $a, b \in U$, with $a \neq b$, such that $\{a\} \in P, \{b\} \in P$.
3. $A \in P$ and $B \in P$ implies $A \cup B \in P$ and $A \times B \in P$.

It is easy to see that this set of axioms guarantees that for any $n \in \mathbb{N}$ there exists a set $C \in P$ such that $|C| = n$. Additionally, all finite unions and finite products of sets are well defined in P. The idea behind this formalization is that we can interpret the elements of P as collections of objects (so, we will call *piles* to the elements of P) and use some relations between sets (or properties of sets) as models for some characteristics of this "knowledge domain", consider for example:

- The subset relation (generally denoted by $A \subseteq B$) shall be used to model the relation " A is a subcollection of B".
- The bijection relation (the existence of a bijective function $f : A \to B$) to model the relation "counting the objects of the collection A produces the same result as counting the objects of the collection B".
- The injection relation (the existence of an injective function $f : A \to B$) to model the relation "The collection A has fewer objects than the collection B".

And so on. Aditionally, notice that in the "real" domain of knowledge, we seem to perceive that, when we are going to join two collections, they are always disjoint. It seems to be undefined when the intersection of its arguments is non empty. A sound approach to model this partial operation would be to define the ternary relation denoted by $\sqcup$:

$$\sqcup(X, Y, Z) \Leftrightarrow X \cap Y = \emptyset \text{ and } X \cup Y = Z \tag{5.1}$$

So, the relation "$\sqcup$" holds between piles A,B,C if and only if A and B are disjoint and $C = A \cup B$. To have an alternative way of modelling such operation (join of collections) with an operation (and not with a relation as $\sqcup$) we proppose to emulate the observed behavior in the formalization by defining in P the binary operation *join of piles* denoted by $\uplus$ and given by

$$A \uplus B = A \times \{a\} \cup B \times \{b\}. \tag{5.2}$$

Now that we have detailed the semantic part of this domain, we should specify its sintactic part. In other words, we have to determine the language that shall be associated to the structure previously described. Even though a language with fewer symbols would suffice to make all constructions we need, for the sake of clarity, we choose the language determined by the symbol set $S_P = \{\emptyset, a, b, \cup, \uplus, \sqcup, \times, \subseteq\}$ where $\emptyset, a, b$ are constant symbols, $\cup, \uplus, \times$ are binary function symbols and $\in, \subseteq$ are binary relation symbols and $\sqcup$ is a ternary relation symbol.

Therefore the pile domain is formalized as the domain (without structure) given by $\mathscr{P} = (S_P, (P, \rho))$ where P is a pile system and the associations given by ρ are as follows: $\rho(\emptyset) = \emptyset$, $\rho(a) = a$, $\rho(b) = b$, $\rho(\cup)$ is the union of sets in P, $\rho(\uplus)$ is the disjoint union of sets defined in 5.2, $\rho(\sqcup)$ is the ternary relation defined in 5.1, $\rho(\times)$ is the product of sets in P, $\rho(\in)$ is the pertenence relation in P and $\rho(\subseteq)$ is the subset relation in P. Observe that here we have abused of the language by denoting some symbols in S_P with the same letter that we have denoted the correspondent objects (or concepts) in P.

Notice that the language gives us the ability to select and isolate certain structure in each domain. Therefore, in the examples, we are going to use formulas and terms from the language to select the appropriate semantic relations and operations to endow this domain with structure. If we are too rigurous in the explicitation of the respective formulas and terms in the domain we shall obscure [1] the presentation unnecessarily. Thus, we will informally characterize the formulas (or terms) we use and denote them in more suggestive ways.

Let us introduce now, as an example, some formulas which are going to be used often in the

[1] Notice that if we want to express in the language of the pile domain: "z is the ordered pair of x and y" an associated formula with x, y, z as free variables might be the awkward looking expression

$$\forall u (u \in z \leftrightarrow (\forall v (v \in u \leftrightarrow v = x) \vee \forall v (v \in u \leftrightarrow (v = x \vee v = y)))) \tag{5.3}$$

thus, we are going to avoid the explicit construction of such formulae rest of the section.

1. Denote by $X \hookrightarrow Y$ a formula in $Form(\mathscr{L}^{S_P})$ wich has as free variables $X, Y \in V$ and for every assignment $\sigma : V \to P$ it holds that $\sigma \models X \hookrightarrow Y$ iff there exists an injective function $f : \sigma(X) \to \sigma(Y)$.
2. Denote by $X \equiv Y$ a formula in $Form(\mathscr{L}^{S_P})$ wich has as free variables $X, Y \in V$ and for every assignment $\sigma : V \to P$ it holds that $\sigma \models X \equiv Y$ iff there exists a bijective function $f : \sigma(X) \to \sigma(Y)$.

We must also give a mathematical model for the target domain involved in this metaphor. To this aim, consider the symbol set $S_{a<} = \{0, 1, +, \cdot, \leq\}$ (wich is standard in arithmetic and related fields) where 0 and 1 are constants, $+, \cdot$ are binary function symbols and $\leq$ is a binary relation symbol. The domain of natural numbers is formalized as $\mathscr{N} = (S_{a<}, (\mathbb{N}, \alpha))$ where $\mathbb{N}$ are the natural numbers, the mapping α is the usual mapping of symbols to the corresponding constants, operations and relations of $\mathbb{N}$.

The idea underlying the following discussion is to use the developed models to formally describe how these domains are related and how the model explains some of the properties of the phenomenon.

Our proposal here is to use the pair $\mathscr{P} = (S_P, (P, \rho))$ as a mathematical representation for the "object collection" domain and the pair $\mathscr{N} = (S_{a<}, (\mathbb{N}, \alpha))$ as a mathematical model for the "natural numbers" domain. In the following, we will show how the model of metaphor can be used to partially project the structure of the source domain $\mathscr{P}$ into the target domain $\mathscr{N}$. Examples 9, 10 are aimed to formalizing the introduction of the addition operation and the $\leq$ relation, examples 15, 16 are directed to formalizing the introduction of the commutative property of the addition, example 17 formalizes the introduction of the associative property of the addition, example 18 is aimed to explaining the product operation and its commutative property, example 19 captures all the previously mentioned properties in one example. The last example (20) is aimed to showing how the consequence relationship between formulas is preserved by the metaphor from the source domain to the target domain.

In the following examples, recall that $Form(\mathscr{L}^{S_P})$ and $Term(\mathscr{L}^{S_P})$ are the formulae set and the term set (respectively) of the language determined by S_P (wich is associated to the pile system P),

$Form(\mathscr{L}^{S_{a<}})$ and $Term(\mathscr{L}^{S_{a<}})$ are the formulae set and the term set (respectively) of the language determined by $S_{a<}$ (wich in turn is associated to the natural numbers set $\mathbb{N}$).

In every example, we are going to need a set of formulas and a set of terms to select the appropriate structure in each domain. In order to do this assume that $\Sigma \subseteq Form(\mathscr{L}^{S_P})$, $\Pi \subseteq Term(\mathscr{L}^{S_P})$, $\Omega \subseteq Form(\mathscr{L}^{S_{a<}})$, $\Psi \subseteq Term(\mathscr{L}^{S_{a<}})$. Aditionally, all the examples uses the mapping given by

$$h: \ P \to \mathbb{N}.$$

This is the mapping that associates to every pile $A \in P$ its cardinality $h(A) = |A| \in \mathbb{N}$.

Example 9. This example shows that the developed model can be used to formalize (or formally express) the way in which the metaphor introduces the "less than" relation. To achieve this, we are going to use the "subcollection" relation and the "join of piles" operation -respectively- in the "object collections" domain. Notice that our first concern will be to select the appropriate structure to model these relations and operations in each domain. We do it by putting $\Sigma = \{X \subseteq Y\}$, $\Omega = \{X \leq Y\}$, $\Pi = \emptyset$, $\Psi = \emptyset$. With those sets we can define the domains with structure $(\Sigma, \Pi, \mathscr{P})$ and $(\Omega, \Psi, \mathscr{N})$ wich shall be used to state this particular example. Now, we should relate these two structured domains by means of a metaphor. We can do this by detailing how the objects of the base domain are related to the elements of the target domain and determining the correspondence between the formulas and terms of each domain. The first point is made by considering the mapping $h: P \to \mathbb{N}$ wich associates to each pile its cardinality. The second point is made by defining $F: \Sigma \to \Omega$ and $T: \Pi \to \Psi$ as the obvious (and unique) mappings that can be considered as a formula translation and a term translation respectively. It is easy to see that the triple of mappings

$$(h, F, T): (P, \Sigma, \Pi) \to (\mathbb{N}, \Omega, \Psi)$$

satisfy the conditions to be considered a metaphor from the domain with structure (P, Σ, Π) to the domain with structure $(\mathbb{N}\Omega, \Psi)$. That is to say, if we take any pair of piles A and B and $A \subseteq B$ holds, this implies necessarily that $h(A) \leq h(B)$. On the other hand, for any two numbers a, b such that $a \leq b$ *there exists* two piles A, B such that $h(A) = a, h(B) = b$ and $A \subseteq B$.

Consider the proposition $3 \leq 5$. Its truthness implies that it is always possible to find a pile with tree elements wich is a subcollection of certain *specific* pile with five elements.

Example 10. This example is similar to the previous one, except that it makes use of the relation "injection" instead of the relation "subcollection" to introduce structure (the relation "less than") into the natural numbers. Thus, we select these appropriate structures in each domain by considering the following sets of formulas and terms: $\Sigma = \{X \hookrightarrow Y\}$,$\Omega = \{X \leq Y\}$, $\Pi = \emptyset$, $\Psi = \emptyset$. If $F : \Sigma \to \Omega$ and $T : \Pi \to \Psi$ are the obvious mappings that can be considered as a formula translation and a term translation respectively, then the triple of mappings

$$(h, F, T) : (P, \Sigma, \Pi) \to (\mathbb{N}, \Omega, \Psi)$$

satisfy the conditions to be considered an homomorphism from the domain (P, Σ, Π) to the domain $(\mathbb{N}, \Omega, \Psi)$. Observe that the conditions to be a homomorphism are a little stronger than the conditions to be a metaphor. In this case, the homomorphism condition means that

$$A \hookrightarrow B \text{ if and only if } h(A) \leq h(B)$$

That is to say, if we take any pair of piles A and B and $A \hookrightarrow B$ holds, this implies necessarily that $h(A) \leq h(B)$. On the other hand, for any two numbers a, b such that $a \leq b$ and *any* two piles such that $h(A) = a,\ h(B) = b$ it holds that $A \hookrightarrow B$.

Consider the proposition $3 \leq 5$. Its truthness implies that any pile with tree elements can be injected into *any* pile with five elements.

Example 11. This example shows how the developed model can be used to formalize (or formally express) how the metaphor introduces the"sum" operation in the natural numbers. To achieve this, we are going to use the "join of piles" relation in the "object collections" domain. Our first concern will be to select the appropriate structure to model these relations in each domain. We do it by putting $\Sigma = \{\sqcup(X, Y, Z)\}$, $\Pi = \emptyset$, $\Omega = \{X + Y = Z\}$, $\Psi = \emptyset$. Consider $F : \Sigma \to \Omega$ and $T : \Pi \to \Psi$ as the unique maps that can be considered as a formula translation and a term translation respectively. Notice that, if we take any two collections A, B such that $A \cap B = \emptyset$ we have that $\sqcup(A, B, A \cup B)$ holds and therefore $h(A) + h(B) = h(A \cup B)$ holds. On the other hand, for any tree numbers a, b, c such that $a + b = c$ we can take any pair of piles A, B such that $h(A) = a,\ h(B) = b$ and in the case they were not disjoint we can disjoint them (by means of applying a product for example) and therefore it is always possible to find two disjoint piles A', B' such that $h(A') = a, h(B') = b$ and $h(A' \cup B') = c$.

Therefore, the triple

$$(h,F,T):(P,\Sigma,\Pi)\to(\mathbb{N},\Omega,\Psi)$$

satisfy the conditions to be considered a metaphor.

Example 12. This example is an alternative modelling to example 11. It uses the "join of piles" operation in the "object collections" domain. Put $\Sigma=\emptyset$, $\Omega=\emptyset$, $\Pi=\{\emptyset,U\uplus V\}$, $\Psi=\{0,U+V\}$. Consider the mapping $h:P\to\mathbb{N}$ wich associates to each pile its cardinality and define $F:\Sigma\to\Omega$ and $T:\Pi\to\Psi$ as the unique maps that can be considered as a formula translation and a term translation respectively. Notice that $h(\emptyset)=0$ and if we take any pair of piles A and B it holds that

$$h(A\uplus B)=h(A)+h(B)$$

implying that the triple of mappings

$$(h,F,T):(P,\Sigma,\Pi)\to(\mathbb{N},\Omega,\Psi)$$

determines a metaphor.

Next examples use the model to formally describe how this metaphor could be used to infer some properties about the natural numbers and their operations. It is important to introduce the way in which the commutativity 5.4 and associativity 5.5 properties are expressed when we are considering the "$\sqcup$" relation.

$$Commutativity(A,B)\Leftrightarrow\exists Z\ (\sqcup(A,B,Z)\wedge\sqcup(B,A,Z)).\tag{5.4}$$

$$Associativity(A,B,C)\Leftrightarrow\exists U,\exists V,\exists W\ (\sqcup(A,B,U)\wedge\sqcup(B,C,V)\wedge\sqcup(U,C,W)\wedge\sqcup(A,V,W))\tag{5.5}$$

The below diagram explains why the associativity of the "join of piles" has been written as in the expression 5.5.

$$\overbrace{\underbrace{(A\cup B)}_{U}\cup C}^{W}\quad=\quad\overbrace{A\cup\underbrace{(B\cup C)}_{V}}^{W}$$

Example 13. In this example we are going to model a metaphor wich introduces the commutativity property of the sum operation using the relation $\sqcup$ and the commutativity property as defined in 5.4.

To make it formal, let

$$\Sigma = \{\exists Z(\sqcup(X,Y,Z) \wedge \sqcup(Y,X,Z))\}, \Omega = \{X+Y=Y+X\}, \Pi = \emptyset, \Psi = \emptyset.$$

If $F : \Sigma \to \Omega$ and $T : \Pi \to \Psi$ are unique maps that can be defined as formula translation and term translation respectively, then $(h,F,T) : (P,\Sigma,\Pi) \to (\mathbb{N},\Omega,\Psi)$ is a homomorphism from the domain (P,Σ,Π) to the domain $(\mathbb{N},\Omega,\Psi)$. In other words, if we take any pair of piles A,B such that $\exists Z\ (\sqcup(A,B,Z) \wedge \sqcup(B,A,Z))$ holds, then $h(A)+h(B) = h(B)+h(A)$ holds. On the other hand, for any pair of numbers $a,b \in \mathbb{N}$ and *any* two piles A,B such that $h(A) = a, h(B) = b$ it holds that $\exists Z\ (\sqcup(A,B,Z) \wedge \sqcup(B,A,Z))$.

Example 14. Similarly, this example shows how the associativity of the sum would be introduced. Let

$$\Sigma = \{\exists U, \exists V, \exists W\ (\sqcup(X,Y,U) \wedge \sqcup(Y,Z,V) \wedge \sqcup(U,Z,W) \wedge \sqcup(X,V,W))\},$$

$$\Omega = \{(X+Y)+Z = X+(Y+Z)\}, \Pi = \emptyset, \Psi = \emptyset.$$

Define $F : \Sigma \to \Omega$ and $T : \Pi \to \Psi$ as the obvious translations between those sets. Therefore, reasoning as in the example 13, it is plain to see that the triple $(h,F,T) : (P,\Sigma,\Pi) \to (\mathbb{N},\Omega,\Psi)$ determines a homomorphism of domains.

Example 15. As an alternative to example 13 to introduce the notion of commutativity in the natural numbers, this model of metaphor would exploit the commutativity of the "grouping two collections" operation in the "object collections" domain. To describe it formally put $\Sigma = \{X \cup Y = Y \cup X\}$, $\Omega = \{X+Y = Y+X\}$, $\Pi = \emptyset$ and $\Psi = \emptyset$. Consider $F : \Sigma \to \Omega$ as the only mapping that can be defined as a formula translation and T as the empty function. Notice that $(h,F,T) : (P,\Sigma,\Pi) \to (\mathbb{N},\Omega,\Psi)$ is a homomorphism between domains. This means that for any pair of piles A,B it holds that

$$A \cup B = B \cup A \text{ implies } h(A)+h(B) = h(B)+h(A)$$

and for any pair of natural numbers a,b and any pair of piles A,B such that $h(a) = A, h(b) = B$

$$a+b = b+a \text{ implies } A \cup B = B \cup A$$

holds. Wich is an easy consequence of the commutativity of the union of sets (wich holds in P) and the commutativity of addition (wich holds in $\mathbb{N}$).

As an alternative to the relation $\sqcup(X,Y,Z)$ we have defined the binary operation *join of piles* by $A \uplus B = A \times \{a\} \cup B \times \{b\}$. We are going to use it in the following examples. We suggest to bear on mind that the operation "$\uplus$" hardly represents the real world operation "join of piles". This operation ("$\uplus$") changes the nature of elements of the resultant pile, it is not commutative and it is not associative. However, these problems can be easily overcomed, the reader shall benefit by introducing the operation $+$ in the natural numbers by another operation (instead of a relation) and it shall become clear the flexibility of the model to represent the phenomenon in various ways.

Example 16. In this example we are going to model a metaphor wich introduces the commutativity property of the addition operation. To do this we are going to use the operation $\uplus$ and a weak-commutativity property wich is defined similarly to the commutativity property but using the bijection relation instead of the equality relation. Thus, given A and B two piles , we are going to say that this relation holds if and only if there exists a bijective function defined from the pile $A \uplus B$ to the pile $B \uplus A$. In other words, we are taking advantage of the equivalence relation wich holds in the pile domain iff the two collections have the same cardinality. To make it formal, let

$$\Sigma = \{X \uplus Y \equiv Y \uplus X\}, \Omega = \{X+Y = Y+X\}, \Pi = \{\emptyset, X \uplus Y\}, \Psi = \{0, X+Y\}.$$

Let $F : \Sigma \to \Omega$ and $T : \Pi \to \Psi$ are the obvious mappings that can be defined as formula translation and term translation respectively. Let us show that the triple

$$(h,F,T) : (P,\Sigma,\Pi) \to (\mathbb{N},\Omega,\Psi)$$

determines a homomorphism.

We use now terminology borrowed from logics. Let $\sigma : V \to P$ an assignment of variables and notice that a reasoning similar to the one perfomed in example 15 let us to conclude that

$$\sigma \models_P X \uplus Y \equiv Y \uplus X \text{ implies } h \circ \sigma \models_{\mathbb{N}} X+Y = Y+X.$$

On the other hand, any assigment of variables with form $h \circ \sigma : V \to \mathbb{N}$ gives the following:

$$h \circ \sigma \models_{\mathbb{N}} X+Y = Y+X \text{ implies } \sigma \models_P X \uplus Y \equiv Y \uplus X$$

These properties could be depicted in a commutative diagram as:

$$\begin{tikzcd}
& V \arrow[dl, "\sigma"'] \arrow[dr, "h\circ\sigma"] & \\
P \arrow[rr, "h"] \arrow[dr, "\models_P"'] & & \mathbb{N} \arrow[dl, "\models_{\mathbb{N}}"] \\
& \{T,F\} &
\end{tikzcd}$$

Observe that if we chose $\Sigma = \{X \uplus Y = Y \uplus X\}$ then (F,T,h) would not be a homomorphism between those domains and neither it would be a metaphor.

Example 17. In the same way, this example shows how the associativity of the addition would be introduced by using the previously mencioned equivalence relation in the "object collections" domain. Let

$$\Sigma = \{(X \uplus Y) \uplus Z \equiv X \uplus (Y \uplus Z)\}, \Omega = \{(X+Y)+Z = X+(Y+Z)\}, \Pi = \{\emptyset, U \uplus V\}, \Psi = \{0, U+V\}.$$

If the mappings $F : \Sigma \to \Omega$ and $T : \Pi \to \Psi$ are chosen as the obvious translations between those sets, then $(F,T,h) : (\Sigma,\Pi,P) \to (\Omega,\Psi,\mathbb{N})$ is a homomorphism from the domain with structure $(\Sigma,\Pi,\mathscr{P})$ to the domain with structure $(\Omega,\Psi,\mathscr{N})$. We omit the details because this case is analog to the previous example 16. As before, observe that if we chose $\Sigma = \{(X \uplus Y) \uplus Z = X \uplus (Y \uplus Z)\}$, then (F,T,h) would not be a homomorphism between those domains and neither it would be a metaphor.

Example 18. Using the technique applied in the last example, we are going to give a model for the introduction of the multiplication and its commutative property. As before, we must select the appropriate structures to work with. In this case we select the product of "object collections" $(\times)$ wich was detailed in the description of this metaphor (item 4). To formalize this, take

$$\Sigma = \{X \times Y \equiv Y \times X\}, \Omega = \{X \cdot Y \equiv Y \cdot X\}, \Pi = \{\emptyset, U \times V\}, \Psi = \{0, U \cdot V\}.$$

Consider the mappings $F : \Sigma \to \Omega$, $T : \Pi \to \Psi$ as the obvious translations. The details of the commutative property are not given because it is similar to previously given examples. Aditonally notice that for any pair of piles A, B it holds that

$$h(A \times B) = h(A) \cdot h(B).$$

Therefore, the triple $(h, F, T) : (P, \Sigma, \Pi) \to (\mathbb{N}, \Omega, \Psi)$ determines a homomorphism.

It is worth it observing that if we chose Σ as $\{(X \times Y) = (Y \times X)\}$, then (F, T, h) would not be a homomorphism between those domains and neither it would be a metaphor.

Example 19. This example intends to gather information from previous examples and select the appropriate structures in the "domain of piles" wich are useful to explain some structure of the natural numbers. As in previous examples this selection is done by setting $\Sigma = \{(X \uplus Y) \uplus Z \equiv X \uplus (Y \uplus Z),\ X \times Y \equiv Y \times X,\ X \hookrightarrow Y\}$, $\Omega = \{(X+Y)+Z = X+(Y+Z),\ X \cdot Y \equiv Y \cdot X,\ X \leq Y\}$, $\Pi = \{\emptyset, U \uplus V, U \times V\}$, $\Psi = \{0, U+V, U \cdot V\}$. If the mappings $F : \Sigma \to \Omega$, $T : \Pi \to \Psi$ are the unique translations between those sets, then $(h, F, T) : (P, \Sigma, \Pi) \to (\mathbb{N}, \Omega, \Psi)$ satisfies the conditions to determine a homomorphism. Therefore, this example deals with the operations sum and multiplication, the relation less or equal and with the properties: associativity of sum and commutativity of multiplication, all at once.

The next example is aimed to show that our model of metaphor is able to express (to some extent) one of the key features of metaphor phenomenon. This feature is refered to as "preservation of inferences" in the cognitive literature. Since we are working with formal models, we need to give an exact meaning to this property. In this case, by "inference" we will understand the usual consequence relation, wich is often defined in mathematical logic and was defined for this context in chapter (4). Intuitively, a formula φ is consequence of a formula ψ if φ holds in the semantic component of the domain whenever ψ holds (written $\psi \models \varphi$). Notice that the preservation of consequence by metaphors is guaranteed by theorem of consequence (42) presented in chapter (4).

Example 20. Given piles A, B and C in the "object collections domain" it holds that: if A can be "injected" in B, then $A \uplus C$ can be "injected" in $B \uplus C$. Our aim is to show that this particular structure can be used to explain through a metaphor the implication: "if $n \leq m$ then $n+k \leq m+k$" wich holds in the set of natural numbers for any k. We have chosen this particular case of "implication" or "consequence" to exemplify how the "preservation of inference" property of metaphors can be expressed mathematically in this context. Let

$$\Sigma = \{X \hookrightarrow Y, X \uplus Z \hookrightarrow Y \uplus Z\}, \Omega = \{\ X \leq Y, X+Z \leq Y+Z\}, \Pi = \emptyset, \Psi = \emptyset.$$

If we choose $F : \Sigma \to \Omega$ as the intuitive translation and T as the empty mapping, it is not difficult to show that $(h,F,T) : (P,\Sigma,\Pi) \to (\mathbb{N},\Omega,\Psi)$ determines a homomorphism. It is worth observing that in the first domain, the consequence $X \hookrightarrow Y \models_P X \uplus Z \hookrightarrow Y \uplus Z$ holds and then, by applying the theorem of consequence, we can conclude that in the domain of natural numbers the consequence $X \leq Y \models_{\mathbb{N}} X+Z \leq Y+Z$ holds.

A rephrasing of the last example would be: in the "object collections domain" it holds that for any pile C, if a pile A can be injected into a pile B, we can conclude that the pile $A \uplus C$ can be injected into the pile $B \uplus C$. Therefore, using the object collection metaphor, we can preserve that inference to conclude in the set of natural numbers that, for any k, if $m \leq n$ then $m+k \leq n+k$ holds.

5.2 Two-Pan Balance Metaphor for Linear Equations

The *two-pan balance metaphor* for linear equations is another well known metaphor and it is used world wide for mathematical training. There is in [120] a good introduction to the debate on the pros and cons of using this metaphor (and others). A discussion about some teaching methods for linear equations is presented in [97]. Additionally, in [1] is suggested that this metaphor is a grounding metaphor in the sense described by Lakoff and Nuñez in [71].

Learning to solve linear equations seems to be a slow, difficult process which requires practice and a little of patience. The study performed in [3] compares a traditional (or symbolic) teaching method against a metaphor based teaching method, providing evidence of a better student performance for the metaphor based method. For further evidence, but sustented in cases, see [107]. It seems that the differences between teaching methods can make a substantial difference on learning results, emphasizing the importance of investigating and understanding the processes in wich those different methodologies are based.

