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BLOWING-UP PATTERNS IN SEMILINEAR ELLIPTIC EQUATIONS OF CRITICAL TYPE

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

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Resumen

Hay dos partes en mi tesis. La primera parte se dedica principalmente a la construcción de soluciones burbujeantes de algunos problemas elípticos con no linealidad exponencial en $\mathbb{R}^2$. En la segunda parte se considera la existencia de soluciones de punta para ecuaciones elípticas en variedades de Riemann.

En la primera parte, utilizamos el método de reducción de Lyapunov-Schmidt para obtener la existencia de soluciones burbujeantes en el problema de contorno Dirichlet

$$\begin{cases} \Delta u + \lambda u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{en } \Omega; \\ u = 0 & \text{en } \partial\Omega, \end{cases}$$

donde Ω es un dominio suave en $\mathbb{R}^2$, $\lambda > 0$ pequeño. Se estudia el problema para $0 < p < 2$ en dominios acotados y para $p = 1$ en dominios no acotados.

A continuación, se considera la existencia de solucions con concentración mixta en el interior y la frontra para el siguiente problema de Neumann

$$\begin{cases} -\Delta u + u = \lambda u^{p-1} e^{u^p}, \quad u > 0 & \text{en } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 & \text{en } \partial\Omega, \end{cases}$$

donde Ω es un dominio suave en $\mathbb{R}^2$, $\lambda > 0$ es un parámetro pequeño, $0 < p < 2$, y ν denota el vector normal exterior a $\partial\Omega$.

Además, construimos las soluciones burbujeantes para el siguiente problema de Neumann

$$\begin{cases} -\Delta u + u = 0 & \text{en } \Omega; \\ \frac{\partial u}{\partial \nu} = \lambda u^{p-1} e^{u^p} & \text{en } \partial\Omega, \end{cases}$$

donde ν es el vector normal exterior de $\partial\Omega$, $\lambda > 0$ es un parámetro pequeño y $0 < p \leq 2$.

Por último, se estudia la existencia de puntos críticos para el funcional de traza de Trudinger-Moser.

En la segunda parte, se considera la existencia de soluciones de punta para ecuaciones elípticas en variedades de Riemann compactas.

Abstract

There are two parts in my thesis. One is mainly devoted to construct bubbling solutions to some elliptic problems with exponential nonlinearity in $\mathbb{R}^2$. The other one is to consider the existence of peak solutions for elliptic equations on Riemannian manifolds.

In the first part, using Lyapunov-Schmidt reduction we get the existence of bubbling solutions to the Dirichlet boundary value problem

$$\begin{cases} \Delta u + \lambda u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth domain in $\mathbb{R}^2$, $\lambda > 0$ small. We study this problem in a bounded domain for $0 < p < 2$ and in an unbounded domain for $p = 1$.

Next, we consider the existence of mixed interior and boundary bubbling solutions for the following Neumann problem

$$\begin{cases} -\Delta u + u = \lambda u^{p-1} e^{u^p}, \quad u > 0 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth domain in $\mathbb{R}^2$, $\lambda > 0$ is a small parameter, $0 < p < 2$, and ν denotes the outer normal vector to $\partial\Omega$.

Moreover, we construct the bubbling solutions to the following Neumann problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = \lambda u^{p-1} e^{u^p} & \text{on } \partial\Omega, \end{cases}$$

where ν is the outer normal vector of $\partial\Omega$, $\lambda > 0$ is a small parameter and $0 < p \leq 2$.

Last, we study the existence of critical points for the super critical Trudinger-Moser trace functional.

In the second part, we consider the existence of peak solutions for elliptic equations on compact Riemannian manifolds.

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Contents

Chapter 1

Introduction

This thesis consists of two parts. One is mainly devoted to construct bubbling solutions to some elliptic equations in $\mathbb{R}^2$ with exponential nonlinearity. The other one is to consider the existence of peak solutions to some elliptic equations on compact Riemannian manifold.

1.1 Dirichlet Problem in $\mathbb{R}^2$

Consider the following boundary value problem

$$\begin{cases} \Delta u + \lambda u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter and $0 < p \leq 2$. This problem is the Euler-Lagrange equation for the functional

$$J_\lambda^p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{p}\int_\Omega e^{u^p}, \quad u \in H_0^1(\Omega), \tag{1.2}$$

which is well defined. Because for a planar domain Ω, the analogue of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$ in dimensions greater than 3, is the Orlicz space embedding

$$H_0^1(\Omega) \ni u \longmapsto e^{u^2} \in L^s(\Omega) \quad \forall\, s \geq 1,$$

which is connected to the critical Trudinger-Moser inequality [94]

$$C(\Omega) = \sup\left\{ \int_\Omega e^{4\pi u^2} \;/\; u \in H_0^1(\Omega),\; \int_\Omega |\nabla u|^2 = 1 \right\} < +\infty.$$

If $p = 1$, the problem (1.1) becomes

$$\begin{cases} \Delta u + \lambda e^u = 0, & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which can be called the Liouville equation after [78]. This kind of problem with exponential nonlinearity appears in many fields of mathematics, such as the study of prescribed Gaussian curvature on a compact Riemannian surface, Chern-Simons gauge theories, the vortex theory for the turbulent Euler flow, and so on, and it has attracted many authors for more than decades.

There are many results about the asymptotic behavior and existence of solution to (1.3).

Proposition 1.1. *(***Asymptotic Analysis of solutions to (1.3)***: Nagasaki-Suzuki [95]) Let u_λ be an unbounded family of solutions to (1.3). Then as $\lambda \to 0$, $\lambda \int_\Omega e^{u_\lambda}$ accumulates to only values $8k\pi$ for $k \in \{0\} \cup \mathbb{N} \cup \{+\infty\}$. According to these cases the solutions $\{u_\lambda\}$ behave as follows*

(i) If $k = 0$, then $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$;

(ii) If $k \in \mathbb{N}$, then there exists $k-$point blow up, i.e. there are k distinct points ξ_j, $j = 1, \cdots, k$, in Ω, separated uniformly from each other and from he boundary $\partial\Omega$, such that, as $\lambda \to 0$, u_λ peaks to infinity in each one of them, and remains bounded away from them, that is, the solutions u_λ to problem (1.3) remain uniformly bounded on $\Omega \backslash \bigcup_{j=1}^{k} B_\delta(\xi_j)$ and

$$\sup_{B_\delta(\xi_j)} u_\lambda \to +\infty, \qquad \text{as} \quad \lambda \to 0,$$

for any $\delta > 0$.

(iii) If $k = +\infty$, then $u_\lambda(x) \to +\infty$ for all $x \in \Omega$.
Moreover, in the case (ii), we have

$$u_\lambda(x) \to \sum_{k=1}^{k} G_\Omega(x, \xi_j) \quad \text{in } C^2_{loc}(\bar{\Omega} \backslash \{\xi_1, \cdots, \xi_k\}) \quad \text{as } \lambda \to 0,$$

where the location of the blow-up points $\xi_1, \cdots, \xi_k$ is such that, after passing to a subsequence, converges to a critical point of the function

$$\varphi_k(\xi_1, \cdots, \xi_k) = \sum_{j=1}^{k} H_\Omega(\xi_j, \xi_j) + \sum_{i \neq j} G_\Omega(\xi_i, \xi_j), \tag{1.4}$$

where $G_\Omega(x, y)$ is the standard Green's function of the problem

$$\begin{cases} -\Delta_x G_\Omega(x, y) = 8\pi \delta_y(x), & x \in \Omega; \\ G_\Omega(x, y) = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

and $H_\Omega(\cdot,\cdot)$ its regular part defined as

$$H_\Omega(x,y) = G_\Omega(x,y) - 4\log\frac{1}{|x-y|}. \tag{1.6}$$

For the proof, the authors in [95] used complex function theory, more precisely, a representation formula of solutions to (1.3), called the Liouville integral formula was a key ingredient. For the other proof of Proposition 1.5 by using real analysis and PDE theory, see H. Brezis and F. Merle [15], Y. Y. Li and I. Shafrir [71], L. Ma and J. Wei [83].

Conversely, many authors constructed blow-up solutions to problem (1.3) with property

$$\lim_{\lambda\to 0}\lambda\int_\Omega e^{u_\lambda} = 8k\pi. \tag{1.7}$$

In order to state the existence of bubbling solutions to (1.3). Let us recall some definitions.

Definition 1.2. *([46] [69]) Let $D\subset\mathbb{R}^N$ and $f\in\mathcal{C}^1(D,\mathbb{R})$. A bounded set K is called C^1-stable critical set of f provided for all $\sigma>0$ there exists $\delta>0$ such that if $g\in\mathcal{C}^1(D,\mathbb{R})$ with the property that*

$$\max_{dist(x,K)\le\sigma}(|g(x)-f(x)|+|\nabla g(x)-\nabla f(x)|)\le\delta,$$

then g has at least one critical point x with $dist(x,K)\le\sigma$.

Remark 1.3. *It is known that a bounded subset K of critical point of f is stable critical set if one of the following conditions is satisfied:*

(i) K is a strict local minimum set of f, namely, $f(x)=f(y)$ for any $x,y\in K$, and for some open neighborhood $\mathcal{O}$ of K, $f(x)<f(y)$ for all $x\in K, y\in\mathcal{O}\backslash K$;

(ii) K is a strict local maximum set of f;

(iii) If the Brower degree $deg(\nabla f,\mathcal{O},0)\neq 0$ for any $\varepsilon>0$ small, where $\mathcal{O}$ is an neighborhood of K.

Definition 1.4. *(M. del Pino, M. Kowalczyk, M. Musso [36])We say that φ_k links in $\mathcal{D}$ at critical level $\mathcal{C}$ relative to B and B_0 if B and B_0 are closed subsets of $\bar{\mathcal{D}}$ with B connected and $B_0\subset B$ such that the following conditions hold: Let us set Γ to be the class of the maps $\Phi\in C(B,\mathcal{D})$ with the property that there exists a function $\Psi\in C([0,1]\times B,\mathcal{D})$ such that*

$$\Psi(0,\cdot)=Id_B,\quad \Psi(1,\cdot)=\Phi,\quad \Psi(t,\cdot)|_{B_0}=Id_{B_0}\ \text{ for }\forall t\in[0,1].$$

We assume

$$\sup_{\xi\in B_0}\varphi_k(\xi)<\mathcal{C}:=\inf_{\Phi\in\Gamma}\sup_{\xi\in B}\varphi_k(\Phi(\xi)), \tag{1.8}$$

and for all $\xi \in \partial\mathcal{D}$ such that $\varphi_k(\xi) = \mathcal{C}$, there exists a vector τ tangent to $\partial\mathcal{D}$ at ξ such that

$$\nabla\varphi_k(\xi) \cdot \tau \neq 0. \tag{1.9}$$

Under these conditions a critical point $\bar{\xi} \in \mathcal{D}$ with $\varphi_k(\bar{\xi}) = \mathcal{C}$ exists, as a standard deformation argument involving the negative gradient flow of φ_k shows. Condition (1.8) is a general way of describing a change of topology in the level sets $\{\varphi_k \le c\}$ in $\mathcal{D}$ taking place at $c = \mathcal{C}$, while (1.9) prevents intersection of the level set $\mathcal{C}$ with the boundary. It is easy to check that the above conditions hold if

$$\inf_{\xi\in\mathcal{D}} \varphi_k(\xi) < \inf_{\xi\in\partial\mathcal{D}} \varphi_k(\xi), \qquad \text{or} \qquad \sup_{\xi\in\mathcal{D}} \varphi_k(\xi) > \sup_{\xi\in\partial\mathcal{D}} \varphi_k(\xi),$$

namely the case of (possibly degenerate) local minimum or maximum points of φ_k. The level $\mathcal{C}$ may be taken in these cases respectively as that of the minimum and the maximum of φ_k in $\mathcal{D}$. These hold also if φ_k is C^1-close to a function with a non-degenerate critical point in $\mathcal{D}$. We call $\mathcal{C}$ a non-trivial critical level of φ_k in $\mathcal{D}$.

Proposition 1.5. *(***Existence of bubbling solutions to (1.3)***) Let φ_k be defined by (1.4). There exists a solution u_λ to (1.3) such that u_λ blows up on points $\xi_1, \dots, \xi_k$, provided one of the following conditions*

(i) φ_k has a nondegenerate critical point $(\xi_1, \dots, \xi_k)$ (S. Baraker and F. Pacard [10]), or,

(ii) there exists a stable set K for φ_k (P. Esposito, M. Grossi, A. Pistoia [46]), or

(iii) φ_k has a topologically non trivial critical value if Ω is not simply connected (M. del Pino, M. Kowalczyk, M. Musso [36]).

If $p = 2$, problem (1.1) becomes

$$\begin{cases} \Delta u + \lambda u e^{u^2} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

This problem is the Euler-Lagrange equation for the functional J_λ^2 (see (1.2)). Construction of bubbling solutions for problem (1.10) is somehow different from the case $p = 1$. This has been treated in [39]. In order to state this result, let us introduce the following function of k distinct points $\xi_1, \cdots, \xi_k \in \Omega$ and k positive numbers $m_1, \cdots, m_k$,

$$\varphi_{k,2}(\xi, m) = a\sum_{j=1}^{k} m_j^2 + 2\sum_{j=1}^{k} m_j^2 \log m_j^2 + \sum_{j=1}^{k} m_j^2 H_\Omega(\xi_j, \xi_j) + \sum_{i\neq j} m_i m_j G_\Omega(\xi_i, \xi_j), \tag{1.11}$$

where $a > 0$ is an absolute constant, and $G_\Omega(x, y)$ is the Green's function and $H_\Omega(\cdot, \cdot)$ its regular part defined by (1.5) and (1.6) respectively. The authors in [39] established that, if $\varphi_{k,2}$ has a *topologically non trivial critical value* (see Definition 1.4), with corresponding critical point $(\xi_1,\dots,\xi_k,m_1,\dots m_k)\in\Omega^k\times\mathbb{R}^k_+$, then there exists a solution u_λ of (1.10) with the shape

$$u_\lambda(x)=\sqrt{\lambda}\left[\sum_{j=1}^{k}m_jG_\Omega(x,\xi_j)+o(1)\right],\quad \text{as}\quad \lambda\to 0, \tag{1.12}$$

where $o(1)\to 0$ as $\lambda\to 0$ uniformly on compact sets of $\Omega\setminus\{\xi_1,\dots,\xi_k\}$. Furthermore,

$$J^2_\lambda(u_\lambda)=2k\pi+\alpha\lambda+4\pi\lambda\varphi_{k,2}(\xi,m)+\lambda o(1)$$

where α is an absolute constant, $\varphi_{k,2}$ is defined in (1.11) and $o(1)\to 0$ as $\lambda\to 0$. In particular, in the case Ω is not simply connected they constructed the solution u_λ of (1.10), with two bubbling points, namely satisfying

$$u_\lambda(x)=\sqrt{\lambda}\left[\sum_{j=1}^{2}m_jG_\Omega(x,\xi_j)+o(1)\right],\quad \text{as}\quad \lambda\to 0,$$

where (m_1,m_2,ξ_1,ξ_2) is a critical point of $\varphi_{2,2}$ defined in (1.11), and $o(1)\to 0$ as $\lambda\to 0$ uniformly on compact sets of Ω.

A natural problem is: what does happen to problem (1.1) for p between **1** and **2**?

We will consider this in Chapter two. In fact, we can get the results for p in all region, that is, $p\in(0,2)$. Let k be an integer, and define

$$\mathcal{M}=\left\{(\xi_1,\cdots,\xi_k)\in\Omega^k\ :\ \text{dist}(\xi_j,\partial\Omega)\geq\delta,\quad |\xi_i-\xi_j|\geq\delta\quad\text{for } i\neq j\right\}$$

for some $\delta>0$. Let $\varepsilon>0$ be a parameter, which depends on λ, defined as

$$p\lambda\left(-\frac{4}{p}\log\varepsilon\right)^{\frac{2(p-1)}{p}}\varepsilon^{\frac{2(p-2)}{p}}=1. \tag{1.13}$$

Observe that, as $\lambda\to 0$, then $\varepsilon\to 0$, and $\lambda=\varepsilon^2$ if $p=1$. We obtain the following result.

Theorem 1.6. *Let $0<p<2$ and k an integer with $k\geq 1$. If Ω is not simply connected, then there exists $\lambda_0>0$ so that, for any $0<\lambda<\lambda_0$ problem (1.1) has a solution u_λ. Moreover*

$$\lim_{\lambda\to 0}\varepsilon^{\frac{2(2-p)}{p}}\int_\Omega e^{u_\lambda^p}=8k\pi, \tag{1.14}$$

where ε satisfies (1.13). Furthermore, there exists a $k-$tuple $\xi^\lambda=(\xi^\lambda_1,\cdots,\xi^\lambda_k)\in\mathcal{M}$ such that as $\lambda\to 0$

$$\nabla\varphi_k(\xi^\lambda_1,\cdots,\xi^\lambda_k)\to 0,$$

and

$$u_\lambda(x)=p^{-\frac{1}{2}}\sqrt{\lambda}\,\varepsilon^{\frac{p-2}{p}}\left(\sum_{j=1}^{k}G_\Omega(x,\xi^\lambda_j)+o(1)\right) \tag{1.15}$$

where φ_k *defined as (1.4),* $G_\Omega(\cdot,\cdot)$ *given by (1.5), and* $o(1) \to 0$*, as* $\lambda \to 0$*, uniformly on each compact subset of* $\bar{\Omega} \backslash \{\xi_1^\lambda, \cdots, \xi_k^\lambda\}$*. Furthermore*

$$J_\lambda^p(u_\lambda) = \lambda \varepsilon^{\frac{2(p-2)}{p}} \left[\frac{8k\pi}{(2-p)p} [-2 + p \log 8] - \frac{16k\pi}{p} \log \varepsilon - \frac{4\pi}{2-p} \varphi_k(\xi^\lambda) + O(|\log \varepsilon|^{-1}) \right] \tag{1.16}$$

where $O(1)$ *uniformly bounded as* $\lambda \to 0$*.*

In [10, 36, 46], the authors considered the existence of bubbling solution to (1.3) in a smooth *bounded* domain in $\mathbb{R}^2$. In particular, the authors in [36] obtained that there exists a solution to (1.3) provided Ω is *not simply connected and bounded domain in* $\mathbb{R}^2$.

In Chapter three, we consider the existence of bubbling solution to (1.3) in an open, connected and unbounded domain in $\mathbb{R}^2$. We define the domain as follows.

Let $\varphi : \mathbb{R} \to [1, +\infty)$ be a smooth function, satisfying

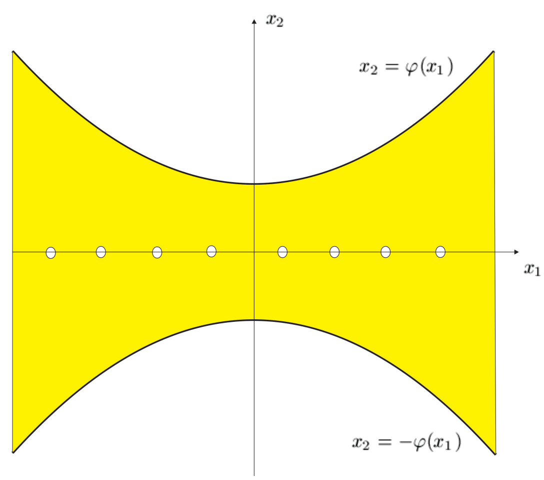
(a) $\varphi(0) = 1$, $x_1 \varphi'(x_1) > 0$ for $x_1 \neq 0$;

(b) $\varphi(x_1) \to +\infty$ as $x_1 \to \pm\infty$, and

(c) $\varphi'(x_1) \to a > 0$ as $x_1 \to +\infty$, and $\varphi'(x_1) \to b < 0$ as $x_1 \to -\infty$.

Define

$$\Omega = \{x = (x_1, x_2) \ : \ |x_2| < \varphi(x_1)\} \tag{1.17}$$



We observe that Ω is symmetric with respect to line $x_2 = 0$, and has two open directions. Moreover, the domain is not necessary symmetric with respect to $x_1 = 0$. We would like to construct bubbling solutions to problem (1.3) in the domain Ω, with the location of blow-up points on the symmetry line of Ω.

Let $\delta > 0$ small but fixed. Let $k > 1$ be an integer. Given k different points on the symmetry line of Ω, we write these points as

$$\xi_j = (t_j, 0), \quad j = 1, \dots, k, \tag{1.18}$$

with $t_1 < t_2 < \dots < t_k$, satisfies

$$t_{i+1} - t_i > \delta, \quad i = 1, 2, \dots, k-1. \tag{1.19}$$

We have the following result.

Theorem 1.7. *Let Ω be an open, connected and unbounded domain of $\mathbb{R}^2$ defined by (1.17), let $k > 1$ be an integer. For $\lambda > 0$ small enough, problem (1.3) has at least one solution u_λ, which blow-up at k points $\xi_1^*, \dots, \xi_k^*$ defined as (1.18), $\xi_j^* = (t_j^*, 0)$ and $\varphi_k(\xi^*) = \max \varphi_k(\xi)$ with φ_k defined by (1.4). Moreover,*

$$u_\lambda(x) = \sum_{j=1}^{k} G_\Omega(x; \xi_j^*) + o(1) \tag{1.20}$$

where $o(1) \to 0$, as $\lambda \to 0$, on each compact subset of $\bar{\Omega} \backslash \{\xi_1^*, \dots, \xi_k^*\}$, and $G_\Omega(\cdot;\cdot)$ is the Green's function in Ω with Dirchlet boundary condition, defined by (1.5).

1.2 Neumann problem in $\mathbb{R}^2$

Consider the following boundary value problem

$$\begin{cases} -\Delta u + u = \lambda u^{p-1} e^{u^p}, \quad u > 0, & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.21}$$

which is equivalent to the stationary Keller-Segel system from chemotaxis, where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter, $0 < p < 2$, and ν denotes the outer normal vector to $\partial\Omega$. This problem is the Euler-Lagrange equation for the functional

$$I_\lambda^p(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + u^2) - \frac{\lambda}{p} \int_\Omega e^{u^p}, \quad u \in H^1(\Omega). \tag{1.22}$$

If $p = 1$, Senba-Suzuki, in [109, 110], have analyzed the asymptotic behavior of solutions to problem (1.21). The blow-up for u takes place as a finite sum of Dirac measures at points with masses 8π or 4π, respectively, depending on whether they are located inside the domain or at the boundary. More precisely, if u_λ is a family of solutions to problem (1.21) when $p = 1$, then there exist non-negative integers $k, l \geq 1$, such that

$$\lim_{\lambda\to 0} \lambda \int_\Omega e^{u_\lambda} = 4\pi(2k + l). \tag{1.23}$$

Let $m = k + l$. Up to subsequences, there exist points ξ_j, $j = 1, \ldots, m$ with $\xi_j \in \Omega$ for $j \leq k$ and $\xi_j \in \partial\Omega$ for $k < j \leq m$, for which

$$u_\lambda(x) \to \sum_{j=1}^{k} 8\pi\tilde{G}(x, \xi_j) + \sum_{j=k+1}^{m} 4\pi\tilde{G}(x, \xi_j), \quad \text{as} \quad \lambda \to 0, \tag{1.24}$$

uniformly on compact subset of $\bar{\Omega}\backslash\{\xi_1, \ldots, \xi_m\}$. Moreover, the $m-$tuple $(\xi_1, \ldots, \xi_m)$ can be characterized as critical point of a functional defined on $\Omega^k \times (\partial\Omega)^l$, given by

$$\tilde{\varphi}_m(\xi) = \tilde{\varphi}_m(\xi_1, \ldots, \xi_m) = \sum_{j=1}^{m} c_j^2 \tilde{H}(\xi_j, \xi_j) + \sum_{l\neq j} c_l c_j \tilde{G}(\xi_l, \xi_j), \tag{1.25}$$

where

$$c_j = 8\pi \;\; \text{for} \;\; j = 1, \ldots, k, \quad \text{and} \;\; c_j = 4\pi \;\; \text{for} \;\; j = k+1, \ldots, m,$$

and $\tilde{G}(x, y)$ is the Green's function of the problem

$$\begin{cases} -\Delta_x \tilde{G}(x, y) + \tilde{G}(x, y) = \delta_y(x), & \text{in } \Omega; \\ \frac{\partial \tilde{G}(x,y)}{\partial \nu_x} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.26}$$

and $\tilde{H}(\cdot, \cdot)$ its regular part, namely,

$$\tilde{H}(x, y) = \begin{cases} \tilde{G}(x, y) + \frac{1}{2\pi} \log|x - y|, & \text{if } y \in \Omega; \\ \tilde{G}(x, y) + \frac{1}{\pi} \log|x - y|, & \text{if } y \in \partial\Omega. \end{cases} \tag{1.27}$$

Conversely, del Pino-Wei, in [41], constructed bubbling solutions u_λ to problem (1.21) when $p = 1$ with the above properties (1.23) and (1.24). Moreover, the location of the bubbling points corresponds to critical points of the function φ_m defined by (1.25).

Question: Does exist blow-up solutions to (1.21) for p between **0** and **2**, such that that solution blow-up inside the domain and on the boundary?

In Chapter four, we will give a positive answer fot this. Namely, we construct bubbling solutions to problem (1.21) with bubbling profiles at points inside Ω and on the boundary of Ω when p is between 0 and 2. In particular, we recover the result in [41] when $p = 1$.

Theorem 1.8. *Let $0 < p < 2$, and $k, l, m \geq 1$ be integers with $m = k + l$. There exists $\lambda_0 > 0$ so that, for any $0 < \lambda < \lambda_0$, problem (1.21) has a solution u_λ, with the following properties:*

(1) u_λ has m local maximum points ξ_j^*, $j = 1, \ldots, m$ such that $\xi_j^* \in \Omega$ for $1 \leq j \leq k$, and $\xi_j^* \in \partial\Omega$ for $k+1 \leq j \leq m$. Furthermore

$$\lim_{\lambda \to 0} \tilde{\varphi}_m(\xi_1^*, \ldots, \xi_m^*) = \min_{\Omega^k \times (\partial\Omega)^l} \tilde{\varphi}_m,$$

where $\tilde{\varphi}_m$ is defined by (1.25). In particular

(2) One has

$$u_\lambda(x) = p^{-\frac{1}{2}} \sqrt{\lambda}\, \varepsilon^{\frac{p-2}{p}} \left[\sum_{j=1}^{k} 8\pi \tilde{G}(x, \xi_j^*) + \sum_{j=k+1}^{m} 4\pi \tilde{G}(x, \xi_j^*) + o(1) \right] \tag{1.28}$$

where ε satisfies (1.13), and $o(1) \to 0$, as $\lambda \to 0$, on each compact subset of $\bar{\Omega} \backslash \{\xi_1^*, \ldots, \xi_m^*\}$, and $\tilde{G}(\cdot, \cdot)$ is the Green's function given in (1.26).

(3) Moreover

$$\lim_{\lambda \to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_\Omega e^{u_\lambda^p} = 4\pi(2k + l). \tag{1.29}$$

Furthermore

$$I_\lambda^p(u_\lambda) = \lambda \varepsilon^{\frac{2(p-2)}{p}} \left[-4\pi(2k+l) \frac{2 - p\log 8}{(2-p)p} - \frac{8\pi}{p}(2k+l)\log\varepsilon - \frac{1}{2(2-p)} \varphi_m(\xi^*) + O(|\log\varepsilon|^{-1}) \right] \tag{1.30}$$

where $O(1)$ uniformly bounded as $\lambda \to 0$.

Consider the following Neumann boundary value problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = \lambda u^{p-1} e^{u^p} & \text{on } \partial\Omega, \end{cases} \tag{1.31}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, ν is the outer normal vector of $\partial\Omega$, $\lambda > 0$ is a small parameter and $0 < p \leq 2$. This problem is the Euler-Lagrange equation for the functional $J_{N,\lambda}^p : H^1(\Omega) \to \mathbb{R}$ defined as

$$J_{N,\lambda}^p(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + u^2 \right) - \frac{\lambda}{p} \int_{\partial\Omega} e^{u^p}.$$

In [27], Dávila-del Pino-Musso have analyzed the asymptotic behavior of solution to problem (1.31) when $p = 1$. Namely, they considered the following problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = \lambda e^u & \text{on } \partial\Omega. \end{cases} \tag{1.32}$$

Suppose that u_λ is a family solution of (1.32), with the property $\lambda \int_{\partial\Omega} e^{u_\lambda}$ bounded, then there is an integer $k \geq 1$, up to subsequences, such that

$$\lim_{\lambda\to 0} \lambda \int_{\partial\Omega} e^{u_\lambda} = 2k\pi. \tag{1.33}$$

Moreover, there are k distinct points ξ_j, $j = 1, \ldots, k$, on the boundary of Ω, such that λe^{u_λ} approaches the sum of k Dirac masses at these points ξ_j. The location of points can be characterized as critical points of a functional of k points of the boundary given by

$$\varphi_{N,k}(\xi_1, \ldots, \xi_k) = -\left[\sum_{j=1}^{k} H_N(\xi_j, \xi_j) + \sum_{l\neq j} G_N(\xi_l, \xi_j)\right], \tag{1.34}$$

where $G_N(x, y)$ is Green's function of the problem

$$\begin{cases} -\Delta_x G_N(x, y) + G_N(x, y) = 0 & x \in \Omega; \\ \frac{\partial G_N(x,y)}{\partial \nu_x} = 2\pi\delta_y(x) & x \in \partial\Omega, \end{cases} \tag{1.35}$$

and H_N its regular part

$$H_N(x, y) = G_N(x, y) - 2\log\frac{1}{|x - y|}. \tag{1.36}$$

The authors in [27] also described the existence of solution with above properties. More precisely, if $\partial\Omega$ has more than one component, they showed that the function φ_k has *topologically nontrivial* critical point $(\xi_1, \ldots, \xi_k)$, then there is a family solution to problem (1.32) with peaks at these points.

In chapter five, we will consider the existence of solution to (1.31) when $0 < p < 2$. Let ε be a parameter, which depends on λ, satisfies,

$$p\lambda\left(-\frac{2}{p}\log\varepsilon\right)^{\frac{2(p-1)}{p}} \varepsilon^{\frac{p-2}{p}} = 1. \tag{1.37}$$

Observe that, as $\lambda \to 0$, then $\varepsilon \to 0$, and $\varepsilon = \lambda$ if $p = 1$. We have

Theorem 1.9. *For $0 < p < 2$, let $k \geq 1$, assume that φ_k defined by (1.34) has a C^0-stable critical point $\xi^* = (\xi_1^*, \ldots, \xi_k^*) \in (\partial\Omega)^k$ with*

$$|\xi_l^* - \xi_j^*| > \delta, \quad \textit{for } \; l \neq j,$$

for some small but fixed number $\delta > 0$. Then the problem (1.31) has a family solutions u_λ for λ small enough, such that

$$\lim_{\lambda\to 0} \varepsilon^{\frac{2-p}{p}} \int_{\partial\Omega} e^{u_\lambda^p} = 2k\pi, \tag{1.38}$$

where ε satisfies (1.37). Moreover, for $\lambda \to 0$

$$\nabla\varphi_{N,k}(\xi_1^*, \dots, \xi_k^*) = 0,$$

and

$$u_\lambda(x) = p^{-\frac{1}{2}} \sqrt{\lambda}\, \varepsilon^{\frac{p-2}{2p}} \left[\sum_{j=1}^{k} G_N(x, \xi_j^*) + o(1) \right] \tag{1.39}$$

where $o(1) \to 0$ on each compact subset of $\bar{\Omega}\backslash\{\xi_1^*, \dots, \xi_k^*\}$, $G_N(\cdot,\cdot)$ defined by (1.35). Furthermore

$$J_{N,\lambda}^p(u_\lambda) = \lambda\varepsilon^{\frac{p-2}{p}} \left[-\frac{2k\pi}{p} + \frac{2k\pi}{p} \log\frac{1}{\varepsilon} + \frac{\pi}{2-p}\varphi_k(\xi) + O(|\log\varepsilon|^{-1}) \right] \tag{1.40}$$

where $O(1)$ uniformly bounded as $\lambda \to 0$.

If $p = 2$, problem (1.31) becomes

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = \lambda u e^{u^2} & \text{on } \partial\Omega, \end{cases} \tag{1.41}$$

For functions $u \in H^1(\Omega)$, due to the Trudinger trace embedding (in the sense of Orlicz spaces) [103, 114]

$$H^1(\Omega) \ni u \longmapsto e^{u^2} \in L^s(\partial\Omega) \quad \forall\, s \geq 1.$$

This optimal embedding is related to the critical Trudinger-Moser trace inequality

$$C_\pi(\Omega) = \sup\left\{ \int_{\partial\Omega} e^{\pi u^2} \ / \ u \in H^1(\Omega),\ \int_\Omega [|\nabla u|^2 + u^2] = 1 \right\} \ < \ +\infty \ ,$$

[74]. It has been proven [124] that for any bounded domain Ω in $\mathbb{R}^2$, with smooth boundary, the supremum $C_\pi(\Omega)$ is attained by a function $u \in H^1(\Omega)$ with $\int_\Omega [|\nabla u|^2 + u^2] = 1$. Furthermore, for any $\alpha \in (0, \pi)$, the supremum $C_\alpha(\Omega)$ is finite and it is attained, while $C_\alpha(\Omega) = \infty$ as soon as $\alpha > \pi$. See also [24, 72, 73, 75] for generalizations.

In Chapter six, we construct bubbling solutions to problem (1.41). To state our result, let us introduce the following function $\overline{\varphi}_k : (\partial\Omega)^k \times (\mathbb{R}^+)^k \to \mathbb{R}$, $\overline{\varphi}_k(\xi, m) = \varphi_k(\xi_1, \dots, \xi_k, m_1, \dots, m_k)$ defined by

$$\overline{\varphi}_k(\xi, m) = 2(\log 2 - 1) \sum_{j=1}^{k} m_j^2 + 2\sum_{j=1}^{k} m_j^2 \log(m_j^2) - \sum_{j=1}^{k} m_j^2 H_N(\xi_j, \xi_j) - \sum_{i\neq j} m_i m_j G_N(\xi_i, \xi_j), \tag{1.42}$$

where G_N is the Green function for the Neumann problem defined by (1.35).

Theorem 1.10. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and let $k \geq 1$ be an integer. Then, for all small $\lambda > 0$ there exists a pair solution u_λ^1, u_λ^2 of problem (1.41) such that*

$$\frac{1}{2}\int_\Omega \left[|\nabla u_\lambda^i|^2 + (u_\lambda^i)^2\right] - \frac{\lambda}{2}\int_{\partial\Omega} e^{(u_\lambda^i)^2} \;=\; \frac{k}{2}\pi \,+\, o(1) \quad i = 1,2$$

where $o(1) \to 0$ as $\lambda \to 0$. Moreover, for any $i = 1,2$, passing to a subsequence, there exists $(\xi^i, m^i) = (\xi_1^i, \ldots, \xi_k^i, m_1^i, \ldots, m_k^i) \in (\partial\Omega)^k \times (\mathbb{R}^+)^k$, with $\xi^1 \neq \xi^2$, such that $\nabla\overline{\varphi}_k(\xi^i, m^i) = 0$ and

$$u_\lambda(x) \;=\; \sqrt{\lambda}\left(\sum_{j=1}^{k} m_j^i G_N(x, \xi_j^i) \;+\; o(1)\right) \tag{1.43}$$

where $o(1) \to 0$ on each compact subset of $\bar{\Omega} \setminus \{\xi_1^i, \ldots, \xi_k^i\}$.

In Chapter seven, we study the existence of critical points of the Trudinger-Moser trace functional

$$E_\alpha(u) = \int_{\partial\Omega} e^{\alpha u^2}, \tag{1.44}$$

constrained to functions

$$u \in M := \left\{u \in H^1(\Omega) \;:\; \|u\|^2 = 1\right\} \tag{1.45}$$

in the super critical regime

$$\alpha > \pi.$$

We will get the following results.

Theorem 1.11. *Let Ω be a bounded domain in $\mathbb{R}^2$. Then there exists $\alpha_0 > \pi$, such that for any $\alpha \in (0, \alpha_0)$, there exists a function $u_\alpha \in M$ which locally maximizes of E_α on M.*

Moreover, we will show the existence of critical points for E_α constrained to M, for $\alpha \in (k\,\pi, \alpha_k)$, for any $k \geq 1$ integer and for some α_k slightly to the right of $k\,\pi$.

Theorem 1.12. *Let Ω be any bounded domain in $\mathbb{R}^2$ with smooth boundary. Fix a positive integer $k \geq 1$. Then there exists $\alpha_k > k\pi$ such that for $\alpha \in (k\pi, \alpha_k)$, the functional $E_\alpha(u)$ restricted to M has at least two critical points u_α^1 and u_α^2. Furthermore, for any $i = 1,2$ there exist numbers $m_{j,\alpha}^i > 0$ and points $\xi_{j,\alpha}^i \in \partial\Omega$, for $j = 1, \ldots, k$ such that*

$$\lim_{\alpha \to k\,\pi} m_{j,\alpha}^i = m_j^i \in (0, \infty), \tag{1.46}$$

$$\xi_{j,\alpha}^i \to \xi_j^i \in \partial\Omega, \quad \textit{with} \quad \xi_j^i \neq \xi_l^i \quad \textit{for} \quad j \neq l, \quad \textit{as} \quad \alpha \to k\,\pi \tag{1.47}$$

and

$$u_\alpha^i(x) = \sqrt{\frac{\alpha - k\pi}{\alpha}} \sum_{j=1}^{k} \left[m_{j,\alpha}^i G_N(x, \xi_{j,\alpha}^i) + o(1)\right], \quad i = 1,2, \tag{1.48}$$

where $o(1) \to 0$ *uniformly on compact sets of* $\bar{\Omega}\backslash\{\xi_1^i, \ldots, \xi_k^i\}$, *as* $\alpha \to k\pi$. *Moreover, for any* $i = 1, 2$, *for any* $\delta > 0$ *small, for any* $j = 1, \ldots, k$,

$$\sup_{x \in B(\xi_j^i, \delta)} u_\alpha^i(x) \to +\infty, \quad as \quad \alpha \to k\,\pi. \tag{1.49}$$

1.3 Elliptic equations on Riemannian manifolds

Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $n \geq 3$. In Chapter seven, we are concerned with the following asymptotically critical elliptic problem

$$\Delta_g u + a(x)u = u^{2^*-1-\varepsilon}, \qquad u > 0 \ \text{ in } \ \mathcal{M}, \tag{1.50}$$

where $\Delta_g = -div_g(\nabla)$ is the Laplace-Beltrami operator on $\mathcal{M}$, $a(x)$ is a $\mathcal{C}^1$ function on $\mathcal{M}$, $2^* = \frac{2n}{n-2}$ denotes the Sobolev critical exponent, ε is a small real parameter such that ε goes to 0.

Recently, nonlinear elliptic equations on compact Riemannian manifold have been brought much attention. Consider the following problem

$$\varepsilon^2 \Delta_g u + u = |u|^{p-2}u \quad \text{in} \ \ \mathcal{M}, \tag{1.51}$$

where $(\mathcal{M}, g)$ is a compact, connected, Riemannian manifold of class C^∞ with Riemannian metric g, $\dim \mathcal{M} = n \geq 3$, $2 < p < 2^*$ and ε is a positive parameter. In [17], the authors proved that the problem (1.51) has a mountain pass solution u_ε which exhibits a spike layer. In particular, they proved that the maximum point of u_ε converges to a maximum point of the scalar curvature Scal_g as ε goes to zero. Multiple solutions were obtained in [12] for the problem (1.51), the authors showed that multiplicity of solutions to (1.51) depends on the topological properties of the manifold $\mathcal{M}$. More precisely, they showed that problem (1.51) has at least $cat(\mathcal{M}) + 1$ nontrivial solutions provided ε is small enough. Here $cat(\mathcal{M})$ denotes the Lusternik-Schnirelmann category of $\mathcal{M}$. While for zero mass case, similar result was obtained in [117]. And in [65] the author constructed an interesting example of two manifolds having the same topology, for which the number of solutions to the problem (1.51) is different.

In [87] the authors showed that for any stable critical point of the scalar curvature it is possible to construct a single peak solution, whose peak approaches such a point as ε goes to zero. In [26] the authors proved that for any fixed positive integer k, problem (1.51) has a $k-$peak solution, whose peaks collapse, as ε goes to zero, to an isolated local minimum point of the scalar curvature. Recently in [89] the authors proved that the existence of positive or sign changing multi-peak solutions of (1.51), whose both positive and negative peaks approach different stable critical points of the scalar curvature as ε goes to zero.

The asymptotically critical case on Riemannian manifold in [90] the authors proved problem (1.50) exists blowing-up families of positive solutions provide the graph of $a(x)$ is distinct at some point from the graph of $\frac{n-2}{4(n-1)}\text{Scal}_g$.

If $a \equiv \frac{n-2}{4(n-1)}\text{Scal}_g$, problem (1.50) is the intensively studied Yamabe problem

$$\Delta_g u + \frac{n-2}{4(n-1)}\text{Scal}_g u = u^{2^*-1-\varepsilon} \quad \text{in} \ \ \mathcal{M} \quad u > 0 \ \text{ in } \ \mathcal{M}, \tag{1.52}$$

is just the so called prescribed scalar curvature problem with $\varepsilon = 0$. The existence of a conformal metric with constant scalar curvature on compact Riemannian manifolds was studied by Yamabe [116], Trudinger [115], Aubin [8] and Schoen [108]. If u is a solution, then $\frac{4(n-1)}{n-2}$ is the scalar curvature of the conformal metric $\tilde{g} = u^{\frac{1}{n-2}} g$. On the compact manifold $(\mathcal{M}, g)$, the coercivity of the operator $\Delta_g + a$ is a necessary condition for the existence of a solution to problem (1.52). In [43] the author consider (1.13) with $\varepsilon \geq 0$, for any smooth, compact Riemannian manifold of dimensional $n \geq 3$ and any smooth function a on $\mathcal{M}$ such that $\Delta_g + a$ is coercive and $a(\xi) < \frac{n-1}{4(n-2)}\text{Scal}_g(\xi)$, then (1.50) exists a solution.

In order to state our main result, it is useful to recall some definitions and results. First, Let us introduce the definition of $\mathcal{C}^1$ stable critical set.

Definition 1.13. *([69]) Let $f \in \mathcal{C}^1(\mathcal{M}, \mathbb{R})$, for any given integer $k \geq 2$, set $\bar{\xi} = (\xi_1, \xi_2, \cdots, \xi_k)$, let $C_1, C_2, \cdots, C_k \subset \mathcal{M}$ be k mutually disjoint closed subsets of critical points of f, we say that $(C_1, C_2, \cdots, C_k) \subset \mathcal{M}^k$ is a $\mathcal{C}^1$ stable critical set of function $F(\bar{\xi}) := \sum_{j=1}^{k} f(\xi_j)$, if for any $\sigma > 0$ there exists $\gamma > 0$ such that if $\Phi \in \mathcal{C}^1(\mathcal{M}^k, \mathbb{R})$ with*

$$\max_{d_g(\xi_j, C_j) \leq \sigma, 1 \leq j \leq k} \left(|F(\bar{\xi}) - \Phi(\bar{\xi})| + |\nabla_g F(\bar{\xi}) - \nabla_g \Phi(\bar{\xi})| \right) \leq \gamma,$$

then Φ has at least one critical point $\bar{\xi}$ in $\mathcal{M}^k$ with $d_g(\xi_j, C_j) \leq \sigma$.

Next, we introduce the following equation which correspond to limiting equation to problem (1.50).

$$\Delta U = U^{2^*-1} \qquad \text{in} \quad \mathbb{R}^n, \tag{1.53}$$

where $\Delta = -div(\nabla)$ is the Laplace-Beltrami operator associated with the Euclidean metric. It is known that [8, 115] the functions $\lambda^{(2-n)/2} U(\lambda^{-1} z)$ satisfy equation (1.53), where

$$U(z) = U(|z|) = \left(\frac{\sqrt{n(n-2)}}{1+|z|^2} \right)^{(n-2)/2}. \tag{1.54}$$

Let us define a smooth cut-off function χ_r satisfies

$$\chi_r(z) := \begin{cases} 1 & \text{if} \quad z \in B(0, \frac{r}{2}); \\ \in (0,1) & \text{if} \quad z \in B(0,r) \backslash B(0, \frac{r}{2}); \\ 0 & \text{if} \quad z \in \mathbb{R}^n \backslash B(0,r), \end{cases} \tag{1.55}$$

and $|\nabla\chi_r(z)| \le \frac{2}{r}$, $|\nabla^2\chi_r(z)| \le \frac{2}{r^2}$. For any point ξ in $\mathcal{M}$ and for any positive real number λ, we define the function $W_{\lambda,\xi}$ on $\mathcal{M}$ by

$$W_{\lambda,\xi}(x) := \begin{cases} \chi_r\left(\exp_\xi^{-1}(x)\right)\lambda^{\frac{2-n}{2}}U\left(\lambda^{-1}\exp_\xi^{-1}(x)\right) & \text{if } x \in B_g(\xi,r); \\ 0 & \text{otherwise.} \end{cases} \tag{1.56}$$

We assume that the operator $\Delta_g + a$ is coercive, we can provide the Hilbert space $H_g^1(\mathcal{M})$ with the inner product

$$\langle u,v\rangle_a = \int_{\mathcal{M}} \left(\langle\nabla u,\nabla v\rangle_g + a(x)uv\right)\, d\mu_g,$$

which induces the norm

$$\|u\|_a^2 = \int_{\mathcal{M}} \left(|\nabla_g u|^2 + a(x)u^2\right)\, d\mu_g.$$

Let

$$\psi(\xi) = a(\xi) - \frac{n-1}{4(n-2)}\mathrm{Scal}_g(\xi). \tag{1.57}$$

In Chapter eight, we construct a family of solutions of equation (1.50), whose peaks approach different stable critical points of $\psi(\xi)$ with ε small enough, which blow-up and concentrate at some points in $\mathcal{M}$, in the sense of the following definition.

Definition 1.14. *For $k \ge 2$ be a positive integer, let u_ε be a family of solution of (1.50), we say that u_ε blow-up and concentrates at point $\bar\xi^0 = (\xi_1^0,\cdots,\xi_k^0) \in \mathcal{M}^k$ if there exist $\bar\xi^\varepsilon = (\xi_1^\varepsilon,\cdots,\xi_k^\varepsilon) \in \mathcal{M}^k$ and $(\lambda_1(\varepsilon),\cdots,\lambda_k(\varepsilon)) \in (\mathbb{R}^+)^k$ with $\lambda_j(\varepsilon) > 0$ such that*

$$\xi_j^\varepsilon \to \xi_j^0, \qquad \lambda_j(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{for} \quad j = 1,2,\cdots,k.$$

and

$$\left\| u_\varepsilon - \sum_{j=1}^k W_{\lambda_j(\varepsilon),\xi_j^\varepsilon} \right\|_a \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Our main result is the following.

Theorem 1.15. *Let $(\mathcal{M},g)$ be a smooth compact Riemannian manifold of dimension $n \ge 6$, let $a(x)$ be a $\mathcal{C}^1$ positive function on $\mathcal{M}$ such that the operator $\Delta_g + a$ is coercive, and for any given integer $k \ge 2$, set $\bar\xi^0 = (\xi_1^0,\cdots,\xi_k^0)$, let ξ_j^0 be an isolated critical point of $\psi(\xi_j)$ with $\deg(\nabla_g\psi, B_g(\xi_j^0,\varrho),0) \ne 0$ for some $\varrho > 0$ and $j = 1,\cdots,k$, we have*

(i) If $\psi(\xi_j^0) > 0$ and ε is small enough, there exists a family of solutions of the subcritical problem, which blow-up and concentrates at $\bar\xi^0$.

(ii) If $\psi(\xi_j^0) < 0$ and ε is small enough, there exists a family of solutions of the supercritical problem, which blow-up and concentrates at $\bar\xi^0$.

In Chapter nine, Let (M,g) be a smooth, compact Riemannian manifold of dimension $n \geq 7$. We consider the Paneitz-Branson type equation

$$\Delta_g^2 u - \operatorname{div}_g(Adu) + au = |u|^{2^\sharp - 2 - \varepsilon} u, \quad \text{in} \ \ M,$$

where $\Delta_g = -\operatorname{div}_g \nabla$ is the Laplace-Beltrami operator, A is a smooth symmetrical $(2,0)$-tensor fields, a is a smooth function on M, $2^\sharp = \frac{2n}{n-4}$ is the critical exponent for the Sobolev embedding and ε is a small positive parameter.

In 1983 Paneitz [102] introduced a conformally fourth order operator defined on 4-dimensional Riemannian manifolds. Branson [14] generalized the definition to $n-$dimensional Riemannian manifolds.

We let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 5$. We also let $H_2^2(M)$ be the Sobolev space consisting of functions in $L^2(M)$ with two derivatives in $L^2(M)$. We consider the geometric Paneitz equation

$$P_g^n u = |u|^{2^\sharp - 2} u \quad \text{in } M. \tag{1.58}$$

Here $2^\sharp = \frac{2n}{n-4}$ is the critical exponent for the Sobolev embedding, P_g^n is the Paneitz-Branson operator which is given by

$$P_g^n u = \Delta_g^2 u - \operatorname{div}_g(A_g du) + \frac{n-4}{2} Q_g u \tag{1.59}$$

where $\Delta_g = -\operatorname{div}_g \nabla$ is the Laplace-Beltrami operator, Q_g is the $Q-$curvature of g, A_g is the smooth symmetrical $(2,0)-$tensor field

$$A_g = \frac{(n-2)^2+4}{2(n-1)(n-2)} S_g g - \frac{4}{n-2} Rc_g, \tag{1.60}$$

where Rc_g and S_g are respectively the Ricci curvature and the Scalar curvature of g.

The Paneitz operator is conformally invariant in the sense that if $\tilde{g} = \phi^{\frac{4}{n-2}} g$ is conformal to g then $P_{\tilde{g}}^n u = \phi^{-\frac{n+4}{n-4}} P_g^n(\phi u)$ for any $u \in C^\infty(M)$. From the viewpoint of conformal geometry equation (1.58) turns out to be the natural fourth order analogue of the second order Yamabe problem. That is why we are led to study extensions to this operator of some classical problems.

Using a terminology introduced by Hebey, we refer to a *Paneitz-Branson type operator with general coefficients* as an operator of the form

$$P_g u = \Delta_g^2 u - \operatorname{div}_g(Adu) + au \tag{1.61}$$

where $A \in \Lambda_{(2,0)}^\infty(M)$ is a smooth symmetric $(2,0)-$tensor field and $a \in C^\infty(M)$ and we refer to *Paneitz-Branson type operator with constant coefficients* as an operator of the form

$$P_g u = \Delta_g^2 u + b\Delta_g u + cu \tag{1.62}$$

where b and c are real numbers.

The Paneitz-Branson operator (1.59) is as in (1.61) whatever (M, g) is. In particular, when (M, g) is Einstein, i.e. $Rc_g = \lambda g$ for some $\lambda \in \mathbb{R}$, the Paneitz-Branson operator (1.59) has constant coefficients as in (1.62) with $b = \frac{n^2-2n-4}{2(n-1)}\lambda$ and $c = \frac{n(n-4)(n^2-4)}{16(n-1)^2}\lambda^2$.

Equation

$$P_g u = \Delta_g^2 u + b\Delta_g u + cu = |u|^{2^\sharp-2}u \quad \text{in } M, \tag{1.63}$$

when P_g is a Paneitz-Branson type operator with constant coefficients as in (1.62), was widely studied. Examples of compact manifolds including locally conformally flat manifold for which equations (1.63) have non constant solutions are in [42, 51]. Compactness of problem (1.63) was studied in [61, 62, 63, 64]. Recently, in [63] Hebey and Robert also studied the stability of problem (1.63). They introduce the following definition of stability. Equation (1.63) is said to be *stable* if for any sequences $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ of real numbers converging to b and c and for any sequence $(u_\alpha)_\alpha$ of solutions to

$$\Delta_g^2 u_\alpha + b_\alpha \Delta_g u_\alpha + c_\alpha u_\alpha = |u_\alpha|^{2^\sharp-2}u_\alpha \quad \text{in } M,$$

bounded in $H_2^2(M)$, there holds that, up to a subsequence, $u_\alpha \to u$ in $C^4(M)$ where u is a smooth solution of (1.63). In other words, problem (1.63) is stable if arbitrary bounded sequences in $H_2^2(M)$ of solutions of equations close to (1.63) do not blow up in one or more points of the manifold. In particular, they prove that if (M, g) is locally conformally flat and the Paneitz-Branson type operator is coercive then problem (1.63) is stable provided $b \neq \frac{1}{n}Tr_g A_g$ if $n \geq 9$ or $n = 7$ and $b < \frac{1}{8}Tr_g A_g$ if $n = 8$. Here and in what follows, if A denotes a smooth $(2,0)-$tensor field, we let $Tr_g A = g^{ij}A_{ij}$ be the trace of A with respect to g. It is easily seen that if A_g is defined in (1.60) then

$$Tr_g A_g = \frac{n^2-2n-4}{2(n-1)}S_g. \tag{1.64}$$

As far as we know, a few results are known about problem

$$P_g u = \Delta_g^2 u - \text{div}_g\,(Adu) + au = |u|^{2^\sharp-2}u \quad \text{in } M, \tag{1.65}$$

when P_g is a Paneitz-Branson type operator with general coefficients as in (1.61). In [50] among other existence results, Esposito and Robert proved that problem (1.65) when $n \geq 8$ has a non constant solution provided $\min_M Tr_g(A - A_g) < 0$. In [108] Sandeep proved that problem (1.65) is stable provided $A - A_g$ is either positive or negative definite. We would like to point out that in the quoted results the quantity $Tr_g A$ plays a crucial role in studying existence of solutions and stability of problems (1.63) and (1.65).

We will show how stability of the problem (1.65) actually depends on the trace of A_g. In particular, by building blowing-up solutions of the slightly subcritical problem (1.66), we will show that problem (1.65) is not stable if $\max_M Tr_g(A - A_g) > 0$ and $n \geq 8$ or if $\min_M Tr_g(A - A_g) > 0$ and $n \geq 7$.

More precisely, we consider the following Paneitz-Branson type equation with slightly sub-critical growth

$$\Delta_g^2 u - \operatorname{div}_g((A_g + B)du) + au = |u|^{2^\sharp - 2 - \varepsilon}u, \text{ in } M, \tag{1.66}$$

where A_g is given in (1.60), $B \in \Lambda^\infty_{(2,0)}(M)$ is a smooth symmetric $(2,0)-$tensor field, $a \in C^\infty(M)$ and ε is a small positive parameter.

Let $P_{g,B}(u) := \Delta_g^2 u - \operatorname{div}_g((A_g + B)u) + au$. We will assume that $P_{g,B}$ is coercive, i.e. there exists $c > 0$ such that

$$\int_M (P_{g,B}u)u d\mu_g \geq c \int_M u^2 d\mu_g \quad \text{for any } u \in H_2^2(M).$$

Coercivity was studied in [61].

Given a C^1-function φ on M, we say that a critical point ξ_0 of φ is $\mathcal{C}^1-$stable if there exists an open neighborhood Ω of ξ_0 such that for any point $\xi \in \overline{\Omega}$ there holds $\nabla\varphi(\xi) = 0$ if and only if $\xi = \xi_0$ and such that the Brouwer degree

$$\deg(\nabla_g \varphi, \Omega, 0) \neq 0.$$

If ξ_0 is a strict local minimum point or a strict local maximum point of φ then ξ_0 is a $\mathcal{C}^1-$stable critical point of φ. Moreover, if φ is a $\mathcal{C}^2-$function on M, then any non degenerate critical point of φ is $\mathcal{C}^1-$stable.

Theorem 1.16. *Assume*

- *$n \geq 8$ and ξ_0 is a $\mathcal{C}^1-$stable critical point of $Tr_g B$ with $Tr_g B(\xi_0) > 0$,*
- *$n \geq 7$, $Tr_g B$ is not constant and $\min\limits_M Tr_g B > 0$.*

Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ equation (1.66) admits a solution u_ε such that the family $(u_\varepsilon)_\varepsilon$ is bounded in $H_2^2(M)$ and the $u_\varepsilon' s$ blow up at the point ξ_0 if $n \geq 8$ or at a global minimum point of $Tr_g B$ if $n = 7$, as ε goes to zero.

We will use a very well known Lyapunov-schmidt reduction method to construct bubbling solutions for the above problems, which was introduced in [9, 52] and already used in many different contexts, see for instance [27, 28, 29, 36, 39, 40, 49, 46, 47, 48, 49] for Dirichlet problem in $\mathbb{R}^2$, [19, 27, 41, 93] for Neumann problem in $\mathbb{R}^2$, and in [11, 25, 37, 38, 58, 91, 92] considered the multi-peak solutions involving the critical Sobolev exponent. In [34, 35] considered the super-critical case, and in [70, 88, 57, 59, 121, 122, 123] considered a peak or multi-peak solutions to a singularly perturbed Neumann problem. In [87, 89, 90] considered elliptic equations on Riemannian manifold, and so on. The main idea is to try to guess the form of the solution (using the shape of the "standard bubble"), then linearize the equation at this approximate solution and use a Lyapunov-Schmidt reduction to arrive at a reduced finite dimensional variational problem, whose critical points yield actual solutions of the equation.

Let us just mention that through out the thesis, the symbol C denotes a generic positive constant independent of the small parameters, it could be changed from one line to another. The symbols $O(t)$ (respectively $o(t)$) will denote quantities for which $\frac{O(t)}{|t|}$ stays bounded (respectively, $\frac{o(t)}{|t|}$ tends to zero) as the small parameter goes to zero. In particular, we will often use the notation $o(1)$ stands for a quantity which tends to zero as the small parameter goes to zero.

Chapter 2

Bubbling solutions for an exponential nonlinearity in $\mathbb{R}^2$

2.1 Introduction

In this Chapter, we consider the following boundary value problem

$$\begin{cases} \Delta u + \lambda u^{p-1} e^{u^p} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter and $0 < p < 2$. This problem is the Euler-Lagrange equation for the functional

$$J_\lambda^p(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{\lambda}{p} \int_\Omega e^{u^p}, \quad u \in H_0^1(\Omega). \tag{2.2}$$

If $p = 1$, the problem (2.1) becomes

$$\begin{cases} \Delta u + \lambda e^{u} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

which can be called the Liouville equation after [78]. This problem is related to Berger's problem concerning the existence of metrics in a given Riemannian surface with prescribed Gaussian curvatures. We refer the reader to the book of T. Aubin [7] for the description of the links between this equation and possible geometric applications.

There are many results about the behavior and existence of solution to (2.3). Thanks to the works of H. Brezis and F. Merle [15], Y. Y. Li and I. Shafrir [71], L. Ma and J. Wei [83],

K. Nagasaki and T. Suzuki [95], the asymptotic behavior of solutions to problem (2.3) has been well understood. More precisely, it is by now known that if u_λ is an unbounded family of solutions to (2.1) for which $\lambda\int_\Omega e^{u_\lambda}$ remains uniformly bounded, then there is an integer $k\geq 1$, such that necessarily

$$\lim_{\lambda\to 0}\lambda\int_\Omega e^{u_\lambda}=8k\pi. \tag{2.4}$$

Moreover, there are k distinct points ξ_j, $j=1,\cdots,k$, in Ω, separated uniformly from each other and from he boundary $\partial\Omega$, such that, as $\lambda\to 0$, u_λ peaks to infinity in each one of them, and remains bounded away from them, that is, the solutions u_λ to problem (2.3) remain uniformly bounded on $\Omega\backslash\bigcup_{j=1}^{k}B_\delta(\xi_j)$ and

$$\sup_{B_\delta(\xi_j)} u_\lambda\to+\infty,\qquad \text{as}\ \ \lambda\to 0,$$

for any $\delta>0$. The location of the blow-up points $\xi_1,\cdots,\xi_k$ is such that, after passing to a subsequence, converges to a critical point of the function

$$\varphi_k(\xi_1,\cdots,\xi_k)=\sum_{j=1}^{k}H_\Omega(\xi_j,\xi_j)+\sum_{i\neq j}G_\Omega(\xi_i,\xi_j), \tag{2.5}$$

where $G_\Omega(x,y)$ is the standard Green's function of the problem

$$\begin{cases}-\Delta_x G_\Omega(x,y)=8\pi\delta_y(x), & x\in\Omega;\\ G_\Omega(x,y)=0, & x\in\partial\Omega,\end{cases} \tag{2.6}$$

and $H_\Omega(\cdot,\cdot)$ its regular part defined as

$$H_\Omega(x,y)=G_\Omega(x,y)-4\log\frac{1}{|x-y|}. \tag{2.7}$$

Conversely, many authors constructed blow-up solutions to problem (2.3) with property (2.4). In [10], S. Baraker and F. Pacard considered problem (2.3) in an open bounded subset Ω of $\mathbb{C}$, and they showed that given a non-degenerate critical point $(\xi_1,\ldots,\xi_k)$ of the function φ_k defined in (2.5), there is a sequence u_λ of solutions to (2.3), that converges to a function u^* in $C^{2,\alpha}_{loc}(\Omega\backslash\{\xi_1,\cdots,\xi_k\})$, where u^* is the solution of

$$\begin{cases}-\Delta u^*=\sum\limits_{j=1}^{k}8\pi\delta_{\xi_j}, & \text{in }\Omega;\\ u^*=0 & \text{on }\partial\Omega.\end{cases}$$

P. Esposito, M. Grossi, A. Pistoia [46] generalized this result relaxing the assumption of non degenerate critical point for φ_k to that of *stable critical point* for φ_k. By using the notion of *topologically non trivial critical value* for φ_k, that we will recall later on, M. del Pino, M. Kowalczyk, M. Musso [36] could establish the following general result: *If the domain Ω is not simply connected, and given any integer $k \geq 1$, there exist k points $\xi_1, \ldots, \xi_k$ in Ω and a family of solutions u_λ to (2.3), satisfying (2.4) and bubbling at exactly those k points. The shape of these solutions is given by*

$$u_\lambda(x) = \sum_{j=1}^{k} G_\Omega(x, \xi_j) + o(1), \quad \text{as} \quad \lambda \to 0 \tag{2.8}$$

where $o(1) \to 0$ as $\lambda \to 0$ uniformly in compact sets contained in $\Omega \backslash \{\xi_1, \ldots, \xi_k\}$. Furthermore

$$J^1_\lambda(u_\lambda) = -16k\pi + 8k\pi \log 8 - 8k\pi \log \lambda - 4\pi\varphi_k(\xi) + o(1) \tag{2.9}$$

where φ_k is defined in (2.5) and $o(1) \to 0$ as $\lambda \to 0$.

If $p = 2$, problem (2.1) becomes

$$\begin{cases} \Delta u + \lambda u e^{u^2} = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

This problem is the Euler-Lagrange equation for the functional J^2_λ (see (2.2)) which corresponds to the free energy associated to the critical Trudinger embedding (in the sense of Orlicz spaces) [125, 103, 114]

$$H^1_0(\Omega) \ni u \longmapsto e^{u^2} \in L^p(\Omega) \quad \forall\, p \geq 1,$$

which is connected to the critical Trudinger-Moser inequality

$$C(\Omega) = \sup\left\{ \int_\Omega e^{4\pi u^2} \;/\; u \in H^1_0(\Omega),\; \int_\Omega |\nabla u|^2 = 1 \right\} \;<\; +\infty\,,$$

[94]. Observe that, in general, critical points of the above constrained variational problem satisfy, after a simple scaling, an equation of the form (2.10). The Trudinger-Moser embedding is critical, involving loss of compactness in $H^1_0(\Omega)$ for the functionals J^2_λ which translates into the presence of non-convergent Palais-Smale (PS) sequences. Let us consider for instance a sequence $\lambda_n \to \lambda_0 \geq 0$, and a sequence u_n with $\nabla J^2_{\lambda_n}(u_n) \to 0$, $J^2_{\lambda_n}(u_n) \to c$. For the Trudinger-Moser functional J^2_λ, a classification of all PS sequences for J_λ does not seem possible after the results in [3]. Actually PS holds as long as $c < 2\pi$, see [1, 31]. On the other hand, for *solutions* more is known. From the result in [44] (see also [2, 44, 100]), we have the following fact:

Assume that u_n solves problem (2.10) *for $\lambda = \lambda_n$, with $J^2_{\lambda_n}(u_n)$ bounded and $\lambda_n \to 0$. Then, passing to a subsequence, there is an integer $k \geq 0$ such that*

$$J^2_{\lambda_n}(u_n) = 2k\pi + o(1). \tag{2.11}$$

When $k = 1$ a more precise answer is obtained in [2]: the solution u_n has for large n only one isolated maximum, which blows up around a point $x_0 \in \Omega$ which is characterized as a critical point of Robin's function $x \mapsto H_\Omega(x, x)$. When $k > 1$, such a description for the behavior of u_n is not known and it seems to be still an open problem.

It is natural to ask whether or not solutions satisfying (2.11) exist. From the result in [3], it follows that there is a $\lambda_0 > 0$ such that a solution to (2.10) exists whenever $0 < \lambda < \lambda_0$ (this is in fact true for a larger class of nonlinearities with *critical exponential growth*). By construction this solution falls, as $\lambda \to 0$, into the bubbling category (2.11) with $k = 1$. Struwe in [112] built in the case of a domain with a sufficiently small hole a solution taking advantage of the presence of topology. M. del Pino, M. Musso and B. Ruf in [39] established a general result concerning existence and multiplicity of solutions of problem (2.10).

In order to state this result, let us introduce the following function of k distinct points $\xi_1, \xi_2, \cdots, \xi_k \in \Omega$ and k positive numbers $m_1, m_2, \cdots, m_k$,

$$\varphi_{k,2}(\xi, m) = a\sum_{j=1}^{k} m_j^2 + 2\sum_{j=1}^{k} m_j^2 \log m_j^2 + \sum_{j=1}^{k} m_j^2 H_\Omega(\xi_j, \xi_j) + \sum_{i\neq j} m_i m_j G_\Omega(\xi_i, \xi_j), \tag{2.12}$$

where $a > 0$ is an absolute constant, and $G_\Omega(x, y)$ is the Green's function defined in (2.6) and $H_\Omega(\cdot, \cdot)$ its regular part. The authors in [39] established that, if $\varphi_{k,2}$ has a *topologically non trivial critical value*, with corresponding critical point $(\xi_1, \dots, \xi_k, m_1, \dots m_k) \in \Omega^k \times \mathbb{R}_+^k$, then there exists a solution u_λ of (2.10) with the shape

$$u_\lambda(x) = \sqrt{\lambda}\left[\sum_{j=1}^{k} m_j G_\Omega(x, \xi_j) + o(1)\right], \quad \text{as} \quad \lambda \to 0, \tag{2.13}$$

where $o(1) \to 0$ as $\lambda \to 0$ uniformly on compact sets of $\Omega \setminus \{\xi_1, \dots, \xi_k\}$. Furthermore,

$$J_\lambda^2(u_\lambda) = 2k\pi + \alpha\lambda + 4\pi\lambda\varphi_{k,2}(\xi, m) + \lambda o(1)$$

where α is an absolute constant, $\varphi_{k,2}$ is defined in (2.12) and $o(1) \to 0$ as $\lambda \to 0$. In particular, in the case Ω is not simply connected they constructed the solution u_λ of (2.10), with two bubbling points, namely satisfying

$$u_\lambda(x) = \sqrt{\lambda}\left[\sum_{j=1}^{2} m_j G_\Omega(x, \xi_j) + o(1)\right], \quad \text{as} \quad \lambda \to 0,$$

where (m_1, m_2, ξ_1, ξ_2) is a critical point of $\varphi_{2,2}$ defined in (2.12), and $o(1) \to 0$ as $\lambda \to 0$ uniformly on compact sets of $\Omega \setminus \{\xi_1, \xi_2\}$.

The above result show a difference between the behavior of finite-energy solutions to problem (2.3) (or problem (2.1) with $p = 1$) and those to problem (2.10) (or problem (2.1) with $p = 2$): far away from the concentration points $\xi_1, \dots, \xi_k$, solutions to (2.3) are at main order sums of Green's functions centered at ξ_j (see (2.8)), while solutions to (2.10) are at main order sum of Green's functions centered at ξ_j but with different positive weights $\sqrt{\lambda} m_j$ whose values depend on the location of the concentration points $\xi_1, \dots, \xi_k$ (see (2.13)). In other words: To construct solutions to (2.10), one not only needs to choose carefully the concentration points $\xi_1, \dots . \xi_k$, as for problem (2.3), but one has to carefully choose the correct weights $m_1, \dots, m_k$. This shows that, in some sense, problem (2.3) has a *subcritical behavior* while problem (2.10) has a *critical behavior.*

This chapter is motivated to understand the solutions to problem (2.1) when p is between 1 and 2. In fact, we obtain existence results for (2.1) in the whole range $0 < p < 2$, and we find that in this range problem (2.1) has a subcritical behavior, in the sense described above. Let us state our result.

Let us define

$$\mathcal{M} = \left\{ (\xi_1, \cdots, \xi_k) \in \Omega^k \ : \ \mathrm{dist}(\xi_j, \partial\Omega) \geq \delta, \quad |\xi_i - \xi_j| \geq \delta \quad \text{for } i \neq j \right\}$$

for some $\delta > 0$. Let ε be a parameter, which depends on λ, defined as

$$p\lambda \left(-\frac{4}{p} \log \varepsilon \right)^{\frac{2(p-1)}{p}} \varepsilon^{\frac{2(p-2)}{p}} = 1. \tag{2.14}$$

Observe that, as $\lambda \to 0$, then $\varepsilon \to 0$, and $\lambda = \varepsilon^2$ if $p = 1$.

Our result states as follows.

Theorem 2.1. *Let $0 < p < 2$ and k an integer with $k \geq 1$. If Ω is not simply connected, then there exists $\lambda_0 > 0$ so that, for any $0 < \lambda < \lambda_0$ problem (2.1) has a solution u_λ. Furthermore*

$$\lim_{\lambda \to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_\Omega e^{u_\lambda^p} = 8k\pi, \tag{2.15}$$

where ε satisfies (2.14). Moreover, there exists an $k-$tuple $\xi^\lambda = (\xi_1^\lambda, \cdots, \xi_k^\lambda) \in \mathcal{M}$ such that as $\lambda \to 0$

$$\nabla \varphi_k(\xi_1^\lambda, \cdots, \xi_k^\lambda) \to 0,$$

and

$$u_\lambda(x) = p^{-\frac{1}{2}} \sqrt{\lambda} \, \varepsilon^{\frac{p-2}{p}} \left(\sum_{j=1}^k G_\Omega(x, \xi_j^\lambda) + o(1) \right) \tag{2.16}$$

where $o(1) \to 0$, as $\lambda \to 0$, on each compact subset of $\bar{\Omega} \backslash \{\xi_1^\lambda, \cdots, \xi_k^\lambda\}$. Furthermore

$$J_\lambda^p(u_\lambda) = \lambda \varepsilon^{\frac{2(p-2)}{p}} \left[\frac{8k\pi}{(2-p)p} [-2 + p \log 8] - \frac{16k\pi}{p} \log \varepsilon - \frac{4\pi}{2-p} \varphi_k(\xi^\lambda) + O(|\log \varepsilon|^{-1}) \right] \tag{2.17}$$

where $O(1)$ uniformly bounded as $\lambda \to 0$.

In [101], T. Ogawa and T. Suzuki investigated the asymptotic behaviour of the blow-up solutions for problem (2.1) when $0 < p \leq 2$ and $\Omega = B(0,1)$. Every smooth positive solution of this problem must be radially symmetric and decreasing in $|x|$ by the result of Gidas-Ni-Nirenberg [54], then $u(0) = \|u\|_{L^\infty}$. Suppose u_λ is a solution satisfying $\|u_\lambda\|_{L^\infty} \to \infty$ as $\lambda \to 0$, then $u_\lambda(x) \to 0$ locally uniformly on $\bar{B}\backslash\{0\}$, as $\lambda \to 0$. Thus, if we consider problem (2.1) in the unit disk of $\mathbb{R}^2$, suppose u is the solution of (2.1), then u blow-up at origin as $\lambda \to 0$.