Now, we shall present a brief description of this metaphor as we understand it. As in the case of the object collection metaphor, we warn the reader that our intention here is just to give an overall idea, instead of giving an exhaustive description as complete and accurate as those that might be found in a cognitive science treaty dealing with this phenomenon.

We will call the base domain of the "two-pan balance metaphor" as the "scales domain". It is usually structured by our own experience, probably aquired while our body interacts with some ordinary concepts such as equilibrium, weight, movement, etc. Let us make a few comments in order to describe the scales domain's structure necessary to model this metaphor. These comments also will enhance our understanding of this domain of knowledge.

1. We are going to assume that the elements of this "scales domain" are scales like the one depicted in figure 5.4 below. Those scales can be balanced or not. In this domain there are two different types of weights and therefore the objects on the pans can be classified into these two different types (in the picture 5.4 those types are the blue boxes and the green circles).

Figure 5.4: Scales of the base domain for this metaphor.

2. There are two operations in this domain that are important for our purposes: Take any balanced scale, if we put objects with same weight on each pan, the equilibrium is not broken. This kind of action suggests the definition of two operations (one for each type of objects). For expositive purposes, we are going to identify those operations as the "box operation" and the "circle operation". The "box" operation simply puts a single box in each plate of the scale and the "circle" operation puts a single circle in each plate of the scale. For example, the scale depicted on the right side of figure 5.5 is the result of adding two blue boxes and one green circle on each plate of the left side scale. In other words, the scale at the right side is the result of applying the "box" operation two times and the "circle" operation one time to the scale depicted in the left side of the figure.

3. The defined operations allow us to define an equivalence relation in this domain: two scales are equivalent if both can be transformed into another balance *b* by finite applications (zero or more times) of the two operations. For example, take the pair of scales drawn in the picture 5.5, the scale at the right side is the result of adding two blue squares and one green circle on each plate of the scale at the left side. Therefore, they are equivalent.

Figure 5.5: Equivalent scales in this domain.

The previous comments have been made in order to give an idea of the minimal structure that (we assume) exists inside of the learner's mind. So, we shall build a formal model for this domain where these crucial concepts can be represented and interpreted. But before that, let us give a glance to the way in which this metaphor works.

The educational aim of this metaphor is to provide learners with some tools to deal with linear equations. Thus, the target domain of this metaphor is the set of linear equations of the form $ax+b=cx+d$ where $a,b,c,d \in \mathbb{N}$. We will denote by E the set of linear equations and we shall refer to it as the "equations domain".

We are going to explain how this metaphor works using the terminology depicted in figure 5.4 i.e. we shall call to the objects of the first type "blue boxes" and "green circles" to the objects of the second type. In the same way, we will say that an scale is in "equilibrium" if the scale is balanced (the weight of the objects on each pan is equal) and "equivalent" meaning that two scales are related by the equivalence relation defined previously in item 3 above.

Denote by B the set of scales, our approach suggests modelling the "two-pan balance metaphor" as a mapping from the scales domain to the equations domain. This can be viewed as a function $e : B \to E$ wich associates equations to scales. Consider one scale (balanced or not) wich has "a" blue boxes and "b" green circles on the first plate, while in the second plate there are "c" blue boxes and "d" green circles (in figure 5.4 we would have $a=2, b=3, c=4, d=1$). To perform the analysis we shall associate the linear equation $ax+b=cx+d$ to such scale. This correspondence defines a bijective function wich we denote by $e : B \to E$. A pictorical representation of this mapping is given in figure 5.6.

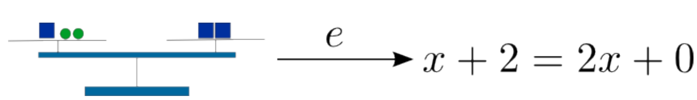

Figure 5.6: Representation of the mapping e between the base domain B and the target domain E.

Intuitively, this mapping relates scale's plates with equation's sides, number of objects of different kind with coeficients of the equated expressions and the "equilibrium" concept in the base domain with the "equality" concept in the target domain. The mapping $e : B \to E$ determines some structural similarity between the domains wich we are going to describe by enumerating the appropriate relations and operations in the scales domain wich are used by the metaphor to introduce (or explain) the appropiate relations and operations in the equations domain:

1. We can add an expression to both sides of any equation to obtain a new equation wich is equivalent to the first one in the sense that both equations share the same solution set. This property suggests the definition of operations in the target domain wich are analog to the operations "box" and "circle" in the base domain. Corresponding to the box operation we define $b : E \to E$, an unary operation wich receives an equation as argument. b adds x to both sides of the equation on wich it acts. On the other hand, analog to the circle operation we define $c : E \to E$, an operation wich takes an equation as argument and computes its value adding 1 to each side of the equation. With this, the following diagrams commute (that is $c \circ e = e \circ circle$ and $b \circ e = e \circ box$):

$$\begin{tikzcd} B \arrow[r, "box"] \arrow[d, "e"'] & B \arrow[d, "e"'] & B \arrow[r, "circle"] \arrow[d, "e"'] & B \arrow[d, "e"'] \\ E \arrow[r, "b"] & E & E \arrow[r, "c"] & E \end{tikzcd}$$

   We may interpret these commutative diagrams as the possibility of performing operations in the domain of equations by just performing operations in the scales domain and applying the metaphor. The sketch drawn in fig (5.7) is formally stated in examples 22 and 47.

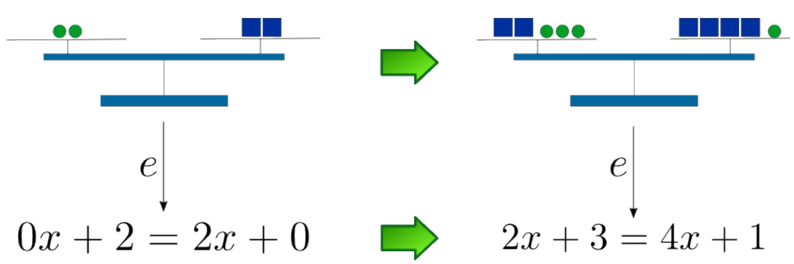

Figure 5.7: Pictorical interpretation of the commutative diagrams

2. If we apply the operations "box" or "circle" to any balanced scale, the result is a balanced

scale. i.e. the scale's equilibrium is not affected. An analog property is obtained (in the equations domain) for the operations "b" and "c" previously defined where the set of solutions of the equation is preserved by these operations. These are two different properties wich are defined on different domains, however they are related in the following way: By definition, the pair of scales $b_1, box(b_1) \in B$ are equivalent, then the pair of equations $e(b_1), b \circ e(b_1) \in E$ are equivalent in the sense that both equations share the same solution set. In other words, the equivalence relation in the scales domain sheds some light on the equivalence relation in the equations domain with help of the mapping e. This is formalized by our method in examples 22, 47 and 48.

3. Notice that in the scales domain it holds that: if the scales b_1 and b_2 are equivalent, then the balances $box(b_1)$ and $box(b_2)$ are equivalent. This implication can be brought to the domain of equations to ensure that if two equations e_1 and e_2 are equivalent then the equations $b(e_1)$ and $b(e_2)$ are equivalent. One attractive feature of our approach is its capacity to express the preservation of some of these sort of inferences. In particular we can use the theorem (42) from chapter (4) to bring some implications from the scales domain to the equations domain. The carrying of this "inference" is formally stated in the examples 23 and 49.

Before the end of this section, it is worth making the following observation: There exists two different kinds of objects allowed to be on the plates of the scales: the "blue boxes" and the "green circles". Somehow, our intuition suggests that such kind of domain could result of a mixture between a domain of scales wich has "blue boxes" only and a domain of scales wich has "green circles" only. For a pictorical sketch of this intuition see fig. (5.8).

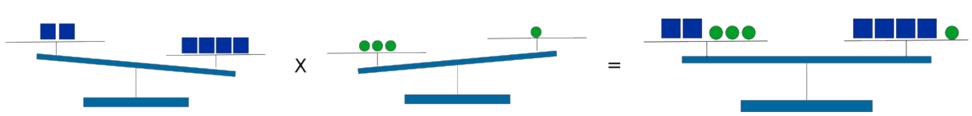

Figure 5.8: Intuitive sketch of the domain "product domain".

In the following sections we shall show that this intuition can be formallized by using the concept of "product domain" wich becomes useful in order to describe domains as combinations of simpler domains. In this particular situation (the scales domain), we will refer to each of those simpler domains where the scales in consideration have just one kind of objects on their plates as "simple scales domains" (left side of Figure 5.8). The construction is formallized along the two following sections and the examples 21 and 46.

There seems to be some links between these kind of mathematical constructions and some cognitive abilities. They suggest that this kind of constructions -products and coproducts- could be considered as models for reasoning and learning processes. For example, the development of some cognitive abilities in children around the age of five is studied in [94]. Some experimental work is performed there in order to claim that: "These results point to a fundamental cognitive principle under development during childhood that is the capacity to compute products in the categorical sense". Additionally, we believe that the cognitive concept of "blending" could be related to these kind of constructions, although -as far as we know-, there is no advances on the field in such direction.

Formal Model Based on Domains

The Simple Scales Domain

Our aim for this and the next section is to describe formally the "scales domain" while showing that it can be formally viewed as a product of two "simple scales domains". In this part, we concentrate in the formal description of these simpler domains. We begin with a semantic description of the domain and after that we shall describe the "simple scales domain" sintactically (the formal language attached to it).

To model the semantic component of the "simple scales domain" our first step is to look for a mathematical structure appropriate to represent the concepts involved. Our proposal here is to use the set $\mathbb{N} \times \mathbb{N}$ and interpret each pair (n,m) as the representant of one (simple) scale where n represents the number of objects on the first plate and m represents the number of objects on the second plate. To avoid name confusion we rename the operation "box" (or "circle") by "Add" and define it recursively by:

$$A(m,n) = (m+1, n+1) \tag{5.6}$$

Now define "$\sim$", an equivalence relation wich -we shall see- plays an important role in this domain:

$$(a_1,a_2) \sim (b_1,b_2) \text{ if and only if } \exists\, n\ (A^n(a_1,a_2) = (b_1,b_2)) \vee (A^n(b_1,b_2) = (a_1,a_2)) \tag{5.7}$$

This equivalence relation means: one of the scales -let us say (a_1,a_2)- can be transformed into the other scale -in this case (b_1,b_2)- by a finite number of repeated applications of the operation "Add". This relation is a kind of generalization of the equilibrium relation wich exists in the scales "real world" and it will allow us to see the "scales domain" as the product of two "simple scales domain".

Now, let us describe the sintactic part of this domain. In order to do this, we are going to determine the language for this domain. Consider the symbol set $S_B = \{0, A, I, \sim\}$ where "0" is a constant symbol, "A", "I" are unary function symbols and "$\sim$" is a binary relation symbol. Our formalization of this "simple scales domain" is determined by the pair $\mathscr{B} = (S_B, (\mathbb{N}^2, \alpha))$ where α maps the symbol "0" to the empty scale $(0,0)$, "A" to the operation "Add" defined in 5.6, "I" to the identity operation and "$\sim$" to the relation of similarity defined by 5.7.

Finally, we just have to endow the structure $(S_B, (\mathbb{N}^2, \alpha))$ with some operations. Put $\Sigma = \{X \sim Y\}$ and $\Pi = \{0, A(Z), I(Z)\}$ to work with the equivalence relation defined in 5.7, the empty balance, the identity operation and the "Add" operation. Notice that $(\mathbb{N}^2, \Sigma, \Pi)$ can be viewed as a formal model for the "simple scales domain". In the next section we will show that $(\mathbb{N}^2, \Sigma, \Pi)$ can be viewed as the basis to build the "scales domain" wich is used in the formalization of the "two-pan balance metaphor".

The Scales Domain

In the previous section we proposed the domain with structure $(\mathbb{N}^2, \Sigma, \Pi)$ as a formal model for the "simple scales domain" . Here we deal with the more general "scales domain", and our proposal for its formal model is the product domain given by $(\mathbb{N}^2, \Sigma, \Pi) \otimes (\mathbb{N}^2, \Sigma, \Pi)$. In the rest of this section we provide an interpretation of its structure adequate for our purposes.

Notice that underlying our new product domain, there is the set of $B = \mathbb{N}^2 \times \mathbb{N}^2$. Therefore, the objects of such set have the form $((b_1,b_2),(c_1,c_2))$, where the pair (b_1,c_1) can be interpreted as the first plate of the scale and the pair (b_2,c_2) can be interpreted as the second plate of the scale (see fig. 5.9). Notice that in each plate there are objects of type "b" (boxes) and objects of type "c" (circles).

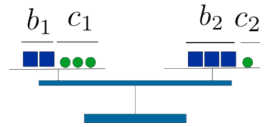

Figure 5.9: Interpretation of the elements of $\mathscr{B}$

Let us interpret the remaining structure of the product domain $(\mathbb{N}^2, \Sigma, \Pi) \otimes (\mathbb{N}^2, \Sigma, \Pi,)$:

From what was seen in the product domain (chapter 4), and without loss of generality, we can assume that the set $\{0, AA, AI, IA, II, \sim\}$ is the symbol set that determines the language of the product domain. Such symbol set has one constant symbol, four unary function symbols and one binary relation symbol wich are mapped by α' as follows:

1. The symbol "0" is mapped to the empty balance $((0,0),(0,0))$.
2. The symbol AA is mapped to the unary operation wich adds to each plate of the scale one object of each type and its defined by

$$AA((b_1,b_2),(c_1,c_2)) = ((b_1+1, b_2+1),(c_1+1, c_2+1))$$

3. The symbol AI is mapped to the unary operation wich adds one "box" to each plate of the scale, and it is given by

$$AI((b_1,b_2),(c_1,c_2)) = ((b_1+1, b_2+1),(c_1, c_2))$$

4. The symbol IA is mapped to the unary operation wich adds one "circle" to each plate of the scale, and it is given by

$$IA((b_1,b_2),(c_1,c_2)) = ((b_1, b_2),(c_1+1, c_2+1))$$

5. The symbol II is mapped to the identity operation given by

$$II((a_1,a_2),(b_1,b_2)) = ((a_1,a_2),(b_1,b_2))$$

6. By the definition of the product domain, the symbol $\sim$ is mapped to the binary relation given by

$$((b_1,b_2),(c_1,c_2)) \sim ((b'_1,b'_2),(c'_1,c'_2)) \text{ if and only if}$$

$$[\exists n(b_1,b_2)\pm(n,n)=(b'_1,b'_2)]\wedge[\exists m(c_1,c_2)\pm(m,m)=(c'_1,c'_2)]$$

   This equivalence relation can be understood as: two scales in this domain are equivalent iff there exists another scale b such that, it is possible to transform both scales into b by a finite number of applications of II, IA, AI or AA.

Additionally, notice that $\Sigma\cdot\Sigma=\{X\sim Y\}$ and $\Pi\cdot\Pi=\{0,II(Z),IA(Z),AI(Z),AA(Z)\}$ provides information to the model about the "interesting structure" to work with. In this case, it is characterized by the items described above.

The Equations Domain

Before describing similarity between domains using the "two-pan balance metaphor" we need to describe formally the "equations domain" wich plays the role of the "target domain" in this metaphor. As we did in the case of the "simple scales domain", we are going to begin with a semantic description of this domain and next, we shall describe the sintactic component (formal language) of this domain.

The objects (or elements) in this domain are first degree linear equations. However, for our purposes, the interesting structure are the ways to manipulate these equations in this domain. Particular interest has to be placed on the transformation processes from one equation to another equivalent equation (in the sense that both equations have the same solution set). To begin with, let us define our set of equations as $E=\{ax+b\equiv cx+d \mid a,b,c,d\in\mathbb{N}\}$. Consider $p_1=(a_1x+b_1\equiv c_1x+d_1)$ and $p_2=(a_2x+b_2\equiv c_2x+d_2)$ two elements of E. Define the following operation:

$$sum(p_1,p_2)=(a_1+a_2)x+(b_1+b_2)\equiv(c_1+c_2)x+(d_1+d_2). \tag{5.8}$$

Define R, an equivalence relation over E wich holds iff one of the equations can be obtained by adding an expression to both sides of the another equation. It is well known that the set of solutions of two equations are the same whenever two equation are equivalent (by R).

Let us define the language wich shall be used for this domain. Consider $S_E = \{e_1, e_2, \oplus, R\}$ as its symbol set where e_1, e_2 are constants, $\oplus$ is a binary function symbol and R is a binary relation symbol. Let us consider the structure given by $\{S_E, (E, \gamma)\}$ where γ maps the symbols in the following way: the symbol e_1 is associated to the equation $1x+0 \equiv 1x+0$, to the symbol e_2 is associated the equation $0x+1 \equiv 0x+1$, the symbol $\oplus$ is associated to the operation *sum* defined in equation 5.8, the symbol R is associated to the equivalence relation "R" defined above.

Also, let $\Omega = \{R(X,Y)\}$ and $\Psi = \{Z \oplus e_1, Z \oplus e_2\}$ to formalize the domain of equations by the domain with structure (E, Ω, Ψ).

Examples

In this section we provide examples aimed to describe formally the "two-pan balance metaphor" for linear equations. These examples are based in concepts given in previous chapters and the domains built in this chapter.

Our approach is to model this metaphor as a mapping wich assigns an equation to each scale (see figure 5.10). Consider the bijective mapping given by:

$$\varepsilon : \mathbb{N}^2 \times \mathbb{N}^2 \to E \text{ defined by}$$

$$\varepsilon(\ (b_1, b_2), (c_1, c_2)\ ) = b_1 x + c_1 \equiv b_2 x + c_2.$$

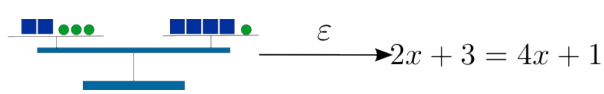

Figure 5.10: mapping ε.

The different domains that shall be used in this section were formalized above. However, the first example just summarizes the key definitions and notations necessary to follow the rest of examples.

Example 21. First, consider the "simple scales domain". To build a formal model for it, we have used the language $\mathscr{L}^{S_B}$ wich is determined by the symbol set $S_B = \{0, A, I, \sim\}$. We propose the domain with structure $(\mathbb{N}, \Sigma, \Pi)$ to model this domain, where $\Sigma = \{X \sim Y\}$ and $\Pi = \{0, A(Z), I(Z)\}$.

Second, we have considered the "scales domain" and formalized it by the product domain given by $(\mathbb{N}^2, \Sigma, \Pi) \otimes (\mathbb{N}^2, \Sigma, \Pi) = (B, \Sigma \otimes \Sigma, \Pi \otimes \Pi)$ where $\Sigma \otimes \Sigma = \{X \sim Y\}$ and $\Pi \otimes \Pi = \{0, II(Z), IA(Z), AI(Z), AA(Z)\}$. It uses the language determined by the symbol set $\{0, AA, AI, IA, II, \sim\}$.

Finally, the "equations domain" was formalized as the domain with structure given by (E, Ω, Ψ) where $\Omega = \{R(X,Y)\}$ and $\Psi = \{Z \oplus e_1, Z \oplus e_2\}$ are determining the structure wich shall be taken as important. The language used by this domain is determined by the symbol set $S_E = \{e_1, e_2, \oplus, R\}$.

Example 22. This example shows that the "scales domain" can be described as similar to the "equations domain". We are going to modify the previously described model for "scales domain" (example 21) in order to focus our attention in some structure useful to describe such similarity. Consider the term set $\Delta = \{IA(Z), AI(Z)\}$ and consider the domain with structure $(B, \Sigma \otimes \Sigma, \Delta)$. On the other hand, consider the model for the "equations domain" and define the following mappings:

$$F : \Sigma \cdot \Sigma \to \Omega \text{ defined by } F(X \sim Y) = R(X,Y)$$

$$T : \Delta \to \Psi \text{ defined by } T(\ AI(Z)\ ) = Z \oplus e_1,\ T(\ IA(Z)\ ) = Z \oplus e_2.$$

Observe that for any pair of scales, $b_1, b_2 \in B$ it holds that $b_1 \sim b_2$ iff $R(\varepsilon(b_1), \varepsilon(b_2))$. Additionally notice that $\varepsilon(AI(b)) = Z \oplus e_1(\varepsilon(b))$ and $\varepsilon(IA(b)) = Z \oplus e_2(\varepsilon(b))$ and therefore the tuple $(\varepsilon, F, T) : (B, \Sigma \otimes \Sigma, \Delta \to (E, \Omega, \Psi)$ is an isomorphism (and thus a metaphor) between $(B, \Sigma \cdot \Sigma, \Delta)$ and (E, Ω, Ψ).

Therefore, these two domains are indistinguishable with respect to the selected structure. It is remarkable that the mapping ε would be interpreted as some kind of "inference tool" because it permits concluding an equivalence between two equations from an equivalence between two scales. Furthermore, it permits operating with equations just by applying operations on scales.

Two observations are in order here to complement last example: First, observe that we would change the "target" domain of the metaphor, to see that the "two-pan balance metaphor" is not strong enough to deal with all first degree equations. Consider the set $E_{\mathbb{Z}} = \{ax + b \equiv cx + d \mid a, b, c, d \in \mathbb{Z}\}$ and observe that $E \subset E_{\mathbb{Z}}$. If we adapt the example to deal with this set, it is obvious that there exists equations (belonging to $E_{\mathbb{Z}}$) wich can not be worked out with this metaphor (those whose coefficients are not natural numbers). Second, this metaphor could be adapted to deal with equations whose all coefficients are negative numbers. It can be achieved by defining the mapping $\varepsilon'(b) = -\varepsilon(b)$. It is straightforward the adaptation of the involved domains in an appropriate way (defining an operation wich substracts expressions instead of adding them).

Example 23 (Inference)**.** This example is aimed to show that our model expresses the fact that this metaphor borrows some reasoning methods from the base domain to the target domain.

When we apply the "circle" (or "box") operation to any scale, the resultant scale is equivalent to the first. When the scale is balanced, the resultant scale still remains balanced. This is an inference (or reasoning) proper of the "scales domain". However, such "inference" can be borrowed from the base domain to be applied succesfully into the "equations domain": If we add some expression to both sides of an equation, then the resultant equation preserves the same set of solutions. Such "inference preservation" would be easily formalized by setting:

$$\Gamma = \{X \sim IA(X)\} \text{ and } \Omega = \{R(X, X \oplus e_2)\}$$

where Γ and Ω are formalizations of the reasonings described above. It is straightforward that considering the obvious translations and $\varepsilon : \mathbb{N} \times \mathbb{N} \to E$ we achieve the formalization of such "inference preservation".

However, we would like to show an application of theorem (42). Let us formalize something a little different, let

$$\Gamma = \{X \sim Y,\ IA(X) \sim IA(Y)\}, \Delta = \emptyset, \Omega = \{R(X,Y),\ R(X \oplus e_2, Y \oplus e_2)\} \text{ and } \Psi = \emptyset.$$

and consider the domain with structure given by $(\Gamma, \Delta, \mathscr{B} \times \mathscr{B})$ as our model for the "scales domain" and the domain with structure given by $(\Omega, \Psi, \mathscr{E})$ as our model for the "equations domain". Consider $T : \Delta \to \Psi$ as the empty function and the mapping $F : \Gamma \to \Omega$ defined by:

$$F(X \sim Y) = R(X,Y) \text{ and } F(IA(X) \sim IA(Y)) = R(X \oplus e_2, Y \oplus e_2)$$

It is straightforward that the tuple $(\varepsilon, F, T) : (B, \Gamma, \Delta) \to (B, \Omega, \Psi)$ determines a metaphor (actually it is an isomorphism) from (B, Γ, Δ) to (E, Ω, Ψ).

Observe that in the scales domain it holds that

$$X \sim Y \models_B IA(X) \sim IA(Y)$$

(wich can be interpreted as $X \sim Y \implies IA(X) \sim IA(Y)$) and thus we can apply the theorem (42) of consequence to conclude that in the domain of equations it holds that

$$R(X,Y) \models_E R(X \oplus e_2, Y \oplus e_2).$$

A comment is in order here to close this section. Notice that the "two-pan balance metaphor" deals just with the transformation of equations into other equivalent equations by adding values to both sides of the equation. However, in most of the cases, the equations will still remain unsolved i.e. in general, there is no way to use this metaphor to transform all equations to equations with form $x = a$. In practice, this kind of resolution for equations can be addresed with another metaphor wich uses a box to represent the variable and candies to represent numbers. It works for equations with form $ax + 0 = 0x + d$. See [102] and [3] for further details.

5.3 Tiled Path Metaphor for Integers

The "object collection metaphor" studied in the first section of this chapter is unfitted to deal with negative numbers, essentially because it is based on piles wich always have zero or more objects. However, the "tiled path metaphor" for integer numbers provides a natural way to introduce the integer numbers' structure into the mind of the learner.

This metaphor takes advantage of our body's navegational system to build a first numerical line representation into the learner's mind. There are claims stated by researchers from the cognitive field who affirm that our numerical representation can be viewed as a spacial numerical line (see [17], [104]). Thus, this metaphor takes our knowledge about how to move in the space (base domain) and project it into our own representation of the integer numbers to structure it in a coherent way.

For the sake of completeness of this work, we are going to describe briefly this metaphor and in a way to enlighten our purposes. Notice that from a cognitive point of view, a more accurate and complete description can be found in [71] under the name of "Motion Along a Path" metaphor.

The base domain of this metaphor is known as *tiled path domain*. We are going to model its structure by considering a two-directional infinite path T (if we call "tiles" to its elements, they would be seen as if they were ordered one after another forming an infinite path -see figure 5.11-). Attempts to formalize this path would end up with some set of axioms for integer numbers. Instead of that, we shall identify T with $\mathbb{Z}$ to describe mathematically this domain while giving a plausible interpretation (from the cognitive point of view) of its structure.