We will prove Theorem 2.1 as consequence of a more general theorem, in a spirit similar to the one used in [36]. To do so, we need to recall the notion of *topologically non-trivial critical level* for φ_k. Let us consider an open set $\mathcal{D}$ compactly contained in the domain of the functional φ_k. We recall that φ_k *links in* $\mathcal{D}$ *at critical level* $\mathcal{C}$ *relative to* B and B_0 if B and B_0 are closed subsets of $\bar{\mathcal{D}}$ with B connected and $B_0 \subset B$ such that the following conditions hold: Let us set Γ to be the class of the maps $\Phi \in C(B, \mathcal{D})$ with the property that there exists a function $\Psi \in C([0,1] \times B, \mathcal{D})$ such that

$$\Psi(0,\cdot) = Id_B, \quad \Psi(1,\cdot) = \Phi, \quad \Psi(t,\cdot)|_{B_0} = Id_{B_0} \ \text{ for } \forall t \in [0,1].$$

We assume

$$\sup_{\xi \in B_0} \varphi_k(\xi) < \mathcal{C} := \inf_{\Phi \in \Gamma} \sup_{\xi \in B} \varphi_k(\Phi(\xi)), \tag{2.18}$$

and for all $\xi \in \partial\mathcal{D}$ such that $\varphi_k(\xi) = \mathcal{C}$, there exists a vector τ tangent to $\partial\mathcal{D}$ at ξ such that

$$\nabla\varphi_k(\xi) \cdot \tau \neq 0. \tag{2.19}$$

Under these conditions a critical point $\bar{\xi} \in \mathcal{D}$ with $\varphi_k(\bar{\xi}) = \mathcal{C}$ exists, as a standard deformation argument involving the negative gradient flow of φ_k shows. Condition (2.18) is a general way of describing a change of topology in the level sets $\{\varphi_k \leq c\}$ in $\mathcal{D}$ taking place at $c = \mathcal{C}$, while (2.19) prevents intersection of the level set $\mathcal{C}$ with the boundary. It is easy to check that the above conditions hold if

$$\inf_{\xi \in \mathcal{D}} \varphi_k(\xi) < \inf_{\xi \in \partial\mathcal{D}} \varphi_k(\xi), \qquad \text{or} \qquad \sup_{\xi \in \mathcal{D}} \varphi_k(\xi) > \sup_{\xi \in \partial\mathcal{D}} \varphi_k(\xi),$$

namely the case of (possibly degenerate) local minimum or maximum points of φ_k. The level $\mathcal{C}$ may be taken in these cases respectively as that of the minimum and the maximum of φ_k in $\mathcal{D}$. These hold also if φ_k is C^1-close to a function with a non-degenerate critical point in $\mathcal{D}$. We call $\mathcal{C}$ a non-trivial critical level of φ_k in $\mathcal{D}$.

Theorem 2.2. *For $0 < p < 2$, let $k \geq 1$, assume that φ_k defined by (2.5) has a topologically non trivial critical level $\mathcal{C}$ in $\mathcal{D}$, then the problem (2.1) has a family solutions u_λ for λ small enough, such that*

$$\lim_{\lambda \to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_\Omega e^{u_\lambda^p} = 8k\pi, \tag{2.20}$$

where ε satisfies (2.14). Moreover, there exists an $k-$tuple $\xi^\lambda = (\xi_1^\lambda, \cdots, \xi_k^\lambda) \in \mathcal{M}$ such that as $\lambda \to 0$

$$\varphi_k(\xi_1^\lambda, \cdots, \xi_k^\lambda) \to c,$$

and

$$u_\lambda(x) = p^{-\frac{1}{2}} \sqrt{\lambda}\, \varepsilon^{\frac{p-2}{p}} \left(\sum_{j=1}^{k} G_\Omega(x, \xi_j^\lambda) + o(1) \right) \tag{2.21}$$

where $o(1) \to 0$ on each compact subset of $\bar{\Omega} \backslash \{\xi_1^\lambda, \cdots, \xi_k^\lambda\}$. Furthermore

$$J_\lambda^p(u_\lambda) = \lambda \varepsilon^{\frac{2(p-2)}{p}} \left[\frac{8k\pi}{(2-p)p} [-2 + p \log 8] - \frac{16k\pi}{p} \log \varepsilon - \frac{4\pi}{2-p} \varphi_k(\xi^\lambda) + O(|\log \varepsilon|^{-1}) \right] \tag{2.22}$$

where $O(1)$ uniformly bounded as $\lambda \to 0$.

The proof of our result relies on a Lyapunov-Schmidt reduction procedure, introduced in [9, 52] and used in many different contexts, see for instance [36, 39, 47, 48, 49, 46]. The key step is to find the ansatz for the solution. Usually, the ansatz is built as a sum of terms, which turns out to be solutions of the associate limit problem, which are properly scaled and translated. For our problem, our approximate solution is built by using the following "basic cells": the radially symmetric solutions of the following Liouville equation

$$\Delta w + e^w = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^w < +\infty,$$

which are given by

$$w_\mu(z) := \log \frac{8\mu^2}{(\mu^2 + |z|^2)^2}, \quad w_\mu(z - \xi) := \log \frac{8\mu^2}{(\mu^2 + |z - \xi|^2)^2} \tag{2.23}$$

where μ is any positive number and ξ any point in $\mathbb{R}^2$ (see [21]). If we use a sum of the above basic cells, properly scaled, and centered at several points of the domain as our approximate solution, we get a very good approximation of a solution in a region far away from the points, which unfortunately turns out to be not good enough close to these points. Thus we need to improve the approximation, at least near the points, and we do this adding two other terms in the expansion of the solution. This can be done in a very natural way, which has first been used in [47] for studying the following problem

$$\begin{cases} \Delta u + u^p = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.24}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, and p is a large exponent. Later on, this method has been applied in other contexts, see [19, 48, 49, 93]. Observe that this method allows to improve the approximation near the points, but it is not useful to improve the approximation far away from this points. Nevertheless, as already mentioned, the approximation we build for this problem is sufficiently accurate in a regime far from the points. After the approximate solution is build, we find an actual solution to (2.1) as a small perturbation of the approximation.

This chapter is organized as follows: Section 2.2 is devoted to describing a first approximation solution to problem (2.1) and estimating the error. Furthermore, problem (2.1) is written as a fixed point problem, which involving a linear operator. In Section 2.3, we study the invertibility of the linear problem. In Section 2.4, we study the nonlinear problem. In Section 2.5, we study the variational reduction, we prove Theorems 2.1 and 2.2 in Section 2.6.

2.2 Preliminaries and ansatz for the solution

In this section we describe the approximate solution for problem (2.1) and then we estimate the error of such approximation in appropriate norms.

Let us consider k distinct points $\xi_1, \xi_2, \cdots, \xi_k$ in Ω, we choose a sufficiently small but fixed number $\delta > 0$ and assume that for $j = 1, 2, \cdots, k$,

$$\operatorname{dist}(\xi_j, \partial\Omega) \geq \delta, \qquad |\xi_i - \xi_j| \geq \delta \quad \text{for } i \neq j, \tag{2.1}$$

Furthermore, we consider k positive numbers μ_j such that

$$\delta < \mu_j < \delta^{-1}, \quad \text{for all} \quad j = 1, \ldots, k. \tag{2.2}$$

The parameters μ_j will be chosen properly later on. Define the function

$$U_{\mu_j,\xi_j}(x) = \log \frac{8\mu_j^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2}.$$

Let us denote $PU_{\mu_j,\xi_j}(x)$ the projection of U_{μ_j,ξ_j} into the space $H_0^1(\Omega)$, in other words, $PU_{\mu_j,\xi_j}(x)$ is the unique solution of

$$\begin{cases} \Delta PU_{\mu_j,\xi_j} = \Delta U_{\mu_j,\xi_j}, & \text{in } \Omega; \\ PU_{\mu_j,\xi_j} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Lemma 2.3. *Assume (2.1) and (2.2). We have*

$$PU_{\mu_j,\xi_j}(x) = U_{\mu_j,\xi_j}(x) + H_\Omega(x, \xi_j) - \log(8\mu_j^2) + O(\mu_j^2\varepsilon^2) \tag{2.4}$$

in $C^1(\bar{\Omega})$ as $\varepsilon \to 0$, and

$$PU_{\mu_j,\xi_j}(x) = G_\Omega(x, \xi_j) + O(\mu_j^2\varepsilon^2) \tag{2.5}$$

in $C^1_{loc}(\bar{\Omega}\backslash\{\xi_j\})$ as $\varepsilon \to 0$, where $G_\Omega(\cdot,\cdot)$ and $H_\Omega(\cdot,\cdot)$ are Green's function and its the regular part as defined in (2.6) and (2.7).

Proof. Let $z(x) = PU_{\mu_j,\xi_j}(x) - U_{\mu_j,\xi_j}(x) + \log(8\mu_j^2)$, then $z(x)$ satisfies

$$\begin{cases} \Delta z(x) = 0 & \text{in } \Omega; \\ z(x) = 2\log(\mu_j^2\varepsilon^2 + |x-\xi_j|^2) & \text{on } \partial\Omega. \end{cases}$$

On the other hand, we note that $\eta(x) = H_\Omega(x, \xi_j)$ satisfies

$$\begin{cases} \Delta \eta(x) = 0 & \text{in } \Omega; \\ \eta(x) = 2\log|x-\xi_j|^2 & \text{on } \partial\Omega. \end{cases}$$

Then we get

$$\begin{cases} \Delta(z(x) - \eta(x)) = 0 & \text{in } \Omega; \\ z(x) - \eta(x) = -2\log\frac{|x-\xi_j|^2}{\mu_j^2\varepsilon^2+|x-\xi_j|^2} & \text{on } \partial\Omega. \end{cases}$$

Since $|x - \xi_j| > \delta$ for $x \in \partial\Omega$, then by the maximum principle we get

$$\max_{\bar{\Omega}} |z(\cdot) - \eta(\cdot)| = \max_{x\in\partial\Omega} |z(\cdot) - \eta(\cdot)| = O(\mu_j^2\varepsilon^2),$$

as $\varepsilon \to 0$, uniformly in Ω. Then we obtain the C^0-estimate in (2.4). Analogous computations give the C^1-closeness and hence the validity of (2.4). By (2.4) we deduce (2.5). □

We shall show later on that $PU_{\mu_j,\xi_j}(x)$ is a good approximation for a solution to (2.1) far from the points ξ_j, but unfortunately it is not good enough for our construction close to the points ξ_j. This is the reason why we need to further adjust $PU_{\mu_j,\xi_j}(x)$. To do this, we need to introduce the following functions w_j^0 and w_j^1.

Let w_{μ_j} be defined as (2.23). Define the function w_j^i to be radial solution of

$$\Delta w_j^i + e^{w_{\mu_j}} w_j^i = e^{w_{\mu_j}} f^i \quad \text{in } \mathbb{R}^2, \quad \text{for} \quad i = 0, 1, \tag{2.6}$$

and

$$f^0 = -\left(w_{\mu_j} + \frac{1}{2}(w_{\mu_j})^2\right), \tag{2.7}$$

$$\begin{aligned} f^1 \;=\; -\Bigg(& w_j^0 + \frac{p-2}{2(p-1)}(w_{\mu_j})^2 + \frac{1}{2}(w_j^0)^2 + \frac{1}{8}(w_{\mu_j})^4 \\ & + 2w_{\mu_j}w_j^0 + \frac{1}{2}(w_{\mu_j})^3 + \frac{1}{2}w_j^0(w_{\mu_j})^2\Bigg). \end{aligned} \tag{2.8}$$

In fact, as shown in [47] (see also [20]), there exists radially symmetric solutions with the properties that

$$w_j^i(y) = C_{ij}\log\frac{|y|}{\mu_j} + O(\frac{1}{|y|}) \quad \text{as } |y| \to \infty, \tag{2.9}$$

for some explicit constants C_{ij}, which can be explicitly computed. In particular, when $i = 0$, the constant C_{0j} is given by

$$\begin{aligned} C_{0j} &= -8\int_0^{+\infty} t\frac{t^2-1}{(t^2+1)^3}\left[\log\frac{8\mu_j^{-2}}{(1+t^2)^2}+\frac{1}{2}\left(\log\frac{8\mu_j^{-2}}{(1+t^2)^2}\right)^2\right]dt \\ &= -4\int_0^{+\infty} \frac{t^2-1}{(t^2+1)^3}\left[\log\frac{8\mu_j^{-2}}{(1+t^2)^2}+\frac{1}{2}\left(\log\frac{8\mu_j^{-2}}{(1+t^2)^2}\right)^2\right]d(t^2) \\ &\quad \text{set } \ r = t^2+1 \\ &= -4\int_1^{+\infty}\frac{r-2}{r^3}\Big[\log(8\mu_j^{-2})-2\log r+\frac{1}{2}\left(\log(8\mu_j^{-2})\right)^2 \\ &\qquad -2\log(8\mu_j^{-2})\log r+2(\log r)^2\Big]\,dr. \end{aligned}$$

Since

$$\int_1^{+\infty}\frac{r-2}{r^3}\,dr = 0, \qquad \int_1^{+\infty}\frac{r-2}{r^3}\log r\,dr = \frac{1}{2},$$

and

$$\int_1^{+\infty}\frac{r-2}{r^3}(\log r)^2\,dr = \frac{3}{2}.$$

Hence

$$C_{0j} = 4\log 8 - 8 - 8\log\mu_j. \tag{2.10}$$

Let us define

$$w^0_{\mu_j,\xi_j}(x) := w^0_j\left(\frac{x-\xi_j}{\varepsilon}\right), \qquad w^1_{\mu_j,\xi_j}(x) := w^1_j\left(\frac{x-\xi_j}{\varepsilon}\right) \qquad \text{for } x\in\Omega.$$

Let $Pw^0_{\mu_j,\xi_j}$ and $Pw^1_{\mu_j,\xi_j}$ denote the projections into $H^1_0(\Omega)$ of $w^0_{\mu_j,\xi_j}$ and $w^1_{\mu_j,\xi_j}$, respectively. By (2.9), we have that

$$\begin{aligned} P\left(w^i_{\mu_j,\xi_j}(x)\right) &= P\left(w^i_j\left(\frac{y-\xi_j'}{\mu_j}\right)\right) \\ &= w^i_j\left(\frac{y-\xi_j'}{\mu_j}\right)-\frac{C_{ij}}{4}H_\Omega(x,\xi_j)+C_{ij}\log(\mu_j\varepsilon)+O(\mu_j\varepsilon) \end{aligned} \tag{2.11}$$

in $C^1(\bar\Omega)$ as $\varepsilon\to 0$, and

$$P\left(w^i_{\mu_j,\xi_j}(x)\right) = P\left(w^i_j\left(\frac{y-\xi_j'}{\mu_j}\right)\right) = -\frac{C_{ij}}{4}G_\Omega(x,\xi_j)+O(\mu_j\varepsilon) \tag{2.12}$$

in $C^1_{loc}(\bar\Omega\backslash\{\xi_j\})$ as $\varepsilon\to 0$.

We define

$$U_\lambda(x) = \frac{1}{p\gamma^{p-1}} \sum_{j=1}^{k} \left(PU_{\mu_j,\xi_j}(x) + \frac{p-1}{p}\frac{1}{\gamma^p} Pw^0_{\mu_j,\xi_j}(x) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} Pw^1_{\mu_j,\xi_j}(x) \right). \tag{2.13}$$

From (2.5) and (2.12), one has, away from the points ξ_j,

$$U_\lambda(x) = \frac{1}{p\gamma^{p-1}} \sum_{j=1}^{k} G_\Omega(x,\xi_j) \left(1 - \frac{p-1}{p}\frac{1}{\gamma^p}\frac{C_{0j}}{4} - \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\frac{C_{1j}}{4} + O(\varepsilon^2) \right). \tag{2.14}$$

Consider now the change of variables

$$v(y) = p\gamma^{p-1}u(\varepsilon y) - p\gamma^p, \qquad \text{with } \ \gamma^p = -\frac{4}{p}\log\varepsilon.$$

By (2.14), then problem (2.1) reduces to

$$\begin{cases} \Delta v + g(v) = 0, \quad v > 0 & \text{in } \Omega_\varepsilon; \\ v = -p\gamma^p & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.15}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$, and

$$g(v) = (1 + \frac{v}{p\gamma^p})^{p-1} e^{\gamma^p[(1+\frac{v}{p\gamma^p})^p - 1]}. \tag{2.16}$$

Let us define the first approximation solution to (2.15) as

$$V_\lambda(y) = p\gamma^{p-1}U_\lambda(\varepsilon y) - p\gamma^p, \tag{2.17}$$

with U_λ defined by (2.13). We write $y = \varepsilon^{-1}x$, $\xi_j' = \varepsilon^{-1}\xi_j$. For $|x - \xi_j| < \delta$ with δ sufficiently small but fixed, by using (2.4), (2.5), (2.11), (2.12) and the fact that $U_{\mu_j,\xi_j}(\varepsilon y) - p\gamma^p = w_j(y - \xi_j')$, we have

$$\begin{aligned} V_\lambda(y) &= PU_{\mu_j,\xi_j}(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p} Pw^0_{\mu_j,\xi_j}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} Pw^1_{\mu_j,\xi_j}(\varepsilon y) - p\gamma^p \\ &\quad + \sum_{i\neq j}^{k} \left(PU_{\mu_i,\xi_i}(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p} Pw^0_{\mu_i,\xi_i}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} Pw^1_{\mu_i,\xi_i}(\varepsilon y) \right) \\ &= U_{\mu_j,\xi_j}(\varepsilon y) + H_\Omega(\varepsilon y,\xi_j) - \log(8\mu_j^2) + O(\mu_j^2\varepsilon^2) - p\gamma^p \\ &\quad + \frac{p-1}{p}\frac{1}{\gamma^p}\left[w^0_j\left(\frac{y-\xi'}{\mu_j}\right) - \frac{C_{0j}}{4} H_\Omega(\varepsilon y,\xi_j) + C_{0j}\log(\mu_j\varepsilon) + O(\mu_j\varepsilon) \right] \\ &\quad + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\left[w^1_j\left(\frac{y-\xi'}{\mu_j}\right) - \frac{C_{1j}}{4} H_\Omega(\varepsilon y,\xi_j) + C_{1j}\log(\mu_j\varepsilon) + O(\mu_j\varepsilon) \right] \\ &\quad + \sum_{i\neq j}^{k} G_\Omega(\xi_i,\xi_j)\left[1 - \frac{C_{0j}}{4}\frac{p-1}{p}\frac{1}{\gamma^p} - \frac{C_{1j}}{4}\left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} \right] + O(\varepsilon^2) \end{aligned}$$

$$
\begin{aligned}
&= w_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p} w_j^0(y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} w_j^1(y) - \log(8\mu_j^2) \\
&\quad + \left[1 - \frac{C_{0j}}{4}\frac{p-1}{p}\frac{1}{\gamma^p} - \frac{C_{1j}}{4}\left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\right]\left(H_\Omega(\xi_j,\xi_j) + \sum_{i\neq j}^k G_\Omega(\xi_i,\xi_j)\right) \\
&\quad + \left[C_{0j}\frac{p-1}{p}\frac{1}{\gamma^p} + C_{1j}\left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\right](\log(\mu_j) + \log\varepsilon) \\
&\quad + O(\varepsilon|y-\xi'|) + O(\varepsilon^2),
\end{aligned}
\tag{2.18}
$$

where

$$
w_j(y) := w_{\mu_j}(y-\xi_j'), \quad w_j^0(y) := w_j^0\left(\frac{y-\xi_j'}{\mu_j}\right), \quad w_j^1(y) := w_j^1\left(\frac{y-\xi_j'}{\mu_j}\right).
$$

We now choose the parameters μ_j: we assume they are defined by the relation

$$
\begin{aligned}
\log(8\mu_j^2) &= \left(H_\Omega(\xi_j,\xi_j) + \sum_{i\neq j}^k G_\Omega(\xi_i,\xi_j)\right) - \frac{p-1}{4}C_{0j} \\
&\quad - \frac{p-1}{p}\frac{1}{\gamma^p}\frac{C_{0j}}{4}\left(H_\Omega(\xi_j,\xi_j) + \sum_{i\neq j}^k G_\Omega(\xi_i,\xi_j) + 4\log(\mu_j) - (p-1)\frac{C_{1j}}{C_{0j}}\right) \\
&\quad - \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\frac{C_{1j}}{4}\left(H_\Omega(\xi_j,\xi_j) + \sum_{i\neq j}^k G_\Omega(\xi_i,\xi_j) + 4\log(\mu_j)\right).
\end{aligned}
\tag{2.19}
$$

Taking into account the explicit expression (2.10) of the constant C_{0j}, we observe that μ_j bifurcates, as λ goes to zero, from the value $\bar{\mu_j}$ defined by

$$
\bar{\mu_j} = 8^{-\frac{p}{2(2-p)}} e^{\frac{p-1}{2-p}} e^{\frac{1}{2(2-p)}\left[H_\Omega(\xi_j,\xi_j) + \sum_{i\neq j}^k G_\Omega(\xi_i,\xi_j)\right]}
\tag{2.20}
$$

solution of equation

$$
\log(8\mu_j^2) = \left(H_\Omega(\xi_j,\xi_j) + \sum_{i\neq j}^k G_\Omega(\xi_i,\xi_j)\right) - \frac{p-1}{4}C_{0j}.
\tag{2.21}
$$

Thus, μ_j is a perturbation of order $\frac{1}{\gamma^p}$ of the value $\bar{\mu_j}$, namely

$$
\begin{aligned}
\log(8\mu_j^2) &= \left[\frac{2(p-1)}{2-p}(1-\log 8)\right. \\
&\quad \left. + \frac{1}{2-p}\left(H_\Omega(\xi_j,\xi_j) + \sum_{i\neq j}^k G_\Omega(\xi_i,\xi_j)\right)\right]\left(1 + O\left(\frac{1}{\gamma^p}\right)\right).
\end{aligned}
\tag{2.22}
$$

Then, by this choice of the parameters μ_j, we deduce that, if $|y-\xi'_j| < \delta/\varepsilon$ with δ sufficiently small but fixed, we can rewrite

$$V_\lambda(y) = w_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}w_j^1(y) + \theta(y), \tag{2.23}$$

with

$$\theta(y) = O(\varepsilon|y-\xi'_j|) + O(\varepsilon^2).$$

We will look for solutions to (2.15) of the form

$$v = V_\lambda + \phi,$$

where V_λ is defined as in (2.17), and ϕ represents a lower order correction. We aim at finding a solution for ϕ small provided that the points ξ_j are suitably chosen. For small ϕ, we can rewrite problem (2.15) as a nonlinear perturbation of its linearization, namely,

$$\begin{cases} L(\phi) = -[E_\lambda + N(\phi)], & x \in \Omega_\varepsilon; \\ \phi = 0, & x \in \partial\Omega_\varepsilon, \end{cases} \tag{2.24}$$

where

$$L(\phi) := \Delta\phi + g'(V_\lambda)\phi, \tag{2.25}$$

$$E_\lambda := \Delta V_\lambda + g(V_\lambda), \tag{2.26}$$

$$N(\phi) := g(V_\lambda + \phi) - g(V_\lambda) - g'(V_\lambda)\phi. \tag{2.27}$$

We recall that $g(t) = (1 + \frac{t}{p\gamma^p})^{p-1} e^{\gamma^p[(1+\frac{t}{p\gamma^p})^p - 1]}$.

In order to solve the problem (2.24), first we have to study the invertibility properties of the linear operator L. In order to do this, we introduce a weighted L^∞-norm defined as

$$\|h\|_* := \sup_{y\in\Omega_\varepsilon} \left(\sum_{j=1}^k (1+|y-\xi'_j|)^{-3} + \varepsilon^2\right)^{-1} |h(y)| \tag{2.28}$$

for any $h \in L^\infty(\Omega_\varepsilon)$. With respect to this norm, the error term E_λ given in (2.26) can be estimated in the following way.

Lemma 2.4. *Let $\delta > 0$ be a small but fixed number and assume that the points ξ_j satisfy (2.1). There exists $C > 0$, such that we have*

$$\|E_\lambda\|_* \le \frac{C}{\gamma^{3p}} = \frac{C}{|\log\varepsilon|^3} \tag{2.29}$$

for all λ small enough.

Proof. Far away from the points ξ_j, namely for $|x-\xi_j|>\delta$, i.e. $|y-\xi_j'|>\frac{\delta}{\varepsilon}$, for all $j=1,\cdots,k$, from (2.5) and (2.12) we have that

$$\Delta V_\lambda(y) = p\gamma^{p-1}\varepsilon^2\Delta U(\varepsilon y) = O(\gamma^{p-1}\varepsilon^4).$$

On the other hand, in this region we have

$$1+\frac{V_\lambda(y)}{p\gamma^p} = 1+\frac{4\log\varepsilon+O(1)}{p\gamma^p} = \frac{O(1)}{|\log\varepsilon|} \tag{2.30}$$

where $O(1)$ denotes a smooth function, uniformly bounded, as $\varepsilon\to 0$, in the considered region. Hence

$$\begin{aligned} g(V_\lambda) &= \left(1+\frac{V_\lambda}{p\gamma^p}\right)^{p-1} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]} \\ &= \begin{cases} C\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2; \\ C\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}}e^{\gamma^p\frac{O(1)}{|\log\varepsilon|^p}} & \text{if } 0<p<1. \end{cases} \\ &= \begin{cases} C\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2; \\ C\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}}e^{\frac{O(1)}{|\log\varepsilon|^{p-1}}} & \text{if } 0<p<1. \end{cases} \end{aligned}$$

Thus if we are far away from the points ξ_j, or equivalently for $|y-\xi_j'|>\frac{\delta}{\varepsilon}$, the size of the error, measured with respect to the $\|\cdot\|_*$-norm, is relatively small. In other words, if we denote by 1_{outer} the characteristic function of the set $\{y : |y-\xi_j'|>\frac{\delta}{\varepsilon}\, j=1,\dots,k\}$, then in this region we have

$$\begin{aligned} \|E_\lambda 1_{\mathsf{outer}}\|_* &\le \begin{cases} C\frac{\varepsilon^{\frac{2(2-p)}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2; \\ C\frac{\varepsilon^{\frac{2-p}{p}}}{|\log\varepsilon|^{p-1}}e^{\log\varepsilon^{\frac{2-p}{p}}+\frac{C}{|\log\varepsilon|^{p-1}}} & \text{if } 0<p<1. \end{cases} \\ &= \begin{cases} C\frac{\varepsilon^{\frac{2(2-p)}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2; \\ C\frac{\varepsilon^{\frac{2-p}{p}}}{|\log\varepsilon|^{p-1}}e^{-\frac{2-p}{p}|\log\varepsilon|+C|\log\varepsilon|^{1-p}} & \text{if } 0<p<1. \end{cases} \\ &\le \begin{cases} C\frac{\varepsilon^{\frac{2(2-p)}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2; \\ C\frac{\varepsilon^{\frac{2-p}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 0<p<1. \end{cases} \end{aligned} \tag{2.31}$$

Here we used that $-\frac{2-p}{p}|\log\varepsilon|+C|\log\varepsilon|^{1-p}<0$ for $0<p<1$ and ε small. Let us now fix the index j in $\{1,\cdots,k\}$, for $|y-\xi_j'|<\frac{\delta}{\varepsilon}$, we have

$$\Delta V_\lambda(y) = -e^{w_j(y)}+\frac{p-1}{p}\frac{1}{\gamma^p}\Delta w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\Delta w_j^1(y)+O(\varepsilon^2). \tag{2.32}$$

On the other hand, for any $R>0$ large but fixed, in the ball $|y-\xi_j'|<R_\varepsilon := R|\log\varepsilon|^\alpha$, with $\alpha\ge 3$, we can use Taylor expansion to first get

$$\begin{aligned}(1+\frac{V_\lambda}{p\gamma^p})^{p-1} &= 1+\frac{p-1}{p}\frac{1}{\gamma^p}w_j+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}[w_j^0+\frac{p-2}{2(p-1)}(w_j)^2]\\ &\quad +(\frac{p-1}{p})^3\frac{1}{\gamma^{3p}}(\log|y|),\end{aligned}$$

$$\begin{aligned}\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1] &= w_j+(\frac{p-1}{p})\frac{1}{\gamma^p}[w_j^0+\frac{(w_j)^2}{2}]\\ &\quad +(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}(w_j^1+w_jw_j^0)+\frac{1}{\gamma^{3p}}(\log|y|)\end{aligned}$$

and

$$e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]}=e^{w_j}\left[1+(\frac{p-1}{p})\frac{1}{\gamma^p}[w_j^0+\frac{(w_j)^2}{2}]\right.$$

$$\left.+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}[w_j^1+w_jw_j^0+\frac{1}{2}(w_j^0+(w_j)^2)^2]+\frac{1}{\gamma^{3p}}(\log|y|)\right]$$

Thus we obtain

$$\begin{aligned}&g(V_\lambda):=(1+\frac{V_\lambda}{p\gamma^p})^{p-1}e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]}\\ =\ & e^{w_j}\left\{1+(\frac{p-1}{p})\frac{1}{\gamma^p}\left[w_j^0+\frac{(w_j)^2}{2}+w_j\right]\right.\\ &+(\frac{p-1}{p}^2)\frac{1}{\gamma^{2p}}\left[w_j^1+2w_jw_j^0+\frac{1}{2}(w_j^0+\frac{(w_j)^2}{2})^2+w_j^0+\frac{p-2}{2(p-1)}w_j^2+\frac{w_j^3}{2}\right]\\ &\left.+O\left(\frac{\log|y-\xi_j'|}{\gamma^{3p}}\right)\right\}.\end{aligned}$$

Thus, thanks to the fact that we have improved our original approximation with the terms w_j^0 and w_j^1, and the definition of $*$-norm, we get that

$$\|E_\lambda 1_{B(\xi_j',R_\varepsilon)}\|_*\le\frac{C}{\gamma^{3p}}=\frac{C}{|\log\varepsilon|^3},\quad \text{for any}\quad j=1,\dots,k. \tag{2.33}$$

Here $1_{B(\xi_j',R_\varepsilon)}$ denotes the characteristic function of $B(\xi_j,R_\varepsilon)$. Finally, in the remaining region, namely where $R_\varepsilon<|y-\xi_j'|<\frac{\delta}{\varepsilon}$, for any $j=1,\dots,k$, we have from one hand that $|\Delta V_\lambda(y)|\le Ce^{w_j(y)}$, and also $|g(V_\lambda(y))|\le Ce^{w_j(y)}$ as consequence of (2.18). This fact, together with (2.33) and (2.31) we obtain estimate (2.29). □

As above computation, we find that very close to the point ξ_j in Ω, we have

$$\|g'(V_\lambda)-e^{w_j}\|_*\to 0\quad \text{as } \lambda\to 0, \tag{2.34}$$

and there exists some positive constant D_0 such that

$$g'(V_\lambda) \leq D_0 \sum_{j=1}^{k} e^{w_j}. \tag{2.35}$$

Moreover, we can get

$$\|g''(V_\lambda)\|_* \leq C. \tag{2.36}$$

Proof of (2.34) and (2.35): We have

$$\begin{aligned} g'(V_\lambda) &= \frac{p-1}{p}\frac{1}{\gamma^p}(1+\frac{V_\lambda}{p\gamma^p})^{p-2}e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]} \\ &\quad +(1+\frac{V_\lambda}{p\gamma^p})^{2(p-1)}e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]} \\ &:= I_a + I_b. \end{aligned}$$

Far away from the points ξ_j, namely for $|x-\xi_j|>\delta$, i.e. $|y-\xi_j'|>\frac{\delta}{\varepsilon}$, for all $j=1,\cdots,k$, a consequence of (2.30) is that

$$I_a = \frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}}O(1), \quad \text{and} \quad I_b = \frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{2(p-1)}}O(1)$$

Then we have

$$g'(V_\lambda)1_{\mathsf{outer}} = \frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}}O(1) \tag{2.37}$$

On the other hand, fix the index j in $\{1,\cdots,k\}$, for $|y-\xi_j'|<R_\varepsilon$ with $R_\varepsilon = R|\log\varepsilon|$, for any $R>0$ large but fixed, we use Taylor expansion to get

$$\begin{aligned} I_a &= \frac{p-1}{p}\frac{1}{\gamma^p}\left(1+\frac{1}{p\gamma^p}\left(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right)\right)^{p-2} \\ &\quad \times e^{\gamma^p\left[\left(1+\frac{1}{p\gamma^p}\left(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right)\right)^p-1\right]} \\ &= \frac{p-2}{p}\frac{1}{\gamma^p}\left[\frac{p-1}{p-2}+\frac{p-1}{p}\frac{1}{\gamma^p}w_j(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^0(y)\right. \\ &\qquad \left. +\left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}w_j^1(y)+\frac{p-1}{p}\frac{1}{\gamma^p}\theta(y)\right] \\ &\quad \times e^{w_j(y)}e^{\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)}e^{\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)}e^{\theta(y)}e^{\frac{1}{2}\frac{p-1}{p}\frac{1}{\gamma^p}\left[w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right]^2}, \end{aligned}$$

and

$$I_b = \left(1+\frac{1}{p\gamma^p}\left(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right)\right)^{2(p-1)}$$

$$
\begin{aligned}
&\times e^{\gamma^p\left[\left(1+\frac{1}{p\gamma^p}\left(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right)\right)^p-1\right]}\\
= &\left[1+\frac{2(p-1)}{p}\frac{1}{\gamma^p}w_j(y)+2\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^0(y)\right.\\
&\quad\left.+2\left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}w_j^1(y)+\frac{2(p-1)}{p}\frac{1}{\gamma^p}\theta(y)\right]\\
&\times e^{w_j(y)}e^{\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)}e^{\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)}e^{\theta(y)}e^{\frac{1}{2}\frac{p-1}{p}\frac{1}{\gamma^p}\left[w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right]^2}.
\end{aligned}
$$

By the definition of w_j^0 and w_j^1, we get that

$$
I_a 1_{B(\xi_j',R_\varepsilon)}=\frac{O(1)}{|\log\varepsilon|},\qquad I_b 1_{B(\xi_j',R_\varepsilon)}-e^{w_j(y)}=\frac{O(1)}{|\log\varepsilon|} \tag{2.38}
$$

Finally, in the remaining region, namely where for any $j=1,\dots,k$, we have $R_\varepsilon<|y-\xi_j'|<\frac{\delta}{\varepsilon}$, we have

$$
|I_a|\le\frac{C}{|\log\varepsilon|}e^{w_j(y)},\qquad |I_b|\le Ce^{w_j(y)}. \tag{2.39}
$$

Then, from (2.38) and the definition of $*-$norm, we find that very close to the point ξ_j in Ω, we have

$$
\|g'(V_\lambda)-e^{w_j}\|_*=\frac{O(1)}{|\log\varepsilon|}
$$

which implies (2.34). Combing (2.37), (2.38) with (2.39) we obtain estimate (2.35).

Proof of (2.36): We have

$$
\begin{aligned}
g''(V_\lambda) &= \frac{(p-1)(p-2)}{p^2}\frac{1}{\gamma^{2p}}(1+\frac{V_\lambda}{p\gamma^p})^{p-3}e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]}\\
&= \frac{3(p-1)}{p}\frac{1}{\gamma^p}(1+\frac{V_\lambda}{p\gamma^p})^{2p-3}e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]}\\
&\quad +(1+\frac{V_\lambda}{p\gamma^p})^{3(p-1)}e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]}\\
&:= I_c+I_d+I_e.
\end{aligned}
$$

By a similar computation as above: Far away from the points ξ_j, namely for $|x-\xi_j|>\delta$, i.e. $|y-\xi_j'|>\frac{\delta}{\varepsilon}$, for all $j=1,\cdots,k$, we have

$$
I_c=\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}}O(1),\quad I_d=\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{2(p-1)}}O(1)\quad,\ \text{and}\quad I_e=\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{3(p-1)}}O(1)
$$

Then

$$
g''(V_\lambda)1_{\mathsf{outer}}=\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}}O(1) \tag{2.40}
$$

where again $O(1)$ denotes a function which is uniformly bounded, as $\varepsilon \to 0$, in the considered region. Let us now fix the index j in $\{1, \cdots, k\}$, for $|y - \xi'_j| < R_\varepsilon$ with any $R_\varepsilon := R|\log \varepsilon|$ for some $R > 0$ large but fixed, by Taylor expansion, we have

$$\begin{aligned} I_c &= \frac{(p-1)(p-2)}{p^2}\frac{1}{\gamma^{2p}}\left(1+\frac{1}{p\gamma^p}\left(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right)\right)^{p-3} \\ &\quad \times e^{\gamma^p\left[\left(1+\frac{1}{p\gamma^p}\left(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right)\right)^p-1\right]} \\ &= \frac{(p-2)(p-3)}{p^2}\frac{1}{\gamma^{2p}}\left[\frac{p-1}{p-3}+\frac{p-1}{p}\frac{1}{\gamma^p}w_j(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^0(y)\right. \\ &\qquad\qquad \left.+\left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}w_j^1(y)+\frac{p-1}{p}\frac{1}{\gamma^p}\theta(y)\right] \\ &\quad \times e^{w_j(y)}e^{\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)}e^{\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)}e^{\theta(y)}e^{\frac{1}{2}\frac{p-1}{p}\frac{1}{\gamma^p}\left[w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right]^2}, \end{aligned}$$

$$\begin{aligned} I_d &= \frac{3(p-1)}{p}\frac{1}{\gamma^{p}}\left(1+\frac{1}{p\gamma^p}\left(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right)\right)^{2p-3} \\ &\quad \times e^{\gamma^p\left[\left(1+\frac{1}{p\gamma^p}\left(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right)\right)^p-1\right]} \\ &= \frac{3(2p-3)}{p}\frac{1}{\gamma^{p}}\left[\frac{p-1}{2p-3}+\frac{p-1}{p}\frac{1}{\gamma^p}w_j(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^0(y)\right. \\ &\qquad\qquad \left.+\left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}w_j^1(y)+\frac{p-1}{p}\frac{1}{\gamma^p}\theta(y)\right] \\ &\quad \times e^{w_j(y)}e^{\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)}e^{\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)}e^{\theta(y)}e^{\frac{1}{2}\frac{p-1}{p}\frac{1}{\gamma^p}\left[w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right]^2}, \end{aligned}$$

and

$$\begin{aligned} I_e &= \left(1+\frac{1}{p\gamma^p}\left(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right)\right)^{3(p-1)} \\ &\quad \times e^{\gamma^p\left[\left(1+\frac{1}{p\gamma^p}\left(w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right)\right)^p-1\right]} \\ &= \left[1+\frac{3(p-1)}{p}\frac{1}{\gamma^p}w_j(y)+3\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^0(y)\right. \\ &\qquad \left.+3\left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}w_j^1(y)+\frac{3(p-1)}{p}\frac{1}{\gamma^p}\theta(y)\right] \\ &\quad \times e^{w_j(y)}e^{\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)}e^{\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)}e^{\theta(y)}e^{\frac{1}{2}\frac{p-1}{p}\frac{1}{\gamma^p}\left[w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right]^2}. \end{aligned}$$

Therefore, we get

$$I_c 1_{B(\xi'_j,R_\varepsilon)} = \frac{O(1)}{|\log\varepsilon|}, \qquad I_d 1_{B(\xi'_j,R_\varepsilon)} = \frac{O(1)}{|\log\varepsilon|^2}, \qquad I_e 1_{B(\xi'_j,R_\varepsilon)} = O(1). \tag{2.41}$$

Finally, for $R_\varepsilon < |y-\xi_j'| < \frac{\delta}{\varepsilon}$, for any j, we have

$$|I_c| \le \frac{C}{|\log\varepsilon|}, \qquad |I_d| \le \frac{C}{|\log\varepsilon|^2}, \qquad |I_e| = O(1) + Ce^{w_j(y)}. \tag{2.42}$$

From (2.40), (2.41) with (2.42), by the definition of $*-$norm, we obtain (2.36) holds.

2.3 The linearized problem

In this section, we prove the bounded invertibility of the operator L. We observe that the operator L can be approximately regarded as a superposition of the linear operator

$$L_j(\phi) = \Delta\phi + e^{w_j}\phi = \Delta\phi + \frac{8\mu_j^2}{\left(\mu_j^2 + |y-\xi_j'|^2\right)^2}\phi.$$

The key fact to develop a satisfactory solvability theory for the operator L is the nondegeneracy of w up to the natural invariances of the equation under translations and dilations, which translates into the fact that

$$z_{0j}(y) = \partial_{\mu_j} w_{\mu_j}(y), \qquad z_{ij}(y) = \partial_{y_i} w_{\mu_j}(y), \quad i = 1,2,$$

satisfy the function $\Delta Z + e^{w_j}Z = 0$, see [10] for a proof. Define for $i = 0,1,2$ and $j = 1,2,\cdots,k$,

$$Z_{ij}(y) := z_{ij}\left(y-\xi_j'\right), \quad i = 0,1,2. \tag{2.43}$$

Consider a large but fixed number $R_0 > 0$ and a radial and smooth cut-off function η with $\eta(r) = 1$ if $r < R_0$ and $\eta(r) = 0$ if $r > R_0 + 1$. Write

$$\eta_j(y) = \eta\left(|y-\xi_j'|\right). \tag{2.44}$$

Given $h \in L^\infty(\Omega_\varepsilon)$, we consider the problem of finding a function ϕ such that for certain scalars c_{ij}, $i = 1,2$, $j = 1,2,\cdots,k$, it satisfies

$$\begin{cases} L(\phi) = h + \sum\limits_{i=1}^{2}\sum\limits_{j=1}^{k} c_{ij}Z_{ij}\eta_j, & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \phi Z_{ij}\eta_j = 0 & \text{for } i = 1,2,\ j = 1,\cdots,k. \end{cases} \tag{2.45}$$

Consider the norm

$$\|\phi\|_\infty = \sup_{y\in\Omega_\varepsilon} |\phi(y)|.$$

The main result of this section is the following:

Proposition 2.5. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that for any points ξ_j, $j = 1, \cdots, k$, in Ω, satisfying (2.1), μ_j is given by (2.22), and $h \in L^\infty(\Omega_\varepsilon)$, there is a unique solution $\phi := T_\lambda(h)$ to problem (2.45) for all $\lambda \leq \lambda_0$. Moreover,*

$$\|\phi\|_\infty \leq C \left(\log \frac{1}{\varepsilon} \right) \|h\|_*. \tag{2.46}$$

The proof will be split into a series of lemmas which we state and prove next.

Lemma 2.6. *The operator L satisfies the maximum principle in $\tilde{\Omega}_\varepsilon = \Omega_\varepsilon \setminus \bigcup_{j=1}^{k} B(\xi'_j, R)$ for R large. Namely, if $L(\phi) \leq 0$ in $\tilde{\Omega}_\varepsilon$ and $\phi \geq 0$ on $\partial\tilde{\Omega}_\varepsilon$, then $\phi \geq 0$ in $\tilde{\Omega}_\varepsilon$.*

Proof. Given $a > 0$, we consider the function

$$Z(y) = \sum_{j=1}^{k} z_0\left(a|y - \xi'_j|\right), \qquad y \in \Omega_\varepsilon, \tag{2.47}$$

where $z_0(r) = \frac{r^2-1}{r^2+1}$ is the radial solution in $\mathbb{R}^2$ of

$$\Delta z_0 + \frac{8}{(1+r^2)^2} z_0 = 0.$$

First, we observe that, if $|y - \xi'_j| \geq R$ for $R > \frac{1}{a}$, then $Z(y) > 0$. By the definition of z_0 we have

$$\begin{aligned} -\Delta Z(y) &= \sum_{j=1}^{k} \frac{8a^2(a^2|y-\xi'_j|^2 - 1)}{(1 + a^2|y-\xi'_j|^2)^3} \\ &\geq \sum_{j=1}^{k} \frac{1}{3} \frac{8a^2}{(1 + a^2|y-\xi'_j|^2)^2} \\ &\geq \sum_{j=1}^{k} \frac{4}{27} \frac{8}{a^2|y-\xi'_j|^4} \end{aligned}$$

provided $R > \frac{\sqrt{2}}{a}$. On the other hand, in the same region, we have

$$g'(V_\lambda)Z(x) \leq D_0 \sum_{j=1}^{k} e^{w_j} Z(y) \leq D_0 \sum_{j=1}^{k} \frac{C}{|y - \xi'_j|^4},$$

for some constant $C > 0$ and D_0 satisfies (2.35). Hence if a is taken small and fixed, and $R > 0$ is chosen sufficiently large depending on this a, then we have $L(Z) < 0$ in $\tilde{\Omega}_\varepsilon$, The function $Z(y)$ is what we are looking for. □

Let us fix such a number $R > 0$ which we may take large whenever it is needed. Define the " inner norm " of ϕ in the following way

$$\|\phi\|_i = \sup_{y \in \cup_{j=1}^k B(\xi_j', R)} |\phi(y)|.$$

Lemma 2.7. *There exists a uniform constant $C > 0$ such that if $L(\phi) = h$ in Ω_ε, $\phi = 0$ on $\partial\Omega_\varepsilon$, then*

$$\|\phi\|_\infty \leq C[\|\phi\|_i + \|h\|_*], \tag{2.48}$$

for any $h \in L^\infty(\Omega_\varepsilon)$.

Proof. We will establish this estimate with the use of suitable barriers. Let M be large, such that $\Omega_\varepsilon \subset B(\xi_j', \frac{M}{\varepsilon})$ for all j. Consider the solution ψ_j of the following problem

$$\begin{cases} -\Delta\psi_j = \frac{2}{|y-\xi_j'|^3} + 2\varepsilon^2, & R < |y - \xi_j'| < \frac{M}{\varepsilon}; \\ \psi_j(y) = 0 \ \text{ for } |y - \xi_j'| = R, & |y - \xi_j'| = \frac{M}{\varepsilon}. \end{cases}$$

We observe that by the direct computation we have that

$$\psi_j(r) = \frac{1}{R} - \frac{1}{r} - \varepsilon^2(r - R) - \left[\frac{1}{R} - \frac{1}{r} - \varepsilon^2\left(\frac{M}{\varepsilon} - R\right)\right]\frac{\log\frac{r}{R}}{\log\frac{M}{\varepsilon R}}.$$

Therefore, this function is uniform bound independent of ε as long as $a < R < \frac{1}{2\varepsilon}$.

Define now the function

$$\tilde{\phi}(y) = 2\|\phi\|_i Z(y) + \|h\|_* \sum_{j=1}^k \psi_j(y),$$

where Z is the function defined in (2.47). First, observe that by the definition of Z, choosing R large if necessary,

$$\tilde{\phi}(y) \geq 2\|\phi\|_i Z(y) \geq \|\phi\|_i \geq |\phi(y)| \qquad \text{for } \ |y - \xi_j'| = R, \ \ j = 1, \cdots, k,$$

and, by the positivity of $Z(y)$ and $\psi_j(y)$,

$$\tilde{\phi}(y) \geq 0 = \phi(y) \qquad \text{for } y \in \partial\Omega_\varepsilon.$$

Finally, by the definition of $\|\cdot\|_*$ we have that

$$|h(y)| \leq \left(\sum_{j=1}^k (1 + |y - \xi_j'|)^{-3} + \varepsilon^2\right)\|h\|_*,$$

then

$$\begin{aligned} L(\tilde{\phi}) &= 2\|\phi\|_i L(Z) + \|h\|_* L(\sum_{j=1}^{k} \psi_j) \leq \|h\|_* \sum_{j=1}^{k} (\Delta\psi_j + g'(V_\lambda)\psi_j) \\ &= \|h\|_* \sum_{j=1}^{k} \left(-\frac{2}{|y-\xi_j'|^3} - 2\varepsilon^2 + g'(V_\lambda)\psi_j \right) \\ &\leq \|h\|_* \sum_{j=1}^{k} \left(-\frac{2}{|y-\xi_j'|^3} - 2\varepsilon^2 + \frac{2kD_0}{R} e^{w_j} \right) \\ &\leq -\|h\|_* \left(\sum_{j=1}^{k} (1+|y-\xi_j'|)^{-3} + \varepsilon^2 \right) \\ &\leq -|h(y)| \leq |L(\phi)(y)|, \end{aligned}$$

provided R large enough. Hence, from Lemma 2.6, we obtain that

$$|\phi(y)| \leq \tilde{\phi}(y) \qquad \text{for } y \in \tilde{\Omega}_\varepsilon,$$

and, since $Z(y) \leq 1$ we get

$$\|\phi\|_\infty \leq C[\|\phi\|_i + \|h\|_*].$$

□

Next we prove uniform a priori estimates for the problem (2.45) when ϕ satisfies additionally orthogonality under dilations. Specifically, we consider the problem

$$\begin{cases} L(\phi) = h, & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \eta_j Z_{ij} \phi = 0 & \text{for } i = 0,1,2, \ j = 1, \cdots, k, \end{cases} \tag{2.49}$$

and prove the following estimate.

Lemma 2.8. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that for any points ξ_j, $j = 1, \cdots, k$, in Ω, satisfying (2.1), μ_j is given by (2.22), and $h \in L^\infty(\Omega_\varepsilon)$, and any solution ϕ to problem (2.49), one has*

$$\|\phi\|_\infty \leq C\|h\|_*. \tag{2.50}$$

Proof. We carry out the proof of lemma by a contradiction. If the result was false, then there exist a sequence $\lambda_n \to 0$, points $\xi_j^n \in \Omega$, $j = 1, \cdots, k$ in Ω, satisfying (2.1), function h_n with $\|h_n\|_* \to 0$ and ϕ_n with $\|\phi_n\|_\infty = 1$,

$$\begin{cases} L(\phi_n) = h_n & \text{in } \Omega_{\varepsilon_n}; \\ \phi_n = 0 & \text{on } \partial\Omega_{\varepsilon_n}; \\ \int_{\Omega_\varepsilon} \eta_j Z_{ij} \phi_n = 0 & \text{for all } i = 0,1,2, \ j = 1, \cdots, k. \end{cases} \tag{2.51}$$

Then from lemma 2.7, we see that $\|\phi_n\|_i$ stays away from zero. Up to a subsequence, for one of the indices, say j, we can assume that there exists $R > 0$ such that,

$$\sup_{|y-(\xi_j^n)'|<R} |\phi_n(y)| \geq \kappa > 0 \quad \text{for all } n.$$

Let us set $\hat{\phi}_n(z) = \phi_n((\xi_j^n)' + z)$. Elliptic estimate allow us to assume that $\hat{\phi}_n$ converges uniformly over compact subsets of $\mathbb{R}^2$ to a bounded, nonzero solution $\hat{\phi}$ of

$$\Delta\phi + \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2}\phi = 0.$$

This implies that $\hat{\phi}$ is a linear combination of the functions z_{ij}, $i = 0, 1, 2$. But orthogonality conditions over $\hat{\phi}_n$ pass to the limit thanks to $\|\hat{\phi}_n\|_\infty \leq 1$. By the dominated convergence theorem then yields that $\int_{\mathbb{R}^2} \eta(z) z_{ij}\hat{\phi} = 0$ for $i = 0, 1, 2$, thus a contradiction with $\liminf_{n\to\infty} \|\phi_n\|_i > 0$. □

Now we establish a priori estimates for the problem (2.49) with the orthogonality condition $\int_{\Omega_\varepsilon} \eta_j Z_{0j}\phi = 0$ dropped. We consider the problem

$$\begin{cases} L(\phi) = h & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \eta_j Z_{ij}\phi = 0 & \text{for } i = 1, 2, \ j = 1, \cdots, k, \end{cases} \tag{2.52}$$

Lemma 2.9. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that for any points $\xi_j \in \Omega$, $j = 1, \cdots, k$, satisfying (2.1), μ_j is given by (2.22), and $h \in L^\infty(\Omega_\varepsilon)$, and any solution ϕ to problem (2.52), one has*

$$\|\phi\|_\infty \leq C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*, \tag{2.53}$$

for all $\lambda < \lambda_0$.

Proof. The proof is already contained in [36] but we reproduce it here for sake of completeness. Let $R > R_0 + 1$ be a large and fixed number, and $\hat{z}_0$ be the solution of the problem

$$\begin{cases} \Delta\hat{z}_{0j} + \frac{8\mu_j^2}{\left(\mu_j^2+|y-\xi_j'|^2\right)^2}\hat{z}_{0j} = 0, & \\ \hat{z}_{0j}(y) = z_{0j}(R) & \text{for } |y-\xi_j'| = R, \\ \hat{z}_{0j}(y) = 0 & \text{for } |y-\xi_j'| = \frac{\delta}{3\varepsilon}. \end{cases}$$

By computation, this function is explicitly given by

$$\hat{z}_{0j}(y) = z_{0j}(y)\left[1 - \frac{\int_R^r \frac{ds}{sz_{0j}^2(s)}}{\int_R^{\frac{\delta}{3\varepsilon}} \frac{ds}{sz_{0j}^2(s)}}\right], \qquad r = |y-\xi_j'|.$$

Next we consider the radial smooth cut-off functions χ_1 and χ_2 withe the following properties:

$$0 \le \chi_1 \le 1, \quad \chi_1 \equiv 1 \text{ in } B(0,R), \quad \chi_1 \equiv 0 \text{ in } B(0,R+1)^c; \ \textit{and}$$
$$0 \le \chi_2 \le 1, \quad \chi_2 \equiv 1 \text{ in } B(0,\frac{\delta}{4\varepsilon}), \quad \chi_1 \equiv 0 \text{ in } B\left(0,\frac{\delta}{3\varepsilon}\right)^c,$$

and $|\chi_2'(r)| \le C\varepsilon$, $|\chi_2''(r)| \le C\varepsilon^2$. Then we set

$$\chi_{1j}(y) = \chi_1(|y-\xi_j'|), \qquad \chi_{2j}(y) = \chi_2(|y-\xi_j'|),$$

and define the test function

$$\tilde{z}_{0j} = \chi_{1j} Z_{0j} + (1-\chi_{1j})\chi_{2j}\hat{z}_{0j}.$$

Let ϕ be a solution to (2.52), we will modify ϕ so that the extra orthogonality conditions with respect to Z_{0j} is satisfies. We set

$$\tilde{\phi} = \phi + \sum_{j=1}^{k} d_j \tilde{z}_{0j}$$

with the number d_j is defined as

$$d_j = -\frac{\int_{\Omega_\varepsilon} \eta_j Z_{0j}\phi}{\int_{\Omega_\varepsilon} \eta_j |Z_{0j}|^2}.$$

Then

$$L(\tilde{\phi}) = h + \sum_{j=1}^{k} d_j L(\tilde{z}_{0j}), \tag{2.54}$$

and the orthogonality condition

$$\int_{\Omega_\varepsilon} \eta_j Z_{0i}\tilde{\phi} = 0, \quad \text{for all } i = 0,1,2,$$

hold. Then from the previous lemma we have the following estimate

$$\|\tilde{\phi}\|_\infty \le C[\|h\|_* + \sum_{j=1}^{k} |d_j| \|L(\tilde{z}_{0j})\|_*]. \tag{2.55}$$

Next, we show that

$$\|L(\tilde{z}_{0j})\|_* \le \frac{C}{\log\frac{1}{\varepsilon}}, \quad \text{and} \quad |d_j| \le C\left(\log\frac{1}{\varepsilon}\right)^2 \|h\|_*. \tag{2.56}$$

Indeed, we have

$$\begin{aligned} L(\tilde{z}_{0j}) &= 2\nabla\chi_{1j}\nabla(Z_{0j}-\hat{z}_{0j})+\Delta\chi_{1j}(Z_{0j}-\hat{z}_{0j}) \\ &\quad +2\nabla\chi_{2j}\nabla\hat{z}_{0j}+\Delta\chi_{2j}\hat{z}_{0j}+O(\varepsilon^4). \end{aligned}$$

We consider the following four regions

$$\begin{aligned} \Omega_1 &= \{y\ :\ |y-\xi_j'|\le R\}, \qquad \Omega_2=\{y\ :\ R<|y-\xi_j'|<R+1\}, \\ \Omega_3 &= \{y\ :\ R+1\le|y-\xi_j'|\le\frac{\delta}{4\varepsilon}\}, \qquad \Omega_2=\{y\ :\ \frac{\delta}{4\varepsilon}<|y-\xi_j'|<\frac{\delta}{3\varepsilon}\}. \end{aligned}$$

First, we note that $L(\tilde{z}_0)=O(\varepsilon^4)$ for $y\in\Omega_1\cup\Omega_3$. For $y\in\Omega_2$, we have

$$\hat{z}_{0j}-Z_{0j}=-z_{0j}(r)\frac{\int_R^r\frac{ds}{sz_{0j}^2(s)}}{\int_R^{\frac{\delta}{3\varepsilon}}\frac{ds}{sz_{0j}^2(s)}},$$

so that

$$|\hat{z}_{0j}-Z_{0j}|\le\frac{C}{\log\frac{1}{\varepsilon}}.$$

Similarly, in this region, we have

$$|\hat{z}_{0j}'-Z_{0j}'|\le\frac{C}{\log\frac{1}{\varepsilon}}.$$

On the other hand, for $y\in\Omega_4$, we have

$$\hat{z}_{0j}(r)\le\frac{C}{\log\frac{1}{\varepsilon}},\quad\text{and}\quad\hat{z}_{0j}'(r)\le\frac{C\varepsilon}{\log\frac{1}{\varepsilon}}.$$

Therefore, from the definition of the $*-$norm, we get

$$\|L(\tilde{z}_{0j})\|_*\le\frac{C}{\log\frac{1}{\varepsilon}}, \tag{2.57}$$

where the number C depends in principle of the chosen large constant R.

Next we show the other inequality of (2.56) holds. Testing equation (2.54) against $\tilde{z}_{0l}$ we have

$$\langle\tilde{\phi},L(\tilde{z}_{0l})\rangle=\langle h,\tilde{z}_{0l}\rangle+d_l\langle L(\tilde{z}_{0l}),\tilde{z}_{0l}\rangle,$$

where $\langle f,g\rangle=\int_{\Omega_\varepsilon}fg$. This relation and (2.55) gives us that

$$d_l\langle L(\tilde{z}_{0l}),\tilde{z}_{0l}\rangle\le C\|h\|_*[1+\|L(\tilde{z}_{0l})\|_*]+C\sum_{j=1}^k|d_j|\|L(\tilde{z}_{0l})\|_*^2. \tag{2.58}$$

We want to measure the size of $\langle L(\tilde z_{0l}), \tilde z_{0l}\rangle$. We decompose

$$\langle L(\tilde z_{0l}), \tilde z_{0l}\rangle = \int_{\Omega_2} L(\tilde z_{0l})\tilde z_{0l} + \int_{\Omega_4} L(\tilde z_{0l})\tilde z_{0l} + O(\varepsilon). \tag{2.59}$$

Since

$$\left|\int_{\Omega_4} L(\tilde z_{0l})\tilde z_{0l}\right| \le C\int |\nabla\chi_{2l}||\nabla\hat z_{0l}||\hat z_{0l}| + C\int |\Delta\chi_{2l}||\hat z_{0l}|^2 + O(\varepsilon^2) \le \frac{C}{\left(\log\frac{1}{\varepsilon}\right)^2}. \tag{2.60}$$

Moreover, for $y \in \Omega_2$, we have

$$\begin{aligned}\int_{\Omega_2} L(\tilde z_{0l})\tilde z_{0l} &= 2\int \nabla\chi_{1l}\nabla(Z_{0l} - \hat z_{0l})\hat z_0 + \int \Delta\chi_{1l}(Z_{0l} - \hat z_{0l})\hat z_{0l} + O(\varepsilon)\\ &= \int \nabla\chi_{1l}\nabla(Z_{0l} - \hat z_{0l})\hat z_{0l} - \int \nabla\chi_{1l}(Z_{0l} - \hat z_{0l})\nabla\hat z_{0l} + O(\varepsilon),\end{aligned}$$

from the integration by parts. Now, we observe that in the considered region Ω_2, $|\hat z_{0l} - Z_{0l}| \le \frac{C}{\log\frac{1}{\varepsilon}}$, while $|\hat z_{0l}'| \sim \frac{1}{R^3} + \frac{1}{R}\frac{1}{\log\frac{1}{\varepsilon}}$. Then, for R is large but independent of ε we have

$$\left|\int \nabla\chi_{1l}(Z_{0l} - \hat z_{0l})\nabla\hat z_{0l}\right| \le \frac{C_1}{R^3}\frac{1}{\log\frac{1}{\varepsilon}},$$

with C_1 is a constant to be chosen independent R. Moreover

$$\begin{aligned}\int \nabla\chi_{1l}\nabla(Z_{0l} - \hat z_{0l})\hat z_{0l} &= 2\pi\int_R^{R+1} \chi_{1l}'(z_{0l} - \hat z_{0l})'\hat z_{0l} r\, dr\\ &= \frac{2\pi}{\int_R^{\frac{\delta}{3\varepsilon}}\frac{ds}{sz_{0l}^2}}\int_R^{R+1}\chi_{1l}'\left[1 - \frac{4\mu_l^2 r^2 z_{0l}\int_R^r\frac{ds}{sz_{0l}^2}}{(\mu_l^2 + r^2)^2}\right] dr\\ &= -\frac{C_2}{\log\frac{1}{\varepsilon}}\left[1 + O\left(\frac{1}{\log\frac{1}{\varepsilon}}\right)\right],\end{aligned}$$

where C_2 is a positive constant independent on ε. Thus, choosing R large enough, we get

$$\int_{\Omega_2} L(\tilde z_{0l})\tilde z_{0l} \sim -\frac{C_2}{\log\frac{1}{\varepsilon}}.$$

Combining this and (2.59), (2.60) we get

$$\langle L(\tilde z_{0l}), \tilde z_{0l}\rangle \le -\frac{C_2}{\log\frac{1}{\varepsilon}}\left[1 + O\left(\frac{1}{\log\frac{1}{\varepsilon}}\right)\right]. \tag{2.61}$$

From (2.57), (2.58) and (2.60) we have

$$|d_j| \le C\left(\log\frac{1}{\varepsilon}\right)^2 \|h\|_*.$$

We thus from estimate (2.55) that

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*.$$

□

Proof of Proposition 2.5 We first establish the validity of the a priori estimate (2.46). The previous lemma yields

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\left[\|h\|_* + \sum_{i=1}^{2}\sum_{j=1}^{k}|c_{ij}|\right]. \tag{2.62}$$

Let us consider the cut-off function χ_{2j} defined in previous lemma. We multiply the first equation of (2.45) by $Z_{ij}\chi_{2j}$, we find

$$\langle L(\phi), Z_{ij}\chi_{2j}\rangle = \langle h, Z_{ij}\chi_{2j}\rangle + c_{ij}\int_{\Omega_\varepsilon}\eta_j|Z_{ij}|^2. \tag{2.63}$$

We have

$$L(Z_{ij}\chi_{2j}) = \Delta\chi_{2j}Z_{ij} + 2\nabla Z_{ij}\nabla\chi_{2j} + \varepsilon O((1+r)^{-3}),$$

with $r = |y-\xi_j'|$. Since $\Delta\chi_{2j} = O(\varepsilon^2)$, $\nabla\chi_{2j} = O(\varepsilon)$, and $Z_{ij} = O(r^{-1})$, $\nabla Z_{ij} = O(r^{-2})$, we get

$$L(Z_{ij}\chi_{2j}) = O(\varepsilon^3)\varepsilon O((1+r)^{-3}).$$

Then we have

$$|\langle L(\phi), Z_{ij}\chi_{2j}\rangle| = |\langle \phi, L(Z_{ij}\chi_{2j})\rangle| \le C\varepsilon\|\phi\|_\infty.$$

Combining this with (2.62) and (2.63) we find

$$|c_{ij}| \le C\left[\|h\|_* + \varepsilon\log\frac{1}{\varepsilon}\sum_{l,m}|c_{lm}|\right]. \tag{2.64}$$

Then,

$$|c_{ij}| \le C\|h\|_*.$$

Combining this with (2.62) we obtain the estimate (2.46) holds.

Next prove the solvability of problem (2.45). We consider the Hilbert space

$$\mathbb{H} = \left\{\phi\in H_0^1(\Omega_\varepsilon)\ :\ \int_{\Omega_\varepsilon}\phi Z_{ij}\eta_j = 0\quad \text{for } i=1,2,\ j=1,2,\cdots,k\right\},$$

endowed with the usual inner product $\langle\phi,\psi\rangle = \int_{\Omega_\varepsilon}\nabla\phi\nabla\psi$. Problem (2.45), expressed in a weak form, is equivalent to find $\phi\in\mathbb{H}$ such that

$$\langle\phi,\psi\rangle = \int_{\Omega_\varepsilon}(W\phi - h)\psi\, dx, \qquad \text{for all } \psi\in\mathbb{H},$$

where $W = g'(V_\lambda)$. With the aid of Riesz's representation theorem, this equation gets rewritten in $\mathbb{H}$ in the operator form

$$(Id - K)\phi = \tilde{h}, \tag{2.65}$$

for certain $\tilde{h} \in \mathbb{H}$, where K is a compact operator in $\mathbb{H}$. The homogeneous equation $\phi = K\phi$ in $\mathbb{H}$, which is equivalent to (2.45) with $h \equiv 0$, has only the trivial solution in view of the a priori estimate (2.46). Now, Fredholm's alternative guarantees unique solvability of (2.65) for any $\tilde{h} \in \mathbb{H}$. This finishes the proof.

The result of Proposition 2.5 implies that the unique solution $\phi = T_\lambda(h)$ of (2.45) defines a continuous linear map form the Banach space $\mathcal{C}_*$ of all functions h in L^∞ for which $\|h\|_* < \infty$ into L^∞, with norm bounded uniformly in λ.

Lemma 2.10. *The operator T_λ is differentiable with respect to the variable $\xi_1, \dots, \xi_k$ in Ω satisfying (2.1), one has the estimate*

$$\|\partial_{(\xi'_m)_l} T_\lambda(h)\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)^2 \|h\|_* \quad \text{for } l = 1, 2,\ m = 1, 2, \cdots, k, \tag{2.66}$$

for a given positive C, independent of ε, and for all ε small enough.

Proof. Differentiating equation (2.45), formally $Z := \partial_{(\xi'_m)_l}\phi$ should satisfy

$$L(Z) = -\partial_{(\xi'_m)_l}(g'(V_\lambda))\phi + \sum_{i=1}^{2} c_{im}\partial_{(\xi'_m)_l}(\eta_m Z_{im}) + \sum_{i=1}^{2}\sum_{j=1}^{k} d_{ij} Z_{ij}\eta_j$$

with $d_{ij} = \partial_{(\xi'_m)_l} c_{ij}$, and the orthogonality conditions now become

$$\int_{\Omega_\varepsilon} Z_{im}\eta_m Z = -\int_{\Omega_\varepsilon} \partial_{(\xi'_m)_l}(Z_{lm}\eta_m)\,\phi.$$

We consider the constants b_{im} defined as

$$b_{im}\int_{\Omega_\varepsilon} \eta_m Z_{im}^2 = \int_{\Omega_\varepsilon} \partial_{(\xi'_m)_l}(Z_{im}\eta_m)\,\phi, \qquad \text{for } \ l = 1, 2.$$

Define

$$\tilde{Z} = Z + \sum_{i=1}^{2} b_{im}\eta_m Z_{im},$$

and

$$f = -\partial_{(\xi'_m)_l}(g'(V_\lambda))\,\phi + \sum_{i=1}^{2} c_{im}\partial_{(\xi'_m)_l}(Z_{im}\eta_m) + \sum_{i=1}^{2} b_{im} L(\eta_m Z_{im}).$$

We then have

$$\begin{cases} L(\tilde{Z}) = f + \sum_{i=1}^{2}\sum_{j=1}^{k} b_{im}\eta_m Z_{im} & \text{in } \Omega_\varepsilon; \\ \tilde{Z} = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \eta_m Z_{im}\tilde{Z} = 0 & \text{for } i = 0, 1, 2. \end{cases}$$

Namely, $\tilde{Z} = T_\lambda(f)$. Using the result of Proposition 2.5 we find that

$$\|f\|_* \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*,$$

hence,

$$\|\partial_{(\xi'_m)_l} T_\lambda(h)\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)^2\|h\|_* \ \text{ for } l = 1, 2, \quad m = 1, 2, \cdots, k.$$

□

2.4 The nonlinear problem

In what follows we keep the notation introduced in the previous sections. We recall that our goal is to solve problem (2.45). The strategy is to solve first the following problem

$$\begin{cases} L(\phi) = -[E_\lambda + N(\phi)] + \sum_{i=1}^{2}\sum_{j=1}^{k} c_{ij}\eta_j Z_{ij}, & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \eta_j Z_{ij}\phi = 0 & \text{for all } i = 1, 2, \ j = 1, 2, \cdots, k. \end{cases} \tag{2.67}$$

We have the following result.

Lemma 2.11. *Under the assumptions of Proposition 2.5, there exist positive numbers C and λ_0, such that problem (2.67) has a unique solution ϕ which satisfies*

$$\|\phi\|_\infty \le \frac{C}{|\log\varepsilon|^2},$$

for all $\lambda < \lambda_0$. Moreover, if we consider the map $\xi' \mapsto \phi$ into the space $C(\bar{\Omega}_\varepsilon)$, the derivative $D_{\xi'}\phi$ exists and defines a continuous function of ξ'. Besides, there is a constant $C > 0$, such that

$$\|D_{\xi'}\phi\|_\infty \le \frac{C}{|\log\varepsilon|}. \tag{2.68}$$

Proof. In terms of the operator T_λ defined in Proposition 2.5, problem (2.67) becomes

$$\phi = T_\lambda(-(N(\phi) + E_\lambda)) := A(\phi). \tag{2.69}$$

For a given number $M > 0$, let us consider the region

$$\mathcal{F}_M := \left\{ \phi \in C(\bar{\Omega}) \ : \ \|\phi\|_\infty \leq \frac{M}{|\log \varepsilon|^2} \right\}.$$

From Proposition 2.5, we get

$$\|A(\phi)\|_\infty \leq C \left(\log \frac{1}{\varepsilon} \right) [\|N(\phi)\|_* + \|E_\lambda\|_*] .$$

By the definition of $N(\phi)$ in (2.27), we can write

$$|N(\phi)| \leq C|g''(V_\lambda + s\phi)||\phi|^2 \leq C|g''(V_\lambda + s\phi)|\|\phi\|_\infty^2$$

for some $0 < s < 1$. Thus, using the fact that $\|\phi\|_\infty \to 0$ as $\lambda \to 0$, and (2.36), we obtain

$$\|N(\phi)\|_* \leq C\|\phi\|_\infty^2$$

Thus

$$\|A(\phi)\|_\infty \ \ \leq \ \ C|\log \varepsilon| \left(C\|\phi\|_\infty^2 + \frac{1}{|\log \varepsilon|^3} \right).$$

We then get that $A(\mathcal{F}_M) \subset \mathcal{F}_M$ for a sufficiently large but fixed M and all small λ. Moreover, for any $\phi_1, \phi_2 \in \mathcal{F}_M$, one has

$$\|N(\phi_1) - N(\phi_2)\|_* \leq C \left(\max_{i=1,2} \|\phi_i\|_\infty \right) \|\phi_1 - \phi_2\|_\infty .$$

In fact,

$$\begin{aligned} N(\phi_1) - N(\phi_2) &= g(V_\lambda + \phi_1) - g(V_\lambda + \phi_2) - g'(V_\lambda)(\phi_1 - \phi_2) \\ &= \int_0^1 \left(\frac{d}{dt} g(V_\lambda + \phi_2 + t(\phi_1 - \phi_2)) \right) dt - g'(V_\lambda)(\phi_1 - \phi_2) \\ &= \int_0^1 (g'(V_\lambda + \phi_2 + t(\phi_1 - \phi_2)) - g'(V_\lambda))\, dt\, (\phi_1 - \phi_2). \end{aligned}$$

Thus, for a certain $t^* \in (0, 1)$, and $s \in (0, 1)$

$$\begin{aligned} |N(\phi_1) - N(\phi_2)| &\leq C|g'(V_\lambda + \phi_2 + t^*(\phi_1 - \phi_2)) - g'(V_\lambda)|\|\phi_1 - \phi_2\|_\infty \\ &\leq C|g''(V_\lambda + s\phi_2 + t^*(\phi_1 - \phi_2))|(\|\phi_1\|_\infty + \|\phi_2\|_\infty)\|\phi_1 - \phi_2\|_\infty . \end{aligned}$$

Thanks to (2.36) and the fact that $\|\phi_1\|_\infty$, $\|\phi_2\|_\infty \to 0$ as $\lambda \to 0$, we conclude that

$$\|N(\phi_1) - N(\phi_2)\|_* \ \ \leq \ \ C\|g''(V_\lambda)\|_*(\|\phi_1\|_\infty + \|\phi_2\|_\infty)\|\phi_1 - \phi_2\|_\infty$$

$$\leq C(\|\phi_1\|_\infty + \|\phi_2\|_\infty)\|\phi_1 - \phi_2\|_\infty.$$

Then we have

$$\begin{aligned}\|A(\phi_1) - A(\phi_2)\|_\infty &\leq C|\log\varepsilon|\|N(\phi_1) - N(\phi_2)\|_* \\ &\leq C|\log\varepsilon|\left(\max_{i=1,2}\|\phi_i\|_\infty\right)\|\phi_1 - \phi_2\|_\infty.\end{aligned}$$

Thus the operator A has a small Lipschitz constant in $\mathcal{F}_M$ for all small λ, and therefore a unique fixed point of A exists in this region.