Let us imagine that over some tile $p \in T$ there is a subject wich has the freedom to walk in any of the two possible directions (forward or backward) offered by the path. Any walk that the subject may take can be described by an operator whose argument is the initial position p of the subject and whose value is the position of the subject after he has walked through the path.

More formally: for every $n \in \mathbb{N}$ we define unary operations $ahead_n$ and $behind_n$ wich change the position of our imaginary subject from the actual tile to a new tile wich is determined by walking n tiles in the appropriate direction. Thus, if we take $p \in T$ as the initial position of the subject, the operation $ahead_n(p)$ returns a position wich is n tiles in front of the subject. In the same way $behind_n(p)$ returns a position wich is n tiles behind the subject. In figure 5.11 (below) we sketch the operators $ahead_2(p)$ and $behind_2(p)$ when a fixed position p is given. The set of objects for this domain is denoted by Q and defined by:

$$Q = \{ahead_n : T \to T \mid n \in \mathbb{N}\} \cup \{behind_n : T \to T \mid n \in \mathbb{N}\} \tag{5.9}$$

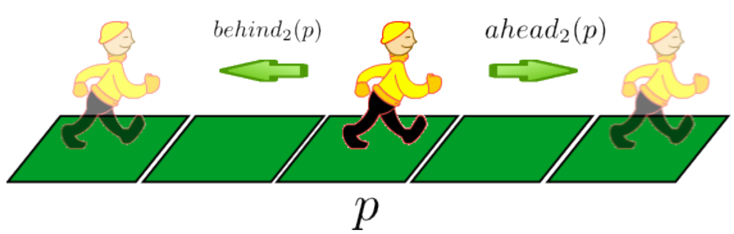

Figure 5.11: Representation of the Tiled Path Domain

The *tiled path metaphor* uses the "tiled path domain" to structure into the learner's mind a

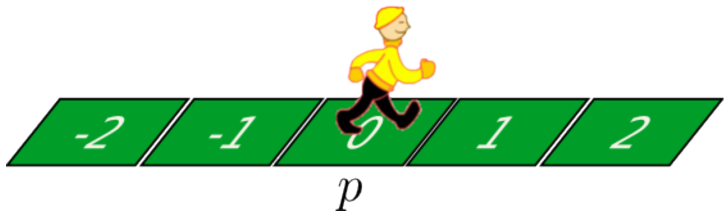

Figure 5.12: Representation of the “Tiled Path Domain Metaphor”

representation of the numerical line including negative numbers. Our approach attempts to formalize this metaphor as a mapping from the set of relative positions Q to the set of integers $\mathbb{Z}$. This mapping $z : Q \to \mathbb{Z}$ is defined as follows:

$$\begin{cases} z(ahead_n) = n, \\ z(behind_n) = -n. \end{cases} \tag{5.10}$$

Notice that the elements of Q can be composed. For example, for every $n, m \in \mathbb{N}$ the operation $behind_n \circ ahead_m(p)$ is well defined, and such operation would force to our imaginary individual to take a forward walk of m tiles and come back n tiles after that. From the definition of $z : Q \to \mathbb{Z}$ (see 5.10) it is clear that

$$z(q_1 \circ q_2) = z(q_1) + z(q_2), \tag{5.11}$$

and therefore $(Q, \circ)$ is isomorphic to $(\mathbb{Z}, +)$ because z is bijective. These properties suggest that this metaphor is well suited to explain the addition operation for integer numbers. These properties are formally stated in examples 24, **??** and 51.

The “tiled path domain” has a great advantage over the “integer numbers domain” because at some extent, every normal brain is expert in driving it’s body through the enviroment. In simpler words, everyone has walked along a path and has some intuition about the concepts “direction” , “opposite direction”, “walk” and their interrelation. The metaphor just make the necessary bridges between these concepts and the structure of $\mathbb{Z}$ as in figure 5.12 wich depicts how this metaphor is used in mathematical training.

It is worth noticing the similarity of figure 5.12 with a linear numbered board game. It was not surprising when we found that a teaching strategy based on this metaphor has been applied succesfully to provide experimental evidence: while playing a linear numbered board game, the learner uses its sensorial knowledge about the "tiled path domain" to structure its own representation of the "integer numbers domain". These studies can be found in [101, 105]. Furthermore, in [104] it is claimed that "representations of numerical magnitude are both correlationally and causally related to arithmetic learning".

It may be posible to interpret the "tiled path domain" as the "object collection metaphor" (as shown in figure 5.13) by considering an "*one directional tiled path*". In this case, it is plain to see that there are also tiles wich are placed before the position p and that our imaginary subject just have to walk backwards to reach them. So, at certain extent, the construction of the *tiled path domain* would be concibed as some sort of combination or join of two unidirectional paths wich are "glued" in the origin. We will show that this intuition can be expressed in our formalism by means of concepts like products and quotients (see examples 24, **??**, 50 and 51). Furthermore, it is suggestive to observe the similarity of these constructions with the classical construction of the group $(\mathbb{Z},+)$ (integer numbers) based on the monoid $(\mathbb{N},+)$ (natural numbers).

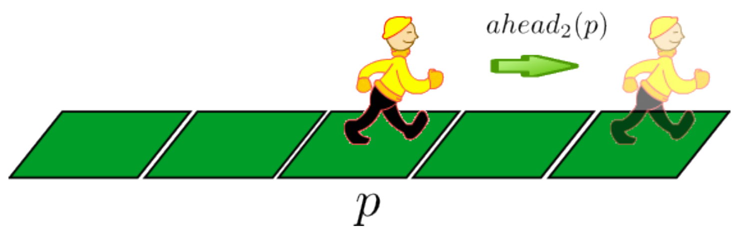

Figure 5.13: The object collection metaphor interpreted in the tiled path domain.

Formal Model based on Domains

The Integers Domain

The formalization of the domain of integers for this metaphor is simple. Recall that $S_{ar} = \{+,\cdot,0,1\}$ and consider the structure given by $(S_{ar},(\mathbb{Z},\gamma))$ (with γ the standard association of symbols). Define the set of terms $\Psi = \{X+Y\}$ and the set of formulas $\Omega = \emptyset$ to build the domain with structure $(\mathbb{Z},\Omega,\Psi)$.

Next, we use concepts developed in previous chapters to build a model for the *tiled path domain*. As we said above, the "tiled path domain" shall be obtained by combining two simpler "one directional path domains". Therefore, our first step shall be to propose a model for the "one directional path domain". After that, we shall take a product of this domain and give arguments to support the idea that a quotient of this domain can be taken as a plausible model for the "tiled path domain".

One Directional Tiled Path Domain

Analogous to the "tiled path domain" we are going to view this domain as a set of operators defined over the tiles of a path (walks over a path), but we want to take advantage of the fact that this set of operators together with the composition operation is isomorphic to the set of natural numbers with the addition operation. Thus, in order to avoid technical complications, we shall adopt the set of natural numbers $\mathbb{N}$ as the underlying set for this domain. The formalization of this domain is built as follows: Let be $S = \{+, \cdot, 0, 1\}$ a symbol set and consider the domain $(\mathbb{N}, \Sigma, \Pi)$. Put $\Sigma = \emptyset$ and $\Pi = \{X + Y\}$.

The Tiled Path Domain

As we have pointed out above, we are going to obtain a model for the "tiled path domain" considering a quotient of the product $(\mathbb{N}, \Sigma, \Pi) \otimes (\mathbb{N}, \Sigma, \Pi)$. In order to abbreviate the present exposition we are not going to detail how $(\Sigma \cdot \Sigma, \Pi \cdot \Pi, \mathscr{N} \times \mathscr{N})$ is built. However, such construction in its generality is exposed in chapter 4.

There is no loss of generality in assuming that the language of this product domain uses $\oplus$ as the only binary function symbol, implying that $\Pi \cdot \Pi = \{X \oplus Y\}$. On the other hand, it is relevant to give an adequate interpretation for the product domain $(\Sigma \cdot \Sigma, \Pi \cdot \Pi, \mathscr{N} \times \mathscr{N})$ in a way to relate its structure to the "tiled path domain".

First, notice that the underlying set of this domain is $\mathbb{N} \times \mathbb{N}$ and therefore every object belonging to this domain is a pair (n_1, n_2). The first coordinate may be interpreted as "a forward walk of length n_1 tiles" and the second coordinate as "a backward walk of length n_2 tiles". This interpretation let us see this product as a structure where every object can be viewed as a pair of walks wich are taken in opposite directions. Recall that the "composition of walks" operation in this domain (wich we will denote by $\oplus$) is given by:

$$(n_1,n_2)\oplus(m_1,m_2)=(n_1+m_1,n_2+m_2).$$

Thus, when two objects of this domain are combined by means of this operation, the resultant object "composes" those walks wich are taken in the same direction.

Notice that some objects of this domain are "equivalent" in the sense that they affect the position of our imaginary subject in the same way. Take for example the objects $(0,0),(1,1),(2,2),\ldots(n,n)$. Each of them make our imaginary subject take a walk of k tiles forward and k tiles backward coming back to the same tile, therefore it is a good idea to identify all those objects as just one. This shall be done by introducing a quotient domain.

Let us consider the map

$$z':\mathbb{N}\times\mathbb{N}\to\mathbb{Z}$$

defined by

$$z'((m,n))=m-n.$$

and notice that z' is a surjective metaphor from the domain $(\mathbb{N},\Sigma,\Pi)\otimes(\mathbb{N},\Sigma,\Pi)$ to the domain $(\mathbb{Z},\Omega,\Psi)$ and then the quotient domain $(\mathbb{N},\Sigma,\Pi)\otimes(\mathbb{N},\Sigma,\Pi)/z'$ is well defined. Our proposal is to use this quotient domain as a formal model for the "tiled path domain".

Examples

Here, we formalize the "Tiled Path metaphor" for integer numbers and provide examples for it.

Example 24. Recall that our model for the "Tiled Path domain" is the quotient domain $(\mathbb{N},\Sigma,\Pi)\otimes(\mathbb{N},\Sigma,\Pi)/z'$ and notice that its "objects" (equivalent pairs of walks) are equivalence classes determined by the fibers of the homorphism z'. The first observation is that this map is stronger than a homomorphism, it is a metaphor. The second observations is that because of the way we have defined the "Tiled Path domain" we know that it is isomorphic to the Integers domain.

Take the domain $(\mathbb{Z}, \Omega, \Psi,)$ as our model for the "integer numbers domain" and define the following translations:

$$F : \Sigma \cdot \Sigma \to \Omega \text{ as the empty function.}$$

$$T : \Pi \cdot \Pi \to \Psi$$

given by

$$T(X \oplus Y) = X + Y$$

Notice that the map z from the quotient of $\mathbb{N} \times \mathbb{N}$ by the fibers of z' to the set of integers $\mathbb{Z}$ which is defined by $z(\overline{(m,n)}) = m - n$ is bijective.

It is remarkable how this construction of the "Tiled Path domain" is very similar to the very well known and classical construction of the integer numbers (the group $(\mathbb{Z}, +)$) based on the natural numbers (the monoid $(\mathbb{N}, +)$). However, our contribution is to offer a cognitive interpretation of these ideas and a formal way of express it.

Chapter 6

Conclusions

The main goal of this thesis was the study of analogies (or metaphors) that occur often in the learning and teaching of mathematics. As methodology, we chose the use of formal models to represent analogies and analize mathematically such representations. However, at the begining of this research, the literature search did not provide us with mathematical models to work with.

The formal mathematical models we found ([41, 40, 55] and [84]) could not be used by us for the purposes of this thesis that are related to cognition and learning. And, to face this situation, we built from scratch a new mathematical model for metaphor. The formulation presented here is based on theories of structure such as universal algebra, formal logic, graph theory, unification theory and theory of categories. This might seem natural when one considers that metaphor is a process that pinpoints commonalities between two structures.

In order to make the phenomenon of analogy susceptible of analisys, many assumptions underlie the model presented here. And yet, its usefulness was exemplified in Chapter 1 where we performed a deep analysis of the metaphor underlying the game "The Great Race"[1]. Furthermore, by using the insights that our model provides we were able to propose a mechanism for "learning by analogy" which explains the fact that this game is beneficial for learning numbers. This indicates that our model can be used to understand and simulate learning by analogy. This sketches the

[1]This game was developed by the cognitive scientist Robert Siegler and his collaborators to enhance children's numerical knowledge.

main contribution of this work: the creation of a formal model of metaphor that would potentially be used in cognitive contexts.

Also, this study makes a small (although meaningful, we think) contribution to mathematics: once the first definitions that set up our model—such as the formalization of "domains of knowledge"—were ready, a challenging problem of mathematical nature appeared: how to characterize the existence of a map between two first order languages which preserves the structure determined by the syntactical operation named substitution[2]. On that matter, Chapter 2 and Chapter 3 provide conditions characterizing the existence of such structure-preserving maps. For such study we used graph theory to develop new representations for terms and formulas of first order languages. These representations yielded the wanted conditions but expressed in a graph-based formalism, and thereafter, these conditions were translated back to the context of first order languages and their grammar by applying unification theory.

Another goal of this thesis is to expose some arguments to convince the reader that our formal model is able to reproduce, to a certain extent, some properties observed in analogies and metaphor. In particular we highlight that our model accounts for the carrying of inferences from the source to the target of the metaphor. Results of this kind might help to end the discussion named *the problem of justification of reasoning by analogy* which has been raised in the field of philosophy [76, 122] and, even when many scholars have discussed it, has remained without a satisfactory solution[56]. Our results[3] provide arguments to justify that some conclusions about the target can be drawn by the metaphor (see some examples in section 1.3.7). And the hypotheses of such results might provide conditions to be checked on those inferences that are going to be "justified" within our formalism.

It captivates the attention that some of our results might seem paradoxical at first sight: on one hand, we obtained a family of results ensuring the uniqueness of a metaphor, and—on the other hand—it is well known that a variety of metaphors between two domains can co-exist. Such contradiction dissapears when an accurate interpretation is given. For example, Lemma 25 ensures that a metaphor extending a partial map between the source domain and the target domain must be unique[4]. The hidden fact is that the source domain (and the target domain) can be formalized in multiple ways[5], and thus, it is possible that different formulations lead to different metaphors (even

[2]Recall that this operation generates such structure by recursive nesting of strings of symbols.
[3]Those would be Theorems 40 41, 42, and Proposition 26.
[4]For the sake of readability, the hypotheses of the Lemma are not explicit here.
[5]A single (human) domain of knowledge can be seen in multiple ways from our formal point of view. For example,

when the partial map is the same!). In other words, our results on the existence and uniqueness of metaphor should be always put in the context of how the source and the target domains were formalized and the structure that was considered on them.

A further interpretation of Lemma 25 might be worth mentioning. The lemma suggests strict conditions[6] under which an analogy is uniquely determined (see section 2.8.2). With these kind of results, future research into our model might shed light on how to pose complex questions in form of an analogy in such a way that a well determined and unique answer can be expected. This might have some consequences into the existent debates about including or not batteries of analogy-items (such as the Miller Analogies Test[7]) in standardized tests for college acceptance.

Some of our results have a sort of "cognitive interpretation" which is linked to cognitive concepts such as "Learning by Analogy" or "Reasoning by Analogy". Because most of these results assume a recursive mechanism for combining sequences of symbols, we believe that some recursive (or iterative) mechanisms are necessary for mimicking the properties of metaphor. The refered assumption might be caused because there is a grammar that is embedded directly into the model. However, we also have to consider that when we ignored the recursive structure provided by the grammar of the formal language, we were unable to reproduce similar results[8]. In turn, this led us to speculate that *some of the brain mechanisms for recursion might be key for the processing of analogies and metaphor*. For the readers interested on this topic, an expanded discussion is provided in Appendix B where we also suggest that the brain's mechanisms for recursion might be a key clue for understanding the strong relationship between language, metaphor and theory of mind that has been observed in humans [85].

The contributions of this work to the field of mathematics education are described next. This thesis has introduced two perspectives to understand analogies in the context of learning, namely the learner's metaphor problem and the teacher's metaphor problem. With respect to the learner's problem, Chapter 1 provides an interpretation of Theorem 31 which suggests that our model can

two formal domains can share the same "objects", but they might differ at the selection of relations or operations between such objects.

[6]Those conditions should be translated as knowing exactly, in each domain, which elements and relations are playing a role in the metaphor and knowing the partial map between the source and the target.

[7]There are claims that the MAT(Miller Analogies Test) outperforms the undergraduate GPA(Grade Point Average) and the GRE (Graduate Record Examinations) as predictor for first-year graduate GPA, overall graduate GPA and comprehensive exam scores [67, 69]. Another comparison between standardized tests can be found in [68].

[8]Such study is presented in Appendix C.

be used to enlighten and, perhaps, to solve such problem.

A deeper analysis of the learner's problem is outside of the scope of this work. Yet our arguments show that our model is a useful framework to analyze metaphors in the context of mathematics learning. In this direction, as future work, it would be interesting to use our model as a basis for building bayesian models of inductive learning (see [111, 108]). Also, by adding a mechanism to emulate the acquisition of knowledge when time is passing by, our model might even attempt to capture the dynamics of learning.

This thesis does not address directly the teacher's metaphor problem. Recall that such problem was introduced with the aim of organizing the kinds of analysis that should be performed on metaphor within the context of learning. The reason why the results of this thesis can not be applied directly to analyze such problem is that the gist of the problem lies on the searching of a *relational* matching between the source and the target. And most results of this thesis assume that such matching is given beforehand. However, the problem also requires a sistematic way to determine a *(non-relational)* matching between the objects of the source and the objects of the target. And then, a little contribution is made: if the relational matching is given, Theorem 30 provides conditions ensuring that a partially defined map can be extended in a unique way from the source's objects to the target's objects. And the proof of the theorem also provides the extension.

For example, let us imagine that we are facing the challenge of designing the Siegler's TGR game for improving numerical knowledge (such situation is an instance of the metaphor's teacher problem). Assume that, for such purpose, we choose the metaphor described by Nunez and Lakoff [86] as "ARITHMETIC is MOTION ALONG A PATH" and that we are about to design the board of the game. Also assume that we are convinced that the number one must be printed on the first square of the board and, that somehow, we have figured out a suitable relational matching[9].

In such case, Theorem 30 assures us that there exists exactly one metaphor which is consistent with the described conditions. Indeed, we can read—between the lines of the proof of the Theorem—how the rest of numbers must be printed out in the rest of the squares of the board[10]. This link between our formal model of metaphor and the design of learning materials shall be

[9](a) the operation "moving one square to the right" matched with the operation "next" between numbers, (b) the relation of "being at the left of" matched with the numbers relation "less than", (c)... , etc.

[10]This might not sound very impressive. However, as far as we know, it is the first formal explanation of *why* such matching is so intuitive and effective for learning.

further discussed in the next section.

Design of learning materials

Our formal model helped us to analize the TGR game developed by the psychologist Robert Siegler and his collaborators to enhance children's numerical knowledge (See introduction). In their seminal study [101] they wrote about the game:

> There is no reason to believe that the present version of The Great Race cannot be improved. Many features of the intervention were based on hunches and guesses rather than empirical research.

The above paragraph highlights the extent to which the design of games for learning (and other didactical materials) is often a handcraft whose efficacy depends heavily on the skills of the craftsman. The core difference between an engineering design and handcrafting is at the heart of this thesis: an engineer works with formal models that endow him with predicting power about the behavior of his designs. Along these lines, this work provides some arguments supporting that *the handcrafting of didactical materials for learning can become a task of systematic design* and offers our model as a tool.

Let us give another[11] example in support of the last claim. First, recall from the introduction of this work, that the linearity of the TGR board was a vital feature of the game since it was shown in [105] that a similar version with a circular board does not improve learning. This shows how an innocent decision concerning the board's design causes a huge effect on learning. Often, experimental studies (like the one performed by Siegler) might be out of the scope for a teacher or for a designer of learning materials. In what follows, we argue that our model is useful for identifying a family of designs with a positive impact in learning.

Let us follow with the TGR example by observing that our model is able to make a prediction consistent with Siegler's results. To this aim, notice that in our formalism, (a) the linear board version of the TGR game satisfies the conditions to be called a metaphor and (b) the circular board

[11] The first example was given in the last paragraph of the previous section.

version of the TGR game does not really deserve to be called metaphor since the order is not preserved from the source to the target and the distance is not preserved either (for more details see the last paragraph of section 1.1). Thus, if we accept the evidence suggesting that analogies have the power of facilitating learning [3, 29, 10, 101, 105, 98], we conclude that the linear board version of the TGR game shall have a beneficial effect on learning. Also, we can conclude that the linear board shall preserve more structure of numbers than the circular board version of the game. Then, our model suggests to make use of a linear board rather than a circular board to implement the game.

Along these lines, in Appendix A, we present an exploratory study whose results are consistent with the core claim of this part. We used the insights provided by our model to design a card game aimed to teach a complex mathematical concept: exponential functions. Encouragingly, the results were positive.

With respect to the question about how to cross the bridge (if it can be crossed) connecting handcrafted learning materials and learning materials designed systematically, this work offers more questions than answers. However, here we offer our formal model of metaphor as a little step on the direction of crossing that bridge.

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Part II

Appendices

Appendix A

Analogy-based games for learning high school mathematics

Introduction

During the last fifteen years, the chilean government has been making large investments in education. However, returns are far below from expected [90]. For example, the mathematics' average score was 421 below from the OCDE's average score of 496. As another example, 22% of the chilean students do not acquire the most elemental mathematical skills. In contrast, 8% of the OCDE's students (in average) and only 1% of Shangai's students are in the same situation.

Unfortunately, learning in chilean schools is an activity in which the students are forced to attend, be quiet and pay attention for many hours at a day; and this for 12 years or more! and, as if this were not enough, this is a monotonous activity where the conventional instruction of mathematics consists of a blackboard filled up with abstract concepts, symbolic representations and little focus on understanding. One alternative for challenging this status quo might be to design educational games that are entertaining, promote cooperative learning and helps the student to interpret and understand these abstract concepts and symbolic representations. There are claims suggesting that well designed games provide environments that are stimulant, highly contextualized, and emotionally safe where a highly engaged and individualized learning can occur[125]. In fact, various initiatives have been promoted for researching gaming and its applications for learning, see for example the “Games for Learning Institute”.

However, the design of games with the characteristics above mentioned is far from trivial; subtle differences in the design of the game might have huge effects in the expected learning. For instance, Robert Siegler et al. created a game named “The Great Race” and then showed that children who played this game obtained huge gains on their numerical knowledge[106]. Crucially, Siegler noticed that the linearity of the board of the game was a vital feature to explain the gains of knowledge because the same game—but constructed on a circular board—did not improve numerical knowledge on children[105].

One explanation for such effect might be that the linear path recalls the “mental number line”[104] and this converts the linear game into a great analogy (or conceptual metaphor[71, 89]) for numbers learning, while a circular game would not be. This sort of argument is supported by many empirical studies that highlight the importance of analogy in teaching mathematics [98, 62, 3, 53].

Psychological literature[29, 72] claims that analogy is a cognitive mechanism used by the human mind for the understanding of an abstract concept called *target*, e.g. numbers, in terms of a more concrete concept called *source*, e.g. the linear board of a game. The work of Siegler suggests that if we are able to build an analogy whose source is determined by certain game and whose target is the mathematical concept that is to be taught; such game play might facilitate the understanding of that mathematical concept. To illustrate the idea, in the next section we sketch the developing of a game with these characteristics. Also, to estimate the effect of the game on learning, we report an exploratory study whose results are encouragingly positive.

The Alien Invasion Game

The educational goal of the Alien Invasion Game (AIG) is the introduction of exponential and logarithmic functions basics. We used literature related to analogy[29, 72] as a guide to implement our idea: the “world of the game” is the source of an analogy whose target contains the abstract concepts to be taught. A brief description of the AIG development is provided next.

The AIG is an adaptation of a famous[1] strategy game which is played with a deck of cards.

[1] Magic the Gathering.

Figure A.1: The AIG's cards.

Our game is acclimated on a fiction scenary where aliens are invading our world. Our scientists use biological weapons to defend the earth population. In this context, the game goal is to infect the aliens with bacteria in order to make them vulnerable to capture. The strength of an infection depends on the amount of energy consumed by the biological weapon. An infection is less effective when a shield is protecting the alien. However, biological weapons can be combined in order to produce stronger weapons.

We built an analogy whose source is the scenary above described and whose target is a set of exponential functions (the functions 2^x, 3^x and 5^x together with their inverse functions) in the following way: One biological weapon corresponds to one exponential function. The amount of energy consumed by the weapon correspond to the argument passed to the exponential function. The strength of the infection produced by the weapon corresponds to the evaluation of the exponential function. The combination of two weapons corresponds to the multiplication of two exponential functions. A shield protecting an alien is related to a logarithmic function.

Method, Materials and Procedure

Participants were 51 high school students between seventeen and nineteen years old. They attended to the same level in the same high-school and they had no previous knowledge of exponential functions. There were two groups of them: the first group was assigned to the game condition (27 students, 9 female) and the second one to the control condition(24 students, 13 female). This is an exploratory study because of this non-random assignment. However, the high-school teachers

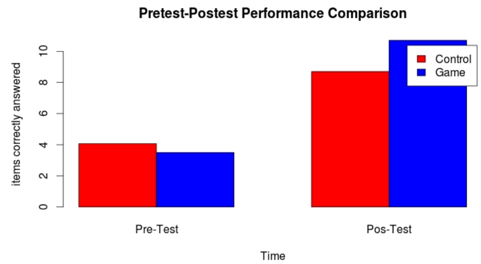

Figure A.2: Non normalized data shows a difference between the two groups after the intervention.

reported that there were no performance differences between these two groups. The experimenters were two female high-school teachers supervised by a male graduate student.

Each student was handed a deck of twenty cards that represent the weapons and other elements of the game (see Figure A.1). The intervention was realized through five 90-minute sessions within a 3-week period. At sessions one and five all students did a test of 14 items about exponential functions. Students in the control condition were taught exponential and logarithmic functions in the high-school traditional manner (blackboard with symbolic explanations and examples). In the game condition, students played the AIG during the three first sessions where they organized themselves in groups of two, three or four players. This game group received the explanations of the mathematical concepts using some AIG's strategies as examples during the last two sessions. Both groups were exposed to the material during the same amount of time. The same teacher gave the lessons for both groups.

Results

The data was normalized before preforming the analysis. With regards to condition, no significant differences were present between the pretest scores of the control group $(M = 0.14, SD = 0.95)$ and the game group $(M = -0.16, SD = 1.04)$. However, in Figure A.2 we observe that the post test scores of the control group were lower than the scores of the game group $(-0.41 \text{ vs } 0.47, t = 3.51, df = 47.9, p < 0.001)$. Additionally, a linear regression analysis confirmed a strong effect of the condition on the post test score.

Brief Discussion

The results are consistent with the hypothesis that playing the AIG game facilitates the acquiring of knowledge of exponential functions in high school students. This in turn suggests that analogy theory and metaphor theory can be used as a guide for designing educational games. It might be worth observing that the cognitive approach presented here might be adapted to teach almost any concept of high school mathematics. Thus the path is open to explore the scope of the approach presented here. However, this study leave many unanswered questions. One of them is related to the emotional impact of the AIG game on the students. The answer might help us to develop techniques for improving the students attitude toward mathematics.

Appendix B

Analogy and Recursion

Combinatory play seems to be the
essential feature in productive thought.

A. Einstein

In his article "The Uniqueness of Human Recursive Thinking"[13] (see also [14]), the psychologist Michael Corballis argues that recursion expresses itself in several key human mental faculties including the faculty of language and theory of mind. In the following discussion we argue that metaphor and analogy should be included in that list. This claim, namely that *some of the mind's recursion mechanisms play a key role in the processing of an analogy (or metaphor)*, has not been made explicitly in the literature dedicated to metaphor or analogy (as far as we know at least). Also, the argument we present here suggests that recursion might be a major clue for understanding the strong relationship between language, metaphor and theory of mind that has been observed (see for example in [85, 47, 48, 45, 16] among many others).

We believe that our model can be used as a guide to obtain some insight into the processing of an analogy because of two reasons. First, its results are consistent with some properties observed in metaphor. For example, those results that can be interpreted as "learning by analogy" (Theorem 30 and Theorem 31) and those ensuring that inferences can be preserved from the source to the target (Proposition 26, Theorem 40, Theorem 41, and Theorem 42) among others. Second, the hypotheses that our model makes can be grounded in psychological theories and empirical research (see sections 1.3, 1.3.2, 1.3.3). The picture that emerges suggests that the assumptions that our model makes, implies some of the properties of the phenomenon. Thus, a rough interpretation of our model would give us a hunch about the necessary and sufficient conditions to process an analogy.

A simple observation on some of the results of our model is the genesis of our claim. A simple review of the results metioned above (those with a cognitive interpretation) is enough to realize that *all of them* share a remarkable aspect: as a hypothesis, each one of them needs a recursive mechanism for combining sequences of symbols. And in fact, when the recursive structure provided by the formal language and its sintax was removed from our model, we were unable to reproduce similar results[1]. This might indicate that the mechanisms providing recursion are necessary for simulating the behavior of a metaphor or an analogy. This observation is consistent with our claim.

In the search for evidence relating recursion and metaphor we found no studies performed with this aim. Seems to us that recursion has not been studied in the context of metaphor (or viceversa) and we believe that this is because, in the current theories of analogy and metaphor, recursion by itself does not play any rol of importance. Consequently, there would be no good reasons to study the relation between analogy and recursion mechanisms. However, in order to persuade our reader that recursion and analogies are strongly linked we shall use an indirect strategy. But first, let us point out why recursion should play a rol of importance in metaphor and analogy.

One of the most influential theoretical frameworks for studying analogy is "the structure-mapping theory[30]" which is a seminal work postulated by the widely acknowledged psychologist Dedre Gentner. In such work she assumes explicitly that "domains and situations are psychologically viewed as systems of objects, objects-attributes and relations between objects". For many, such relational representation of the information, is the kernel of this theory. We are going to argue that such representation of the information is a constrained version of a more general and useful principle that is based on recursion.

Our model represents information by means of three components: a) a way of representing individuals[2]—the Gentner's objects—, b) the ability of using symbolic representations for referencing relations between such individuals[3]—the Gentner's relations and attributes—, and c) the capacity

[1]Such study is presented in Appendix C of this thesis.
[2]a "zero order" representation.
[3]a representation of "first order

of using recursion on these symbolic representations to create hierarchical representations of the information[4]—there is no analogous component in Gentner's Theory. Thus our model is based in Gentner's theory with the addition of c) as a key component of the problem.

The subtle differences between the Gentner's proposal and our own, can be highlighted by means of an example. To this aim, let us recall the TGR game developed by Siegler (see section 1.1). The metaphor underlying such board game is that arithmetic is motion along a path [71] and observe that under the Gentner's conception, the Figure B.1 might be a good depiction of the source and the target of the metaphor.

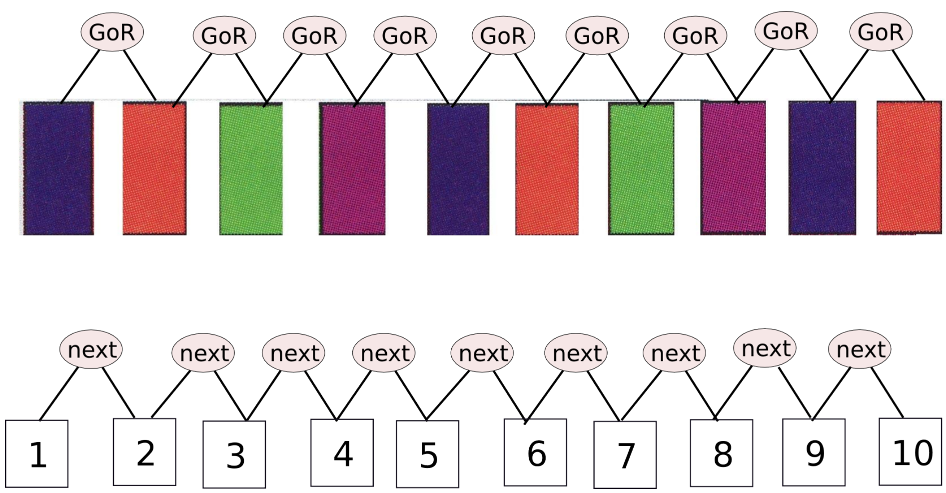

Figure B.1: Above (below), the representation of the source (target) of the TGR metaphor by using the ideas of Dedre Gentner. In the source of this metaphor, each square of the board is represented as an object and its relational structure is determined by which squares are adjacent to each other. In such representation GoR means "Go to the square at the Right".

Actually, such conception explains succesfully the metaphor as the "geometry" of the source being preserved into the "geometry" of the target. However, such representation fails to acknowledge the recursive patterns that are present, as for example, that 2 is next(1), 3 is next(next(1)), 4 is next(next(next(1))) and so on. And such recursive pattern is the key that our model exploits for obtaining those results that we have interpreted as "learning by analogy[5]".

In this example, the symbolic recursion endowed to our model has at least one advantage: one

[4]representations of second, third and "higher" orders.

[5]The interpretation of Theorem 30 and Theorem 31 are given in the introduction of this work, section 1.3.5.

of the issues of finding an analogical mapping, under the Gentner's Theory, is that it needs to check all the posibilities. In our model, where a recursive pattern is expected, if such pattern is pinned down, then there is a clear guidance on how the analogical mapping between the squares and the numbers should be done (first square $\rightarrow$ 1, GoR(first square) $\rightarrow$ next(1), GoR(GoR(first square)) $\rightarrow$ next(next(1)) and so on). Actually, this might have implications with respect to learning by analogy because this will provide a clear guidance to reshape or augmenting (if necessary) the target of the metaphor by using the recursive structure of the source. This might have other implications in generalizing learning (as for example, what is the number that follows 10?).

The use of metaphors such as "love is a journey" in daily speech, hardly reveals the use of recursion. Then, recursion might be difficult to observe directly in daily metaphors[6]. Nevertheless, recursion can be observed directly in other cognitive phenomena and we shall focus on two particular cases: the faculty of language and the theory of mind (ToM)[7].

In what follows we want to convince the reader that these two cognitive faculties have been consistently related to metaphor in the literature. This with the aim to argue for our affirmation as follows. Such revision will show us that there is, at least, a kind of a weak correlation between these three faculties (language, ToM and metaphor). This correlation must be caused by the brain mechanisms that are shared between these three faculties. Since recursion mechanisms are key components of the faculty of language and ToM, these mechanisms should be the main contributors for the appearance of such correlation. This might indicate that recursion should be strongly involved in the processing of metaphors and analogies.

The last affirmation is not a firm conclusion since the argument is based on a correlation that might be triggered by many other mechanisms, such as working memory, executive functions, metarepresentation and other cognitive processes that are usually theorized in the cognitive literature. However, we believe that we posit here a hypothesis that can be tested empirically and, after that, it can be rejected or strengthened.

[6]However, observe that when *explaining* a concept to other person by means of an analogy, the use of recursion is a little more evident since the hierarchical structure must be explicitly pointed out. This difference might be consistent with the following pattern. The utterer uses a daily metaphor when he assumes that the receiver knows the source and the target of the metaphor. On the other hand, the utterer uses an analogy, when he wants the receiver to make an inference of a target that the receiver does not know completely.

[7]Theory of mind is the ability to attribute mental states—beliefs, intents, desires, pretending, knowledge, etc.—to oneself and others and to understand that others have beliefs, desires, and intentions that are different from one's own.

Before entering into the review of the literature that supports our argument, let us pinpoint why recursion might be necessary for faculties such as language, theory of mind and analogy. In the case of language, most linguists agree that one key characteristic of human language is its recursive nature. In this way, recursion provides to the human mind with the capacity to generate an infinite range of expressions by combining a finite set of elements[49].

For the ToM case,let us consider a person affirming "You must be thinking that this is a good idea"[8]. In such case, there is little need for recursion. However, for such person to emit the utterance "I think Alice must be thinking that I think this is a good idea" is necessary that such utterer is able to think that Alice is thinking about the thoughts of him[9]. Thus revealing, through language, how a recursive process is triggered to assign different states to three different representations of the involved individuals: the state of "thinking about Alice" is assigned to the utterer and , *under* this label, the state of "thinking about the utterer" is given to the mind of Alice. Even deeper, the state of "thinking that the idea is good" is given to the utterer, even when he thinks that the idea is bad. It is crucial to observe how these representations have a hierarchical structure which permits, for example, that the utterer is labeled under different states without generating conflicts. These kind of representations are generated necessarily through mechanisms of recursive nature. Actually, many models of theory of mind assume that recursion is a natural component [109, 110].

Finally, besides of the reasons given by our formal model, a similar explanation makes sense for arguing that a normal processing of an analogy would need recursion mechanisms. These mechanisms would ensure that the information that the process receives as input, can be represented hierarchically. That was exemplified above by Alice and the utterer. Recursion allows us to represent knowledge by means of recursive structures (various levels). And then, consistent with current thinking in the cognitive sciences, an analogy would result from the comparison of two hierarchical structures representing the source domain and the target domain[36, 86, 30, 29].

Our argument begins by acknowledging that language and ToM are strongly related to recursion. As we mentioned at the beginning of this appendix, the psychologist Michael Corballis argues that recursion expresses itself in the faculty of language and the theory of mind[10]. In the earlier case, there is no doubt that recursion is a key component of language[49]. For the latter case, there is an agreement among theorists that recursion underlies theory of mind[110]; this is reflected in

[8] This would be a first order representation.
[9] This would be a third order representation.
[10] Observe that he does not include metaphor or analogy among them.

the fact that some ToM tasks have been developed by probing recursively ToM understanding (first level: inferences about a belief, second level: inferences about a belief about a belief, and so on, up to fifth level or even more [78, 61]).

More evidence supporting the view that recursion may be one general ability serving both: language and theory of mind; comes from the fact that the children's ability to use embedded syntactical structures comes on line shortly before the ability to solve false belief tasks[16]. Furthermore, the training in a specific linguistic construction, sentential complements[11] (such as John *said* that Fred went shopping), improves performance in theory of mind tasks[45]. This might explain why verbal ability and performance on ToM tasks are highly correlated [48].

Next, we need to argue for two points: (1) that language and metaphor are strongly related and (2) that ToM and metaphor are strongly related. In support for the first point, we present the point of view of the cognitive scientist Dedre Gentner (who has dedicated her entire life to the study of analogy, similarity, metaphor, learning and other related cognitive phenomena). In her article named "Why We're so Smart"[31], she arguments in favor of her thesis:

> ... what makes humans smart is (1) our exceptional ability to learn by analogy, (2) the possession of symbol systems such as language and mathematics, and (3) a relation of mutual causation between them whereby our analogical prowess [powers] is multiplied by the possession of relational language.

Thus arguing in favor of a strong link between analogy and language. Also, we believe that the mind's recursion mechanisms should be the "relation of mutual causation" which is sostained by Gentner's thesis. In such case, her thesis is consistent with our hypothesis.

Also, in [5] (see also [48]) it is suggested that analogical reasoning is strongly dependent on language. In such study they compared the relational reasoning performance of diferent groups of patients with some disabilities in language. They report that "the most severely aphasic patients (i.e., Broca's, global, and Wernicke's aphasics) had the lowest scores, approaching chance performance on the relational-reasoning items. Additionally, has been reported that complex hierarchical

[11] Sentential complements are recursive sentences in the sense that they allow for the embedding of tensed propositions under a main verb.

structures mimicking those of natural languages mainly activate Broca's area[26][12]. This might suggest that recursion mechanisms are implemented in the Broca's area and its failure affects analogical reasoning. And this is what happens in children with language impairments. They have deficits in understanding metaphorical language[100].

Now we need to support the second point, namely that analogies and ToM are strongly related. let us begin with the observation of the psychologist Francesca Happé who claimed that first order ToM understanding is necessary for comprehension of metaphorical expressions [47]. In another study, the role of theory of mind in comprehension of metaphor was examined in patients of schizophrenia during remission, the comprehension of metaphor was significantly correlated with second-order false belief understanding[83]. The results presented by Happé were challenged by the results of [85]. In such study there was no correlation between metaphor understanding and first order ToM understanding. However, they report a strong correlation between second order ToM capabilities and metaphor understanding. Consistent with our hypothesis, these results might be pointing that the influence on metaphor understanding is not coming from the ability of Theory of Mind directly, but from the recursion mechanisms in which such faculty relies on.

We want to end this discussion by presenting other kind of evidence supporting our hypothesis. The researchers Hauser, Chomsky and Fitch argue in [49] that the mechanisms necessary for recursion evolved for reasons other than language, that is to say, to solve other computational problems such as navigation, number quantification, or social relationships. They suggest that in order to enlighten the intricacies of language, it might be worth carrying studies for finding evidence of recursion in animals, but in a noncommunicative domain. If they are right, and there is recursion in animals, it is plausible to assume that the performance of such circuitry is dependent of its evolutionary state[13] and then the following reasoning might serve in support of our hypothesis.

If we assume that there exists such recursion circuitry in animals and that it is shared between language, ToM and analogy (among other faculties). Also, if we furthermore assume that such

[12]Natural languages mainly activate left Brodmann area (BA) 44/45, and hierarchically structured mathematical formulae, moreover, strongly recruit more anteriorly located region BA 47.[26]

[13]A clarification is needed here because this assumption might seem contradictory with what Hauser, Chomsky and Fitch postulated in [49] where they argue that such recursion mechanisms (FLN) is the unique component that makes human language qualitatively different from any other animal system of communication. However, in the context of our assumption, what would make human language different from other systems of comunication is the *performance* of such recursion mechanisms which in the human case might be more complex than those encountered in the recursive mechanisms of any non-human animal.

recursion circuitry is high performance for humans, average performance for apes and low performance for monkeys. Then, since such recursion mechanisms are key for language, ToM and analogy, it is reasonable to expect that the performance of recursion sets a bound on the performance of each one of these three cognitive abilities. Furthermore, if these assumptions are correct, it is reasonable to expect a positive correlation between the performance of such recursive mechanisms and the performances of the mentioned three faculties. Some evidence consistent with such reasoning is presented next in the form of a series of studies on analogy, language and ToM in animals.

In first place, let us consider our species. A normal person, without effort, is able to handle its highly recursive native language (unless it is the piraha language where there is a debate about its recursivity), around age of six, normal people can perform most of the typical ToM tasks, understand metaphors and create analogies to express himself. Then, there might seem to be at the very least some sort of weak correlation between these three factors and high performance mechanisms for recursion.

In second place, let us consider let us consider apes (Pan troglodytes). It is well known that some of them have been trained to use, at certain extent, sign languages such as the American Sign Language[14]. Apes have not fully mastered those languages, but a chimpanzee called Washoe could use 68 gestures after three years of training, eventually getting to 150 gestures. Their trainers reported that Washoe combined spontaneously such gestures to communicate, thus revealing some traits of recursion [27]. Another chimpanzee called Sarah, using a token language, could reliably produce phrases with grammatical form such as “Mary give apple Sarah” and to learn imperative sentences with a grammar as imperative or negative [99]. This shows that the language that an ape is able to handle can have some combinatory structure but not as complex as the human languages have.

With respect to the theory of mind, there is a debate about if chimpanzees have theory of mind, but the studies that support this claim suggest that chimpanzees, at most, have a rudimentary theory of mind [49]. A study performed by Call and colleagues lends credence to the latter hypothesis, providing evidence that chimpanzees are also sensitive to human intentions. Specifically, chimpanzees remained in a testing area longer and exhibited fewer frustration behaviors when an exper-

[14]First studies claimed that there was no recursivity on such languages. However, later was revealed that the way in which recursion was implemented in such languages was by means of non-manual markers of relative clauses as consisting of raised brows, a backward head tilt, and a tensed upper lip.

imenter behaved as if he intended to give food but was unable to do so, than when the experimenter behaved as if he had no intention of giving food [11]. Finally, for the case of analogies Gillan, Premack and Woodruff [38] reported that a chimpanzee (with a history of succesfull relational matchings) named Sarah succeded also in completing partially constructed analogies involving either complex geometric forms or functional relationships between common objects. Such study was critiziced because of the posibility that her successes might be attributable to simpler nonanalogical strategies. However, a reanalyses is performed in [91] where they conclude that "the tasks Sarah performed were demanding, including tests of not only her ability to comprehend analogy problems, but also her ability to construct analogies by arranging items in a systematic manner on a board. These reanalyses confirm not only that Sarah can solve analogy problems, but also that she does so preferentially even in situations in which a simpler associative strategy would suffice".

Finally, lest us present the case of capuchino monkeys (Cebus appella) where the current evidence is controversial. The current research suggests that these three faculties or are not present, or are present but in a weaker way than in chimpanzees. For the case of language, there is no available evidence supporting that capuchino monkeys can handle complex languages. However, in [2] is suggested that capuchino monkeys can use tokens as symbols to represent and combine quantities. With respect to the theory of mind, the study performed by Call and colleagues for chimpanzees was extended to capuchin monkeys in [95] and they suggest that, like chimpanzees, capuchin monkeys distinguish between different goal-directed acts, vacating an enclosure sooner when an experimenter acts unwilling to give food than when she acts unable to give food. However, this result is unique since previous experiments with captive capuchins have produced negative results with respect to theory of mind. Regarding analogies, there is a study [59] where one capuchino monkey (out of four) was able to perform a very basic relational matching. Again, this evidence is contrary to previous arguments which claim that monkeys can not solve analogical problems.

To end this section, it might be relevant to mention that Conway and Christiansen, by the recopilation of evidence about primate behaviour, suggest that "The limitations on primate hierarchical learning might thus be one of the key reasons that they have not developed advanced language abilities." [12] Thus providing support for what we argued before, namely that recursion (necessary for representing information in a hierarchical way) might be relevant not only for language but for theory of mind and analogy as well.

As future work, along the lines of human learning, it would be interesting to research if the “performance” of this “recursion module” embedded into our mind can be measured[15]. In such case, we could design a within subjects experiment to test if such “recursiveness” can be correlated positively to the capacity for learning and reasoning by analogy. The proposal is challenging since there is not a standardized way to measure the capacity for reasoning by analogy. Another question of interest should be to answer if this “recursion module” can be trained[16] or else if it is determined genetically.

[15]The idea of measuring the performance of this “recursive structure” through language and ToM tasks might be interesting but it have to adress the issue of the existence of a wide variety of factors influencing language and ToM. For example, the working memory capacity or the strength of the inhibitory system.

[16]A suggestion for initiating such inquiry: There exists an study [63] where a group of students who practiced substitution—which is a kind of recursive task— outperformed in symbolizing algebraic story problems to a group of students who practiced on more similar looking algebraic story problems

Appendix C

Relational Spaces

C.1 Introduction

Our aim in this chapter is to propose yet another representation to model and study metaphor. In previous chapters we have been representing knowledge domains -where our mental faculties take place- by one structure which has two components: a syntactic component and a semantic component. In this section we intend to inquire what are the conclusions of such model if we leave aside its syntactic aspects and endow with more structure its semantic part.

In his influential work “An investigation of the Laws of Thought” [8], George Boole analizes the nature of human mind in an attempt to discover the laws which command the processes of reasoning. As an abstraction of that work, the mathematical structures called Boolean Algebras were created by Boole’s succesors in the tradition of algebraic logic. In this work, we suggest that it might be fruitful to introduce boolean algebras as a key ingredient of our model, because of their natural interpretation (as an abstraction of the laws of thought) and their mathematical properties.

In this chapter, we shall provide two formal models for a metaphor between two “domains of knowledge”. First, we shall introduce Relational Spaces (which are just particular instances of boolean algebras). Next, we shall define concepts like the generator of a relational space, structural mappings between relational spaces, products and quotients of relational spaces. At the end of this chapter, we shall weaken the concept of a relational space in order to provide a formalization of metaphor whose properties are strongly related to the definition of metaphor in Chapter 4.

Before going into the details of the developed theory, we want to mention some particular metaphors (and their respective knowledge domains) that this work intends to formalize. Those metaphors are usually found in the context of education and are often used for mathematical teaching. In Chapter 5, a detailed description and analysis of these metaphors shall be given.

- *"Object Collection" Metaphor For Natural Numbers.* This is a metaphor which is often used by teachers of mathematics helping students to learn the addition operation for natural numbers. As an example of this, the teacher shows to the students two separate groups of beans and tells to the 5 years old students that addition is grouping and if he joins these two piles of beans, the first having 5 beans and the second 4 beans, the resulting pile (she counts 1, 2, 3, ..., 9) has nine beans, therefore 5 plus 4 is nine.
- *"Motion Along a Path" Metaphor for Integer Numbers.* The appearence of negative numbers, makes adding integers a little harder than adding whole numbers. The metaphor mentioned above is not useful anymore because it is based on cardinalities which are always positive or zero. The skilled teacher switch the view from piles of objects to steps forward (positive) or backward (negative), being able to add signed numbers by just walking the appropriate number of steps (in the appropriate direction) corresponding to each number.
- *"Two-Pan Balance" Metaphor for Linear Equations.* This metaphor helps students to solve equations of the form $ax+b=cx+d$. It is designed to take advantage of the fact that children seem to "just know" that by substracting the same weight to both plates of a balance in equilibrium, the equilibrium is not broken. This metaphor provides students with tools to understand the linear equation associated to the balance.

C.2 Relational Spaces

In this chapter, the concept of relational spaces will play the chief role that domains have been performing in previous chapters, that is to say, mathematical models for conceptual domains or knowledge domains. In the applications chapter (Chapter 5) we shall use domains and relational spaces to model every knowledge domain which is involved in the metaphors previously mentioned. When referring to relational spaces throughout this chapter, we will often write domain or domain of knowledge instead.

The ideas about domains and metaphors underlying this model are partially taken from the theory of embodied cognition (as presented and developed in [71]) and the Structure Mapping Theory [30]. In the development of the present theory, we will make the assumption that *relations* between the objects of the domain and the *logical operations* acting on these relations are the essential elements of the model.

We assume that the collection of objects of a knowledge domain can be represented by a set X. Let n be a natural number, any n-ary relation which holds on such domain can be represented as a subset of X^n. Notice that any $(n-1)$-ary operation $\Lambda : X^{n-1} \to X$ defined in that domain can be represented as a n-ary relation in a standard way. Our model for domains of knowledge will be a set X, together with a collection $\{R_\alpha\}_{\alpha \in I}$ of n-ary relations defined over X.

Our key assumption is that the family $\{R_\alpha\}_{\alpha \in I}$ of n-ary relations reflects the nature of the domain. For instance, let $R, S \subseteq X^n$ be n-ary relations defined over X. Every n-uple $(x_1, ..., x_n)$ belonging to R (or S) expresses an assertion relating the objects $x_1, .., x_n$ in the domain of knowledge. However, there are ways to combine such assertions (R and S) to form new assertions. For example: "$(x_1, ..., x_n) \in R$ and $(x_1, ..., x_n) \in S$", "if $(x_1, ..., x_n) \in R$ then $(x_1, ..., x_n) \in S$" are new n-ary relations which depend on the relations R and S. To deal with this way of combining assertions, we shall require that collection $\{R_\alpha\}_{\alpha \in I}$ being a boolean algebra.

A relational space can be introduced in terms of a *field of sets*, which is a standard and well known concept (see [39]). Thus, let us introduce it first. A pair $(Y, \mathscr{F})$ where Y is a set and $\mathscr{F}$ is a non empty subset of $\mathscr{P}(Y)$ closed under the intersection and union of pairs of sets and under complements of individual sets is called a *field of sets*. In other words, a field of sets $\mathscr{F}$ is a subalgebra of the power set $\mathscr{P}(Y)$. The elements of Y are called *points* and those of $\mathscr{F}$ are called *complexes*.