We shall next analyze the differentiability of the map $\xi' = (\xi'_1, \cdots, \xi'_k) \mapsto \phi$. Assume for instance that the partial derivative $\partial_{(\xi'_j)_i}\phi$ exists for $i = 1, 2$. Since $\phi = T_\lambda(-(N(\phi) + E_\lambda))$, formally that

$$\partial_{(\xi'_j)_i}\phi = (\partial_{(\xi'_j)_i}T_\lambda)\left(-(N(\phi) + E_\lambda)\right) + T_\lambda\left(-(\partial_{(\xi'_j)_i}N(\phi) + \partial_{(\xi'_j)_i}E_\lambda)\right).$$

From Lemma 2.10, we have

$$\|\partial_{(\xi'_j)_i}T_\lambda\left(-(N(\phi) + E_\lambda)\right)\|_\infty \leq C|\log\varepsilon|^2\|N(\phi) + E_\lambda\|_* \leq C\frac{1}{|\log\varepsilon|}.$$

On the other hand,

$$\begin{aligned}\partial_{(\xi'_j)_i}N(\phi) &= [g'(V_\lambda + \phi) - g'(V_\lambda) - g''(V_\lambda)\phi]\partial_{(\xi'_j)_i}V_\lambda + \partial_{(\xi'_j)_i}[g'(V_\lambda) - e^{w_j}]\phi \\ &\quad + [g'(V_\lambda + \phi) - g'(V_\lambda)]\partial_{(\xi'_j)_i}\phi + [g'(V_\lambda) - e^{w_j}]\partial_{(\xi'_j)_i}\phi.\end{aligned}$$

Then,

$$\|\partial_{(\xi'_j)_i}N(\phi)\|_* \leq C\left\{\|\phi\|_\infty^2 + \frac{1}{|\log\varepsilon|}\|\phi\|_\infty + \|\partial_{(\xi'_j)_i}\phi\|_\infty\|\phi\|_\infty + \frac{1}{|\log\varepsilon|}\|\partial_{(\xi'_j)_i}\phi\|_\infty\right\}.$$

Since $\|\partial_{(\xi_j)_i}E_\lambda\|_* \leq \frac{C}{|\log\varepsilon|^3}$, and by Proposition 2.5 we then have

$$\|\partial_{(\xi'_j)_i}\phi\|_\infty \leq \frac{C}{|\log\varepsilon|},$$

for all $i = 1, 2$, $j = 1, \cdots, k$. Then, the regularity of the map $\xi' \mapsto \phi$ can be proved by standard arguments involving the implicit function theorem and the fixed point representation (2.69). This concludes proof of the Lemma. □

2.5 Variational reduction

We have solved the nonlinear problem (2.67). In order to find a solution to the original problem we need to find ξ' such that

$$c_{ij}(\xi') = 0 \quad \text{for all } i = 1, 2, \quad j = 1, \cdots, k. \tag{2.70}$$

This problem is variational: indeed it is equivalent to finding critical points of a function of $\xi = \varepsilon\xi'$. Associated to (2.1), let us consider the energy functional J_λ given by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{\lambda}{p}\int_\Omega e^{u^p}\, dx, \qquad u \in H_0^1(\Omega), \tag{2.71}$$

and the finite-dimensional restriction

$$F_\lambda(\xi) = J_\lambda\left(\left(U_\lambda + \tilde{\phi}\right)(x,\xi)\right), \tag{2.72}$$

where

$$\left(U_\lambda + \tilde{\phi}\right)(x,\xi) = \gamma + \frac{1}{p\gamma^{p-1}}\left((V_\lambda + \phi)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right) \tag{2.73}$$

with V_λ defined in (2.17), ϕ is the unique solution to problem (2.67) given by Lemma 2.11. Critical points of F_λ correspond to solutions of (2.70) for a small λ, as the following result states.

Lemma 2.12. *Under the assumptions of Proposition 2.5, the functional $F_\lambda(\xi)$ is of class C^1. Moreover, for all $\lambda > 0$ sufficiently small, if $D_\xi F_\lambda(\xi) = 0$, then ξ satisfies (2.70).*

Proof. A direct consequence of the results obtained in Lemma 2.11 and the definition of function U_λ is the fact the map $\xi \mapsto F_\lambda(\xi)$ is of class C^1. Define

$$I_\lambda(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2\, dy - \int_{\Omega_\varepsilon} e^{\gamma^p[(1+\frac{v}{p\gamma^p})^p - 1]}\, dy. \tag{2.74}$$

Let us differentiate the function $F_\lambda(\xi)$ with the respect to ξ. Since

$$J_\lambda\left(\left(U_\lambda + \tilde{\phi}\right)(x,\xi)\right) = \frac{1}{p^2\gamma^{2(p-1)}} I_\lambda\left((V_\lambda + \phi)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right), \tag{2.75}$$

we can differentiate directly $I_\lambda\left(V_\lambda(\xi) + \phi(\xi)\right)$ under the integral sign. Let $m \in \{1, \ldots, k\}$ and $l \in 1, 2$. We have

$$\begin{aligned}
& \partial_{\xi_{m,l}} F_\lambda(\xi) \\
= \;& \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1} DI_\lambda\left(V_\lambda(\xi) + \phi(\xi)\right)\left[\partial_{(\xi'_m)_l} V_\lambda(\xi) + \partial_{(\xi'_m)_l}\phi(\xi)\right] \\
= \;& \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1} \sum_{i=1}^{2}\sum_{j=1}^{k} \int_{\Omega_\varepsilon} c_{ij}\eta_j Z_{ij}\left[\partial_{(\xi'_m)_l} V_\lambda(\xi) + \partial_{(\xi'_m)_l}\phi(\xi)\right] \\
= \;& \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1}\left[\sum_{i=1}^{2}\sum_{j=1}^{k}\int_{\Omega_\varepsilon} c_{ij}\eta_j Z_{ij}\partial_{(\xi'_m)_l} V_\lambda(\xi) + \sum_{i=1}^{2}\sum_{j=1}^{k}\int_{\Omega_\varepsilon} c_{ij}\eta_j Z_{ij}\partial_{(\xi'_m)_l}\phi(\xi)\right]
\end{aligned}$$

By the expansion of V_λ, we have

$$\partial_{(\xi'_m)_l} V_\lambda$$

$$
\begin{aligned}
&= \partial_{(\xi'_m)_l}\left(\sum_{m=1}^{k}\left(PU_{\mu_m,\xi_m}(\varepsilon y)+\frac{p-1}{p}\frac{1}{\gamma^p}Pw^0_{\mu_m,\xi_m}(\varepsilon y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}Pw^1_{\mu_m,\xi_m}(\varepsilon y)\right)-p\gamma^p\right)\\
&= \partial_{(\xi'_m)_l}\left(w_m(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w^0_m(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w^1_m(y)+\theta(y)\right)\\
&= \partial_{(\xi'_m)_l}w_m(y)+\frac{p-1}{p}\frac{1}{\gamma^p}\partial_{(\xi'_m)_l}w^0_m(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\partial_{(\xi'_m)_l}w^1_m(y)+\partial_{(\xi'_m)_l}\theta(y)\\
&= -Z_{lm}+\frac{p-1}{p}\frac{1}{\gamma^p}\partial_{(\xi'_m)_l}w^0_m(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\partial_{(\xi'_m)_l}w^1_m(y)+\partial_{(\xi'_m)_l}\theta(y).
\end{aligned}
$$

Hence, for $j\neq m$, we have

$$
\int_{\Omega_\varepsilon}\eta_j Z_{ij}\partial_{(\xi'_m)_l}V_\lambda(\xi)=\left(-\int_{B(\xi'_j,R)}\eta_j Z_{ij}Z_{lm}\right)(1+O(\frac{1}{\gamma^p}))=O(\varepsilon),
$$

while for $j=m$ and $i\neq l$, by symmetry we get

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}\eta_j Z_{ij}\partial_{(\xi'_m)_l}V_\lambda(\xi)\\
&= \left(-\int_{B(\xi'_j,R)}\eta_j Z_{ij}(Z_{lm}+\frac{p-1}{p}\frac{1}{\gamma^p}\partial_{(\xi'_m)_l}w^0_m(y))\right)(1+O(\frac{1}{\gamma^{2p}}))=O(\frac{1}{\gamma^p}).
\end{aligned}
$$

If now $j=m$ and $i=l$, we get

$$
\int_{\Omega_\varepsilon}\eta_j Z_{ij}\partial_{(\xi'_m)_l}V_\lambda(\xi)=\left(-\int_{B(\xi'_m,R)}\eta_m Z_{lm}Z_{lm}\right)(1+O(\frac{1}{\gamma^p})).
$$

We thus conclude that

$$
\sum_{i=1}^{2}\sum_{j=1}^{k}\int_{\Omega_\varepsilon}c_{ij}\eta_j Z_{ij}\partial_{(\xi'_m)_l}V_\lambda(\xi)=-c_{lm}\int_{B(\xi'_m,R)}\eta_m Z_{lm}Z_{lm}+O(\frac{1}{\gamma^p}).
$$

On the other hand, given (2.68), we have that

$$
\left|\sum_{i=1}^{2}\sum_{j=1}^{k}\int_{\Omega_\varepsilon}c_{ij}\eta_j Z_{ij}\partial_{(\xi'_m)_l}\phi(\xi)\right|\leq C\sum_{i,j}|c_{ij}|\|\partial_{(\xi'_m)_l}\phi\|_\infty\leq o(1)\sum_{i,j}|c_{ij}|.
$$

Thus, if $D_\xi F_\lambda(\xi)=0$, for $i,l=1,2, j=1,2,\cdots,k$, we then have

$$
c_{lm}(\int_{\Omega_\varepsilon}\eta_m Z_{lm}Z_{lm})(1+o(1))=0,\quad m=1,\dots,k,\quad l=1,2. \tag{2.76}
$$

This concludes the proof of the Lemma. □

Next we give an asymptotic estimate of $F_\lambda(\xi)$ defined in (2.72). We have the following result.

Lemma 2.13. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that μ_j are given by (2.17), the following expansion holds*

$$\begin{aligned}\lambda^{-1}\varepsilon^{\frac{2(2-p)}{p}}F_\lambda(\xi) &= \frac{8k\pi}{(2-p)p}[-2+p\log 8] - \frac{16k\pi}{p}\log\varepsilon \\ &\quad -\frac{4\pi}{2-p}\varphi_k(\xi) + |\log\varepsilon|^{-1}\theta_\lambda(\xi)\end{aligned} \tag{2.77}$$

uniformly for any points ξ_j, $j = 1, \cdots, k$ in Ω, satisfying (2.1), where

$$\varphi_k(\xi) = \varphi_k(\xi_1, \cdots, \xi_k) = \sum_{j=1}^{k} H_\Omega(\xi_j, \xi_j) + \sum_{i\neq j} G_\Omega(\xi_i, \xi_j). \tag{2.78}$$

Furthermore

$$\lambda^{-1}\varepsilon^{\frac{2(2-p)}{p}}\nabla_{(\xi_m)_l}F_\lambda(\xi) = -\frac{4\pi}{(2-p)p}\nabla_{(\xi_m)_l}\varphi_k(\xi) + |\log\varepsilon|^{-1}\theta_\lambda(\xi). \tag{2.79}$$

In (2.77) and (2.79), the function θ_λ denotes a smooth function of the points ξ, which is uniformly bounded, as $\lambda \to 0$, for points ξ satisfying (2.1).

Proof. We have

$$\begin{aligned}F_\lambda(\xi) &= J_\lambda\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right) \\ &= \frac{1}{2}\int_\Omega |\nabla\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right)|^2\, dx - \frac{\lambda}{p}\int_\Omega e^{\left(U_\lambda(\xi)+\tilde{\phi}(\xi)\right)^p}\, dx.\end{aligned} \tag{2.80}$$

Using the change of variables (4.3), namely $\left(U_\lambda + \tilde{\phi}\right)(x,\xi) = \gamma + \frac{1}{p\gamma^{p-1}}\left((V_\lambda + \phi)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right)$, together with (2.74) and (2.75), we have that

$$J_\lambda\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right) - J_\lambda\left(U_\lambda(\xi)\right) = \frac{1}{p^2\gamma^{2(p-1)}}\left[I_\lambda(V_\lambda + \phi) - I_\lambda(V_\lambda)\right]$$

Since by construction $I'_\lambda(V_\lambda + \phi)[\phi] = 0$, we have

$$J_\lambda\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right) - J_\lambda\left(U_\lambda(\xi)\right) = \frac{1}{p^2\gamma^{2(p-1)}}\int_0^1 D^2I_\lambda(V_\lambda + t\phi)\phi^2(1-t)\, dt$$

$$= \frac{1}{p^2\gamma^{2(p-1)}}\int_0^1\left[\int_{\Omega_\varepsilon}(E_\lambda + N(\phi))\phi + \int_{\Omega_\varepsilon}[g'_\lambda(V_\lambda) - g'_\lambda(V_\lambda + t\phi)]\phi^2\right](1-t)\, dt$$

Since $\|E_\lambda\|_* \le \frac{c}{|\log\varepsilon|^3}$, $\|\phi\|_\infty \le \frac{c}{|\log\varepsilon|^2}$, $\|N(\phi)\|_* \le \frac{c}{|\log\varepsilon|^4}$ and (2.36), we get that

$$\left|J_\lambda\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right) - J_\lambda\left(U_\lambda(\xi)\right)\right| \le \frac{C}{\gamma^{2(p-1)}|\log\varepsilon|^3} \tag{2.81}$$

Next we expand

$$J_\lambda\left(U_\lambda(\xi)\right)=\frac{1}{2}\int_\Omega|\nabla\left(U_\lambda(\xi)\right)|^2\,dx-\frac{\lambda}{p}\int_\Omega e^{(U_\lambda(\xi))^p}\,dx. \tag{2.82}$$

First we expand the term $\int_\Omega|\nabla U_\lambda|^2$. By (2.23) we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega|\nabla\left(U_\lambda(\xi)\right)|^2\\
=\ &\frac{1}{2}\frac{1}{p^2\gamma^{2(p-1)}}\left\{\sum_{j=1}^k\int_\Omega|\nabla PU_{\mu_j,\xi_j}|^2+\sum_{l\neq j}\int_\Omega\nabla PU_{\mu_l,\xi_l}\nabla PU_{\mu_j,\xi_j}\right.\\
&+\frac{p-1}{p}\frac{1}{\gamma^p}\sum_{j=1}^k\int_\Omega\nabla PU_{\mu_j,\xi_j}(x)\nabla Pw^0_{\mu_j,\xi_j}(x)\\
&+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\sum_{j=1}^k\int_\Omega\nabla PU_{\mu_j,\xi_j}\nabla Pw^1_{\mu_j,\xi_j}\\
&+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left[\sum_{j=1}^k\int_\Omega|\nabla Pw^0_{\mu_j,\xi_j}|^2+\sum_{l\neq j}\int_\Omega\nabla Pw^0_{\mu_l,\xi_l}\nabla Pw^0_{\mu_j,\xi_j}\right]\\
&+\left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}\sum_{j=1}^k\int_\Omega\nabla Pw^0_{\mu_j,\xi_j}\nabla Pw^1_{\mu_j,\xi_j}\\
&\left.+\left(\frac{p-1}{p}\right)^4\frac{1}{\gamma^{4p}}\left[\sum_{j=1}^k\int_\Omega|\nabla w^1_{\mu_j,\xi_j}|^2+\sum_{l\neq j}\int_\Omega\nabla Pw^1_{\mu_l,\xi_l}\nabla Pw^1_{\mu_j,\xi_j}\right]\right\}.
\end{aligned} \tag{2.83}$$

Let us estimate the first two terms. We observe that the remaining terms are $O(\frac{1}{\gamma^{2(p-1)}\gamma^p})$. First, we note that PU_{μ_j,ξ_j} satisfies

$$-\Delta PU_{\mu_j,\xi_j}=\varepsilon^2e^{U_{\mu_j,\xi_j}},\quad \text{in }\Omega,\quad PU_{\mu_j,\xi_j}=0\quad\text{on }\partial\Omega.$$

Then we have

$$\begin{aligned}
&\int_\Omega|\nabla PU_{\mu_j,\xi_j}(x)|^2=\varepsilon^2\int_\Omega e^{U_{\mu_j,\xi_j}}PU_{\mu_j,\xi_j}(x)\\
=\ &\varepsilon^2\int_\Omega e^{U_{\mu_j,\xi_j}}\left(U_{\mu_j,\xi_j}(x)+H_\Omega(x,\xi_j)-\log(8\mu_j^2)+O(\mu_j^2\varepsilon^2)\right)\\
=\ &\int_\Omega\frac{8\varepsilon^2\mu_j^2}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2}\left(\log\frac{1}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2}+H_\Omega(x,\xi_j)+O(\mu_j^2\varepsilon^2)\right)\\
=\ &\int_\Omega\frac{8\varepsilon^{-2}\mu_j^{-2}}{(1+|\frac{x-\xi_j}{\varepsilon\mu_j}|^2)^2}\left(\log\frac{\varepsilon^{-4}\mu_j^{-4}}{(1+|\frac{x-\xi_j}{\varepsilon\mu_j}|^2)^2}+H_\Omega(x,\xi_j)+O(\mu_j^2\varepsilon^2)\right)\\
=\ &\int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|z|^2)^2}\left(\log\frac{1}{(1+|z|^2)^2}+H_\Omega(\xi_j+\varepsilon\mu_jz,\xi_j)-4\log(\varepsilon\mu_j)\right)+O(\mu_j^2\varepsilon^2)
\end{aligned}$$

$$\begin{aligned} &= \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2} \log \frac{1}{(1+|z|^2)^2} + \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2} \left(H_\Omega(\xi_j+\varepsilon\mu_j z, \xi_j) - H_\Omega(\xi_j,\xi_j)\right) \\ &\quad + \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|z|^2)^2} H_\Omega(\xi_j,\xi_j) - 4\log(\varepsilon\mu_j) \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} + O(\mu_j^2\varepsilon^2). \end{aligned} \tag{2.84}$$

But

$$\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} = 8\pi + O(\varepsilon), \tag{2.85}$$

and

$$\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} \log \frac{1}{(1+|y|^2)^2} = -16\pi + O(\varepsilon). \tag{2.86}$$

Moreover,

$$\begin{aligned} &\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} \left(H_\Omega(\xi_j+\varepsilon\mu_j y, \xi_j) - H_\Omega(\xi_j,\xi_j)\right) \\ = &\int_{\Omega_{\varepsilon\mu_j}} \frac{1}{(1+|y|^2)^2} O\left(\varepsilon^\alpha |y|^\alpha\right) = O(\varepsilon). \end{aligned} \tag{2.87}$$

Therefore from (2.84)-(2.87), we have

$$\begin{aligned} &\int_\Omega |\nabla PU_{\mu_j,\xi_j}(x)|^2 \, dx \\ = &-16\pi + 8\pi H_\Omega(\xi_j,\xi_j) - 32\pi\log\varepsilon - 16\pi\log(8\mu_j^2) \\ &+16\pi\log(8) + O\left(\frac{1}{\gamma^p}\right). \end{aligned} \tag{2.88}$$

Now, we calculate that

$$\begin{aligned} &\sum_{l\neq j} \int_\Omega \nabla PU_{\mu_l,\xi_l} \nabla PU_{\mu_j,\xi_j} \, dx = \sum_{l\neq j} \int_\Omega \varepsilon^2 e^{U_{\mu_l,\xi_l}} PU_{\mu_j,\xi_j} \\ = &\sum_{l\neq j} \int_\Omega \frac{8\varepsilon^2\mu_l^2}{(\varepsilon^2\mu_l^2+|x-\xi_l|^2)^2} \left(\log \frac{8\mu_j^2}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2} + H_\Omega(x,\xi_j) - \log(8\mu_j^2) + O(\mu_j^2\varepsilon^2)\right) \\ = &\sum_{l\neq j} \int_\Omega \frac{8\varepsilon^2\mu_l^2}{(\varepsilon^2\mu_l^2+|x-\xi_l|^2)^2} \left(\log \frac{1}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2} + H_\Omega(x,\xi_j) + O(\mu_j^2\varepsilon^2)\right) \\ = &\sum_{l\neq j} \int_{\Omega_{\varepsilon\mu_l}} \frac{8}{(1+|z|^2)^2} \left(\log \frac{1}{(\varepsilon^2\mu_j^2+|\varepsilon\mu_l z+\xi_l-\xi_j|^2)^2} + H_\Omega(\xi_l+\varepsilon\mu_l z, \xi_j)\right) + O(\mu_j^2\varepsilon^2) \\ = &\sum_{l\neq j} \int_{\Omega_{\varepsilon\mu_l}} \frac{8}{(1+|z|^2)^2} G(\xi_l,\xi_j) + O(\mu_j^2\varepsilon^2) \end{aligned}$$

$$= \quad 8\pi \sum_{l\neq j} G(\xi_l, \xi_j) + O(\mu_j^2 \varepsilon^2). \tag{2.89}$$

Thus, from (2.83), (2.88), (2.89) and (2.22) we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla U_\lambda(x)|^2\, dx \\ = \;& \frac{1}{p^2\gamma^{2(p-1)}}\Big\{-8k\pi - 16k\pi\log\varepsilon + 8k\pi\log(8) - 8k\pi\frac{2(p-1)}{2-p}(1-\log 8) \\ &- \frac{4p\pi}{2-p}\left(\sum_{j=1}^k H_\Omega(\xi_j,\xi_j) + \sum_{i\neq j}^k G_\Omega(\xi_i,\xi_j)\right) + O\left(\frac{1}{|\log\varepsilon|}\right)\Big\}. \end{aligned} \tag{2.90}$$

Finally, let us evaluate the second term in the energy

$$\begin{aligned} \frac{\lambda}{p}\int_\Omega e^{(U_\lambda)^p}\, dx \;&=\; \frac{\lambda}{p}\int_\Omega e^{\gamma^p\left(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon})\right)^p}\, dx \\ &=\; \frac{\lambda}{p}\sum_{j=1}^k \int_{B(\xi_j,\tilde\delta)} e^{\gamma^p\left(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon})\right)^p}\, dx \\ &\quad + \frac{\lambda}{p}\int_{\Omega\setminus\bigcup_{j=1}^k B(\xi_j,\tilde\delta)} e^{\gamma^p\left(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon})\right)^p}\, dx \\ &:=\; I + II. \end{aligned} \tag{2.91}$$

First we observe that

$$II = \lambda\Theta_\lambda(\xi) \tag{2.92}$$

with $\Theta_\lambda(\xi)$ a function, uniformly bounded, as $\lambda\to 0$. On the other hand,

$$\begin{aligned} I \;&=\; \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^k \int_{B(\xi_j',\tilde\delta/\varepsilon)} e^{\gamma^p\left[\left(1+\frac{1}{p\gamma^p}(V_\lambda)(y)\right)^p-1\right]}\, dy \\ &=\; \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^k \int_{B(\xi_j',\tilde\delta/\varepsilon)} e^{\left\{w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_j^0(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_j^1(y)+\theta(y)\right\}}(1+O(\frac{1}{\gamma^p}))dy \\ &=\; \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^k \int_{B(0,\frac{\tilde\delta}{\mu_j\varepsilon})} \frac{8}{(1+|y|^2)^2}\left(1+O(\frac{1}{\gamma^p})\right)\, dy \\ &=\; \frac{1}{p^2\gamma^{2(p-1)}}8k\pi\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right), \end{aligned} \tag{2.93}$$

with $\Theta_\lambda(\xi)$ a function, uniformly bounded, as $\lambda\to 0$. From (2.91)-(2.93) we get

$$\frac{\lambda}{p}\int_\Omega e^{(U_\lambda)^p}\, dx = \frac{1}{p^2\gamma^{2(p-1)}}8k\pi\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right), \tag{2.94}$$

Therefore, from (2.80), (2.81), (2.82), (2.90), (2.94) and (2.14) and by the choice of the parameters μ_j in (2.22), and (2.14), we can write the whole asymptotic expansion of $F_\lambda(\xi)$, namely (2.77) holds.

Let us now prove the validity of (2.79). Fix $m \in \{1, \dots, k\}$ and $l \in \{1, 2\}$. Arguing as in the proof of Lemma 2.12, we have

$$\partial_{(\xi_m)_l} F_\lambda(\xi) = \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1}\left[\sum_{i=1}^{2}\sum_{j=1}^{k} c_{ij}\int_{\Omega_\varepsilon}\eta_j Z_{ij}\partial_{(\xi'_m)_l}V_\lambda\right](1+O(\frac{1}{\gamma^p})). \tag{2.95}$$

On the one hand, if we multiply equation in (2.67) against $\partial_{(\xi'_m)_l}V_\lambda$, we get

$$\int_{\Omega_\varepsilon}(\Delta v_\xi + g(v_\xi))\partial_{(\xi'_m)_l}V_\lambda = \sum_{i=1}^{2}\sum_{j=1}^{k} c_{ij}\int_{\Omega_\varepsilon}\eta_j Z_{ij}\partial_{(\xi'_m)_l}V_\lambda$$

where $v_\xi = (V_\lambda + \phi)(y, \xi') = (V_\lambda + \phi)(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon})$. On the other hand, we have that

$$\partial_{(\xi_m)_l}U_\lambda(x) = \frac{\varepsilon^{-1}}{p\gamma^{p-1}}\partial_{(\xi'_m)_l}V_\lambda(\frac{x}{\varepsilon}).$$

Putting together these information, we have that

$$\partial_{(\xi_m)_l}F_\lambda(\xi) = \left(\int_\Omega\left[\Delta(U_\lambda+\tilde{\phi}) + \lambda(U_\lambda+\tilde{\phi})^{p-1}e^{(U_\lambda+\tilde{\phi})^p}\right]\partial_{(\xi_m)_l}U_\lambda\right)(1+o(1)).$$

Furthermore, since $\|\tilde{\phi}\|_\infty \le \frac{C}{\gamma^{p-1}}\|\phi\|_\infty$, by definition of U_λ we have that

$$(U+\tilde{\phi})(x) = U_\lambda(x)\left(1+O(\frac{1}{\gamma^p})\right) \quad \text{in} \quad \Omega.$$

Hence, by means of integrations by parts, and the boundary conditions satisfied by U_λ, we get that

$$\partial_{(\xi_m)_l}F_\lambda(\xi) = \left(\int_\Omega\left[\Delta U_\lambda + \lambda U_\lambda^{p-1}e^{U_\lambda^p}\right]\partial_{(\xi_m)_l}U_\lambda\right)(1+O(\frac{1}{\gamma^p})),$$

where $O(1)$ here denotes a smooth function of the points ξ, which is uniformly bounded as $\lambda \to 0$. We thus conclude that

$$\begin{aligned}\partial_{(\xi_m)_l}F_\lambda(\xi) &= \left(\int_\Omega\left[-\nabla U_\lambda\nabla\partial_{(\xi_m)_l}U_\lambda + \lambda U_\lambda^{p-1}e^{U_\lambda^p}\partial_{(\xi_m)_l}U_\lambda\right]\right)(1+O(\frac{1}{\gamma^p}))\\ &= -\partial_{(\xi_m)_l}J_\lambda(U_\lambda)(1+O(\frac{1}{\gamma^p})).\end{aligned}$$

Computations analogous to the ones we performed to get expansion (2.77) give us the validity of (2.79). This concludes the proof of the Lemma. □

2.6 Proof of the main results

In this section, we will prove the main result.

Proof of Theorem 2.2: From Lemma 2.12, the function

$$U_\lambda(\xi) + \tilde{\phi}(\xi) = \frac{1}{p\gamma^{p-1}}\Big(p\gamma^p + (V_\lambda + \phi)(\frac{x}{\varepsilon})\Big)$$

where V_λ defined by (2.17) and $\phi(\xi)$ is the unique solution of problem (2.67), is a solution of problem (2.1) if we adjust ξ so that it is a critical point of $F_\lambda(\xi)$ defined by (2.72). This is equivalent to finding a critical point of

$$\widetilde{F}_\lambda(\xi) := A\lambda^{-1}\varepsilon^{\frac{2(2-p)}{p}}F_\lambda(\xi) + B + C\log\varepsilon,$$

for suitable constants A, B and C. On the other hand, from Lemmas 2.13, for $\xi \in \mathcal{M}$, we have that,

$$\widetilde{F}_\lambda(\xi) = \varphi_k(\xi) + O(|\log\varepsilon|^{-1})\Theta_\lambda(\xi),$$

where φ_k is given by (2.5), and $\Theta_\lambda(\xi)$ is uniformly bounded in consider region as $\lambda \to 0$.

Let us observe that if $M > \mathcal{C}$, then assumptions (2.18), (2.19) still hold for the function $\min\{M, \varphi_k(\xi)\}$ as well as for $\min\{M, \varphi_k(\xi) + O(|\log\varepsilon|^{-1})\Theta_\lambda(\xi)\}$. It follows that the function $\min\{M, \tilde{F}(\xi)\}$ satisfies for all λ small assumptions (2.18),(2.19) in $\mathcal{D}$ and therefore has a critical value $\mathcal{C}_\lambda < M$ which is close to the value $\mathcal{C}$ in this region. If $\xi_\lambda \in \mathcal{D}$ is a critical point at this level for $\tilde{F}_\lambda(\xi) + \beta$, then since

$$\tilde{F}_\lambda(\xi_\lambda) \le \mathcal{C}_\lambda < M$$

we have that there exists a $\delta > 0$ such that $|\xi_{\lambda,j} - \xi_{\lambda,i}| > \delta$, $dist(\xi_{\lambda,j}, \partial\Omega) > 0$. This implies C^1-closeness of $\tilde{F}_\lambda(\xi)$ and $\varphi_k(\xi)$ at this level, hence $\nabla\varphi_k(\xi_\lambda) \to 0$. The function $u_\lambda = U(\xi_\lambda) + \tilde{\phi}(\xi_\lambda)$ is therefore a solution as predicted by the theorem. □

Expansion (2.20) follows from (2.14) and (2.94), while (2.16) holds as a direct consequence of the construction of U_λ. Expansion (2.17) is consequence of (2.77)

Proof of Theorem 2.1: According to the result of Theorem 2.2, the proof of Theorem 2.2 reduces to show that, for any $k \ge 1$ the function φ_k has a non trivial critical values in some open set $\mathcal{D}$, compactly contained in Ω^k. This fact has already been established in [36] for the function $(-\varphi_k)$ in the context of construction of solutions to the Liouville problem

$$\Delta u + \varepsilon^2 e^u = 0, \quad \text{in} \quad \Omega, \quad u = 0, \quad \text{on} \quad \partial\Omega$$

for a not simply connected domain Ω in $\mathbb{R}^2$. For completeness, we recall here the principal ingredients employed in the proof of the existence of a non trivial critical value for $(-\varphi_k)$ and we refer the reader to [36] for a complete proof of each step.

Let $\mathcal{D}$ be given by

$$\mathcal{D} = \{x \in \Omega^k \, : \, \mathrm{dist}(x, \partial\Omega^k) > \delta\}$$

for some positive and small δ to be chosen. Let Ω_1 be a bounded non empty component of $\mathbb{R}^2 \setminus \bar{\Omega}$ and let γ be a closed, smooth Jordan curve contained in Ω which encloses Ω_1. Let S be the image of γ, $B_0 = \emptyset$ and $B = S^k$. Define

$$\mathcal{C} = \inf_{\Phi \in \Gamma} \sup_{z \in B} (-\varphi_k) \left(\Phi(z)\right) \tag{2.96}$$

where

$$\Gamma = \{\Phi(z) = \Psi(1, z) \, : \, \Psi : [0,1] \times B \to \mathcal{D} \, \text{continuous and } \, \Psi(0, z) = z\}.$$

Observe that, since $\sum_j H_\Omega(\xi_j, \xi_j)$ is bounded in $\mathcal{D}$ and $\sum_{i \neq j} G_\Omega(\xi_i, \xi_j)$ is bounded below, the function $(-\varphi_k)$ is bounded above in $\mathcal{D}$.

With an argument based on degree theory, in Lemma 7.1 in [36], it is proven that:

There exists $K > 0$, independent of δ in the definition of $\mathcal{D}$, such that $\mathcal{C} \geq -K$.

This fact ensures the validity of (2.18).

A delicate analysis of the behavior of H and G contained in Lemma 7.2 and Lemma 7.3 in [36] is the key step to show the validity of the following result

Given $K > 0$, there exists $\delta > 0$ such that, if $(\xi_1, \ldots, \xi_k) \in \partial\mathcal{D}$, and $|\varphi_k(\xi_1, \ldots, \xi_k)| \leq K$, then there exists a vector τ, tangent to $\partial\mathcal{D}$, such that $\nabla\varphi_k(\xi_1, \ldots, \xi_k) \cdot \tau \neq 0$.

This fact is proved in Lemma 7.4 in [36] and it shows the validity of (2.19). Having established (2.18) and (2.19), we conclude that φ_k has a non trivial critical value in $\mathcal{D}$, which gives the proof of Theorem 2.1.

Chapter 3

Bubbling solutions for Liouville equation in unbounded domain

3.1 Introduction

Let us consider the following boundary value problem

$$\begin{cases} \Delta u + \varepsilon^2 e^u = 0, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is an open, connected and unbounded domain in $\mathbb{R}^2$, and $\varepsilon > 0$ is a small parameter.

Let $G(x; y)$ be the Green's function for the negative Laplacian with Dirichlet boundary condition in Ω, namely

$$\begin{cases} -\Delta_x G(x; y) = 8\pi\delta_y(x), & x \in \Omega; \\ G(x; y) = 0, & x \in \partial\Omega, \end{cases} \tag{3.2}$$

and $H(x; y)$ its regular part, given by

$$H(x; y) = G(x; y) - 4\log\frac{1}{|x-y|}. \tag{3.3}$$

For every $x \in \Omega$, the leading term of the regular part of the Green's function

$$\mathcal{R}(x) = H(x; x)$$

is called the Robin function of Ω at the point x.

Define the function

$$\varphi_k(\xi_1, \cdots, \xi_k) = \sum_{j=1}^{k} H(\xi_j, \xi_j) + \sum_{i \neq j} G(\xi_i, \xi_j),$$

In this chapter, we consider problem (3.1) on *unbounded domain, which is open, connected in* $\mathbb{R}^2$, we define it as follows. For $x \in \mathbb{R}^2$, we write $x = (x_1, x_2)$. Let $\varphi : \mathbb{R} \to [1, +\infty)$ be a smooth function, satisfying

(1) $\varphi(0) = 1$, $x_1\varphi'(x_1) > 0$ for $x_1 \neq 0$;

(2) $\varphi(x_1) \to +\infty$ as $x_1 \to \pm\infty$, and

(3) $\varphi'(x_1) \to a > 0$ as $x_1 \to +\infty$, and $\varphi'(x_1) \to b < 0$ as $x_1 \to -\infty$.

Define the domain(see figure 1):

$$\Omega = \{x = (x_1, x_2) \ : \ |x_2| < \varphi(x_1)\} \tag{3.4}$$

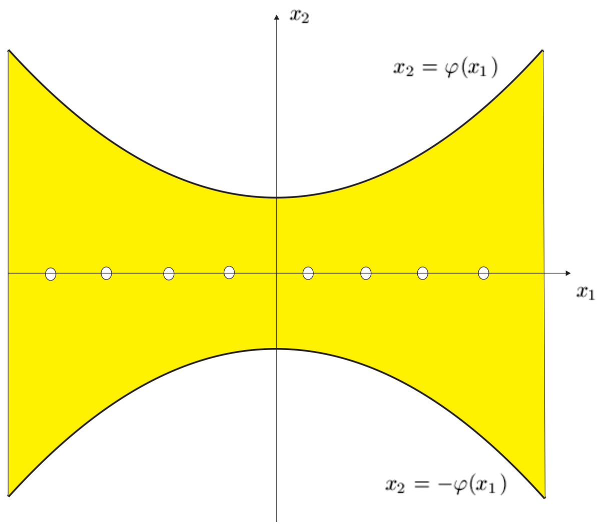


Figure 1: $\Omega = \{x = (x_1, x_2) \ : \ |x_2| \leq \varphi(x_1)\}$

We observe that Ω is symmetric with respect to line $x_2 = 0$, and has two open directions. Moreover, the domain is not necessary symmetric with respect to $x_1 = 0$. We would like to construct bubbling solutions to problem (3.1) in domain Ω, the location of blow-up points on the symmetric line of Ω.

Let $\delta > 0$ small but fixed. Let $k > 1$ be an integer. Given k different points on the symmetry line of Ω, we write these points as

$$\xi_j = (t_j, 0), \quad j = 1, \ldots, k, \tag{3.5}$$

with $t_1 < t_2 < \ldots < t_k$, satisfies

$$t_{i+1} - t_i > \delta, \quad i = 1, 2, \ldots, k-1. \tag{3.6}$$

Ore results sates as follows.

Theorem 3.1. *Let Ω be an open, connected and unbounded domain of $\mathbb{R}^2$ defined by (3.4), let $k > 1$ be an integer. For $\varepsilon > 0$ small enough, problem (3.1) has at least one solution u_ε, which blow-up at k points $\xi_1^*, \ldots, \xi_k^*$ defined as (3.5) satisfies (3.6). Moreover,*

$$u_\varepsilon(x) = \sum_{j=1}^{k} G(x; \xi_j^*) + o(1) \tag{3.7}$$

where $o(1) \to 0$, as $\varepsilon \to 0$, on each compact subset of $\bar{\Omega} \backslash \{\xi_1^*, \ldots, \xi_k^*\}$, and $G(\cdot; \cdot)$ is the Green's function given in (3.2).

Remark 3.2. *Let $R > 0$ be a large number, if we scaling the domain Ω, set $\Omega_R = \frac{\Omega}{R}$, then domain Ω_R approximates two sectors in plane. In fact, we can construct bubbling solutions to (3.1) in any open, connected and unbounded domain, which has multiplicity ends. For instance, in $\tilde{\Omega}$ (see Yellow area in figure 2), which has four ends and is symmetric with respect to axis. By the same proof of Theorem 3.1, we can obtain that there exists a solution to (3.1)in $\tilde{\Omega}$, that blows-up at k peaks for any k, the location of bubbling points on the symmetry lines $x_1 = 0$ and $x_2 = 0$.*

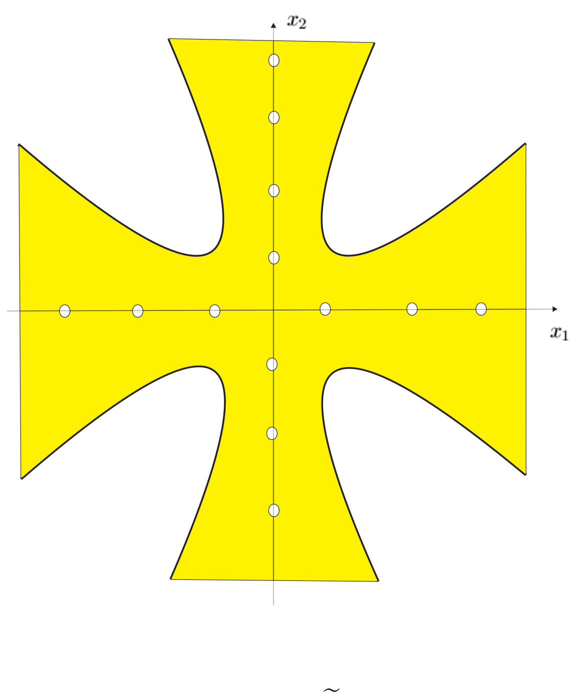


Figure 2: $\widetilde{\Omega}$

This chapter is organized as follows. In Section 3.2, we give the behavior of Green function $G(x;y)$ of domain Ω. In Section 3.3, we describe a first approximation solution to problem (3.1) and estimating the error. We give the proof of the main result in Section 3.4. Section 3.5 is devoted to give the asymptotic expansion of the reduced energy.

3.2 The asymptotic behavior of the Green function of Ω

In this section. we are devoted to study the behavior of the Green function $G(x;y)$ of Ω. In order to do this, we first consider the Green function in a sector. For $x \in \mathbb{R}^2$, we write it in the polar coordinate as $x = (r, \theta)$ with $r = |x|$. Let $\alpha > 0$, define the sector in $\mathbb{R}^2$ as

$$D_\alpha = \left\{(r,\theta) \in \mathbb{R}^2 \ : \ 0 < r < +\infty, -\alpha \le \theta \le \alpha\right\}. \tag{3.8}$$

Let G_{D_α} be the Green function in the sector D_α with Dirichlet boundary condition, that is

$$\begin{cases} -\Delta_y G_{D_\alpha}(x;y) = 8\pi\delta_x(y), & y \in D_\alpha; \\ G_{D_\alpha}(x;y) = 0, & y \in \partial D_\alpha. \end{cases} \tag{3.9}$$

We write it as

$$G_{D_\alpha}(x;y) = H_{D_\alpha}(x;y) + 4\log\frac{1}{|x-y|},$$

where $H_{D_\alpha}(x;y)$ denotes its regular part. Let $\mathcal{R}_{D_\alpha}$ be the Robin function in D_α.

Lemma 3.3. *Let $x = (r,\theta), y = (t,\eta) \in D_\alpha$, we have*

(a)

$$G_{D_\alpha}(x;y) = 2\ln\frac{r^{\frac{\pi}{\alpha}} + 2r^{\frac{\pi}{2\alpha}}t^{\frac{\pi}{2\alpha}}\cos\left(\pi\left[\frac{\theta+\eta}{2\alpha}\right]\right) + t^{\frac{\pi}{\alpha}}}{r^{\frac{\pi}{\alpha}} - 2r^{\frac{\pi}{2\alpha}}t^{\frac{\pi}{2\alpha}}\cos\left(\pi\left[\frac{\theta-\eta}{2\alpha}\right]\right) + t^{\frac{\pi}{\alpha}}}.$$

(b) For point x on the symmetry line of D_α, i.e. $x = (\xi, 0)$, we have

$$\mathcal{R}_{D_\alpha}(x) = 4\ln\left(\frac{\alpha}{\pi}|\xi|\right). \tag{3.10}$$

Proof. (a) We set $\bar{y} = (t, -\eta - 2\alpha)$. The conformal map of D_α into the unit disk is

$$w(x,y) = \frac{x^{\frac{\pi}{2\alpha}} - y^{\frac{\pi}{2\alpha}}}{x^{\frac{\pi}{2\alpha}} - \bar{y}^{\frac{\pi}{2\alpha}}}$$

We note that: (i) Since $x^{\frac{\pi}{2\alpha}}$ is analytic, and the function of analytic function is analytic, so $w(x,y)$ is a analytic function of x;

(ii) $w(x,x)=0$ and $w(x,y)\neq 0$ for $x\neq y$, and $w(x,y)=1$ for $y\in\partial D_\alpha$;

(iii) $\frac{\partial w}{\partial x}(x,y)$ everywhere in D_α.
Hence by the method of conformal mapping,

$$-4\ln|w(x;y)|$$

is the Green function of D_α. That is

$$\begin{aligned} G_{D_\alpha}(x;y) &= 4\ln\frac{|x^{\frac{\pi}{2\alpha}}-\bar{y}^{\frac{\pi}{2\alpha}}|}{|x^{\frac{\pi}{2\alpha}}-y^{\frac{\pi}{2\alpha}}|} = 2\ln\frac{|x^{\frac{\pi}{2\alpha}}-\bar{y}^{\frac{\pi}{2\alpha}}|^2}{|x^{\frac{\pi}{2\alpha}}-y^{\frac{\pi}{2\alpha}}|^2} \\ &= 2\ln\frac{r^{\frac{\pi}{\alpha}}+2r^{\frac{\pi}{2\alpha}}t^{\frac{\pi}{2\alpha}}\cos\left(\pi\left[\frac{\theta+\eta}{2\alpha}\right]\right)+t^{\frac{\pi}{\alpha}}}{r^{\frac{\pi}{\alpha}}-2r^{\frac{\pi}{2\alpha}}t^{\frac{\pi}{2\alpha}}\cos\left(\pi\left[\frac{\theta-\eta}{2\alpha}\right]\right)+t^{\frac{\pi}{\alpha}}}. \end{aligned}$$

We have that

(i) $G_{D_\alpha}(x;y)=G_{D_\alpha}(y;x)$,

(ii) $G_{D_\alpha}(x;y)\big|_{y\in\partial D_\alpha}=0$, and

(iii) $G_{D_\alpha}(x;y)\to 0$ as $|x-y|\to+\infty$ satisfies $\frac{t}{r}\to+\infty$.

(b) For x,y on the symmetry line of D_α, we write $x=(r,0), y=(t,0)$, assume that $r>t$, from (a) we have

$$G_{D_\alpha}((r,0);(t,0)) = 2\ln\frac{1+2(\frac{t}{r})^{\frac{\pi}{2\alpha}}+(\frac{t}{r})^{\frac{\pi}{\alpha}}}{1-2(\frac{t}{r})^{\frac{\pi}{2\alpha}}+(\frac{t}{r})^{\frac{\pi}{\alpha}}} = 4\ln\frac{r^{\frac{\pi}{\alpha}}+t^{\frac{\pi}{\alpha}}}{r^{\frac{\pi}{\alpha}}-t^{\frac{\pi}{\alpha}}}.$$

We recall that

$$G_{D_\alpha}(x;y)=H_{D_\alpha}(x;y)+4\log\frac{1}{|x-y|},\quad x,y\in D_\alpha.$$

Thus

$$\begin{aligned} H_{D_\alpha}((r,0);(t,0)) &= G_{D_\alpha}((r,0);(t,0))-4\log\frac{1}{r-t} \\ &= 4\ln\left(\frac{r^{\frac{\pi}{\alpha}}+t^{\frac{\pi}{\alpha}}}{r^{\frac{\pi}{\alpha}}-t^{\frac{\pi}{\alpha}}}(r-t)\right). \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{R}_{D_\alpha}(x) &= H_{D_\alpha}((r,0);(r,0)) = \lim_{t\to r} H_{D_\alpha}((r,0);(t,0)) \\ &= 4\lim_{t\to r}\ln\left(\frac{r^{\frac{\pi}{\alpha}}+t^{\frac{\pi}{\alpha}}}{r^{\frac{\pi}{\alpha}}-t^{\frac{\pi}{\alpha}}}(r-t)\right) = 4\lim_{\epsilon\to 0}\ln\left(\frac{r^{\frac{\pi}{\alpha}}+(r-\epsilon)^{\frac{\pi}{\alpha}}}{r^{\frac{\pi}{\alpha}}-(r-\epsilon)^{\frac{\pi}{\alpha}}}\epsilon\right) \\ &= 4\ln\left(\frac{\alpha}{\pi}r\right). \end{aligned}$$

□

For the Green function $G(x;y)$ in Ω, we have the following result.

Lemma 3.4. *For $x=(x_1,x_2), y=(y_1,y_2)\in\Omega$, one has*

$$G(x;y)\sim G_{D_\sigma}(x;y) \quad \text{for } x_1,y_1>0, \tag{3.11}$$

and

$$G(x;y)\sim G_{D_{\varrho,2\pi-\varrho}}(x;y) \quad \text{for } x_1,y_1<0, \tag{3.12}$$

where $G_{D_\sigma}(x;y)$ is the Green function of the sector D_σ defined as (3.8) with $\sigma=\arctan(a)\in(0,\frac{\pi}{2})$, and

$$D_{\varrho,2\pi-\varrho}=\left\{(r,\theta)\in\mathbb{R}^2 \;:\; \varrho<\theta<2\pi-\varrho\right\}$$

with $\varrho=\arctan(b)\in(\frac{\pi}{2},\pi)$.

Proof. We may assume that $x_1,y_1>0$. Given $R>0$ large, making the scaling of domain, set $\Omega_R=\frac{\Omega}{R}$. By the definition of Ω, according to property (3) of φ, $\varphi'(x_1)\to a>0$ as $x_1\to+\infty$. Then domain Ω_R approaches to a sector D_σ if R goes to infinity, with $\sigma=\arctan(a)\in(0,\frac{\pi}{2})$. Thus, $\tilde{x}=\frac{x}{R}=(\tilde{r},0)\in D_\sigma\subset\Omega_R$, and the ray $\{(\tilde{r},0):\tilde{r}\geq 0\}$ is the symmetry line of D_σ.

Let us denote $G_R(\tilde{x};\tilde{y})$ and $G_{D_\sigma}(\tilde{x};\tilde{y})$ are Green function in Ω_R and D_σ with Dirichlet boundary condition. We note that

$$G(x;y)=G(R\tilde{x};R\tilde{y})=G_R(\tilde{x};\tilde{y}) \tag{3.13}$$

We observe that the function

$$v(\tilde{r},\rho)=\tilde{r}^{\frac{\pi}{2\sigma}}\cos\left(\frac{\pi}{2\sigma}\rho\right)$$

is harmonic in sector D_σ and satisfies Dirichlet boundary condition. By Phragmèn-Lindelöf principle [104], we have that $G_{D_\sigma}(\tilde{x};\tilde{y})>0$ in $D_\sigma(\tilde{x};\tilde{y})$. For $r_0>0$, we define

$$m_1=r_0^{-\frac{\pi}{2\sigma}}\inf_{\tilde{r}=r_0}\frac{G_R(\tilde{x},\tilde{y})-G_{D_\sigma}(\tilde{x},\tilde{y})}{\cos\left(\frac{\pi}{2\sigma}\rho\right)},$$

and

$$m_2=r_0^{-\frac{\pi}{2\sigma}}\sup_{\tilde{r}=r_0}\frac{G_{D_\sigma}(\tilde{x},\tilde{y})-G_R(\tilde{x},\tilde{y})}{\cos\left(\frac{\pi}{2\sigma}\rho\right)}.$$

Set

$$\psi(\tilde{x})=G_R(\tilde{x};\tilde{y})-G_{D_\sigma}(\tilde{x};\tilde{y})-m_1v(\tilde{r},\rho).$$

Then we have

$$\begin{cases}-\Delta\psi(\tilde{x})=0, & \tilde{x}\in D_\sigma\cap\{\tilde{x}=(\tilde{r},\theta):\tilde{r}\leq r_0\};\\ \psi(\tilde{x})\geq 0, & \tilde{x}\in\partial\left(D_\sigma\cap\{\tilde{z}=(\tilde{r},\theta):\tilde{r}\leq r_0\}\right).\end{cases}$$

Then by the maximum principle we get

$$\psi(\tilde{x}) \geq 0 \quad \text{for} \;\; \tilde{x} \in D_\sigma \cap \{\tilde{x} = (\tilde{r}, \theta) : \tilde{r} \leq r_0\}.$$

From (3.13), we then obtain, for $x, y \in \Omega$,

$$G(x; y) \geq G_{D_\sigma}(x; y) - m_1 \left(\frac{x}{R}\right)^{\frac{\pi}{2\sigma}} \cos\left(\frac{\pi}{2\sigma}\rho\right) \tag{3.14}$$

with $\rho \in (-\sigma, \sigma)$. On the other hand, set

$$\widetilde{\psi}(\tilde{x}) = G_{D_\sigma}(\tilde{x}; \tilde{y}) - G_R(\tilde{x}; \tilde{y}) - m_2 v(\tilde{r}, \rho).$$

By the same way, we have

$$\begin{cases} -\Delta\widetilde{\psi}(\tilde{x}) = 0, & \tilde{x} \in D_\sigma \cap \{\tilde{z} = (\tilde{r}, \theta) : \tilde{r} \leq r_0\}; \\ \widetilde{\psi}(\tilde{x}) \leq 0, & \tilde{x} \in \partial\left(D_\sigma \cap \{\tilde{z} = (\tilde{r}, \theta) : \tilde{r} \leq r_0\}\right). \end{cases}$$

Then by the maximum principle we get

$$\widetilde{\psi}(\tilde{x}) \leq 0 \quad \text{for} \;\; \tilde{x} \in D_\sigma \cap \{\tilde{x} = (\tilde{r}, \theta) : \tilde{r} \leq r_0\}.$$

From (3.13), we also obtain, for $x, y \in \Omega$,

$$G(x; y) \leq G_{D_\sigma}(x; y) - m_2 \left(\frac{x}{R}\right)^{\frac{\pi}{2\sigma}} \cos\left(\frac{\pi}{2\sigma}\rho\right) \tag{3.15}$$

with $\rho \in (-\sigma, \sigma)$. From (3.14) and (3.15), letting $R \to +\infty$, we obtain that, for $x, y \in D_\sigma \subset \Omega$,

$$G(x; y) \sim G_{D_\sigma}(x; y)$$

that is (3.11) holds.

On the other hand, if $x_1, y_1 < 0$. Since $\varphi'(x_1) \to b < 0$ as $x_1 \to -\infty$. We have that, if $R \to +\infty$, Ω_R approaches to a sector

$$D_{\varrho, 2\pi-\varrho} = \left\{(r, \theta) \in \mathbb{R}^2 \;\;:\;\; \varrho < \theta < 2\pi - \varrho\right\}$$

with $\varrho = \arctan(b) \in (\frac{\pi}{2}, \pi)$. Then (3.12) follows from the same argument. □

It is consequence of above Lemma 3.4 and (3.10), we have

Corollary 3.5. *For points on the symmetry line of Ω, then we have the asymptotic behavior of Robin function $\mathcal{R}(x)$ for $x = (\xi, 0)$,*

$$\mathcal{R}(x) \sim 4 \ln\left(\frac{\sigma}{\pi}|\xi|\right), \quad \textit{with some } \sigma > 0. \tag{3.16}$$

3.3 The first approximation solution

In this Section, we build the first approximation solution and to estimate its error. Let us introduce the radially symmetric solutions of the following limit equation

$$\Delta w + e^w = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^w < +\infty,$$

which are given by the one parameter family of functions

$$w_\mu(z) = \log \frac{8\mu^2}{(\mu^2 + |z|^2)^2}. \tag{3.17}$$

Let $\delta > 0$ small but fixed, and $k > 1$ be an integer. Let $\xi = (\xi_1, \dots, \xi_k)$ given by (3.5) satisfies (3.6). Moreover, consider k positive numbers μ_j such that

$$\delta < \mu_j < \delta^{-1}, \quad \text{for all } j = 1, \dots, k. \tag{3.18}$$

The parameters μ_j will be chosen properly later on. Define the function

$$\begin{aligned} u_j(x) &= \log \frac{1}{(\mu_j^2 \varepsilon^2 + |x - \xi_j|^2)^2} \\ &= w_{\mu_j}\left(\frac{x - \xi_j}{\varepsilon}\right) + 4 \log \frac{1}{\varepsilon} - \log(8\mu_j^2). \end{aligned} \tag{3.19}$$

Given a radial smooth cut-off function $\eta_\delta : \mathbb{R} \mapsto [0,1]$ such that $\eta_\delta(x) = 1$ for $|x| \le \frac{\delta}{2}$, $0 < \eta_\delta(x) < 1$ for $\frac{\delta}{2} < |x| < \delta$, and $\eta_\delta(x) = 0$ for $|x| \ge \delta$. Set

$$U_j(x) = (u_j(x) + H(x, \xi_j))\, \eta_\delta(x - \xi_j) + (1 - \eta_\delta(x - \xi_j)) G(x, \xi_j).$$

We now define the first ansatz is given by

$$U(x) = \sum_{j=1}^{k} U_j(x). \tag{3.20}$$

Consider now the change of variables

$$v(y) = u(\varepsilon y) - 4 \log \frac{1}{\varepsilon}.$$

If u is a solutions of problem (3.1), then v satisfies the following problen

$$\begin{cases} \Delta v + e^v = 0, & \text{in } \Omega_\varepsilon; \\ v = -4 \log \frac{1}{\varepsilon}, & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{3.21}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. We also write $\xi_j' = \varepsilon^{-1}\xi_j$ and define the first approximation solutions to (3.21) as

$$V(y) = U(\varepsilon y) - 4\log\frac{1}{\varepsilon}. \tag{3.22}$$

We will look for solutions to (3.21) of the form

$$v = V + \phi,$$

where V is defined as in (3.22), and ϕ represents a lower order correction. We aim at finding a solution for ϕ small provided that the points ξ_j are suitably chosen. For small ϕ, we can rewrite problem (3.21) as a nonlinear perturbation of its linearization, namely,

$$\begin{cases} \Delta\phi + e^{V(y)}\phi = -[E + N(\phi)], & x \in \Omega_\varepsilon; \\ \phi = 0, & x \in \partial\Omega_\varepsilon, \end{cases} \tag{3.23}$$

where

$$E := \Delta V(y) + e^{V(y)}, \tag{3.24}$$

$$N(\phi) := e^{V+\phi} - e^{V(y)} - e^{V(y)}\phi. \tag{3.25}$$

We aim to choose suitable μ_j, $j = 1, \dots, k$ such that the error term is small.

In fact, we observe that $e^{V(y)} = \varepsilon^4 e^{U(x)}$ with $x = \varepsilon y$. If $\delta > 0$ small but fixed, for $|x - \xi_j| \geq \frac{\delta}{2}$, that is $|y - \xi_j'| \geq \frac{\delta}{2\varepsilon}$, we have

$$\begin{aligned} u_j(x) + H(x,\xi_j) &= \log\frac{1}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2} + H(x,\xi_j) \\ &= G(x,\xi_j) + O(\mu_j^2\varepsilon^2). \end{aligned}$$

Then we have that $U_j(x) = G(x,\xi_j) + O(\mu_j^2\varepsilon^2)$. Hence, for all $j = 1, \dots, k$,

$$e^{V(y)} = \varepsilon^4 e^{U(x)} = \varepsilon^4 e^{\sum_{j=1}^{k} G(x,\xi_j) + O(\mu_j^2\varepsilon^2)} = O(\varepsilon^4) \text{ if } |y - \xi_j'| \geq \frac{\delta}{2\varepsilon}. \tag{3.26}$$

Moreover, $\Delta V(y) = \varepsilon^2 \Delta U(x)$ and then we obtain

$$\Delta V(y) = O(\varepsilon^4) \quad \text{if } |y - \xi_j'| \geq \frac{\delta}{2\varepsilon}. \tag{3.27}$$

On the other hand, fix $j \in \{1, \dots, k\}$, for $|y - \xi_j'| \leq \frac{\delta}{2\varepsilon}$, we have $U_j(x) = u_j(x) + H(x,\xi_j)$. We write $y = \xi_j' + z$, then

$$e^{V(y)} = \varepsilon^4 e^{U(x)} = \varepsilon^4 e^{\sum_{j=1}^{k}[u_j(x) + H(x,\xi_j)]}$$

$$
\begin{aligned}
&= \varepsilon^4 e^{u_j(x)} e^{H(x,\xi_j)+\sum_{i\neq j}^{k}[u_i(x)+H(x,\xi_i)]} \\
&= e^{w_{\mu_j}(z)} e^{-\log(8\mu_j^2)+H(x,\xi_j)+\sum_{i\neq j}^{k}[u_i(x)+H(x,\xi_i)]} \\
&= e^{w_{\mu_j}(z)} e^{-\log(8\mu_j^2)+H(\xi_j,\xi_j)+O(\varepsilon|y-\xi_j'|)+\sum_{i\neq j}^{k}[u_i(x)+H(x,\xi_i)]}.
\end{aligned}
$$

since

$$
\begin{aligned}
&\sum_{i\neq j}^{k}[u_i(x)+H(x,\xi_i)] \\
= &\sum_{i\neq j}^{k}\left[\log\frac{1}{(\mu_j^2\varepsilon^2+|\xi_j-\xi_i+\varepsilon z|^2)^2}+H(\xi_j+\varepsilon z,\xi_i)\right] \\
= &\sum_{i\neq j}^{k}[G(\xi_j,\xi_i)+O(\varepsilon|z|)]
\end{aligned}
$$

If we choose μ_j satisfies

$$
\log(8\mu_j^2)=\mathcal{R}(\xi_j)+\sum_{i\neq j}^{k}G(\xi_j,\xi_i). \tag{3.28}
$$

Then we have

$$
e^{V(y)} = e^{w_{\mu_j}(z)}\times O(\varepsilon|y-\xi_j'|) \quad \text{if } |y-\xi_j'|\leq\frac{\delta}{2\varepsilon}. \tag{3.29}
$$

And, we have

$$
\Delta V(y)=e^{w_{\mu_j}(z)} \quad \text{if } |y-\xi_j'|\leq\frac{\delta}{2\varepsilon}.
$$

Therefore

$$
|E(y)|\leq C\varepsilon\sum_{j=1}^{k}\frac{1}{1+|y-\xi_j'|^3}. \tag{3.30}
$$

3.4 The Existence result

Let us define

$$
z_{0j}(y)=\partial_{\mu_j}w_{\mu_j}(y), \qquad z_{ij}(y)=\partial_{y_i}w_{\mu_j}(y), \quad i=1,2,
$$

It is known [10] that the solutions of $\Delta Z+e^{w_j}Z=0$ are given by z_{0j},z_{1j},z_{2j}. Define for $i=0,1,2$ and $j=1,2,\cdots,k$,

$$
Z_{ij}(y):=z_{ij}\left(y-\xi_j'\right), \quad i=0,1,2. \tag{3.31}
$$

Consider a large but fixed number $R_0 > 0$ and a radial and smooth cut-off function χ with $\chi(r) = 1$ if $r < R_0$ and $\chi(r) = 0$ if $r > R_0 + 1$. Write

$$\chi_j(y) = \chi\left(|y - \xi_j'|\right). \tag{3.32}$$

Given h of class $C^{0,\alpha}(\Omega_\varepsilon)$, we consider the linear problem of finding a function ϕ and scalars c_{ij}, $i = 1, 2$, $j = 1, ..., k$ such that

$$\begin{cases} \Delta V(y) + e^{V(y)}\phi = -[E + N(\phi)] + \sum_{i=1}^{2}\sum_{j=1}^{k} c_{ij} Z_{ij}\chi_j & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \phi Z_{ij}\chi_j = 0 \qquad \text{for } i = 1, 2, \ j = 1, \cdots, k. \end{cases} \tag{3.33}$$

Consider the norms

$$\|\phi\|_\infty = \sup_{y\in\Omega_\varepsilon} |\phi(y)|, \quad \|h\|_* = \sup_{y\in\Omega_\varepsilon} \left(\sum_{j=1}^{k} (1 + |y - \xi_j'|)^{-3} + \varepsilon^2 \right)^{-1} |h(y)|.$$

By the same argument in [36], we have the following result.

Proposition 3.6. *There exist positive constant ε_0 and C, such that, for ξ_j, $j = 1, \ldots, k$ given by (3.5), then there is a unique solution ϕ to problem (3.33) for all $\varepsilon < \varepsilon_0$. Moreover*

$$\|\phi\|_\infty \le C\varepsilon|\log\varepsilon|.$$

Furthermore, the map $\xi' \mapsto \phi \in H_0^1(\Omega_\varepsilon)$ is $\mathcal{C}^1$, and

$$\|D_{\xi'}\phi\|_\infty \le C\varepsilon|\log\varepsilon|^2.$$

After problem (3.33) has been solved, we find a solution to problem (3.23), if we can find a point $\xi' = \frac{\xi}{\varepsilon} = (\xi_1', \ldots, \xi_k')$ such that coefficients $c_{ij}(\xi')$ in (3.33) satisfy

$$c_{ij}(\xi') = 0 \quad \text{for all } i = 1, 2, \quad j = 1, \cdots, k. \tag{3.34}$$

We now introduce the finite dimensional restriction $\mathcal{I}_\varepsilon(\xi)$, given by

$$\mathcal{I}_\varepsilon(\xi) = J\left(\left(U + \tilde{\phi}\right)(x, \xi)\right) \tag{3.35}$$

where J is the energy function of (3.1), that is

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \varepsilon^2 \int_\Omega e^u,$$

and

$$\left(U + \tilde{\phi}\right)(x, \xi) = (V + \phi)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right) \tag{3.36}$$

with V defined in (3.22), ϕ is the unique solution to problem (3.33) given by Proposition 3.6.

The next result, whose proof is postponed until Section 3.5.

Proposition 3.7. *(i) The functional $\mathcal{I}_\varepsilon(\xi)$ is of class C^1. Moreover, for all $\varepsilon > 0$ sufficiently small, if $D_\xi \mathcal{I}_\varepsilon(\xi) = 0$, then ξ satisfies (3.34).*

(ii) Let $\delta > 0$ small but fixed, then there exist positive numbers ε_0 and C, such that for any points ξ_j, $j = 1, \ldots, k$ given by (3.5), μ_j are given by (3.28), the following expansion holds

$$\begin{aligned}\mathcal{I}_\varepsilon(\xi) &= 8k\pi(\log 8 - 2) - 16k\pi \log \varepsilon + 4\pi\Phi(\xi) \\ &\quad + \delta \log \delta \Theta(\xi) + \delta \Upsilon(\xi) + \varepsilon\Theta(\xi) + O(\varepsilon^2)|\Omega|. \end{aligned} \tag{3.37}$$

where $|\Omega|$ denotes the measure of Ω, and

$$\Phi(\xi) = -\left[\sum_{j=1}^{k} \mathcal{R}(\xi_j) + \sum_{i \neq j} G(\xi_i, \xi_j)\right]. \tag{3.38}$$

and $\Theta(\xi)$ is a smooth function in the consider region, and $\Upsilon(\xi)$ is a smooth function so that $|\Upsilon(\xi)| \leq C\Phi(\xi)$.

Proof of Theorem 3.1: According to Proposition 3.7, we have a solution to (3.1) if we find a critical point ξ of $\mathcal{I}_\varepsilon(\xi)$, it is equivalent to finding a critical point of the function $\Phi(\xi)$, given points ξ_j, $j = 1, \ldots, k$ as (3.5), thus it is suffice to find a critical point of $\Phi(\xi)$, which defined by

$$\Phi(\xi) = -4\pi\left[\sum_{j=1}^{k} \mathcal{R}(\xi_j) + \sum_{i \neq j} G(\xi_i, \xi_j)\right].$$

By the properties of Green function, we know that if $\delta \to 0$, then points ξ_j, $j = 1, \ldots, k$ given by (3.5) satisfies (3.6), we have

$$\Phi(\xi) \to -\infty \quad \text{as } \delta \to 0.$$

On the other hand, we note that points ξ_j (given by (3.5))on the symmetric line $x_2 = 0$, from Corollary 3.5, we have

$$\Phi(\xi) \to -\infty \qquad \text{for some } \ |t_j| \to +\infty, \ i = 1, 2, 3.$$

Thus $\Phi(\xi)$ has a maximum point, denote it by ξ^*, that is, there exists critical points $\xi^* = (\xi_1^*, \ldots, \xi_k^*)$ of $\Phi(\xi)$.

Moreover, while (3.7) holds as a direct consequence of the construction of U.

3.5 Expansion of energy

Proof of Proposition 3.7: (i) The proof is the standard way, see [36].

(ii) According to the same proof of Lemma 5.2 in [36], we have

$$\mathcal{I}_\varepsilon(\xi) = J(U) + \theta_\varepsilon(\xi), \tag{3.39}$$

where $|\theta_\varepsilon(\xi)| + |\nabla\theta_\varepsilon(\xi)| \to 0$ uniformly on points given by (3.5). We now give the expansion of energy $J(U)$, we have

$$J(U) = \frac{1}{2}\int_\Omega |\nabla U|^2 - \varepsilon^2 \int_\Omega e^U := I - II \tag{3.40}$$

We first claim

$$I = 8k\pi(\log 8 - 1) - 16k\pi\log\varepsilon + 4\pi\Phi(\xi) + \delta\log\delta\Theta(\xi) + \delta\Upsilon(\xi), \tag{3.41}$$

where $\Phi(\xi)$ given by (3.38), $\Theta(\xi)$ is a smooth function in the consider region, and $\Upsilon(\xi)$ is a smooth function so that $|\Upsilon(\xi)| \le C\Phi(\xi)$.

Proof of (3.41): We can write it as

$$\begin{aligned} I &= \frac{1}{2}\sum_{j=1}^k \int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla U|^2 + \frac{1}{2}\int_{\Omega\setminus\cup_{j=1}^k B_{\frac{\delta}{2}}(\xi_j)} |\nabla U|^2 \\ &:= \frac{1}{2}\left(\sum_{j=1}^k I_j + I_2\right) \end{aligned} \tag{3.42}$$

where

$$\begin{aligned} I_j &= \int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla U|^2 = \int_{B_{\frac{\delta}{2}}(\xi_j)} \left|\nabla\left(\sum_{l=1}^k U_l(x)\right)\right|^2 \\ &= \sum_{l=1}^k \int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla U_l(x)|^2 + \sum_{i\neq l}\int_{B_{\frac{\delta}{2}}(\xi_j)} \nabla U_i(x)\nabla U_l(x) \\ &= \int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla U_j(x)|^2 + \sum_{l\neq j}^k \int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla U_l(x)|^2 \\ &\quad + \sum_{i\neq j}\int_{B_{\frac{\delta}{2}}(\xi_j)} \nabla U_i(x)\nabla U_j(x) + \sum_{i\neq l\neq j}\int_{B_{\frac{\delta}{2}}(\xi_j)} \nabla U_i(x)\nabla U_l(x) \\ &:= I_{j1} + I_{j2} + I_{j3} + I_{j4}. \end{aligned} \tag{3.43}$$

Estimate I_{j1}: We observe that $U_j(x) = u_j(x) + H(x,\xi_j)$ for $x \in B_{\frac{\delta}{2}}(\xi_j)$, and integrating by parts, using $-\Delta H(x,\xi_j) = 0$ in Ω, we have

$$I_{j1} = \int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla U_j(x)|^2 = \int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla(u_j(x) + H(x,\xi_j))|^2$$

$$\begin{aligned}
&= \int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla u_j(x)|^2 + 2\int_{B_{\frac{\delta}{2}}(\xi_j)} \nabla u_j(x)\nabla H(x,\xi_j) + \int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla H(x,\xi_j)|^2 \\
&= \int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla u_j(x)|^2 + 2\int_{\partial B_{\frac{\delta}{2}}(\xi_j)} u_j(x)\frac{\partial H(x,\xi_j)}{\partial \nu} \\
&\quad + \int_{\partial B_{\frac{\delta}{2}}(\xi_j)} H(x,\xi_j)\frac{\partial H(x,\xi_j)}{\partial \nu} \\
&= \int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla u_j(x)|^2 + 2\int_{\partial B_{\frac{\delta}{2}}(\xi_j)} \log\frac{1}{(\mu_j^2\varepsilon^2+|\frac{\delta}{2}|^2)^2}\frac{\partial H(x,\xi_j)}{\partial \nu} \\
&\quad + \int_{\partial B_{\frac{\delta}{2}}(\xi_j)} H(x,\xi_j)\frac{\partial H(x,\xi_j)}{\partial \nu} \\
&= \int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla u_j(x)|^2 + \delta\log\delta\Theta(\xi) + \delta\Upsilon(\xi).
\end{aligned} \tag{3.44}$$

where now, and the rest of the proof, ν will denote the out normal vector of Ω, $\Theta(\xi)$ is a smooth function in the consider region, and $\Upsilon(\xi)$ is a smooth function so that $|\Upsilon(\xi)| \le C\Phi(\xi)$. Moreover

$$\begin{aligned}
&\int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla u_j(x)|^2 = 16\int_{B_{\frac{\delta}{2}}(\xi_j)} \frac{|x-\xi_j|^2}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^2}dx \\
&= 16\int_{B_{\frac{\delta}{2\mu_j\varepsilon}}(\xi_j)} \frac{|y|^2}{(1+|y|^2)^2}dy \\
&= 16\pi\left[-2\log(\mu_j\varepsilon) - 1 + \log[(\mu_j\varepsilon)^2 + (\frac{\delta}{2})^2] + \frac{(\mu_j\varepsilon)^2}{(\mu_j\varepsilon)^2+(\frac{\delta}{2})^2}\right] \\
&= 16\pi\Big[-\log(8\mu_j^2) - 2\log(\varepsilon) + \log 8 - 1 \\
&\qquad + \log[(\mu_j\varepsilon)^2 + (\frac{\delta}{2})^2] + \frac{(\mu_j\varepsilon)^2}{(\mu_j\varepsilon)^2+(\frac{\delta}{2})^2}\Big]
\end{aligned} \tag{3.45}$$

Estimate I_{j2}: We note that for $\delta > 0$ arbitrarily small, and $\xi = (\xi_1, \cdots, \xi_k)$ given as (3.5), then we have $U_l(x) = G(x,\xi_l)$ for $x \in B_{\frac{\delta}{2}}(\xi_j)$ with $\delta > 0$ small.

$$\begin{aligned}
I_{j2} &= \sum_{l\neq j}^{k}\int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla U_l(x)|^2 = \sum_{l\neq j}^{k}\int_{B_{\frac{\delta}{2}}(\xi_j)} |\nabla G(x,\xi_l)|^2 \\
&= \sum_{l\neq j}^{k}\int_{\partial B_{\frac{\delta}{2}}(\xi_j)} G(x,\xi_l)\frac{\partial G(x,\xi_l)}{\partial \nu}
\end{aligned}$$

$$= \delta\Upsilon(\xi) \tag{3.46}$$

where Υ denotes a smooth function of ξ so that $|\Upsilon(\xi)| \leq C\Phi(\xi)$.