As far as we know, there is not standard terminology in the literature to deal with the discussion that follows and then, we will introduce our own terminology in the following presentation.

Definition 43. Let X be a nonempty set and $n \geq 1$ an integer. If the pair $(X^n, \mathscr{S})$ is a field of sets, we shall say that $\mathscr{S}$ is a *n-ary relational family* on X. The pair $(X, \mathscr{S})$ is called a *n-relational space*. If the field $\mathscr{S}$ is clear from the context, we shall write "n-relational space X", instead. After the standard terminology for the field of sets $(X^n, \mathscr{S})$, we will refer to the elements of $\mathscr{S}$ as *complexes* of X.

It is worth noticing that each n-ary relational family $\mathscr{S}$ satisfies the axioms of a boolean algebra (see Section C.3) when the empty set is the distinguished element zero, X^n is the distinguished element 1 and the operations of sets: union, intersection and complementation ($'$) are taken as the operations join, meet and negation of the boolean algebra.

Example 25. If X is any nonempty set, $\mathscr{P}(X^n)$ and $\{\emptyset, X^n\}$ are n-ary relational families over X. They are called the *discrete* n-ary relational family and the *trivial* n-ary relational family, respectively.

Example 26. Let us take the set $\mathbb{Z}$ of integers and consider $\mathscr{S} = \{\mathbb{Z}^2, \leq, >, \emptyset\}$. The pair $(\mathbb{Z}, \mathscr{S})$ is a 2-ary relational space.

To help us interpret the mathematical concept of a n-ary relational space $(X, \mathscr{S})$ as a knowledge domain, we can think of X as the collection of objects of the domain, the complexes of X (the elements of $\mathscr{S}$) as the relations between objects which hold in the domain of knowledge. The structure determined by the boolean algebra, endows the structure with the ability to operate logically with the complexes of X. In words of George Boole, the proposed structure handles the "laws of thought" inside of the knowledge domain.

Take for example the *Object Collection Metaphor for Natural Numbers*. Observe that the objects of the base domain are collections of things and thus, X becomes a set whose elements are "collections of things". Next, we should identify relations in this domain which are interesting for our study: the grouping of two collections (a ternary relation), the binary relation of "being a subcollection of", etc. On the other hand, the target domain of this metaphor will take the set $\mathbb{N}$ of natural numbers as its set of objects. The complexes of this domain will be well known relations defined in the set of natural numbers, as for example the addition (viewed as a relation) and the binary relation of a number being less than or equal to other number ($\leq$).

It is easy to verify that the intersection of any non-empty family $\{B_i\}_{i \in I}$ of n-ary relational families on X is again a n-ary relational family on X. The intersection of the empty family is, by convention, the n-ary relational family $\mathscr{P}(X^n)$. It follows that if $\mathscr{E}$ is any subset of $\mathscr{P}(X^n)$, there is a unique smallest n-ary relational family $\mathscr{R}(\mathscr{E})$ on X containing $\mathscr{E}$. We say that $\mathscr{R}(\mathscr{E})$ is the n-ary relational family *generated* by $\mathscr{E}$. If $\mathscr{E}$ is finite, we say that $\mathscr{R}(\mathscr{E})$ is finitely generated. The following observation is often useful:

Lemma 47. *If $\mathscr{E} \subseteq \mathscr{R}(\mathscr{F})$ then $\mathscr{R}(\mathscr{E}) \subseteq \mathscr{R}(\mathscr{F})$*

Example 27. Let X be a nonempty set, if $\mathscr{E}$ is empty then the n-ary relational family generated by $\mathscr{E}$ is the smallest possible boolean subalgebra of $\mathscr{P}(X^n)$, namely $\{\emptyset, X^n\}$.

Example 28. The n-ary relational family on X given by

$$\mathscr{S} = \{B \subseteq X^n \mid B \text{ is finite or } B' \text{ is finite }\},$$

i.e. the set of finite and cofinite subsets of X^n. It is easy to show that $\mathscr{S}$ is generated by the set of singletons of X^n.

Notice that the definition of the n-ary relational family generated by $\mathscr{E} \subseteq \mathscr{P}(X^n)$ is non constructive, and thus it gives no hint of which elements belong to that set. By treating the n-ary relational family $\mathscr{R}(\mathscr{E})$ as the boolean subalgebra of $\mathscr{P}(X^n)$ generated by $\mathscr{E}$, the result is an approach that gives us the precise information about its elements. We suggest to the interested reader to review [39] for details. In the case when $\mathscr{E}$ is a finite set, any element $r \in \mathscr{R}(\mathscr{E})$ can be written as

$$r = \bigcup_{i=1}^{n} \bigcap_{e \in \mathscr{E}} p(e),$$

where n is an integer and $p(e)$ is equal to e or its complement e'. Below, we paraphrase the theorem which addresses the case when $\mathscr{E}$ is not finite:

Theorem 48. *The complexes of the n-ary relational family generated by a set $\mathscr{E}$ are exactly those which can be written as a finite union of finite intersections of elements and complements of elements from $\mathscr{E}$.*

C.3 Structural Mappings

The overall purpose of this chapter is to provide a formal model able to express some properties atributed to metaphor. The main property sought to be described is the way in which a metaphor relates two different domains of knowledge by preserving some structure between them. In this section, we describe our proposal to model this phenomenon as a structure preserving mapping between two different domains of knowledge, each of them represented by relational spaces.

Definition 44. Let $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ be two n-relational spaces. Given a mapping $f : X \to Y$, let us consider the mapping $\tilde{f} : X^n \to Y^n$ defined by:

$$\tilde{f}((x_1, ..., x_n)) = (f(x_1), ..., f(x_n)).$$

We say that f is *structural* from X to Y when for every complex $V \in \mathscr{T}$, $\tilde{f}^{-1}(V) \in \mathscr{S}$.

The idea underlying the definition of structurality is that the mapping of relations between domains is induced by the mapping of objects between domains. Observe that when $f : X \to Y$ and $g : Y \to Z$ are structural, the composition $g \circ f$ is a structural mapping. Thus, (since the identity mapping is structural) we can define the category whose objects are the n-relational spaces and whose arrows are structural mappings.

Example 29. Let us consider the following relations:

$$\leq_{\mathbb{Z}} = \{(a,b) \in \mathbb{Z}^2 \mid a \leq b\}, \quad \leq_{\mathbb{N}} = \{(m,n) \in \mathbb{N}^2 \mid m \leq n\}.$$

When considering $\mathbb{Z}$ endowed with the 2-relational family generated by $\leq_{\mathbb{Z}}$ and $\mathbb{N}$ endowed with the 2-relational family generated by $\leq_{\mathbb{N}}$, the injection mapping $i : \mathbb{N} \to \mathbb{Z}$ defined by $i(n) = n$ is structural.

Example 30. Any mapping $f : X \to Y$ is structural, when Y is endowed with the trivial n-ary relational family $\{\emptyset, Y^n\}$.

In the case of the "*Object Collection*" metaphor for natural numbers, let us consider the mapping f which associates to any collection of objects, its cardinality. Let us take V as the relation "less than or equal to" of natural numbers. The set $\tilde{f}^{-1}(V)$ is the relation "injection" which holds between two collections of objects when the first one of them can be injected into the second one. Notice that the injection relation from the base domain is "preserved" by f and becomes the "less than or equal to" relation in the natural numbers domain. More details about this example, and its respective formalization can be found in chapter 5 which is devoted to applications.

Additionally, let $(X,\mathscr{S})$, $(Y,\mathscr{T})$ be two n-relational spaces where $\mathscr{T}$ is generated by a family of complexes $\mathscr{E}$. From Theorem 48 and the fact that $\tilde{f}^{-1}$ commutes with unions, intersections and complements, it is clear that the mapping $f : X \to Y$ is structural if and only if $\tilde{f}^{-1}(V) \in \mathscr{S}$ for every $V \in \mathscr{E}$. When $f : X \to Y$ is a surjective structural mapping, the n-ary relational family on X given by

$$\mathscr{S}' = \{\tilde{f}^{-1}(E) \mid E \in \mathscr{T}\}$$

is a subalgebra of the boolean algebra $\mathscr{S}$ and the mapping $F : \mathscr{S}' \to \mathscr{T}$ given by $F(B) = \tilde{f}(B)$ for $B \in \mathscr{S}'$ is a homomorphism between the boolean algebras $\mathscr{S}'$ and $\mathscr{T}$. When f is bijective, it is clear that F is an isomorphism of boolean algebras.

We shall see in Chapter 5 that the definition of structurality is enough to model most of the metaphors mentioned at the begining of this chapter. However, we shall weaken this definition later, in order to get the definition of *metaphor* which enable us formalize some instances of the phenomenon which can not be modelled properly with structural mappings. Before that, let us introduce some constructions on relational spaces which can be built thanks to the concept of structurality.

Products

From a categorical point of view, the properties which characterize a mathematical object as a product are well known (see [79]). Basically, the product of a family of objects is the "most general" object which admits a morphism to each of the given objects. As we have pointed out before, relational spaces together with structural mappings as arrows constitute a category. We use such category as the natural context where the product of relational spaces can be defined.

It is important to emphasize that our interest for defining products of relational spaces it is not purely theoretical. We shall see that these constructions are useful as models for the phenomenon that we are investigating. For example, in the *two-pan balance metaphor for linear equations* the base knowledge domain (whose objects are scales with two kinds of weights on their plates) shall be formalized as the product of two simpler knowledge domains (domains whose objects are scales with just one kind of weights on their plates). Another example would be the *motion along a path metaphor for integer numbers* where the modelling of the (infinite bi-directional) "tiled path" domain shall make use of a product of two simpler (infinite uni-directional) tiled paths.

Additionally, we hope that this kind of construction could be useful to represent other cognitive mechanisms of the human mind (see [94] for further research in this direction). The product of two domains would be interpreted as a cognitive operation which receives two domains of knowledge A, B and returns a new domain of knowledge $A \times B$. Such operation resembles the cognitive mechanism called *conceptual blending* which we hope could be associated to this kind of construction. An introduction for conceptual blending can be found in [114], the phenomenon is described and analyzed in [21, 22] and some comments about it are given in [87, 41].

Definition 45. Let $\{X_\alpha\}_{\alpha\in A}$ be an indexed collection of nonempty sets, $\mathscr{S}_\alpha$ a n-ary relational family on X_α, $X = \prod_{\alpha\in A} X_\alpha$ and $\pi_\alpha : X \to X_\alpha$ the coordinate maps. The *n-ary relational product family* (of $\{\mathscr{S}_\alpha\}_{\alpha\in A}$) on X is the n-ary relational family generated by:

$$\{\tilde{\pi}_\alpha^{-1}(E) \ : \ E \in \mathscr{S}_\alpha, \alpha \in A\}$$

We denote this n-ary relational family by $\otimes_{\alpha\in A}\mathscr{S}_\alpha$. The *$n$-ary relational product space* (of $\{(X_\alpha, \mathscr{S}_\alpha)\}_{\alpha\in A}$) is the pair $(\Pi_{\alpha\in A}X_\alpha, \otimes_{\alpha\in A}\mathscr{S}_\alpha)$.

We give here an alternative, more intuitive characterization of the n-ary relational product family in the case of countably many factors. First, we need to define a natural bijective mapping

$$\Gamma : \prod_{\alpha\in A}(X_\alpha)^n \to (\prod_{\alpha\in A} X_\alpha)^n$$

To this aim, notice that any $c \in \prod_{\alpha\in A}(X_\alpha)^n$ can be viewed as a function $c : A \to \bigcup_{\alpha\in A}(X_\alpha)^n$ such that $c(\beta) = (c(\beta)_1, ..., c(\beta)_n) \in (X_\beta)^n$. Moreover, any $d \in (\prod_{\alpha\in A}(X_\alpha))^n$ is a tuple $(d_1, ..., d_n)$ where each $d_i : A \to \bigcup_{\alpha\in A}(X_\alpha)$ is a function such that $d_i(\gamma) \in X_\gamma$. The mapping Γ is defined by:

$$\Gamma(c) = (d_1, ..., d_n) \text{ such that for any } \beta \in A \text{ and } 1 \le i \le n, \ \ d_i(\beta) = c(\beta)_i.$$

For example, if $n=2, A=\{1,2,3\}$ we have that: $\Gamma : X_1^2 \times X_2^2 \times X_3^2 \to (X_1 \times X_2 \times X_3)^2$ is defined by $\Gamma((x_1,y_1),(x_2,y_2),(x_3,y_3)) = ((x_1,x_2,x_3),(y_1,y_2,y_3))$.

Proposition 29. *If A is countable, then $\bigotimes_{\alpha\in A}\mathscr{S}_\alpha$ is the n-ary relational family generated by*

$$\Upsilon = \{\Gamma(\prod_{\alpha\in A} E_\alpha) : E_\alpha \in \mathscr{S}_\alpha, E_\beta = (X_\beta)^n \textit{ for all but finitely many } \beta \in A\}.$$

Proof. Let $E_\alpha \in \mathscr{S}_\alpha$. Since A is countable, we can apply Γ to obtain that $\tilde{\pi}_\alpha^{-1}(E_\alpha) = \Gamma(\prod_{\beta\in A} E_\beta)$ where $E_\beta = (X_\beta)^n$ for $\beta \neq \alpha$. Therefore, $\{\tilde{\pi}_\alpha^{-1}(E) \ : \ E \in \mathscr{S}_\alpha, \alpha \in A\} \subseteq \Upsilon$ and then $\bigotimes_{\alpha\in A}\mathscr{S}_\alpha \subseteq \mathscr{R}(\Upsilon)$. On the other hand, let us consider some $\Gamma(\prod_{\alpha\in A} E_\alpha)$ belonging to Υ. Since $E_\alpha = (X_\alpha)^n$ for all but finitely many $\alpha \in A$, we can consider the finite index set $I = \{\alpha |\ E_\alpha \neq (X_\alpha)^n\}$ and since A is countable we obtain that $\Gamma(\prod_{\alpha\in A} E_\alpha) = \bigcap_{i\in I} \tilde{\pi}_i^{-1}(E_i)$. Therefore, $\Upsilon \subseteq \bigotimes_{\alpha\in A}\mathscr{S}_\alpha$ and the result follows from Lemma 47. $\square$

Proposition 30. *Suppose that for each $\alpha \in A$, $\mathscr{S}_\alpha$ is generated by $\mathscr{E}_\alpha$. Then $\bigotimes_{\alpha\in A}\mathscr{S}_\alpha$ is generated by*

$$\mathscr{F}_1 = \{\tilde{\pi}_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathscr{E}_\alpha, \alpha \in A\}.$$

Additionally, if A is countable and $(X_\alpha)^n \in \mathscr{E}_\alpha$ for all α, then $\bigotimes_{\alpha\in A}\mathscr{S}_\alpha$ is generated by

$$\mathscr{F}_2 = \{\Gamma(\prod_{\alpha\in A} E_\alpha) : E_\alpha \in \mathscr{E}_\alpha, E_\beta = (X_\beta)^n \textit{ for all but finitely many } \beta \in A\}.$$

Proof. Clearly, $\mathscr{R}(\mathscr{F}_1) \subseteq \bigotimes_{\alpha\in A}\mathscr{S}_\alpha$. On the other hand, for each α, the collection $\{E \subseteq X_\alpha^n : \tilde{\pi}_\alpha^{-1}(E) \in \mathscr{R}(\mathscr{F}_1)\}$ is an n-ary relation family on X_α that contains $\mathscr{E}_\alpha$ and hence $\mathscr{S}_\alpha$. In other words, $\tilde{\pi}_\alpha^{-1}(E) \in \mathscr{R}(\mathscr{F}_1)$ for all $E \in \mathscr{S}_\alpha$, $\alpha \in A$, and hence $\bigotimes_{\alpha\in A}\mathscr{S}_\alpha \subseteq \mathscr{R}(\mathscr{F}_1)$. The second assertion follows from the first as in the proof of Proposition 29. $\square$

The following property is a categorical characterization of the n-ary relational product space:

Theorem 49. *Let $\{(X_\alpha, S_\alpha)\}_{\alpha\in A}$ be a family of relational spaces and let $(Z,\mathscr{U})$ be a relational space. For every family $\{f_\alpha : Z \to X_\alpha\}_{\alpha\in A}$ of structural mappings, there exists a unique mapping h :*

$(Z,\mathscr{U}) \to (\Pi_{\alpha\in A}X_\alpha, \otimes_{\alpha\in A}S_\alpha)$ *which is structural and makes commutative the following diagram.*

$$\begin{tikzcd}
(Z,\mathscr{U}) \arrow[d, dashed, "h"] \arrow[dr, "f_\alpha"] & \\
(\Pi_{\alpha\in A}X_\alpha, \otimes S_\alpha) \arrow[r, "\pi_\alpha"] & (X_\alpha, S_\alpha)
\end{tikzcd}$$

Proof. Let us define the mapping $h : Z \to \Pi_{\alpha\in A}X_\alpha$ by $h(z)_\beta = f_\beta(z)$, where $z \in Z$ and $\beta \in A$. To prove that h is structural, we must show that for an arbitrary $V = \tilde{\pi}_\beta^{-1}(E_\beta)$ (where $E_\beta \in S_\beta$), the set $\tilde{h}^{-1}(V)$ belongs to $\mathscr{U}$. To this aim, notice that the affirmation $a \in \tilde{h}^{-1}(V)$ is equivalent to each one of the following assertions:

$$\begin{aligned}
a &\in \tilde{h}^{-1}(\tilde{\pi}_\beta^{-1}(E_\beta)) \\
\tilde{\pi}_\beta \circ \tilde{h}(a) &\in E_\beta \\
\tilde{f}_\beta(a) &\in E_\beta \\
a &\in \tilde{f_\beta}^{-1}(E_\beta)
\end{aligned}$$

Therefore, $\tilde{h}^{-1}(V)$ is equal to $\tilde{f}_\beta^{-1}(E_\beta)$ which belongs to $\mathscr{U}$ because f_β is structural. The uniqueness of h comes from the fact that if the diagram above commutes, then $h(z)_\beta = f_\beta(z)$ for all $z \in Z$. $\square$

Quotients

In this section we introduce quotient relational spaces. We use them as representations of domains of knowledge, while modelling metaphor. For example the base domain of knowledge for the *motion along a path metaphor for integer numbers*, will be formalized as the quotient of a certain relational space(see example 51 in chapter 5).

Let X be a non empty set and "$\sim$" an equivalence relation defined on X. We will write $X/\sim$ for the quotient set of X by $\sim$ and $[x]$ for the equivalence class of x. Additionally, for a given pair $(a_1,...,a_n),(b_1,...,b_n) \in X^n$, we shall write $(a_1,...,a_n) \sim (b_1,...,b_n)$, meaning that for every $i \in \{1,...,n\}$, $a_i \sim b_i$.

Definition 46. Let $(X,\mathscr{S})$ be a n-relational space and "$\sim$" an equivalence relation defined on X. We will say that "$\sim$" is *compatible* with $\mathscr{S}$, when for every complex $A \in \mathscr{S}$ and every pair $(a_1,...,a_n),(b_1,...,b_n) \in X^n$, if $(a_1,...,a_n) \sim (b_1,...,b_n)$ then:

$$(a_1,...,a_n) \in A, \text{ implies that } (b_1,...,b_n) \in A. \tag{C.1}$$

Observe that because of the simmetry in the definition, we can reeplace the implication C.1 by the equivalence

$$(a_1,...,a_n) \in A \Leftrightarrow (b_1,...,b_n) \in A.$$

Proposition 31. *Let $(X,\mathscr{S})$ be a n-relational space and "$\sim$" an equivalence relation defined on X. Assume that $\mathscr{S}$ is generated by $\mathscr{E}$ and that for every complex $E \in \mathscr{E}$ and each pair $(a_1,...,a_n),(b_1,...,b_n) \in X^n$ verifying $(a_1,...,a_n) \sim (b_1,...,b_n)$, it holds that*

$$(a_1,...,a_n) \in E \;\Leftrightarrow\; (b_1,...,b_n) \in E.$$

Then, "$\sim$" is compatible with $\mathscr{S}$.

Proof. Let us take any $A \in \mathscr{S}$ and notice that $A = \bigcup_i \bigcap_j E_{ij}$ where $E_{ij} \in \mathscr{E}$ or ${E_{ij}}' \in \mathscr{E}$. Let $(a_1,...,a_n),(b_1,...,b_n) \in X^n$ such that $(a_1,...,a_n) \sim (b_1,...,b_n)$. Clearly, $(a_1,...,a_n) \in \bigcup_i \bigcap_j E_{ij}$ is equivalent to $(b_1,...,b_n) \in \bigcup_i \bigcap_j E_{ij}$ and then,

$$(a_1,...,a_n) \in A \text{ if and only if } (b_1,...,b_n) \in A.$$

$\square$

Let us take a n-relational space $(X,\mathscr{S})$ and some equivalence relation on X, namely "$\sim$". For any complex $A \in \mathscr{S}$, we define the set $[A] = \{([a_1],....,[a_n]) \in (X/\sim)^n \mid (a_1,....,a_n) \in A]\}$ in order to define the following family of sets:

$$\mathscr{S}/\sim = \;\{[A] \subseteq (X/\sim)^n \mid A \in \mathscr{S}\}. \tag{C.2}$$

In this context, the following technical lemma shall be used.

Lemma 50. *Let $(X,\mathscr{S})$ be a n-relational space and let $\sim$ some equivalence relation on X. The following equivalence holds:*

$$([a_1],....,[a_n]) \in [A] \Leftrightarrow [\exists (b_1,...,b_n) \in X^n][(a_1,....,a_n) \sim (b_1,...,b_n) \wedge (b_1,...,b_n) \in A]. \quad \text{(C.3)}$$

If additionally $\sim$ is compatible with $\mathscr{S}$, it holds that:

$$([a_1],....,[a_n]) \in [A] \Leftrightarrow [\forall (b_1,...,b_n) \in X^n][(a_1,....,a_n) \nsim (b_1,...,b_n) \vee (b_1,...,b_n) \in A]. \quad \text{(C.4)}$$

Proof. The equivalence in C.3 is straightforward. Let us show equivalence C.4. To this aim, let us take $([a_1],....,[a_n]) \in [A]$, by C.3 there exists $(b_1,...,b_n) \in X^n$ such that $(a_1,....,a_n) \sim (b_1,...,b_n)$ and $(b_1,...,b_n) \in A$. Because $\sim$ is compatible with $\mathscr{S}$, it follows that $(a_1,...,a_n) \in A$. Let us take any $(c_1,...,c_n) \in X^n$ and let us assume that $(c_1,...,c_n) \sim (a_1,...,a_n)$. By applying again the compatibility of $\sim$ and $\mathscr{S}$, we obtain that $(c_1,...,c_n) \in A$. To show the other direction, let us assume that the right side of C.4 holds. Since $(a_1,...,a_n) \sim (a_1,...,a_n)$, we obtain that $(a_1,...,a_n) \in A$. Therefore, by applying C.3, we obtain that $([a_1],....,[a_n]) \in [A]$. □

Proposition 32. *Let $(X,\mathscr{S})$ is a n-relational space and let "$\sim$" be an equivalence relation on X. If "$\sim$" is compatible with $\mathscr{S}$, then $\mathscr{S}/\sim$ is a n-relational family on $(X/\sim)$ which satisfies for every $A,B \in \mathscr{S}$, $[A]' = [A']$, $[A] \cup [B] = [A \cup B]$ and $[A] \cap [B] = [A \cap B]$.*

Proof. First, let us show that for any $A \in \mathscr{S}$, $[A'] = [A]'$. Let us take $([a_1],....,[a_n]) \in [A]'$, by equivalence C.4 of Lemma 50, it is equivalent to the existence of some $(b_1,...,b_n) \in X^n$, such that $(b_1,...,b_n) \sim (a_1,...,a_n)$ and $(b_1,...,b_n) \in A'$. By applying equivalence C.3 of Lemma 50 we obtain that $([a_1],....,[a_n]) \in [A']$. This shows that $[A'] = [A]'$. An easy application of Lemma 50 shows that $[A \cup B] = [A] \cup [B]$ and that $[A \cap B] = [A] \cap [B]$. Since $\mathscr{S}$ is a n-relational family on X, these assertions show that the family of sets $\mathscr{S}/\sim$ is a n-relational family on $(X/\sim)$. □

Definition 47. Let $(X,\mathscr{S})$ be a n-relational space and "$\sim$" an equivalence relation compatible with $\mathscr{S}$. The n-ary relational space given by the pair $(X/\sim, \mathscr{S}/\sim)$ is called the *quotient relational space of* $(X,\mathscr{S})$ *by "$\sim$"* and we denote it by $(X,\mathscr{S})/\sim$.

Proposition 33. *Let $(X,\mathscr{S})$ be a n-relational space and "$\sim$" an equivalence relation compatible with $\mathscr{S}$. Assume that $\theta : X \to X/\sim$ is the mapping defined by $\theta(a) = [a]$. Then, the mapping θ is a structural mapping which satisfies: for every $E \in \mathscr{S}$, $\tilde{\theta}^{-1}([E]) = E$.*

Proof. Let us take $E \in \mathscr{S}$. Clearly, by definition of $\tilde{\theta}$ and $[E]$, it follows that $E \subseteq \tilde{\theta}^{-1}([E])$. On the other hand, let us take $(a_1,...,a_n) \in \tilde{\theta}^{-1}([E])$, that is to say $([a_1],...,[a_n]) \in [E]$. By applying the equivalence C.4 of Lemma 50, it follows that $(a_1,...,a_n) \in E$, and then $\tilde{\theta}^{-1}([E]) \subseteq E$. This shows that θ is a structural mapping. □

Notice that the two earlier propositions show that the n-relational families $\mathscr{S}$ and $\mathscr{S}/\sim$ are isomorphic as boolean algebras. Actually, the following property holds:

Proposition 34. *Let $(X,\mathscr{S})$ be a n-relational space and let $\sim$ be an equivalence relation on X. The two following assertions are equivalent:*

- *The relation "$\sim$" is compatible with $\mathscr{S}$.*
- *The mapping $\vartheta : \mathscr{S} \to \mathscr{S}/\sim$ defined by $\vartheta(A) = [A]$, is an isomorphism of boolean algebras.*

Proof. Let us assume that $\sim$ is compatible with $\mathscr{S}$. Let us show that $\vartheta : \mathscr{S} \to \mathscr{S}/\sim$ is an invertible mapping. We claim that the mapping $\vartheta^{-1} : \mathscr{S}/\sim \to \mathscr{S}$, which is defined by $\vartheta^{-1}(C) = \tilde{\theta}^{-1}(C)$, is the inverse of ϑ. Notice that by Proposition 33, for every $A \in \mathscr{S}$, $\vartheta^{-1}([A]) = A$. On the other hand, by Lemma 50, $C \in \mathscr{S}/\sim$ if and only if there exists $B \in \mathscr{S}$ such that $C = [B]$. Then, $[\vartheta^{-1}(C)] = [\vartheta^{-1}([B])] = [B] = C$. Therefore, ϑ is an invertible mapping and Proposition 32 shows that it is an isomorphism between boolean algebras.

On the other hand, let us assume that $\vartheta : \mathscr{S} \to \mathscr{S}/\sim$ is an isomorphism. To show that $\sim$ is compatible with $\mathscr{S}$, let us take $A \in \mathscr{S}, (a_1,...,a_n),(b_1,...,b_n) \in X^n$ such that $(a_1,...,a_n) \sim (b_1,...,b_n)$. Let us show by using contradiction that if $(a_1,...,a_n) \in A$, then $(b_1,...,b_n) \in A$. Let us assume that $(a_1,...,a_n) \in A$ but $(b_1,...,b_n) \in A'$. Last assumption implies that $([b_1],...,[b_n]) \in [A']$, and since ϑ is an isomorphism of boolean algebras, it means that $([b_1],...,[b_n]) \in [A]'$. Because $(a_1,...,a_n) \sim (b_1,...,b_n)$, it follows that $([a_1],...,[a_n]) \in [A]'$, which is clearly a contradiction because $(a_1,...,a_n) \in A$ implies that $([a_1],...,[a_n]) \in [A]$. □

The following property of quotient relational spaces holds:

Proposition 35. *Let $(X,\mathscr{S})$ and $(Y,\mathscr{T})$ be two n-relational spaces, $f : X \to Y$ a structural mapping and "$\simeq$" an equivalence relation on Y compatible with $\mathscr{T}$. Denote by "$\sim$" the equivalence relation* *on X defined by*

$$a_1 \sim a_2 \text{ if and only if } f(a_1) \simeq f(a_2).$$

If "$\sim$" is compatible with $\mathscr{S}$, then there exists a unique structural mapping $f_* : (X/\sim) \to (Y/\simeq)$ which satisfies $\theta \circ f = f_* \circ \theta$ i.e. the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\theta"'] & Y \arrow[d, "\theta"] \\ (X/\sim) \arrow[r, dashed, "f_*"] & (Y/\simeq) \end{tikzcd}$$

Proof. Let us assume that "$\sim$" is compatible with $\mathscr{S}$. In order to make the above diagram commute, the mapping $f_* : (X/\sim) \to (Y/\simeq)$ must be determined by $f_*([x]_\sim) = [f(x)]_\simeq$. Since by to hypothesis $x_1 \sim x_2$ implies $f(x_1) \simeq f(x_2)$, then the mapping f_* is well defined. Since $\sim$ is compatible with $\mathscr{S}$, the quotient relational space $(X, \mathscr{S})/\sim$ is well defined. To show that $f_* : (X/\sim) \to (Y/\simeq)$ is structural, let us take a complex $[E]$ of $(Y, \mathscr{T})/\simeq$ (where $E \in \mathscr{T}$) and compute:

$$\begin{aligned} \tilde{f}_*^{-1}([E]) &= \{([x_1], ..., [x_n]) \in (X/\sim)^n \mid \tilde{f}_*([x_1], ..., [x_n]) \in [E]\} \\ &= \{([x_1], ..., [x_n]) \in (X/\sim)^n \mid (f_* \circ \theta(x_1, ..., x_n) \in [E]\} \end{aligned}$$

by applying the commutativity of the diagram we obtain:

$$\begin{aligned} \tilde{f}_*^{-1}([E]) &= \{([x_1], ..., [x_n]) \in (X/\sim)^n \mid \theta \circ f(x_1, ..., x_n) \in [E]\} \\ &= \{([x_1], ..., [x_n]) \in (X/\sim)^n \mid f(x_1, ..., x_n) \in \tilde{\theta}^{-1}([E])\} \end{aligned}$$

and by applying Proposition 33, it follows that:

$$\begin{aligned} \tilde{f}_*^{-1}([E]) &= \{([x_1], ..., [x_n]) \in (X/\sim)^n \mid (x_1, ..., x_n) \in f^{-1}(E)\} \\ &= [f^{-1}(E)] \in \mathscr{S}/\sim . \end{aligned}$$

and thus, f_* is a structural mapping. □

Metaphors

As we pointed out before, structurality might be too strong to formalize some examples of metaphor. In this section we shall weaken such concept in order to formalize some metaphors that shall be considered in the chapter devoted to applications (Chapter 5).

Definition 48. Let $(Y,\mathscr{T})$ be a n-ary relational space, X a non-empty set and $f: X \to Y$ a mapping from X to Y. Given $A \in \mathscr{T}$, the *metaphorical set for A respect to f* is the set $\mathscr{M}_f(A) = \{E \subseteq X^n \mid \tilde{f}(E) = A\}$.

We shall write $\mathscr{M}(A)$ instead of $\mathscr{M}_f(A)$ when f is clear from the context and it does not lead into confusion. Next, we present some simple properties of this definition which can be related to the cognitive phenomenon of metaphor when adequate interpretations are given. Notice that,

$$\text{if } E \in \mathscr{M}_f(A) \text{ and } E^* \in \mathscr{M}_f(A'), \text{ then } E \cap E^* = \emptyset.$$

That is to say, if we are using a relation E as a metaphor for the relation A and E^* as a metaphor for the complement of A, the two relations (E and E^*) must be disjoint.

The following straightforward characterization of elements of $\mathscr{M}(A)$ holds:

Lemma 51. *Let $(Y,\mathscr{T})$ be a n-ary relational space, X a non empty set and $f: X \to Y$ a mapping. Given any $A \in \mathscr{T}$, the following sentences are equivalent:*

- $E \in \mathscr{M}_f(A)$
- *If $(e_1,...,e_n) \in E$ then $\tilde{f}((e_1,...,e_n)) \in A$ and for any $(a_1,...,a_n) \in A$ there exists $(c_1,....,c_n) \in E$ such that $\tilde{f}((c_1,...,c_n)) = (a_1,....,a_n)$.*

In the cognitive literature which deals with metaphor it is claimed that one of the main properties of this phenomenon is the preservation of some "inferences" from one domain to another. For the purposes of this study, we are going to choose a mathematical interpretation to represent the concept of inference. We shall formalize inferences as statements of the sort "if the relation E_1 holds, then the relation E_2 holds", that is to say, the truthness of the condition given by E_1 ensures the truthness of the condition given by E_2. Mathematically, this statement can be formalized by $E_1 \subseteq E_2$. The following straightforward corollary of Lemma 51, shows that the definition of metaphor captures the idea of preserving inferences.

Corollary 52 (Inference)**.** *Let X be a non-empty set, $(Y,\mathscr{T})$ a n-ary relational space and $f : X \to Y$ a mapping between them. Assume that $A_1, A_2 \in \mathscr{T}$, $E_1 \in \mathscr{M}_f(A_1)$ and $E_2 \in \mathscr{M}_f(A_2)$. If $E_1 \subseteq E_2$ then $A_1 \subseteq A_2$.*

Let $(Y,\mathscr{T})$ be a n-ary relational space, X be a non-empty set and $f : X \to Y$ be a mapping from X to Y. We will consider the partial order on $\mathscr{M}_f(A)$ which is given by the inclusion of sets. If we denote by $\emptyset$ the empty n-ary relation, $\mathscr{M}(A) \cup \{\emptyset\}$ is a lattice. Nevertheless, before proving this, it is important to notice that the meet (infimum) on that lattice is *not* necesarily given by the intersection of sets.

Consider the following example: Take $X = \{a,b,c,d\}$, $Y = \{1,2\}$ and $f : X \to Y$ such that $f(a) = f(b) = 1$ and $f(c) = f(d) = 2$. Notice that the relations E_1 and E_2 depicted in figures (C.1- a) and (C.1- b) respectively, are metaphors for the identity relation $1_Y = \{(1,1),(2,2)\}$ on Y, i.e. $\tilde{f}(E_1) = 1_Y$ and $\tilde{f}(E_2) = 1_Y$. Observe however that the intersection $E_1 \cap E_2$ is not a metaphor for 1_Y, in this case the meet between E_1 and E_2 is the empty relation $\emptyset$.

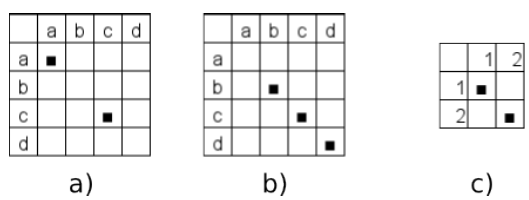

Figure C.1: (a): Relation E_1 on X, (b) Relation E_2 on X (c) Relation 1_Y on Y.

Last example motivates the following lemma:

Lemma 53. *Let X be a non-empty set, $(Y,\mathscr{T})$ a n-ary relational space and $f : X \to Y$ a mapping between them. Suppose that $\{E_\lambda\}_{\lambda\in\Delta}$ is a non-empty collection of sets belonging to $\mathscr{M}_f(A)$, then:*

1. $\bigcup_{\lambda\in\Delta} E_\lambda \in \mathscr{M}_f(A)$.
2. *either* $\bigcap_{\lambda\in\Delta} E_\lambda \in \mathscr{M}_f(A)$ *or* $\{F \in \mathscr{M}_f(A) \mid (\forall\lambda\in\Delta) F \subseteq E_\lambda\} = \emptyset$.

Proof. To show 1, notice that $\tilde{f}(\bigcup_{\lambda\in\Delta} E_\lambda) = \bigcup_{\lambda\in\Delta} \tilde{f}(E_\lambda) = \bigcup_{\lambda\in\Delta} A = A$. Therefore, $\bigcup_{\lambda\in\Delta} E_\lambda \in \mathscr{M}_f(A)$. To show 2, let us assume that $\bigcap_{\lambda\in\Delta} E_\lambda \notin \mathscr{M}_f(A)$, then $\tilde{f}(\bigcap_{\lambda\in\Delta} E_\lambda)$ is a proper subset of A. On the other hand, if we assume that there exists $F \in \mathscr{M}_f(A)$ such that $(\forall\lambda \in \Delta)F \subseteq E_\lambda$, it follows that $F \subseteq \bigcap_{\lambda\in\Delta} E_\lambda$, and then $\tilde{f}(F) \subseteq \tilde{f}(\bigcap_{\lambda\in\Delta} E_\lambda)$. Therefore, $\tilde{f}(F) \subset A$ and then $F \notin \mathscr{M}_f(A)$ which is contradictory. Therefore, the set $\{F \in \mathscr{M}_f(A) \mid (\forall\lambda \in \Delta)F \subseteq E_\lambda\}$ must be empty. □

The following result is a straightforward corollary from last lemma:

Proposition 36. *Let X be a non-empty set, the pair $(Y, \mathscr{T})$ a n-ary relational space and $f : X \to Y$ a mapping. For every $A \in \mathscr{T}$, the set $\mathscr{M}_f(A) \cup \{\emptyset\}$ is a bounded complete lattice.*

Proof. If $\mathscr{M}(A)$ is empty, the result is trivial. Therefore, let us assume that $\mathscr{M}(A)$ is non empty. We consider the set $\mathscr{M}(A) \cup \{\emptyset\}$ with the partial order given by the inclusion of sets. In such case, Lemma 53 ensures that every collection $\{E_\lambda\}_{\lambda\in\Delta} \subseteq \mathscr{M}(A)$ has a supremum and an infimum. Then, $\mathscr{M}(A) \cup \{\emptyset\}$ is a bounded complete lattice whose least element is the empty n-ary relation $\emptyset$ and whose greatest element is the element given by $\tilde{f}^{-1}(A)$. □

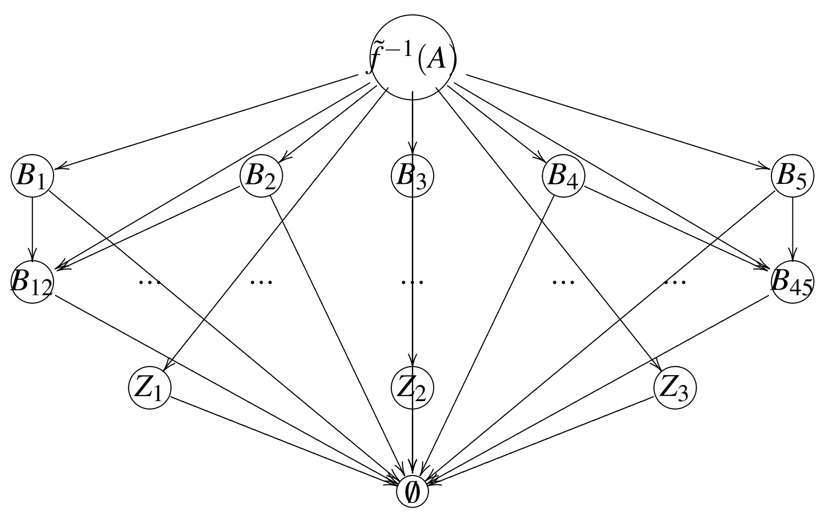

Figure C.2: Sketch of the lattice $\mathscr{M}(A) \cup \{\emptyset\}$.

Observe that when a mapping $f : X \to Y$ is structural, for every coimplex A of Y, the greatest element of the lattice $\mathscr{M}_f(A)$ is a complex of X. However, such is not always the case in the real world. For example, in one of the versions of the *object collection metaphor for natural numbers*, the relation "subset or equal to" ($\subseteq$) is used to induce the target relation "less than or equal to" ($\leq$).

Notice that the relation given by $\tilde{f}^{-1}(\leq)$ is the injection [1] ($\hookrightarrow$) relation. In what follows, we shall modify slightly some definitions given in the previous sections in order to capture the behavior above described. Additionally, we shall explore some implications of such modifications. Next definition is a first step in such direction by modifying the concept of a n-ary relational family.

Definition 49. Let X be a non-empty set, a *n-ary metaphorical family* on X is a subset $\mathscr{F} \subseteq \mathscr{P}(X^n)$ which is closed under finite unions and finite intersections, together with a mapping $*: \mathscr{F} \to \mathscr{F}$ such that:

1. For every $A \in \mathscr{F}$, $(A^*)^* = A$. ($*$ is idempotent).
2. For every $A, B \in \mathscr{F}$, $(A \cap B)^* = (A^* \cup B^*)$.

Notice that when $\mathscr{F}$ is a n-ary metaphorical family on a nonempty set X, straightforward computations show that the equation $(A \cup B)^* = A^* \cap B^*$ holds for any $A, B \in \mathscr{F}$. In complete analogy with the previous concepts we will call the pair $(X, \mathscr{F})$ a n-ary metaphorical space and the *complexes* of X will be the elements of $\mathscr{F}$.

Example 31. Any n-ary relational family $\mathscr{S}$ is a n-ary metaphorical family where the operation $*$ is given by the complementation of sets.

Example 32. Let us assume that $X = \{a, b, c\}$. If $\mathscr{F} = \{\{a, b\}, \{a, c\}, \{a, b, c\}, \{a\}\}$ and the operation $*$ is given by $\{a, b\}^* = \{a, c\}$ and $\{a, b, c\}^* = \{a\}$. Then $\mathscr{F}$ is a 1-ary metaphorical family on X. This example shows that $*$ does not always coincides with the complement operation.

Example 33. Let $(X, \mathscr{S})$ be a n-ary relational space and $D \subset X^n$. Define $\mathscr{S}' = \{A \in \mathscr{S} \mid D \subseteq A\}$. Clearly, $\mathscr{S}'$ is closed for finite unions and finite intersections. Notice that $\mathscr{S}'$ is not an algebra of sets because for any $U \in \mathscr{S}'$ the complement U' does not necessarily belongs to $\mathscr{S}'$. However, if for any $A \in \mathscr{S}'$ we define $*$ by $A^* = A' \cup D$ it holds that

$$A^{**} = (A' \cup D)^* = (A' \cup D)' \cup D = (A \cap D') \cup D = (A \cup D) \cap (D \cup D') = A.$$

by closing $\mathscr{S}'$ with respect to $*$, we obtain a n-ary metaphorical family on X since straightforward set-theoretic computations show that $(A \cap B)^* = A^* \cup B^*$.

[1]The injection relation is defined by: $a \hookrightarrow b$ if and only if there exists an injective function $f: a \to b$.

Example 34. Let $(X,\mathscr{S})$ be a n-ary relational space and $Z \subset X^n$. Put $\mathscr{S}' = \{A \cap Z \mid A \in \mathscr{S}\}$. It is clear that $\mathscr{S}'$ is closed under finite unions and finite intersections. Let us define $*$ by $A^* = A' \cap Z$. Then, it is easily shown that $A^{**} = A$ and $(A \cap B)^* = A^* \cup B^*$. Therefore, $\mathscr{S}'$ is a n-ary metaphorical family on X.

Example 35. Let us take $X = \{a,b,c\}$. If $\mathscr{F} = \{\{a,b\},\{a,b,c\},\{a\}\}$ and $*$ is the idempotent operation defined by $\{a,b\}^* = \{a,b\}$ and $\{a,b,c\}^* = \{a\}$. In such case, $\mathscr{F}$ is a 1-ary metaphorical family on X. To see this, compute:

$$(\{a,b\} \cap \{a\})^* = \{a\}^* = \{a,b,c\} = \{a,b\}^* \cup \{a\}^*$$
$$(\{a,b\} \cap \{a,b,c\})^* = \{a,b\}^* = \{a,b\} = \{a,b\}^* \cup \{a,b,c\}^*$$
$$(\{a,b,c\} \cap \{a\})^* = \{a\}^* = \{a,b,c\} = \{a,b,c\}^* \cup \{a\}^*$$

We shall see in Chapter 5 that metaphorical spaces can be used to formalize some metaphors that are not susceptible of being modelled by relational spaces and structural mappings. We shall prove such assertion by modelling the *object collection metaphor for natural numbers*, which uses the set-relation "subset of or equal to" to induce the number-relation "less than or equal to". To do this, we shall require a n-ary metaphorical space as a model for the base domain, while the target domain shall be formalized by a n-ary relational space. To explore some properties of the family of such models, we make the following definition:

Definition 50. Let $(X,\mathscr{H})$ be a n-ary metaphorical space and $(Y,\mathscr{T})$ a n-ary relational space. A mapping $f : X \to Y$ is called *metaphorical* if $F : \mathscr{H} \to \mathscr{T}$ determined by $F(A) = \tilde{f}(A)$, is a well defined mapping and satisfies $F(A^*) = F(A)'$ for each $A \in \mathscr{H}$.

Example 36. Let $(X,\mathscr{S})$, $(Y,\mathscr{T})$ be n-ary relational spaces. Let $f : X \to Y$ be a structural mapping. The mapping f is metaphorical by considering the n-ary metaphorical space $(X,\mathscr{H})$, where $\mathscr{H} = \{\tilde{f}^{-1}(E) \mid E \in \mathscr{T}\} \subseteq \mathscr{S}$.

In order to display accurately the following result, we need to introduce a set of axioms for boolean algebras. A *boolean algebra* is a non empty set A, together with two binary operations $\wedge$ and $\vee$ (meet and join), a unary operation $'$ (complementation), and two distinguished elements 0 and 1, satisfying the following axioms (for every $p,q,r \in A$):

1. $p \wedge 1 = p$, $p \vee 0 = p$. (Identity laws)

2. $p \wedge p' = 0$, $p \vee p' = 1$. (Complement laws)

3. $p \wedge q = q \wedge p$, $p \vee q = q \vee p$. (Commutativity)

4. $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$. (Distributivity)

By observing example 35, it becomes clear that $\{a,b\} \cup \{a,b\}^* = \{a,b\}$ and $\{a\} \cup \{a\}^* = \{a,b,c\}$. Thus, the complement laws do not necessarily hold in a metaphorical family. It means that a metaphorical family is not necessarily a boolean algebra with operations $\cap, \cup$ and $*$. Despite this fact, next theorem shows that the existence of a metaphorical mapping forces to a metaphorical family to become a boolean algebra. Actually, a metaphorical mapping preserves the structure of a boolean algebra.

Theorem 54. *Let $(X, \mathscr{H})$ be a n-ary metaphorical space and $(Y, \mathscr{T})$ a n-ary relational space. If $f : X \to Y$ is metaphorical, then $\mathscr{H}$ is a boolean algebra and the mapping $F : \mathscr{H} \to \mathscr{T}$ (see Definition 50) is a boolean algebra homomorphism.*

Proof. We are going to consider $\cap$, $\cup$ and $*$ as the meet $(\wedge)$, join $(\vee)$ and complementation $(')$ of $\mathscr{H}$. All above axioms of a boolean algebra are straightforward with the proposed join, meet and complement, except perhaps the following two:

- For any $A \in \mathscr{H}$: $A \cup A^* = 1$,
- For any $A \in \mathscr{H}$: $A \cap A^* = 0$.

where 1 and 0, elements of $\mathscr{H}$, are yet to be defined. We will show that for any $A \in \mathscr{H}$, $A \cap A^* = \emptyset$. Therefore, 0 should be defined as $\emptyset$ and 1 as $\emptyset^*$, and with such definitions, both axioms will hold. The uniqueness of the structure is clear, then. Let us show now that if $A \in \mathscr{H}$, then $A \cap A^* = \emptyset$. Since $\tilde{f}(A \cap A^*) \subseteq \tilde{f}(A) \cap \tilde{f}(A^*) = \tilde{f}(A) \cap \tilde{f}(A)' = \emptyset$, then $\tilde{f}(A \cap A^*) = \emptyset$, which implies that $A \cap A^* = \emptyset$. Notice that, from this, $A \cup A^* = (A^* \cap A)^* = 0^* = 1$. Finally, proving that $F : \mathscr{H} \to \mathscr{T}$ is a boolean algebra homomorphism requires showing that $F(A^*) = F(A)'$, $F(A \cup B) = F(A) \cup F(B)$ and that $F(A \cap B) = F(A) \cap F(B)$. The first condition is direct because f is metaphorical. The second one follows directly from the definition of F and for the third: $F(A \cap B) = F((A^* \cup B^*)^*) = F(A^* \cup B^*)' = (F(A^*) \cup F(B^*))' = (F(A)' \cup F(B)')' = F(A) \cap F(B)$. $\square$

We shall close this chapter investigating how the structure of a metaphorical space can be generated in a way that a given map becomes metaphorical for a fixed target domain. Let us consider the following lemma:

Lemma 55. *Let $f : (X, \mathscr{H}) \to (Y, \mathscr{T})$ be a metaphorical mapping. And let* 1 *be the distinguished element of the boolean algebra $\mathscr{H}$.*

- *For every $H \in \mathscr{H}$, $1 \cap \tilde{f}^{-1}(\tilde{f}(H)) = H$.*
- *$\mathscr{H} \subseteq \{1 \cap \tilde{f}^{-1}(E) \mid E \in \mathscr{T}\}$.*

Proof. For the first assertion, let us take $H \in \mathscr{H}$. Clearly, $H \subseteq 1 \cap \tilde{f}^{-1}(\tilde{f}(H))$. To show the reciprocal inclusion, let us take $h \in 1 \cap \tilde{f}^{-1}(\tilde{f}(H))$. Then $h \in (H \cup H^*) \cap \tilde{f}^{-1}(\tilde{f}(H))$, by applying distributivity it follows that $h \in H \cup (H \cap H^*)$. It is easy to see that $H \cap H^* = \emptyset$ since $H \cap H^* \subseteq \tilde{f}^{-1}(\tilde{f}(H) \cap \tilde{f}^{-1}(\tilde{f}(H^*) = \tilde{f}^{-1}(\tilde{f}(H) \cap [\tilde{f}^{-1}(\tilde{f}(H)]' = \emptyset$. Therefore, $1 \cap \tilde{f}^{-1}(\tilde{f}(H)) \subseteq H$. The second assertion follows directly from the first. □

Let $(Y, \mathscr{T})$ be a n-relational space and $f : X \to Y$ a mapping. A set $Z \subseteq X^n$ is called *equilibrated* respect to f when for every $E \in \mathscr{T}$, $Z \cap \tilde{f}^{-1}(E) \in \mathscr{M}_f(E)$, or equivalently $\tilde{f}(Z \cap \tilde{f}^{-1}(E)) = E$. A set $X' \subseteq X$ is said to be *centered* respect to f, when $(X')^n$ contains a equilibrated subset respect to f.

Next proposition shows that for a target domain (formalized by a relational space $(Y, \mathscr{T})$) and a given mapping $f : X \to Y$, the existence of a metaphorical mapping is characterized by the existence of a set $Z \subseteq X^n$ equilibrated respecto to f.