Estimate I_{j3}: since δ is arbitrarily small, in $B_{\frac{\delta}{2}}(\xi_j)$ we have $U_j(x) = u_j(x) + H(x, \xi_j)$, and $U_i(x) = G(x, \xi_i)$ for $i \neq j$, integrating by parts, we have

$$\begin{aligned} I_{j3} &= \sum_{i\neq j} \int_{B_{\frac{\delta}{2}}(\xi_j)} \nabla U_i(x)\nabla U_j(x) = \sum_{i\neq j} \int_{B_{\frac{\delta}{2}}(\xi_j)} \nabla G(x,\xi_i)\nabla[u_j(x) + H(x,\xi_j)] \\ &= \sum_{i\neq j} \int_{\partial B_{\frac{\delta}{2}}(\xi_j)} \left[\log \frac{1}{(\mu_j^2\varepsilon^2 + |\frac{\delta}{2}|^2)^2} + H(x,\xi_j)\right] \frac{\partial G(x,\xi_i)}{\partial\nu} \\ &= \delta \log \delta\Theta(\xi), \end{aligned} \tag{3.47}$$

where Θ denotes again a smooth function of ξ.

Estimate I_{j4}: Using again δ is arbitrarily small and by a Taylor expansion, if $i \neq l \neq j$, on $B_{\frac{\delta}{2}}(\xi_j)$ we have $U_i(x) = G(x, \xi_i)$ and $U_l(x) = G(x, \xi_l)$, then

$$\begin{aligned} I_{j4} &= \sum_{i\neq l\neq j} \int_{B_{\frac{\delta}{2}}(\xi_j)} \nabla U_i(x)\nabla U_l(x) = \sum_{i\neq j} \int_{B_{\frac{\delta}{2}}(\xi_j)} \nabla G(x,\xi_i)\nabla G(x,\xi_l) \\ &= \sum_{l\neq l;i,l\neq j} \int_{\partial B_{\frac{\delta}{2}}(\xi_j)} G(x,\xi_i)\frac{\partial G(x,\xi_l)}{\partial\nu} \\ &= \delta\Upsilon(\xi), \end{aligned} \tag{3.48}$$

where Υ denotes again a smooth function of ξ so that $|\Upsilon(\xi)| \leq C\Phi(\xi)$.

Thus, from (3.43) to (3.48), we obtain

$$\begin{aligned} I_j &= -16\pi \log(8\mu_j^2) - 32k\pi \log\varepsilon + 16k\pi(\log 8 - 1) \\ &\quad + \delta \log \delta\Theta(\xi) + \delta\Upsilon(\xi) \end{aligned} \tag{3.49}$$

with $\Phi(\xi)$ given by (3.38), $\Theta(\xi)$ is a smooth function in the consider region, and $\Upsilon(\xi)$ is a smooth function so that $|\Upsilon(\xi)| \leq C\Phi(\xi)$.

Next we estimate I_2.

$$I_2 = \int\limits_{\Omega\setminus\cup_{j=1}^k B_\delta(\xi_j)} |\nabla U|^2 + \int\limits_{\cup_{j=1}^k (B_\delta(\xi_j)\setminus B_{\frac{\delta}{2}}(\xi_j))} |\nabla U|^2 := I_{2A} + I_{2B}. \tag{3.50}$$

We first estimate the first term of above. Since δ is arbitrarily small, and points ξ_j given by (3.5). We have that on $\Omega\setminus \cup_{j=1}^k B_\delta(\xi_j)$, $U(x) = \sum\limits_{l=1}^k G(x, \xi_l)$. Then

$$I_{2A} = \sum_{l=1}^k \int\limits_{\Omega\setminus\cup_{j=1}^k B_{\frac{\delta}{2}}(\xi_j)} |\nabla U_l|^2 + \sum_{i\neq l}^k \int\limits_{\Omega\setminus\cup_{j=1}^k B_{\frac{\delta}{2}}(\xi_j)} \nabla U_i \nabla U_l$$

$$
\begin{aligned}
&= \sum_{l=1}^{k} \int_{\Omega\setminus\cup_{j=1}^{k} B_{\frac{\delta}{2}}(\xi_j)} |\nabla G(x,\xi_l)|^2 + \sum_{i\neq l}^{k} \int_{\Omega\setminus\cup_{j=1}^{k} B_{\delta}(\xi_j)} \nabla G(x,\xi_i)\nabla G(x,\xi_l) \\
&= \int_{\Omega\setminus\cup_{j=1}^{k} B_{\delta}(\xi_j)} |\nabla G(x,\xi_j)|^2 + \sum_{l\neq j}^{k} \int_{\Omega\setminus\cup_{j=1}^{k} B_{\delta}(\xi_j)} |\nabla G(x,\xi_l)|^2 \\
&\quad + \sum_{i\neq l}^{k} \int_{\Omega\setminus\cup_{j=1}^{k} B_{\delta}(\xi_j)} \nabla G(x,\xi_i)\nabla G(x,\xi_l) \\
&= \sum_{j=1}^{k} \int_{\Omega\setminus B_{\delta}(\xi_j)} |\nabla G(x,\xi_j)|^2 + \sum_{j=1}^{k}\sum_{l\neq j}^{k} \int_{\Omega\setminus B_{\delta}(\xi_j)} |\nabla G(x,\xi_l)|^2 \\
&\quad + \sum_{i\neq j}^{k} \int_{\Omega\setminus B_{\delta}(\xi_j)} \nabla G(x,\xi_i)\nabla G(x,\xi_j) + \sum_{i\neq l\neq j}^{k} \int_{\Omega\setminus B_{\delta}(\xi_j)} \nabla G(x,\xi_i)\nabla G(x,\xi_l) \\
&= I_{2a} + I_{2b} + I_{2c} + I_{2d}.
\end{aligned}
\tag{3.51}
$$

Using the fact $-\Delta G(x,\xi_j) = 8\pi\delta_{\xi_j}(x)$ in Ω and $G(x,\xi_j) = 0$ for x on the boundary of Ω, integrating by parts, we have

$$
\begin{aligned}
I_{2a} &= \sum_{j=1}^{k} \int_{\Omega\setminus B_{\delta}(\xi_j)} |\nabla G(x,\xi_j)|^2 \\
&= \sum_{j=1}^{k} \int_{\partial(\Omega\setminus B_{\delta}(\xi_j))} G(x,\xi_j)\frac{\partial G(x,\xi_j)}{\partial\nu} - \sum_{j=1}^{k} \int_{\Omega\setminus B_{\delta}(\xi_j)} G(x,\xi_j)\Delta G(x,\xi_j) \\
&= \sum_{j=1}^{k} \int_{\partial(\Omega\setminus B_{\delta}(\xi_j))} G(x,\xi_j)\frac{\partial G(x,\xi_j)}{\partial\nu} + 8\pi\sum_{j=1}^{k} \int_{\Omega\setminus B_{\delta}(\xi_j)} G(x,\xi_j)\delta_{\xi_j}(x) \\
&= \sum_{j=1}^{k} \int_{\partial(\Omega\setminus B_{\delta}(\xi_j))} G(x,\xi_j)\frac{\partial G(x,\xi_j)}{\partial\nu} \\
&= \sum_{j=1}^{k} \int_{\partial B_{\delta}(\xi_j)} G(x,\xi_j)\frac{\partial G(x,\xi_j)}{\partial\nu}.
\end{aligned}
\tag{3.52}
$$

We observe that on $\partial B_\delta(\xi_j)$, we have $G(x,\xi_j) = -4\log\delta + H(x,\xi_j)$ and $\frac{\partial G(x,\xi_j)}{\partial\nu} = \frac{4}{\delta} + \nabla H(x,\xi_j)\cdot\nu$, and by a Taylor expansion, we have

$$
\int_{\partial B_{\delta}(\xi_j)} G(x,\xi_j)\frac{\partial G(x,\xi_j)}{\partial\nu}
$$

$$
\begin{aligned}
&= \int\limits_{\partial B_\delta(\xi_j)} [-4\log\delta + H(x,\xi_j)]\left[\frac{4}{\delta} + \nabla H(x,\xi_j)\cdot\nu\right] \\
&= \int\limits_{\partial B_\delta(\xi_j)} [-4\log\delta + \mathcal{R}(\xi_j) + H(x,\xi_j) - H(\xi_j,\xi_j)] \\
&\qquad \times \left[\frac{4}{\delta} + \nabla H(x,\xi_j)\cdot\nu\right] \\
&= 8\pi\mathcal{R}(\xi_j) - 32\pi\log\frac{\delta}{2} + \delta\log\delta\Theta(\xi) + \delta\Theta(\xi). \qquad (3.53)
\end{aligned}
$$

And

$$
\begin{aligned}
I_{2b} &= \sum_{j=1}^{k}\sum_{l\neq j}^{k} \int\limits_{\Omega\setminus B_\delta(\xi_j)} |\nabla G(x,\xi_l)|^2 \\
&= \sum_{l\neq j}^{k} \int\limits_{\partial B_{\frac{\delta}{2}}(\xi_j)} G(x,\xi_l)\frac{\partial G(x,\xi_l)}{\partial\nu} = \delta\Upsilon(\xi). \qquad (3.54)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
I_{2c} &= \sum_{i\neq j}^{k} \int\limits_{\Omega\setminus B_\delta(\xi_j)} \nabla G(x,\xi_i)\nabla G(x,\xi_j) \\
&= \sum_{i\neq j}^{k} \int\limits_{\partial(\Omega\setminus B_\delta(\xi_j))} G(x,\xi_i)\frac{G(x,\xi_j)}{\partial\nu} - \sum_{i\neq j}^{k} \int\limits_{\Omega\setminus B_\delta(\xi_j)} G(x,\xi_i)\Delta G(x,\xi_j) \\
&= \sum_{i\neq j}^{k} \int\limits_{\partial B_\delta(\xi_j)} G(x,\xi_i)\frac{G(x,\xi_j)}{\partial\nu} \\
&= \sum_{i\neq j}^{k} \int\limits_{\partial B_\delta(\xi_j)} [G(\xi_i,\xi_j) + O(\delta)][\frac{4}{\delta} + \nabla H(x,\xi_j)\cdot\nu] \\
&= 8\pi\sum_{i\neq j}^{k} G(\xi_i,\xi_j) + \delta\Upsilon(\xi). \qquad (3.55)
\end{aligned}
$$

Moreover,

$$
I_{2d} = \sum_{i\neq l\neq j}^{k} \int\limits_{\Omega\setminus B_\delta(\xi_j)} \nabla G(x,\xi_i)\nabla G(x,\xi_l)
$$

$$
\begin{aligned}
&= \sum_{i\neq l\neq j}^{k} \int_{\partial B_\delta(\xi_j)} G(x,\xi_i)\frac{\partial G(x,\xi_l)}{\partial \nu} \\
&= \delta\Upsilon(\xi). \qquad (3.56)
\end{aligned}
$$

We observe that on $B_\delta(\xi_j)\backslash B_{\frac{\delta}{2}}(\xi_j)$, we have $U_j(x) = G(x,\xi_j) + O(\mu_j^2\varepsilon^2)$ and $U_l(x) = G(x,\xi_l)$ for $l \neq j$. Then we have

$$
\begin{aligned}
I_{2B} &= \int_{\cup_{j=1}^k (B_\delta(\xi_j)\backslash B_{\frac{\delta}{2}}(\xi_j))} |\nabla U|^2 = \int_{\cup_{j=1}^k (B_\delta(\xi_j)\backslash B_{\frac{\delta}{2}}(\xi_j))} \left|\nabla\left(\sum_{l=1}^k U_j\right)\right|^2 \\
&= \int_{\cup_{j=1}^k (B_\delta(\xi_j)\backslash B_{\frac{\delta}{2}}(\xi_j))} |\nabla U_j|^2 + \sum_{l\neq j} \int_{\cup_{j=1}^k (B_\delta(\xi_j)\backslash B_{\frac{\delta}{2}}(\xi_j))} |\nabla U_l|^2 \\
&= \sum_{j=1}^k \int_{B_\delta(\xi_j)\backslash B_{\frac{\delta}{2}}(\xi_j)} |\nabla(G(x,\xi_j) + O(\mu_j^2\varepsilon^2))|^2 \\
&\quad + \sum_{j=1}^k \sum_{l\neq j} \int_{B_\delta(\xi_j)\backslash B_{\frac{\delta}{2}}(\xi_j)} |\nabla G(x,\xi_l)|^2 \\
&= \delta\Upsilon(\xi). \qquad (3.57)
\end{aligned}
$$

Thus, from (3.50) to (3.57), we obtain

$$
I_2 = 8\pi\left[\sum_{j=1}^k \mathcal{R}(\xi_j) + \sum_{i\neq j}^k G(\xi_i,\xi_j)\right] + \delta\log\delta\Theta(\xi) + \delta\Upsilon(\xi) \qquad (3.58)
$$

Therefore (3.41) follows from (3.49), (3.58) and the choice of μ_j in (3.28).

Finally, let is estimate the second term II in the energy.

$$
II = \varepsilon^2\int_\Omega e^U = \sum_{j=1}^k \varepsilon^2 \int_{B_{\frac{\delta}{2}}(\xi_j)} e^U + \varepsilon^2 \int_{\Omega\backslash\cup_{j=1}^k B_{\frac{\delta}{2}}(\xi_j)} e^U \qquad (3.59)
$$

where

$$
\begin{aligned}
\varepsilon^2 \int_{B_{\frac{\delta}{2}}(\xi_j)} e^U &= \varepsilon^2 \int_{B_{\frac{\delta}{2}}(\xi_j)} e^{U_j + \sum_{i\neq j} U_i} \\
&= \varepsilon^2 \int_{B_{\frac{\delta}{2}}(\xi_j)} e^{u_j(x)} e^{H(x,\xi_j) + \sum_{i\neq j} U_i}
\end{aligned}
$$

$$
\begin{aligned}
&= \varepsilon^2 \int\limits_{B_{\frac{\delta}{2}}(\xi_j)} \frac{1}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2} e^{H(x,\xi_j)+\sum\limits_{i\neq j} U_i} \\
&= \int\limits_{B_{\frac{\delta}{2}}(\xi_j)} \frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2} e^{-\log(8\mu_j^2)+H(x,\xi_j)+\sum\limits_{i\neq j} U_i} \\
&= \int\limits_{B_{\frac{\delta}{2}}(\xi_j)} \frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2} e^{-\log(8\mu_j^2)+\mathcal{R}(\xi_j)+\sum\limits_{i\neq j} G(\xi_i,\xi_j)+O(|x-\xi_j|)} \\
&= 8\pi + \varepsilon\Theta_\varepsilon(\xi) \qquad (3.60)
\end{aligned}
$$

Moreover,

$$
\varepsilon^2 \int\limits_{\Omega\setminus\cup_{j=1}^k B_{\frac{\delta}{2}}(\xi_j)} e^U = \varepsilon^2 \int\limits_{\Omega\setminus\cup_{j=1}^k B_{\frac{\delta}{2}}(\xi_j)} e^{O(1)} = O(\varepsilon^2)|\Omega|. \qquad (3.61)
$$

Thus

$$
II = 8k\pi + O(\varepsilon^2)|\Omega| + \varepsilon\Theta(\xi). \qquad (3.62)
$$

Therefore, (3.37) follows (3.39), (3.40), (3.41) and (3.62). This completes the proof.

Chapter 4

Mixed interior and boundary bubbling solutions for Neumann problem in $\mathbb{R}^2$

[1]

4.1 Introduction

Consider the following boundary value problem

$$\begin{cases} -\Delta u + u = \lambda u^{p-1} e^{u^p}, \quad u > 0, & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter, $0 < p < 2$, and ν denotes the outer normal vector to $\partial\Omega$. This problem is the Euler-Lagrange equation for the functional

$$J_\lambda^p(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + u^2) - \frac{\lambda}{p} \int_\Omega e^{u^p}, \quad u \in H^1(\Omega). \tag{4.2}$$

If $p = 1$, Senba-Suzuki, in [109, 110], have analyzed the asymptotic behavior of solutions to problem (4.1). If u_λ is a family of solutions to problem (4.1) when $p = 1$, then there exist non-negative integers $k, l \geq 1$, such that

$$\lim_{\lambda \to 0} \lambda \int_\Omega e^{u_\lambda} = 4\pi(2k + l). \tag{4.3}$$

[1]The main result of this chapter was published in *Journal of Differential Equations*, Volume 253, Issue 2, 15 July 2012, 727-763.

Let $m = k + l$. Up to subsequences, there exist points ξ_j, $j = 1, \dots, m$ with $\xi_j \in \Omega$ for $j \le k$ and $\xi_j \in \partial\Omega$ for $k < j \le m$, for which

$$u_\lambda(x) \to \sum_{j=1}^{k} 8\pi G(x, \xi_j) + \sum_{j=k+1}^{m} 4\pi G(x, \xi_j), \quad \text{as} \quad \lambda \to 0, \tag{4.4}$$

uniformly on compact subset of $\bar{\Omega} \setminus \{\xi_1, \dots, \xi_m\}$. Moreover, the $m-$tuple $(\xi_1, \dots, \xi_m)$ can be characterized as critical point of a functional defined on $\Omega^k \times (\partial\Omega)^l$, given by

$$\varphi_m(\xi) = \varphi_m(\xi_1, \dots, \xi_m) = \sum_{j=1}^{m} c_j^2 H(\xi_j, \xi_j) + \sum_{l \ne j} c_l c_j G(\xi_l, \xi_j), \tag{4.5}$$

where

$$c_j = 8\pi \ \text{ for } \ j = 1, \dots, k, \quad \text{ and } \ c_j = 4\pi \ \text{ for } \ j = k+1, \dots, m,$$

and $G(x, y)$ is the Green's function of the problem

$$\begin{cases} -\Delta_x G(x, y) + G(x, y) = \delta_y(x), & \text{in } \Omega; \\ \frac{\partial G(x,y)}{\partial \nu_x} = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

and $H(\cdot, \cdot)$ its regular part, namely,

$$H(x, y) = \begin{cases} G(x, y) + \frac{1}{2\pi} \log |x - y|, & \text{if } y \in \Omega; \\ G(x, y) + \frac{1}{\pi} \log |x - y|, & \text{if } y \in \partial\Omega. \end{cases} \tag{4.7}$$

Conversely, del Pino-Wei, in [41], constructed bubbling solutions u_λ to problem (4.1) when $p = 1$ with the above properties (4.3) and (4.4). Moreover, the location of the bubbling points corresponds to critical points of the function φ_m defined by (4.5). Furthermore, they obtained the following expansion of the energy functional

$$J_\lambda^1(u_\lambda) = -4\pi(2k + l)(2 - \log 8) - 8\pi(2k + l) \log \varepsilon - \frac{1}{2} \varphi_m(\xi) + o(1),$$

where $o(1) \to 0$ as $\lambda \to 0$.

This chapter is devoted to construct solutions to problem (4.1) with bubbling profiles at points inside Ω and on the boundary of Ω when p is between 0 and 2. In particular, we recover the result in [41] when $p = 1$.

Let ε be a parameter, which depends on λ, defined as

$$p\lambda \left(-\frac{4}{p} \log \varepsilon \right)^{\frac{2(p-1)}{p}} \varepsilon^{\frac{2(p-2)}{p}} = 1. \tag{4.8}$$

Observe that, as $\lambda \to 0$, then $\varepsilon \to 0$, and $\lambda = \varepsilon^2$ if $p = 1$. Our result states as follows.

Theorem 4.1. *Let $0 < p < 2$, and $k, l, m \geq 1$ be integers with $m = k+l$. There exists $\lambda_0 > 0$ so that, for any $0 < \lambda < \lambda_0$, problem (4.1) has a solution u_λ, with the following properties:*

(1) u_λ has m local maximum points ξ_j^*, $j = 1, \ldots, m$ such that $\xi_j^* \in \Omega$ for $1 \leq j \leq k$, and $\xi_j^* \in \partial\Omega$ for $k+1 \leq j \leq m$. Furthermore

$$\lim_{\lambda \to 0} \varphi_m(\xi_1^*, \ldots, \xi_m^*) = \min_{\Omega^k \times (\partial\Omega)^l} \varphi_m,$$

where φ_m is defined by (4.5). In particular

(2) One has

$$u_\lambda(x) = p^{-\frac{1}{2}} \sqrt{\lambda}\, \varepsilon^{\frac{p-2}{p}} \left[\sum_{j=1}^{k} 8\pi G(x, \xi_j^*) + \sum_{j=k+1}^{m} 4\pi G(x, \xi_j^*) + o(1) \right] \tag{4.9}$$

where ε satisfies (4.8), and $o(1) \to 0$, as $\lambda \to 0$, on each compact subset of $\bar{\Omega} \backslash \{\xi_1^*, \ldots, \xi_m^*\}$, and $G(\cdot, \cdot)$ is the Green's function given in (4.6).

(3) Moreover

$$\lim_{\lambda \to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_\Omega e^{u_\lambda^p} = 4\pi(2k + l). \tag{4.10}$$

Furthermore

$$J_\lambda^p(u_\lambda) = \lambda \varepsilon^{\frac{2(p-2)}{p}} \left[-4\pi(2k+l) \frac{2 - p \log 8}{(2-p)p} - \frac{8\pi}{p}(2k+l) \log \varepsilon - \frac{1}{2(2-p)} \varphi_m(\xi^*) + O(|\log \varepsilon|^{-1}) \right] \tag{4.11}$$

where $O(1)$ uniformly bounded as $\lambda \to 0$.

This chapter is organized as follows. In Section 4.2, describing a first approximation solution to problem (4.1) and estimating the error. We describe the proof of the main result in Section 4.3. Section 4.4 is devoted to perform the finite dimensional reduction. Section 4.5 contains the asymptotic expansion of the reduced energy.

4.2 Preliminaries and ansatz for the solution

In this section we describe the approximate solution for problem (4.1) and then we estimate the error of such approximation in appropriate norms.

let us introduce the following limit problem

$$\Delta w + e^w = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^w \, dx < +\infty. \tag{4.12}$$

It is well known that the solutions to (4.12) can be all written in the following form

$$w_\mu(z) = \log \frac{8\mu^2}{(\mu^2 + |z|^2)^2}, \quad \text{and} \quad w_{\mu,\xi}(z) := w_\mu(z-\xi) \tag{4.13}$$

where μ is any positive number and ξ any point in $\mathbb{R}^2$ (see [21]).

We choose a sufficiently small but fixed number $\delta > 0$ and define

$$\mathcal{M}_\delta := \left\{ \xi := (\xi_1, \dots, \xi_m) \in \Omega^k \times (\partial\Omega)^l \;\middle|\; \min_{i=1,\dots,k} \operatorname{dist}(\xi_i, \partial\Omega) \geq \delta, \;\; \min_{i\neq j} |\xi_i - \xi_j| \geq \delta \right\}. \tag{4.14}$$

Let us consider m distinct points $(\xi_1, \dots, \xi_m) \in \mathcal{M}_\delta$, with $\xi_1, \dots, \xi_k$ in Ω and $\xi_{k+1}, \dots, \xi_m$ on $\partial\Omega$. Moreover we consider m positive numbers μ_j such that

$$\delta < \mu_j < \delta^{-1}, \quad \text{for all} \quad j = 1, \dots, m. \tag{4.15}$$

We define the function

$$u_j(x) = \log \frac{8\mu_j^2}{(\mu_j^2 \varepsilon^2 + |x - \xi_j|^2)^2},$$

and a correction term defined as the solution of

$$\begin{cases} -\Delta H_j + H_j = -u_j, & \text{in } \Omega; \\ \frac{\partial H_j}{\partial \nu} = -\frac{\partial u_j}{\partial \nu}, & \text{on } \partial\Omega. \end{cases} \tag{4.16}$$

Lemma 4.2. *For any $0 < \alpha < 1$, $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{M}_\delta$, then we have*

$$H_j(x) = c_j H(x, \xi_j) - \log(8\mu_j^2) + O(\varepsilon^\alpha), \tag{4.17}$$

uniformly in $\bar{\Omega}$ as $\varepsilon \to 0$, where $H(\cdot,\cdot)$ is the regular part of Green's function defined in (4.7).

Proof. First, on the boundary, we have

$$\frac{\partial H_j}{\partial \nu} = -\frac{\partial u_j}{\partial \nu} = 4 \frac{(x - \xi_j) \cdot \nu(x)}{\mu_j^2 \varepsilon^2 + |x - \xi_j|^2}.$$

Thus,

$$\lim_{\varepsilon \to 0} \frac{\partial H_j}{\partial \nu} = 4 \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2}, \qquad \forall\, x \in \partial\Omega \backslash \{\xi_j\}.$$

On the other hand, the regular part of Green's function $H(x, y)$ satisfies

$$\begin{cases} -\Delta_x H(x,y) + H(x,y) = -\frac{4}{c_j} \log \frac{1}{|x-y|}, & \text{in } \Omega; \\ \frac{\partial H(x,y)}{\partial \nu_x} = \frac{4}{c_j} \frac{(x-y)\cdot\nu(x)}{|x-y|^2}, & \text{on } \partial\Omega. \end{cases} \tag{4.18}$$

Set $z(x) = H_j(x) - c_j H(x,\xi_j) + \log(8\mu_j^2)$, then we get

$$\begin{cases} -\Delta z(x) + z(x) = \log \frac{1}{|x-\xi_j|^4} - \log \frac{1}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2}, & \text{in } \Omega; \\ \frac{\partial z(x)}{\partial \nu} = \frac{\partial H_j(x)}{\partial \nu} - 4\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}, & \text{on } \partial\Omega. \end{cases}$$

A direct computation shows that, there is a positive constant C such that

$$\left\| \frac{\partial H_j(x)}{\partial \nu} - 4\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2} \right\|_{L^q(\partial\Omega)} \le C\varepsilon^{1/q}, \quad \forall\ q > 1, \tag{4.19}$$

and

$$\left\| \log \frac{1}{|x-\xi_j|^4} - \log \frac{1}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2} \right\|_{L^q(\Omega)} \le C\varepsilon, \quad \text{for any } \ 1 < q < 2.$$

Then by elliptic regularity theory, we obtain

$$\|z_\varepsilon\|_{W^{1+s,q}(\Omega)} \le \left(\left\| \frac{\partial z_\varepsilon}{\partial \nu} \right\|_{L^q(\partial\Omega)} + \|\Delta z_\varepsilon\|_{L^q(\Omega)} \right) \le C\varepsilon^{1/q} \tag{4.20}$$

for any $0 < s < \frac{1}{q}$. By the Morrey embedding we obtain

$$\|z_\varepsilon\|_{C^\beta(\bar{\Omega})} \le C\varepsilon^{1/q}$$

for any $0 < \beta < \frac{1}{2} + \frac{1}{q}$. Then we obtain that (4.17) holds with $\alpha = \frac{1}{q}$. □

We now define the first ansatz is given by

$$U(x) = \frac{1}{p\gamma^{p-1}} \sum_{j=1}^{m} \left[u_j(x) + H_j(x) \right],$$

with some number γ, to be fixed later on. We want to show that $U(x)$ is a good approximation for a solution to (4.1) far from the points ξ_j, but unfortunately it is not good enough for our construction close to the points ξ_j. Thus we need to further adjust this ansatz. In order to do this, we set

$$w_{\mu_j}(y) = w_{\mu_j}(y - \xi_j') = \log \frac{8\mu_j^2}{(\mu_j^2 + |y-\xi_j'|^2)^2}.$$

Define the function w_{ij} to be the radial solution of

$$\Delta w_{ij} + e^{w_{\mu_j}} w_{ij} = e^{w_{\mu_j}} f^i \quad \text{in } \mathbb{R}^2, \quad \text{for} \quad i = 0, 1, \tag{4.21}$$

where

$$f^0 = -\left(w_{\mu_j} + \frac{1}{2}(w_{\mu_j})^2 \right),$$

$$\begin{aligned} f^1 &= -\Bigg(2w_{\mu_j}w_{0j} + \frac{1}{2}\left[w_{0j} + \frac{(w_{\mu_j})^2}{2}\right]^2 \\ &\qquad + w_{0j} + \frac{p-2}{2(p-1)}(w_{\mu_j})^2 + \frac{(w_{\mu_j})^3}{2}\Bigg). \end{aligned}$$

In fact, as shown in [47] (see also [20]), there exists radially symmetric solutions with the properties that

$$w_{ij}(y) = C_{ij}\log\frac{|y-\xi_j'|}{\mu_j} + O(\frac{1}{|y-\xi_j'|}) \quad \text{as } |y-\xi_j| \to \infty, \tag{4.22}$$

for some explicit constants C_{ij}, which can be explicitly computed. In particular, when $i = 0$, the constant C_{0j} is given by

$$\begin{aligned} C_{0j} &= -8\int_0^{+\infty} t\frac{t^2-1}{(t^2+1)^3}\left[\log\frac{8\mu_j^{-2}}{(1+t^2)^2} + \frac{1}{2}\left(\log\frac{8\mu_j^{-2}}{(1+t^2)^2}\right)^2\right]dt \\ &= -4\int_0^{+\infty} \frac{t^2-1}{(t^2+1)^3}\left[\log\frac{8\mu_j^{-2}}{(1+t^2)^2} + \frac{1}{2}\left(\log\frac{8\mu_j^{-2}}{(1+t^2)^2}\right)^2\right]d(t^2) \\ &= -4\int_1^{+\infty} \frac{r-2}{r^3}\Big[\log(8\mu_j^{-2}) - 2\log r + \frac{1}{2}\left(\log(8\mu_j^{-2})\right)^2 \\ &\qquad -2\log(8\mu_j^{-2})\log r + 2(\log r)^2\Big]\,dr. \end{aligned}$$

Since

$$\int_1^{+\infty}\frac{r-2}{r^3}\,dr = 0,$$

$$\int_1^{+\infty}\frac{r-2}{r^3}\log r\,dr = \frac{1}{2},$$

and

$$\int_1^{+\infty}\frac{r-2}{r^3}(\log r)^2\,dr = \frac{3}{2}.$$

Hence

$$C_{0j} = 4\log 8 - 8 - 8\log\mu_j. \tag{4.23}$$

Let H_{ij}, for $i = 0, 1$, be a new correction defined as the solution of

$$\begin{cases} -\Delta H_{ij} + H_{ij} = -w_{ij}(x/\varepsilon), & \text{in } \Omega; \\ \frac{\partial H_{ij}}{\partial\nu} = -\frac{\partial w_{ij}}{\partial\nu}, & \text{on } \partial\Omega. \end{cases} \tag{4.24}$$

Lemma 4.3. *For any* $0 < \alpha < 1$, *for* $i = 0, 1$, *one has*

$$H_{ij}(x) = -\frac{C_{ij}c_j}{4} H(x, \xi_j) + C_{ij} \log(\mu_j) + C_{ij} \log \varepsilon + O(\varepsilon^\alpha) \tag{4.25}$$

uniformly in $\bar{\Omega}$ *as* $\varepsilon \to 0$, *where* H *is the regular part of Green's function defined in (4.7).*

Proof. The proof is the same as Lemma 4.2. First we note that, on the boundary, we have

$$\lim_{\varepsilon \to 0} \frac{\partial H_{ij}}{\partial \nu} = -C_{ij} \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2}, \quad \forall\, x \in \partial\Omega \backslash \{\xi_j\}.$$

Define $\tilde{z}(x) = H_{ij}(x) + \frac{C_{ij}c_j}{4} H(x, \xi_j) - C_{ij} \log(\mu_j \varepsilon)$, by using (4.18), then we can get

$$\begin{cases} -\Delta \tilde{z}(x) + \tilde{z}(x) = -C_{ij} \log \frac{1}{|x-\xi_j|} - w_{ij}, & \text{in } \Omega; \\ \frac{\partial \tilde{z}(x)}{\partial \nu} = \frac{\partial H_{ij}(x)}{\partial \nu} + 4C_{ij} \frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}, & \text{on } \partial\Omega. \end{cases}$$

From (4.22), we can get

$$\left\| C_{ij} \log \frac{1}{|x - \xi_j|} - w_{ij} \right\|_{L^q(\Omega)} \le C\varepsilon, \quad \text{for any } \ 1 < q < 2,$$

for some constant $C > 0$, and

$$\left\| \frac{\partial H_{ij}(x)}{\partial \nu} + 4C_{ij} \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} \right\|_{L^q(\partial\Omega)} \le C\varepsilon^{1/q}, \quad \forall\, q > 1,$$

Then by the same procedure as proof of Lemma 4.2, we obtain that (4.25) holds. □

Now we define the first approximation solution to (4.1) as

$$\begin{aligned} U_\lambda(x) \ &= \ \frac{1}{p\gamma^{p-1}} \sum_{j=1}^m \Bigg[u_j(x) + H_j(x) + \frac{p-1}{p} \frac{1}{\gamma^p} \left(w_{0j}(x) + H_{0j}(x) \right) \\ &\quad + \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \left(w_{1j}(x) + H_{1j}(x) \right) \Bigg]. \end{aligned} \tag{4.26}$$

From Lemma 4.2 and Lemma 4.3, one has, away from the points ξ_j,

$$U_\lambda(x) = \frac{1}{p\gamma^{p-1}} \sum_{j=1}^m c_j G(x, \xi_j) \left[1 - \frac{p-1}{p} \frac{1}{\gamma^p} \frac{C_{0j}}{4} - \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \frac{C_{1j}}{4} + O(\varepsilon^\alpha) \right]. \tag{4.27}$$

Consider now the change of variables

$$v(y) = p\gamma^{p-1} u(\varepsilon y) - p\gamma^p, \quad \text{with } \ \gamma^p = -\frac{4}{p} \log \varepsilon.$$

By the choice of ε in (4.8), then problem (4.1) reduces to

$$\begin{cases} -\Delta v + \varepsilon^2 v = f(v) - p\gamma^p\varepsilon^2, \quad v > 0, & \text{in } \Omega_\varepsilon; \\ \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{4.28}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$, and

$$f(v) = (1 + \frac{v}{p\gamma^p})^{p-1} e^{\gamma^p[(1+\frac{v}{p\gamma^p})^p - 1]}. \tag{4.29}$$

Let us define the first approximation solution to (4.28) as

$$V_\lambda(y) = p\gamma^{p-1}U_\lambda(\varepsilon y) - p\gamma^p, \tag{4.30}$$

with U_λ defined by (4.26).

We write $y = \varepsilon^{-1}x,\ \xi_j' = \varepsilon^{-1}\xi_j$. For $|x - \xi_j| < \delta$ with δ sufficiently small but fixed, by using Lemma 4.2, Lemma 4.3 and (4.27), and the fact that $u_j(\varepsilon y) - p\gamma^p = w_{\mu_j}(y - \xi_j')$, we have

$$\begin{aligned}
V_\lambda(y) &= u_j(\varepsilon y) + H_j(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p}\left(w_{0j}(\varepsilon y) + H_{0j}(\varepsilon y)\right) \\
&\quad + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\left(w_{1j}(\varepsilon y) + H_{1j}(\varepsilon y)\right) - p\gamma^p \\
&\quad + \sum_{l\neq j}^{m} \Bigg[u_l(\varepsilon y) + H_l(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p}\left(w_{0l}(\varepsilon y) + H_{0l}(\varepsilon y)\right) \\
&\qquad + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\left(w_{1l}(\varepsilon y) + H_{1l}(\varepsilon y)\right)\Bigg] \\
&= w_{\mu_j}(y - \xi_j') + c_j H(\varepsilon y, \xi_j) - \log(8\mu_j^2) + O(\varepsilon^\alpha) \\
&\quad + \frac{p-1}{p}\frac{1}{\gamma^p}\left[w_{0j}(\varepsilon y) - \frac{C_{0j}c_j}{4}H(x,\xi_j) + C_{0j}\log(\mu_j) + C_{0j}\log\varepsilon + O(\varepsilon^\alpha)\right] \\
&\quad + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\left[w_{1j}(\varepsilon y) - \frac{C_{1j}c_j}{4}H(x,\xi_j) + C_{1j}\log(\mu_j) + C_{1j}\log\varepsilon + O(\varepsilon^\alpha)\right] \\
&\quad + \sum_{l\neq j}^{m} c_l\ G(\xi_l,\xi_j)\left[1 - \frac{C_{0l}}{4}\frac{p-1}{p}\frac{1}{\gamma^p} - \frac{C_{1l}}{4}\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\right] \\
&= w_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}w_{0j}(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_{1j}(y) + O(\varepsilon|y - \xi_j'|) + O(\varepsilon^\alpha) \\
&\quad - \log(8\mu_j^2) + c_j H(\xi_j,\xi_j) + \sum_{l\neq j} c_l G(\xi_l,\xi_j) - \frac{p-1}{4}C_{0j} \\
&\quad - \frac{p-1}{p}\frac{1}{\gamma^p}\left[\frac{C_{0j}}{4}\left(c_j H(\xi_j,\xi_j) + \sum_{l\neq j} c_l G(\xi_l,\xi_j) - 4\log\mu_j\right) + \frac{(p-1)C_{1j}}{4}\right]
\end{aligned}$$

$$-\left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\left[\frac{C_{1j}}{4}\left(c_j H(\xi_j,\xi_j)+\sum_{l\neq j} c_l G(\xi_l,\xi_j)-4\log\mu_j\right)\right], \tag{4.31}$$

where

$$w_j(y)=w_{\mu_j}(y-\xi_j),\quad w_{0j}(y)=w_{0j}(y-\xi_j),\quad w_{1j}(y)=w_{1j}(y-\xi_j).$$

We now choose the parameters μ_j: we assume they are defined by the relation

$$\begin{aligned}\log(8\mu_j^2) &= c_j H(\xi_j,\xi_j)+\sum_{l\neq j} c_l G(\xi_l,\xi_j)-\frac{p-1}{4}C_{0j}\\ &\quad -\frac{p-1}{p}\frac{1}{\gamma^p}\left[\frac{C_{0j}}{4}\left(c_j H(\xi_j,\xi_j)+\sum_{l\neq j} c_l G(\xi_l,\xi_j)-4\log\mu_j\right)+\frac{(p-1)C_{1j}}{4}\right]\\ &\quad -\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left[\frac{C_{1j}}{4}\left(c_j H(\xi_j,\xi_j)+\sum_{l\neq j} c_l G(\xi_l,\xi_j)-4\log\mu_j\right)\right]. \end{aligned}\tag{4.32}$$

Taking into account the explicit expression (4.23) of the constant C_{0j}, we observe that μ_j bifurcates, as λ goes to zero, from the value

$$\bar\mu_j = 8^{-\frac{p}{2(2-p)}} e^{\frac{p-1}{2-p}} e^{\frac{1}{2(2-p)}\left[c_j H(\xi_j,\xi_j)+\sum_{l\neq j} c_l G(\xi_l,\xi_j)\right]} \tag{4.33}$$

solution of equation

$$\log(8\mu_j^2)=c_j H(\xi_j,\xi_j)+\sum_{l\neq j} c_l G(\xi_l,\xi_j)-\frac{p-1}{4}C_{0j}. \tag{4.34}$$

Thus, μ_j is a perturbation of order $\frac{1}{\gamma^p}$ of the value $\bar\mu_j$, namely

$$\begin{aligned}\log(8\mu_j^2) &= \left[\frac{2(p-1)}{2-p}(1-\log 8)+\frac{1}{2-p}\left(c_j H(\xi_j,\xi_j)+\sum_{l\neq j} c_l G(\xi_l,\xi_j)\right)\right]\\ &\quad \times\left(1+O\left(\frac{1}{\gamma^p}\right)\right).\end{aligned}\tag{4.35}$$

Then, by this choice of the parameters μ_j, we deduce that, if $|y-\xi_j'|<\delta/\varepsilon$ with δ sufficiently small but fixed, we can rewrite

$$V_\lambda(y)=w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_{0j}(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_{1j}(y)+\theta(y), \tag{4.36}$$

with

$$\theta(y)=O(\varepsilon|y-\xi_j'|)+O(\varepsilon^\alpha).$$

In the rest of this chapter, we will look for solutions for problem (4.28) in the form $v = V_\lambda + \phi$, where ϕ will represent a lower order correction. For small ϕ, we can rewrite problem (4.28) as a nonlinear perturbation of its linearization, namely,

$$\begin{cases} L(\phi) = E_\lambda + N(\phi), & x \in \Omega_\varepsilon; \\ \frac{\partial \phi}{\partial \nu} = 0, & x \in \partial\Omega_\varepsilon, \end{cases} \tag{4.37}$$

where

$$L(\phi) := -\Delta\phi + \varepsilon^2\phi - W\phi, \quad \text{with} \;\; W = f'(V_\lambda), \tag{4.38}$$

$$E_\lambda = \Delta V_\lambda + f(V_\lambda) - \varepsilon^2 V_\lambda + 4\varepsilon^2 \log\varepsilon, \tag{4.39}$$

and

$$N(\phi) = f(V_\lambda + \phi) - f(V_\lambda) - f'(V_\lambda)\phi. \tag{4.40}$$

For any $h \in L^\infty(\Omega_\varepsilon)$, let us define a weighted L^∞-norm defined as

$$\|h\|_* := \sup_{y\in\Omega_\varepsilon} \left(\sum_{j=1}^{m} (1 + |y - \xi_j'|)^{-2-\sigma} + \varepsilon^2 \right)^{-1} |h(y)| \tag{4.41}$$

where we fix $0 < \sigma < 1$. With respect to this norm, the error term E_λ given in (4.39) can be estimated in the following way.

Lemma 4.4. *Let $\delta > 0$ be a small but fixed number and assume that the points $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{M}_\delta$. There exists $C > 0$, such that we have*

$$\|E_\lambda\|_* \leq \frac{C}{\gamma^{3p}} = \frac{C}{|\log\varepsilon|^3} \tag{4.42}$$

for all λ small enough.

Proof. First we observe that

$$-\varepsilon^2 V_\lambda + 4\varepsilon^2 \log\varepsilon = O(\varepsilon^2). \tag{4.43}$$

Far away from the points ξ_j, namely for $|x - \xi_j| > \delta$, i.e. $|y - \xi_j'| > \frac{\delta}{\varepsilon}$, for all $j = 1, \ldots, m$, from (4.27) we have that

$$\Delta V_\lambda(y) = p\gamma^{p-1}\varepsilon^2 \Delta U(\varepsilon y) = O(\gamma^{p-1}\varepsilon^4).$$

On the other hand, in this region we have

$$1 + \frac{V_\lambda(y)}{p\gamma^p} = 1 + \frac{4\log\varepsilon + O(1)}{p\gamma^p} = \frac{O(1)}{|\log\varepsilon|} \tag{4.44}$$

where $O(1)$ denotes a smooth function, uniformly bounded, as $\varepsilon \to 0$, in the considered region. Hence

$$
\begin{aligned}
f(V_\lambda) &= \left(1+\frac{V_\lambda}{p\gamma^p}\right)^{p-1} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]} \\
&= \begin{cases} C\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2; \\ C\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}} e^{\gamma^p\frac{O(1)}{|\log\varepsilon|^p}} & \text{if } 0<p<1. \end{cases} \\
&= \begin{cases} C\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2; \\ C\frac{\varepsilon^{\frac{4}{p}}}{|\log\varepsilon|^{p-1}} e^{\frac{O(1)}{|\log\varepsilon|^{p-1}}} & \text{if } 0<p<1. \end{cases}
\end{aligned}
$$

Thus if we are far away from the points ξ_j, or equivalently for $|y-\xi_j'|>\frac{\delta}{\varepsilon}$, the size of the error, measured with respect to the $\|\cdot\|_*$-norm, is relatively small. In other words, if we denote by 1_{outer} the characteristic function of the set $\{y : |y-\xi_j'|>\frac{\delta}{\varepsilon},\ j=1,\dots,m\}$, then in this region we have

$$
\begin{aligned}
\|E_\lambda 1_{\mathsf{outer}}\|_* &\le \begin{cases} C\frac{\varepsilon^{\frac{2(2-p)}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2; \\ C\frac{\varepsilon^{\frac{2-p}{p}}}{|\log\varepsilon|^{p-1}} e^{\log\varepsilon^{\frac{2-p}{p}}+\frac{C}{|\log\varepsilon|^{p-1}}} & \text{if } 0<p<1. \end{cases} \\
&= \begin{cases} C\frac{\varepsilon^{\frac{2(2-p)}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2; \\ C\frac{\varepsilon^{\frac{2-p}{p}}}{|\log\varepsilon|^{p-1}} e^{-\frac{2-p}{p}|\log\varepsilon|+C|\log\varepsilon|^{1-p}} & \text{if } 0<p<1. \end{cases} \\
&\le \begin{cases} C\frac{\varepsilon^{\frac{2(2-p)}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2; \\ C\frac{\varepsilon^{\frac{2-p}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 0<p<1. \end{cases}
\end{aligned} \tag{4.45}
$$

Here we used that $-\frac{2-p}{p}|\log\varepsilon|+C|\log\varepsilon|^{1-p}<0$ for $0<p<1$ and ε small. Let us now fix the index j in $\{1,\dots,m\}$, for $|y-\xi_j'|<\frac{\delta}{\varepsilon}$, we have

$$
\Delta V_\lambda(y) = -e^{w_j(y)} + \frac{p-1}{p}\frac{1}{\gamma^p}\Delta w_{0j}(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\Delta w_{1j}(y) + O(\varepsilon^2). \tag{4.46}
$$

On the other hand, for any $R>0$ large but fixed, in the ball $|y-\xi_j'|<R_\varepsilon := R|\log\varepsilon|^\alpha$, with $\alpha\ge 3$, we can use Taylor expansion to first get

$$
\begin{aligned}
(1+\frac{V_\lambda}{p\gamma^p})^{p-1} &= 1+\frac{p-1}{p}\frac{1}{\gamma^p}w_j + (\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}[w_{0j}+\frac{p-2}{2(p-1)}(w_j)^2] \\
&\quad +(\frac{p-1}{p})^3\frac{1}{\gamma^{3p}}(\log|y-\xi_j'|),
\end{aligned}
$$

$$
\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1] = w_j + (\frac{p-1}{p})\frac{1}{\gamma^p}[w_{0j}+\frac{(w_j)^2}{2}]
$$

$$+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}(w_{1j}+w_jw_{0j})+\frac{1}{\gamma^{3p}}(\log|y-\xi_j'|),$$

and

$$e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]}=e^{w_j}\left\{1+(\frac{p-1}{p})\frac{1}{\gamma^p}\left[w_{0j}+\frac{(w_j)^2}{2}\right]\right.$$
$$\left.+(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}\left[w_{1j}+w_jw_{0j}+\frac{1}{2}(w_{0j}+(w_j)^2)^2\right]+\frac{1}{\gamma^{3p}}(\log|y-\xi_j'|)\right\}.$$

Thus we obtain

$$\begin{aligned} f(V_\lambda) &= (1+\frac{V_\lambda}{p\gamma^p})^{p-1}e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p-1]} \\ &= e^{w_j}\left\{1+(\frac{p-1}{p})\frac{1}{\gamma^p}\left[w_{0j}+\frac{(w_j)^2}{2}+w_j\right]\right. \\ &\quad +(\frac{p-1}{p})^2\frac{1}{\gamma^{2p}}\left[w_{1j}+2w_jw_{0j}+\frac{1}{2}\left(w_{0j}+\frac{(w_j)^2}{2}\right)^2\right. \\ &\quad \left.+w_{0j}+\frac{p-2}{2(p-1)}(w_j)^2+\frac{(w_j)^3}{2}\right] \\ &\quad \left.+O\left(\frac{\log|y-\xi_j'|}{\gamma^{3p}}\right)\right\}. \end{aligned}$$

Thus, thanks to the fact that we have improved our original approximation with the terms w_{0j} and w_{1j}, and the definition of $*$-norm, we get that

$$\|E_\lambda 1_{B(\xi_j',R_\varepsilon)}\|_*\le\frac{C}{\gamma^{3p}}=\frac{C}{|\log\varepsilon|^3},\quad\text{for any}\quad j=1,\dots,m. \tag{4.47}$$

Here $1_{B(\xi_j',R_\varepsilon)}$ denotes the characteristic function of $B(\xi_j,R_\varepsilon)$. Finally, in the remaining region, namely where $R_\varepsilon<|y-\xi_j'|<\frac{\delta}{\varepsilon}$, for any $j=1,\dots,m$, we have from one hand that $|\Delta V_\lambda(y)|\le Ce^{w_j(y)}$, and also $|f(V_\lambda(y))|\le Ce^{w_j(y)}$ as consequence of (4.31). This fact, together with (4.47) and (4.45), (4.43) we obtain estimate (4.42). □

As the proof of (2.34), (2.35) and (2.36), we have that very close to the point ξ_j in Ω,

$$\|f'(V_\lambda)-e^{w_j}\|_*\to 0\quad\text{ as }\lambda\to 0, \tag{4.48}$$

and there exists some positive constant D_0 such that

$$f'(V_\lambda)\le D_0\sum_{j=1}^m e^{w_j}. \tag{4.49}$$

Moreover, we can get

$$\|f''(V_\lambda)\|_*\le C. \tag{4.50}$$

4.3 The existence result

The operator L defined in (4.38) can be seen as a superposition of linear operators,

$$\mathcal{L}_*(\phi) = -\Delta\phi - \frac{8}{(1+|z|^2)^2}\phi,$$

namely, equation $-\Delta w - e^w = 0$ linearized around the radial solution $w(y) = \log \frac{8}{(1+|y|^2)^2}$. The key face to develop a satisfactory solvability theory for the operator L is the nondegeneracy of w up to the natural invariances of the equation under translations and dilations. In fact, the functions

$$z_{0j}(y) = \partial_{\mu_j} w_{\mu_j}(y), \qquad z_{ij}(y) = \partial_{y_i} w_{\mu_j}(y), \quad i = 1,2,$$

satisfy the function $\Delta Z + e^{w_{\mu_j}} Z = 0$, where w_{μ_j} defined by (4.13), see [10] for a proof.

Let us consider a large but fixed number $R_0 > 0$ and a radial and smooth cut-off function η with $\eta(r) = 1$ if $r < R_0$ and $\eta(r) = 0$ if $r > R_0 + 1$, $0 \le \eta \le 1$.

The interior bubble case: for $j = 1, \dots, k$, we define

$$\eta_j(y) = \eta\left(|y - \xi_j'|\right), \qquad Z_{ij}(y) := z_{ij}\left(y - \xi_j'\right), \quad i = 0,1,2, \quad j = 1,2,\dots,k. \tag{4.51}$$

The boundary bubble case: for $j = k+1, \dots, m$, we first strengthen the boundary. Namely, at the boundary point $\xi_j \in \partial\Omega$, without loss of generality, we assume that $\xi_j = 0$ and the unit outward normal at ξ_j is $-e_2 = (0,-1)$. Let $G(x_1)$ be the defining function for the boundary $\partial\Omega$ in a neighbourhood $B_\rho(\xi_j)$ of ξ_j, that is, $\Omega \cap B_\rho(\xi_j) = \{(x_1,x_2) \;\Big|\; x_2 > G(x_1),\ (x_1,x_2) \in B_\rho(\xi_j)\}$. Then, let $F_j : B_\rho(\xi_j) \cap \Omega \to \mathbb{R}^2$ be defined by

$$F_j = (F_{j,1}, F_{j,2}), \qquad \text{with } F_{j,1} = x_1 + \frac{x_2 - G(x_1)}{1 + |G'(x_1)|^2} G'(x_1), \quad F_{j,2} = x_2 - G(x_1). \tag{4.52}$$

Then we set $F_j^\varepsilon(y) = \frac{1}{\varepsilon} F_j(\varepsilon y)$, and define

$$\eta_j(y) = \eta\left(|F_j^\varepsilon(y)|\right), \qquad Z_{ij}(y) := z_{ij}\left(F_j^\varepsilon(y)\right), \quad i = 0,1, \quad j = k+1,\dots,m. \tag{4.53}$$

It is important to observe that F_j preserves the Neumann boundary condition and

$$\Delta Z_{0j} + \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2} Z_{0j} = O\left(\frac{\varepsilon^\alpha}{(1 + |y - \xi_j'|)^3}\right). \tag{4.54}$$

Define the norm

$$\|\phi\|_\infty = \sup_{y \in \Omega_\varepsilon} |\phi(y)|.$$

Consider the problem of finding a function ϕ such that for certain scalars c_{ij}, it satisfies

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi - W\phi = [E_\lambda + N(\phi)] + \sum_{j=1}^{m}\sum_{i=1}^{J_j} c_{ij} Z_{ij}\eta_j, & \text{in } \Omega_\varepsilon; \\ \frac{\partial\phi}{\partial\nu} = 0, & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \phi Z_{ij}\eta_j = 0, & \text{for } i = 1, J_j,\ j = 1, \dots, m, \end{cases} \tag{4.55}$$

where $J_j = 2$ if $j = 1, \dots, k$ and $J_j = 1$ if $j = k+1, \dots, m$.

Equation (4.55) is solved in the following Proposition, whose proof is postponed to Section 4.4.

Proposition 4.5. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that for any points ξ_j, $j = 1, \dots, m$, in $\mathcal{M}_\delta$, μ_j is given by (4.35), then problem (4.55) has a unique solution ϕ which satisfies*

$$\|\phi\|_\infty \le \frac{C}{|\log\varepsilon|^2},$$

for all $\lambda < \lambda_0$. Moreover, if we consider the map $\xi' \mapsto \phi$ into the space $C(\bar{\Omega}_\varepsilon)$, the derivative $D_{\xi'}\phi$ exists and defines a continuous function of ξ'. Besides, there is a constant $C > 0$, such that

$$\|D_{\xi'}\phi\|_\infty \le \frac{C}{|\log\varepsilon|}. \tag{4.56}$$

In order to find a solution to the original problem we need to find ξ' such that

$$c_{ij}(\xi') = 0 \quad \text{for all } i = 1, J_j, \quad j = 1, \dots, m. \tag{4.57}$$

This problem is indeed variational: it is equivalent to finding critical points of a function of $\xi = \varepsilon\xi'$. Associated to (4.1), let us consider the energy functional J_λ given by

$$J_\lambda^p(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2)\, dx - \frac{\lambda}{p}\int_\Omega e^{u^p}\, dx, \tag{4.58}$$

and the finite-dimensional restriction

$$F_\lambda(\xi) = J_\lambda^p\left(\left(U_\lambda + \tilde{\phi}\right)(x,\xi)\right), \tag{4.59}$$

where

$$\left(U_\lambda + \tilde{\phi}\right)(x,\xi) = \gamma + \frac{1}{p\gamma^{p-1}}\left((V_\lambda + \phi)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right)$$

with V_λ defined in (4.30), ϕ is the unique solution to problem (4.55) given by Proposition 4.5.

The next result, whose proof is postponed until Section 4.5.

Proposition 4.6. *(i) The functional $F_\lambda(\xi)$ is of class C^1. Moreover, for all $\lambda > 0$ sufficiently small, if $D_\xi F_\lambda(\xi) = 0$, then ξ satisfies (4.57).*

(ii) Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that for any points ξ_j, $j = 1, \ldots, m$ in $\mathcal{M}_\delta$, μ_j are given by (4.35), the following expansion holds

$$\begin{aligned}\lambda^{-1}\varepsilon^{\frac{2(2-p)}{p}}F_\lambda(\xi) &= -4\pi(2k+l)\frac{2-p\log 8}{(2-p)p} - \frac{8\pi}{p}(2k+l)\log\varepsilon \\ &\quad -\frac{1}{2(2-p)}\varphi_m(\xi) + O(|\log\varepsilon|^{-1}),\end{aligned} \tag{4.60}$$

where

$$\varphi_m(\xi) = \varphi_m(\xi_1, \ldots, \xi_m) = \sum_{j=1}^{m} c_j^2 H(\xi_j, \xi_j) + \sum_{l \neq j} c_l c_j G(\xi_l, \xi_j). \tag{4.61}$$

Proof of Theorem 4.1: First, from the same argument as Lemma 6.1 in [41], we have that

$$\min_{\partial\mathcal{M}_\delta} \varphi_m(\xi) \to +\infty, \qquad \text{as } \delta \to 0. \tag{4.62}$$

We state it here for completeness. Let $\xi = (\xi_1, \ldots, \xi_m) \in \partial\mathcal{M}_\delta$. There are two possibilities: either there exists $j_0 \leq k$ such that $d(\xi_{j_0}, \partial\Omega) = \delta$, or exists $i_0 \neq j_0$, $|\xi_{i_0} - \xi_{j_0}| = \delta$.

In the first case, a consequence of the properties of the Green's function is that for all $\xi \in \Omega$

$$H(\xi, \xi) \geq C\frac{1}{d(\xi, \partial\Omega)}. \tag{4.63}$$

In the second case, we may assume that there exists a fixed constant C such that $d(\xi_i, \partial\Omega) \geq C$, $i = 1, \ldots, k$, as otherwise it follows into the first case. But then it is easy to see that

$$G(\xi_i, \xi_j) \geq C\frac{1}{|\xi_i - \xi_j|}. \tag{4.64}$$

Then by (4.63) and (4.64) we obtain (4.62).

From (i) of Proposition 4.6, the function

$$\left(U_\lambda + \tilde{\phi}\right)(x, \xi) = \gamma + \frac{1}{p\gamma^{p-1}}\left((V_\lambda + \phi)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right)$$

where V_λ defined by (4.30) and $\phi(\xi)$ is the unique solution of problem (4.55), is a solution of problem (4.1) if we adjust ξ so that it is a critical point of $F_\lambda(\xi)$ defined by (4.59). This is equivalent to finding a critical point of

$$\widetilde{F}_\lambda(\xi) := a\lambda^{-1}\varepsilon^{\frac{2(2-p)}{p}}F_\lambda(\xi) + b + c\log\varepsilon,$$

for suitable constants a, b and c. On the other hand, from (ii) of Proposition 4.6, for $\xi \in \mathcal{M}_\delta$, we have that,

$$\widetilde{F}_\lambda(\xi) = \varphi_m(\xi) + O(|\log \varepsilon|^{-1})\Theta_\lambda(\xi),$$

where φ_m is given by (4.61), and $\Theta_\lambda(\xi)$ is uniformly bounded in consider region as $\lambda \to 0$.

From (4.62), the function φ_m is C^1, bounded from below in $\mathcal{M}_\delta$, we have that, for δ is arbitrarily small, φ_m has an absolute minimum in $\mathcal{M}_\delta$. This implies that $\widetilde{F}_\lambda$ also has an absolute minimum $(\xi_1^*, \dots, \xi_m^*) \in \mathcal{M}_\delta$ such that

$$\lim_{\lambda \to 0} \varphi_m(\xi_1^*, \dots, \xi_m^*) = \min_{\mathcal{M}_\delta} \varphi_m.$$

Moreover, while (4.9) holds as a direct consequence of the construction of U_λ, and (3) of Theorem holds from (ii) of Proposition 4.6.

Remark 4.7. *Using Ljusternik-Schnirelmann theory, one can get a second, distinct solution satisfying Theorem 4.1. The proof is similar to [27].*

4.4 The finite dimensional reduction

This section is devoted to the proof of Proposition 4.5. Given $h \in L^\infty(\Omega_\varepsilon)$, we first consider the problem of finding a function ϕ such that for certain scalars c_{ij}, it satisfies

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi - W\phi = h + \sum_{j=1}^{m}\sum_{i=1}^{J_j} c_{ij} Z_{ij}\eta_j, & \text{in } \Omega_\varepsilon; \\ \frac{\partial \phi}{\partial \nu} = 0, & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \phi Z_{ij}\eta_j = 0, & \text{for } i = 1, J_j, \quad j = 1, \dots, m. \end{cases} \tag{4.65}$$

First we show that the following result:

Proposition 4.8. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that for any points ξ_j, $j = 1, \dots, m$, in $\mathcal{M}_\delta$, μ_j is given by (4.35), and $h \in L^\infty(\Omega_\varepsilon)$, there is a unique solution $\phi := T_\lambda(h)$ to problem (4.65) for all $\lambda \le \lambda_0$. Moreover,*

$$\|\phi\|_\infty \le C\left(\log \frac{1}{\varepsilon}\right)\|h\|_*. \tag{4.66}$$

The proof will be spit into a series of lemmas which we state and prove next.

Lemma 4.9. *There exist constants $R_1 > 0$, $C > 0$ such that for $\lambda > 0$ small enough and for any points $\xi_j \in \bar{\Omega}$, $j = 1, \dots, m$, in $\mathcal{M}_\delta$, set $\tilde{\Omega}_\varepsilon = \Omega_\varepsilon \setminus \bigcup_{j=1}^{m} B(\xi_j', R_1)$, we have*

$$\psi : \ \tilde{\Omega}_\varepsilon \to [1, \infty)$$

smooth and positive verifying

$$L(\psi) := -\Delta\psi + \varepsilon^2\psi - W\psi \geq \sum_{j=1}^{m} \frac{1}{|y-\xi_j'|^{2+\sigma}} + \varepsilon^2 \qquad \text{in } \tilde{\Omega}_\varepsilon,$$

with

$$\frac{\partial\psi}{\partial\nu} \geq 0 \quad \text{on } \partial\tilde{\Omega}_\varepsilon, \qquad \psi > 0 \ \text{ in } \ \tilde{\Omega}_\varepsilon.$$

Moreover ψ is bounded uniformly,

$$1 \leq \psi \leq C \qquad \text{in } \tilde{\Omega}_\varepsilon.$$

Proof. We take

$$\psi_{1j}(r) = 1 - \frac{1}{r^\sigma}, \qquad \text{where } \ r = |y-\xi_j'|. \tag{4.67}$$

A direct computation shows that, we have

$$-\Delta\psi_{1j} = \sigma^2\frac{1}{r^{2+\sigma}}.$$

If $\xi_j' \in \Omega_\varepsilon$, then we have

$$\frac{\partial\psi_{1j}}{\partial\nu} = O(\varepsilon^{1+\sigma}).$$

If $\xi_j' \in \Omega_\varepsilon$ and $|y-\xi_j'| > R_1$, we have

$$\frac{\partial\psi_{1j}}{\partial\nu} = \sigma\frac{(y-\xi_j')\cdot\nu}{r^{2+\sigma}}.$$

We write the boundary $\partial\Omega_\varepsilon$ near point ξ_j' as the graph $\{(y_1,y_2) : y_2 = G_\varepsilon(y_1)\}$ with $G_\varepsilon(y_1) = \frac{1}{\varepsilon}G(\varepsilon y_1)$ and G a smooth function such that $G(0)=0$ and $G'(0)=0$. Fix $\delta>0$ small. Then for $R_1 < r < \delta/\varepsilon$ we have that r is comparable with y_1, $G_\varepsilon'(y_1) = O(\varepsilon r)$ and $G_\varepsilon(y_1) = O(\varepsilon r^2)$. Then and $G'(0)=0$. Then

$$\begin{aligned}
\frac{\partial\psi_{1j}}{\partial\nu} &= \frac{\sigma}{r^{2+\sigma}}\frac{1}{\sqrt{G_\varepsilon'(y_1)^2+1}}\left(-y_1G_\varepsilon'(y_1) + G_\varepsilon(y_1)\right) \\
&= \frac{\sigma}{r^{2+\sigma}}\frac{1}{\sqrt{O(\delta^2)+1}}O(\varepsilon r^2) \\
&= O(\frac{\varepsilon}{r^\sigma}), \qquad \forall\ R_1 < r < \frac{\delta}{\varepsilon}.
\end{aligned}$$

Hence, we obtain that

$$\frac{\partial\psi_{1j}}{\partial\nu} = o(\varepsilon), \qquad \text{on } \partial\Omega_\varepsilon.$$

Next, let us define

$$\psi = \sum_{j=1}^{m} \psi_{1j} + C\psi_0,$$

where ψ_0 is the solution of the following problem

$$-\Delta\psi_0 + \varepsilon^2\psi_0 = \varepsilon^2 \quad \text{in } \Omega_\varepsilon; \qquad \frac{\partial\psi_0}{\partial\nu} = \varepsilon \quad \text{on } \partial\Omega_\varepsilon.$$

It is directly checked that $\frac{4}{\sigma^2}\psi$ satisfies the required condition. □

Lemma 4.10. *The operator L satisfies the maximum principle in $\tilde{\Omega}_\varepsilon$ for $R \geq R_1$ large but independent of λ, with R_1 in Lemma 4.9. Namely, if $L(\phi) \geq 0$ in $\tilde{\Omega}_\varepsilon$ and $\phi \geq 0$ on $\partial\tilde{\Omega}_\varepsilon$, then $\phi \geq 0$ in $\tilde{\Omega}_\varepsilon$.*

Proof. Given $a > 0$, we consider the function

$$Z(y) = \sum_{j=1}^{m} z_0\left(a|y - \xi_j'|\right), \qquad y \in \Omega_\varepsilon, \tag{4.68}$$

where $z_0(r) = \frac{r^2-1}{r^2+1}$ is the radial solution in $\mathbb{R}^2$ of

$$\Delta z_0 + \frac{8}{(1+r^2)^2} z_0 = 0.$$

First, we observe that, if $|y - \xi_j'| \geq R$ for $R > \frac{1}{a}$, then $Z(y) > 0$. By the definition of z_0 we have

$$\begin{aligned} -\Delta Z(y) + \varepsilon^2 Z(y) &= \sum_{j=1}^{m} \frac{(8a^2+\varepsilon^2)(a^2|y-\xi_j'|^2 - 1)}{(1+a^2|y-\xi_j'|^2)^3} \\ &\geq \sum_{j=1}^{m} \frac{1}{3} \frac{8a^2+\varepsilon^2}{(1+a^2|y-\xi_j'|^2)^2} \\ &\geq \sum_{j=1}^{m} \frac{1}{3} \frac{8a^2}{(1+a^2|y-\xi_j'|^2)^2} \geq \sum_{j=1}^{m} \frac{4}{27} \frac{8}{a^2|y-\xi_j'|^4} \end{aligned}$$

provided $R > \frac{\sqrt{2}}{a}$. On the other hand, from (4.48), in the same region, we have

$$f'(V_\lambda)Z(y) \leq D_0 \sum_{j=1}^{m} e^{w_j} Z(y) \leq \sum_{j=1}^{m} \frac{C}{|y-\xi_j'|^4}.$$

Hence if a is taken small and fixed, and $R > 0$ is chosen sufficiently large depending on this a, then we have $L(Z) > 0$ in $\tilde{\Omega}_\varepsilon$. Thus the function $Z(y)$ is what we are looking for. □

Let us fix such a number $R > 0$ which we may take large whenever it is needed. Define the following inner norm of ϕ in the following way

$$\|\phi\|_i = \sup_{y \in \cup_{j=1}^m B(\xi_j', R)} |\phi(y)|.$$

Lemma 4.11. *There exists a uniform constant $C > 0$ such that if $L(\phi) = h$ in Ω_ε, $\phi = 0$ on $\partial\Omega_\varepsilon$, then*

$$\|\phi\|_\infty \leq C[\|\phi\|_i + \|h\|_*], \tag{4.69}$$

for any $h \in L^\infty(\Omega_\varepsilon)$.

Proof. Define now the function

$$\tilde{\phi}(y) = 2\|\phi\|_i Z(y) + \|h\|_* \psi(y),$$

where Z is the function defined in (4.68), and the function ψ satisfying the properties of lemma 4.9. First, observe that by the definition of Z, choosing R large if necessary,

$$\tilde{\phi}(y) \geq 2\|\phi\|_i Z(y) \geq \|\phi\|_i \geq |\phi(y)| \qquad \text{for} \quad |y - \xi_j'| = R,$$

and, by the positivity of $Z(y)$ and $\psi(y)$,

$$\tilde{\phi}(y) \geq 0 = \phi(y) \qquad \text{for } y \in \partial\Omega_\varepsilon.$$

Finally, by the definition of $\|\cdot\|_*$ we have that

$$|h(y)| \leq \left(\sum_{j=1}^m (1 + |y - \xi_j'|)^{-2-\sigma} + \varepsilon^2 \right) \|h\|_*,$$

we then have

$$\begin{aligned} L(\tilde{\phi}) &= 2\|\phi\|_i L(Z) + \|h\|_* L(\psi) \\ &\geq \|h\|_* \left(\sum_{j=1}^m (1 + |y - \xi_j'|)^{-2-\sigma} + \varepsilon^2 \right) \\ &\geq |h(y)| \geq L(\phi)(y), \end{aligned}$$

provided R large enough. Hence, from Lemma 4.10, we obtain that

$$|\phi(y)| \leq \tilde{\phi}(y) \qquad \text{for } y \in \tilde{\Omega}_\varepsilon,$$

and, since $Z(y) \leq 1$ and from lemma 4.9 we get

$$\|\phi\|_\infty \leq C[\|\phi\|_i + \|h\|_*].$$

□

Next we prove uniform a priori estimates for the problem (4.65) when ϕ satisfies additionally orthogonality under dilations. Specifically, we consider the problem

$$\begin{cases} L(\phi) = h, & \text{in } \Omega_\varepsilon; \\ \frac{\partial \phi}{\partial \nu} = 0, & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \phi Z_{ij}\eta_j = 0, & \text{for } i = 0, \dots, J_j,\ j = 1, \dots, m, \end{cases} \tag{4.70}$$

and prove the following estimate.

Lemma 4.12. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that for any points ξ_j, $j = 1, \dots, m$, in $\mathcal{M}_\delta$, μ_j is given by (4.35), and $h \in L^\infty(\Omega_\varepsilon)$, and any solution ϕ to problem (4.70), one has*

$$\|\phi\|_\infty \le C\|h\|_*. \tag{4.71}$$

Proof. We carry out the proof of lemma by a contradiction. If the result was false, then there exist a sequence $\lambda_n \to 0$, points ξ_j^n, $j = 1, \dots, m$ in $\mathcal{M}_\delta$, function h_n with $\|h_n\|_* \to 0$ and ϕ_n with $\|\phi_n\|_\infty = 1$,

$$\begin{cases} L(\phi_n) = h_n, & \text{in } \Omega_{\varepsilon_n}; \\ \frac{\partial \phi_n}{\partial \nu} = 0, & \text{on } \partial\Omega_{\varepsilon_n}; \\ \int_{\Omega_{\varepsilon_n}} \phi_n Z_{ij}\eta_j = 0, & \text{for all } i = 0, \dots, J_j,\ j = 1, \dots, m. \end{cases} \tag{4.72}$$

Then from lemma 4.11, we see that $\|\phi_n\|_i$ stays away from zero. Up to a subsequence, for one of the indices, say j, we can assume that there exists $R > 0$ such that,

$$\sup_{|y - (\xi_j^n)'| < R} |\phi_n(y)| \ge \kappa > 0 \quad \text{for all } n.$$

Let us set $\hat{\phi}_n(z) = \phi_n((\xi_j^n)' + z)$. Elliptic estimate allow us to assume that $\hat{\phi}_n$ converges uniformly over compact subsets of $\mathbb{R}^2$ to a bounded, nonzero solution $\hat{\phi}$ of

$$\Delta\phi + \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2}\phi = 0.$$

This implies that $\hat{\phi}$ is a linear combination of the functions z_{ij}, $i = 0, \dots, J_j$. But orthogonality conditions over $\hat{\phi}_n$ pass to the limit thanks to $\|\hat{\phi}_n\|_\infty \le 1$. By the dominated convergence theorem then yields that $\int_{\mathbb{R}^2} \eta(|z|) z_{ij}\hat{\phi} = 0$ for $i = 0, \dots, J_j$, thus a contradiction with $\liminf_{n\to\infty} \|\phi_n\|_i > 0$. □

Now we establish a priori estimates for the problem (4.70) with the orthogonality condition $\int_{\Omega_\varepsilon} \eta_j Z_{0j}\phi = 0$ dropped. We consider the problem

$$\begin{cases} L(\phi) = h, & \text{in } \Omega_\varepsilon; \\ \frac{\partial \phi}{\partial \nu} = 0, & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \eta_j Z_{ij}\phi = 0, & \text{for } i = 1, J_j,\ j = 1, \dots, m, \end{cases} \tag{4.73}$$

Lemma 4.13. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that for any points ξ_j, $j = 1, \ldots, m$, in $\mathcal{M}_\delta$, μ_j is given by (4.35), and $h \in L^\infty(\Omega_\varepsilon)$, and any solution ϕ to problem (4.73), one has*

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*, \tag{4.74}$$

for all $\lambda < \lambda_0$.

Proof. Let ϕ satisfies (4.73). We modify ϕ to $\tilde{\phi}$, such that $\tilde{\phi}$ satisfies all orthogonality condition. For this, let us set $R > R_0 + 1$ large and fixed. Set

$$a_{0j} = \frac{1}{\mu_j\left(\frac{4}{c_j}\log\frac{1}{\varepsilon R} + H(\xi_j, \xi_j)\right)}.$$

Define

$$\hat{Z}_{0j}(y) = Z_{0j}(y) - \frac{1}{\mu_j} + a_{0j}G(\xi_j, \varepsilon y).$$

We note that the function $\hat{Z}_{0j}$ satisfies the Neumann boundary condition. Let χ be a radial smooth cut-off function on $\mathbb{R}^2$ so that $0 \le \chi \le 1$, $|\nabla\chi| \le C$ in $\mathbb{R}^2$, $\chi \equiv 1$ in $B_R(0)$ and $\chi \equiv 0$ in $\mathbb{R}^2 \backslash B_{R+1}(0)$. Set

$$\chi_j(y) = \chi(|y - \xi_j'|) \quad \text{for } j = 1, \ldots, k; \qquad \chi_j(y) = \chi(F_j^\varepsilon(y)) \quad \text{for } j = k+1, \ldots, m. \tag{4.75}$$

Now, we define

$$\tilde{Z}_{0j} = \chi_j Z_{0j} + (1 - \chi_j)\hat{Z}_{0j}.$$

Given ϕ satisfying (4.73), we set

$$\tilde{\phi} = \phi + \sum_{j=1}^m d_j\tilde{Z}_{0j}, \qquad \text{where } d_j = -\frac{\int_{\Omega_\varepsilon}\eta_j Z_{0j}\phi}{\int_{\Omega_\varepsilon} Z_{0j}^2\eta_j}.$$

Therefore, our result is a direct consequence of the following claim.

Claim:

$$|d_j| \le C|\log\varepsilon|\|h\|_* \quad \forall\, j = 1, \ldots, m. \tag{4.76}$$

First, using the notation $L = -\Delta + \varepsilon^2 I - W$, we observe that $\tilde{\phi}$ satisfies

$$\begin{cases} L(\tilde{\phi}) = h + \sum\limits_{j=1}^m d_j L(\tilde{Z}_{0j}), & \text{in } \Omega_\varepsilon; \\ \frac{\partial\tilde{\phi}}{\partial\nu} = 0, & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{4.77}$$

Thus by Lemma 4.12, we have

$$\|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} \leq C\sum_{j=1}^m |d_j|\|L(\tilde{Z}_{0j})\|_* + C\|h\|_*. \tag{4.78}$$

Multiplying the first equation in (4.77) by $\tilde{Z}_{0s}$, for $s = 1, \ldots, m$, integrating by parts and using the second equation in (4.77), we find

$$\begin{aligned}\sum_{j=1}^m d_j \int_{\Omega_\varepsilon} L(\tilde{Z}_{0j})\tilde{Z}_{0s} &\leq C\|h\|_* \left(1 + \sum_{j=1}^m \|L(\tilde{Z}_{0j})\|_*\right) \\ &\quad + C\sum_{j=1}^m |d_j|\|L(\tilde{Z}_{0j})\|_*^2. \end{aligned}\tag{4.79}$$

Next we estimate the size of $\|L(\tilde{Z}_{0j})\|_*$. From (4.54), we have

$$\begin{aligned} L(\hat{Z}_{0j}) &= e^{w_j}Z_{0j} - W\hat{Z}_{0j} + O(\varepsilon(1+|y-\xi_j'|)^3) \\ &= e^{w_j}\left(\frac{1}{\mu_j} - a_{0j}G(\xi_j, \varepsilon y)\right) + O(\varepsilon(1+|y-\xi_j'|)^3). \end{aligned}$$

Thus, we have

$$\|(1-\chi_j)L(\hat{Z}_{0j})\|_* \leq \frac{C}{\log(1/\varepsilon)},$$

where C is a constant, which depends on the chosen large constant R. Hence

$$\begin{aligned} L(\tilde{Z}_{0j}) &= \chi_j L(Z_{0j}) + (1-\chi_j)L(\hat{Z}_{0j}) + 2\nabla\chi_j\nabla(Z_{0j} - \hat{(Z)}_{0j}) + \Delta\chi_j(Z_{0j} - \hat{Z}_{0j}) \\ &= O(\varepsilon^{2+\alpha}) + (1-\chi_j)e^{w_j}\left(\frac{1}{\mu_j} - a_{0j}G(\xi_j,\varepsilon y)\right) \\ &\quad + 2\nabla\chi_j\nabla(Z_{0j} - \hat{(Z)}_{0j}) + \Delta\chi_j(Z_{0j} - \hat{Z}_{0j}). \end{aligned}\tag{4.80}$$

Since, for $r = |y - \xi_j'| \in (R, R+1)$, we have

$$\begin{aligned} \hat{Z}_{0j} - Z_{0j} &= a_{0j}G(\xi_j, \varepsilon y) - \frac{1}{\mu_j} \\ &= a_{0j}\left(\frac{4}{c_j}\log\frac{1}{\varepsilon|y-\xi_j'|} + H(\xi_j,\varepsilon y)\right) - \frac{1}{\mu_j}. \end{aligned}$$

Therefore, for $r = |y - \xi_j'| \in (R, R+1)$, we have

$$\hat{Z}_{0j} - Z_{0j} = \frac{C}{\log(1/\varepsilon)}\log\frac{1}{r} + O\left(\frac{\varepsilon^\alpha}{\log(1/\varepsilon)}\right), \tag{4.81}$$

and

$$\nabla(\hat{Z}_{0j} - Z_{0j}) = -\frac{C}{\log(1/\varepsilon)}\frac{1}{r} + O\left(\frac{\varepsilon^\alpha}{\log(1/\varepsilon)}\right). \tag{4.82}$$

From (4.80), (4.81) and (4.82) we obtain

$$\|L(\tilde{Z}_{0j})\|_* \leq \frac{C}{\log(1/\varepsilon)}. \tag{4.83}$$

Now we estimate the left term of (4.79). From (4.80), we see that for $j \neq s$,

$$\begin{aligned}\int_{\Omega_\varepsilon} L(\tilde{Z}_{0j})\tilde{Z}_{0s} &= O(\varepsilon^\alpha) + \int_{\Omega_\varepsilon} O\left(\frac{1}{\log(1/\varepsilon)}(|\nabla\chi_j| + |\Delta\chi_j|)\right)\tilde{Z}_{0s} \\ &= O\left(\left(\frac{1}{\log(1/\varepsilon)}\right)^2\right).\end{aligned}$$

Moreover, for $j = s$, we have

$$\int_{\Omega_\varepsilon} L(\tilde{Z}_{0s})\tilde{Z}_{0s} = I_1 + I_2 + O(\varepsilon),$$

where

$$\begin{aligned}I_2 &= \int_{\Omega_\varepsilon} O(\varepsilon^{2+\alpha}) + (1-\chi_s)e^{w_j}\left(\frac{1}{\mu_s} - a_{0s}G(\xi_j, \varepsilon y)\right) \\ &= O(\varepsilon^\alpha) + O\left(\frac{1}{\log(1/\varepsilon)}\right)\end{aligned}$$

and

$$\begin{aligned}I_1 &= \int_{\Omega_\varepsilon}\left[2\nabla\chi_s\nabla(Z_{0s} - \hat{Z}_{0s}) + \Delta\chi_s(Z_{0s} - \hat{Z}_{0s})\right]\tilde{Z}_{0s} \\ &= \int_{\Omega_\varepsilon}\nabla\chi_s\nabla(Z_{0s} - \hat{Z}_{0s})\hat{Z}_{0s} - \int_{\Omega_\varepsilon}\nabla\chi_s(Z_{0s} - \hat{Z}_{0s})\nabla\hat{Z}_{0s} + O(\varepsilon).\end{aligned}$$

We observe that in the consider region, $r \in (R, R+1)$ with $r = |y - \xi_j'|$, $|Z_{0s} - \hat{Z}_{0s}| \leq \frac{C}{\log(1/\varepsilon)}$ while $|\nabla Z_{0s}'| \leq \frac{1}{R^3} + \frac{C}{\log(1/\varepsilon)}$. Thus

$$\left|\int_{\Omega_\varepsilon}\nabla\chi_s\nabla(Z_{0s} - \hat{Z}_{0s})\hat{Z}_{0s}\right| \leq \frac{D}{R^3}\frac{1}{\log(1/\varepsilon)},$$

where D may be chosen independent on R. Now we have

$$\int_{\Omega_\varepsilon}\nabla\chi_s(Z_{0s} - \hat{Z}_{0s})\nabla\hat{Z}_{0s} = -\frac{E}{\log(1/\varepsilon)}\left[1 + O\left(\frac{1}{R}\right)\right]$$

where E may be chosen independent on ε. Thus we choose R large enough, we then have $I_1 \sim -\frac{E}{\log(1/\varepsilon)}$. Therefore, we have

$$\int_{\Omega_\varepsilon} L(\tilde{Z}_{0s})\tilde{Z}_{0s} = -\frac{E}{\log(1/\varepsilon)}\left[1 + O\left(\frac{1}{R}\right)\right],$$

and

$$\int_{\Omega_\varepsilon} L(\tilde{Z}_{0j})\tilde{Z}_{0s} = O\left(\frac{1}{R}\frac{1}{\log(1/\varepsilon)}\right) \quad \text{for } \ j \neq s.$$

Thus, we obtain that (4.76) holds. This finishes the proof of Lemma. □

Proof of Proposition 4.8 We first establish the validity of the a priori estimate (4.66). The previous lemma yields

$$\|\phi\|_\infty \leq C\left(\log\frac{1}{\varepsilon}\right)\left[\|h\|_* + \sum_{i=1}^{J_j}\sum_{j=1}^{m}|c_{ij}|\right]. \tag{4.84}$$

Let χ_j be a smooth cut-off function defined as (4.75). We multiply the first equation of (4.65) by $Z_{ij}\chi_j$, we find

$$\langle L(\phi), Z_{ij}\chi_j\rangle = \langle h, Z_{ij}\chi_j\rangle + c_{ij}\int_{\Omega_\varepsilon}\eta_j\chi_j|Z_{ij}|^2. \tag{4.85}$$

We have

$$-L(Z_{ij}\chi_j) = \Delta\chi_j Z_{ij} + 2\nabla Z_{ij}\nabla\chi_j + \varepsilon O((1+r)^{-3}),$$

with $r = |y - \xi_j'|$. Since $\Delta\chi_j = O(\varepsilon^2)$, $\nabla\chi_j = O(\varepsilon)$, and $Z_{ij} = O(r^{-1})$, $\nabla Z_{ij} = O(r^{-2})$, we get

$$-L(Z_{ij}\chi_j) = O(\varepsilon^3)\varepsilon O((1+r)^{-3}).$$

Then we have

$$|\langle L(\phi), Z_{ij}\chi_j\rangle| = |\langle\phi, L(Z_{ij}\chi_j)\rangle| \leq C\varepsilon\|\phi\|_\infty.$$

Combining this with (4.84) and (4.85) we find

$$|c_{ij}| \leq C\left[\|h\|_* + \varepsilon\left(\log\frac{1}{\varepsilon}\right)\sum_{a,b}|c_{ab}|\right].$$

Then, $|c_{ij}| \leq C\|h\|_*$. Combining this with (4.84) we obtain the estimate (4.66) holds.

Next prove the solvability of problem (4.65). To this purpose we consider the space

$$\mathbb{H} = \left\{\phi \in H^1(\Omega_\varepsilon) \ : \ \int_{\Omega_\varepsilon}\phi Z_{ij}\eta_j = 0 \quad \text{for } i = 1, J_j, \ j = 1, 2, \ldots, m\right\},$$

endowed with the usual inner product $\langle\phi,\psi\rangle = \int_{\Omega_\varepsilon}(\nabla\phi\nabla\psi + \varepsilon^2\phi\psi)$. Problem (4.65), expressed in a weak form, is equivalent to find $\phi \in \mathbb{H}$ such that

$$\langle\phi,\psi\rangle = \int_{\Omega_\varepsilon}(W\phi + h)\psi \ dx, \qquad \text{for all } \psi \in \mathbb{H},$$

With the aid of Riesz's representation theorem, this equation gets rewritten in $\mathbb{H}$ in the operator form

$$(Id - K)\phi = \tilde{h}, \tag{4.86}$$

for certain $\tilde{h} \in \mathbb{H}$, where K is a compact operator in $\mathbb{H}$. The homogeneous equation $\phi = K\phi$ in $\mathbb{H}$, which is equivalent to (4.65) with $h \equiv 0$, has only the trivial solution in view of the a priori estimate (4.66). Now, Fredholm's alternative guarantees unique solvability of (4.86) for any $\tilde{h} \in \mathbb{H}$. This finishes the proof.

The result of Proposition 4.8 implies that the unique solution $\phi = T_\lambda(h)$ of (4.65) defines a continuous linear map form the Banach space $\mathcal{C}_*$ of all functions h in L^∞ for which $\|h\|_* < \infty$ into L^∞, with norm bounded uniformly in λ.

Lemma 4.14. *The operator T_λ is differentiable with respect to the variable ξ_a in $\bar{\Omega}$ with $\xi \in \mathcal{M}_\delta$, one has the estimate*

$$\|\partial_{(\xi_a')_b} T_\lambda(h)\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)^2 \|h\|_* \quad \text{for } b = 1, J_j,\ a = 1, 2, \dots, m. \tag{4.87}$$

Proof. Differentiating equation (4.65), formally $Z := \partial_{(\xi_a')_b}\phi$ should satisfy

$$L(Z) = -\partial_{(\xi_a')_b} W\phi + \sum_{i=1}^{J_j} c_{ia}\partial_{(\xi_a')_b}\left(\eta_a Z_{ia}\right) + \sum_{i=1}^{J_j}\sum_{j=1}^{m} d_{ij} Z_{ij}\eta_j$$

with $d_{ij} = \partial_{(\xi_a')_b} c_{ij}$, and the orthogonality conditions now become

$$\int_{\Omega_\varepsilon} Z_{ij}\eta_j Z = -\int_{\Omega_\varepsilon} \partial_{(\xi_a')_b}\left(Z_{ij}\eta_j\right)\phi.$$

We consider the constants b_{ia} defined as

$$b_{ia}\int_{\Omega_\varepsilon} \eta_a Z_{ia}^2 = \int_{\Omega_\varepsilon} \partial_{(\xi_a')_b}\left(Z_{ia}\eta_a\right)\phi.$$

Define

$$\tilde{Z} = Z + \sum_{i=1}^{J_j} b_{ia}\eta_a Z_{ia},$$

and

$$g = -\partial_{(\xi_a')_b} W\phi + \sum_{i=1}^{J_j} c_{ia}\partial_{(\xi_a')_b}\left(Z_{ia}\eta_{2a}\right) + \sum_{i=1}^{J_j} b_{ia} L(\eta_{2a} Z_{ia}).$$

We then have

$$\begin{cases} L(\tilde{Z}) = g + \sum\limits_{i=1}^{J_j}\sum\limits_{j=1}^{m} b_{ia}\eta_a Z_{ia}, & \text{in } \Omega_\varepsilon; \\ \tilde{Z} = 0, & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} Z_{ia}\eta_a \tilde{Z} = 0, & \text{for } i = 0, \dots, J_j,\ a = 1, \dots, m. \end{cases}$$

Furthermore, $\tilde{Z} = T_\lambda(g)$. Using the result of Proposition 4.8 we find that

$$\|g\|_* \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*,$$

hence,

$$\|\partial_{(\xi'_a)_b} T_\lambda(h)\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)^2\|h\|_* \ \text{ for } b = 1, J_j, \quad a = 1, 2, \ldots, m.$$

□

Next, we will prove Proposition 4.5.

Proof of Proposition 4.5 In terms of the operator T_λ defined in Proposition 4.5, problem (4.55) becomes

$$\phi = T_\lambda(N(\phi) + E_\lambda) := A(\phi). \tag{4.88}$$

For a given number $M > 0$ let us consider the region

$$\mathcal{F}_M := \left\{\phi \ : \ \|\phi\|_\infty \le \frac{M}{|\log\varepsilon|^2}\right\}.$$

From Proposition 4.8, we get

$$\|A(\phi)\|_\infty \le C|\log\varepsilon|\left[\|N(\phi)\|_* + \|E_\lambda\|_*\right].$$

From Lemma 4.4, we have $\|E_\lambda\|_* \le \frac{C}{|\log\varepsilon|^3}$. And, by the definition of $N(\phi)$ in (4.40), and from (4.50) then we have

$$\|N(\phi)\|_* \le C\|\phi\|_\infty^2$$

Thus

$$\|A(\phi)\|_\infty \ \ \le \ \ C|\log\varepsilon|\left(C\|\phi\|_\infty^2 + \frac{1}{|\log\varepsilon|^3}\right).$$

We then get that $A(\mathcal{F}_M) \subset \mathcal{F}_M$ for a sufficiently large but fixed M and all small λ. Moreover, for any $\phi_1, \phi_2 \in \mathcal{F}_M$, one has

$$\|N(\phi_1) - N(\phi_2)\|_* \le C\left(\max_{i=1,2}\|\phi_i\|_\infty\right)\|\phi_1 - \phi_2\|_\infty.$$

In fact,

$$\begin{aligned} N(\phi_1) - N(\phi_2) &= f(V_\lambda + \phi_1) - f(V_\lambda + \phi_2) - f'(V_\lambda)(\phi_1 - \phi_2) \\ &= \int_0^1\left(\frac{d}{dt}f(V_\lambda + \phi_2 + t(\phi_1 - \phi_2)\right)dt - f'(V_\lambda)(\phi_1 - \phi_2) \end{aligned}$$

$$= \int_0^1 (f'(V_\lambda + \phi_2 + t(\phi_1 - \phi_2) - f'(V_\lambda))\, dt\, (\phi_1 - \phi_2).$$

Thus, for a certain $t^* \in (0,1)$, and $s \in (0,1)$

$$\begin{aligned}|N(\phi_1) - N(\phi_2)| &\leq C|f'(V_\lambda + \phi_2 + t^*(\phi_1 - \phi_2) - f'(V_\lambda)|\|\phi_1 - \phi_2\|_\infty \\ &\leq C|f''(V_\lambda + s\phi_2 + t^*(\phi_1 - \phi_2))|(\|\phi_1\|_\infty + \|\phi_2\|_\infty)\|\phi_1 - \phi_2\|_\infty.\end{aligned}$$

Thanks to (4.50) and the fact that $\|\phi_1\|_\infty$, $\|\phi_2\|_\infty \to 0$ as $\lambda \to 0$, we conclude that

$$\begin{aligned}\|N(\phi_1) - N(\phi_2)\|_* &\leq C\|f''(V_\lambda)\|_*(\|\phi_1\|_\infty + \|\phi_2\|_\infty)\|\phi_1 - \phi_2\|_\infty \\ &\leq C(\|\phi_1\|_\infty + \|\phi_2\|_\infty)\|\phi_1 - \phi_2\|_\infty.\end{aligned}$$

Then we have

$$\begin{aligned}\|A(\phi_1) - A(\phi_2)\|_\infty &\leq C|\log\varepsilon|\|N(\phi_1) - N(\phi_2)\|_* \\ &\leq C|\log\varepsilon|\left(\max_{i=1,2}\|\phi_i\|_\infty\right)\|\phi_1 - \phi_2\|_\infty.\end{aligned}$$

Thus the operator A has a small Lipschitz constant in $\mathcal{F}_M$ for all small λ, and therefore a unique fixed point of A exists in this region.

We shall next analyze the differentiability of the map $\xi' = (\xi'_1, \ldots, \xi'_m) \mapsto \phi$. Assume for instance that the partial derivative $\partial_{(\xi'_j)_i}\phi$ exists for $i = 1, J_j$. Since $\phi = T_\lambda (N(\phi) + E_\lambda)$, formally that

$$\partial_{(\xi'_j)_i}\phi = (\partial_{(\xi'_j)_i}T_\lambda)(N(\phi) + E_\lambda) + T_\lambda\left(\partial_{(\xi'_j)_i}N(\phi) + \partial_{(\xi'_j)_i}E_\lambda\right).$$

From Lemma 4.14, we have

$$\|\partial_{(\xi'_j)_i}T_\lambda(N(\phi) + E_\lambda)\|_\infty \leq C|\log\varepsilon|^2\|N(\phi) + E_\lambda\|_* \leq C\frac{1}{|\log\varepsilon|}.$$

On the other hand,

$$\begin{aligned}\partial_{(\xi'_j)_i}N(\phi) &= [f'(V_\lambda + \phi) - f'(V_\lambda) - f''(V_\lambda)\phi]\partial_{(\xi'_j)_i}V_\lambda + \partial_{(\xi'_j)_i}[f'(V_\lambda) - e^{w_j}]\phi \\ &\quad + [f'(V_\lambda + \phi) - f'(V_\lambda)]\partial_{(\xi'_j)_i}\phi + [f'(V_\lambda) - e^{w_j}]\partial_{(\xi'_j)_i}\phi.\end{aligned}$$

Then,

$$\|\partial_{(\xi'_j)_i}N(\phi)\|_* \leq C\left\{\|\phi\|_\infty^2 + \frac{1}{|\log\varepsilon|}\|\phi\|_\infty + \|\partial_{(\xi'_j)_i}\phi\|_\infty\|\phi\|_\infty + \frac{1}{|\log\varepsilon|}\|\partial_{(\xi'_j)_i}\phi\|_\infty\right\}.$$

Since $\|\partial_{(\xi_j)_i}E_\lambda\|_* \leq \frac{C}{|\log\varepsilon|^3}$, and by Proposition 4.8 we then have

$$\|\partial_{(\xi'_j)_i}\phi\|_\infty \leq \frac{C}{|\log\varepsilon|},$$

for all $i = 1, J_j$, $j = 1, \ldots, m$. Then, the regularity of the map $\xi' \mapsto \phi$ can be proved by standard arguments involving the implicit function theorem and the fixed point representation (4.88). This concludes proof of Proposition 4.5.

4.5 Variational reduction

In this Section, we prove Proposition 4.6.

Proof of (i) of Proposition 4.6 A direct consequence of the results obtained in Proposition 4.5 and the definition of function U_λ is the fact the map $\xi \mapsto F_\lambda(\xi)$ is of class C^1.

Define

$$I_\lambda(v) = \frac{1}{2}\int_{\Omega_\varepsilon} \left(|\nabla v|^2 + \varepsilon^2 v^2\right)\ dy - \int_{\Omega_\varepsilon} e^{\gamma^p[(1+\frac{v}{p\gamma^p})^p - 1]}\ dy. \tag{4.89}$$

Let us differentiate the function $F_\lambda(\xi)$ with the respect to ξ. Since

$$J_\lambda^p\left(\left(U_\lambda + \tilde{\phi}\right)(x,\xi)\right) = \frac{1}{p^2\gamma^{2(p-1)}} I_\lambda\left(\left(V_\lambda + \phi\right)(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon})\right), \tag{4.90}$$

we can differentiate directly $I_\lambda\left(V_\lambda(\xi) + \phi(\xi)\right)$ under the integral sign, for $a \in \{1,\dots,m\}$ and $b \in \{1, J_j\}$, so that

$$\begin{aligned}
&\partial_{(\xi_a)_b} F_\lambda(\xi)\\
= &\ \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1} DI_\lambda\left(V_\lambda(\xi) + \phi(\xi)\right)\left[\partial_{(\xi_a')_b} V_\lambda(\xi) + \partial_{(\xi_a')_b}\phi(\xi)\right]\\
= &\ \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1} \sum_{i=1}^{J_j}\sum_{j=1}^{m}\int_{\Omega_\varepsilon} c_{ij}\eta_j Z_{ij}\left[\partial_{(\xi_a')_b} V_\lambda(\xi) + \partial_{(\xi_a')_b}\phi(\xi)\right]\\
= &\ \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1}\left[\sum_{i=1}^{J_j}\sum_{j=1}^{m}\int_{\Omega_\varepsilon} c_{ij}\eta_j Z_{ij}\partial_{(\xi_a')_b} V_\lambda(\xi) + \sum_{i=1}^{J_j}\sum_{j=1}^{m}\int_{\Omega_\varepsilon} c_{ij}\partial_{(\xi_a')_b}\left(\eta_j Z_{ij}\right)\phi(\xi)\right],
\end{aligned}$$

since $\int_{\Omega_\varepsilon}\eta_j Z_{ij}\phi(\xi) = 0$. By the expansion of V_λ, we have

$$\begin{aligned}
\partial_{(\xi_a')_b} V_\lambda &= \partial_{(\xi_a')_b} w_a(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\partial_{(\xi_a')_b} w_{0a}(y)\\
&\quad + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\partial_{(\xi_a')_b} w_{1m}(y) + \partial_{(\xi_a')_b}\theta(y)\\
&= -Z_{ba} + \frac{p-1}{p}\frac{1}{\gamma^p}\partial_{(\xi_a')_b} w_{0a}(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\partial_{(\xi_m')_l} w_{1a}(y) + \partial_{(\xi_a')_b}\theta(y).
\end{aligned}$$

Moreover,

$$\int_{\Omega_\varepsilon} c_{ij}\partial_{(\xi_a')_b}\left(\eta_j Z_{ij}\right)\phi(\xi) = o(1)\int_{\Omega_\varepsilon} c_{ij}\eta_j Z_{ij}\partial_{(\xi_a')_b}\left(V_\lambda\right)$$

Then, if $D_\xi F_\lambda(\xi) = 0$, for $i, b = 1, J_j;\ j, a = 1,\dots,m$, we then have

$$\sum_{i=1}^{J_j}\sum_{j=1}^{m} c_{ij}\int_{\Omega_\varepsilon}\eta_j Z_{ij}(Z_{ba} + o(1)) = 0. \tag{4.91}$$

This is a strictly diagonal dominant system. It implies that $c_{ij} = 0$ for $i = 1, J_j$; $j = 1, \dots, m$. This concludes the proof of (i) of Proposition 4.6.

Proof of (ii) of Proposition 4.6 We have

$$\begin{aligned} F_\lambda(\xi) &= J_\lambda^p\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right) \\ &= \frac{1}{2}\int_\Omega \left[|\nabla\left(U_\lambda + \tilde{\phi}\right)|^2 + \left(U_\lambda + \tilde{\phi}\right)^2\right] - \frac{\lambda}{p}\int_\Omega e^{\left(U_\lambda + \tilde{\phi}\right)^p}. \end{aligned}$$

From (4.90) we have that

$$J_\lambda^p\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right) - J_\lambda^p\left(U_\lambda(\xi)\right) = \frac{1}{p^2\gamma^{2(p-1)}}\left[I_\lambda(V_\lambda + \phi) - I_\lambda(V_\lambda)\right].$$

Since by construction $I'_\lambda(V_\lambda + \phi)[\phi] = 0$, we have

$$\begin{aligned} & J_\lambda^p\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right) - J_\lambda^p\left(U_\lambda(\xi)\right) \\ = & \frac{1}{p^2\gamma^{2(p-1)}}\int_0^1 D^2 I_\lambda(V_\lambda + t\phi)\phi^2(1-t)\,dt \\ = & \frac{1}{p^2\gamma^{2(p-1)}}\int_0^1 \left[\int_{\Omega_\varepsilon}(E_\lambda + N(\phi))\phi + \int_{\Omega_\varepsilon}[f'(V_\lambda) - f'(V_\lambda + t\phi)]\phi^2\right](1-t)\,dt \end{aligned}$$

Since $\|E_\lambda\|_* \le \frac{C}{|\log\varepsilon|^3}$, $\|\phi\|_\infty \le \frac{C}{|\log\varepsilon|^2}$, $\|N(\phi)\|_* \le \frac{C}{|\log\varepsilon|^4}$ and (4.50), we get that

$$\left|J_\lambda^p\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right) - J_\lambda^p\left(U_\lambda(\xi)\right)\right| \le \frac{C}{\gamma^{2(p-1)}|\log\varepsilon|^3} \tag{4.92}$$

Next we expand

$$J_\lambda^p\left(U_\lambda(\xi)\right) = \frac{1}{2}\int_\Omega \left[|\nabla\left(U_\lambda(\xi)\right)|^2 + U_\lambda(\xi)^2\right] - \frac{\lambda}{p}\int_\Omega e^{(U_\lambda(\xi))^p}. \tag{4.93}$$

Now we write

$$U_j(x) := u_j(x) + H_j(x), \qquad U_{0j} := w_{0j}(x) + H_{0j}(x), \qquad U_{1j} := w_{1j}(x) + H_{1j}(x).$$

By (4.26),

$$U_\lambda(x) = \frac{1}{p\gamma^{p-1}}\sum_{j=1}^m\left(U_j(x) + \frac{p-1}{p}\frac{1}{\gamma^p}U_{0j}(x) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}U_{1j}(x)\right).$$

We have

$$\frac{1}{2}\int_\Omega \left[|\nabla\left(U_\lambda(\xi)\right)|^2 + U_\lambda(\xi)^2\right]$$

$$
\begin{aligned}
= \quad & \frac{1}{p^2\gamma^{2(p-1)}}\Bigg\{\frac{1}{2}\sum_{j=1}^{m}\int_\Omega(|\nabla U_j|^2+U_j^2)+\sum_{l\neq j}\int_\Omega(\nabla U_l\nabla U_j+U_lU_j) \\
& +\frac{p-1}{p}\frac{1}{\gamma^p}\sum_{j=1}^{m}\int_\Omega(\nabla U_j\nabla U_{0j}+U_jU_{0j})+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\sum_{j=1}^{m}\int_\Omega(\nabla U_j\nabla U_{1j}+U_jU_{1j}) \\
& +\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left[\frac{1}{2}\sum_{j=1}^{m}\int_\Omega(|\nabla U_{0j}|^2+U_{0j}^2)+\sum_{l\neq j}\int_\Omega(\nabla U_{0l}\nabla U_{0j}+U_{0l}U_{0j})\right] \\
& +\left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}\sum_{j=1}^{m}\int_\Omega(\nabla U_{0j}\nabla U_{1j}+U_{0j}U_{1j}) \\
& +\left(\frac{p-1}{p}\right)^4\frac{1}{\gamma^{4p}}\left[\frac{1}{2}\sum_{j=1}^{m}\int_\Omega(|\nabla U_{1j}|^2+U_{1j}^2)+\sum_{l\neq j}\int_\Omega(\nabla U_{1l}\nabla U_{1j}+U_{1l}U_{1j})\right]\Bigg\}. \quad (4.94)
\end{aligned}
$$

Let us estimate the first two terms. We observe that the remaining terms are $O(\frac{1}{\gamma^{2(p-1)}\gamma^p})$. First, we note that U_j satisfies

$$
-\Delta U_j+U_j=\varepsilon^2e^{u_j},\quad \text{in }\Omega,\quad \frac{\partial U_j}{\partial\nu}=0\quad\text{on }\partial\Omega.
$$

Then we have

$$
\begin{aligned}
& \int_\Omega(|\nabla U_j(x)|^2+U_j(x)^2)\,dx \\
= \quad & \varepsilon^2\int_\Omega e^{u_j}U_j(x)=\varepsilon^2\int_\Omega e^{u_j}(u_j(x)+H_j(x)) \\
= \quad & \varepsilon^2\int_\Omega\frac{8\mu_j^2}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2}\left(\log\frac{8\mu_j^2}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2}+c_jH(x,\xi_j)-\log(8\mu_j^2)+O(\varepsilon^\alpha)\right) \\
= \quad & \varepsilon^2\int_\Omega\frac{8\mu_j^2}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2}\left(\log\frac{1}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2}+c_jH(x,\xi_j)+O(\varepsilon^\alpha)\right) \\
= \quad & \int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|y|^2)^2}\left(\log\frac{1}{(1+|y|^2)^2}+c_jH(\xi_j+\varepsilon\mu_jy,\xi_j)-4\log(\varepsilon\mu_j)\right)+O(\varepsilon^\alpha) \\
= \quad & \int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|y|^2)^2}\log\frac{1}{(1+|y|^2)^2}+c_j\int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|y|^2)^2}\left(H(\xi_j+\varepsilon\mu_jy,\xi_j)-H(\xi_j,\xi_j)\right) \\
& +c_j\int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|y|^2)^2}H(\xi_j,\xi_j)-4\log(\varepsilon\mu_j)\int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|y|^2)^2}+O(\varepsilon^\alpha). \quad (4.95)
\end{aligned}
$$

But

$$
\int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|y|^2)^2}=c_j+O(\varepsilon), \quad (4.96)
$$

$$
\int_{\Omega_{\varepsilon\mu_j}}\frac{8}{(1+|y|^2)^2}\log\frac{1}{(1+|y|^2)^2}=-2c_j+O(\varepsilon^\alpha). \quad (4.97)
$$

Moreover, for $0 < \alpha < 1$,

$$\begin{aligned}&\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2}\left(H(\xi_j+\varepsilon\mu_j y,\xi_j)-H(\xi_j,\xi_j)\right)\\ =& \int_{\Omega_{\varepsilon\mu_j}} \frac{1}{(1+|y|^2)^2} O\left(\varepsilon^\alpha |y|^\alpha\right) = O(\varepsilon^\alpha).\end{aligned} \tag{4.98}$$

Therefore from (4.95)-(4.98), we have

$$\begin{aligned}&\int_\Omega (|\nabla U_j(x)|^2 + U_j(x)^2)\ dx\\ =& -2c_j + c_j^2 H(\xi_j,\xi_j) - 4c_j\log\varepsilon - 4c_j\log\mu_j + O(\varepsilon^\alpha)\\ =& -2c_j + c_j^2 H(\xi_j,\xi_j) - 4c_j\log\varepsilon - 2c_j\log(8\mu_j^2) + 2c_j\log(8) + O(\varepsilon^\alpha).\end{aligned} \tag{4.99}$$

Now, we calculate that

$$\begin{aligned}&\sum_{l\neq j}\int_\Omega (\nabla U_l\nabla U_j + U_l U_j)\ dx\\ =& \ \varepsilon^2\sum_{l\neq j}\int_\Omega e^{u_l}U_j = \varepsilon^2\sum_{l\neq j}\int_\Omega e^{u_l}(u_j+H_j)\ dx\\ =& \ \varepsilon^2\sum_{l\neq j}\int_\Omega \frac{8\mu_l^2}{(\varepsilon^2\mu_l^2+|x-\xi_l|^2)^2}\left(\log\frac{8\mu_j^2}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2} + c_j H(x,\xi_j) - \log(8\mu_j^2) + O(\varepsilon^\alpha)\right)\\ =& \ \varepsilon^2\sum_{l\neq j}\int_\Omega \frac{8\mu_l^2}{(\varepsilon^2\mu_l^2+|x-\xi_l|^2)^2}\left(\log\frac{1}{(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2} + c_j H(x,\xi_j) + O(\varepsilon^\alpha)\right)\\ =& \sum_{l\neq j}\int_{\Omega_{\varepsilon\mu_l}} \frac{8}{(1+|y|^2)^2}\left(\log\frac{1}{(\varepsilon^2\mu_j^2+|\varepsilon\mu_l y+\xi_l-\xi_j|^2)^2} + c_j H(\xi_l+\varepsilon\mu_l y,\xi_j)\right) + O(\varepsilon^\alpha)\\ =& \sum_{l\neq j}\int_{\Omega_{\varepsilon\mu_l}} \frac{8}{(1+|y|^2)^2} c_j G(\xi_l,\xi_j) + O(\varepsilon^\alpha)\\ =& \sum_{l\neq j} c_l c_j G(\xi_l,\xi_j) + O(\varepsilon^\alpha).\end{aligned} \tag{4.100}$$

Thus, from (4.94), (4.99) and (4.100) we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega (|\nabla U_\lambda(x)|^2 + U_\lambda(x)^2)\ dx\\ =& \ \frac{1}{p^2\gamma^{2(p-1)}}\left\{-4\pi(2k+l)\frac{p}{2-p}(1-\log 8) - 8\pi(2k+l)\log\varepsilon\right.\\ &\left. -\frac{p}{2(2-p)}\sum_{j=1}^m\left[c_j^2 H(\xi_j,\xi_j) + \sum_{l\neq j} c_l c_j G(\xi_l,\xi_j)\right] + O(|\log\varepsilon|^{-1})\right\}.\end{aligned} \tag{4.101}$$

Finally, let us evaluate the second term in the energy

$$\begin{aligned}\frac{\lambda}{p}\int_\Omega e^{(U_\lambda)^p}\,dx &= \frac{\lambda}{p}\int_\Omega e^{\gamma^p\left(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon})\right)^p}\,dx\\ &= \frac{\lambda}{p}\sum_{j=1}^k\int_{B(\xi_j,\tilde\delta)} e^{\gamma^p\left(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon})\right)^p}\,dx\\ &\quad +\frac{\lambda}{p}\int_{\Omega\setminus\bigcup_{j=1}^k B(\xi_j,\tilde\delta)} e^{\gamma^p\left(1+\frac{1}{p\gamma^p}(V_\lambda)(\frac{x}{\varepsilon})\right)^p}\,dx\\ &:= I+II. \end{aligned}\tag{4.102}$$

First we observe that

$$II=\lambda\Theta_\lambda(\xi)\tag{4.103}$$

with $\Theta_\lambda(\xi)$ a function, uniformly bounded, as $\lambda\to 0$. On the other hand,

$$\begin{aligned} I &= \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^m\int_{B(\xi_j',\tilde\delta/\varepsilon)} e^{\gamma^p\left[\left(1+\frac{1}{p\gamma^p}(V_\lambda)(y)\right)^p-1\right]}\,dy\\ &= \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^m\int_{B(\xi_j',\tilde\delta/\varepsilon)} e^{\left\{w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}w_{0j}(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}w_{1j}(y)+\theta(y)\right\}}(1+O(\frac{1}{\gamma^p}))dy\\ &= \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^m\int_{B(0,\frac{\tilde\delta}{\mu_j\varepsilon})}\frac{8}{(1+|y|^2)^2}\left(1+O(\frac{1}{\gamma^p})\right)\,dy\\ &= \frac{1}{p^2\gamma^{2(p-1)}}4\pi(2k+l)\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right), \end{aligned}\tag{4.104}$$

with $\Theta_\lambda(\xi)$ a function, uniformly bounded, as $\lambda\to 0$. From (4.102)-(4.104) we get

$$\frac{\lambda}{p}\int_\Omega e^{(U_\lambda)^p}\,dx=\frac{1}{p^2\gamma^{2(p-1)}}4\pi(2k+l)\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right).\tag{4.105}$$

Thus from (4.92), (4.93), (4.101) and (4.105), we obtain that

$$\begin{aligned}F_\lambda(\xi)=\frac{1}{p^2\gamma^{2(p-1)}}&\left\{-4\pi(2k+l)\frac{2-p\log 8}{2-p}-8\pi(2k+l)\log\varepsilon\right.\\ &\left.-\frac{p}{2(2-p)}\sum_{j=1}^m c_j^2H(\xi_j,\xi_j)+\sum_{l\neq j}c_lc_jG(\xi_l,\xi_j)+O(|\log\varepsilon|^{-1})\right\},\end{aligned}$$

which implies (4.60) by (4.8). This concludes the proof of Proposition 4.6.

Chapter 5

Bubbling solutions for elliptic equation with exponential Neumann data in $\mathbb{R}^2$

[1]

5.1 Introduction

Consider the following boundary value problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = \lambda u^{p-1} e^{u^p} & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, ν is the outer normal vector of $\partial\Omega$, $\lambda > 0$ is a small parameter and $0 < p < 2$.

In [27], Dávila-del Pino-Musso have analyzed the asymptotic behavior of solution to problem (5.1) when $p = 1$. Namely, they considered the following problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = \lambda e^{u} & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Suppose that u_λ is a family solution of (5.2), with the property $\lambda \int_{\partial\Omega} e^{u_\lambda}$ bounded, then there

[1]The main result of this chapter was worked with Monica Musso, to appear in *Annali Scuola Normale Superiore di Pisa*, DOI Number: 10.2422/2036-2145.201204−007.

is an integer $k \geq 1$, up to subsequences, such that

$$\lim_{\lambda \to 0} \lambda \int_{\partial\Omega} e^{u_\lambda} = 2k\pi. \tag{5.3}$$

Moreover, there are k distinct points ξ_j, $j = 1, \dots, k$, on the boundary of Ω, such that λe^{u_λ} approaches the sum of k Dirac masses at these points ξ_j. The location of points can be characterized as critical points of a functional of k points of the boundary given by

$$\varphi_k(\xi_1, \dots, \xi_k) = -\left[\sum_{j=1}^{k} H(\xi_j, \xi_j) + \sum_{l \neq j} G(\xi_l, \xi_j)\right], \tag{5.4}$$

where $G(x, y)$ is Green's function of the problem

$$\begin{cases} -\Delta_x G(x, y) + G(x, y) = 0 & x \in \Omega; \\ \frac{\partial G(x,y)}{\partial \nu_x} = 2\pi \delta_y(x) & x \in \partial\Omega, \end{cases} \tag{5.5}$$

and H its regular part

$$H(x, y) = G(x, y) - 2 \log \frac{1}{|x - y|}. \tag{5.6}$$

The authors in [27] also described the existence of solution with above properties. More precisely, if $\partial\Omega$ has more than one component, they showed that the function φ_k has *topologically nontrivial* critical point $(\xi_1, \dots, \xi_k)$, then there is a family solution to problem (5.2) with peaks at these points.

In this chapter, we will consider the existence of solution to (5.1) when $0 < p < 2$. This problem is the Euler-Lagrange equation for the functional $J_\lambda : H^1(\Omega) \to \mathbb{R}$ defined as

$$J_\lambda(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + u^2\right) - \frac{\lambda}{p} \int_{\partial\Omega} e^{u^p}.$$

By Young's and Hödler inequalities, we know that J_λ corresponds to the critical Trudinger-Moser trace embedding

$$H^1(\Omega) \ni u \mapsto e^{u^2} \in L^r(\partial\Omega) \quad \forall\, r \geq 1,$$

which is connected to the following critical Trudinger-Moser trace inequalities

$$S_\alpha := \sup \left\{ \int_{\partial\Omega} e^{\alpha u^2} \;:\; u \in H^1(\Omega) \backslash \{0\}, \;\; \|u\|_{H^1} \leq 1, \; \int_{\partial\Omega} u = 0 \right\} < \infty \tag{5.7}$$

for any $\alpha \leq \pi$, see [6]. By multiplying a suitable test function, we can find that smallness of λ is necessary for the existence of a solution. From (5.7), there is a minimizer solution near zero. On the other hand, there is a second solution exists for (5.1) by Mountain Pass Theorem.

In this chapter, we will establish the existence of solution to (5.1) by Lyapunov-Schmidt reduction procedure. In order to state our result, let us first introduce the following definition.

Definition 5.1. *We say that ξ is a C^0-stable critical point of $\varphi : \mathcal{M} \to \mathbb{R}$ if for any sequence of function $\varphi_n : \mathcal{M} \to \mathbb{R}$ such that $\varphi_n \to \varphi$ uniformly on compact sets of $\mathcal{M}$, φ_n has a critical point ξ^n such that $\varphi_n(\xi^n) \to \varphi(\xi)$.*

In particular, if ξ is a strict local minimum or maximum point of φ, then ξ is C^0-stable critical point.

Let ε be a parameter, which depends on λ, satisfies,

$$p\lambda \left(-\frac{2}{p} \log \varepsilon \right)^{\frac{2(p-1)}{p}} \varepsilon^{\frac{p-2}{p}} = 1. \tag{5.8}$$

Observe that, as $\lambda \to 0$, then $\varepsilon \to 0$, and $\varepsilon = \lambda$ if $p = 1$.

Our result states as follows.

Theorem 5.2. *For $0 < p < 2$, let $k \geq 1$, assume that φ_k defined by (5.4) has a C^0-stable critical point $\xi^* = (\xi_1^*, \dots, \xi_k^*) \in (\partial\Omega)^k$ with*

$$|\xi_l^* - \xi_j^*| > \delta, \quad \textit{for} \ \ l \neq j,$$

for some small but fixed number $\delta > 0$. Then the problem (5.1) has a family solutions u_λ for λ small enough, such that

$$\lim_{\lambda \to 0} \varepsilon^{\frac{2-p}{p}} \int_{\partial\Omega} e^{u_\lambda^p} = 2k\pi, \tag{5.9}$$

where ε satisfies (5.8). Moreover, for $\lambda \to 0$

$$\nabla \varphi_k(\xi_1^*, \dots, \xi_k^*) = 0,$$

and

$$u_\lambda(x) = p^{-\frac{1}{2}} \sqrt{\lambda}\, \varepsilon^{\frac{p-2}{2p}} \left[\sum_{j=1}^{k} G(x, \xi_j^*) + o(1) \right] \tag{5.10}$$

where $o(1) \to 0$ on each compact subset of $\bar{\Omega} \backslash \{\xi_1^*, \dots, \xi_k^*\}$. Furthermore

$$J_\lambda(u_\lambda) = \lambda \varepsilon^{\frac{p-2}{p}} \left[-\frac{2k\pi}{p} + \frac{2k\pi}{p} \log \frac{1}{\varepsilon} + \frac{\pi}{2-p} \varphi_k(\xi) + O(|\log \varepsilon|^{-1}) \right] \tag{5.11}$$

where $O(1)$ uniformly bounded as $\lambda \to 0$.

The proof of our result relies on a very well known Lyapunov-Schmidt reduction procedure, introduced in [9, 52] and used in many different contexts, see for instance [19, 27, 29, 36, 39, 46, 47, 48, 49, 93]. We use Lyapunov-Schmidt reduction method to reduce the problem to a finite dimensional one, with some reduced energy. Then, the solutions in Theorem turn out to be generated by critical points of the reduced energy functionals. The key step is to find the ansatz for the solution. Usually, the ansatz is built as a sum of terms, which turns out to be solutions of the associate limit problem, which are properly scaled and translated. For our problem, let us introduce the following limit problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2_+; \\ \frac{\partial v}{\partial \nu} = e^v & \text{on } \partial\mathbb{R}^2_+; \\ \int_{\partial\mathbb{R}^2_+} e^v < \infty. \end{cases} \tag{5.12}$$

A family solutions to (5.12) is given by

$$w_{t,\mu}(x) = w_{t,\mu}(x_1, x_2) = \log \frac{2\mu}{(x_1 - t)^2 + (x_2 + \mu)^2}, \tag{5.13}$$

where $t \in \mathbb{R}$ and $\mu > 0$ are parameters. Set

$$w_\mu(x) := w_{0,\mu}(x) = \log \frac{2\mu}{x_1^2 + (x_2 + \mu)^2}. \tag{5.14}$$

If we use above solution, properly scaled, and centered at several points on the boundary of domain as our approximate solution, we get a very good approximation of a solution in a region far away from the points, which unfortunately turns out to be not good enough close to these points. Thus we need to improve the approximation, at least near the points, and we do this adding two other terms in the expansion of the solution. This can be done in a very natural way, which has first been used, for instance, in [47] for studying the following problem

$$\begin{cases} \Delta u + u^p = 0, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.15}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, and p is a large exponent. Later on, this method has been applied in other contexts, see [19, 48, 49, 93]. In particular, H. Castro in [19] used this method to study the following Neumann problem

$$\begin{cases} -\Delta u + u = 0, \quad u > 0 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases} \tag{5.16}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, ν is the outer normal vector to $\partial\Omega$ and p is a large exponent. They showed that, if $p > 1$ is a large parameter, for any integer $k \geq 1$, there exists at least two families of solution u_p, which developing exactly k peaks $\xi_j \in \partial\Omega$, and in the sense that $pu_p^p \rightharpoonup 2e\pi \sum_{j=1}^k \delta_{\xi_j}$ as $p \to +\infty$.

This chapter is organized as follows: Section 5.2 is devoted to describing a first approximation solution to problem (5.1) and estimating the error. Furthermore, problem (5.1) is written as a fixed point problem, which involving a linear operator. In Section 5.3, we study the invertibility of the linear problem. In Section 5.4, we study the nonlinear problem. In Section 5.5, we study the variational reduction, we prove the main Theorem 5.2 in Section 5.6. We will give some estimates in Section 5.7.

5.2 Preliminaries and ansatz for the solution

For any parameter $\varepsilon > 0$, we can produce a solution to

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+; \\ \frac{\partial u}{\partial \nu} = \varepsilon e^u & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{5.1}$$

by taking

$$u(x) = w_\mu(x/\varepsilon) - 2\log\varepsilon = \log\frac{2\mu}{x_1^2 + (x_2 + \varepsilon\mu)^2},$$

where w_μ defined by (5.14). Based on this, we choose a sufficiently small but fixed number $\delta > 0$ and assume that for any points ξ_j, $j = 1, \dots, k$, on $\partial\Omega$, satisfying

$$|\xi_l - \xi_j| > \delta, \quad \text{for } l \neq j. \tag{5.2}$$

Furthermore, we consider k positive numbers μ_j such that

$$\delta < \mu_j < \delta^{-1}, \quad \text{for all} \quad j = 1, \dots, k. \tag{5.3}$$

We define

$$u_j(x) = \log\frac{2\mu_j}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2}.$$

We note that

$$u_j(x) = w_{\mu_j}\left(A_j\left(\frac{x - \xi_j}{\varepsilon}\right)\right) - 2\log\varepsilon,$$

where $A_j : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ a rotation map, satisfies

$$A\nu_\Omega(\xi) = \nu_{\mathbb{R}^2_+}(0),$$

without lost of generality, in the follows, we will denote that $A_j = I$.

We define the first ansatz is given by

$$U(x) = \frac{1}{p\gamma^{p-1}}\sum_{j=1}^{k}\left[u_j(x) + H_j^\varepsilon(x)\right],$$

with some number γ, to be fixed later on, where H_j^ε is a correction term defined as the solution of

$$\begin{cases} -\Delta H_j^\varepsilon + H_j^\varepsilon = -u_j & \text{in } \Omega; \\ \frac{\partial H_j^\varepsilon}{\partial \nu} = \varepsilon e^{u_j} - \frac{\partial u_j}{\partial \nu} & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

Lemma 5.3. *Assume (5.2) and (5.3), for any $0 < \alpha < 1$, one has*

$$H_j^\varepsilon(x) = H(x, \xi_j) - \log(2\mu_j) + O(\varepsilon^\alpha) \tag{5.5}$$

uniformly in $\bar{\Omega}$, where H is the regular part of Green's function defined (5.5).

Proof. On the boundary, we have

$$\frac{\partial H_j^\varepsilon}{\partial \nu} = \varepsilon e^{u_j} - \frac{\partial u_j}{\partial \nu} = 2\varepsilon\mu_j \frac{1 - \nu(\xi_j) \cdot \nu(x)}{|x - \xi_j - \varepsilon\mu_j \nu(\xi_j)|^2} + 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j - \varepsilon\mu_j \nu(\xi_j)|^2}.$$

Thus,

$$\lim_{\varepsilon \to 0} \frac{\partial H_j^\varepsilon}{\partial \nu} = 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2}, \qquad \forall\, x \neq \xi_j.$$

Let $z_\varepsilon(x) = H_j^\varepsilon(x) + \log(2\mu_j) - H(x, \xi_j)$, then from the definition of $H(x, \xi_j)$ and H_j^ε, we have

$$\begin{cases} -\Delta z_\varepsilon + z_\varepsilon = \log \frac{1}{|x-\xi_j|^2} - \log \frac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2} & \text{in } \Omega; \\ \frac{\partial z_\varepsilon}{\partial \nu} = \frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2} & \text{on } \partial\Omega, \end{cases}$$

First, we claim that there is a positive constant C such that

$$\left\| \frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} \right\|_{L^q(\partial\Omega)} \le C\varepsilon^{1/q}, \qquad \forall\, q > 1, \tag{5.6}$$

In fact,

$$\begin{aligned} & \frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} \\ = \; & 2\varepsilon\mu_j \frac{1 - \nu(\xi_j) \cdot \nu(x)}{|x - \xi_j - \varepsilon\mu_j \nu(\xi_j)|^2} + 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j - \varepsilon\mu_j \nu(\xi_j)|^2} - 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} \\ = \; & 2\varepsilon\mu_j \frac{1 - \nu(\xi_j) \cdot \nu(x)}{|x - \xi_j - \varepsilon\mu_j \nu(\xi_j)|^2} + 2\varepsilon\mu_j \frac{(x - \xi_j) \cdot \nu(x)\,[2(x - \xi_j) \cdot \nu(x) - \varepsilon\mu_j]}{|x - \xi_j|^2 |x - \xi_j - \varepsilon\mu_j \nu(\xi_j)|^2}. \end{aligned}$$

Now, we observe that

$$|1 - \nu(\xi_j) \cdot \nu(x)| \le C|x - \xi_j|^2, \qquad |(x - \xi_j) \cdot \nu(x)| \le C|x - \xi_j|^2, \qquad \forall\, x \in \partial\Omega.$$

Hence,

$$\left|\frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}\right| \le C\varepsilon + C\frac{\varepsilon|2(x-\xi_j)\cdot\nu(\xi_j)-\varepsilon\mu_j|}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}. \tag{5.7}$$

For $\rho > 0$ small, we have

$$\left|\frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}\right| \le C\varepsilon, \qquad \text{for } \ \forall\ |x-\xi_j| \ge \rho,\ x\in\partial\Omega. \tag{5.8}$$

Now let $q > 1$, we have

$$\begin{aligned}
&\int_{B_\rho(\xi_j)\cap\partial\Omega}\left|\frac{\varepsilon|2(x-\xi_j)\cdot\nu(\xi_j)-\varepsilon\mu_j|}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}\right|^q dx\\
=\ & C\varepsilon\int_{B_{\rho/\varepsilon}(0)\cap\partial\Omega_\varepsilon}\left|\frac{2y\cdot\nu(0)-\mu_j}{y-\mu_j\nu(0)}\right|^q dy\\
\le\ & C\varepsilon\int_0^{\rho/\varepsilon}\frac{1}{(1+s)^q}ds \le C\varepsilon.
\end{aligned} \tag{5.9}$$

Combining (5.7) with (5.8) and (5.9) we conclude that (5.6) holds.

Next, we show that

$$\left\|\log\frac{1}{|x-\xi_j|^2} - \log\frac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}\right\|_{L^q(\Omega)} \le C\varepsilon, \qquad \text{for any } \ 1<q<2. \tag{5.10}$$

In fact, for $q \ge 1$, we write

$$\begin{aligned}
&\left\|\log\frac{1}{|x-\xi_j|^2} - \log\frac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}\right\|^q_{L^q(\Omega)}\\
=\ & \int_{B_{10\varepsilon\mu_j}(\xi_j)\cap\Omega}\dots + \int_{\Omega\setminus B_{10\varepsilon\mu_j}(\xi_j)}\dots := I_1 + I_2.
\end{aligned} \tag{5.11}$$

Next we estimate I_1 and I_2. For I_1, we observe that

$$\int_{B_{10\varepsilon\mu_j}(\xi_j)\cap\Omega}\left|\log\frac{1}{|x-\xi_j|^2}\right|^q dx \le C\int_0^{C\varepsilon}|\log r|^q r dr \le C\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^q,$$

and the same bound is true for the integral of $|\log\frac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}|^q$ in $B_{10\varepsilon\mu_j}(\xi_j)\cap\Omega$. Hence, we have

$$|I_1| \le C\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^q. \tag{5.12}$$

For I_2, if $|x-\xi_j| \ge 10\varepsilon\mu_j$, we have

$$|x-\xi_j| \le |x-\xi_j-t\varepsilon\mu_j\nu(\xi_j)| + \mu_j\varepsilon \le |x-\xi_j-t\varepsilon\mu_j\nu(\xi_j)| + \frac{1}{10}|x-\xi_j|$$

for any $t \in [0,1]$, then we have $|x-\xi_j| \le C|x-\xi_j - t\varepsilon\mu_j\nu(\xi_j)|$. Using this fact, we can obtain

$$\begin{aligned} & \left|\log \frac{1}{|x-\xi_j|^2} - \log \frac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}\right| \\ \le \quad & C \sup_{0\le t\le 1} \frac{C\varepsilon}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|} \le \frac{C\varepsilon}{|x-\xi_j|}. \end{aligned}$$

Thus for $1<q<2$,

$$|I_1| \le C\varepsilon^q \int_{10\varepsilon\mu_j}^{D} r^{1-q}dr \le C\varepsilon^q, \tag{5.13}$$

where D is the diameter of Ω. Thus, combining (5.11) with (5.12) and (5.13) we obtain that (5.10) holds.

Therefore by elliptic regularity theory, we obtain

$$\|z_\varepsilon\|_{W^{1+s,q}(\Omega)} \le \left(\left\| \frac{\partial z_\varepsilon}{\partial \nu} \right\|_{L^q(\partial\Omega)} + \|\Delta z_\varepsilon\|_{L^q(\Omega)} \right) \le C\varepsilon^{1/q} \tag{5.14}$$

for any $0<s<\frac{1}{q}$. By the Morrey embedding we obtain

$$\|z_\varepsilon\|_{C^\beta}(\bar{\Omega}) \le C\varepsilon^{1/q}$$

for any $0<\beta<\frac{1}{2}+\frac{1}{q}$. This proves the Lemma with $\alpha=\frac{1}{q}$. □

We shall show later on that $U(x)$ is a good approximation for a solution to (5.1) far from the points ξ_j, but unfortunately it is not good enough for our construction close to the points ξ_j. This is the reason why we need to further adjust this ansatz. In order to do this, let us first introduce the following result, whose proof is given in [27].

Proposition 5.4. *Any bounded solution of the following problem*

$$\begin{cases} \Delta\phi = 0 & in \ \ \mathbb{R}^2_+; \\ \frac{\partial\phi}{\partial\nu} - e^{w_\mu}\phi = 0 & on \ \ \partial\mathbb{R}^2_+, \end{cases} \tag{5.15}$$

is a linear combination of

$$z_{0\mu}(x) = -\frac{1}{\mu}\left(x\cdot\nabla w_\mu(x)+1\right) = \frac{1}{\mu} - 2\frac{x_2+\mu}{x_1^2+(x_2+\mu)^2}, \tag{5.16}$$

and

$$z_{1\mu}(x) = \frac{\partial w_\mu}{\partial x_1} = -2\frac{x_1}{x_1^2+(x_2+\mu)^2}. \tag{5.17}$$

Now, let us consider the following problem

$$\begin{cases} \Delta\phi = 0 & \text{in } \ \mathbb{R}^2_+; \\ \frac{\partial\phi}{\partial\nu} - e^{w_\mu}\phi = e^{w_\mu} g & \text{on } \ \partial\mathbb{R}^2_+, \end{cases} \tag{5.18}$$

with w_μ defined in (5.14). In [19] it is showed that

Proposition 5.5. *Let g be a $C^1(\partial\mathbb{R}^2_+)$ function such that, for $\mu > 0$, $k \geq 0$, satisfies*

$$g(x) = O(\log^k(1 + |x|)) \quad as\ |x| \to \infty, \tag{5.19}$$

and

$$\int_{\partial\mathbb{R}^2_+} e^{w_\mu} g z_{0\mu} = 0 = \int_{\partial\mathbb{R}^2_+} e^{w_\mu} g z_{1\mu}. \tag{5.20}$$

Then (5.18) has a solution $\phi \in C^\alpha(\mathbb{R}^2_+)$. Moreover, for any $0 < \alpha < 1$, and $|x| \to \infty$,

$$|\phi(x)| \leq C\frac{1}{|x|^\alpha}, \quad |\nabla\phi(x)| \leq C\frac{1}{|x|^{1+\alpha}}, \quad |\nabla^2\phi(x)| \leq C\frac{1}{|x|^{2+\alpha}}, \tag{5.21}$$

where C is a positive constant, which depends on $\|g\|_{L^p(\partial\mathbb{R}^2_+)}$, for some $p = p(\alpha) > 1$.

Let us define ϕ_{1j} the solution of the problem

$$\begin{cases} \Delta\phi_{1j} = 0 & \text{in } \ \mathbb{R}^2_+; \\ \frac{\partial\phi_{1j}}{\partial\nu} - e^{w_{\mu_j}}\phi_{1j} = e^{w_{\mu_j}} g_1 & \text{on } \ \partial\mathbb{R}^2_+, \end{cases} \tag{5.22}$$

where $w_{\mu_j}(y) = \log\frac{2\mu_j}{y_1^2 + (y_2+\mu_j)^2}$ and

$$g_1 = \alpha_{1j}(w_{\mu_j} - 1) + w_{\mu_j} + \frac{1}{2}(w_{\mu_j})^2$$

with α_{1j} is a constant to be fixed, which depend on μ_j. Let us observe that the function g_1 satisfies (5.19) by the definition. We now choose α_{1j} such that the orthogonality condition (5.20) hold. First we observe that g_1 is a symmetric function for any choice of α_{1j}, hence

$$\int_{\partial\mathbb{R}^2_+} e^{w_{\mu_j}} g_1 z_{1\mu_j} = 0.$$

Next, we choose parameter α_{1j} such that the other orthogonality condition satisfies. Since

$$\begin{aligned} &\int_{\partial\mathbb{R}^2_+} e^{w_{\mu_j}} g_1 z_{0\mu_j} \\ = \ &-\frac{1}{\mu_j}\int_{\partial\mathbb{R}^2_+} e^{w_{\mu_j}}\left(\alpha_{1j}(w_{\mu_j} - 1) + w_{\mu_j} + \frac{1}{2}(w_{\mu_j})^2\right)\left(y\cdot\nabla w_{\mu_j}(y) + 1\right) \end{aligned}$$

$$
\begin{aligned}
= & -\frac{1}{\mu_j}\alpha_{1j}\int_{-\infty}^{\infty}\left(e^{w_{\mu_j}(y_1,0)}\left(w_{\mu_j}(y_1,0)-1\right)\frac{\partial w_{\mu_j}}{\partial y_1}(y_1,0)y_1+e^{w_{\mu_j}(y_1,0)}\left(w_{\mu_j}(y_1,0)-1\right)\right)dy_1 \\
& -\frac{1}{\mu_j}\int_{-\infty}^{\infty}\left(e^{w_{\mu_j}(y_1,0)}w_{\mu_j}(y_1,0)\frac{\partial w_{\mu_j}}{\partial y_1}(y_1,0)y_1+e^{w_{\mu_j}(y_1,0)}w_{\mu_j}(y_1,0)\right)dy_1 \\
& -\frac{1}{\mu_j}\int_{-\infty}^{\infty}\left(e^{w_{\mu_j}(y_1,0)}\frac{(w_{\mu_j})^2}{2}(y_1,0)\frac{\partial w_{\mu_j}}{\partial y_1}(y_1,0)y_1+e^{w_{\mu_j}(y_1,0)}\frac{(w_{\mu_j})^2}{2}(y_1,0)\right)dy_1 \\
= & -\frac{1}{\mu_j}\left[\alpha_{1j}\int_{-\infty}^{\infty}e^{w_{\mu_j}(y_1,0)}dy_1+\int_{-\infty}^{\infty}e^{w_{\mu_j}(y_1,0)}w_{\mu_j}(y_1,0)dy_1\right].
\end{aligned}
$$

Thus we need to choose α_{1j} such that

$$
\alpha_{1j}\int_{-\infty}^{\infty}e^{w_{\mu_j}(y_1,0)}dy_1+\int_{-\infty}^{\infty}e^{w_{\mu_j}(y_1,0)}w_{\mu_j}(y_1,0)dy_1=0.
$$

Since

$$
\int_{-\infty}^{\infty}e^{w_{\mu_j}(y_1,0)}dy_1=\int_{-\infty}^{\infty}\frac{2\mu_j}{y_1^2+\mu_j^2}\,dy_1=2\int_{-\infty}^{\infty}\frac{1}{t^2+1}\,dt=2\pi,
$$

and

$$
\begin{aligned}
& \int_{-\infty}^{\infty}e^{w_{\mu_j}(y_1,0)}w_{\mu_j}(y_1,0)dy_1 \\
= & \int_{-\infty}^{\infty}\frac{2\mu_j}{y_1^2+\mu_j^2}\log\frac{2\mu_j}{y_1^2+\mu_j^2}\,dy_1 \\
= & \ 2\int_{-\infty}^{\infty}\frac{1}{t^2+1}\left[\log\frac{1}{t^2+1}+\log(2\mu_j^{-1})\right]\,dt=-2\pi\log(2\mu_j).
\end{aligned}
$$

Here we use the following fact (See the proof in the Appendix)

$$
\int_{-\infty}^{\infty}\frac{1}{t^2+1}\log\frac{1}{t^2+1}dt=-2\pi\log 2. \tag{5.23}
$$

Therefore, we choose α_{1j} satisfies

$$
\alpha_{1j}=\log(2\mu_j). \tag{5.24}
$$

Then we get the existence of ϕ_{1j} by Proposition 5.5. With this function, we define

$$
w_{1j}(y)=\phi_{1j}(y)+\alpha_{1j}w_{\mu_j}(y).
$$

We observe that w_{1j} satisfies

$$
\begin{cases}
\Delta w_{1j}=0 & \text{in } \ \mathbb{R}^2_+; \\
\frac{\partial w_{1j}}{\partial\nu}-e^{w_{\mu_j}}w_{1j}=e^{w_{\mu_j}}\left(w_{\mu_j}+\frac{1}{2}(w_{\mu_j})^2\right) & \text{on } \ \partial\mathbb{R}^2_+.
\end{cases}
$$

Next, we consider ϕ_{2j}, a solution of

$$\begin{cases} \Delta \phi_{2j} = 0 & \text{in } \mathbb{R}^2_+; \\ \frac{\partial \phi_{2j}}{\partial \nu} - e^{w_{\mu_j}} \phi_{2j} = e^{w_{\mu_j}} g_2 & \text{on } \partial \mathbb{R}^2_+, \end{cases} \tag{5.25}$$

where

$$\begin{aligned} g_2 &= \alpha_{2j}(w_{\mu_j} - 1) + w_{1j} + \frac{p-2}{2(p-1)}(w_{\mu_j})^2 + \frac{1}{2}(w_{1j})^2 \\ &\quad + \frac{1}{8}(w_{\mu_j})^4 + w_{\mu_j} w_{1j} + \frac{1}{2}(w_{\mu_j})^3 + \frac{1}{2} w_{1j}(w_{\mu_j})^2 \end{aligned}$$

with some parameter α_{2j} such that the orthogonality condition (5.20) satisfies as above, and we note that g_2 satisfies (5.19) by the definition. Then we have the existence of function ϕ_{2j} by Proposition 5.5.

For $\xi_j \in \partial\Omega$, let $\delta > 0$ be a fixed small radius, depending only in the geometry of Ω, such that

$$F_j : B_\delta(0) \cap (\Omega - \xi_j) \to M \cap \mathbb{R}^2_+, \tag{5.26}$$

is a C^2 diffeomorphism, and M an open neighborhood of the origin such that

$$F_j \left(B_\delta(0) \cap (\partial\Omega - \xi_j) \right) \subseteq M \cap \partial \mathbb{R}^2_+,$$

We can select F_j so that it preserves area. For $i = 1, 2$, define

$$\begin{aligned} \tilde{w}_{ij}(x) &= \phi_{ij}\left(\frac{F_j(x - \xi_j)}{\varepsilon} \right) + \alpha_{ij} w_{\mu_j} \left(\frac{(x - \xi_j)}{\varepsilon} \right) \\ &:= \tilde{\phi}_{ij}(y) + \alpha_{ij} \tilde{w}_j(y), \end{aligned}$$

where

$$\tilde{w}_j(y) := w_{\mu_j}\left(y - \xi_j' \right) = \log \frac{2\mu_j}{|y - \xi_j' - \mu_j \nu(\xi_j')|^2},$$

with $\xi_j' = \xi_j / \varepsilon$ and where we will write ν for the exterior normal unit vector to $\partial\Omega$ and $\partial\Omega_\varepsilon$. Then, let us define the first approximation solution to (5.1) is

$$\begin{aligned} U_\lambda(x) &= \frac{1}{p\gamma^{p-1}} \sum_{j=1}^{k} \Bigg[u_j(x) + H_j^\varepsilon(x) + \frac{p-1}{p} \frac{1}{\gamma^p} \left(\tilde{w}_{1j}(x) + H_{1j}^\varepsilon(x) \right) \\ &\quad + \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \left(\tilde{w}_{2j}(x) + H_{2j}^\varepsilon(x) \right) \Bigg] \end{aligned} \tag{5.27}$$

where H_{ij}^{ε}, $i = 1, 2$, is a new correction term, that is, which is the solution of

$$\begin{cases} -\Delta H_{ij}^{\varepsilon} + H_{ij}^{\varepsilon} = -\alpha_{ij}\tilde{w}_j(x/\varepsilon) & \text{in } \Omega; \\ \frac{\partial H_{ij}^{\varepsilon}}{\partial \nu} = \alpha_{ij}\left(\varepsilon e^{u_j} - \frac{\partial u_j}{\partial \nu}\right) & \text{on } \partial\Omega. \end{cases} \tag{5.28}$$

By the same arguments as Lemma 5.3, we have the following result.

Lemma 5.6. *For any $0 < \alpha < 1$, for $i = 1, 2$, one has*

$$H_{ij}^{\varepsilon}(x) = \alpha_{ij} H(x, \xi_j) - \alpha_{ij}\log(2\mu_j) - 2\alpha_{ij}\log\varepsilon + O(\varepsilon^{\alpha}) \tag{5.29}$$

uniformly in $\bar{\Omega}$, where H is the regular part of Green's function defined (5.5).