Proposition 37. *Let $(Y, \mathscr{T})$ be a n-relational space and $f : X \to Y$ a mapping. The two following assertions are equivalent:*

1. *There exists a set $Z \subseteq X^n$ which is equilibrated respect to f.*
2. *There exists a n-ary metaphorical family $\mathscr{H}$ such that $f : (X, \mathscr{H}) \to (Y, \mathscr{T})$ is metaphorical and the associated mapping $F : \mathscr{H} \to \mathscr{T}$ is surjective.*

Proof. 1. $\implies$ 2.
Let us assume that $Z \subseteq X^n$ is equilibrated respect to f. Notice that $\mathscr{S} = \{\tilde{f}^{-1}(E) \mid E \in \mathscr{T}\}$ is a n-ary relational family on X and then, by using the structure of Example 34, the family $\mathscr{H} = \{Z \cap \tilde{f}^{-1}(E) \mid E \in \mathscr{T}\}$ is a n-ary metaphorical family on X. Notice that since Z is equilibrated, $\tilde{f}(Z \cap \tilde{f}^{-1}(E)) = E$, for every $E \in \mathscr{T}$. This shows that $F : \mathscr{H} \to \mathscr{T}$ is a surjective well defined mapping. To show that $f : X \to Y$ is metaphorical, we must prove that $F((Z \cap \tilde{f}^{-1}(E))^*) = F(Z \cap \tilde{f}^{-1}(E))'$. To this aim, straightforward set-theoretic computations show that $(Z \cap \tilde{f}^{-1}(E))^* = Z \cap \tilde{f}^{-1}(E')$, and then

$$F((Z \cap \tilde{f}^{-1}(E))^*) = F(Z \cap \tilde{f}^{-1}(E')) = \tilde{f}(Z \cap \tilde{f}^{-1}(E')) = E' = F(Z \cap \tilde{f}^{-1}(E))'.$$

2. $\implies$ 1.
We shall show that $\mathscr{H} = \{1 \cap \tilde{f}^{-1}(E) \mid E \in \mathscr{T}\}$, implying that the set 1 is equilibrated respect to f. Because of Lemma 55, we obtain that $\mathscr{H} \subseteq \{1 \cap \tilde{f}^{-1}(E) \mid E \in \mathscr{T}\}$. We are going to show the reciprocal inclusion. Let us take a element with form $1 \cap \tilde{f}^{-1}(E)$, where $E \in \mathscr{T}$. Since the mapping $F : \mathscr{H} \to \mathscr{T}$ is surjective, there is a set $H \in \mathscr{H}$ such that $\tilde{f}(H) = E$. By Lemma 55, $1 \cap \tilde{f}^{-1}(\tilde{f}(H)) = H$, that is $1 \cap \tilde{f}^{-1}(E) = H \in \mathscr{H}$. It shows that $\{1 \cap \tilde{f}^{-1}(E) \mid E \in \mathscr{T}\} \subseteq \mathscr{H}$. $\square$

Corollary 56. *Let $(Y, \mathscr{T})$ be a n-relational space and $f : X \to Y$ a mapping. If there exists a set $A \subseteq X$ which is centered respect to f, then there exists a n-ary metaphorical family $\mathscr{H}$ such that $f_{|A} : (A, \mathscr{H}) \to (Y, \mathscr{T})$ is metaphorical and the associated mapping $F_{|A} : \mathscr{H} \to \mathscr{T}$ is surjective.*

Proof. Because A is centered respect to f, there is a set $Z \subseteq A^n \subseteq X^n$ which is equilibrated respect to f. It implies that Z is equilibrated respect to $f_{|A}$. By applying Proposition 37, the result follows. $\square$

Relational Spaces Model of the Object Collection Metaphor for Natural Numbers

The examples in this section shall use relational spaces instead of (formal) domains. First, let us introduce the appropriate relational spaces that we are going to use in this section. In what follows, we denote with $(')$ the complement operation for sets.

In the examples that are going to be presented, the "object collections" domain is modelled as an

n-relational space $(P,\mathscr{S})$ for some fixed n, where P is a pile system (the definition of a pile system given in def. (42) is still useful) and $\mathscr{S}$ is an n-ary relational family defined over P. Usually we will assume that $\mathscr{S}$ is generated by a set S of complexes i.e. $\mathscr{R}(S) = \mathscr{S}$. This way, the relational spaces used in the examples can be determined just by specifying a set S of complexes (S can be thought as the set of relations that are interesting for our study). For example, the relation "subcollection or equal to" (wich shall be denoted with the symbol $\subseteq$) is defined in P by $A \subseteq B$ if and only if every object belonging to pile A belongs to pile B. Thus, if $S = \{\subseteq\}$ is our generating set, it determines the relational family $\mathscr{S} = \{\subseteq, \subseteq', \emptyset, P^2\}$.

Here are some binary relations wich shall often be used in the examples and can be defined in P:

- The *subset or equal* relation denoted by $\subseteq$. We shall write $A \subseteq B$ meaning "A is a subset or equal to B".
- The *superset* relation denoted by $\supset$. We shall write $A \supset B$ meaning "A is a superset of B" and it is defined by $A \supset B$ if and only if $A \neq B$ and every element of B belongs to A.
- The *injection* relation denoted by $\hookrightarrow$. We will write $A \hookrightarrow B$ meaning "A is injected to B" and it is defined by $A \hookrightarrow B$ if and only if there exists an injective function $f : A \to B$.
- The *bijection* relation denoted by $\equiv$. We will write $A \equiv B$ meaning "A is in bijection with B" and it will be defined by $A \equiv B$ if and only if there exists a bijective function $f : A \to B$. bijective function $f : A \to B$.

Additionally, let us consider some ternary relations that can be defined over P:

- The *join of piles* relation denoted by "join". We shall write $join(A,B,C)$ meaning "the join of A and B is C" wich is defined by $join(A,B,C)$ if and only if $A \cup B = C$ and $A \cap B = \emptyset$.
- The *disjoint join of piles* relation denoted by "disjoin". We shall write $disjoin(A,B,C)$ meaning "the disjoint join of A and B is in bijection with C" wich is defined by $disjoin(A,B,C)$ if and only if $A \uplus B \equiv C$.

- The *product of piles* relation wich is denoted by "prod". We shall write $prod(A,B,C)$ meaning "the product of the piles A and B is in bijection with C" wich is defined by $prod(A,B,C)$ if and only if $A \times B \equiv C$.

- The *injection* relation in its 3-ary form [2] denoted by $\hookrightarrow_3$. We shall write $\hookrightarrow_3 (A,B,C)$ meaning "A is injected in B". Such relation is defined by $\hookrightarrow_3 (A,B,C)$ if and only if exists an injective function $f : A \to B$ (notice that this relation is independent of C).

Now we can build a more elaborated example. Consider $n = 2$ and let $S = \{\subseteq, \hookrightarrow, \equiv\}$ that is, the set with the three binary relations: subset or equal, injection and bijection. This set of relations or complexes determines the 2-relational family characterized by being the smallest set of binary relations over P wich contains S and is closed by complements, finite unions and finite intersections.

In the examples we shall formalize the "natural numbers" knowledge domain using an n-relational space $(\mathbb{N}, \mathscr{T})$ where $\mathbb{N}$ is the set of natural numbers and $\mathscr{T}$ is an n-ary relational family (for some fixed n) generated by a set T of interesting relations over $\mathbb{N}$ i.e. $\mathscr{R}(T) = \mathscr{T}$. As before, in each example it will suffice to specify T in order to determine $\mathscr{T}$. Let us describe some (binary and ternary) relations in the natural numbers set that will often be used in the examples.

- The natural numbers *identity* relation wich is denoted by "$=$".

- The relation *sum(m, n, k)* will mean "the sum of m and n is k". That is to say, $sum(m,n,k)$ if and only if $m+n = k$.

- The relation *mult(m, n, k)* will mean "k is the result of the multiplication of m and n". That is to say, $mult(m,n,k)$ if and only if $m \cdot n = k$.

- The relation $m \leq n$ will mean "m is less than or equal to n".

- The *less than or equal to* relation in its 3-ary form denoted by $\leq_3$. We will write $\leq_3 (m,n,k)$ instead of "m is less than or equal to n". This relation is defined by $\leq_3 (m,n,k)$ if and only if $m \leq n$.

[2] This kind of definitions are useful in case we want to model some domain where there are binary relations as well as ternary relations working together.

Let us give a brief overview of the examples: the examples 37, 38 and 39 are different ways of formalizing the introduction of the relation $\leq$ in the natural numbers through a metaphor. Examples 40 and 41 are two ways of formalizing the introduction of the addition operation through a metaphor, but the first one can perhaps be seen as more "natural" and the last one as more "mathematical". Example 42 suggests that the adequate relation in P to introduce the equality in the natural numbers $\mathbb{N}$ should be the "bijection" relation. Example 43 is oriented to explain the introduction of multiplication in natural numbers through a metaphor using the operation "product" defined in $\mathscr{P}$. Example 44 is aimed to show the preservation of the "inferential structure" from the source domain to the target domain. The last example (example 45) shows how a metaphorical space can be created from a relational space.

Our goal here is to suggest how some relations can be "transferred" from the base domain (the object collections) to the target domain (the natural numbers) through the mapping

$$h: \ P \to \mathbb{N}$$

wich associates to every pile $A \in P$ its cardinality $h(A) = |A| \in \mathbb{N}$. Recall that for any n, h determines the mapping $\tilde{h} : P^n \to \mathbb{N}^n$ defined by: $\tilde{h}(a_1, ..., a_n) = (h(a_1), ..., h(a_n))$.

Example 37. On the base domain consider the relational family generated by the relation "subset of or equal to" i.e. let $S = \{\subseteq\}$. On the target domain consider the relational family generated by the relation "less than or equal to" i.e. let $T = \{\leq\}$. Denote their respective 2-ary relational spaces by $(P, \mathscr{S})$ and $(\mathbb{N}, \mathscr{T})$. Notice that $\tilde{h}(\subseteq) = \{(n,m) \in \mathbb{N}^2 \mid n = h(a),\ m = h(b),\ a \subseteq b\}$ wich is in turn equal to "$\leq$" and therefore h acts in "$\subseteq$" as a metaphor for "$\leq$".

Notice that in the previous example one might think that the mapping h introduces the "$\leq$" relation in $\mathbb{N}$ via the "$\subseteq$" relation in the piles domain. However, notice that the relation $\subseteq'$ (the complement relation of "subset of or equal to") would not be an appropriate relation to introduce the concept of "greater than" (the complement relation of "less than or equal to" $\leq$) into the natural numbers through the metaphor. Additionally, it is worth it observing that the mapping h is not structural (because the relation $\tilde{f}^{-1}(\leq)$ does not belong to $\mathscr{S}$) neither it is metaphorical (because $\tilde{h}(\subseteq') \neq \leq'$ and therefore $H : \mathscr{S} \to \mathscr{T}$ is not well defined). Nevertheless, we can build a metaphorical space based on $(P, \mathscr{S})$ the relational space from last example in such a way that the mapping h becomes metaphorical. The following example explores the details.

Example 38. We are going to consider the relation "superset" ($\supset$) to introduce the notion "greater than" into the natural numbers through a metaphor. Denote by K the relation given by the union of relations $\subseteq$ and $\supset$. Consider the 2-metaphorical space given by $(P, \mathscr{M})$, where $\mathscr{M} = \{\subseteq, \supset, \emptyset, K\}$ with $(\subseteq)^* = \supset$, $(\emptyset)^* = K$. Additionally, consider the 2-relational space $(\mathbb{N}, \mathscr{T})$ where $\mathscr{T}$ is generated by $T = \{\leq\}$. Notice that $h : P \to \mathbb{N}$ is such that

$$\tilde{h}(\subseteq) = \leq \text{ and } \tilde{h}(\supset) = \leq'$$

(We shall write $>$ for the complement relation of $\leq$, this relation is known as "greater than"), thus h acts in "$\subseteq$" as a metaphor for "$\leq$" and acts in "$\supset$" as metaphor for "$>$". Therefore, the mapping $H : \mathscr{M} \to \mathscr{T}$ is well defined and h is a metaphorical mapping. Additionally, notice that

$$\delta_h(\subseteq) = \tilde{h}(\subseteq') \cap \leq \, = \, \leq$$

i.e. the deviation set of $\leq$ has an infinite number of elements and therefore the metaphorical mapping h has infinite deviation.

In the last example, the infinite deviation of the metaphorical mapping would be interpreted as "this metaphor is prone to induce errors". Therefore it might be better if we could introduce the relations "less than or equal to" and "greater than" through a metaphor with zero deviation. This can be achieved by using different relational spaces. The next example displays that idea in detail using the "injection" relation to introduce the concept of "less than or equal to" in the natural numbers through a metaphor. Thus, suggesting that this "injection" relation would be better relation to introduce the concept of "less than or equal to" into the natural numbers through a metaphor.

Example 39. Let us consider the relational family generated by the "injection" relation i.e. let $S = \{\hookrightarrow\}$ and consider $T = \{\leq\}$. Let us denote by $(P, \mathscr{S})$ and $(\mathbb{N}, \mathscr{T})$ their respective generated 2-relational spaces. Notice that h acts in "$\hookrightarrow$" as metaphor for "$\leq$" (i.e. $\tilde{h}(\hookrightarrow) = \leq$) and acts in "$\hookrightarrow'$" as metaphor for "$>$" (i.e. $\tilde{h}(\hookrightarrow') = >$). Therefore, $H : \mathscr{S} \to \mathscr{T}$ is well defined, thus h is a metaphorical mapping. Notice that

$$\delta_h(\hookrightarrow) = \tilde{h}(\hookrightarrow') \cap \leq \, = \, \leq' \cap \leq = \emptyset$$

$$\delta_h(\hookrightarrow') = \tilde{h}(\hookrightarrow) \cap > \, = \, \leq' \cap \leq = \emptyset$$

and thus it has zero deviance. Observe additionally that $\tilde{h}^{-1}(\leq) = \hookrightarrow$ and $\tilde{h}^{-1}(\leq') = \hookrightarrow'$ belong to

$\mathscr{S}$, therefore this mapping is structural and its associated metaphorical mapping has zero deviation.

Up to now, we have been dealing with the introduction of the order relation in the natural numbers through a metaphor. Now we want to analyze how a metaphor would introduce the sum operation in the natural numbers. Recall that the models of this section are not able to treat the addition directly as an operation. Therefore we have defined (at the beginning of this section) some ternary relations wich will be used in the following examples. Notice that we shall need 3-relational spaces or 3-metaphorical spaces to work with those relations.

Example 40. In the base domain, let us work with the relational space generated by the relation "disjoint join of piles". In other words, let $S = \{disjoin\}$ and $T = \{sum\}$. Consider their respective 3-ary domains $(P, \mathscr{S})$ and $(\mathbb{N}, \mathscr{T})$. First observe that $\tilde{h}^{-1}(sum) = disjoin$ and $\tilde{h}^{-1}(sum') = disjoin'$ and therefore h is a structural mapping. Additionally h acts in $disjoin$ as a metaphor for sum and acts in $disjoin'$ as a metaphor for sum', therefore the associated mapping $H : \mathscr{S} \to \mathscr{T}$ is well defined, implying that h is metaphorical. It is straightforward that h has zero deviation.

The last example is a mathematically simple way to present a metaphor dealing with the introduction of the addition in the natural numbers. However, it might not reflect accuratelly the reality essentially because the "disjoint join of piles" is not what is observed in the real world "object collection domain". The "disjoint join of piles" can be thought as a mathematical idealization of the operation "join of piles". The next example examines the relation "join of piles" defined above as a way to introduce the sum through a metaphor.

Example 41. It is true and appealing to say that h acts in "$join$" as a metaphor for "sum" because $\tilde{h}(join) = sum$, but unfortunately h does not act in "$join'$" as metaphor for "sum'" because $\tilde{h}(join') \neq sum'$. The key idea here is to use an appropriate relation instead of $join'$. To make this possible we have to build a metaphorical space, so let us denote by K the relation given by the union of "$join$" and "$disjoin'$". Consider the 3-metaphorical space given by $(P, \mathscr{M})$, where $\mathscr{M} = \{\emptyset, join, disjoin', K\}$ with $(join)^* = disjoin'$, $(\emptyset)^* = K$. Let $T = \{sum\}$ and consider the associated 3-ary relational space $(\mathbb{N}, \mathscr{T})$. Notice that h acts as a metaphor in $join$ for sum and in $disjoin'$ for sum'. Additionally $H : \mathscr{M} \to \mathscr{T}$ is well defined and therefore h is metaphorical. Notice that

$$\delta_h(join) = \tilde{h}(join') \cap sum = sum$$

and therefore h is a metaphorical mapping with infinite deviation.

It is worth observing that associativity and commutativity properties are shared by the addition of natural numbers and the join of piles, so it would be rather straightforward to define a metaphor to deal with those concepts, because the involved n-ary relational families would be the trivial families $\{P^n,\emptyset\}$ and $\{\mathbb{N}^n,\emptyset\}$ for $n=2$ in the case of commutativity and $n=3$ in the case of associativity.

The next example suggests (or predicts) that the appropriate concept to introduce the identity relation (equality) in the natural numbers through this metaphor is the concept wich relates two piles "having the same number of objects". Notice that it is appealing to use the "equality of piles" (denoted here by $=_P$) to introduce the "equality of numbers" through a metaphor, because it holds that h acts in "$=_P$" as a metaphor for "$=$" (since $\tilde{h}(=_P)$ is the identity relationship set $\{(m,n)\in\mathbb{N}^2 \mid m=n\ \}$). Unfortunately again, the mapping does not act in the complementary relation ($=_P'$) as metaphor for the adequate complementary relation ($='$ or $\neq$).

Example 42. Let us consider the relational space generated by the "bijection" relation, that is to say, let $S=\{\equiv\}$, $T=\{=\}$. Denote their respective 2-relational spaces by $(P,\mathscr{S})$, $(\mathbb{N},\mathscr{T})$. In this example $h:P\to\mathbb{N}$ is structural because $\tilde{h}^{-1}(=)$ is the set $\{(A,B)\in P^2 \mid A \text{ is in bijection with } B\}$ (denoted by $\equiv$), $\tilde{h}^{-1}(=')$ is the relation determined by $\equiv'$ and therefore its associated metaphorical mapping has zero deviation.

Now that we have suggested that the "bijection relation" seems to be an appropriate relation to introduce the identity relationship through this metaphor, we can use that relationship combined with the product of piles to obtain a way to introduce the multiplication into the natural numbers through this metaphor.

Example 43. Let us work with the 3-ary relational spaces generated by $S=\{prod\}$ and $T=\{mult\}$. Denote them by $(P,\mathscr{S})$, $(\mathbb{N},\mathscr{T})$. From the above definition of *prod* it is straightforward that h acts in "*prod*" as a metaphor for "*mult*" and acts in "*prod*$'$" as a metaphor for "*mult*$'$". The associated mapping $H:\mathscr{S}\to\mathscr{T}$ is well defined, and therefore h is metaphorical and it is easy to see that it has zero deviation. Notice that $\tilde{h}^{-1}(mult)=prod$ and $\tilde{h}^{-1}(mult')=prod'$, so h is structural from $(P,\mathscr{S})$ to $(\mathbb{N},\mathscr{T})$.

As we observed above, some authors from the field of cognitive science claim that one of the key properties of metaphor is the "preservation of inferences" from the base domain to the target domain (see [71, 30, 36]). The following example shows how a kind of "preservation of inference" between domains can be formalized in this context. The idea underlying the following example is to interpret "inference" as the mathematical concept of implication. Let us explain this idea in more detail: let A and B be relations in the base domain; $\overline{A}$ and $\overline{B}$ relations in the target domain. Assume that the metaphorical mapping under consideration acts in A as metaphor for $\overline{A}$ and acts in B as metaphor for $\overline{B}$. This particular idea of "preservation of inference" can be understood as the property of metaphorical mappings (corollary 52) wich states: if A implies B then $\overline{A}$ implies $\overline{B}$. This way, the implication relation observed in the base domain between A and B is preserved by the metaphorical mapping to hold in the target domain between the relations $\overline{A}$ and $\overline{B}$. In the following example we need the interaction between binary relations and ternary relations, so we are going to use 3-ary relational spaces and binary relations modified to be ternary relations (as $\hookrightarrow_3$ and $\leq_3$).

Example 44. In the natural numbers it is well known that for any $k \in \mathbb{N}$ "if $n \leq m$ then $n+k \leq m+k$". The aim of this example is to induce this implication from the "object collections" domain through a metaphor, while showing how this preservation can be formalized in this context. To this end denote by $\trianglelefteq$ the ternary relation in P given by

$$\trianglelefteq(A,B,C) \text{ iff the pile } A \uplus C \text{ can be injected into the pile } B \uplus C.$$

(i.e. $A \uplus C \hookrightarrow B \uplus C$). In the same way define a relation in $\mathbb{N}$ denoted by the symbol $\triangleleft$ and defined by:

$$\triangleleft(a,b,c) \text{ iff } a+c \leq b+c.$$

Now, put $T = \{\leq_3, \triangleleft\}$ and consider its associated 3-ary relational space $(\mathbb{N}, \mathscr{T})$. On the other hand, put $S = \{\hookrightarrow_3, \trianglelefteq\}$ and consider its associated 3-ary relational space $(P, \mathscr{S})$. Notice that $h : P \to \mathbb{N}$ is a structural mapping because

$$\tilde{h}^{-1}(\leq_3) \ = \hookrightarrow_3, \tilde{h}^{-1}(\leq_3') \ = \hookrightarrow_3', \tilde{h}^{-1}(\triangleleft) = \trianglelefteq, \tilde{h}^{-1}(\triangleleft') = \trianglelefteq'.$$

Therefore, h is a metaphorical mapping with zero deviation. Observe that in P, it holds that for any pile C: if the pile A can be injected into a pile B then the pile $A \uplus C$ can be injected into the pile $B \uplus C$ ($A \hookrightarrow_3 B \Longrightarrow A \uplus C \hookrightarrow_3 B \uplus C$). Because h is a structural mapping the corollary (52) from chapter (C) ensures that $a \leq b \Longrightarrow a+c \leq b+c$ holds in the domain of the natural numbers.

The previous example might look a little artificial because the addition in the natural numbers is introduced by some kind of disjoint join wich is not really observed in the "real life" object collections domain. A more representative model of the phenomenom would induce the addition in the natural numbers by using the join of piles relation. That is the aim of the following example.

Example 45. We are going to use a metaphorical space defined over P. Denote by K the subset of P^3 given by:

$$K = join \cup disjoin'$$

and consider the 3-ary metaphorical space defined by

$$\mathscr{H} = \{E \cap K \mid E \in \mathscr{S}\}$$

where $\mathscr{S} = \mathscr{R}(\mathscr{S})$ is the 3-ary relational space determined by $\mathscr{S} = \{\hookrightarrow_3, disjoin\}$. Recall that in this case the operation $*$ is defined by $E^* = (E' \cap K)$. Thus

$$(join)^* = (disjoin \cap K)^* = disjoin',$$

$$(\hookrightarrow_3 \cap K)^* = (\hookrightarrow'_3 \cap K).$$

Consider the relational space $(\mathbb{N}, \mathscr{T})$ where $T = \{\leq, sum\}$. It is straightforward that $h : P \to \mathbb{N}$ acts in *join* as a metaphor for *sum*, in $disjoin'$ as a metaphor for sum', in $\hookrightarrow_3 \cap K$ as a metaphor for $\leq$ and in $\hookrightarrow'_3 \cap K$ as a metaphor for $\leq'$. Therefore, h is a metaphorical mapping and it is straightforward that it has infinite deviation.

The metaphor from the last example induces the sum of natural numbers via the *join* relation wich represents the "real life" join of piles. It is interesting to observe that we should not use the relation $\subseteq$ to induce the $\leq$ relation while using the *join* relation to induce the *sum* relation. Together, they are somehow unable to generate a metaphorical space. To see the nature of the problem, let us assume for a while that we can define the 3-ary metaphorical family given by $\mathscr{H} = \{\subseteq_3, \hookrightarrow'_3, join, disjoin'\}$ where $(\subseteq_3)^* = \hookrightarrow'_3$, $(join)^* = disjoin'$. On the other hand consider $T = \{\leq, sum\}$ and its associated relational space $(\mathscr{T}, \mathbb{N})$. In this case $h : P \to \mathbb{N}$ acts on $\subseteq_3$ as a metaphor for $\leq_3$, on *join* as a metaphor for *sum*, on $\hookrightarrow'_3$ as a metaphor for $\leq'_3$ and on $disjoint'$ as a metaphor for sum'. Therefore, the mapping $H : \mathscr{H} \to \mathscr{T}$ would be well defined and it has to be a homomorphism of boolean algebras (by theorem 54 from chapter C). Thus it should preserve the intersection of relations, in particular it should holds that $H(\subseteq_3 \cap join) = H(\subseteq_3) \cap H(join) \ = \leq_3$

$\cap sum$. Notice however that

$$\subseteq_3 \cap join = \{(\emptyset, B, B) \in P^3 \mid B \in P\}$$

therefore

$$H(\subseteq_3 \cap join) = \tilde{h}(\subseteq_3 \cap join) = \{(0, j, j) \in \mathbb{N}^3 \mid j \in \mathbb{N}\}$$

but the relation $\leq_3 \cap sum = \{(m,n,k) \in \mathbb{N}^3 \mid m \leq n, m+n = k\}$ is clearly different from the expected result.

The problem was the definition of the operation $*$ on $\mathscr{H} = \{\subseteq_3, \hookrightarrow'_3, join, disjoin'\}$ wich does not satisfies the conditions (see chapter C) to determine a metaphorical family.

Two-Pan Balance Metaphor Model Based on Relational Spaces

Our aim in this section is to develop a formal model for the "Two-pan balance metaphor" similarly as we did before, but this time we are going to use relational spaces instead of formal domains. Thus, in first place we have to define a relational space wich is appropriate to represent the "simple scales domain", and then, we shall show that the product of simple scales domain is well suited to represent the "scales domain".