Proof. The proof follows from the same arguments as those to prove Lemma 5.3. First, on the boundary, we have

$$\lim_{\varepsilon \to 0} \frac{\partial H_j^{\varepsilon}}{\partial \nu}(x) = 2\alpha_{ij}\frac{(x - \xi_j)\cdot \nu(x)}{|x - \xi_j|^2}, \qquad \forall\, x \neq \xi_j.$$

The regular part of Green's function satisfies

$$\begin{cases} -\Delta_x H(x, \xi_j) + H(x, \xi_j) = -\log\frac{1}{|x-\xi_j|^2} & \text{in } \Omega; \\ \frac{\partial H(x,\xi_j)}{\partial \nu_x} = 2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2} & \text{on } \partial\Omega. \end{cases}$$

Set $\tilde{z}_{\varepsilon} = H_j^{\varepsilon}(x) + \alpha_{ij}\log(2\mu_j\varepsilon^2) - \alpha_{ij}H(x, \xi_j)$, we have

$$\begin{aligned} -\Delta\tilde{z}_{\varepsilon} + \tilde{z}_{\varepsilon} &= -\Delta H_j^{\varepsilon} + H_j^{\varepsilon} + \alpha_{ij}\log(2\mu_j\varepsilon^2) - \alpha_{ij}\left[-\Delta H(x,\xi_j) + H(x,\xi_j)\right] \\ &= -\alpha_{ij}\tilde{w}_j + \alpha_{ij}\log(2\mu_j\varepsilon^2) - \alpha_{ij}\left[-\Delta H(x,\xi_j) + H(x,\xi_j)\right] \\ &= \alpha_{ij}\left[\log\frac{1}{|x-\xi_j|^2} - \log\frac{1}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2}\right], \qquad \text{in } \Omega. \end{aligned}$$

On the other hand, on the boundary, we have

$$\frac{\partial \tilde{z}_{\varepsilon}}{\partial \nu} = \alpha_{ij}\left[\frac{\partial H_j^{\varepsilon}}{\partial \nu} - 2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}\right].$$

From (5.6) and (5.14), by the same procedure as proof of Lemma 5.3, we obtain that (5.29) holds. ☐

Consider now the change of variables

$$v(y) = p\gamma^{p-1}u(\varepsilon y) - p\gamma^{p}, \qquad \text{with } \gamma^p = -\frac{2}{p}\log\varepsilon.$$

Then under the choice of ε in (5.8), problem (5.1) reduces to

$$\begin{cases} -\Delta v + \varepsilon^2 v = 2\varepsilon^2 \log \varepsilon & \text{in } \Omega_\varepsilon; \\ \frac{\partial v}{\partial \nu} = f(v) & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{5.30}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$, and

$$f(v) = (1 + \frac{v}{p\gamma^p})^{p-1} e^{\gamma^p[(1+\frac{v}{p\gamma^p})^p - 1]}.$$

Let us define the first approximation solution to (5.30) as

$$V_\lambda(y) = p\gamma^{p-1} U_\lambda(\varepsilon y) - p\gamma^p, \tag{5.31}$$

with U_λ defined by (5.27). We write $y = \varepsilon^{-1}x,\ \xi_j' = \varepsilon^{-1}\xi_j$. For $|x - \xi_j| < \delta$ with δ sufficiently small but fixed, by Lemma 5.3 and 5.6, and the fact $u_j(\varepsilon y) - p\gamma^p = \tilde{w}_j(y)$, we have

$$\begin{aligned}
& V_\lambda(y) \\
= \;& u_j(\varepsilon y) + H_j^\varepsilon(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p}\left(\tilde{w}_{1j}(\varepsilon y) + H_{1j}^\varepsilon(\varepsilon y)\right) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\left(\tilde{w}_{2j}(\varepsilon y) + H_{2j}^\varepsilon(\varepsilon y)\right) - p\gamma^p \\
& + \sum_{l\neq j}^k \left(\tilde{w}_l(\varepsilon y) + H_l^\varepsilon(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p}(\tilde{w}_{1l}(\varepsilon y) + H_{1l}^\varepsilon(\varepsilon y)) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}(\tilde{w}_{2l}(\varepsilon y) + H_{2l}^\varepsilon(\varepsilon y))\right) \\
= \;& \tilde{w}_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\tilde{w}_{1j}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\tilde{w}_{2j}(\varepsilon y) + O(\varepsilon|y - \xi_j'|) + O(\varepsilon^\alpha) \\
& - \log(2\mu_j) + \left[1 + \alpha_{1j}\frac{p-1}{p}\frac{1}{\gamma^p} + \alpha_{2j}\left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\right]\left(H(\xi_j,\xi_j) + \sum_{l\neq j}^k G(\xi_l,\xi_j)\right) \\
& - \left[\alpha_{1j}\frac{p-1}{p}\frac{1}{\gamma^p} + \alpha_{2j}\left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\right](\log(2\mu_j) + 2\log\varepsilon).
\end{aligned} \tag{5.32}$$

We now choose the parameters μ_j: we assume they are defined by the relation

$$\begin{aligned}
\log(2\mu_j) \;=\; & \left(H(\xi_j,\xi_j) + \sum_{l\neq j}^k G(\xi_l,\xi_j)\right) + (p-1)\alpha_{1j} \\
& + \alpha_{1j}\frac{p-1}{p}\frac{1}{\gamma^p}\left(H(\xi_j,\xi_j) + \sum_{l\neq j}^k G(\xi_l,\xi_j) - \log(2\mu_j) + (p-1)\frac{\alpha_{2j}}{\alpha_{1j}}\right) \\
& + \alpha_{2j}\left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\left(H(\xi_j,\xi_j) + \sum_{l\neq j}^k G(\xi_l,\xi_j) - \log(2\mu_j)\right).
\end{aligned} \tag{5.33}$$

Taking into account the explicit expression (5.24) of the constant α_{1j}, we observe that μ_j bifurcates, as λ goes to zero, from the value

$$\bar{\mu}_j = \frac{1}{2} e^{\frac{1}{2-p}\left[H(\xi_j,\xi_j) + \sum_{l\neq j}^k G(\xi_l,\xi_j)\right]} \tag{5.34}$$

solution of equation

$$\log(2\mu_j) = \left(H(\xi_j, \xi_j) + \sum_{l \neq j}^{k} G(\xi_l, \xi_j) \right) + (p-1)\alpha_{1j}. \tag{5.35}$$

Thus, μ_j is a perturbation of order $\frac{1}{\gamma^p}$ of the value $\bar{\mu}_j$, namely

$$\log(2\mu_j) = \frac{1}{2-p} \left(H(\xi_j, \xi_j) + \sum_{l \neq j}^{k} G(\xi_l, \xi_j) \right) \left(1 + O\left(\frac{1}{\gamma^p} \right) \right). \tag{5.36}$$

Then, by this choice of the parameters μ_j, we deduce that, if $|y - \xi'_j| < \delta/\varepsilon$ with δ sufficiently small but fixed, we can rewrite

$$V_\lambda(y) = \tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(\varepsilon y) + \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(\varepsilon y) + \theta(y), \tag{5.37}$$

with

$$\theta(y) = O(\varepsilon |y - \xi'_j|) + O(\varepsilon^\alpha).$$

We will look for solutions to (5.30) of the form

$$v = V_\lambda + \phi,$$

where V_λ is defined as in (5.31), and ϕ represents a lower order correction. We aim at finding a solution for ϕ small provided that the points ξ_j is suitably chosen. For small ϕ, we can rewrite problem (5.30) as a nonlinear perturbation of its linearization, namely,

$$\begin{cases} -\Delta \phi + \varepsilon^2 \phi = 0 & x \in \Omega_\varepsilon; \\ L(\phi) = E_\lambda + N(\phi) & x \in \partial\Omega_\varepsilon, \end{cases} \tag{5.38}$$

where

$$L(\phi) := \frac{\partial \phi}{\partial \nu} - f'(V_\lambda)\phi, \tag{5.39}$$

$$E_\lambda := f(V_\lambda) - \frac{\partial V_\lambda}{\partial \nu}, \tag{5.40}$$

$$N(\phi) := f(V_\lambda + \phi) - f(V_\lambda) - f'(V_\lambda)\phi. \tag{5.41}$$

We recall that $f(t) = (1 + \frac{t}{p\gamma^p})^{p-1} e^{\gamma^p[(1+\frac{t}{p\gamma^p})^p - 1]}$.

In order to solve the problem (5.38), first we have to study the invertibility properties of the linear operator L. In order to do this, we introduce a weighted L^∞-norm defined as

$$\|h\|_{*,\partial\Omega_\varepsilon} := \sup_{y\in\partial\Omega_\varepsilon} \left(\sum_{j=1}^{k} (1+|y-\xi_j'|)^{-1-\sigma} + \varepsilon \right)^{-1} |h(y)| \tag{5.42}$$

for any $h \in L^\infty(\partial\Omega_\varepsilon)$, where we fix $0 < \sigma < 1$ will be made later on. With respect to this norm, the error term E_λ given in (5.40) can be estimated in the following way.

Lemma 5.7. *Let $\delta > 0$ be a small but fixed number, assume (5.2) and (5.3). Then there exists $C > 0$ such that*

$$\|E_\lambda\|_{*,\partial\Omega_\varepsilon} \le \frac{C}{\gamma^{3p}} = \frac{C}{|\log\varepsilon|^3} \tag{5.43}$$

for λ small enough.

Proof. Far away from the points ξ_j, namely for $|x-\xi_j| > \delta$, i.e. $|y-\xi_j'| > \frac{\delta}{\varepsilon}$, for all $j = 1, \dots, k$, we have that

$$\frac{\partial V_\lambda}{\partial \nu} = p\gamma^{p-1}\varepsilon \frac{\partial U_\lambda(\varepsilon y)}{\partial \nu} = O(\gamma^{p-1}\varepsilon^2).$$

On the other hand, in this region we have

$$1 + \frac{V_\lambda(y)}{p\gamma^p} = 1 + \frac{2\log\varepsilon + O(1)}{p\gamma^p} = \frac{O(1)}{|\log\varepsilon|}$$

where $O(1)$ denotes a smooth function, uniformly bounded, as $\varepsilon \to 0$, in the considered region. Hence

$$\begin{aligned} f(V_\lambda) &= \left(1+\frac{V_\lambda}{p\gamma^p}\right)^{p-1} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p - 1]} \\ &= \begin{cases} C\frac{\varepsilon^{\frac{2}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1 \le p < 2; \\ C\frac{\varepsilon^{\frac{2}{p}}}{|\log\varepsilon|^{p-1}} e^{\gamma^p \frac{O(1)}{|\log\varepsilon|^p}} & \text{if } 0 < p < 1. \end{cases} \\ &= \begin{cases} C\frac{\varepsilon^{\frac{2}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1 \le p < 2; \\ C\frac{\varepsilon^{\frac{2}{p}}}{|\log\varepsilon|^{p-1}} e^{\frac{O(1)}{|\log\varepsilon|^{p-1}}} & \text{if } 0 < p < 1. \end{cases} \end{aligned}$$

Hence if we are far away from the points ξ_j, or equivalently for $|y-\xi_j'| > \frac{\delta}{\varepsilon}$, the size of the error, measured with respect to the $\|\cdot\|_{*,\partial\Omega_\varepsilon}$-norm, is relatively small. Namely, if we denote by 1_{outer} the characteristic function of the set $\{y : |y-\xi_j'| > \frac{\delta}{\varepsilon},\ j=1,\dots,k\}$, then in this region we have

$$
\begin{aligned}
\|E_\lambda 1_{\mathsf{outer}}\|_{*,\partial\Omega_\varepsilon} &\le \begin{cases} C\dfrac{\varepsilon^{\frac{2-p}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2;\\ C\dfrac{\varepsilon^{\frac{2-p}{2p}}}{|\log\varepsilon|^{p-1}}e^{\log\varepsilon^{\frac{2-p}{2p}}+\frac{C}{|\log\varepsilon|^{p-1}}} & \text{if } 0<p<1.\end{cases}\\
&= \begin{cases} C\dfrac{\varepsilon^{\frac{2-p}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2;\\ C\dfrac{\varepsilon^{\frac{2-p}{2p}}}{|\log\varepsilon|^{p-1}}e^{-\frac{2-p}{2p}|\log\varepsilon|+C|\log\varepsilon|^{1-p}} & \text{if } 0<p<1.\end{cases}\\
&\le \begin{cases} C\dfrac{\varepsilon^{\frac{2-p}{p}}}{|\log\varepsilon|^{p-1}} & \text{if } 1\le p<2;\\ C\dfrac{\varepsilon^{\frac{2-p}{2p}}}{|\log\varepsilon|^{p-1}} & \text{if } 0<p<1.\end{cases}
\end{aligned}
\tag{5.44}
$$

Here we used that $-\frac{2-p}{2p}|\log\varepsilon| + C|\log\varepsilon|^{1-p} < 0$ for $0<p<1$ and ε small. Let us now fix the index j in $\{1,\dots,k\}$, for $|y-\xi_j'| < \frac{\delta}{\varepsilon}$, we have

$$
\begin{aligned}
\frac{\partial V_\lambda}{\partial\nu} &= e^{\tilde w_j(y)} + \frac{p-1}{p}\frac{1}{\gamma^p}\frac{\partial \tilde w_{1j}(x)}{\partial\nu} + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\frac{\partial\tilde w_{2j}(x)}{\partial\nu} + O(\varepsilon^2)\\
&= e^{\tilde w_j(y)} + \frac{p-1}{p}\frac{1}{\gamma^p}\left(\frac{\partial\tilde\phi_{1j}(y)}{\partial\nu} + \alpha_{1j}e^{\tilde w_j}\right)\\
&\quad + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left(\frac{\partial\tilde\phi_{2j}(y)}{\partial\nu} + \alpha_{2j}e^{\tilde w_j}\right) + O(\varepsilon^2).
\end{aligned}
$$

On the other hand, for any $R>0$ large but fixed, in the ball $|y-\xi_j'| < R_\varepsilon := R|\log\varepsilon|^\alpha$, with $\alpha\ge 3$, we can use Taylor expansion to obtain

$$
\begin{aligned}
&f(V_\lambda)\\
=&\left(1+\frac{1}{p\gamma^p}\left(\tilde w_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\tilde w_{1j}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde w_{2j}(\varepsilon y) + \theta(y)\right)\right)^{p-1}\\
&\times e^{\gamma^p\left[\left(1+\frac{1}{p\gamma^p}\left(\tilde w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}\tilde w_{1j}(\varepsilon y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde w_{2j}(\varepsilon y)+\theta(y)\right)\right)^p-1\right]}\\
=&\left(1+\frac{p-1}{p}\frac{1}{\gamma^p}\tilde w_j(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde w_{1j}(\varepsilon y) + \left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}\tilde w_{2j}(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p}\theta(y)\right)\\
&\times e^{\tilde w_j(y)}e^{\frac{p-1}{p}\frac{1}{\gamma^p}\tilde w_{1j}(\varepsilon y)}e^{\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde w_{2j}(\varepsilon y)}e^{\theta(y)}e^{\frac12\frac{p-1}{p}\frac{1}{\gamma^p}\left[\tilde w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}\tilde w_{1j}(\varepsilon y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde w_{2j}(\varepsilon y)+\theta(y)\right]^2}\\
=&\ e^{\tilde w_j(y)} + \frac{p-1}{p}\frac{1}{\gamma^p}\left\{e^{\tilde w_j}\tilde w_{1j}(\varepsilon y) + e^{\tilde w_j}\left[\tilde w_j + \frac12(\tilde w_j)^2\right]\right\} + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\\
&\times\left\{e^{\tilde w_j}\tilde w_{2j}(\varepsilon y) + e^{\tilde w_j}\left[\tilde w_{1j} + \frac{p-2}{2(p-1)}(\tilde w_j)^2 + \frac12(\tilde w_{1j})^2 + \frac18(\tilde w_j)^4 + \tilde w_j\tilde w_{1j} + \frac12(\tilde w_j)^3 + \frac12\tilde w_{1j}(\tilde w_j)^2\right]\right\}
\end{aligned}
$$

$$+\frac{p-1}{p}\frac{1}{\gamma^p}e^{\tilde{w}_j}\theta(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}e^{\tilde{w}_j}\left[\tilde{w}_j+\tilde{w}_{1j}(\varepsilon y)\right]\theta(y)+\left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}\Big\{e^{\tilde{w}_j}\tilde{w}_{2j}(\varepsilon y)$$
$$+e^{\tilde{w}_j}\Big[\frac{1}{6}(\tilde{w}_{1j})^3(\varepsilon y)+\frac{1}{48}(\tilde{w}_j)^6+\tilde{w}_j\tilde{w}_{1j}(\varepsilon y)+\frac{1}{2}(\tilde{w}_{1j})^2(\varepsilon y)+2\tilde{w}_j\tilde{w}_{2j}(\varepsilon y)+\frac{1}{2}\tilde{w}_{1j}(\varepsilon y)(\tilde{w}_j)^3$$
$$+\frac{2p-3}{2(p-1)}\tilde{w}_j(\tilde{w}_{1j})^2(\varepsilon y)+\frac{1}{8}(\tilde{w}_j)^5+(\tilde{w}_{1j})^2(\varepsilon y)+\frac{1}{2}\tilde{w}_{1j}(\varepsilon y)(\tilde{w}_j)^2+\frac{p-2}{4(p-1)}(\tilde{w}_j)^4$$
$$+\tilde{w}_{1j}(\varepsilon y)\tilde{w}_{2j}(\varepsilon y)+\frac{1}{8}\tilde{w}_{1j}(\varepsilon y)(\tilde{w}_j)^4+\frac{1}{4}(\tilde{w}_j\tilde{w}_{1j}(\varepsilon y))^2+\frac{1}{2}\tilde{w}_{2j}(\varepsilon y)(\tilde{w}_j)^2\Big]\Big\}+O\left(\frac{\log|y-\xi_j'|}{\gamma^{3p}}\right).$$

Thus, thanks to the fact that we have improved our original approximation with the terms $\tilde{w}_{1j}(\varepsilon y)$ and $\tilde{w}_{2j}(\varepsilon y)$, and the definition of $*$-norm, we get that

$$\|E_\lambda 1_{B(\xi_j',R_\varepsilon)}\|_{*,\partial\Omega_\varepsilon}\le\frac{C}{\gamma^{3p}}=\frac{C}{|\log\varepsilon|^3}. \tag{5.45}$$

Here $1_{B(\xi_j',R_\varepsilon)}$ denotes the characteristic function of $B(\xi_j',R_\varepsilon)$. Finally, in the remaining region, namely where $R_\varepsilon<|y-\xi_j'|<\frac{\delta}{\varepsilon}$, for any $j=1,\dots,k$, we have from one hand that $|\frac{\partial V_\lambda(y)}{\partial\nu}|\le Ce^{\tilde{w}_j(y)}$, and also $|f(V_\lambda(y))|\le Ce^{\tilde{w}_j(y)}$ as consequence of (5.32). This fact, together with (5.44) and (5.45) we obtain estimate (5.43). □

As the proof of (2.34), (2.35) and (2.36), we have the following two Lemmas.

Lemma 5.8. *For very close to the point ξ_j on $\partial\Omega$, we have*

$$f'(V_\lambda)\approx e^{\tilde{w}_j}\qquad \text{as } \lambda\to 0, \tag{5.46}$$

and there exists some positive constant D_0 such that

$$f'(V_\lambda)\le D_0\sum_{j=1}^k e^{\tilde{w}_j}. \tag{5.47}$$

Lemma 5.9. *We have*

$$\|f''(V_\lambda)\|_{*,\partial\Omega_\varepsilon}\le C \tag{5.48}$$

for some positive constant.

5.3 The linearized problem

In this section, we prove the bounded invertibility of the operator L. First of all, we will solve the following linear problem. Given $h\in C(\partial\Omega_\varepsilon)$, find a function ϕ such that

$$\begin{cases}-\Delta\phi+\varepsilon^2\phi=0 & \text{in } \Omega_\varepsilon;\\ L(\phi)=h+\sum_{j=1}^k c_j\chi_j Z_{1j} & \text{on } \partial\Omega_\varepsilon;\\ \int_{\Omega_\varepsilon}\chi_j Z_{1j}\phi=0 & \text{for } j=1,\dots,k,\end{cases} \tag{5.49}$$

for certain scalars c_j, where operator L defined as (5.39), and Z_{1j}, χ_j are defined as follows: Define

$$F_j^\varepsilon(y) = \frac{1}{\varepsilon} F_j(\varepsilon y), \tag{5.50}$$

with F_j is given by (5.26). Set

$$Z_{ij}(y) = z_{ij}(F_j^\varepsilon(y)), \quad i = 0, 1, \quad j = 1, \ldots, k.$$

with z_{0j} and z_{1j} defined as (5.16) and (5.17).

Next, let us consider a large but fixed number $R_0 > 0$ and a nonnegative radial and smooth cut-off function χ with $\chi(r) = 1$ if $r < R_0$ and $\chi(r) = 0$ if $r > R_0 + 1$, $0 \leq \chi \leq 1$. Then set

$$\chi_j(y) = \chi(|F_j^\varepsilon(y)|).$$

Equation (5.49) is solved in the following Proposition.

Proposition 5.10. *Let $\delta > 0$ be a small but fixed number, assume (5.2) and (5.3), and μ_j is given by (5.36). Then there exist positive numbers λ_0 and C, such that problem (5.49) has a unique solution $\phi = T_\lambda(h)$ which satisfies*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \left(\log \frac{1}{\varepsilon}\right) \|h\|_{*, \partial\Omega_\varepsilon}, \tag{5.51}$$

for all $\lambda < \lambda_0$.

We carry the proof out in the following steps.

Step 1: Constructing a suitable barrier.

Lemma 5.11. *There exist positive constants R_1 and C, independent of λ, such that if λ small enough, there exists $\psi : \Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j') \to \mathbb{R}$, smooth and positive, satisfies*

$$\begin{cases} -\Delta\psi + \varepsilon^2\psi \geq \sum\limits_{j=1}^k \frac{1}{|y-\xi_j'|^{2+\sigma}} + \varepsilon^2 & \text{in } \Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j'); \\ \frac{\partial\psi}{\partial\nu} - f'(V_\lambda)\psi \geq \sum\limits_{j=1}^k \frac{1}{|y-\xi_j'|^{1+\sigma}} + \varepsilon & \text{on } \partial\Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j'); \\ \psi \geq 1 & \text{on } \Omega_\varepsilon \cap (\cup_{j=1}^k \partial B_{R_1}(\xi_j')). \end{cases}$$

Moreover, we have a uniform bound

$$0 < \psi \leq C \quad \text{in } \Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j').$$

Proof. Let $\eta_j \in C_0^\infty(\mathbb{R}^2)$ be such that $0 \le \eta_j \le 1$, $\eta_j \equiv 1$ in $\Omega_\varepsilon \cap B_{\delta/2\varepsilon}(\xi_j')$, $\eta_j \equiv 0$ in $\Omega_\varepsilon \backslash B_{\delta/\varepsilon}(\xi_j')$, $|\nabla \eta_j| \le C\varepsilon$ in Ω_ε, $|\Delta \eta_j| \le C\varepsilon^2$ in Ω. Let $\psi_0(y) = \tilde{\psi}(\varepsilon y)$, where $\tilde{\psi}$ is the solution to

$$\begin{cases} -\Delta \tilde{\psi} + \tilde{\psi} = 1 & \text{in } \Omega; \\ \frac{\partial \tilde{\psi}}{\partial \nu} = 1 & \text{on } \partial\Omega, \end{cases}$$

so that

$$-\Delta \psi_0 + \varepsilon^2 \psi = \varepsilon^2 \quad \text{in } \Omega_\varepsilon, \quad \text{and} \quad \frac{\partial \psi_0}{\partial \nu} = \varepsilon \quad \text{on } \partial\Omega_\varepsilon.$$

In particular, ψ_0 is uniformly bounded in Ω_ε. Take the function

$$\psi = \sum_{j=1}^{k} \eta_j \left[\frac{(y - \xi_j') \cdot \nu(\xi_j')}{r^{1+\sigma}} + C \frac{1}{r^\sigma} \right] + C\psi_0,$$

where $r = |y - \xi_j' - \mu_j \nu(\xi_j')|$. It is directly checked that ψ satisfies the required condition. □

Step 2: Transferring a linear equation. We study first the linear equation

$$\begin{cases} -\Delta \phi + \varepsilon^2 \phi = h_1 & \text{in } \Omega_\varepsilon; \\ \frac{\partial \phi}{\partial \nu} - f'(V_\lambda)\phi = h & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{5.52}$$

where h_1, h are in suitable weight spaces: we consider for h the norm defined in (5.42) and for h_1

$$\|h_1\|_{**,\Omega_\varepsilon} := \sup_{y \in \Omega_\varepsilon} \left(\sum_{j=1}^{k} (1 + |y - \xi_j'|)^{-2-\sigma} + \varepsilon^2 \right)^{-1} |h_1(y)|. \tag{5.53}$$

For the solution of (5.52) under some orthogonality conditions, we have an a priori estimate.

Lemma 5.12. *There are $R_0 > 0$ and $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ and any solution of (5.52) with the orthogonality conditions*

$$\int_{\Omega_\varepsilon} \chi_j Z_{ij} \phi = 0 \quad \forall\ i = 0, 1; \quad j = 1, \dots, k, \tag{5.54}$$

we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C \left(\|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon} \right) \tag{5.55}$$

where C is independent of λ.

Proof. We take $R_0 = 2R_1$, with R_1 being the constant of Lemma 5.11. Thanks to the barrier ψ of that Lemma we deduce the following maximum principle holds in $\Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j')$: if $\phi \in H^1(\Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j'))$ satisfies

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi \geq 0 & \text{in } \Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j'); \\ \frac{\partial\phi}{\partial\nu} - f'(V_\lambda)\phi \geq 0 & \text{on } \partial\Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j'); \\ \phi \geq 0 & \text{on } \Omega_\varepsilon \cap (\cup_{j=1}^k \partial B_{R_1}(\xi_j')), \end{cases}$$

then $\phi \geq 0$ in $\Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j')$.

Let h_1, h be bounded and ϕ a solution to (5.52) satisfying (5.54). Define the inner norm of ϕ as

$$\|\phi\|_i = \sup_{\Omega_\varepsilon \cap (\cup_{j=1}^k B_{R_1}(\xi_j'))} |\phi|,$$

and set

$$\tilde{\phi} = C_1\psi \left(\|\phi\|_i + \|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon}\right)$$

with C_1 a constant independent of λ, and ψ is the function given in Lemma 5.11. By the above maximum principle we deduce that $\phi \leq \tilde{\phi}$ and $-\phi \leq \tilde{\phi}$ in $\Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j')$. Since ψ is uniformly bounded, then we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \left(\|\phi\|_i + \|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon}\right) \tag{5.56}$$

for some constant C independent of ϕ and λ.

We prove the Lemma by contradiction. Assume that there exist a sequence $\lambda_n \to 0$, and points $\xi_1^n, \dots, \xi_k^n$ on $\partial\Omega$ satisfies (5.2) and functions ϕ_n, f_n and h_n with $\|\phi_n\|_{L^\infty(\Omega_{\varepsilon_n})} = 1$, $\|h_1\|_{**,\Omega_{\varepsilon_n}} \to 0$, $\|h\|_{*,\partial\Omega_{\varepsilon_n}} \to 0$, such that for each n, ϕ_n solves (5.52) satisfying (5.54). By (5.56) we see that $\|\phi_n\|_i$ stays away from zero. For of the indices, say j, we can assume that $\sup_{B_{R_1}(\xi_j')} |\phi_n| \geq c > 0$ for all n. Consider $\hat{\phi}_n = \phi_n(z - \xi_j')$, and let us translate and rotate Ω_{ε_n} such that Ω_{ε_n} approaches the upper half-plane and $\xi_j' = 0$. Then by elliptic estimate $\hat{\phi}_n$ converges uniformly on compact sets to a nontrivial solution $\hat{\phi}$ of (5.15). By Proposition 5.4 $\hat{\phi}$ is a linear combination of z_{0j} and z_{1j}. On the other hand, we can take the limit in the orthogonality relation (5.54), we find that $\int_{\mathbb{R}_+^2} \chi\hat{\phi}z_{ij} = 0$ for $i = 0, 1$. This contradicts the fact that $\hat{\phi} \not\equiv 0$. □

Step 3: Establishing an a priori estimate. In what follows, we will establish an a priori estimate for solution to (5.52) with the orthogonality condition $\int_{\Omega_\varepsilon} \chi_j Z_{1j}\phi = 0$ only.

Lemma 5.13. *For λ small enough, if ϕ is a solution of (5.52) and satisfies*

$$\int_{\Omega_\varepsilon} \chi_j Z_{1j}\phi = 0 \quad \forall\ j = 1, \dots, k, \tag{5.57}$$

then there holds

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C|\log\varepsilon| \left(\|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon}\right), \tag{5.58}$$

where C is independent of λ.

Proof. Let ϕ satisfies (5.52) and (5.57). In order to use Lemma 5.12, we will modify ϕ to $\tilde{\phi}$ so that satisfy all orthogonality with respect to Z_{ij} for $i = 0, 1$. Let $R > R_0 + 1$ be large but fixed, $\delta > 0$ be small and fixed. Set

$$\hat{Z}_{0j}(y) = \psi Z_{0j}(y),$$

where

$$\psi(y) = \tilde{h}(|F_j^\varepsilon(y)|), \qquad \tilde{h}(x) = \frac{\log(\delta/\varepsilon) - \log|x|}{\log(\delta/\varepsilon) - \log R}$$

with F_j^ε is the change of variables defined in (5.50). We observes that $\tilde{h}$ is just the solution to

$$\begin{cases} \Delta\tilde{h} = 0 & \text{in } B_{\delta/\varepsilon}(0)\backslash\bar{B}_R(0); \\ \tilde{h} = 1 & |x| = R; \\ \tilde{h} = 0 & |x| = \delta/\varepsilon. \end{cases}$$

Let $\bar{\eta}_{1j}$, $\bar{\eta}_{2j}$ be radial smooth cut-off functions on $\mathbb{R}^2$ such that

$$\begin{aligned} &0 \le \bar{\eta}_{1j} \le 1, \ |\nabla\bar{\eta}_{1j}| \le C \quad \text{in } \mathbb{R}^2, \\ &\bar{\eta}_{1j} \equiv 1 \quad \text{in } B_R(0), \quad \bar{\eta}_{1j} = 0 \quad \text{in } \mathbb{R}^2\backslash B_{R+1}(0), \end{aligned}$$

and

$$\begin{aligned} &0 \le \bar{\eta}_{2j} \le 1, \ |\nabla\bar{\eta}_{2j}| \le C\varepsilon/\delta, \ |\nabla^2\bar{\eta}_{2j}| \le C\varepsilon^2/\delta^2 \quad \text{in } \mathbb{R}^2, \\ &\bar{\eta}_{2j} \equiv 1 \quad \text{in } B_{\frac{\delta}{4\varepsilon}}(0), \quad \bar{\eta}_{2j} = 0 \quad \text{in } \mathbb{R}^2\backslash B_{\frac{\delta}{3\varepsilon}}(0). \end{aligned}$$

Now, we write

$$\eta_{1j}(y) = \bar{\eta}_{1j}(F_j^\varepsilon(y)), \quad \eta_{2j}(y) = \bar{\eta}_{2j}(F_j^\varepsilon(y)). \tag{5.59}$$

Define

$$\tilde{Z}_{0j} = \eta_{1j}Z_{0j} + (1 - \eta_{1j})\eta_{2j}\hat{Z}_{0j}.$$

Given ϕ satisfying (5.52) and (5.57), we set

$$\tilde{\phi} = \phi + \sum_{j=1}^{k} d_j\tilde{Z}_{0j}, \qquad \text{where } d_j = -\frac{\int_{\Omega_\varepsilon}\chi_j Z_{0j}\phi}{\int_{\Omega_\varepsilon} Z_{0j}^2\chi_j}.$$

Therefore, our result is a direct consequence of the following claim.

Claim:

$$|d_j| \le C|\log\varepsilon|\left(\|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon}\right) \quad \forall\ j = 1,\dots,k. \tag{5.60}$$

First, using the notation $\widetilde{L} = -\Delta + \varepsilon^2 I$, we observe that $\tilde{\phi}$ satisfies

$$\begin{cases} \widetilde{L}(\tilde{\phi}) = h_1 + \sum\limits_{j=1}^{k} d_j \widetilde{L}(\tilde{Z}_{0j}) & \text{in } \Omega_\varepsilon; \\ \frac{\partial\tilde{\phi}}{\partial\nu} - f'(V_\lambda)\tilde{\phi} = h + \sum\limits_{j=1}^{k} d_j\left(\frac{\partial\tilde{Z}_{0j}}{\partial\nu} - f'(V_\lambda)\tilde{Z}_{0j}\right) & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{5.61}$$

Thus by Lemma 5.12, we have

$$\begin{aligned} \|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} \ \le\ & C\sum_{j=1}^{k}|d_j|\left(\left\|\frac{\partial\tilde{Z}_{0j}}{\partial\nu} - f'(V_\lambda)\tilde{Z}_{0j}\right\|_{*,\partial\Omega_\varepsilon} + \|\widetilde{L}(\tilde{Z}_{0j})\|_{**,\Omega_\varepsilon}\right) \\ & + C\|h\|_{*,\partial\Omega_\varepsilon} + C\|h_1\|_{**,\Omega_\varepsilon}. \end{aligned} \tag{5.62}$$

Multiplying the first equation in (5.61) by $\tilde{Z}_{0l}$, integrating by parts and using the second equation in (5.61), we find

$$\begin{aligned} & d_l\left[\int_{\Omega_\varepsilon}\widetilde{L}(\tilde{Z}_{0l})\tilde{Z}_{0l} + \int_{\partial\Omega_\varepsilon}\tilde{Z}_{0l}\left(\frac{\partial\tilde{Z}_{0l}}{\partial\nu} - f'(V_\lambda)\tilde{Z}_{0l}\right)\right] \\ =\ & -\int_{\partial\Omega_\varepsilon} h\tilde{Z}_{0l} - \int_{\Omega_\varepsilon} h_1\tilde{Z}_{0l} + \int_{\partial\Omega_\varepsilon}\tilde{\phi}\left(\frac{\partial\tilde{Z}_{0l}}{\partial\nu} - f'(V_\lambda)\tilde{Z}_{0l}\right) + \int_{\Omega_\varepsilon}\tilde{\phi}\widetilde{L}(\tilde{Z}_{0l}). \end{aligned} \tag{5.63}$$

Thus by (5.62), we deduced that

$$\begin{aligned} & d_l\left[\int_{\Omega_\varepsilon}\widetilde{L}(\tilde{Z}_{0l})\tilde{Z}_{0l} + \int_{\partial\Omega_\varepsilon}\tilde{Z}_{0l}\left(\frac{\partial\tilde{Z}_{0l}}{\partial\nu} - f'(V_\lambda)\tilde{Z}_{0l}\right)\right] \\ \le\ & C\|h\|_{*,\partial\Omega_\varepsilon} + C\|h_1\|_{**,\Omega_\varepsilon} + \|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)}\left\|\frac{\partial\tilde{Z}_{0l}}{\partial\nu} - f'(V_\lambda)\tilde{Z}_{0l}\right\|_{*,\partial\Omega_\varepsilon} \\ & + \|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)}\|\widetilde{L}(\tilde{Z}_{0l})\|_{**,\Omega_\varepsilon} \\ \le\ & C\left(\|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon}\right)\left(1 + \left\|\frac{\partial\tilde{Z}_{0l}}{\partial\nu} - f'(V_\lambda)\tilde{Z}_{0l}\right\|_{*,\partial\Omega_\varepsilon} + \|\widetilde{L}(\tilde{Z}_{0l})\|_{**,\Omega_\varepsilon}\right) \\ & + C\sum_{j=1}^{k}\left(\left\|\frac{\partial\tilde{Z}_{0j}}{\partial\nu} - f'(V_\lambda)\tilde{Z}_{0j}\right\|_{*,\partial\Omega_\varepsilon} + \|\widetilde{L}(\tilde{Z}_{0j})\|_{**,\Omega_\varepsilon}\right). \end{aligned} \tag{5.64}$$

To achieve the claim by proving the following estimates: for some constant $C > 0$ independent of λ,

$$\int_{\Omega_\varepsilon} \widetilde{L}(\tilde{Z}_{0j})\tilde{Z}_{0j} + \int_{\partial\Omega_\varepsilon} \tilde{Z}_{0j}\left(\frac{\partial \tilde{Z}_{0j}}{\partial\nu} - f'(V_\lambda)\tilde{Z}_{0j}\right) \geq \frac{C}{|\log\varepsilon|}, \tag{5.65}$$

$$\|\widetilde{L}(\tilde{Z}_{0j})\|_{**,\Omega_\varepsilon} \leq \frac{C}{|\log\varepsilon|}, \tag{5.66}$$

$$\left\|\frac{\partial \tilde{Z}_{0j}}{\partial\nu} - f'(V_\lambda)\tilde{Z}_{0j}\right\|_{*,\partial\Omega_\varepsilon} \leq \frac{C}{|\log\varepsilon|}. \tag{5.67}$$

In [27] it is showed that estimates (5.65), (5.66) and (5.67) hold. □

Step 4: In proving the solvability of (5.49), we may first solve the following problem: for given $h \in L^\infty(\Omega_\varepsilon)$ and find $\phi \in L^\infty(\Omega_\varepsilon)$ and $d_1, \dots, d_k \in \mathbb{R}$, such that

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi = \sum\limits_{j=1}^{k} d_j\chi_j Z_{1j} & \text{in } \Omega_\varepsilon; \\ \frac{\partial\phi}{\partial\nu} - f'(V_\lambda)\phi = h & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon}\chi_j Z_{1j}\phi = 0 & \text{for } \ j = 1, \dots, k, \end{cases} \tag{5.68}$$

First we prove that for any ϕ, $d_1, \dots, d_k$ solution to (5.68) the bound

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C|\log\varepsilon|\|h\|_{*,\partial\Omega_\varepsilon} \tag{5.69}$$

holds. In fact, by Lemma 5.13, we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C|\log\varepsilon|\left(\|h\|_{*,\partial\Omega_\varepsilon} + \sum_{j=1}^{k}|d_j|\right) \tag{5.70}$$

and therefore it is enough to prove that $|d_j| \leq C\|h\|_{*,\partial\Omega_\varepsilon}$.

Let η_{2j} be the cut-off function defined in (5.59), and multiply (5.68) by $\eta_{2l}Z_{1l}$. Integrating by parts we get

$$\begin{aligned} d_l\int_{\Omega_\varepsilon}\chi_l Z_{1l}^2 &= -\int_{\partial\Omega_\varepsilon} h\eta_{2l}Z_{1l} + \int_{\partial\Omega_\varepsilon}\phi\frac{\partial\eta_{2l}}{\partial\nu}Z_{1l} + \int_{\partial\Omega_\varepsilon}\phi\eta_{2l}Z_{1l}\left(\frac{\partial Z_{1l}}{\partial\nu} - f'(V_\lambda)Z_{1l}\right) \\ &\quad + \int_{\Omega_\varepsilon}\phi(-\Delta(\eta_{2l}Z_{1l}) + \varepsilon^2\eta_{2l}Z_{1l}). \end{aligned} \tag{5.71}$$

Since, $Z_{1l} = O(\frac{1}{1+r})$ and $\nabla \eta_{2l} = O(\varepsilon)$, we have

$$\left| \int_{\partial\Omega_\varepsilon} \phi \frac{\partial \eta_{2l}}{\partial \nu} Z_{1l} \right| \le C\varepsilon \log \frac{1}{\varepsilon}.$$

On the other hand, we can estimate that

$$\frac{\partial Z_{1l}}{\partial \nu} - f'(V_\lambda) Z_{1l} = O\left(\frac{\varepsilon}{1+r}\right) + O\left(\frac{\varepsilon^\alpha}{1+r^2}\right), \quad |y| < \frac{\delta}{\varepsilon}, \quad y \in \partial\Omega_\varepsilon,$$

and which implies that

$$\int_{\partial\Omega_\varepsilon} \left| \frac{\partial Z_{1l}}{\partial \nu} - f'(V_\lambda) Z_{1l} \right| = O(\varepsilon^\alpha). \tag{5.72}$$

Moreover, we implies that

$$\int_{\partial\Omega_\varepsilon} \left| -\Delta(\eta_{2l} Z_{1l}) + \varepsilon^2 \eta_{2l} Z_{1l} \right| = O\left(\varepsilon \log \frac{1}{\varepsilon}\right). \tag{5.73}$$

Thus from (5.71)-(5.73), we conclude that

$$d_l \int_{\Omega_\varepsilon} \chi_l Z_{1l}^2 \le C\|h\|_{*,\partial\Omega_\varepsilon} + C\varepsilon^\alpha \|\phi\|_{L^\infty(\Omega_\varepsilon)}. \tag{5.74}$$

Combing (5.70) and (5.74) we have

$$|d_l| \le C\left(\|h\|_{*,\partial\Omega_\varepsilon} + C\varepsilon^\alpha \log \frac{1}{\varepsilon^2} \sum_{j=1}^{k} |d_j| \right).$$

This implies that

$$|d_l| \le C\|h\|_{*,\partial\Omega_\varepsilon} \tag{5.75}$$

which proves (5.69).

Now consider the Hilbert space

$$\mathbb{H} = \left\{ \phi \in H^1(\Omega_\varepsilon) \ : \ \int_{\Omega_\varepsilon} \chi_j Z_{1j} \phi = 0 \quad \forall\ j = 1, \dots, k \right\},$$

endowed the norm $\|\phi\|_{H^1}^2 = \int_{\Omega_\varepsilon} |\nabla\phi|^2 + \varepsilon^2 \phi^2$. Problem (5.68), expressed in a weak form, is equivalent to find $\phi \in \mathbb{H}$ such that

$$\int_{\Omega_\varepsilon} \nabla\phi\nabla\psi + \varepsilon^2 \phi\psi - \int_{\partial\Omega_\varepsilon} f'(V_\lambda)\psi = \int_{\partial\Omega_\varepsilon} h\psi, \qquad \text{for all } \psi \in \mathbb{H},$$

With the aid of Fredholm's alternative guarantees unique solvability of (5.68), which is guarantees by (5.69).

Step 5: In order to solve (5.49), let $Y_i \in L^\infty(\Omega_\varepsilon)$, $d_{ij} \in \mathbb{R}$ be the solution of (5.68) with $h = \chi_i Z_{1i}$, that is

$$\begin{cases} -\Delta Y_i + \varepsilon^2 Y_i = \sum_{j=1}^{k} d_j \chi_j Z_{1j} & \text{in } \Omega_\varepsilon; \\ \frac{\partial Y_i}{\partial \nu} - f'(V_\lambda) Y_i = \chi_i Z_{1i} & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \chi_j Z_{1j} Y_i = 0 & \text{for } \ j = 1, \dots, k, \end{cases} \tag{5.76}$$

From Step 4, there is a unique solution $Y_i \in L^\infty(\Omega_\varepsilon)$ of (5.76), and

$$\|Y_i\|_{L^\infty(\Omega_\varepsilon)} \leq C|\log \varepsilon|, \qquad |d_{ij}| \leq C \tag{5.77}$$

for some constant C independent on λ.

Multiplying (5.76) by $\eta_{2j} Z_{1j}$, and integrates by parts, we have

$$d_{ij} \int_{\Omega_\varepsilon} \chi_j Z_{1j}^2 + \delta_{ij} \int_{\partial\Omega_\varepsilon} \chi_j Z_{1j}^2 \;\; = \;\; \int_{\partial\Omega_\varepsilon} \left(\frac{\partial Z_{1j}}{\partial \nu} - f'(V_\lambda) Z_{1j} \right) \eta_{2j} Y_i + \int_{\partial\Omega_\varepsilon} \frac{\partial \eta_{2j}}{\partial \nu} Z_{1j} Y_i,$$

where δ_{ij} is Kronecker's delta. From (5.72), (5.73) and (5.75) we obtain

$$d_{ij} \int_{\Omega_\varepsilon} \chi_j Z_{1j}^2 + \delta_{ij} \int_{\partial\Omega_\varepsilon} \chi_j Z_{1j}^2 = O\left(\varepsilon^\alpha \log \frac{1}{\varepsilon} \right)$$

Then we get

$$d_{ij} = A\delta_{ij} + O\left(\varepsilon^\alpha \log \frac{1}{\varepsilon} \right) \tag{5.78}$$

with $A > 0$ is independent of ε. Hence the matrix D with entries d_{ij} in invertible for small ε and $\|D^{-1}\| \leq C$ uniformly in ε. Then, given $h \in L^\infty(\partial\Omega_\varepsilon)$ we find ϕ_1, $d_1, \dots, d_k$, the solution to (5.76) and define

$$\phi = \phi_1 + \sum_{i=1}^{k} c_i Y_i,$$

where c_i satisfies

$$\sum_{i=1}^{k} c_i d_{ij} = -d_j, \quad \forall \ j = 1, \dots, k.$$

Then ϕ satisfies (5.49) and we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \;\; \leq \;\; \|\phi_1\|_{L^\infty(\Omega_\varepsilon)} + \log \frac{1}{\varepsilon} \sum_{i=1}^{k} |c_i|$$

$$
\begin{aligned}
&\leq \quad C \log \frac{1}{\varepsilon} \|h\|_{*,\partial\Omega_\varepsilon} + \log \frac{1}{\varepsilon} \sum_{i=1}^{k} |d_i| \\
&\leq \quad C \log \frac{1}{\varepsilon} \|h\|_{*,\partial\Omega_\varepsilon}
\end{aligned}
$$

by (5.75). This finishes the proof of Proposition 5.10.

Remark 5.14. *A slight modification of the proof above also shows that for any $h \in L^\infty(\partial\Omega_\varepsilon)$ and $h_1 \in L^\infty(\Omega_\varepsilon)$, the equation*

$$
\begin{cases}
-\Delta\phi + \varepsilon^2\phi = h_1 & \textit{in } \Omega_\varepsilon; \\
L(\phi) = h + \sum\limits_{j=1}^{k} c_j \chi_j Z_{1j} & \textit{on } \partial\Omega_\varepsilon; \\
\int_{\Omega_\varepsilon} \chi_j Z_{1j} \phi = 0 & \text{for} \quad j = 1, \ldots, k,
\end{cases}
$$

has a unique solution ϕ, $c_1, \ldots, c_k$ and that satisfy

$$
\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \log \frac{1}{\varepsilon} \left(\|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon} \right),
$$

$$
|c_j| \leq C \left(\|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon} \right), \quad \forall\, j = 1, \ldots, k
$$

holds for C independent of λ.

The result of Proposition 5.10 implies that the unique solution $\phi = T_\lambda(h)$ of (5.49) defines a continuous linear map form the Banach space $\mathcal{C}_*$ of all functions h in L^∞ for which $\|h\|_* < \infty$ into L^∞, with norm bounded uniformly in λ.

Lemma 5.15. *The operator T_λ is differentiable with respect to the variable $\xi_1, \ldots, \xi_k$ on $\partial\Omega$ satisfying 5.2, one has the estimate*

$$
\|\partial_{\xi_l'} T_\lambda(h)\|_{L^\infty(\Omega_\varepsilon)} \leq C \left(\log \frac{1}{\varepsilon} \right)^2 \|h\|_{*,\partial\Omega_\varepsilon} \quad \text{for} \quad l = 1, \ldots, k, \tag{5.79}
$$

for a given positive C, independent of λ, and for all λ small enough.

Proof. Differentiating equation (5.49), formally $Z := \partial_{\xi_l'}\phi$ should satisfy in Ω_ε the equation

$$
-\Delta Z + \varepsilon^2 Z = 0 \quad \text{in } \Omega_\varepsilon,
$$

and on the boundary $\partial\Omega_\varepsilon$

$$
L(Z) = -\partial_{\xi_l'}(f'(V_\lambda))\phi + c_l \partial_{\xi_l'}(\chi_l Z_{1l}) + \sum_{j=1}^{k} d_j Z_{1j} \chi_j
$$

with $d_j = \partial_{\xi_l'} c_j$, and the orthogonality conditions now become

$$\int_{\Omega_\varepsilon} Z_{1j}\chi_j Z = 0 \quad \text{if } j \neq l.$$

$$\int_{\Omega_\varepsilon} Z_{1l}\chi_l Z = -\int_{\Omega_\varepsilon} \partial_{\xi_l'}\left(Z_{1l}\chi_l\right)\phi.$$

We consider the constants b_l defined as

$$b_l \int_{\Omega_\varepsilon} \chi_l^2 |Z_{1l}^2 = \int_{\Omega_\varepsilon} \partial_{\xi_l'}\left(Z_{1l}\chi_l\right)\phi, \quad \text{for } \ l = 1, \dots, k.$$

Define

$$\tilde{Z} = Z + b_l \chi_l Z_{1l}.$$

We then have

$$\begin{cases} -\Delta \tilde{Z} + \varepsilon^2 \tilde{Z} = a & \text{in } \Omega_\varepsilon; \\ L(\tilde{Z}) = b + \sum_{j=1}^{k} d_j Z_{1j}\chi_j & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \chi_j Z_{1j} \tilde{Z} = 0 & \text{for } j = 1, \dots, k, \end{cases}$$

where

$$a = b_l(-\Delta(\chi_l Z_{1l}) + \varepsilon^2 \chi_l Z_{1l}),$$
$$b = -\partial_{\xi_l'}\left(f'(V_\lambda)\right)\phi + c_l \partial_{\xi_l'}\left(Z_{1l}\chi_l\right) + L(\chi_l Z_{1l}),$$

and we have

$$\|a\|_{**,\Omega_\varepsilon} \leq C \log\frac{1}{\varepsilon}\|h\|_{*,\partial\Omega_\varepsilon}, \qquad \|b\|_{*,\partial\Omega_\varepsilon} \leq C \log\frac{1}{\varepsilon}\|h\|_{*,\partial\Omega_\varepsilon}.$$

Hence, using the result of Proposition 5.10 we obtain that

$$\|\partial_{\xi_l'} T_\lambda(h)\|_{L^\infty(\Omega_\varepsilon)} \leq C\left(\log\frac{1}{\varepsilon}\right)^2 \|h\|_{*,\partial\Omega_\varepsilon} \quad \text{for } \ l = 1, \dots, k.$$

□

5.4 The nonlinear problem

Let us now introduce the following auxiliary nonlinear problem

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi = 0 & \text{in } \Omega_\varepsilon; \\ L(\phi) = E_\lambda + N(\phi) + \sum_{j=1}^{k} c_j \chi_j Z_{1j} & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \chi_j Z_{1j}\phi = 0 & \text{for } \ j = 1, \dots, k, \end{cases} \tag{5.80}$$

Proposition 5.16. *Under the condition of Proposition 5.10, there exist positive numbers λ_0 and C, such that problem (5.80) has a unique solution ϕ which satisfies*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{C}{|\log \varepsilon|^2}, \tag{5.81}$$

for all $\lambda < \lambda_0$. Moreover, if we consider the map $\xi' \mapsto \phi$ into the space $C(\bar{\Omega}_\varepsilon)$, the derivative $D_{\xi'}\phi$ exists and defines a continuous function of ξ'. Besides, there is a constant $C > 0$, such that

$$\|D_{\xi'}\phi\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{C}{|\log \varepsilon|}. \tag{5.82}$$

Proof. In terms of the operator T_λ defined in Proposition 5.10, problem (5.80) becomes

$$\phi = T_\lambda \left(N(\phi) + E_\lambda\right) := A(\phi). \tag{5.83}$$

For a given number $M > 0$, let us consider the region

$$\mathcal{F}_M := \left\{ \phi \in C(\bar{\Omega}_\varepsilon) \;:\; \|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{M}{|\log \varepsilon|^2} \right\}.$$

From Proposition 5.10, we get

$$\|A(\phi)\|_{L^\infty(\Omega_\varepsilon)} \leq C \left(\log \frac{1}{\varepsilon}\right) \left[\|N(\phi)\|_{*,\partial\Omega_\varepsilon} + \|E_\lambda\|_{*,\partial\Omega_\varepsilon}\right].$$

From (5.43) and (5.48), by the definition of $N(\phi)$ in (5.41), we have

$$\|A(\phi)\|_{L^\infty(\Omega_\varepsilon)} \;\leq\; C|\log \varepsilon| \left(C\|\phi\|^2_{L^\infty(\Omega_\varepsilon)} + \frac{1}{|\log \varepsilon|^3} \right).$$

We then get that $A(\mathcal{F}_M) \subset \mathcal{F}_M$ for a sufficiently large but fixed M and all small λ. Moreover, for any $\phi_1, \phi_2 \in \mathcal{F}_M$, one has

$$\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega_\varepsilon} \leq C \left(\max_{i=1,2} \|\phi_i\|_{L^\infty(\Omega_\varepsilon)} \right) \|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)},$$

In fact,

$$\begin{aligned} N(\phi_1) - N(\phi_2) &= f(V_\lambda + \phi_1) - f(V_\lambda + \phi_2) - f'(V_\lambda)(\phi_1 - \phi_2) \\ &= \int_0^1 \left(\frac{d}{dt} f(V_\lambda + \phi_2 + t(\phi_1 - \phi_2) \right) dt - f'(V_\lambda)(\phi_1 - \phi_2) \\ &= \int_0^1 \left(f'(V_\lambda + \phi_2 + t(\phi_1 - \phi_2) - f'(V_\lambda)\right) dt\, (\phi_1 - \phi_2). \end{aligned}$$

Thus, for a certain $t^* \in (0,1)$, and $s \in (0,1)$

$$|N(\phi_1) - N(\phi_2)| \leq C|f'(V_\lambda + \phi_2 + t^*(\phi_1 - \phi_2) - f'(V_\lambda)| \|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)}$$

$$\leq C|f''(V_\lambda + s\phi_2 + t^*(\phi_1 - \phi_2))| \left(\|\phi_1\|_{L^\infty(\Omega_\varepsilon)} + \|\phi_2\|_{L^\infty(\Omega_\varepsilon)}\right) \|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)}.$$

Thanks to (5.48) and the fact that $\|\phi_1\|_{L^\infty(\Omega_\varepsilon)}$, $\|\phi_2\|_{L^\infty(\Omega_\varepsilon)} \to 0$ as $\lambda \to 0$, we conclude that

$$\begin{aligned}
& \|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega_\varepsilon} \\
\leq\ & C\|f''(V_\lambda)\|_{*,\partial\Omega_\varepsilon} \left(\|\phi_1\|_{L^\infty(\Omega_\varepsilon)} + \|\phi_2\|_{L^\infty(\Omega_\varepsilon)}\right) \|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)} \\
\leq\ & C(\|\phi_1\|_{L^\infty(\Omega_\varepsilon)} + \|\phi_2\|_{L^\infty(\Omega_\varepsilon)})\|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)}.
\end{aligned}$$

Then we have

$$\begin{aligned}
& \|A(\phi_1) - A(\phi_2)\|_{L^\infty(\Omega_\varepsilon)} \\
\leq\ & C|\log\varepsilon| \|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega_\varepsilon} \\
\leq\ & C|\log\varepsilon| \left(\max_{i=1,2} \|\phi_i\|_{L^\infty(\Omega_\varepsilon)}\right) \|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)}.
\end{aligned}$$

Thus the operator A has a small Lipschitz constant in $\mathcal{F}_M$ for all small λ, and therefore a unique fixed point of A exists in this region.

We shall next analyze the differentiability of the map $\xi' = (\xi_1', \ldots, \xi_k') \mapsto \phi$. Assume for instance that the partial derivative $\partial_{\xi_l'}\phi$ exists, for $l = 1, \ldots, k$. Since $\phi = T_\lambda\left(N(\phi) + E_\lambda\right)$, formally that

$$\partial_{\xi_l'}\phi = \left(\partial_{\xi_l'} T_\lambda\right)\left(N(\phi) + E_\lambda\right) + T_\lambda\left(\partial_{\xi_l'} N(\phi) + \partial_{\xi_l'} E_\lambda\right).$$

From (5.79), we have

$$\|\partial_{\xi_l'} T_\lambda\left(N(\phi) + E_\lambda\right)\|_{L^\infty(\Omega_\varepsilon)} \ \leq\ C|\log\varepsilon|^2 \|N(\phi) + E_\lambda\|_{*,\partial\Omega_\varepsilon} \leq \frac{C}{|\log\varepsilon|}.$$

On the other hand,

$$\begin{aligned}
\partial_{\xi_m'} N(\phi) \ =\ & [f'(V_\lambda + \phi) - f'(V_\lambda) - f''(V_\lambda)\phi]\partial_{\xi_l'} V_\lambda + \partial_{\xi_l'}[f'(V_\lambda) - e^{w_{\mu_j}}]\phi \\
& + [f'(V_\lambda + \phi) - f'(V_\lambda)]\partial_{\xi_l'}\phi + [f'(V_\lambda) - e^{w_{\mu_j}}]\partial_{\xi_l'}\phi.
\end{aligned}$$

Then,

$$\begin{aligned}
& \|\partial_{\xi_l'} N(\phi)\|_{*,\partial\Omega_\varepsilon} \\
\leq\ & C\left\{\|\phi\|^2_{L^\infty(\Omega_\varepsilon)} + \frac{1}{|\log\varepsilon|}\|\phi\|_{L^\infty(\Omega_\varepsilon)} + \|\partial_{\xi_l'}\phi\|_{L^\infty(\Omega_\varepsilon)}\|\phi\|_{L^\infty(\Omega_\varepsilon)} + \frac{1}{|\log\varepsilon|}\|\partial_{\xi_l'}\phi\|_{L^\infty(\Omega_\varepsilon)}\right\}.
\end{aligned}$$

Since $\|\partial_{\xi_l'} E_\lambda\|_{*,\partial\Omega_\varepsilon} \leq \frac{C}{|\log\varepsilon|^3}$, and by Proposition 5.10 we then have

$$\|\partial_{\xi_l'}\phi\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{C}{|\log\varepsilon|}$$

for all $l = 1, \ldots, k$. Then, the regularity of the map $\xi' \mapsto \phi$ can be proved by standard arguments involving the implicit function theorem and the fixed point representation (5.83). This concludes proof of the Proposition. □

5.5 Variational reduction

After problem (5.80) has been solved, in order to find a solution to the original problem we need to find ξ' such that

$$c_j(\xi') = 0 \quad \text{for all } \ j = 1, \ldots, k. \tag{5.84}$$

This problem is indeed variational: it is equivalent to finding critical points of a function of $\xi = \varepsilon\xi'$. Associated to (5.1), let us introduce the energy functional $J_\lambda : H^1(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2) - \frac{\lambda}{p}\int_{\partial\Omega} e^{u^p}, \tag{5.85}$$

and the finite-dimensional restriction

$$F_\lambda(\xi) = J_\lambda\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right), \tag{5.86}$$

where $\tilde{\phi} = \tilde{\phi}(\xi) = \tilde{\phi}(x,\xi)$ is the function defined in Ω from the relation $\tilde{\phi}(x,\xi) = \phi(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon})$, with ϕ is the unique solution to problem (5.80) given by Proposition 5.10.

Lemma 5.17. *The functional $F_\lambda(\xi)$ is of class C^1. Moreover, for all $\lambda > 0$ sufficiently small, if $D_\xi F_\lambda(\xi) = 0$, then ξ satisfies (5.84).*

Proof. A direct consequence of the results obtained in Proposition 5.16 and the definition of function U_λ is the fact the map $\xi \mapsto F_\lambda(\xi)$ is of class C^1. Define

$$I_\lambda(v) = \frac{1}{2}\int_{\Omega_\varepsilon} \left(|\nabla v|^2 + \varepsilon^2 v^2\right) - \int_{\partial\Omega_\varepsilon} e^{\gamma^p[(1+\frac{v}{p\gamma^p})^p - 1]}.$$

Let us differentiate the function $F_\lambda(\xi)$ with the respect to ξ. Since

$$J_\lambda\left(\left(U_\lambda + \tilde{\phi}\right)(x,\xi)\right) = \frac{1}{p^2\gamma^{2(p-1)}} I_\lambda\left(\left(V_\lambda + \phi\right)\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right)\right), \tag{5.87}$$

we can differentiate directly $I_\lambda\left(V_\lambda(\xi) + \phi(\xi)\right)$ under the integral sign, so that

$$\begin{aligned}
\partial_{\xi_l} F_\lambda(\xi) &= \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1} DI_\lambda\left(V_\lambda(\xi) + \phi(\xi)\right)\left[\partial_{\xi_l'} V_\lambda(\xi) + \partial_{\xi_l'}\phi(\xi)\right] \\
&= \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1}\sum_{j=1}^k \int_{\partial\Omega_\varepsilon} c_j\chi_j Z_{1j}\left[\partial_{\xi_l'} V_\lambda(\xi) + \partial_{\xi_l'}\phi(\xi)\right] \\
&= \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1}\left[\sum_{j=1}^k \int_{\partial\Omega_\varepsilon} c_j\chi_j Z_{1j}\partial_{\xi_l'} V_\lambda(\xi) + \sum_{j=1}^k \int_{\partial\Omega_\varepsilon} c_j\partial_{\xi_l'}\left(\chi_j Z_{1j}\right)\phi(\xi)\right],
\end{aligned}$$

since $\int_{\Omega_\varepsilon} \chi_j Z_{1j}\phi = 0$. By the expansion of V_λ, we have

$$\begin{aligned}
\partial_{\xi_l'} V_\lambda &= \partial_{\xi_l'}\left(\tilde{w}_l(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\tilde{w}_{1l}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde{w}_{2l}(\varepsilon y) + \theta(y)\right) \\
&= \partial_{\xi_l'}\tilde{w}_l(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\partial_{\xi_l'}\tilde{w}_{1l}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\partial_{\xi_l'}\tilde{w}_{2l}(\varepsilon y) + \partial_{\xi_l'}\theta(y) \\
&= -Z_{1l} + \frac{p-1}{p}\frac{1}{\gamma^p}\partial_{\xi_l'}\tilde{w}_{1l}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\partial_{\xi_l'}\tilde{w}_{2l}(\varepsilon y) + \partial_{\xi_l'}\theta(y).
\end{aligned}$$

Moreover,

$$\int_{\partial\Omega_\varepsilon} c_j \partial_{\xi_l'}\left(\chi_j Z_{1j}\right)\phi(\xi) = o(1)\int_{\partial\Omega_\varepsilon} c_j\chi_j Z_{1j}\partial_{\xi_l'}V_\lambda.$$

Then, if $D_\xi F_\lambda(\xi) = 0$, for $j = 1, 2, \ldots, k$, we then have

$$\sum_{j=1}^{k} c_j \int_{\partial\Omega_\varepsilon} \chi_j Z_{1j}(Z_{1l} + o(1)) = 0. \tag{5.88}$$

This is a strictly diagonal dominant system. It implies that $c_j = 0$ for $j = 1, \ldots, k$. This concludes the proof of the Lemma. □

Next, we will write the expansion of J_λ as λ goes to zero.

Lemma 5.18. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that μ_j are given by (5.36), the following expansion holds*

$$\lambda^{-1}\varepsilon^{\frac{2-p}{p}}F_\lambda(\xi) = -\frac{2k\pi}{p} + \frac{2k\pi}{p}\log\frac{1}{\varepsilon} + \frac{\pi}{2-p}\varphi_k(\xi) + O(|\log\varepsilon|^{-1}) \tag{5.89}$$

uniformly for any points ξ_j, $j = 1, \ldots, k$ on $\partial\Omega$, where

$$\varphi_k(\xi) = \varphi_k(\xi_1, \ldots, \xi_k) = -\left[\sum_{j=1}^{k} H(\xi_j, \xi_j) + \sum_{l\neq j} G(\xi_l, \xi_j)\right]. \tag{5.90}$$

Proof. We have

$$\begin{aligned}
F_\lambda(\xi) &= J_\lambda\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right) \\
&= \frac{1}{2}\int_\Omega \left[|\nabla\left(U_\lambda + \tilde{\phi}\right)|^2 + \left(U_\lambda + \tilde{\phi}\right)^2\right] - \frac{\lambda}{p}\int_{\partial\Omega} e^{\left(U_\lambda + \tilde{\phi}\right)^p}.
\end{aligned}$$

From (5.87) we have that

$$J_\lambda\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right) - J_\lambda\left(U_\lambda(\xi)\right) = \frac{1}{p^2\gamma^{2(p-1)}}\left[I_\lambda(V_\lambda + \phi) - I_\lambda(V_\lambda)\right].$$

Since by construction $I'_\lambda(V_\lambda+\phi)[\phi]=0$, we have

$$J_\lambda\left(U_\lambda(\xi)+\tilde\phi(\xi)\right)-J_\lambda\left(U_\lambda(\xi)\right)=\frac{1}{p^2\gamma^{2(p-1)}}\int_0^1 D^2I_\lambda(V_\lambda+t\phi)\phi^2(1-t)\,dt$$

$$= \frac{1}{p^2\gamma^{2(p-1)}}\int_0^1\left[\int_{\partial\Omega_\varepsilon}(E_\lambda+N(\phi))\phi+\int_{\partial\Omega_\varepsilon}[f'_\lambda(V_\lambda)-f'_\lambda(V_\lambda+t\phi)]\phi^2\right](1-t)\,dt$$

Since $\|E_\lambda\|_{*,\partial\Omega_\varepsilon}\le\frac{C}{|\log\varepsilon|^3}$, $\|\phi\|_{L^\infty(\Omega_\varepsilon)}\le\frac{C}{|\log\varepsilon|^2}$, $\|N(\phi)\|_{*,\partial\Omega_\varepsilon}\le\frac{C}{|\log\varepsilon|^4}$ and (5.48), we get that

$$\left|J_\lambda\left(U_\lambda(\xi)+\tilde\phi(\xi)\right)-J_\lambda\left(U_\lambda(\xi)\right)\right|\le\frac{C}{\gamma^{2(p-1)}|\log\varepsilon|^3} \tag{5.91}$$

Next we expand

$$J_\lambda\left(U_\lambda(\xi)\right)=\frac12\int_\Omega\left[|\nabla\left(U_\lambda(\xi)\right)|^2+U_\lambda(\xi)^2\right]-\frac{\lambda}{p}\int_{\partial\Omega}e^{(U_\lambda(\xi))^p}. \tag{5.92}$$

Now we write

$$U_j(x):=u_j(x)+H_j^\varepsilon(x),\qquad U_{1j}:=\tilde w_{1j}(x)+H_{1j}^\varepsilon(x),\qquad U_{2j}:=\tilde w_{2j}(x)+H_{2j}^\varepsilon(x)$$

By (5.27),

$$U_\lambda(x)=\frac{1}{p\gamma^{p-1}}\sum_{j=1}^k\left(U_j(x)+\frac{p-1}{p}\frac{1}{\gamma^p}U_{1j}(x)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}U_{2j}(x)\right)$$

Then we have

$$\begin{aligned}
&\frac12\int_\Omega\left[|\nabla\left(U_\lambda(\xi)\right)|^2+U_\lambda(\xi)^2\right]\\
=\;&\frac{1}{p^2\gamma^{2(p-1)}}\Big\{\frac12\sum_{j=1}^k\int_\Omega(|\nabla U_j|^2+U_j^2)+\sum_{l\neq j}\int_\Omega(\nabla U_l\nabla U_j+U_lU_j)\\
&+\frac{p-1}{p}\frac{1}{\gamma^p}\sum_{j=1}^k\int_\Omega(\nabla U_j\nabla U_{1j}+U_jU_{1j})+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\sum_{j=1}^k\int_\Omega(\nabla U_j\nabla U_{2j}+U_jU_{2j})\\
&+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left[\frac12\sum_{j=1}^k\int_\Omega(|\nabla U_{1j}|^2+U_{1j}^2)+\sum_{l\neq j}\int_\Omega(\nabla U_{1l}\nabla U_{1j}+U_{1l}U_{1j})\right]\\
&+\left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}\sum_{j=1}^k\int_\Omega(\nabla U_{1j}\nabla U_{2j}+U_{1j}U_{2j})\\
&+\left(\frac{p-1}{p}\right)^4\frac{1}{\gamma^{4p}}\left[\frac12\sum_{j=1}^k\int_\Omega(|\nabla U_{2j}|^2+U_{2j}^2)+\sum_{l\neq j}\int_\Omega(\nabla U_{2l}\nabla U_{2j}+U_{2l}U_{2j})\right]\Big\}.
\end{aligned} \tag{5.93}$$

Let us estimate the first two terms. We observe that the remaining terms are $O(\frac{1}{\gamma^{2(p-1)}\gamma^p})$. We have

$$\begin{aligned}\int_\Omega(|\nabla U_j|^2+U_j^2) &= \int_\Omega|\nabla u_j|^2+\int_\Omega u_j^2+\int_\Omega|\nabla H_j^\varepsilon|^2+\int_\Omega(H_j^\varepsilon)^2 \\ &+2\int_\Omega\nabla u_j\nabla H_j^\varepsilon+2\int_\Omega u_jH_j^\varepsilon. \end{aligned}\tag{5.94}$$

Multiplying (5.4) by H_j^ε, it yields

$$\begin{aligned}\int_\Omega|\nabla H_j^\varepsilon|^2+\int_\Omega(H_j^\varepsilon)^2 &= -\int_\Omega u_jH_j^\varepsilon+\int_{\partial\Omega}\frac{\partial H_j^\varepsilon}{\partial\nu}H_j^\varepsilon \\ &= -\int_\Omega u_jH_j^\varepsilon+\varepsilon\int_{\partial\Omega}e^{u_j}H_j^\varepsilon-\int_{\partial\Omega}\frac{\partial u_j}{\partial\nu}H_j^\varepsilon,\end{aligned}$$

Multiplying (5.4) by u_j again, we find

$$\int_\Omega u_j^2+\int_\Omega H_j^\varepsilon u_j = -\int_\Omega\nabla u_j\nabla H_j^\varepsilon+\varepsilon\int_{\partial\Omega}e^{u_j}u_j-\int_{\partial\Omega}\frac{\partial u_j}{\partial\nu}u_j,$$

Then we get

$$\begin{aligned}&\int_\Omega(|\nabla U_j|^2+U_j^2)\\ =&\int_\Omega|\nabla u_j|^2-\int_{\partial\Omega}\frac{\partial u_j}{\partial\nu}u_j+\int_\Omega\nabla u_j\nabla H_j^\varepsilon-\int_{\partial\Omega}\frac{\partial u_j}{\partial\nu}H_j^\varepsilon+\varepsilon\int_{\partial\Omega}e^{u_j}(u_j+H_j^\varepsilon)\\ =&\ \varepsilon\int_{\partial\Omega}e^{u_j}(u_j+H_j^\varepsilon)\\ =&\ \varepsilon\int_{\partial\Omega}\frac{2\mu_j}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}\left(\log\frac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}+H(x,\xi_j)+O(\varepsilon^\alpha)\right).\end{aligned}$$

Taking the changing variables $y=\frac{x-\xi_j}{\varepsilon\mu_j}$, we have

$$\begin{aligned}&\int_\Omega(|\nabla U_j|^2+U_j^2)\\ =&\int_{\partial\Omega_{\varepsilon\mu_j}}\frac{2}{|y-\nu(0)|^2}\left(\log\frac{1}{|y-\nu(0)|^2}+H(\xi_j+\varepsilon\mu_jy,\xi_j)-2\log(\mu_j\varepsilon)\right)+O(\varepsilon^\alpha)\end{aligned}\tag{5.95}$$

Since

$$\int_{\partial\Omega_{\varepsilon\mu_j}}\frac{1}{|y-\nu(0)|^2}=\pi+O(\varepsilon).$$

$$\int_{\partial\Omega_{\varepsilon\mu_j}}\frac{1}{|y-\nu(0)|^2}\log\frac{1}{|y-\nu(0)|^2} = \int_{-\infty}^{\infty}\frac{1}{1+t^2}\log\frac{1}{1+t^2}\ dt+O(\varepsilon^\alpha)$$

$$= -2\pi\log(2) + O(\varepsilon^\alpha).$$

and

$$\begin{aligned}
&\int_{\partial\Omega_{\varepsilon\mu_j}} \frac{2}{|y-\nu(0)|^2}\left(H(\xi_j+\varepsilon\mu_j y,\xi_j) - H(\xi_j,\xi_j)\right) \\
=& \int_{\partial\Omega_{\varepsilon\mu_j}} \frac{2}{|y-\nu(0)|^2} O(\varepsilon^\alpha|y|^\alpha) = O(\varepsilon^\alpha).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{k}\int_\Omega (|\nabla U_j|^2 + U_j^2) \\
=& -2k\pi\log 2 + \pi\sum_{j=1}^{k}[H(\xi_j,\xi_j) - 2\log(\varepsilon\mu_j)] + O(\varepsilon^\alpha) \\
=& -2k\pi\log\varepsilon + \pi\sum_{j=1}^{k}[H(\xi_j,\xi_j) - 2\log(2\mu_j)] + O(\varepsilon^\alpha).
\end{aligned} \tag{5.96}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{l\neq j}\int_\Omega (\nabla U_l\nabla U_j + U_l U_j) \\
=& \sum_{l\neq j}\int_\Omega \nabla u_l\nabla u_j + 2\int_\Omega \nabla u_l\nabla H_j^\varepsilon + \int_\Omega \nabla H_l^\varepsilon\nabla H_j^\varepsilon \\
&+ \int_\Omega u_l u_j + 2\int_\Omega u_l H_j^\varepsilon + \int_\Omega H_l^\varepsilon H_j^\varepsilon.
\end{aligned} \tag{5.97}$$

Multiplying (5.4) by H_l^ε, it yields

$$\int_\Omega \nabla H_j^\varepsilon\nabla H_l^\varepsilon + \int_\Omega H_j^\varepsilon H_l^\varepsilon = -\int_\Omega u_j H_l^\varepsilon + \varepsilon\int_{\partial\Omega} e^{u_j}H_l^\varepsilon - \int_{\partial\Omega}\frac{\partial u_j}{\partial\nu}H_l^\varepsilon. \tag{5.98}$$

Multiplying (5.4) by u_l^ε again, we have

$$\int_\Omega \nabla u_j^\varepsilon\nabla u_l^\varepsilon + \int_\Omega H_j^\varepsilon u_l^\varepsilon = -\int_\Omega \nabla H_j^\varepsilon\nabla u_l + \varepsilon\int_{\partial\Omega} e^{u_j}u_l^\varepsilon - \int_{\partial\Omega}\frac{\partial u_j}{\partial\nu}u_l. \tag{5.99}$$

By (5.97)-(5.99) we find that

$$\begin{aligned}
&\sum_{l\neq j}\int_\Omega (\nabla U_l\nabla U_j + U_l U_j) \\
=& \sum_{l\neq j}\int_\Omega \nabla u_l\nabla u_j + 2\int_\Omega \nabla u_l\nabla H_j^\varepsilon + \int_\Omega \nabla H_l^\varepsilon\nabla H_j^\varepsilon
\end{aligned}$$

$$\begin{aligned} & + \int_\Omega u_l u_j + 2\int_\Omega u_l H_j^\varepsilon + \int_\Omega H_l^\varepsilon H_j^\varepsilon \\ = & \ \pi \sum_{l\neq j} G(\xi_l, \xi_j) + O(\varepsilon^\alpha). \end{aligned} \tag{5.100}$$

Therefore, by (5.93), (5.96) and (5.100), using the choice of μ_j in (5.36) we get

$$\begin{aligned} & \frac{1}{2}\int_\Omega \left[|\nabla (U_\lambda(\xi))|^2 + U_\lambda(\xi)^2\right] \\ = & \ \frac{1}{p^2\gamma^{2(p-1)}}\left[-2k\pi\log\varepsilon - \frac{p}{2-p}\pi\left(\sum_{j=1}^k H(\xi_j,\xi_j) + \sum_{l\neq j}^k G(\xi_l,\xi_j)\right) + O(|\log\varepsilon|^{-1})\right] \end{aligned} \tag{5.101}$$

Finally, let use estimate the second term in (5.92). We have

$$\begin{aligned} \frac{\lambda}{p}\int_{\partial\Omega} e^{U_\lambda^p} &= \frac{\lambda}{p}\int_{\partial\Omega} e^{\gamma^p\left(1+\frac{1}{p\gamma^p}V_\lambda(\frac{x}{\varepsilon})\right)^p} \\ &= \frac{\lambda}{p}\sum_{j=1}^k \int_{\partial\Omega\cap B(\xi_j,\tilde\delta)} e^{\gamma^p\left(1+\frac{1}{p\gamma^p}V_\lambda(\frac{x}{\varepsilon})\right)^p} \\ &\quad + \frac{\lambda}{p}\int_{\partial\Omega\setminus \bigcup_{j=1}^k B(\xi_j,\tilde\delta)} e^{\gamma^p\left(1+\frac{1}{p\gamma^p}V_\lambda(\frac{x}{\varepsilon})\right)^p} \\ &:= I_1 + I_2. \end{aligned} \tag{5.102}$$

First we observe that

$$I_2 = \lambda\Theta_\lambda(\xi) \tag{5.103}$$

with $\Theta_\lambda(\xi)$ a function, uniformly bounded, as $\lambda \to 0$. On the other hand,

$$\begin{aligned} I_1 &= \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^k \int_{\partial\Omega_\varepsilon\cap B(\xi_j',\tilde\delta/\varepsilon)} e^{\gamma^p\left[\left(1+\frac{1}{p\gamma^p}V_\lambda(y)\right)^p-1\right]} \\ &= \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^k \int_{\partial\Omega_\varepsilon\cap B(\xi_j',\tilde\delta/\varepsilon)} e^{\left\{\tilde w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}\tilde w_{1j}(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde w_{2j}(y)+\theta(y)\right\}}(1+O(\frac{1}{\gamma^p})) \\ &= \frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^k \int_{\partial\Omega_\varepsilon\cap B(0,\frac{\tilde\delta}{\mu_j\varepsilon})} \frac{2}{|y-\nu(0)|^2}\left(1+O(\frac{1}{\gamma^p})\right) \\ &= \frac{1}{p^2\gamma^{2(p-1)}}2k\pi\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right), \end{aligned} \tag{5.104}$$

with $\Theta_\lambda(\xi)$ a function, uniformly bounded, as $\lambda \to 0$. From (5.102)-(5.104) we get

$$\frac{\lambda}{p}\int_{\partial\Omega} e^{U_\lambda^p} = \frac{1}{p^2\gamma^{2(p-1)}}\left[2k\pi\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right)\right]. \tag{5.105}$$

By (5.8), (5.91), (5.92), (5.101) and (5.105), we can write the whole asymptotic expansion of $F_\lambda(\xi)$, namely (5.89) holds. $\square$

5.6 Proof of The main Theorem

Proof of Theorem 5.2: From Lemma 5.17, the function

$$U_\lambda(\xi) + \tilde{\phi}(\xi) = \frac{1}{p\gamma^{p-1}}\left(p\gamma^p + (V_\lambda + \phi)(\frac{x}{\varepsilon})\right)$$

where V_λ defined by (5.31) and $\phi(\xi)$ is the unique solution of problem (5.80), is a solution of problem (5.1) if we adjust ξ so that it is a critical point of $F_\lambda(\xi)$ defined by (5.86). This is equivalent to finding a critical point of

$$\widetilde{F}_\lambda(\xi) := A\lambda^{-1}\varepsilon^{\frac{2-p}{p}}F_\lambda(\xi) + B + C\log\varepsilon,$$

for suitable constants A, B and C. On the other hand, from Lemmas 5.18, for $\xi = (\xi_1, \ldots, \xi_k) \in \partial\Omega$ satisfies (5.2), we have that,

$$\widetilde{F}_\lambda(\xi) = \varphi_k(\xi) + O(|\log\varepsilon|^{-1})\Theta_\lambda(\xi),$$

where φ_k is given by (5.4), and $\Theta_\lambda(\xi)$ is uniformly bounded in consider region as $\lambda \to 0$.

By the assumptions φ_k has a C^0- stable critical point $(\xi_1^*, \ldots, \xi_k^*)$, by Definition 5.1 we deduce that if λ is small enough, there exists a critical point ξ^n of $\widetilde{F}_\lambda(\xi)$ such that $\widetilde{F}_\lambda(\xi)(\xi^n) \to \varphi_k(\xi^*)$. Moreover,, up to subsequence, $\xi^n \to \xi$ as λ goes to zero, with $\varphi_k(\xi) = \varphi_k(\xi^*)$ and $\nabla\varphi_k(\xi^*) = 0$.

Expansion (5.9) follows from (5.8) and (5.105), while (5.10) holds as a direct consequence of the construction of U_λ. Expansion (5.11) is consequence of (5.89).

Remark 5.19. *Using Ljusternik-Schnirelmann theory, one can get a second, distinct solution satisfying Theorem 5.2. The proof is similar to [27].*

5.7 Appendix

Proof of (5.23): Since $(\arctan t)' = \frac{1}{1+t^2}$, by integration parts, we have

$$\int_0^\infty \frac{1}{t^2+1}\log\frac{1}{t^2+1}dt = \int_0^\infty \log\frac{1}{t^2+1}d(\arctan t)$$
$$= \left[\arctan t\log\frac{1}{1+t^2}\right]\Big|_0^{+\infty} + \int_0^{+\infty}\frac{2t\arctan t}{1+t^2}dt.$$

Set $t = \tan x$, we have $1 + t^2 = \sec^2 t$, $dt = \sec^2 t dx$, then we have

$$\begin{aligned}
& \int_0^{+\infty} \frac{2t \arctan t}{1+t^2} dt = 2 \int_0^{\frac{\pi}{2}} x \tan x dx \\
= & -2 \int_0^{\frac{\pi}{2}} x d(\log(\cos x)) \\
= & [-2x \log(\cos x)] \Big|_0^{\frac{\pi}{2}} + 2 \int_0^{\frac{\pi}{2}} \log(\cos x) \ dx \\
= & [-2 \arctan t \log[\cos(\arctan t)]] \Big|_0^{+\infty} + 2 \int_0^{\frac{\pi}{2}} \log(\cos x) \ dx \\
= & \big[- \arctan t \log[\cos^2(\arctan t)]\big] \Big|_0^{+\infty} + 2 \int_0^{\frac{\pi}{2}} \log(\cos x) \ dx.
\end{aligned}$$

Since

$$\cos^2(\arctan t) = \cos^2 x = \frac{1}{\sec^2 x} = \frac{1}{1 + \tan^2 x} = \frac{1}{1+t^2}.$$

On the other hand, we note that

$$\int_0^{\frac{\pi}{2}} \log(\cos x) \ dx = \int_0^{\frac{\pi}{2}} \log(\sin x) \ dx.$$

Then we have

$$\begin{aligned}
& 2 \int_0^{\frac{\pi}{2}} \log(\cos x) \ dx \\
= & \int_0^{\frac{\pi}{2}} \log(\cos x) \ dx + \int_0^{\frac{\pi}{2}} \log(\sin x) \ dx \\
= & \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin(2x)}{2}\right) \ dx = \frac{1}{2} \int_0^{\pi} \log\left(\frac{\sin(x)}{2}\right) \ dx \\
= & \frac{1}{2} \int_0^{\pi} \log(\sin \ x) dx - \frac{\pi}{2} \log 2 \\
= & \int_0^{\frac{\pi}{2}} \log(\sin \ x) dx - \frac{\pi}{2} \log 2 \\
= & \int_0^{\frac{\pi}{2}} \log(\cos \ x) dx - \frac{\pi}{2} \log 2.
\end{aligned}$$

Hence we get

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{1}{t^2+1} \log \frac{1}{t^2+1} dt = 2 \int_0^{\infty} \frac{1}{t^2+1} \log \frac{1}{t^2+1} dt \\
= & \ 4 \int_0^{\frac{\pi}{2}} \log(\cos x) \ dx = -2\pi \log 2.
\end{aligned}$$

Chapter 6

New solutions for critical Neumann problems $\mathbb{R}^2$

6.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and $\lambda > 0$. This chapter is concerned with the existence of positive solutions to the boundary value problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega; \quad u > 0 \quad \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = \lambda u e^{u^2} & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where ν denotes the outer unitary normal vector of $\partial\Omega$. Elliptic equations with nonlinear Neumann boundary condition of exponential type arise in conformal geometry (prescribing Gaussian curvature of the domain and curvature of the boundary), see for instance [22, 23, 76] and references therein, and in corrosion modelling, see [16, 66, 84].

Problem (6.1) is the Euler-Lagrange equation for the functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega \left[|\nabla u|^2 + u^2\right] - \frac{\lambda}{2}\int_{\partial\Omega} e^{u^2}, \quad u \in H^1(\Omega). \tag{6.2}$$

For functions $u \in H^1(\Omega)$, the maximal growth of integrability on the boundary is of exponential type, due to the Trudinger trace embedding (in the sense of Orlicz spaces) [103, 114]

$$H^1(\Omega) \ni u \longmapsto e^{u^2} \in L^p(\partial\Omega) \quad \forall\, p \geq 1.$$

This optimal embedding is related to the critical Trudinger-Moser trace inequality

$$C_\pi(\Omega) = \sup\{\int_{\partial\Omega} e^{\pi u^2} \;/\; u \in H^1(\Omega),\; \int_\Omega [|\nabla u|^2 + u^2] = 1\} \;<\; +\infty\,,$$

[74]. It has been proven [124] that for any bounded domain Ω in $\mathbb{R}^2$, with smooth boundary, the supremum $C_\pi(\Omega)$ is attained by a function $u \in H^1(\Omega)$ with $\int_\Omega [|\nabla u|^2 + u^2] = 1$. Furthermore, for any $\alpha \in (0, \pi)$, the supremum $C_\alpha(\Omega)$ is finite and it is attained, while $C_\alpha(\Omega) = \infty$ as soon as $\alpha > \pi$. See also [24, 72, 73, 75] for generalizations. Observe that critical points of the above constrained variational problem satisfy, after a simple scaling, an equation of the form (6.1).

The Trudinger-Moser trace embedding is critical, involving loss of compactness analogous to that related to the Trudinger-Moser embedding for functions u with zero boundary value,

$$H_0^1(\Omega) \ni u \longmapsto e^{u^2} \in L^p(\Omega) \quad \forall\, p \geq 1$$

for which the analogous problem to (6.1) is

$$\begin{cases} \Delta u + \lambda u e^{u^2} = 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.3}$$

whose energy functional is given by $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega e^{u^2}. \tag{6.4}$$

Even though I_λ satisfies the compactness PS-condition for energy levels less that 2π [1], loss of compactness in $H_0^1(\Omega)$ is described by the presence of families of blowing up solutions for Problem (6.3). It has been proven in [44] that if u_n solves problem (6.3) for $\lambda = \lambda_n$, with $I_{\lambda_n}(u_n)$ bounded and $\lambda_n \to 0$, then, passing to a subsequence, there is an integer $k \geq 0$ such that

$$I_{\lambda_n}(u_n) = 2k\pi + o(1)\ . \tag{6.5}$$

[2, 44]. This quantization property is not known for general Palais-Smale sequences associated to I_λ [3]. When $k = 1$ a more precise description of the blowing up behavior of these families of solutions is known [2]. On the other hand, concerning existence of solutions to (6.3), a first observation is that the functional I_λ has the mountain pass structure. In [1, 6] it is shown that there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ the mountain pass level is below 2π where PS-condition holds. Thus a solution to (6.3) exists. As $\lambda \to 0$, the family of mountain pass solutions satisfies (6.5) with $k = 1$. In [112] it is proven that if Ω has a sufficiently small hole, a solution to (6.3), satisfying (6.5), exists. Further results were obtained in [39]: if Ω has a hole of any size, namely Ω is not simply connected, then a solution satisfying property (6.5) with $k = 2$ exists. This solution happens to blow up exactly at 2 points in Ω. General conditions for the existence of solutions of problem (6.3) for small λ, which satisfy the bubbling condition (6.5), for any $k \geq 1$, are provided in [39], together with the precise characterization of their blow up profile. In fact, blowing up solutions satisfying (6.5) happens to blow up at exactly k points which are located in the interior of Ω. See also [5, 32, 40] for related results.

In this chapter, we are concerned with the construction of solutions to (6.1), in the same spirit as the result described above in [39]. Assume Ω is any bounded domain with smooth boundary. For any integer k we find existence of a pair of solutions u_λ to problem (6.1) for small λ, whose energy satisfy the bubbling condition

$$J_\lambda(u_\lambda) = k\frac{\pi}{2} + o(1) \quad \text{as} \quad \lambda \to 0. \tag{6.6}$$

Furthermore, we give a precise description of their bubbling behavior.

To state our result, let us introduce the following function $\varphi_k : (\partial\Omega)^k \times (\mathbb{R}^+)^k \to \mathbb{R}$, $\varphi_k(\xi, m) = \varphi_k(\xi_1, \dots, \xi_k, m_1, \dots, m_k)$ defined by

$$\begin{aligned} \varphi_k(\xi, m) &= 2(\log 2 - 1)\sum_{j=1}^k m_j^2 + 2\sum_{j=1}^k m_j^2 \log(m_j^2) \\ &\quad - \sum_{j=1}^k m_j^2 H(\xi_j, \xi_j) - \sum_{i \neq j} m_i m_j G(\xi_i, \xi_j), \end{aligned} \tag{6.7}$$

where G is the Green function for the Neumann problem

$$\begin{cases} -\Delta_x G(x,y) + G(x,y) = 0 & x \in \Omega; \\ \frac{\partial G(x,y)}{\partial \nu_x} = 2\pi\delta_y(x) & x \in \partial\Omega, \end{cases} \tag{6.8}$$

and H its regular part defined as

$$H(x,y) = G(x,y) - 2\log\frac{1}{|x-y|}. \tag{6.9}$$

Theorem 6.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and let $k \geq 1$ be an integer. Then, for all small $\lambda > 0$ there exists a pair solution u_λ^1, u_λ^2 of problem (6.1) such that*

$$\frac{1}{2}\int_\Omega \left[|\nabla u_\lambda^i|^2 + (u_\lambda^i)^2\right] - \frac{\lambda}{2}\int_{\partial\Omega} e^{(u_\lambda^i)^2} \;=\; \frac{k}{2}\pi \,+\, o(1) \quad i = 1,2$$

where $o(1) \to 0$ as $\lambda \to 0$. Moreover, for any $i = 1,2$, passing to a subsequence, there exists $(\xi^i, m^i) = (\xi_1^i, \dots, \xi_k^i, m_1^i, \dots, m_k^i) \in (\partial\Omega)^k \times (\mathbb{R}^+)^k$, with $\xi^1 \neq \xi^2$, such that $\nabla\varphi_k(\xi^i, m^i) = 0$ and

$$u_\lambda(x) \;=\; \sqrt{\lambda}\left(\sum_{j=1}^k m_j^i G(x, \xi_j^i) \;+\; o(1)\right) \tag{6.10}$$

where $o(1) \to 0$ on each compact subset of $\bar{\Omega} \setminus \{\xi_1^i, \dots, \xi_k^i\}$.

These solutions blow up at points located near $\xi_1, \dots, \xi_k \in \partial\Omega$, while far away from these points the solutions looks like a combination of Green function with positive weights m_1, ...,

m_k. These points and parameters $(\xi_1, \dots, \xi_k, m_1, \dots, m_k)$ correspond to two distinct critical points of φ_k.

We can actually show a stronger version of this result. If $\partial\Omega$ has more than one component, then pairs of families of solutions blowing up at k points on each component happen to exist. In reality, associated to each *topologically nontrivial* critical point situation associated to φ_k (for instance local maxima or saddle points possibly degenerate), a solution with concentration peaks at a corresponding critical point exists. We will not elaborate more on this point, and we refer the interested reader to [27].

It is important to remark the interesting analogy between these results and those known for other problems with exponential non linearity on the boundary, as

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda e^u & \text{on } \partial\Omega, \end{cases} \tag{6.11}$$

and

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda \sinh u & \text{on } \partial\Omega. \end{cases} \tag{6.12}$$

[16, 27, 29, 66, 84, 85]. See also [19, 37] for related problems.

We will just describe the analogy between our Problem (6.1) and the problem of finding positive solutions to (6.11). Similar analogy exists with the problem of finding sign changing solutions to (6.12). But in this case we refer the reader to the results in [16, 29, 66, 84].

In [27], construction of solutions to (6.11) with $\lambda \int_{\partial\Omega} e^{u_\lambda}$ bounded is carried out: for any integer $k \geq 1$, there are at least *two* distinct families of solutions u_λ which approaches the sum of k Dirac masses at the boundary. The location of these possible points of concentration may be further characterized as critical points of the functional of k points $\xi_1, \dots, \xi_k$ of the boundary defined as

$$\Psi_k(\xi_1, \dots, \xi_k) = -\left[\sum_{j=1}^{k} H(\xi_j, \xi_j) + \sum_{i \neq j} G(\xi_i, \xi_j)\right],$$

where G and H are defined in (6.8) and (6.9) respectively. Observe that the function Ψ_k only depends on points on the boundary $\partial\Omega$ and it does not depend on positive parameters $m_1, \dots m_k$, unlike function φ_k which is defined in (6.7) and which determines the bubbling behavior of solutions to (6.1). Furthermore, far from $\xi_1, \dots \xi_k$, the solutions to Problem (6.11) found in [27] look like

$$u_\lambda(x) \;=\; \sum_{j=1}^{k} G(x, \xi_j) \;+\; o(1) \quad \text{as} \quad \lambda \to 0.$$

Thus, also the solutions to Problem (6.11) found in [27] are combinations of Green function, far from the concentration points, but unlike the solutions obtained in Theorem 6.1 for Problem (6.1), the weights in from of the Green functions are always equal to 1. Thus, to construct solutions to Problem (6.1), not only we have to find the location of the bubbling points $\xi_1, \ldots, \xi_k$ on the boundary, but also the weights $m_1, \ldots, m_k$ in front of the Green functions in (6.10).

The solutions predicted in Theorem 6.1 are constructed as a small additive perturbation of an appropriate initial approximation. A linearization procedure leads to a finite dimensional reduction, where the reduced problem corresponds to that of adjusting variationally the location of the concentration points $\xi_1, \ldots, \xi_k$ and of the weights $m_1, \ldots, m_k$. A precise description of the approximation and a detailed outline of the proof and of the organization of this chapter are given in Section 6.2.

6.2 A first approximation and outline of the argument

It is useful for our purpose to consider the change of variables $u = \sqrt{\lambda}\tilde{u}$ so that problem (6.1) gets rewritten as

$$\begin{cases} -\Delta \tilde{u} + \tilde{u} = 0 & \text{in } \Omega; \quad \tilde{u} > 0 \quad \text{in } \Omega; \\ \frac{\partial \tilde{u}}{\partial \nu} = \lambda \tilde{u} e^{\lambda \tilde{u}^2} & \text{on } \partial\Omega. \end{cases} \tag{6.13}$$

The first part of this section is devoted to construct a good approximation for a solution to Problem (6.13) and to estimate its error. To do so, let us introduce the following problem in the entire plane

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2_+; \\ \frac{\partial v}{\partial \nu} = e^v & \text{on } \partial\mathbb{R}^2_+; \quad \int_{\partial\mathbb{R}^2_+} e^v < \infty. \end{cases} \tag{6.14}$$

The positive solutions to Problem (6.14) are the basic elements for our construction. So, let us recall that all positive solutions to (6.14) are given by

$$w_{t,\mu}(x) = w_{t,\mu}(x_1, x_2) = \log \frac{2\mu}{(x_1 - t)^2 + (x_2 + \mu)^2}, \tag{6.15}$$

where t is any real number and $\mu > 0$ is any strictly positive number (see [76, 99, 126]). Set

$$w_\mu(x) := w_{0,\mu}(x) = \log \frac{2\mu}{x_1^2 + (x_2 + \mu)^2}. \tag{6.16}$$

We next describe an approximate solution to (6.13) whose shape is given by the sum of functions w_μ centered at points on the boundary of Ω and properly scaled. Let k be an integer, $\xi_1, \dots, \xi_k$ be points on the boundary of Ω and $m_1, \dots, m_k$ positive numbers. We assume there exists a positive, small number δ such that

$$|\xi_i - \xi_j| > \delta \quad \text{for } i \neq j, \qquad \delta < m_j < \frac{1}{\delta}. \tag{6.17}$$

We thus define the functions

$$u_j(x) = \log \frac{1}{|x - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2}, \quad \text{for any} \quad j = 1, \dots, k$$

and

$$\tilde{U}(x) = \sum_{j=1}^{k} m_j \left[u_j(x) + H_j(x) \right], \tag{6.18}$$

where H_j is the unique solution to the problem

$$\begin{cases} -\Delta H_j + H_j = -u_j & \text{in } \ \Omega; \\ \frac{\partial H_j}{\partial \nu} = 2\varepsilon_j \mu_j e^{u_j} - \frac{\partial u_j}{\partial \nu} & \text{on } \ \partial\Omega. \end{cases} \tag{6.19}$$

In the above definitions, μ_j and ε_j are positive numbers. These numbers μ_j and ε_j will be defined later on in terms of λ, ξ_j and m_j in order to ensure that $\tilde{U}$ is a function very close to a solution for Problem (6.13). Let us just mention that, a posteriori, the parameters ε_j will tend to zero, as $\lambda \to 0$, namely

$$\lim_{\lambda \to 0} \varepsilon_j = 0, \quad \text{for any} \quad j = 1, \dots, k, \tag{6.20}$$

while the numbers μ_j will remain bounded from above and strictly positive, as $\lambda \to 0$. Taking this into account, we easily see that the shape of the function $\tilde{U}$ change depending whether you evaluate it far from the fixed points ξ_j or in a region very close to one of the points ξ_j. Let us then describe carefully the shape of $\tilde{U}$ in these two regions. For this purpose, we need the following

Lemma 6.2. *Assume (6.17) and (6.20). For any $0 < \alpha < 1$, one has*

$$H_j(x) = H(x, \xi_j) + \varepsilon_j^{\alpha} O(1), \quad \textit{as} \quad \lambda \to 0 \tag{6.21}$$

where $O(1)$ denotes a function in Ω which is uniformly bounded as $\lambda \to 0$, and H is the regular part of Green's function defined in (6.8).

Proof. The proof has been done by del Pino- Dávila-Musso in [27], see also the proof of Lemma 5.3. □

A direct consequence of Lemma 6.2 is that, for a given $\delta > 0$ small and fixed, in the region $|x-\xi_j| > \delta$ for all $j = 1, \ldots, k$, the function $\tilde{U}$ looks like

$$\tilde{U}(x) = \sum_{j=1}^{k} m_j \left[G(x, \xi_j) + O(\varepsilon_j^\alpha) \right], \quad \text{as} \quad \lambda \to 0. \tag{6.22}$$

Here and in what follows, with $O(\varepsilon_j^\alpha)$ we denote a general function in Ω of the form $\varepsilon_j^\alpha \Theta(x)$ where $\Theta(x)$ is uniformly bounded in Ω as $\lambda \to 0$.

Let us now examine $\tilde{U}$ in a neighborhood of a given ξ_j. Assume $|x - \xi_j| < \delta$ and set $y = \frac{x}{\varepsilon_j}$, $\xi_j' = \frac{\xi_j}{\varepsilon_j}$. Explicit computations give that

$$\begin{aligned}
\tilde{U}(x) &= m_j \left[u_j(x) + H_j(x) \right] + \sum_{i \neq j} m_i \left[u_i(x) + H_i(x) \right] \\
&= m_j \left[\log \frac{2\mu_j}{|x - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2} - \log(2\mu_j) + H(x, \xi_j) + O(\varepsilon_j^\alpha) \right] \\
&\quad + \sum_{i \neq j} m_i \left[\log \frac{1}{|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2} + H(x, \xi_i) + O(\varepsilon_i^\alpha) \right] \\
&= m_j \left[\log \frac{2\mu_j}{|y - \xi_j' - \mu_j \nu(\xi_j')|^2} + 2 \log \frac{1}{\varepsilon_j} - \log(2\mu_j) + H(\xi_j, \xi_j) + O(|x - \xi_j|) + O(\varepsilon_j^\alpha) \right] \\
&\quad + \sum_{i \neq j} m_i \left[\log \frac{1}{|\xi_i - \xi_j|^2} + H(\xi_j, \xi_i) \right] \\
&\quad + \sum_{i \neq j} m_i \left[\log \frac{1}{|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2} - \log \frac{1}{|\xi_i - \xi_j|^2} + O(|x - \xi_j|) + O(\varepsilon_i^\alpha) \right] \\
&= m_j \left[\log \frac{2\mu_j}{|y - \xi_j' - \mu_j \nu(\xi_j')|^2} + 2 \log \frac{1}{\varepsilon_j} - \log(2\mu_j) + H(\xi_j, \xi_j) + O(|x - \xi_j|) + O(\varepsilon_j^\alpha) \right] \\
&\quad + \sum_{i \neq j} m_i G(\xi_j, \xi_i) + \sum_{i \neq j} m_i \left[\log \frac{1}{|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2} - \log \frac{1}{|\xi_i - \xi_j|^2} + O(|x - \xi_j|) + O(\varepsilon_i^\alpha) \right],
\end{aligned}$$

as $\lambda \to 0$. We set

$$w_j(x) = w_{\mu_j}\left(\frac{x - \xi_j}{\varepsilon_j}\right) = \log \frac{2\mu_j}{|y - \xi_j' - \mu_j \nu(\xi_j')|^2},$$

and

$$\beta_j = -\log(2\mu_j) + H(\xi_j, \xi_j) + \sum_{i \neq j} m_j^{-1} m_i G(\xi_j, \xi_i), \qquad \theta(x) = O(|x - \xi_j|) + \sum_{j=1}^{k} O(\varepsilon_j^\alpha).$$

We thus write the above expansion in the following compact form: for $|x - \xi_j| < \delta$,

$$\tilde{U}(x) = m_j \left(w_j(x) + \log \varepsilon_j^{-2} + \beta_j + \theta(x) \right), \quad \text{as} \quad \lambda \to 0. \tag{6.23}$$

Formulas (6.22) and (6.23) give a precise description of the function $\tilde{U}$.

The solution to (6.13) we are looking for has the form

$$\tilde{u} = \tilde{U} + \phi, \tag{6.24}$$

where $\tilde{U}$ is defined as in (6.18), and ϕ represents a lower order correction. In fact, we aim at finding a solution $\tilde{u}$ for a function ϕ small in some proper sense provided that the points ξ_j and the parameters m_j are suitably chosen. Assuming for the moment that ϕ is small, we rewrite problem (6.13) as follows

$$\begin{cases} -\Delta\phi + \phi = 0 & \text{in } \Omega; \\ L(\phi) = E + N(\phi) & \text{on } \partial\Omega, \end{cases} \tag{6.25}$$

where

$$L(\phi) := \frac{\partial\phi}{\partial\nu} - \left[\sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j}\right] \phi, \tag{6.26}$$

$$E := f(\tilde{U}) - \frac{\partial\tilde{U}}{\partial\nu}, \tag{6.27}$$

and

$$N(\phi) := f(\tilde{U} + \phi) - f(\tilde{U}) - f'(\tilde{U})\phi + \left[f'(\tilde{U}) - \sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j}\right] \phi. \tag{6.28}$$

Here and in what follows f denotes the nonlinearity

$$f(\tilde{u}) = \lambda\tilde{u}e^{\lambda\tilde{u}^2}.$$

It is not hard to believe that having a good approximation $\tilde{U}$ to a solution of Problem (6.13) is reflected into the fact that the function E is small, in some sense to be made precise. It is in this context that we will choose μ_j and ε_j in such a way that the error of approximation E for $\tilde{U}$ is small around each point ξ_j under some appropriate norm.

Let us be more precise. The error E is clearly defined by (6.27). Assume $\delta > 0$ is a small but fixed positive number and $x \in \partial\Omega$ with $|x - \xi_j| < \delta$. In this region, we have that

$$\begin{aligned} f(\tilde{U}) &= \lambda\left[m_j\left(w_j(x) + \log\varepsilon_j^{-2} + \beta_j + \theta(x)\right)\right] e^{\lambda\left[m_j\left(w_j(x)+\log\varepsilon_j^{-2}+\beta_j+\theta(x)\right)\right]^2} \\ &= \left(\lambda m_j(\log\frac{1}{\varepsilon_j^2} + \beta_j) + \lambda m_j(w_j + O(1))\right) \\ &\quad \times e^{\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)^2} e^{2\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)w_j} e^{2\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)\theta(x)} e^{\lambda m_j^2(w_j+\theta(x))^2} \end{aligned}$$

$$\begin{aligned} &= \lambda m_j(\log\frac{1}{\varepsilon_j^2}+\beta_j)\left(1+(\log\frac{1}{\varepsilon_j^2}+\beta_j)^{-1}(w_j+O(1))\right) \\ &\quad\times e^{\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)^2} e^{2\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)w_j} e^{2\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)\theta(x)} e^{\lambda m_j^2(w_j+\theta(x))^2} \end{aligned}$$

as $\lambda \to 0$. We thus choose ε_j to be defined as

$$2\lambda m_j^2\left(\log\frac{1}{\varepsilon_j^2}+\beta_j\right)=1. \tag{6.29}$$

It is immediate to see that, with this definition, (6.20) holds true. Thanks to (6.29), one has

$$\begin{aligned} f(\tilde U) &= \frac{1}{2m_j}\left(1+2\lambda m_j^2(w_j+O(1))\right)e^{\frac12(\log\frac{1}{\varepsilon_j^2}+\beta_j)}e^{w_j}e^{\theta(x)}e^{\lambda m_j^2(w_j+\theta(x))^2} \\ &= \frac{1}{2m_j}\varepsilon_j^{-1}e^{\beta_j/2}\left(1+2\lambda m_j^2(w_j+O(1))\right)e^{w_j}e^{\theta(x)}e^{\lambda m_j^2w_j^2}(1+O(\lambda)w_j). \end{aligned}$$

On the other hand, in the same region, we have

$$\frac{\partial\tilde U}{\partial\nu}=\frac{\partial}{\partial\nu}\left[m_j\left(w_j(x)+\log\varepsilon_j^{-2}+\beta_j+\theta(x)\right)\right]=m_j\varepsilon_j^{-1}e^{w_j}+\sum_{j=1}^k O(\varepsilon_j^2), \quad \text{as} \quad \lambda\to 0.$$

Thus, in order to match at main order the two terms $\frac{\partial\tilde U}{\partial\nu}$ and $f(\tilde U)$ in a region near the point ξ_j, we fix the parameter μ_j such that the number β_j satisfies

$$e^{\beta_j/2}=2m_j^2. \tag{6.30}$$

This condition defines the parameter μ_j as follows

$$\log(2\mu_j)=-2\log(2m_j^2)+H(\xi_j,\xi_j)+\sum_{i\neq j}m_im_j^{-1}G(\xi_i,\xi_j). \tag{6.31}$$

With these choices of μ_j we get

$$\begin{aligned} f(\tilde U) &= m_j\left(1+2\lambda m_j^2(w_j+O(1))\right)\varepsilon_j^{-1}e^{w_j}e^{\lambda m_j^2w_j^2}(1+O(\theta(x)))(1+O(\lambda)w_j) \\ &= m_j\left(1+2\lambda m_j^2(w_j+O(1))\right)\varepsilon_j^{-1}e^{w_j}e^{\lambda m_j^2w_j^2}(1+O(\lambda w_j)). \end{aligned}$$

As a conclusion, the election we made of μ_j and of ε_j gives that in the region $|x-\xi_j|<\delta$, the error of approximation can be described as follows

$$E=m_j\left\{\left(1+2\lambda m_j^2(w_j+O(1))\right)e^{\lambda m_j^2w_j^2}(1+O(\lambda w_j))-1\right\}\varepsilon_j^{-1}e^{w_j}. \tag{6.32}$$

Let us mention now that a direct computation shows that $E(x)\sim\lambda\varepsilon_j^{-1}e^{w_j(x)}$ in the region $|x-\xi_j|=O(\lambda)$; while, in the region $|x-\xi_j|>\delta$ for all j, we have that $|E(x)|\le C\lambda$, for some positive constant C. We thus conclude that the error of approximation satisfies the global bound

$$|E| \leq C\lambda\rho(x),$$

where

$$\rho(x) := \sum_{j=1}^{k} \rho_j(x)\chi_{B_\delta(\xi_j)}(x) + 1.$$

Here $\chi_{B_\delta(\xi_j)}$ is the characteristic function on $B_\delta(\xi_j)\bigcap\partial\Omega$ and

$$\rho_j(x) := \frac{1}{2\lambda m_j^2}\left\{\left(1+2\lambda m_j^2(w_j+O(1))\right)e^{\lambda m_j^2 w_j^2}(1+O(\lambda w_j))-1\right\}\varepsilon_j^{-1}e^{w_j}$$

From now on, let us write

$$\rho_j(x) = c\gamma_j\left\{\left(1+\frac{1}{\gamma_j}(w_j+1)\right)\left(1+\frac{1}{\gamma_j}(1+|w_j|)\right)e^{\frac{w_j^2}{2\gamma_j}}-1\right\}\varepsilon_j^{-1}e^{w_j}, \tag{6.33}$$

where $\gamma_j = \log\varepsilon_j^{-2}$. We define the $L^\infty-$weight norm

$$\|h\|_{*,\partial\Omega} = \sup_{x\in\partial\Omega}\rho(x)^{-1}|h(x)|. \tag{6.34}$$

We thus have the validity of the following key estimate for the error term E

$$\|E\|_{*,\partial\Omega} \leq C\lambda. \tag{6.35}$$

We conclude this section explaining the strategy to solve Problem (6.25), which guarantees the existence of a solution to Problem (6.13) of the form (6.18). In fact, we will solve Problem (6.25) into two steps. The first step consists in solving Problem (6.25) in a *projected* space. Let us be more precise.

Define in $\mathbb{R}^2_+ = \{(x_1,x_2) : x_2 > 0\}$

$$z_{0j}(x_1,x_2) = \frac{1}{\mu_j} - 2\frac{x_2+\mu_j}{x_1^2+(x_2+\mu_j)^2}, \qquad z_{1j}(x_1,x_2) = -2\frac{x_1}{x_1^2+(x_2+\mu_j)^2}.$$

It has been shown in [27] that these functions are all the bounded solutions to the linearized equation around w_{μ_j} (6.16) associated to Problem (6.14), that is they solve

$$\Delta\psi = 0 \quad \text{in } \mathbb{R}^2_+, \quad -\frac{\partial\psi}{\partial x_2} = e^{w_{\mu_j}}\psi \quad \text{on } \partial\mathbb{R}^2_+. \tag{6.36}$$

For $\xi_j \in \partial\Omega$, we define $F_j : B_\delta(\xi_j) \to M$ to be a diffeomorphism, where M is an open neighborhood of the origin in $\mathbb{R}^2_+$ such that $F_j(\Omega\cap B_\delta(\xi_j)) = \mathbb{R}^2_+\cap M$, $F_j(\partial\Omega\cap B_\delta(\xi_j)) = \partial\mathbb{R}^2_+\cap M$. We can select F_j so that it preserves area. Define

$$Z_{ij}(x) = z_{ij}\left(\varepsilon_j^{-1}F_j(x)\right), \quad i=0,1, \quad j=1,\dots,k. \tag{6.37}$$

Next, let us consider a large but fixed number $R_0 > 0$ and a nonnegative radial and smooth cut-off function χ with $\chi(r) = 1$ if $r < R_0$ and $\chi(r) = 0$ if $r > R_0 + 1$, $0 \le \chi \le 1$. Then set

$$\chi_j(x) = \varepsilon_j^{-1}\chi\left(\varepsilon_j^{-1}F_j(x)\right). \tag{6.38}$$

The problem we first solve is to find a function ϕ and numbers c_{ij} such that

$$\begin{cases} -\Delta\phi + \phi = 0 & \text{in } \Omega; \\ L(\phi) = E + N(\phi) + \sum_{i=0,1}\sum_{j=1}^{k} c_{ij}Z_{ij}\chi_j & \text{on } \partial\Omega; \\ \int_\Omega \phi Z_{ij}\chi_j = 0 & \text{for } i = 0,1,\ j = 1,\cdots,k. \end{cases} \tag{6.39}$$

Consider the norm

$$\|\phi\|_\infty = \sup_{x\in\Omega}|\phi(x)|.$$

We prove the following

Proposition 6.3. *Let $\delta > 0$ be a small but fixed number. Assume the points $\xi_1,\ldots,\xi_k \in \partial\Omega$ and the parameters $m_1,\ldots,m_k$ satisfy*

$$|\xi_i - \xi_j| \ge \delta, \quad \forall\ i \ne j, \qquad \delta < m_j < \frac{1}{\delta}. \tag{6.40}$$

Then there exist positive numbers λ_0 and C, such that, for any $0 < \lambda < \lambda_0$, Problem (6.39) has a unique solution ϕ, c_{ij} which satisfies

$$\|\phi\|_\infty \le C\lambda, \quad |c_{ij}| \le C\lambda$$

for all $\lambda < \lambda_0$. Moreover, if we consider the map $(\xi, m) \mapsto \phi$ into the space $C(\bar\Omega)$, the derivative $D_\xi\phi$ and $D_m\phi$ exists and defines a continuous function of (ξ, m). Besides, there is a constant $C > 0$, such that

$$\|D_{\xi_{sl}}\phi\|_\infty \le C\lambda, \qquad \|D_{m_s}\phi\|_\infty \le C\lambda \tag{6.41}$$

for all s, l.

The proof of this result is contained in Section 6.3.

At this stage of our argument, we have solved the nonlinear problem (6.39). In order to find a solution to the original problem we need to find ξ and m such that

$$c_{ij}(\xi, m) = 0 \quad \text{for all } i = 0,1, \quad j = 1,\cdots,k. \tag{6.42}$$

This problem is indeed variational: it is equivalent to finding critical points of a function of ξ and m. Associated to (6.1), let us consider the energy functional J_λ given by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega(|\nabla u|^2 + u^2) - \frac{\lambda}{2}\int_{\partial\Omega} e^{u^2}, \quad u \in H^1(\Omega), \tag{6.43}$$

and the finite-dimensional restriction

$$\mathcal{I}_\lambda(\xi,m) = J_\lambda\left(\sqrt{\lambda}\left(\tilde{U}(\xi,m)+\phi(\xi,m)\right)\right), \tag{6.44}$$

where ϕ is the unique solution to problem (6.39) given by Proposition 6.3. Critical points of $\mathcal{I}_\lambda$ correspond to solutions of (6.42) for a small λ, as the following result states.

Proposition 6.4. *Under the assumptions of Proposition 6.3, the functional $\mathcal{I}_\lambda(\xi,m)$ is of class C^1. Moreover, for all $\lambda>0$ sufficiently small, if $D_{\xi,m}\mathcal{I}_\lambda(\xi,m)=0$, then (ξ,m) satisfies (6.42).*

The proof of the above Proposition, together with the expansion of the functional $I_\lambda(\xi,m)$ is given in Section 6.4. Section 6.5 is devoted to conclude the proof of Theorem 6.1. The final Appendix, Section 6.6, contains the proofs of some estimates we have used through the paper.

6.3 Proof of Proposition 6.3

The proof of Proposition 6.3 is based on a fixed point argument and the invertibility property of the following linear Problem: Given $h\in L^\infty(\partial\Omega)$, find a function ϕ and constants c_{ij} such that

$$\begin{cases} -\Delta\phi+\phi=0 & \text{in } \Omega;\\ L(\phi)=h+\sum\limits_{i=0,1}\sum\limits_{j=1}^{k} c_{ij}\chi_j Z_{ij} & \text{on } \partial\Omega;\\ \int_\Omega \chi_j Z_{ij}\phi = 0 & \text{for } \ i=0,1, \quad j=1,\dots,k. \end{cases} \tag{6.45}$$

We shall prove the validity of the following

Proposition 6.5. *Let $\delta>0$ be a small but fixed number and assume we have $\xi_1,\dots,\xi_k\in\partial\Omega$ and $m_1,\dots,m_k$ with*

$$|\xi_i-\xi_j|\geq\delta, \quad \forall\ i\neq j, \qquad \delta<m_j<\frac{1}{\delta}. \tag{6.46}$$

Then there exist positive numbers λ_0 and C such that, for any $0<\lambda<\lambda_0$ and any $h\in L^\infty(\partial\Omega)$, there is a unique solution $\phi\equiv T_\lambda(h)$, and $c_{ij}\in\mathbb{R}$ to (6.45). Moreover,

$$\|\phi\|_\infty \leq C\|h\|_{*,\partial\Omega}. \tag{6.47}$$

The proof of this result is based on the a-priori estimate for solutions to the following problem

$$\begin{cases} -\Delta\phi + \phi = f & \text{in } \Omega; \\ L(\phi) = h + \sum\limits_{i=0,1}\sum\limits_{j=1}^{k} c_{ij}\chi_j Z_{ij} & \text{on } \partial\Omega; \\ \int_\Omega \chi_j Z_{ij}\phi = 0 & \text{for } \ i=0,1, \ j=1,\dots,k. \end{cases} \tag{6.48}$$

Define

$$\|f\|_{**,\Omega} := \sup_{x\in\Omega}\left(\sum_{j=1}^{k}\frac{\varepsilon_j^\sigma}{(1+|x-\xi_j-\varepsilon_j\mu_j\nu(\xi_j)|)^{2+\sigma}}+1\right)^{-1}|f(x)| \tag{6.49}$$

where $0<\sigma<1$.

Lemma 6.6. *Under the assumptions of Proposition 6.5, if ϕ is a solutions of (6.48) for some $h\in L^\infty(\partial\Omega)$ and for some $f\in L^\infty(\Omega)$ with $\|h\|_{*,\partial\Omega}$, $\|f\|_{**,\Omega}<\infty$ and $c_{ij}\in\mathbb{R}$, then*

$$\|\phi\|_\infty \le C\left[\|h\|_{*,\partial\Omega}+\|f\|_{**,\Omega}\right]. \tag{6.50}$$

Proof. We will carry out the proof of the a priori estimate (6.50) by contradiction. We assume then the existence of sequences $\lambda_n\to 0$, points $\xi_j^n\in\partial\Omega$ and numbers m_j^n, μ_j^n which satisfy relations (6.46) and (6.31), functions h_n, f_n with $\|h_n\|_{*,\partial\Omega}$, $\|f_n\|_{**,\Omega}\to 0$, ϕ_n with $\|\phi_n\|_\infty=1$, constants $c_{ij,n}$,

$$-\Delta\phi_n+\phi_n \;=\; f_n, \qquad \text{in } \Omega, \tag{6.51}$$

$$L(\phi_n) = h_n + \sum_{i=0}^{2}\sum_{j=1}^{k} c_{ij,n}Z_{ij}\chi_j, \qquad \text{on } \partial\Omega, \tag{6.52}$$

$$\int_\Omega Z_{ij}\chi_j\phi_n = 0, \qquad \text{for all } i,\ j. \tag{6.53}$$

We will prove that in reality under the above assumption we must have that $\phi_n\to 0$ uniformly in $\bar\Omega$, which is a contradiction that concludes the result of the Lemma.

Passing to a subsequence we may assume that the points ξ_j^n approach limiting, distinct points ξ_j^* in $\partial\Omega$. We claim that $\phi_n\to 0$ in C^1 local sense on compacts of $\bar\Omega\setminus\{\xi_1^*,\dots,\xi_k^*\}$. Indeed, let us observe that $f_n\to 0$ locally uniformly in $\bar\Omega$, away from the points ξ_j. Away from the ξ_j^*'s we have then $-\Delta\phi_n+\phi_n\to 0$ uniformly on compact subsets on $\bar\Omega\setminus\{\xi_1^*,\dots,\xi_k^*\}$. Since ϕ_n is bounded it follows also that passing to a further subsequence, ϕ_n approaches in C^1 local sense on compacts of $\bar\Omega\setminus\{\xi_1^*,\dots,\xi_k^*\}$ a limit ϕ^* which is bounded and satisfies $-\Delta\phi^*+\phi^*=0$ in $\Omega\setminus\{\xi_1^*,\dots,\xi_k^*\}$. Furthermore, observe that far from $\{\xi_1^*,\dots,\xi_k^*\}$, $h_n\to 0$ locally uniformly on $\partial\Omega\setminus\{\xi_1^*,\dots,\xi_k^*\}$ and so we also have $\frac{\partial\phi_n}{\partial\nu}\to 0$ on $\partial\Omega\setminus\{\xi_1^*,\dots,\xi_k^*\}$. Hence ϕ^* extends smoothly to a function which satisfies $-\Delta\phi^* + \phi^* = 0$ in Ω, and $\frac{\partial \phi^*}{\partial \nu} = 0$ on $\partial\Omega$. We conclude that $\phi^* = 0$, and the claim follows.

For notational convenience, we shall omit the explicit dependence on n in the rest of the proof. We shall next show that

$$|c_{ij}| \leq C(\|\phi\|_\infty + \|h\|_{*,\partial\Omega} + \|f\|_{**,\Omega}). \tag{6.54}$$

Multiplying the first equation of (6.48) by Z_{ij} and integrating over $B(\xi_j, \delta)$, we find

$$\begin{aligned} \sum_{l=0,1} c_{lj} \int_{\partial\Omega \bigcap B(\xi_j,\delta)} \chi_j Z_{lj} Z_{ij} &= -\int_{\partial\Omega \bigcap B(\xi_j,\delta)} h Z_{ij} + \int_{\partial\Omega \bigcap B(\xi_j,\delta)} L(Z_{ij})\phi - \int_{\Omega \bigcap \partial B(\xi_j,\delta)} \frac{\partial \phi}{\partial \nu} Z_{ij} \\ &+ \int_{\Omega \bigcap B(\xi_j,\delta)} (-\Delta Z_{ij} + Z_{ij})\phi - \int_{\Omega \bigcap B(\xi_j,\delta)} f Z_{ij} \end{aligned} \tag{6.55}$$

Having in mind that $\phi_n \to 0$ in C^1 sense in $\Omega \bigcap \partial B(\xi_j, \delta)$, we have that $\int_{\Omega \bigcap \partial B(\xi_j,\delta)} \frac{\partial \phi}{\partial \nu} Z_{ij} \to 0$ as $\lambda \to 0$. Furthermore, a direct computation shows that

$$\int_{\partial\Omega \bigcap B(\xi_j,\delta)} \chi_j Z_{lj} Z_{ij} = M_i \delta_{li} + o(1), \quad \text{as} \quad \lambda \to 0 \tag{6.56}$$

where M_i is some universal constant and $\delta_{li} = 1$ if $i = l$, and $= 0$ if $i \neq l$. On the other hand, we have that

$$\int_{\partial\Omega \bigcap B(\xi_j,\delta)} \left(\frac{\partial Z_{ij}}{\partial \nu} - [\sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j}] Z_{ij} \right) \phi + \int_{\Omega \bigcap B(\xi_j,\delta)} (-\Delta Z_{ij} + Z_{ij})\phi \leq C\|\phi\|_\infty \tag{6.57}$$

and

$$\left| \int_\Omega f Z_{ij} \right| \leq C \|f\|_{**,\Omega}. \tag{6.58}$$

In fact, estimate (6.58) is a direct consequence of the definition of the $\|\cdot\|_{**,\Omega}$-norm. Let us prove the validity of (6.57). Recall that in $\Omega \bigcap B(\xi_j, \delta)$, we have that $Z_{ij}(x) = z_{ij}(\varepsilon_j^{-1} F_j(x))$, where F_j is chosen to preserve area (see (6.37)). Performing the change of variables $y = \varepsilon_j^{-1} F_j(x)$, we get that

$$\int_{\Omega \bigcap B(\xi_j,\delta)} (-\Delta Z_{ij} + Z_{ij})\phi = (1 + o(1)) \int_{\mathbb{R}^2_+ \bigcap B(0,\frac{\delta}{\varepsilon_j})} \left(\mathcal{L} z_{ij} + \varepsilon_j^2 z_{ij}\right) \tilde{\phi} \tag{6.59}$$

where $\tilde{\phi}(y) = \phi(F_j^{-1}(\varepsilon_j y))$ and $\mathcal{L}$ is a second order differential operator defined as follows

$$\mathcal{L} = -\Delta + O(\varepsilon_j |y|)\nabla^2 + O(\varepsilon_j)\nabla, \quad \text{in} \quad \mathbb{R}^2_+ \bigcap B(0, \frac{\delta}{\varepsilon_j}). \tag{6.60}$$

Hence

$$\left|\int_{\Omega\bigcap B(\xi_j,\delta)}(-\Delta Z_{ij}+Z_{ij})\phi\right|\leq C\|\phi\|_\infty.$$

On the other hand, we observe that, after a possible rotation, we can assume that $\nabla F_j(\xi_j) = I$. Hence, using again the change of variables $y=\varepsilon_j^{-1}F_j(x)$, we get

$$\int_{\partial\Omega\bigcap B(\xi_j,\delta)} L(Z_{ij})\phi = (1+o(1))\int_{\partial\mathbb{R}^2_+\bigcap B(0,\frac{\delta}{\varepsilon_j})}(B(z_{ij})-\tilde{W}z_{ij})b(y)\tilde{\phi} \tag{6.61}$$

where $\tilde{W}(y)=\varepsilon_j W(F_j^{-1}(\varepsilon_j y))$ with $W(x)=\sum_{j=1}^k \varepsilon_j^{-1}e^{w_j}$, and $b(y)$ is a positive function, coming from the change of variables, which is uniformly positive and bounded as $\lambda\to 0$. Furthermore B is a differential operator of order one on $\partial\mathbb{R}^2_+$. In fact, we have that

$$B=-\frac{\partial}{\partial y_2}+O(\varepsilon_j|y|)\nabla \quad \text{on} \quad \partial\mathbb{R}^2_+\bigcap B(0,\frac{\delta}{\varepsilon_j}) \tag{6.62}$$

On the other hand, since

$$W(x)=\varepsilon_j^{-1}\frac{2\mu_j\varepsilon_j^2}{|x-\xi_j-\varepsilon_j\mu_j\nu(\xi_j)|^2}\left(1+\sum_{l\neq j}\varepsilon_l\varepsilon_j O(1)\right)$$

we get

$$\tilde{W}(y)=\frac{2\mu_j}{y_1^2+\mu_j^2}+\sum_l\frac{\varepsilon_l^\alpha}{(1+|y|)} \quad \text{on} \quad \partial\mathbb{R}^2_+\bigcap B(0,\frac{\delta}{\varepsilon_j}), \tag{6.63}$$

for some $0<\alpha<1$. Thus we can conclude that

$$\left|\int_{\partial\Omega\bigcap B(\xi_j,\delta)} L(Z_{ij})\phi\right|\leq C\|\phi\|_\infty.$$

This shows the validity of (6.57).

We shall now estimate the term $\int_{\partial\Omega}hZ_{ij}$. Using the definition of the $\|\cdot\|_{*,\partial\Omega}$-norm, we observe that

$$\begin{aligned}
\left|\int_{\partial\Omega}hZ_{ij}\right| &= \int_{\partial\Omega}\rho(x)^{-1}|h|\rho(x)Z_{ij}\leq\|h\|_{*,\partial\Omega}\int_{\partial\Omega}\rho(x)Z_{ij}\\
&= \|h\|_{*,\partial\Omega}\int_{\partial\Omega}\left(\sum_{l=1}^k\rho_l\chi_{B_\delta(\xi_l)}(x)+1\right)Z_{ij}\\
&\leq C\|h\|_{*,\partial\Omega}\sum_{l=1}^k\int_{\partial\Omega\cap B_\delta(\xi_l)}\gamma_l\left\{\left(1+\frac{w_l+1}{\gamma_l}\right)\left(1+\frac{1+|w_l|}{\gamma_l}\right)e^{\frac{w_l^2}{2\gamma_l}}-1\right\}\varepsilon_l^{-1}e^{w_l}
\end{aligned}$$

$$+C\|h\|_{*,\partial\Omega} \int\limits_{\partial\Omega\setminus\bigcup_{l=1}^{k} B_\delta(\xi_l)} Z_{ij}. \tag{6.64}$$

Since Z_{ij} are uniformly bounded, as $\lambda \to 0$, in $\partial\Omega \setminus \bigcup_{l=1}^{k} B_\delta(\xi_l)$, we just need to estimate $\int\limits_{\partial\Omega\cap B_\delta(\xi_j)} \gamma_j \left\{ \left(1+\frac{w_j+1}{\gamma_j}\right)\left(1+\frac{1+|w_j|}{\gamma_j}\right) e^{\frac{w_j^2}{2\gamma_j}} - 1 \right\} \varepsilon_j^{-1} e^{w_j}$. Recall that the functions w_j are defined as

$$w_j(x) = \log \frac{2\mu_j}{|y-\xi_j'-\mu_j\nu(\xi_j')|^2},$$

with $y = \frac{x}{\varepsilon_j}$, $\xi_j' = \frac{\xi_j}{\varepsilon_j}$, and $\gamma_j = -2\log\varepsilon_j$. We decompose $\partial\Omega\cap B_\delta(\xi_j)$ into the union of $\partial\Omega \cap B_{\frac{\delta}{\gamma_j}}(\xi_j)$ and $\partial\Omega\cap\left(B_\delta(\xi_j)\setminus B_{\frac{\delta}{\gamma_j}}(\xi_j)\right)$. We write

$$\begin{aligned}
&\int\limits_{\partial\Omega\cap B_\delta(\xi_j)} \gamma_j \left\{ \left(1+\frac{w_j+1}{\gamma_j}\right)\left(1+\frac{1+|w_j|}{\gamma_j}\right) e^{\frac{w_j^2}{2\gamma_j}} - 1 \right\} \varepsilon_j^{-1} e^{w_j} \\
= &\int\limits_{\partial\Omega\cap B_{\frac{\delta}{\gamma_j}}(\xi_j)} \gamma_j \left\{ \left(1+\frac{w_j+1}{\gamma_j}\right)\left(1+\frac{1+|w_j|}{\gamma_j}\right) e^{\frac{w_j^2}{2\gamma_j}} - 1 \right\} \varepsilon_j^{-1} e^{w_j} \\
&+ \int\limits_{\partial\Omega\cap\left(B_\delta(\xi_j)\setminus B_{\frac{\delta}{\gamma_j}}(\xi_j)\right)} \gamma_j \left\{ \left(1+\frac{w_j+1}{\gamma_j}\right)\left(1+\frac{1+|w_j|}{\gamma_j}\right) e^{\frac{w_j^2}{2\gamma_j}} - 1 \right\} \varepsilon_j^{-1} e^{w_j} \\
= &\; L_1 + L_2.
\end{aligned} \tag{6.65}$$

Using the change of variables $\varepsilon_j y = x - \xi_j$, we have

$$L_1 = \int\limits_{\partial\Omega_{\varepsilon_j}\cap B_{\frac{\delta}{\gamma_j\varepsilon_j}}(0)} \gamma_j \left\{ \left(1+\frac{\bar{w}_j+1}{\gamma_j}\right)\left(1+\frac{1+|\bar{w}_j|}{\gamma_j}\right) e^{\frac{\bar{w}_j^2}{2\gamma_j}} - 1 \right\} e^{\bar{w}_j}$$

and

$$L_2 = \int\limits_{\partial\Omega_{\varepsilon_j}\cap\left(B_{\frac{\delta}{\varepsilon_j}}(0)\setminus B_{\frac{\delta}{\gamma_j\varepsilon_j}}(0)\right)} \gamma_j \left\{ \left(1+\frac{\bar{w}_j+1}{\gamma_j}\right)\left(1+\frac{1+|\bar{w}_j|}{\gamma_j}\right) e^{\frac{\bar{w}_j^2}{2\gamma_j}} - 1 \right\} e^{\bar{w}_j}$$

where $\Omega_{\varepsilon_j} = \frac{\Omega-\xi_j}{\varepsilon_j}$ and

$$\bar{w}_j = \log\frac{2\mu_j}{|y-\mu_j\nu(0)|^2}.$$

First we estimate L_1:

$$
\begin{aligned}
L_1 &= \int\limits_{\partial\Omega_{\varepsilon_j}\cap B_{\frac{\delta}{\gamma_j\varepsilon_j}}(0)} \gamma_j \left\{ \left(1+\frac{\bar{w}_j+1}{\gamma_j}\right)\left(1+\frac{1+|\bar{w}_j|}{\gamma_j}\right) e^{\frac{\bar{w}_j^2}{2\gamma_j}} - 1 \right\} e^{\bar{w}_j} \\
&\leq C \int\limits_{\partial\Omega_{\varepsilon_j}\cap B_{\frac{\delta}{\gamma_j\varepsilon_j}}(0)} e^{\bar{w}_j} = C \int\limits_{\partial\Omega_{\varepsilon_j}\cap B_{\frac{\delta}{\gamma_j\varepsilon_j}}(0)} \frac{1}{|y-\mu_j\nu(0)|^2} \\
&\leq C \int_{\mu_j-\frac{\delta}{\gamma_j\varepsilon_j}}^{\mu_j} \frac{1}{r^2} dr \leq C.
\end{aligned}
$$

On the other hand, using the fact that $\bar{w}_j = -2\log r + O(1)$ with $r = |y-\mu_j\nu(0)|$, the term L_2 can be estimated as follows

$$
\begin{aligned}
L_2 &= \int\limits_{\partial\Omega_{\varepsilon_j}\cap\left(B_{\frac{\delta}{\varepsilon_j}}(0)\setminus B_{\frac{\delta}{\gamma_j\varepsilon_j}}(0)\right)} \gamma_j \left\{ \left(1+\frac{\bar{w}_j+1}{\gamma_j}\right)\left(1+\frac{1+|\bar{w}_j|}{\gamma_j}\right) e^{\frac{\bar{w}_j^2}{2\gamma_j}} - 1 \right\} e^{\bar{w}_j} \\
&\leq C \int\limits_{\partial\Omega_{\varepsilon_j}\cap\left(B_{\frac{\delta}{\varepsilon_j}}(0)\setminus B_{\frac{\delta}{\gamma_j\varepsilon_j}}(0)\right)} \gamma_j e^{\frac{\bar{w}_j^2}{2\gamma_j}} \frac{\gamma_j+\bar{w}_j}{\gamma_j} e^{\bar{w}_j} \\
&\leq C \int_{\frac{\delta}{\gamma_j\varepsilon_j}}^{\frac{\delta}{\varepsilon_j}} \frac{1}{r^2} e^{\frac{(\log r)^2}{|\log\varepsilon_j|}} (\gamma_j - 2\log r) dr \\
&\leq C \int_{\log\frac{\delta}{\gamma_j\varepsilon_j}}^{\log\frac{\delta}{\varepsilon_j}} e^{-t} e^{\frac{t^2}{|\log\varepsilon_j|}} (\gamma_j - t) dt \leq C \int_{\log\frac{\delta}{\gamma_j\varepsilon_j}}^{\log\frac{\delta}{\varepsilon_j}} e^{-\sigma t}(\gamma_j - t) dt \leq C
\end{aligned}
$$

for some positive σ. Therefore we get

$$
\left| \int_{\partial\Omega} h Z_{ij} \right| \leq C\|h\|_{*,\partial\Omega}. \tag{6.66}
$$

Thus, from (6.55)-(6.66) we find the validity of (6.54).

We now conclude our argument by contradiction to prove (6.50). From (6.54), we have that $c_{ij,n}$ is bounded, thus we may assume that $c_{ij,n} \to c_{ij}$ as $n \to \infty$.

Let us fix $R > 0$ large sufficiently but fixed. By the maximum principe and the Hopf Lemma we find that,

$$
\max_{\bar{\Omega}\setminus\bigcup_{j=1}^k B_{R\varepsilon_j}(\xi_{j,n})} |\phi_n| = \max_{\bar{\Omega}\setminus\bigcup_{j=1}^k \partial B_{R\varepsilon_j}(\xi_{j,n})} |\phi_n|.
$$

Thus, from $\|\phi_n\|_\infty = 1$, we can find that there is some fixed $j_0 \in \{1, 2, \cdots, k\}$ such that

$$\max_{\bar{\Omega} \cap \ \partial B_{R\varepsilon_{j_0}}(\xi_{j_0,n})} |\phi_n| = 1. \tag{6.67}$$

Set $\Omega_{\varepsilon_{j_0}} = \frac{\Omega - \xi_{j_0,n}}{\varepsilon_{j_0,n}}$, and consider

$$\hat{\phi}_n(z) = \phi_n(\xi_{j_0,n} + \varepsilon_{j_0,n} z), \quad \hat{h}_n(z) = h_n(\xi_{j_0,n} + \varepsilon_{j_0,n} z), \quad \hat{f}_n(z) = f_n(\xi_{j_0,n} + \varepsilon_{j_0,n} z),$$
$$\hat{Z}_{ij}(z) = Z_{ij}(\xi_{j_0,n} + \varepsilon_{j_0,n} z)$$

Then

$$-\Delta \hat{\phi}_n(z) + \varepsilon_{j_0}^2 \hat{\phi}_n(z) = \varepsilon_{j_0}^2 f_n(z) \quad \text{in } \Omega_{\varepsilon_{j_0}},$$
$$\frac{\partial \hat{\phi}_n}{\partial \nu} - \varepsilon_{j_0}[\sum_{j=1}^k \varepsilon_j^{-1} e^{w_j}]\hat{\phi}_n = \varepsilon_{j_0}\hat{h}_n + \sum_{i=0,1}\sum_{j=1}^k \varepsilon_{j_0} c_{ij,n}\chi_j \hat{Z}_{ij} \quad \text{on } \partial\Omega_{\varepsilon_{j_0}}.$$

Then by elliptic estimate $\hat{\phi}_n$ (up to subsequence) converges uniformly on compact sets to a nontrivial solution $\hat{\phi} \neq 0$ of the problem

$$\begin{cases} \Delta\phi = 0, & \text{in } \mathbb{R}^2_+; \\ \frac{\partial\phi}{\partial\nu} - \frac{2\mu_j}{x_1^2+\mu_j^2}\phi = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

By the nondegeneracy result ([27]), we conclude that $\hat{\phi}$ is a linear combination of z_{0j} and z_{1j}. On the other hand, we can take the limit in the orthogonality relation and we find that $\int_{\partial\mathbb{R}^2_+} \chi\hat{\phi} z_{ij} = 0$ for $i = 0, 1$. This contradicts the fact that $\hat{\phi} \not\equiv 0$. This ends the proof of the Lemma. □

Proof of Proposition 6.5 In proving the solvability of (6.45), we may first solve the following problem: for given $h \in L^\infty(\partial\Omega)$, with $\|h\|_{*,\partial\Omega}$ bounded, find $\phi \in L^\infty(\Omega)$ and $d_{ij} \in \mathbb{R}$, $i = 0, 1$ $j = 1, \ldots, k$ such that

$$\begin{cases} -\Delta\phi + \phi = \sum_{i=0,1}\sum_{j=1}^k d_{ij}\chi_j Z_{ij} & \text{in } \Omega; \\ \frac{\partial\phi}{\partial\nu} - [\sum_{j=1}^k \varepsilon_j^{-1} e^{w_j}]\phi = h & \text{on } \partial\Omega; \\ \int_\Omega \chi_j Z_{ij}\phi = 0 & \text{for } \ i = 0, 1, \ j = 1, \ldots, k. \end{cases} \tag{6.68}$$

First we prove that for any ϕ, d_{ij} solution to (6.68) the bound

$$\|\phi\|_\infty \leq C\|h\|_{*,\partial\Omega} \tag{6.69}$$

holds. In fact, by Lemma 6.6, we have

$$\|\phi\|_\infty \leq C\left(\|h\|_{*,\partial\Omega} + \sum_{i=0,1}\sum_{j=1}^k \varepsilon_j |d_{ij}|\right) \tag{6.70}$$

and therefore it is enough to prove that $\varepsilon_j |d_{ij}| \leq C\|h\|_{*,\partial\Omega}$.

Fix an integer j. To show that $\varepsilon_j |d_{ij}| \leq C\|h\|_{*,\partial\Omega}$, we shall multiply equation (6.68) against a test function, properly chosen. Let us observe that, the proper test function depends whether we are considering the case $i = 0$ or $i = 1$. We start with $i = 0$. We define $\hat{z}_{0j}(y) = h(y) z_{0j}(y)$, where $h(y) = \frac{\log(\frac{\delta}{\varepsilon_j}) - \log|y|}{\log\frac{\delta}{\varepsilon_j} - \log R}$. In fact, we recognize that $\Delta h = 0$ in $B(0, \frac{\delta}{\varepsilon_j}) \setminus B(0, R)$, $h = 1$ on $\partial B(0, R)$ and $h = 0$ on $\partial B(0, \frac{\delta}{\varepsilon_j})$.

Let η_1 and η_2 be two smooth cut-off functions defined in $\mathbb{R}^2$ as

$$\eta_1 \equiv 1 \quad \text{in} \quad B(0, R), \quad \equiv 0 \quad \text{in} \quad \mathbb{R}^2 \setminus B(0, R+1)$$

so that

$$0 \leq \eta_1 \leq 1, \quad |\nabla \eta_1| \leq C$$

and

$$\eta_2 \equiv 1 \quad \text{in} \quad B(0, \frac{\delta}{4\varepsilon_j}), \quad \equiv 0 \quad \text{in} \quad \mathbb{R}^2 \setminus B(0, \frac{\delta}{3\varepsilon_j})$$

so that

$$0 \leq \eta_2 \leq 1, \quad |\nabla \eta_2| \leq C\frac{\varepsilon_j}{\delta}, \quad |\nabla^2 \eta_2| \leq C(\frac{\varepsilon_j}{\delta})^2.$$

We assume that $R > R_0$ (see (6.38)) and we define

$$\tilde{Z}_{0j}(x) = \eta_1(\varepsilon_j^{-1} F_j(x)) Z_{0j}(x) + \left(1 - \eta_1(\varepsilon_j^{-1} F_j(x))\right) \eta_2\left(\varepsilon_j^{-1} F_j(x)\right) \hat{z}_{0j}\left(\varepsilon_j^{-1} F_j(x)\right), \qquad (6.71)$$

for $x \in B(\xi_j, \delta) \bigcap \Omega$.