The Simple Scales Domain

We suggested that $\mathbb{N}^2$ can be useful to represent the collection of simple scales. Thus, we just have to define the adequate relations over this set to generate an appropriate relational space. Consider the following binary relations over $\mathbb{N}^2$:

$$Add = \{((n,m),(n+1,m+1)) \in \mathbb{N}^2 \times \mathbb{N}^2 \mid n,m \in \mathbb{N}\}. \tag{C.5}$$

$$Equiv = \{((n,m),(j,k)) \in \mathbb{N}^2 \times \mathbb{N}^2 \mid j+m = n+k\}. \tag{C.6}$$

$$Id = \{((n,m),(n,m)) \in \mathbb{N}^2 \times \mathbb{N}^2 \mid n,m \in \mathbb{N}\}. \tag{C.7}$$

Let $B = \{Add, Equiv, Id\}$ and consider the 2-ary relational family $\mathscr{B} = \mathscr{R}(B)$. We have named the relations in such a way to suggest their adequate interpretation. The relation Add defined in C.5 can be interpreted as the operation wich adds an object in each one of the plates of the simple scale. The relation $Equiv$ (equivalence) characterizes the pairs of balances wich are equivalent i.e. two scales (n,m) and (j,k) are equivalent if and only if one of them can be transformed into the another by means of the operation Add. In other words $n-j=m-k$ and therefore the definition in C.6 makes sense. The relation Id above in C.7 is just the identity operation in $\mathbb{N}^2$.

As a model for the domain of "simple scales" we are going to use the 2-ary relational space given by $(\mathbb{N}^2, \mathscr{B})$.

The Scales Domain

As we said before, our aim here is to take the product $(\mathbb{N}^2, \mathscr{B}) \times (\mathbb{N}^2, \mathscr{B})$ and show that this structure can be considered as a plausible model for the scales domain.

In first place, notice that objects belonging to this product have the form $((b_1,b_2),(c_1,c_2))$, and therefore the pair (b_1,c_1) can be interpreted as the first plate of the scale and (b_2,c_2) can be interpreted as the second plate of the scale. Notice that in each plate there are objects of type "b" (boxes) and objects of type "c" (circles). In picture 5.4 we would have that $b_1 = 2, b_2 = 4, c_1 = 3, c_2 = 1$. On the other hand, recall that the product of relational families $\mathscr{B} \otimes \mathscr{B}$ is generated by the set

$$\{\tilde{\pi}_\alpha^{-1}(E) \ : \ E \in \mathscr{B}, \alpha \in \{1,2\}\} \tag{C.8}$$

Notice that $\mathscr{B} \otimes \mathscr{B}$ has many complexes. However, only a few of them will be useful for us, and we will only deal with those.

- Denote by AI the complex belonging to $\mathscr{B} \times \mathscr{B}$ wich is given by the following intersection:

$$AI = \tilde{\pi}_1^{-1}(Add) \cap \tilde{\pi}_2^{-1}(Id) \tag{C.9}$$

$$= \{ \ (\ ((b_1,b_2),(c_1,c_2)),((b_1+1,b_2+1),(c_1,c_2)) \ ) \in (\mathbb{N}^2 \times \mathbb{N}^2)^2 \mid b_1,b_2,c_1,c_2 \in \mathbb{N} \ \}$$

Notice that this relation would be interpreted as the operation wich adds one object of type "box" to each plate of the scale.

- Denote by IA the complex given by:

$$IA = \tilde{\pi}_1^{-1}(Id) \cap \tilde{\pi}_2^{-1}(Add) \tag{C.10}$$

$$= \{ \ (\ ((b_1,b_2),(c_1,c_2)),((b_1,b_2),(c_1+1,c_2+1)) \ ) \in (\mathbb{N}^2 \times \mathbb{N}^2)^2 \mid b_1,b_2,c_1,c_2 \in \mathbb{N} \ \}$$

This relation would be interpreted as the operation wich adds one "circle" to each plate of the scale.

- Consider the complex given by:

$$AA = \tilde{\pi}_1^{-1}(Add) \cap \tilde{\pi}_2^{-1}(Add). \tag{C.11}$$

$$= \{ \ (\ ((b_1,b_2),(c_1,c_2)),((b_1+1,b_2+1),(c_1+1,c_2+1))) \in (\mathbb{N}^2 \times \mathbb{N}^2)^2 \mid b_1,b_2,c_1,c_2 \in \mathbb{N} \ \}$$

The interpretation of this relation is obvious, is the operation wich adds one "circle" and one "box" to each plate of the scale.

- Denote by II the complex given by:

$$II = \tilde{\pi}_1^{-1}(Id) \cap \tilde{\pi}_2^{-1}(Id) \tag{C.12}$$

$$= \{ \ (\ ((b_1,b_2),(c_1,c_2)),((b_1,b_2),(c_1,c_2)) \ ) \in (\mathbb{N}^2 \times \mathbb{N}^2)^2 \mid b_1,b_2,c_1,c_2 \in \mathbb{N} \ \} \}$$

This relation takes the place of the identity operation in the scales domain.

- Denote by EE the complex given by:

$$EE = \tilde{\pi}_1^{-1}(Equiv) \cap \tilde{\pi}_2^{-1}(Equiv). \tag{C.13}$$

$$= \{(((m,n),(i,j)),((a,b),(c,d))) \in (\mathbb{N}^2 \times \mathbb{N}^2)^2 \mid n+a=m+b, j+c=i+d, a,b,c,d,i,j,m,n \in \mathbb{N}\}$$

Notice that this relation is exactly what we would expect of the relation of equivalence in the scales domain i.e. two scales are equivalent if and only if both can be transformed into the same scale by a finite number of applications of the operations represented by AA, II, AI and IA.

These remarks suggest that the relational space $(\mathbb{N}^2, \mathscr{B}) \times (\mathbb{N}^2, \mathscr{B}) = (\mathbb{N}^2 \times \mathbb{N}^2, \mathscr{B} \otimes \mathscr{B})$ is a plausible model for the "scales domain".

The Equations Domain

As before, we shall describe formally the domain of "equations" i.e. we shall use relational spaces to build a model for the target domain of the two-pan balance metaphor.

Once again, we will say that an equation will be a string of symbols wich has the form $ax+b \equiv cx+d$ where a,b,c,d are symbols that denote natural numbers. all those equations, but just with those equations whose solution is a natural number.

In this domain, the transformations from one equation to another equivalent equation are performed by means of algorithms, wich in turn can be modeled as operations. However we need to represent these operations as relations in order to work with relational spaces. Consider the set of equations $E = \{ax+b \equiv cx+d \mid a,b,c,d \in \mathbb{N}\}$.

Let us define some binary relations on this domain that we shall use to model this metaphor.

- This relation represents the operation wich adds the expression "x" to each side of the equation.

$$A_1 = \{(ax+b \equiv cx+d, (a+1)x+b \equiv (c+1)x+d) \in E^2 \mid a,b,c,d \in \mathbb{N}\} \tag{C.14}$$

- This relation represents the operation wich adds the expression "1" to each side of the equation.

$$A_2 = \{(ax+b \equiv cx+d, ax+(b+1) \equiv cx+(d+1)) \in E^2 \mid a,b,c,d \in \mathbb{N}\} \tag{C.15}$$

- This relation represents the identity operation in this domain

$$I = \{(e,e) \in E^2 \mid e \in E \ \} \tag{C.16}$$

- This relation represents the operation wich adds the expression "$x+1$" to each side of the equation.

$$A = \{(ax+b \equiv cx+d, (a+1)x+(b+1) \equiv (c+1)x+(d+1)) \in E^2 \mid a,b,c,d \in \mathbb{N}\} \quad \text{(C.17)}$$

Now, in order to express the equivalence of equations, we are going to introduce an operation in this domain. Assume that $e_1 = a_1x+b_1 \equiv c_1x+d_1$ and $e_2 = a_2x+b_2 \equiv c_2x+d_2$ and define:

$$plus(e_1,e_2) = (a_1+a_2)x+(b_1+b_2) \equiv (c_1+c_2)x+(d_1+d_2).$$

- The following relationship can be interpreted as the equivalence relationship in the equations domain.

$$R = \{(e_1,e_2) \in E^2 \mid \exists e = (ax+b \equiv ax+b) \text{ such that } a,b \in \mathbb{Z}, plus(e_1,e) = e_2\} \quad \text{(C.18)}$$

Let $T = \{A, I, A_1, A_2, R\}$ and consider the 2-ary relational family $\mathscr{T} = \mathscr{R}(T)$ i.e. the 2-ary relational family generated by T. Our proposal is to use the 2-ary relational space given by $(E,\mathscr{T})$ as a model for the "equations domain".

Examples

Our goal in this section is to give some examples for concepts from previous chapters. These examples are based on the relational spaces built above. As our approach suggests, we are going to model metaphors between the domains previously described by means of a mapping:

$$\varepsilon : \mathbb{N}^2 \times \mathbb{N}^2 \to E \text{ defined by } \varepsilon(\ (b_1,b_2),(c_1,c_2)\ ) = b_1x+c_1 \equiv b_2x+c_2.$$

Recall that for every $n \in \mathbb{N}$, ε determines a mapping:

$$\tilde{\varepsilon} : (\mathbb{N}^2 \times \mathbb{N}^2)^n \to E^n \text{ defined by } \tilde{\varepsilon}(a_1,...a_n) = (\varepsilon(a_1),...,\varepsilon(a_n)) \text{ where } a_i \in \mathbb{N}^2 \times \mathbb{N}^2$$

Example 46. As before, we are going to use the first example just to summarize notations and definitions for the models proposed in this section.

In first place, our model for the "simple scales domain" is the 2-ary relational space $(\mathbb{N}^2, \mathscr{B})$ where $\mathscr{B} = \mathscr{R}(B)$ and $B = \{Add, Equivalence, Id\}$.

Second, for the "scales domain" we have proposed the relational space $(\mathbb{N}^2, \mathscr{B}) \times (\mathbb{N}^2, \mathscr{B}) = (\mathbb{N}^2 \times \mathbb{N}^2, \mathscr{B} \otimes \mathscr{B})$ as a model. We have emphasized the complexes (or relations) AI, IA, AA, II, EE because of its interpretation.

Third, our model for the "equations domain" shall be the relational space $(E, \mathscr{T})$ where $\mathscr{T} = \mathscr{R}(T)$ and $T = \{A_1, A_2, A, I, R\}$ as previously defined.

Example 47. Consider the relational spaces $(\mathbb{N}^2 \times \mathbb{N}^2, \mathscr{B} \otimes \mathscr{B})$ and $(E, \mathscr{T})$. There is a considerable quantity of complexes in $\mathscr{B} \otimes \mathscr{B}$ wich are "unnecessary" in the sense that they do not have a meaning in this metaphor. For example, consider the complexes given by

$$\tilde{\pi}_1^{-1}(Add) = \{ \ (\ ((b_1, b_2), (c_1, c_2)), ((b_1+1, b_2+1), (c_3, c_4)) \ ) \in (\mathbb{N}^2 \times \mathbb{N}^2)^2 \mid b_1, b_2, c_1, c_2, c_3, c_4 \in \mathbb{N} \ \}$$

$$\tilde{\pi}_1^{-1}(Equiv) = \{ \ (\ ((b_1, b_2), (c_1, c_2)), ((b_3, b_4), (c_3, c_4)) \ ) \in (\mathbb{N}^2 \times \mathbb{N}^2)^2 \mid b_i, c_i \in \mathbb{N} \text{ and } b_2 + b_3 = b_1 + b_4 \ \}$$

and their unions, intersections and complements. Observe that it is difficult to give meaningful interpretations to them in the scales domain. Additionally, their image by $\tilde{\varepsilon}$ are not complexes of $\mathscr{T}$. Despite these observations, it is straightforward that:

$$\tilde{\varepsilon}^{-1}(A_1) = AI$$

$$\tilde{\varepsilon}^{-1}(A_2) = IA$$

$$\tilde{\varepsilon}^{-1}(A) = AA$$

$$\tilde{\varepsilon}^{-1}(I) = II$$

$$\tilde{\varepsilon}^{-1}(R) = EE$$

wich means that $\varepsilon : \mathbb{N}^2 \times \mathbb{N}^2 \to E$ is a structural mapping i.e. $\mathscr{B} \otimes \mathscr{B}$ has a boolean subalgebra wich is isomorphic to the boolean algebra determined by $\mathscr{T}$. This means that the scales domain can explain all the existent logic in the domain in the equations domain. Therefore, its associated metaphorical mapping (wich we explore in the next example) has zero deviation.

the bijective mapping $f : \mathbb{N}^2 \times \mathbb{N}^2 \to E$. Notice that $\tilde{f}^{-1}(A_1) = AI, \tilde{f}^{-1}(A_2) = IA, \tilde{f}^{-1}(A) =$

$AA, \tilde{f}^{-1}(I) = II$ and $\tilde{f}^{-1}(R) = EE$, so the mapping f is structural and its associated metaphorical mapping has zero deviation.

Example 48. The relational space $(\mathbb{N}^2 \times \mathbb{N}^2, \mathscr{B} \otimes \mathscr{B})$ is generated by the family of complexes defined in equation C.8. However, such set has many complexes that are not easily interpretable in the context of the scales domain as we saw in the previous example. This example modify such relational space removing the unnecessary structure ("unnecessary" in the sense of interpretativeness of the model).

Consider the set of complexes $B' = \{AI, IA, AA, II, EE\}$ and consider $\mathscr{B}' = \mathscr{R}(B')$ the 2-ary relational family generated by it. Notice that $\mathscr{B}'$ is a subalgebra of $\mathscr{B} \otimes \mathscr{B}$ (in the sense of boolean algebras) and thus the relational (metaphorical also) space given by $(\mathbb{N}^2 \times \mathbb{N}^2, \mathscr{B}')$ can be seen as another model (with less structure) for the "scales domain".

In that case, the bijective mapping ε is a metaphorical mapping from $(\mathbb{N}^2 \times \mathbb{N}^2, \mathscr{B}')$ to $(E, \mathscr{T})$ wich means that the mapping $\xi : \mathscr{B}' \to \mathscr{T}$ (given by $\xi(A) = \tilde{\varepsilon}(A)$ for $A \in \mathscr{B}'$) is well defined and it is a boolean algebra morphism. Furthermore, $\xi : \mathscr{B}' \to \mathscr{T}$ is an isomorphism (of boolean algebras). These remarks suggest that the pair of domains have the same logic structure. Obviously, the metaphorical mapping ε has deviation zero.

Example 49 (Inference)**.** The aim of this example is showing that our approach let us model this metaphor bringing an inference of the sort "if ... then ..." from the scales domain to the equations domain. The inference that we have selected to preserve in this example is the following: For any scale a, if we add a "box" on each of its plates then, the resultant scale b is equivalent to the original one . Notice that this implication can be modelled by the inclusion $AI \subseteq EE$ i.e. if $(a,b) \in AI \Rightarrow (a,b) \in EE$ ("if b is the result of putting a box on each plate of a, then a and b are equivalent").

Let $\mathscr{B}'$ be the 2-ary relational family generated by $\{AI, IA, AA, II, EE\}$ and $\mathscr{T}$ the 2-ary relational family generated by $\{A_1, A_2, A, I, R\}$. In example 48 we have seen that ε is a metaphorical mapping from $(\mathbb{N}^2 \times \mathbb{N}^2, \mathscr{B})$ to $(E, \mathscr{T})$.

Notice that $\tilde{\varepsilon}(AI) = A_1$ and $\tilde{\varepsilon}(EE) = R$, and thus we can use the inference corollary (corollary 52 in chapter C) to conclude that in the "equation domains" it holds that if $(e_1, e_2) \in A_1$ then $(e_1, e_2) \in R$. i.e. we can conclude that if the equation e_2 is the result of adding "x" to each side of the equation e_1, then they both have the same solution set.

Tiled Path Metaphor for Integers Model

One Directional Tiled Path Domain

As before, we are going to use the set $\mathbb{N}$ of natural numbers to model this domain. The goal of this section is to use relational spaces instead of formal domains to model the same metaphor.

We have to introduce the addition operation into the relational structure of the model. It can be done by considering the addition as a ternary relation. Thus, consider the relation defined by

$$sum = \{(a,b,c) \in \mathbb{N}^3 \mid a+b=c\}$$

and its generated 3-ary relational family $\mathscr{S} = \mathscr{R}(sum)$. The 3-relational space $(\mathbb{N}, \mathscr{S})$ will serve as a model for the "one directional tiled path domain".

The Two Directional Path Domain

The formal model for the tiled path domain (interpreted as a path where walks in two directions are allowed) shall be strongly related to the product $(\mathbb{N}\times\mathbb{N}, \mathscr{S}\otimes\mathscr{S})$. Thus, we are going to describe some structure for this relational space while introducing adequate interpretations in order to relate it to the formal model.

Notice that objects belonging to $(\mathbb{N}\times\mathbb{N}, \mathscr{S}\otimes\mathscr{S})$ are pairs of natural numbers (a,b). The first coordinate might be interpreted as "a forward walk of a tiles" and the second coordinate as "a backward walk of b tiles".

The 3-ary relational family $\mathscr{S}\otimes\mathscr{S}$ is generated by complexes of the form $\tilde{\pi}_i^{-1}(E)$ where $E \in \mathscr{S}$ and $i \in \{1,2\}$. Let us direct our attention to the following complexes belonging to $\mathscr{S}\otimes\mathscr{S}$:

- $\tilde{\pi}_1^{-1}(sum) = \{((a_1,a_2),(b_1,b_2),(c_1,c_2)) \in (\mathbb{N}\times\mathbb{N})^3 \mid a_1+b_1=c_1\}$
- $\tilde{\pi}_2^{-1}(sum) = \{((a_1,a_2),(b_1,b_2),(c_1,c_2)) \in (\mathbb{N}\times\mathbb{N})^3 \mid a_2+b_2=c_2\}$

and then the complex given by

$$add = \tilde{\pi}_1^{-1}(sum) \cap \tilde{\pi}_2^{-1}(sum) = \{((a_1,a_2),(b_1,b_2),(c_1,c_2)) \in (\mathbb{N}\times\mathbb{N})^3 \mid a_1+b_1 = c_1, a_2+b_2 = c_2\} \tag{C.19}$$

belongs to the 3-ary relational family $\mathscr{S}\otimes\mathscr{S}$. Observe that this relation is associated to the addition wich acts component by component. Therefore, such complex can be interpreted as an operation wich takes two pairs of walks and composes them in a new pair of walks as described in C.19. In examples, we are going to denote this complex by "*add*".

domain as only one object. Therefore consider the equivalence relation $\sim$ in $\mathbb{N}\times\mathbb{N}$ given by:

It is worth observing that it would be appealing to introduce the equivalence relation

$$(a_1,a_2) \sim (b_1,b_2) \text{ if and only if } a_1+b_2 = b_1+a_2 \tag{C.20}$$

with the purpose of using a quotient of the relational space $(\mathbb{N}\times\mathbb{N}, \mathscr{S}\otimes\mathscr{S})$ to model this domain. Unfortunately, it is straightforward that "$\sim$" is not compatible with $\mathscr{S}\otimes\mathscr{S}$ and thus we cannot define the adequate "quotient" relational space. We shall explore some alternatives in examples 50 and 51.

The Integers Domain

Our intention here is to formalize the domain of integer numbers as a relational space. In order to acomplish that, we have to define the sum operation as a relation. Consider the relation denoted by "$+$" and defined by

$$+ = \{(a,b,c) \in \mathbb{Z}^3 \mid a+b = c\}$$

and its generated 3-relational family $\mathscr{A} = \mathscr{R}(+)$. Thus, the 3-relational space wich will serve as a model for the integer numbers domain is given by the pair $(\mathbb{Z}, \mathscr{A})$.

C.4 Examples

We are going to model the *tiled path metaphor* for integer numbers by using the mapping

$$z : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$$

wich associates to each pair of “walks” (a,b) in the tiled path domain, the number $a-b$ in the integers domain. That is,

$$z(a,b) = a - b.$$

Also, for any $n \in \mathbb{N}$, we have the induced mapping

$$\tilde{z} : (\mathbb{N} \times \mathbb{N})^n \to \mathbb{Z}^n$$

defined by

$$((a_1,b_1),(a_2,b_2),...,(a_n,b_n)) \to (a_1 - b_1, a_2 - b_2, ..., a_n - b_n)$$

Example 50. ese haces el quotient

The aim of this example is to show that the mapping z is able to introduce the addition structure into the “integer numbers domain” by means of a metaphor. Notice that

$$\tilde{z}(add) = \tilde{z}(\ \{((a_1,a_2),(b_1,b_2),(c_1,c_2)) \in (\mathbb{N} \times \mathbb{N})^3 \mid a_1 + b_1 = c_1, a_2 + b_2 = c_2\}\ ) =$$

$$= \{((a_1 - a_2),(b_1 - b_2),(a_1 - a_2 + b_1 - b_2)) \mid a_1,a_2,b_1,b_2 \in \mathbb{N}\} =$$

$$= \{(x,y,x+y) \mid x,y \in \mathbb{Z}\ \} = +$$

In other words, the image of the relation *add* (by the mapping $\tilde{z}$) is the relation “+” defined over the set of integer numbers. It suggests that, within this formalism, the relation *add* can be used to introduce the addition operation into the integer numbers. In order to achieve that, we shall use the 3-ary metaphorical family over $\mathbb{N} \times \mathbb{N}$ given by $\mathscr{K} = \{add, \tilde{z}^{-1}(+')\}$ where $(add)^* = \tilde{z}^{-1}(+')$. Observe that

$$\tilde{z}(add) = + \quad \text{and} \quad \tilde{z}(\tilde{z}^{-1}(+')) = +'$$

and therefore z is metaphorical from $(\mathbb{N} \times \mathbb{N}, \mathscr{K})$ to $(\mathbb{Z}, \mathscr{A})$. It is straightforward that z has non zero deviation.

Example 51. In the previous example (50) we have used a metaphorical space related to $(\mathbb{N} \times \mathbb{N}, \mathscr{S} \otimes \mathscr{S})$ as a model for the "tiled path domain". The reader might be wondering, why we have not introduced some equivalence relation in order to group all objects in the domain wich are equivalent "as walks"? (for example take the pairs $(1,2)$, $(2,3)$, $(3,4)$, and declare them all equivalent to the pair $(0,1)$). The reason is that such equivalence relation (C.20) is not compatible with the relational family $\mathscr{S} \otimes \mathscr{S}$. This example is aimed to show that an equivalence relation can be introduced in this context to model this metaphor in a way closer to the classical construction of the integer numbers. We shall use a different relational space in order to achieve this.

Observe that the mapping $z : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ is *not* structural from $(\mathbb{N} \times \mathbb{N}, \mathscr{S} \otimes \mathscr{S})$ to $(\mathbb{Z}, \mathscr{A})$. This can be seen by considering the complex given by

$$\tilde{z}^{-1}(+) = \{((a_1,a_2),(b_1,b_2),(x_1,x_2)) \in (\mathbb{N} \times \mathbb{N})^3 \mid x_1 - x_2 = a_1 + b_1 - (a_2 + b_2)\} \tag{C.21}$$

Tiled Path Metaphor for Integers and noticing that it is clearly different from the relation "add" (actually "add" is strictly contained in $\tilde{z}^{-1}(+)$). Thus, it seems to be a good idea to consider the 3-ary relational family generated by the complex $\tilde{z}^{-1}(+)$ (wich is a sort of generalized addition by components) i.e. as a model for the "Tiled Path Domain" we are going to use the relational family $\mathscr{M} = \mathscr{R}(\{\tilde{z}^{-1}(+)\})$ and its associated 3-relational space $(\mathbb{N} \times \mathbb{N}, \mathscr{M})$. In such case, the mapping z is structural from $(\mathbb{N} \times \mathbb{N}, \mathscr{M})$ to $(\mathbb{Z}, \mathscr{A})$ and obviously its associated metaphorical mapping has zero deviance.

The mapping $z : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ is surjective and the equivalence relation wich is determined by the fibers of z (wich is equivalent to the one defined in C.20) is compatible with $\mathscr{M}$:

Let $(a_1,a_2) \sim (d_1,d_2)$, $(b_1,b_2) \sim (e_1,e_2)$, $(c_1,c_2) \sim (f_1,f_2)$ and assume that $((a_1,a_2),(b_1,b_2),(c_1,c_2)) \in \tilde{z}^{-1}(+)$. Then $c_1 - c_2 = a_1 + b_1 - (a_2 + b_2)$ wich holds if and only if $f_1 - f_2 = d_1 + e_1 - (d_2 + e_2)$ wich means that $((d_1,d_2),(e_1,e_2),(f_1,f_2)) \in \tilde{z}^{-1}(+)$.

Therefore, we can define the quotient domain $(\mathbb{N} \times \mathbb{N}, \mathscr{M})/\sim$ wich would be considered as another model for the "Tiled Path Domain" where all the equivalent pairs of walks are identified. It can be easily seen that this domain is isomorphic to the domain $(\mathbb{Z}, \mathscr{A})$ (the integers domain). Additionally, we can use the proposition 35 in chapter C to ensure the existence of a mapping (an isomorphism) $z^* : \mathbb{N} \times \mathbb{N}/\sim \to \mathbb{Z}$ wich is structural from $(\mathbb{N} \times \mathbb{N}, \mathscr{M})/\sim$ to $(\mathbb{Z}, \mathscr{A})$ and makes the following diagram commute.

$$\begin{tikzcd}
(\mathbb{N}\times\mathbb{N},\mathscr{M}) \arrow[r, "z"] \arrow[d, "\vartheta_{\mathbb{N}\times\mathbb{N}}"'] & (\mathbb{Z},\mathscr{A}) \\
(\mathbb{N}\times\mathbb{N},\mathscr{M})/\sim \arrow[ur, dashed, "z^*"] &
\end{tikzcd}$$

Notice that the construction of this model for the tiled path domain strongly resembles the construction of the integer numbers (group $(\mathbb{Z},+)$) based on the natural numbers (monoid $(\mathbb{N},+)$). But, as we are working with relations, it requires to generalize the addition operation as the relation written in C.21. Our work maintains those classical ideas and just places them in a cognitive context.