We multiply equation (6.68) against $\tilde{Z}_{0j}$ and we integrate by parts. We get

$$\sum_{a=0,1} d_{aj} \int_\Omega \chi_j Z_{aj} \tilde{Z}_{0j} = \int_\Omega (-\Delta \tilde{Z}_{0j} + \tilde{Z}_{0j})\phi + \int_{\partial\Omega} h\tilde{Z}_{0,j} + \int_{\partial\Omega} L(\tilde{Z}_{0j})\phi$$

Observe first that, assuming $R > R_0$, we have

$$d_{aj} \int_\Omega \chi_j Z_{aj} \tilde{Z}_{0j} = d_{aj} \int_\Omega \chi_j Z_{aj} Z_{0j} = \varepsilon_j M_0 \delta_{a0} d_{aj}(1 + o(1)), \quad \text{as} \quad \lambda \to 0. \qquad (6.72)$$

Furthermore we have that

$$\left| \int_{\partial\Omega} h\tilde{Z}_{0j} \right| \leq C\|h\|_{*,\partial\Omega}. \qquad (6.73)$$

We claim that

$$\| - \Delta \tilde{Z}_{0j} + \tilde{Z}_{0j}\|_{**,\Omega} \leq \frac{C}{|\log \varepsilon_j|} \qquad (6.74)$$

$$\|L(\tilde{Z}_{0j})\|_{*,\partial\Omega} \leq \frac{C}{|\log \varepsilon_j|} \qquad (6.75)$$

The proof of estimates (6.74) and (6.75) is postponed to the Appendix, Section 6.6. Assuming for the moment the validity of (6.74) and (6.75), from estimates (6.72)–(6.75) we conclude that

$$|\varepsilon_j d_{0j}| \leq C\left(\|h\|_{*,\partial\Omega} + |\log\varepsilon_j|^{-1}\|\phi\|_\infty\right). \tag{6.76}$$

We shall now obtain an estimate similar to (6.76) for $\varepsilon_j d_{1j}$. To do so, we use another test function. Indeed we multiply equation (6.68) against $\eta_2 Z_{1j}$ and we integrate by parts. We get

$$\sum_{a=0,1} d_{aj}\int_\Omega \chi_j Z_{aj}\eta_2 Z_{1j} = \int_\Omega(-\Delta(\eta_2 Z_{1j}) + \eta_2 Z_{1j})\phi - \int_{\partial\Omega} h\eta_2 Z_{1,j}$$
$$+\int_{\partial\Omega} L(Z_{1j})\eta_2\phi + \int_{\partial\Omega} Z_{1j}\frac{\partial\eta_2}{\partial\nu}\phi$$

Observe first that, assuming $R > R_0$, we have

$$d_{aj}\int_\Omega \chi_j Z_{aj}\eta_2 Z_{1j} = d_{aj}\int_\Omega \chi_j Z_{aj} Z_{1j} = M_1\delta_{a1}\varepsilon_j d_{1j}(1+o(1)), \quad \text{as} \quad \lambda\to 0,$$

and $\left|\int_{\partial\Omega} h\eta_2 Z_{1j}\right| \leq C\|h\|_{*,\partial\Omega}$. Using the change of variables $y = \varepsilon_j^{-1}F_j(x)$, we get that

$$\int_{\partial\Omega} Z_{1j}\frac{\partial\eta_2}{\partial\nu}\phi = \int_{\partial\Omega_{\varepsilon_j}} z_{1j}\frac{\partial\eta_2}{\partial\nu}\tilde\phi$$

where $\Omega_{\varepsilon_j} = \frac{\Omega}{\varepsilon_j}$ and $\tilde\phi(y) = \phi(F_j^{-1}(\varepsilon_j^{-1}y))$. But $z_{1j} = O(\frac{1}{1+r})$ and $\nabla\eta_2 = O(\varepsilon_j)$ so $|\int_{\partial\Omega} Z_{1j}\frac{\partial\eta_2}{\partial\nu}\phi| \leq C\varepsilon_j|\log\varepsilon_j|$. Using again the change of variables $y = \varepsilon_j^{-1}F_j(x)$, and proceeding similarly to (6.61), (6.62) and (6.63), one gets

$$\int_{\partial\Omega} L(Z_{ij})\eta_2\phi = (1+o(1))\int_{\partial\Omega_{\varepsilon_j}}[\frac{\partial z_{ij}}{\partial\nu} - \tilde W z_{ij}]\eta_2\tilde\phi$$

where $\tilde\phi(y) = \phi(F_j^{-1}(\varepsilon_j y))$ and $b(y)$ is a positive function, coming from the change of variables, which is uniformly positive and bounded as $\lambda\to 0$. Observe that $\frac{\partial z_{ij}}{\partial\nu} - \tilde W z_{ij} = O(\frac{\varepsilon_j}{1+r}) + O(\frac{\varepsilon_j^\alpha}{1+r^2})$ for $y\in\Omega_{\varepsilon_j}$ and $|y| < \delta\varepsilon_j^{-1}$, and this implies that

$$\int_{\partial\Omega_{\varepsilon_j}}\left|\frac{\partial z_{ij}}{\partial\nu} - \tilde W z_{ij}\right| \leq C\varepsilon_j^\alpha$$

for some $0<\alpha<1$. Thus we can conclude that

$$\left|\int_{\partial\Omega} L(Z_{ij})\,\eta_2\phi\right| \leq C\varepsilon_j^\alpha\|\phi\|_\infty.$$

Consider once again the change of variables $y = \varepsilon_j^{-1}F_j(x)$. Arguing as in (6.59) and (6.60) we get that

$$\int_\Omega(-\Delta(\eta_2 Z_{ij}) + \eta_2 Z_{ij})\phi = (1+o(1))\int_{\Omega_{\varepsilon_j}}\left(-\Delta(\eta_2 z_{ij}) + \varepsilon_j^2\eta_2 z_{ij}\right)\tilde\phi$$

where $\tilde{\phi}(y) = \phi(F_j^{-1}(\varepsilon_j y))$. We thus compute in $y \in \Omega_{\varepsilon_1}$, with $|y| < \delta\varepsilon_j^{-1}$,

$$\Delta(\eta_2 z_{1j}) = \Delta\eta_2\, z_{1j} + 2\nabla\eta_2\nabla z_{1j} + \eta_2\,\Delta z_{1j} = O(\frac{\varepsilon_1^2}{1+r}) + O(\frac{\varepsilon_j}{1+r}) + \eta_2\Delta z_{1j}.$$

On the other hand, in this region we have $-\Delta z_{1j} + \varepsilon_j^2 z_{1j} = O(\frac{\varepsilon_j}{1+r^2}) + O(\frac{\varepsilon_j^2}{1+r})$. Thus

$$\int_{\Omega_{\varepsilon_j}} \left|-\Delta(\eta_2 z_{ij}) + \varepsilon_j^2\eta_2 z_{ij}\right| \le C\varepsilon_j|\log\varepsilon_j|$$

Summarizing all the above information, we get

$$|\varepsilon_j d_{1j}| \le C\left(\|h\|_{*,\partial\Omega} + \varepsilon_j\|\phi\|_\infty\right) \tag{6.77}$$

Estimates (6.76), (6.77) combined with (6.70) yields

$$|\varepsilon_j d_{ij}| \le C\|h\|_{*,\partial\Omega}.$$

which gives the validity of (6.69). Now consider the Hilbert space

$$\mathbb{H} = \left\{\phi \in H^1(\Omega)\ :\ \int_\Omega \chi_j Z_{ij}\phi = 0\ \ \forall\ i=0,1,\ \ j=1,\dots,k\right\},$$

endowed the norm $\|\phi\|_{H^1}^2 = \int_\Omega(|\nabla\phi|^2+\phi^2)$. Problem (6.68), expressed in a weak form, is equivalent to find $\phi \in \mathbb{H}$ such that

$$\int_\Omega(\nabla\phi\nabla\psi + \phi\psi) - \int_{\partial\Omega}[\sum_{j=1}^k \varepsilon_j^{-1}e^{w_j}]\psi = \int_{\partial\Omega} h\psi, \qquad \text{for all } \psi \in \mathbb{H},$$

With the aid of Fredholm's alternative guarantees unique solvability of (6.68), which is guarantees by (6.69).

In order to solve (6.45), let $Y_{ls} \in L^\infty(\Omega_\varepsilon)$, $d_{ij}^{ls} \in \mathbb{R}$ be the solution of (6.68) with $h = \chi_s Z_{ls}$, that is

$$\begin{cases} -\Delta Y_{ls} + Y_{ls} = \sum_{i=0,1}\sum_{j=1}^k d_{ij}^{ls}\chi_j Z_{ij} & \text{in } \Omega; \\ \frac{\partial Y_{ls}}{\partial\nu} - [\sum_{j=1}^k \varepsilon_j^{-1}e^{w_j}]Y_{ls} = \chi_s Z_{ls} & \text{on } \partial\Omega; \\ \int_\Omega \chi_j Z_{ij} Y_{ls} = 0 & \text{for } \ l=0,1,\ s=1,\dots,k, \end{cases} \tag{6.78}$$

Then there is a unique solution $Y_{ls} \in L^\infty(\Omega)$ of (6.78), and

$$\|Y_{ls}\|_\infty \le C, \qquad \varepsilon_j|d_{ij}^{ls}| \le C \tag{6.79}$$

for some constant C independent on λ.

Multiplying (6.78) by Z_{ij}, and integrates by parts, we have

$$\begin{aligned}\sum_{i=0,1}\sum_{j=1}^{k}\int_{B(\xi_j,\delta)} d_{ij,ls}\chi_j(Z_{ij})^2 &= \int_{\partial B(\xi_j,\delta)}\chi_s Z_{ls}Z_{ij} + \int_{B(\xi_j,\delta)}(-\Delta Z_{ij}+Z_{ij})\,Y_{ls} \\ &\quad + \int_{\partial B(\xi_j,\delta)}\left(\frac{\partial Z_{ij}}{\partial\nu} - [\sum_{j=1}^{k}\varepsilon_j^{-1}e^{w_j}]Z_{ij}\right)Y_{ls} \\ &= \delta_{il}\delta_{js}\int_{\partial B(\xi_j,\delta)}\chi_j(Z_{ij})^2 + o(1)\end{aligned}$$

where δ_{il},δ_{js} are Kronecker's delta. Then we get

$$d_{0j,0s} = a\delta_{js} + o(1), \quad d_{1j,1s} = b\delta_{js} + o(1) \tag{6.80}$$

with $a, b > 0$ are independent of ε_j. Hence the matrix D_1 (or D_2) with entries $d_{0j,0s}$ (or $d_{1j,1s}$) in invertible for small ε_j and $\|D_i^{-1}\| \le C(i = 1, 2)$ uniformly in ε_j.

Now, given $h \in L^\infty(\partial\Omega)$ we find ϕ_1, d_{ij}, solution to (6.68). Define constants c_{ls} as

$$\sum_{l=0,1}\sum_{s=1}^{k} c_{ls}d_{ij}^{ls} = -d_{ij}, \quad \forall\ i = 0,1, \quad j = 1,\dots,k.$$

The above linear system is almost diagonal, since arguing as before one can show that $d_{ij}^{ls} = \varepsilon_j^{-1}M_i\delta_{js}\delta_{il}(1+o(1))$, as $\lambda \to 0$, where M_i is a positive universal constant. Then define

$$\phi = \phi_1 + \sum_{l=0,1}\sum_{s=1}^{k} c_{ls}Y_{ls},$$

A direct computation shows that ϕ satisfies (6.45) and furthermore

$$\|\phi\|_\infty \;\le\; \|\phi_1\|_\infty + \sum_{l=0,1}\sum_{s=1}^{k}|c_{ls}| \le C\|h\|_{*,\partial\Omega} + \sum_{i=0,1}\sum_{j=1}^{k}\varepsilon_j|d_{ij}| \le C\|h\|_{*,\partial\Omega}$$

by (6.69). This finishes the proof of Proposition 6.5.

Remark 6.7. *A slight modification of the proof above also shows that for any $h \in L^\infty(\partial\Omega)$ and $f \in L^\infty(\Omega)$, with $\|h\|_{*,\partial\Omega}$, $\|f\|_{**,\Omega} < \infty$, the equation*

$$\begin{cases} -\Delta\phi + \phi = f & \text{in } \Omega; \\ L(\phi) = h + \sum\limits_{i=0,1}\sum\limits_{j=1}^{k} c_{ij}\chi_j Z_{ij} & \text{on } \partial\Omega; \\ \int_\Omega \chi_j Z_{ij}\phi = 0 & \text{for } \ i = 0,1, \quad j = 1,\dots,k, \end{cases}$$

has a unique solution ϕ, c_{ij}, $i = 0, 1$, $j = 1, \ldots, k$ *and that satisfy*

$$\|\phi\|_\infty \le C \left(\|h\|_{*,\partial\Omega} + \|f\|_{**,\Omega} \right),$$

$$|c_{ij}| \le C \left(\|h\|_{*,\partial\Omega} + \|f\|_{**,\Omega} \right), \quad \forall\ i = 0, 1, \quad j = 1, \ldots, k$$

holds for C *independent of* λ.

The result of Proposition 6.5 implies that the unique solution $\phi = T_\lambda(h)$ of (6.45) defines a continuous linear map form the Banach space $\mathcal{C}_*$ of all functions h in $L^\infty(\partial\Omega)$ for which $\|h\|_{*,\partial\Omega} < \infty$ into L^∞, with norm bounded uniformly in λ.

Lemma 6.8. *The operator* T_λ *is differentiable with respect to the variable* $\xi_1, \ldots, \xi_k$ *on* $\partial\Omega$ *satisfying 6.46, and* $m_1, \ldots, m_k$, *one has the estimate*

$$\|D_\xi T_\lambda(h)\|_\infty \le C\|h\|_{*,\partial\Omega}, \qquad \|D_m T_\lambda(h)\|_\infty \le C\|h\|_{*,\partial\Omega}. \tag{6.81}$$

for a given positive C, *independent of* λ, *and for all* λ *small enough.*

Proof. Differentiating equation (6.45), formally $Z := \partial_{\xi_{sl}}\phi$, for all s, l, should satisfy in Ω the equation

$$-\Delta Z + Z = 0 \quad \text{in } \Omega,$$

and on the boundary $\partial\Omega$

$$L(Z) = -\partial_{\xi_{sl}} \left(\sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j} \right) \phi + \sum_{i=0,1} \sum_{j=1}^{k} c_{ij} \partial_{\xi_{sl}} \left(\chi_j Z_{ij} \right) + \sum_{i=0,1} \sum_{j=1}^{k} d_{ij} Z_{ij} \chi_j$$

with $d_{ij} = \partial_{\xi_{sl}} c_{ij}$, and the orthogonality conditions now become

$$\int_\Omega Z_{ij} \chi_j Z = 0 \quad \text{if } s \ne j.$$

$$\int_\Omega Z_{is} \chi_s Z = -\int_\Omega \partial_{\xi_{sl}} \left(Z_{is} \chi_s \right) \phi.$$

We consider the constants α_{ab}, $a = 0, 1$, $b = 1, \ldots, k$, defined as

$$\alpha_{ab} \int_\Omega \chi_b^2 |Z_{ab}|^2 = \int_\Omega \partial_{\xi_{sl}} \left(Z_{ab} \chi_b \right) \phi, \qquad \text{for } \ a = 0, 1, \ b = 1, \ldots, k.$$

Define

$$\tilde{Z} = Z + \sum_{a=0,1} \sum_{b=1}^{k} \alpha_{ab} \chi_b Z_{ab}.$$

We then have

$$\begin{cases} -\Delta\tilde{Z} + \tilde{Z} = f_1 & \text{in } \Omega; \\ L(\tilde{Z}) = h_1 + \sum_{i=0,1}\sum_{j=1}^{k} d_{ij} Z_{ij}\chi_j & \text{on } \partial\Omega; \\ \int_\Omega \chi_j Z_{ij}\tilde{Z} = 0 & \text{for } \ i=0,1, \quad j=1,\dots,k, \end{cases}$$

where

$$f_1 = \sum_{a=0,1}\sum_{b=1}^{k} \alpha_{ab}(-\Delta(\chi_b Z_{ab}) + \chi_b Z_{ab}),$$

$$h_1 = -\partial_{\xi_{sl}}\left(\sum_{j=1}^{k}\varepsilon_j^{-1}e^{w_j}\right)\phi + \sum_{i=0,1}\sum_{j=1}^{k} c_{ij}\partial_{\xi_{ls}}\left(Z_{ij}\chi_j\right) + \sum_{a=0,1}\sum_{b=1}^{k}\alpha_{ab}L(\chi_b Z_{ab}).$$

Hence, using the result of Proposition 6.5 we have that

$$\|\tilde{Z}\|_\infty \le C\left(\|h_1\|_{*,\partial\Omega} + \|f_1\|_{**,\Omega}\right).$$

By the definition of α_{ab}, we get $|\alpha_{ab}| \le C\|\phi\|_\infty$. Since $\|\phi\|_\infty \le C\|h\|_{*,\partial\Omega}$, $|c_{ij}| \le C\|h\|_{*,\partial\Omega}$ we obtain that

$$\|\tilde{Z}\|_\infty \le C\|h\|_{*,\partial\Omega}.$$

Hence we get

$$\|\partial_{\xi_{sl}} T_\lambda(h)\|_\infty \le C\|h\|_{*,\partial\Omega} \quad \text{for all } \ s,l.$$

Analogous computation holds true if we differentiate with respect to m_j. □

We are now in the position to prove Proposition 6.3.

Proof of Proposition 6.3. In terms of the operator T_λ defined in Proposition 6.5, problem (6.39) becomes

$$\phi = T_\lambda\left(E + N(\phi)\right) := A(\phi). \tag{6.82}$$

For a given number $\gamma > 0$, let us consider the region

$$\mathcal{F}_\gamma := \left\{\phi \in C(\bar{\Omega}) \ : \ \|\phi\|_\infty \le \gamma\lambda\right\}.$$

From Proposition 6.5, we get

$$\|A(\phi)\|_\infty \le C\left[\|E\|_{*,\partial\Omega} + \|N(\phi)\|_{*,\partial\Omega}\right].$$

We claim that

$$\left\| f'(\tilde{U}) - \sum_{j=1}^{k}\varepsilon_j^{-1}e^{w_j}\right\|_{*,\partial\Omega} \le C\lambda. \tag{6.83}$$

and

$$\left\| f''(\tilde{U}) \right\|_{*,\partial\Omega} \le C. \tag{6.84}$$

We postpone the proofs of (6.83) and (6.84) to the Appendix, Section 6.6. From (6.35), (6.83) and (6.84), from the definition of $N(\phi)$ in (6.28), it follows that

$$\|A(\phi)\|_\infty \;\le\; C\left(\lambda + \|\phi\|_\infty^2 + \lambda\|\phi\|_\infty\right).$$

We then get that $A(\mathcal{F}_\gamma) \subset \mathcal{F}_\gamma$ for a sufficiently large but fixed γ and all small λ. Moreover, for any $\phi_1, \phi_2 \in \mathcal{F}_\gamma$, one has

$$\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega} \le C\left[\left(\max_{i=1,2} \|\phi_i\|_\infty\right) + \lambda\right] \|\phi_1 - \phi_2\|_\infty,$$

In fact, since

$$\begin{aligned}
& N(\phi_1) - N(\phi_2) \\
= \;& f(\tilde{U} + \phi_1) - f(\tilde{U} + \phi_2) - f'(\tilde{U})(\phi_1 - \phi_2) + \left[f'(\tilde{U}) - \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j}\right](\phi_1 - \phi_2) \\
= \;& \int_0^1 \left(\frac{d}{dt} f(\tilde{U} + \phi_2 + t(\phi_1 - \phi_2))\right) dt - f'(\tilde{U})(\phi_1 - \phi_2) + \left[f'(\tilde{U}) - \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j}\right](\phi_1 - \phi_2) \\
= \;& \int_0^1 \left(f'(\tilde{U} + \phi_2 + t(\phi_1 - \phi_2)) - f'(\tilde{U})\right) dt\,(\phi_1 - \phi_2) + \left[f'(\tilde{U}) - \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j}\right](\phi_1 - \phi_2)
\end{aligned}$$

Thus, for a certain $t^* \in (0,1)$, and $s \in (0,1)$

$$\begin{aligned}
& |N(\phi_1) - N(\phi_2)| \\
\le \;& C\left[|f'(\tilde{U} + \phi_2 + t^*(\phi_1 - \phi_2)) - f'(\tilde{U})| + \left(f'(\tilde{U}) - \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j}\right)\right] \|\phi_1 - \phi_2\|_\infty \\
\le \;& C\Big[|f''(\tilde{U} + s\phi_2 + t^*(\phi_1 - \phi_2))| \left(\|\phi_1\|_{L^\infty(\Omega)} + \|\phi_2\|_\infty\right) \\
& + [f'(\tilde{U}) - \sum_{j=1}^k \varepsilon_j^{-1} e^{w_j}]\Big] \|\phi_1 - \phi_2\|_\infty.
\end{aligned}$$

Thanks to (6.83), (6.84) and the fact that $\|\phi_1\|_\infty, \|\phi_2\|_\infty \to 0$ as $\lambda \to 0$, we conclude that

$$\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega} \;\le\; C\left[\|\phi_1\|_\infty + \|\phi_2\|_\infty + \lambda\right] \|\phi_1 - \phi_2\|_\infty.$$

Then we have

$$\|A(\phi_1) - A(\phi_2)\|_\infty \le C\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega} \le C\left[\max_{i=1,2} \|\phi_i\|_\infty + \lambda\right] \|\phi_1 - \phi_2\|_\infty.$$

Thus the operator A has a small Lipschitz constant in $\mathcal{F}_\gamma$ for all small λ, and therefore a unique fixed point of A exists in this region.

We shall next analyze the differentiability of the map $(\xi, m) = (\xi_1, \dots, \xi_k, m_1, \dots, m_k) \mapsto \phi$. Assume for instance that the partial derivative $\partial_{\xi_{sl}}\phi$ exists, for $s = 1, \dots, k$, $l = 1, 2$. Since $\phi = T_\lambda (N(\phi) + E)$, formally we have that

$$\partial_{\xi_{sl}}\phi = (\partial_{\xi_{sl}} T_\lambda) (N(\phi) + E) + T_\lambda (\partial_{\xi_{sl}} N(\phi) + \partial_{\xi_{sl}} E) .$$

From (6.81), we have

$$\|\partial_{\xi_{sl}} T_\lambda (N(\phi) + E)\|_\infty \leq C\|N(\phi) + E\|_{*,\partial\Omega} \leq C\lambda.$$

On the other hand,

$$\begin{aligned}\partial_{\xi_{sl}} N(\phi) &= [f'(\tilde{U} + \phi) - f'(\tilde{U}) - f''(\tilde{U})\phi]\partial_{\xi_{sl}}\tilde{U} + \partial_{\xi_{sl}} \left(\frac{\partial Z_{ij}}{\partial \nu} - [\sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j}] \right) \phi \\ &\quad + [f'(\tilde{U} + \phi) - f'(\tilde{U})]\partial_{\xi_{sl}}\phi + \left(f'(\tilde{U}) - [\sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_{\mu_j}}] \right) \partial_{\xi_{sl}}\phi.\end{aligned}$$

Then,

$$\|\partial_{\xi_{sl}} N(\phi)\|_{*,\partial\Omega} \leq C \left\{ \|\phi\|_\infty^2 + \lambda\|\phi\|_\infty + \|\phi\|_\infty \|\partial_{\xi_{sl}}\phi\|_\infty + \lambda\|\partial_{\xi_{sl}}\phi\|_\infty \right\} .$$

Since $\|\partial_{\xi_{sl}} E\|_{*,\partial\Omega} \leq \lambda$, Proposition 6.5 guarantees that

$$\|\partial_{\xi_{sl}}\phi\|_\infty \leq C\lambda$$

for all s, l. Analogous computation holds true if we differentiate with respect to m_j. Then, the regularity of the map $(\xi, m) \mapsto \phi$ can be proved by standard arguments involving the implicit function theorem and the fixed point representation (6.82). This concludes proof of the Proposition. □

6.4 Variation Reduction

Up to now we have solved the nonlinear problem (6.39). In order to find a solution to the original problem we need to find ξ and m such that

$$c_{ij}(\xi, m) = 0 \quad \text{for all } i = 0, 1, \quad j = 1, \cdots, k. \tag{6.85}$$

We recall the following definitions: the energy functional associated to Problem (6.1) is

$$J_\lambda(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + u^2) - \frac{\lambda}{2} \int_{\partial\Omega} e^{u^2}, \quad u \in H^1(\Omega), \tag{6.86}$$

and the finite-dimensional restriction

$$\mathcal{I}_\lambda(\xi, m) = J_\lambda\left(\sqrt{\lambda}\left(\tilde{U}(\xi, m) + \phi(\xi, m)\right)\right), \tag{6.87}$$

where ϕ is the unique solution to problem (6.39) given by Proposition 6.3. Critical points of $\mathcal{I}_\lambda$ correspond to solutions of (6.85) for a small λ, as the result of Proposition (6.4) states. We give the proof of this result.

Proof of Proposition 6.4 A direct consequence of the results obtained in Proposition 6.3 and the definition of function $\tilde{U}$ is the fact the map $(\xi, m) \mapsto \mathcal{I}_\lambda(\xi, m)$ is of class C^1.

From Proposition 6.3, we have

$$\begin{aligned}
& D_{\xi,m}\mathcal{I}_\lambda(\xi, m) \\
= \; & D_{\xi,m}J_\lambda\left(\sqrt{\lambda}\left(\tilde{U}(\xi, m) + \phi(\xi, m)\right)\right)\left[\sqrt{\lambda}D_{\xi,m}\tilde{U}(\xi, m) + \sqrt{\lambda}D_{\xi,m}\phi(\xi, m)\right] \\
= \; & D_{\xi,m}J_\lambda\left(\sqrt{\lambda}\left(\tilde{U}(\xi, m) + \phi(\xi, m)\right)\right)\left[\sqrt{\lambda}D_{\xi,m}\tilde{U}(\xi, m)\right](1 + o(1)).
\end{aligned} \tag{6.88}$$

We can rewrite

$$(\tilde{U} + \phi)(\xi, m)(x) \;=\; m_l v_l\left(\frac{x - \xi_l}{\varepsilon_l}\right) + \frac{1}{2\lambda m_l}$$

with

$$v_l(y) := w_{\mu_l}(y) + \sum_{j=1}^{k}\left(O(|\varepsilon_l y + \xi_l - \xi_j|) + O(\varepsilon_j^2)\right) \quad \text{for} \;\; |y| \leq \frac{\delta}{\varepsilon_l}.$$

Since $\tilde{U} + \phi$ is the solution of (6.39), then v_l satisfies

$$-\Delta v_l + \varepsilon_l^2\left(v_l + \frac{1}{2\lambda m_l^2}\right) = 0, \quad \text{in} \;\; \Omega_l$$

and

$$\begin{aligned}
& \frac{\partial v_l}{\partial \nu} - (1 + 2\lambda m_l^2 v_l)e^{v_l}e^{\lambda m_l^2 v_l^2} \\
= \; & m_l^{-1}\varepsilon_l \sum_{i=0,1}\sum_{j=1}^{k} c_{ij}\varepsilon_j^{-1}\chi\left(\frac{F_j(\varepsilon_l y + \xi_l - \xi_j)}{\varepsilon_j}\right) z_{ij}\left(\frac{F_j(\varepsilon_l y + \xi_l - \xi_j)}{\varepsilon_j}\right), \quad \text{on} \;\; \partial\Omega_l
\end{aligned}$$

where $\Omega_l = \frac{\Omega - \xi_l}{\varepsilon_l}$. For any l, we define

$$I_l(v_l) = \frac{1}{2}\int_{\Omega_l}\left[|\nabla v_l|^2 + \varepsilon_l^2\left(v_l + \frac{1}{2\lambda m_l^2}\right)^2\right] - \int_{\partial\Omega_l} e^{v_l}e^{\lambda m_l^2 v_l^2}.$$

We note that

$$\mathcal{I}_\lambda(\xi, m) = \lambda m_l^2 I_l(v_l).$$

We compute the differential $D_{m_s}\mathcal{I}_\lambda(\xi, m)$, $s = 1, \cdots, k$, thus we have

$$\begin{aligned}
& D_{m_s}\mathcal{I}_\lambda(\xi, m) \\
= \; & \lambda m_l^2 D_{m_s} I_l(v_l) = \lambda m_l^2 D I_l(v_l)[D_{m_s} v_l] \\
= \; & \lambda m_l \varepsilon_l \sum_{i=0,1} \sum_{j=1}^{k} \left(\int_{\partial\Omega_l} \varepsilon_j^{-1} \chi \left(\frac{F_j(\varepsilon_l y + \xi_l - \xi_j)}{\varepsilon_j} \right) z_{ij} \left(\frac{F_j(\varepsilon_l y + \xi_l - \xi_j)}{\varepsilon_j} \right) D_{m_s} v_l(y)\; dy \right) c_{ij}.
\end{aligned}$$

Now, fix i and j, we compute the coefficient in front of c_{ij}, we choose $l = j$ and obtain

$$\begin{aligned}
& \int_{\partial\Omega_l} \varepsilon_j^{-1} \chi \left(\frac{F_j(\varepsilon_l y + \xi_l - \xi_j)}{\varepsilon_j} \right) z_{ij} \left(\frac{F_j(\varepsilon_l y + \xi_l - \xi_j)}{\varepsilon_j} \right) D_{m_s} v_l(y)\; dy \\
= \; & \int_{\partial\Omega_l} \varepsilon_j^{-1} \chi\left(y\right) z_{ij}\left(y\right) D_{m_s} \left[w_{\mu_j}(y) + \sum_{j=1}^{k} \left(O(|\varepsilon_j y|) + O(\varepsilon_j^2) \right) \right] \; dy \\
= \; & \frac{\partial \mu_j}{\partial m_s} \int_{\partial\mathbb{R}^2_+} z_{0j}^2(y)\; dy(1 + o(1)).
\end{aligned}$$

Thus we concludes that for any $s = 1, 2, \cdots, k$, we have

$$D_{m_s}\mathcal{I}_\lambda(\xi, m) = \lambda m_l \varepsilon_l \sum_{j=1}^{k} \frac{\partial \mu_j}{\partial m_s} \int_{\partial\mathbb{R}^2_+} z_{0j}^2(y)\; dy c_{0j}(1 + o(1)).$$

Similarly, we get that for all s, l

$$\begin{aligned}
& D_{\xi_{s1}}\mathcal{I}_\lambda(\xi, m) \\
= \; & \lambda m_l \varepsilon_l \left[\sum_{j=1}^{k} \left(\frac{\partial \mu_j}{\partial \xi_{s1}} \int_{\partial\mathbb{R}^2_+} z_{0j}^2(y) dy \right) c_{0j} + \left(\int_{\partial\mathbb{R}^2_+} z_{1s}^2(y) dy \right) c_{1s} \right] (1 + o(1)).
\end{aligned}$$

Thus, we can conclude that $D_{\xi,m}\mathcal{I}_\lambda(\xi, m) = 0$, is equivalent to the following system

$$\left[\sum_{j=1}^{k} \frac{\partial \mu_j}{\partial m_s} c_{0j} \right] (1 + o(1)) = 0, \quad s = 1, 2, \cdots, k, \tag{6.89}$$

$$\left[A \sum_{j=1}^{k} \frac{\partial \mu_j}{\partial \xi_{s1}} c_{0j} + c_{1s} \right] (1 + o(1)) = 0, \quad \text{for all } s, \tag{6.90}$$

for some fixed constant A, with $o(1)$ small in the sense of the L^∞ norm as $\lambda \to 0$. The conclusion of the Lemma follows if we show that the matrix $\frac{\partial \mu_j}{\partial m_s}$ of dimension $k \times k$ is invertible in the range of the points ξ_j and parameters m_j we are considering. Indeed, this fact implies unique solvability of (6.89). Inserting this in (6.90) we get unique solvability of (6.90).

Consider the definition of the μ_j, in terms of m_j's and points ξ_j given in (6.46). These relations correspond to the gradient $D_m F(m,\xi)$ of the function $F(m,\xi)$ defined as follows

$$F(m,\xi) = \frac{1}{2}\sum_{j=1}^{k} m_j^2 \left[-2\log\left(2m_j^2\right) - \log(2\mu_j) + 2 + H(\xi_j,\xi_j)\right] + \sum_{i\neq j} m_i m_j G(\xi_i,\xi_j).$$

We set $s_j = m_j^2$, then the above function can be written as follows

$$F(s,\xi) \quad = \quad \frac{1}{2}\sum_{j=1}^{k} s_j\left[-2\log(2s_j) - \log(2\mu_j) + 2 + H(\xi_j,\xi_j)\right] + \sum_{i\neq j} G(\xi_i,\xi_j)\sqrt{s_i s_j}.$$

This function is strictly convex function of the parameters s_j, for parameters s_j uniformly bounded and uniformly bounded away from 0 and for points ξ_j in Ω uniformly far away from each other and from the boundary. For this reason, the matrix $(\frac{\partial^2 F}{\partial s_i \partial s_j})$ is invertible in the range of parameters and points we are considering. Thus, by the implicit function theorem, relation (6.31) defines a diffeomorphism between μ_j and m_j. This fact gives the invertibility of $(\frac{\partial \mu_j}{\partial m_s})$. This concludes the proof of Lemma.

In order to solve for critical points of the functional $\mathcal{I}_\lambda$, a key step is its expected closeness to the functional $J_\lambda(\sqrt{\lambda}\tilde{U})$. This fact is contained in the following

Lemma 6.9. *The following expansion holds*

$$\mathcal{I}_\lambda(\xi,m) = J_\lambda(\sqrt{\lambda}\tilde{U}) + \vartheta_\lambda(\xi,m),$$

where

$$|\vartheta_\lambda(\xi,m)| + |\nabla\vartheta_\lambda(\xi,m)| = O(\lambda^3),$$

uniformly on points $\xi_1,\dots,\xi_k$ and parameters $m_1,\dots,m_k$ satisfying the constraints in Proposition 6.5.

Proof. Taking into account $DJ_\lambda(\sqrt{\lambda}(\tilde{U}+\phi))[\phi] = 0$, a Taylor expansion gives

$$\begin{aligned} & J_\lambda(\sqrt{\lambda}(\tilde{U}+\phi)) - J_\lambda(\sqrt{\lambda}\tilde{U}) \\ = \; & \lambda\int_0^1 D^2 J_\lambda\left(\sqrt{\lambda}(\tilde{U}+t\phi)\right)[\phi]^2(1-t)\,dt \\ = \; & \lambda\int_0^1\left(\int_{\partial\Omega}[N(\phi)+E]\phi + \int_{\partial\Omega}[f'(\tilde{U}) - f'(\tilde{U}+t\phi)]\phi^2\right)(1-t)\,dt. \end{aligned} \tag{6.91}$$

Since $\|\phi\|_\infty \le C\lambda$, we have

$$J_\lambda(\sqrt{\lambda}(\tilde{U}+\phi)) - J_\lambda(\sqrt{\lambda}\tilde{U}) = \vartheta_\lambda(\xi,m) = O(\lambda^3).$$

Let us differentiate with respect to ξ. We use the representation (6.91) and differentiate directly under the integral sign, we get that, for all j, l

$$\begin{aligned} &\partial_{\xi_{jl}}\left[J_\lambda(\sqrt{\lambda}(\tilde{U}+\phi)) - J_\lambda(\sqrt{\lambda}\tilde{U})\right] \\ = \ &\lambda \int_0^1 \left(\int_{\partial\Omega} \partial_{\xi_{jl}}[(N(\phi)+E)\phi] + \int_{\partial\Omega} \partial_{\xi_{jl}}[(f'(\tilde{U}) - f'(\tilde{U}+t\phi))\phi^2]\right)(1-t)\, dt. \end{aligned}$$

Since $\|\partial_{\xi_{jl}}\phi\|_\infty \leq C\lambda$ and the computations in the proof of Lemma 6.3 we get

$$\partial_{\xi_{jl}}[J_\lambda(\sqrt{\lambda}(\tilde{U}+\phi)) - J_\lambda(\sqrt{\lambda}\tilde{U})] = \partial_{\xi_{jl}}\vartheta_\lambda(\xi, m) = O(\lambda^3).$$

And, in the same argument, we get

$$\partial_{m_j}[J_\lambda(\sqrt{\lambda}(\tilde{U}+\phi)) - J_\lambda(\sqrt{\lambda}\tilde{U})] = O(\lambda^3).$$

The continuity in ξ and m of all these expressions is inherited from that of ϕ and its derivatives in ξ and m in the L^∞ norm. This concludes the proof. □

We end this section with the asymptotic estimate of $J_\lambda(U)$, where

$$U(x) = \sqrt{\lambda}\tilde{U}(x) = \sqrt{\lambda}\sum_{j=1}^k m_j \left[\log \frac{1}{|x - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2} + H_j(x)\right]$$

and J_λ is the energy functional associated to (6.1), whose definition is as follows

$$J_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2) - \frac{\lambda}{2}\int_{\partial\Omega} e^{u^2}.$$

We have the following result.

Lemma 6.10. *Let μ_j be given by (6.31). Then*

$$J_\lambda(U) = \frac{k\pi}{2} - \frac{|\partial\Omega|}{2}\lambda + \pi\varphi_k(\xi, m)\lambda + \lambda^2\Theta_\lambda(\xi, m), \tag{6.92}$$

with $|\partial\Omega|$ denotes the measure of domain $\partial\Omega$, and $\Theta_\lambda(\xi, m)$ is a function, uniformly bounded with its derivatives, as $\lambda \to 0$, for points ξ and parameters m satisfying (6.46). Furthermore the function $\varphi_k(\xi, m) = \varphi_k(\xi_1, \ldots, \xi_k, m_1, \ldots, m_k)$ is defined by

$$\begin{aligned} \varphi_k(\xi, m) \ = \ & 2(\log 2 - 1)\sum_{j=1}^k m_j^2 + 2\sum_{j=1}^k m_j^2 \log(m_j^2) \\ & - \sum_{j=1}^k m_j^2 H(\xi_j, \xi_j) - \sum_{i\neq j} m_i m_j G(\xi_i, \xi_j). \end{aligned}$$

Proof. Let us set

$$U(x)=\sum_{j=1}^{k}U_j(x),\qquad \text{with } \ U_j(x)=\sqrt{\lambda}m_j[u_j(x)+H_j(x)]$$

where

$$u_j(x)=\log\frac{1}{|x-\xi_j-\varepsilon_j\mu_j\nu(\xi_j)|^2}$$

and H_j defined in (6.19). Then

$$\begin{aligned}J_\lambda(U) &= \frac{1}{2}\int_\Omega\left|\sum_{j=1}^{k}\nabla U_j\right|^2+\frac{1}{2}\int_\Omega\left(\sum_{j=1}^{k}U_j\right)^2-\frac{\lambda}{2}\int_{\partial\Omega}e^{U^2}\\ &= \sum_{j=1}^{k}\frac{1}{2}\int_\Omega\left(|\nabla U_j|^2+U_j^2\right)+\frac{1}{2}\sum_{i\neq j}\int_\Omega(\nabla U_i\nabla U_j+U_iU_j)-\frac{\lambda}{2}\int_{\partial\Omega}e^{U^2}\\ &= I_1+I_2+I_3. \qquad (6.93)\end{aligned}$$

First, we write

$$\begin{aligned}\int_\Omega(|\nabla U_j|^2+U_j^2) &= \lambda m_j^2\left[\int_\Omega|\nabla u_j|^2+\int_\Omega u_j^2+\int_\Omega|\nabla H_j|^2+\int_\Omega(H_j)^2\right.\\ &\quad\left.+2\int_\Omega\nabla u_j\nabla H_j+2\int_\Omega u_jH_j\right]. \qquad (6.94)\end{aligned}$$

Multiplying (6.19) by H_j, it yields

$$\begin{aligned}\int_\Omega|\nabla H_j|^2+\int_\Omega(H_j)^2 &= -\int_\Omega u_jH_j+\int_{\partial\Omega}\frac{\partial H_j}{\partial\nu}H_j\\ &= -\int_\Omega u_jH_j+2\varepsilon_j\mu_j\int_{\partial\Omega}e^{u_j}H_j-\int_{\partial\Omega}\frac{\partial u_j}{\partial\nu}H_j,\end{aligned}$$

Multiplying (7.31) by u_j again, we find

$$\int_\Omega u_j^2+\int_\Omega H_ju_j = -\int_\Omega\nabla u_j\nabla H_j+2\varepsilon_j\mu_j\int_{\partial\Omega}e^{u_j}u_j-\int_{\partial\Omega}\frac{\partial u_j}{\partial\nu}u_j,$$

Then we get

$$\begin{aligned}&\int_\Omega(|\nabla U_j|^2+U_j^2)\\ &= \lambda m_j^2\left[\int_\Omega|\nabla u_j|^2-\int_{\partial\Omega}\frac{\partial u_j}{\partial\nu}u_j+\int_\Omega\nabla u_j\nabla H_j-\int_{\partial\Omega}\frac{\partial u_j}{\partial\nu}H_j+2\varepsilon_j\mu_j\int_{\partial\Omega}e^{u_j}(u_j+H_j)\right]\\ &= 2\lambda m_j^2\varepsilon_j\mu_j\int_{\partial\Omega}e^{u_j}(u_j+H_j)\end{aligned}$$

$$= \quad 2\lambda m_j^2 \int_{\partial\Omega} \frac{\varepsilon_j \mu_j}{|x-\xi_j-\varepsilon_j\mu_j\nu(\xi_j)|^2} \left(\log \frac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2} + H(x,\xi_j) + O(\varepsilon_j^\alpha) \right).$$

Taking the change of variables $y = \frac{x-\xi_j}{\varepsilon_j\mu_j}$, we have

$$\begin{aligned}
&\int_\Omega (|\nabla U_j|^2 + U_j^2) \\
= \quad & 2\lambda m_j^2 \left[\int_{\partial\Omega_{\varepsilon_j\mu_j}} \frac{1}{|y-\nu(0)|^2} \left(\log \frac{1}{|y-\nu(0)|^2} + H(\xi_j+\varepsilon_j\mu_j y,\xi_j) - 2\log(\mu_j\varepsilon_j) \right) + O(\varepsilon_j^\alpha) \right] \\
= \quad & 2\lambda m_j^2 \left[\int_{\partial\Omega_{\varepsilon_j\mu_j}} \frac{1}{|y-\nu(0)|^2} \left(\log \frac{1}{|y-\nu(0)|^2} + H(\xi_j,\xi_j) - 2\log(\varepsilon_j) - 2\log(2\mu_j) + 2\log 2 \right) \right. \\
& +2\lambda m_j^2 \left[\int_{\partial\Omega_{\varepsilon_j\mu_j}} \frac{1}{|y-\nu(0)|^2} \left(H(\xi_j+\varepsilon_j\mu_j y,\xi_j) - H(\xi_j,\xi_j) \right) + O(\varepsilon_j^\alpha) \right].
\end{aligned}$$

We have

$$\int_{\partial\Omega_{\varepsilon_j\mu_j}} \frac{1}{|y-\nu(0)|^2} = \pi + O(\varepsilon_j^\alpha).$$

$$\begin{aligned}
\int_{\partial\Omega_{\varepsilon_j\mu_j}} \frac{1}{|y-\nu(0)|^2} \log \frac{1}{|y-\nu(0)|^2} \quad &= \quad \int_{-\infty}^{\infty} \frac{1}{1+t^2} \log \frac{1}{1+t^2} \; dt + O(\varepsilon_j^\alpha) \\
&= \quad -2\pi \log(2) + O(\varepsilon_j^\alpha).
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\partial\Omega_{\varepsilon_j\mu_j}} \frac{1}{|y-\nu(0)|^2} \left(H(\xi_j+\varepsilon\mu_j y,\xi_j) - H(\xi_j,\xi_j) \right) \\
= \quad & \int_{\partial\Omega_{\varepsilon_j\mu_j}} \frac{1}{|y-\nu(0)|^2} O(\varepsilon_j^\alpha |y|^\alpha) = O(\varepsilon_j^\alpha).
\end{aligned}$$

Using the definition of ε_j, we thus conclude that

$$\begin{aligned}
&\int_\Omega (|\nabla U_j|^2 + U_j^2) \\
= \quad & 2\lambda m_j^2 \left[-2\pi\log 2 + \pi(H(\xi_j,\xi_j) - 2\log(\varepsilon_j) - 2\log(2\mu_j) + 2\log 2) + O(\varepsilon_j^\alpha) \right] \\
= \quad & \pi + 2\lambda m_j^2 \left[\pi H(\xi_j,\xi_j) - 2\pi\log(2m_j^2) - 2\pi\log(2\mu_j) + O(\varepsilon_j^\alpha) \right]
\end{aligned}$$

Therefore

$$I_1 \quad = \quad \frac{1}{2} \sum_{j=1}^{k} \int_\Omega (|\nabla U_j|^2 + U_j^2)$$

$$= \frac{k\pi}{2} + \sum_{j=1}^{k} \lambda m_j^2 \left[\pi H(\xi_j,\xi_j) - 2\pi \log(2m_j^2) - 2\pi \log(2\mu_j) + O(\varepsilon_j^{\alpha}) \right]. \qquad (6.95)$$

On the other hand, we have

$$\begin{aligned}
&\sum_{i\neq j} \int_\Omega (\nabla U_i \nabla U_j + U_i U_j) \\
= &\sum_{i\neq j} \lambda m_i m_j \left[\int_\Omega \nabla u_i \nabla u_j + \int_\Omega \nabla u_i \nabla H_j + \int_\Omega \nabla u_j \nabla H_i + \int_\Omega \nabla H_i \nabla H_j \right. \\
&\left. + \int_\Omega u_i u_j + \int_\Omega u_i H_j + \int_\Omega u_j H_i + \int_\Omega H_i H_j \right].
\end{aligned}$$

Multiplying (6.19) by H_i and integrating we find

$$\int_\Omega \nabla H_j \nabla H_i + \int_\Omega H_j H_i = -\int_\Omega u_j H_i + 2\varepsilon_j \mu_j \int_{\partial\Omega} e^{u_j} H_i - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} H_i.$$

Hence

$$\begin{aligned}
&\sum_{i\neq j} \int_\Omega (\nabla U_i \nabla U_j + U_i U_j) \\
= &\sum_{i\neq j} \lambda m_i m_j \left[\int_\Omega \nabla u_i \nabla u_j + \int_\Omega \nabla u_i \nabla H_j + \int_\Omega \nabla u_j \nabla H_i \right. \\
&\left. + \int_\Omega u_i u_j + \int_\Omega u_i H_j + 2\varepsilon_j \mu_j \int_{\partial\Omega} e^{u_j} H_i - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} H_i \right]. \qquad (6.96)
\end{aligned}$$

Multiplying (6.19) by u_i again and integrating we find

$$\int_\Omega \nabla H_j \nabla u_i + \int_\Omega u_j u_i = -\int_\Omega H_j u_i + 2\varepsilon_j \mu_j \int_{\partial\Omega} e^{u_j} u_i - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} u_i. \qquad (6.97)$$

By (6.96)-(6.97) we find that

$$\begin{aligned}
&\sum_{i\neq j} \int_\Omega (\nabla U_i \nabla U_j + U_i U_j) \\
= &\sum_{i\neq j} \lambda m_i m_j \left[\int_\Omega \nabla u_i \nabla u_j - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} u_i + \int_\Omega \nabla u_j \nabla H_i - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} H_i \right. \\
&\left. + 2\varepsilon_j \mu_j \int_{\partial\Omega} e^{u_j}(u_i + H_i) \right].
\end{aligned}$$

Then

$$I_2 = \frac{1}{2} \sum_{i\neq j} \int_\Omega (\nabla U_i \nabla U_j + U_i U_j) = \sum_{i\neq j} \lambda m_i m_j \varepsilon_j \mu_j \int_{\partial\Omega} e^{u_j}(u_i + H_i)$$

$$
\begin{aligned}
&= \sum_{i\neq j} \lambda m_i m_j \int_{\partial\Omega} \frac{\varepsilon_j\mu_j}{|x-\xi_j-\varepsilon_j\mu_j\nu(\xi_j)|^2}\left(\log\frac{1}{|x-\xi_j-\varepsilon_j\mu_j\nu(\xi_j)|^2}+H_i(x)\right)\\
&\qquad set \quad \frac{x-\xi_j}{\varepsilon_j\mu_j}=y\\
&= \sum_{i\neq j} \lambda m_i m_j \int_{\partial\Omega_{\varepsilon_j\mu_j}} \frac{1}{|y-\nu(0)|^2}\left(\log\frac{1}{|x-\xi_j-\varepsilon_j\mu_j\nu(\xi_j)|^2}+H_i(\varepsilon_j\mu_j y+\xi_j)\right)\\
&= \sum_{i\neq j} \lambda\pi m_i m_j \left[G(\xi_i,\xi_j)+O\left(\varepsilon_i^2\log\frac{1}{\varepsilon_i}+\varepsilon_j^2\log\frac{1}{\varepsilon_j}\right)+O(\varepsilon_i^\alpha+\varepsilon_j^\alpha)\right].
\end{aligned}
\tag{6.98}
$$

Finally, let us evaluate the third term in the energy

$$
\begin{aligned}
\frac{\lambda}{2}\int_{\partial\Omega} e^{U^2(x)} &= \frac{\lambda}{2}\sum_{j=1}^{k}\int_{\partial\Omega\cap B(\xi_j,\delta\sqrt{\varepsilon_j})} e^{U^2(x)} + \frac{\lambda}{2}\int_{\partial\Omega\setminus\bigcup_{j=1}^{k}B(\xi_j,\delta\sqrt{\varepsilon_j})} e^{U^2(x)}\\
&:= I+II.
\end{aligned}
\tag{6.99}
$$

Since

$$
\begin{aligned}
&\int_{\partial\Omega\cap B(\xi_j,\delta\sqrt{\varepsilon_j})} e^{U^2(x)}\\
&= \int_{\partial\Omega\cap B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} e^{U^2(x)} + \int_{\partial\Omega\cap(B(\xi_j,\delta\sqrt{\varepsilon_j})\setminus B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|))} e^{U^2(x)} := I_A+I_B,
\end{aligned}
$$

where

$$
\begin{aligned}
I_A &= \int_{\partial\Omega\cap B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} e^{U^2(x)}\\
&= \int_{\partial\Omega\cap B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} e^{\left[\sqrt{\lambda}m_j(-2\log\varepsilon_j+\beta_j+w_j+\theta(x))\right]^2}\\
&= \varepsilon_j^{-1}e^{\frac{\beta_j}{2}}\int_{\partial\Omega\cap B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} e^{w_j}e^{\theta(x)}e^{\lambda m_j^2[w_j^2+2w_j\theta(x)+\theta^2(x)]}\\
&= 2m_j^2\varepsilon_j^{-1}\int_{\partial\Omega\cap B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} \frac{2\mu_j}{|\frac{x-\xi_j}{\varepsilon_j}-\mu_j\nu(\xi_j')|^2}(1+O(\lambda))\\
&= 2m_j^2\int_{\frac{\partial\Omega-\xi_j}{\varepsilon_j\mu_j}\cap B(0,\frac{\delta|\log\varepsilon_j|}{\mu_j})} \frac{2}{|y-\nu(0)|^2}(1+O(\lambda))
\end{aligned}
$$

$$= \quad 4\pi m_j^2 \left(1+\lambda\Theta_\lambda(m,\xi)\right), \tag{6.100}$$

with $\Theta_\lambda(m,\xi)$ a function, uniformly bounded with its derivatives, as $\lambda \to 0$.

$$\begin{aligned} |I_B| &\le C \int_{\delta|\log\varepsilon_j|}^{\delta\varepsilon_j^{-\frac{1}{2}}} \frac{1}{r^2} e^{\frac{\log^2 r}{\gamma_j^2}} r\, dr \\ &\quad \text{set} \quad t = \log r \\ &= C \int_{R_1+\log\gamma_j^2}^{R_2+\frac{\gamma_j^2}{4}} e^{-2t+\frac{4t^2}{\gamma_j^2}} dt \le C \int_{R_1+\log\gamma_j^2}^{R_2+\frac{\gamma_j^2}{4}} e^{-t} dt = O(\lambda). \end{aligned} \tag{6.101}$$

Moreover, we have

$$II = \frac{\lambda}{2}\left[|\partial\Omega| + \sum_{j=1}^{k} \lambda^2 \Theta_\lambda(m,\xi)\right], \tag{6.102}$$

with $|\partial\Omega|$ denotes the measure of domain $\partial\Omega$, and $\Theta_\lambda(m,\xi)$ is a function, uniformly bounded with its derivatives, as $\lambda \to 0$. Then from (6.99)-(6.102), we get

$$I_3 = -2\lambda\pi \sum_{j=1}^{k} m_j^2 \left(1+\lambda\Theta_\lambda(m,\xi)\right). \tag{6.103}$$

Hence from (6.95), (6.98) and (6.103) we obtain

$$\begin{aligned} J_\lambda(U) &= \pi\left[\sum_{j=1}^{k} m_j^2 H(\xi_j,\xi_j) + \sum_{i\neq j} m_i m_j G(\xi_i,\xi_j) - 2\sum_{j=1}^{k} m_j^2 - 2\sum_{j=1}^{k} m_j^2 \log(2m_j^2)\right]\lambda \\ &\quad + \frac{k\pi}{2} - \frac{|\partial\Omega|}{2}\lambda - 2\pi\lambda \sum_{j=1}^{k} m_j^2 \log(2\mu_j) + o(\lambda). \end{aligned}$$

By the choice of μ_j in (6.31), we get that the function $\Theta(\xi,m)$ in the expansion (6.92) is uniformly bounded, as $\lambda \to 0$, for points ξ and parameters m satisfying (6.46). In order to prove that also the derivatives, in ξ and in m, of this function $\Theta(\xi,m)$ are uniformly bounded, as $\lambda \to 0$, in the same region, one argues similarly as for the C^0 expansion of $J_\lambda(U)$. We leave the details to the reader. Thus the proof of Lemma is complete. □

6.5 Proof of Theorem 6.1

In this section, we will prove the main result.

Proof of Theorem 6.1 Let $\mathcal{D}$ be the open set such that

$$\bar{\mathcal{D}} \subset \left\{(\xi, m) \in (\partial\Omega)^k \times \mathbb{R}^k_+ : \ \xi_i \neq \xi_j, \ \forall\, i \neq j)\right\}$$

From Lemma 6.4, the function

$$u_\lambda(x) = \sqrt{\lambda}\left(\tilde{U}(\xi, m) + \phi(\xi, m)\right)$$

where $\tilde{U}(\xi, m)$ defined by (6.18) and ϕ is the unique solution to problem (6.39) given by Proposition 6.3, is a solution of problem (6.1) if we adjust (ξ, m) so that it is a critical point of $\mathcal{I}_\lambda(\xi, m)$ defined by (6.44). This is equivalent to finding a critical point of

$$\tilde{\mathcal{I}}_\lambda(\xi, m) = \frac{1}{\pi\lambda}\left[\mathcal{I}_\lambda(\xi, m) - \frac{k\pi}{2} + \frac{|\partial\Omega|}{2}\lambda\right].$$

On the other hand, from Lemmas 6.9 and 6.10, for $(\xi, m) \in \mathcal{D}$ satisfies (6.17), we have that,

$$\tilde{\mathcal{I}}_\lambda(\xi, m) = \varphi_k(\xi, m) + o(1)\Theta_\lambda(m, \xi), \tag{6.104}$$

where $\Theta_\lambda(m, \xi)$ and $\nabla\Theta_\lambda(m, \xi)$ are uniformly bounded in consider region as $\lambda \to 0$. Thus we need to find a critical point of

$$\varphi_k(\xi, m) = 2(\log 2 - 1)\sum_{j=1}^{k} m_j^2 + 2\sum_{j=1}^{k} m_j^2 \log(m_j^2) - \sum_{j=1}^{k} m_j^2 H(\xi_j, \xi_j) - \sum_{i\neq j} m_i m_j G(\xi_i, \xi_j).$$

We make the change of variables $s_j = m_j^2$, and set $b = 2(\log 2 - 1)$. And we next find critical point of

$$\begin{aligned}\varphi_k(\xi, s) \quad &= \quad b\sum_{j=1}^{k} s_j + 2\sum_{j=1}^{k} s_j \log(s_j) \\ &\quad - \left[\sum_{j=1}^{k} s_j H(\xi_j, \xi_j) + \sum_{i\neq j}^{k} \sqrt{s_i s_j}\, G(\xi_i, \xi_j)\right],\end{aligned} \tag{6.105}$$

which is well defined on $\bar{\mathcal{D}}$. For $j \in \{1, 2, \ldots, k\}$, we have

$$\partial_{s_j}\varphi_k(\xi, s) = b + 2 + 2\log(s_j) - H(\xi_j, \xi_j) - \frac{1}{2}\sum_{i\neq j}^{k}\sqrt{\frac{s_i}{s_j}}\, G(\xi_i, \xi_j),$$

and

$$\partial^2_{s_j s_j}\varphi_k(\xi, s) = \frac{2}{s_j} + \frac{1}{4}\sum_{i\neq j}^{k}\sqrt{\frac{s_i}{s_j}}\frac{1}{s_j} G(\xi_i, \xi_j),$$

$$\partial^2_{s_js_i}\varphi_k(\xi,s)=\frac{1}{4}\sum_{i\neq j}^{k}\frac{1}{\sqrt{s_is_j}}G(\xi_i,\xi_j).$$

We have that $\varphi_k(\xi,s)$ is strictly convex as a function s, and it is bounded below. Hence it has a unique minimum point, which we denote by $\bar{s}=(\bar{s}_1,\dots,\bar{s}_k)$, each component of $\bar{s}$ is a function of points $\xi_1,\dots,\xi_k$, namely

$$\bar{s}_j=\bar{s}_j(\xi_1,\dots,\xi_k)$$

satisfies

$$b+2+2\log(\bar{s}_j)-H(\xi_j,\xi_j)-\frac{1}{2}\sum_{i\neq j}^{k}\sqrt{\frac{\bar{s}_i}{\bar{s}_j}}G(\xi_i,\xi_j)=0. \tag{6.106}$$

We have

(1) $\bar{s}_j$ is a C^1 function with respect to ξ defined in $(\partial\Omega)^k$;

(2) There is a positive constant c_0, such that $\bar{s}_j\geq c_0$ for each $j=1,\dots,k$;

(3) $\bar{s}_j\to+\infty$ as $|\xi_i-\xi_j|\to 0$ for some $i\neq j$.

In fact, (1) directly holds by the implicit function theorem. Moreover, since $G(\xi_i,\xi_j)$ is positive and $H(\xi_j,\xi_j)$ is bounded, from (6.106) we have

$$\bar{s}_j^2>e^{-(b+2)+H(\xi_j,\xi_j)}$$

Then we get (2) holds. Furthermore, for some $i\neq j$ we have $G(\xi_i,\xi_j)\to+\infty$ as $|\xi_i-\xi_j|\to 0$, so (3) holds by (6.106).

A direct computation shows that

$$\Phi_k(\xi):=\varphi_k(\xi,\bar{s})=-2\sum_{j=1}^{k}\bar{s}_j(\xi)$$

for $\xi\in\hat{\Omega}_k=\{(\xi_1,\dots,\xi_k)\in(\partial\Omega)^k\ :\ \xi_i\neq\xi_j \text{ if } i\neq j\}$.

Given one component $\mathcal{C}_0$ of $\partial\Omega$. Let $\Lambda:S^1\to\mathcal{C}_0$ be a continuous bijective function the parametrizes $\mathcal{C}_0$. Set

$$\tilde{\Omega}_k=\{(\xi_1,\dots,\xi_k)\in\mathcal{C}_0^k\ :\ |\xi_i-\xi_j|>\delta \text{ for } i\neq j\}.$$

Next, we find critical point of Φ_k. The function Φ_k is C^1, bounded from above in $\tilde{\Omega}_k$, and from (3) we have

$$\Phi_k(\xi)=\Phi_k(\xi_1,\dots,\xi_k)\to-\infty \text{ as } |\xi_i-\xi_j|\to 0 \text{ for some } i\neq j.$$

Hence, since δ is arbitrarily small, Φ_k has an absolute maximum M in $\tilde{\Omega}_k$.

On the other hand, Using Ljusternik-Schnirelmann theory as the proof in [27], we get that Φ_k has at least two distinct points in $\tilde{\Omega}_k$. Let $cat(\tilde{\Omega}_k)$ be the Ljusternik-Schnirelmann category of $\tilde{\Omega}_k$ relative to $\tilde{\Omega}_k$, which is the minimum number of closed and contractible in $\tilde{\Omega}_k$ sets whose union covers $\tilde{\Omega}_k$. We will estimate the number of critical points for Φ_k below by $cat(\tilde{\Omega}_k)$.

Claim: $cat(\tilde{\Omega}_k) > 1$.

Indeed, by contradiction, suppose that $cat(\tilde{\Omega}_k) = 1$. This means that $\tilde{\Omega}_k$ is contractible in itself, namely there exist a point $\xi^0 \in \tilde{\Omega}_k$ and a continuous function $\Gamma : [0,1] \times \tilde{\Omega}_k \to \tilde{\Omega}_k$, such that, for all $\xi \in \tilde{\Omega}_k$,

$$\Gamma(0,\xi) = \xi, \quad \Gamma(1,\xi) = \xi_0.$$

Define $f : S^1 \to \tilde{\Omega}_k$ to be the continuous function given by

$$f(\bar{\xi}) = \left(\Lambda(\bar{\xi}), \Lambda(e^{2\pi i \frac{1}{k}}\bar{\xi}), \ldots, \Lambda(e^{2\pi i \frac{k-1}{k}}\bar{\xi})\right).$$

Let $\eta : [0,1] \times S^1 \to S^1$ be the well defined continuous map given by

$$\eta(t,\bar{\xi}) = \Lambda^{-1} \circ \pi_1 \circ \Gamma(t, f(\bar{\xi})),$$

where π_1 is the projection on the first component. The function η is a contraction of S^1 to a point and this gives a contradiction, then claim follows.

Therefore we have that $cat(\tilde{\Omega}_k) \geq 2$ for any $k \geq 1$. Define

$$c = \sup_{C \in \Xi} \inf_{\xi \in C} \Phi_k(\xi)$$

where

$$\Xi = \{C \subset \tilde{\Omega}_k \ : \ C \text{ closed and } cat(C) \geq 2\}.$$

Then by Ljusternik-Schnirelmann theory we obtain that c is a critical level.

If $c \neq M$, we conclude that Φ_k has at least two distinct critical points in $\tilde{\Omega}_k$. If $c = M$, there is at least one set C such that $cat(C) \geq 2$, where the function Φ_k reaches its absolute maximum. In this case we conclude that there are infinitely many critical points for Φ_k in $\tilde{\Omega}_k$.

Thus we obtain that the function Φ_k has at least two distinct critical points in $\tilde{\Omega}_k$, denote by ξ^1, ξ^2. Hence $(\xi^1, \bar{s}(\xi^1))$ and $(\xi^2, \bar{s}(\xi^2))$ are two distinct critical points for the function $\varphi_k(\xi, s)$. From (6.104) we then have that $\tilde{\mathcal{I}}_\lambda(\xi, m)$ has at least two critical points. This ends the proof of Theorem.

6.6 Appendix

Proof of (6.74). We shall prove

$$\| -\Delta \tilde{Z}_{0j} + \tilde{Z}_{0j}\|_{**,\Omega} \leq \frac{C}{|\log \varepsilon_j|}$$

where $\tilde{Z}_{0j}$ is defined in (6.71). Performe the change of variables $y = \varepsilon_j^{-1} F_j(x)$ and denote $\tilde{z}_{0j}(y) = \tilde{Z}_{0j}(F_j^{-1}(\varepsilon_j y))$. Then $-\Delta \tilde{Z}_{0j} + \tilde{Z}_{0j} = \big(\mathcal{L}\tilde{z}_{0j} + \varepsilon_j^2 \tilde{z}_{0j}\big)$, where $\mathcal{L}$ is defined in (6.60). We shall show that

$$\big|\big(\mathcal{L}\tilde{z}_{0j} + \varepsilon_j^2 \tilde{z}_{0j}\big)\big| \leq \frac{C}{|\log \varepsilon_j|}\left[\varepsilon_j^2 + \sum_{j=1}^m (1+|y-\xi_j'|)^{-2-\sigma}\right], \quad y \in \frac{\Omega}{\varepsilon_j}.$$

This fact implies (6.74).

Let us first consider the region where $|y| < R$. In this region, $\tilde{z}_{0j} = z_{0j}$. Since $\Delta z_{0j} = 0$ and since (6.60) holds, we have that

$$\big(\mathcal{L}\tilde{z}_{0j} + \varepsilon_j^2 \tilde{z}_{0j}\big) = O(\varepsilon_j) \quad \text{for} \quad |y| < R. \tag{6.107}$$

In the region $R+1 < |y| < \frac{\delta}{4\varepsilon_j}$, we have $\tilde{z}_{0j} = h z_{0j}$. Therefore, in this region,

$$|\Delta \tilde{z}_{0j}| = 2|\nabla h \nabla z_{0j}| \leq \frac{C}{r^3 \log \frac{\delta}{\varepsilon_j}} \qquad R+1 < r < \frac{\delta}{4\varepsilon_j}, \quad r = |y|.$$

For the other terms we find

$$\begin{aligned} |\nabla^2 \tilde{z}_{0j}| &\leq |\nabla^2 h| z_{0j} + 2|\nabla h \nabla z_{0j}| + h|\nabla^2 z_{0j}| \\ &= O(\frac{1}{r^2 \log \frac{\delta}{\varepsilon_j}}) + O(\frac{1}{r^3 \log \frac{\delta}{\varepsilon_j}}) + O(\frac{1}{r^3}) \qquad R+1 < r < \frac{\delta}{4\varepsilon_j} \end{aligned}$$

so

$$O(\varepsilon_j |y|)|\nabla^2 \tilde{z}_{0j}| = O(\frac{\varepsilon_j}{r \log \frac{\delta}{\varepsilon_j}}) + O(\frac{\varepsilon_j}{r^2}) \qquad R+1 < r < \frac{\delta}{4\varepsilon_j}.$$

Also

$$|\nabla \tilde{z}_{0j}| \leq |\nabla h| z_{0j} + h|\nabla z_{0j}| = O(\frac{1}{r \log \frac{\delta}{\varepsilon_j}}) + O(\frac{1}{r^2}) \qquad R+1 < r < \frac{\delta}{4\varepsilon_j}.$$

Hence

$$\big(\mathcal{L}\tilde{z}_{0j} + \varepsilon_j^2 \tilde{z}_{0j}\big) = O(\frac{1}{r^3 \log \frac{\delta}{\varepsilon_j}}) + O(\frac{\varepsilon_j}{r \log \frac{\delta}{\varepsilon_j}}) + O(\frac{\varepsilon_j}{r^2}) + \varepsilon_j^2 \tilde{z}_{0j} \qquad R+1 < r < \frac{\delta}{4\varepsilon_j}. \tag{6.108}$$

In the region $\frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}$ the definition of $\tilde{z}_{0j}$ is $\tilde{z}_{0j} = \eta_2 h z_{0j}$. We will estimate each term of (6.60) using the facts that $\nabla\eta_2 = O(\frac{\varepsilon_j}{\delta})$, $|\nabla^2\eta_2| = O(\frac{\varepsilon_j^2}{\delta^2})$ and that in the considered region $h = O(\frac{1}{\log\frac{\delta}{\varepsilon j}})$ which implies also $\tilde{z}_{0j} = O(\frac{1}{\log\frac{\delta}{\varepsilon j}})$. We obtain

$$\begin{aligned}\Delta\tilde{z}_{0j} &= \Delta\eta_2 h z_{0j} + 2\nabla\eta_2\nabla(hz_{0j}) + \eta_2\Delta(hz_{0j})\\ &= \Delta\eta_2 h z_{0j} + 2\nabla\eta_2\nabla h z_{0j} + 2\nabla\eta_2\nabla z_{0j} h + 2\eta_2\nabla h\nabla z_{0j}\\ &= O(\frac{\varepsilon_j^2}{\delta^2\log\frac{\delta}{\varepsilon j}}) + O(\frac{\varepsilon_j}{r\delta\log\frac{\delta}{\varepsilon j}}) + O(\frac{\varepsilon_j}{r^2\delta\log\frac{\delta}{\varepsilon j}}) + O(\frac{1}{r^3\log\frac{\delta}{\varepsilon j}})\\ &= O(\frac{\varepsilon_j^2}{\delta^2\log\frac{\delta}{\varepsilon j}}) \qquad \frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}.\end{aligned}$$

Next

$$\nabla^2\tilde{z}_{0j} = \nabla^2\eta_2 h z_{0j} + 2\nabla\eta_2\nabla(hz_{0j}) + \eta_2\nabla^2(hz_{0j}) \qquad \frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}.$$

and by the above computations, for $\frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}$,

$$\nabla^2\tilde{z}_{0j} = O(\frac{\varepsilon_j^2}{\delta^2\log\frac{\delta}{\varepsilon j}}) + \eta_2(\nabla^2 h z_{0j} + 2\nabla h\nabla z_{0j} + h\nabla^2 z_{0j}) = O(\frac{\varepsilon_j^2}{\delta^2\log\frac{\delta}{\varepsilon j}}).$$

Similarly, for $\frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}$

$$\nabla\tilde{z}_{0j} = \nabla\eta_2 h z_{0j} + \eta_2\nabla h z_{0j} + \eta_2 h\nabla z_{0j} = O(\frac{\varepsilon_j}{\delta\log\frac{\delta}{\varepsilon j}})$$

This shows that

$$\big(\mathcal{L}\tilde{z}_{0j} + \varepsilon_j^2\tilde{z}_{0j}\big) = O(\frac{\varepsilon^2}{\delta^2\log\frac{\delta}{\varepsilon j}}) \qquad \frac{\delta}{4\varepsilon_j} < r < \frac{\delta}{3\varepsilon_j}. \tag{6.109}$$

Thus we only need to estimate the size of $\mathcal{L}\tilde{z}_{0j} + \varepsilon_j^2\tilde{z}_{0j}$ in the region $R < r < R+1$. In this region we have $\tilde{z}_{0j} = \eta_1 z_{0j} + (1-\eta_{1j})hz_{0j}$ and hence

$$\begin{aligned}\Delta\tilde{z}_{0j} &= \Delta\eta_1(1-h)z_{0j} - 2\nabla\eta_1\nabla h z_{0j} + 2\nabla\eta_1\nabla z_{0j}(1-h) + \eta_1\Delta z_{0j}\\ &\quad + (1-\eta_1)\Delta(hz_{0j})\\ &= O(\frac{1}{\log\frac{\delta}{\varepsilon j}}) + +\eta_1\Delta z_{0j} + (1-\eta_1)\Delta(hz_{0j}) \qquad R < r < R+1.\end{aligned}$$

First we recall that $\Delta z_{0j} = 0$ and, for $R < r < R+1$,

$$\Delta(hZ_{0j}) = 2\nabla h\nabla z_{0j} + O(\varepsilon_j) = O(\frac{1}{\log\frac{\delta}{\varepsilon j}}) + O(\varepsilon_j).$$

Thus

$$\mathcal{L}\tilde{z}_{0j} + \varepsilon_j^2 \tilde{z}_{0j} = O(\frac{1}{\log \frac{\delta}{\varepsilon}}) \qquad R < r < R+1. \tag{6.110}$$

This bound and (6.107), (6.108) and (6.109) imply (6.74).

Proof of (6.75). We shall prove

$$\|L(\tilde{Z}_{0j})\|_{*,\partial\Omega} \leq \frac{C}{|\log \varepsilon_j|}$$

We perform the change of variables $y = \varepsilon_j^{-1} F_j(x)$. We already observed that we can assume that $\nabla F_j(\xi_j) = I$. Hence,

$$L(\tilde{Z}_{0j}) = (1 + o(1)) \left[B(\tilde{z}_{0j}) - \tilde{W} \tilde{z}_{0j} \right]$$

where $\tilde{z}_{0j} = \tilde{Z}_{0j}(F_j^{-1}(\varepsilon_j y))$ and $\tilde{W}(y) = W(F_j^{-1}(\varepsilon_j y))$. B is the differential operator of order one on $\partial\mathbb{R}^2_+$, defined in (6.62) and $\tilde{W}$ is described in (6.63). Thus in the region $y \in \partial\left(\frac{\Omega}{\varepsilon_j}\right)$, with $|y| < R$, we get

$$B(\tilde{z}_{0j}) - \tilde{W}\tilde{z}_{0j} = O(\varepsilon_j) \tag{6.111}$$

Next, in the region $R < |x| < R+1$ we have

$$\begin{aligned}
\nabla \tilde{z}_{0j} &= \nabla(\eta_1(1-h)z_{0j} + hz_{0j}) \\
&= \nabla\eta_1(1-h)z_{0j} - \eta_1\nabla h z_{0j} + \eta_1(1-h)\nabla z_{0j} + \nabla h z_{0j} + h\nabla z_{0j} \\
&= O(\frac{1}{\log \frac{\delta}{\varepsilon j}}) + \eta_1(1-h)\nabla z_{0j} + h\nabla z_{0j}.
\end{aligned}$$

Since h is radial this implies

$$B(\tilde{z}_{0j}) = -h\frac{\partial z_{0j}}{\partial x_2} + O(\frac{1}{R^2 \log \frac{\delta}{\varepsilon j}}) + O(\frac{R\varepsilon_j}{\log \frac{\delta}{\varepsilon j}}) \qquad R < |y| < R+1, y \in \partial\mathbb{R}^2_+.$$

Using (6.63) we see that

$$B(\tilde{z}_{0j}) - \tilde{W}\tilde{z}_{0j} = O(\frac{1}{R^2 \log \frac{\delta}{\varepsilon j}}) + O(\frac{R\varepsilon_j}{\log \frac{\delta}{\varepsilon j}}) \qquad R < |y| < R+1, y \in \partial\mathbb{R}^2_+. \tag{6.112}$$

Using the fact that h has zero normal derivative on $\partial\mathbb{R}^2_+$ we deduce

$$\begin{aligned}
B(\tilde{h}z_{0j}) &= -h\frac{\partial z_{0j}}{\partial x_2} + O(\varepsilon_j r)(\nabla h z_{0j} + h\nabla z_{0j}) \\
&= -h\frac{\partial z_{0j}}{\partial x_2} + O(\frac{\varepsilon_j}{\log \frac{\delta}{\varepsilon j}}) + O(\frac{\varepsilon_j}{r}) \qquad R+1 < r < \frac{\delta}{\varepsilon_j}.
\end{aligned} \tag{6.113}$$

On the other hand, using (6.63) we have in $R+1<r<\frac{\delta}{\varepsilon_j}$

$$B(\tilde z_{0j})-\tilde W\tilde z_{0j}=O(\frac{\varepsilon_j}{\log\frac{\delta}{\varepsilon_j}})+O(\frac{\varepsilon_j^\alpha}{r}) \tag{6.114}$$

for some $0<\alpha<1$. Finally we consider $\frac{\delta}{4\varepsilon_j}<r<\frac{\delta}{3\varepsilon_j}$. Here we have $\tilde z_{0j}=\eta_2 h z_{0j}$ and $h, z_{0j}=O(\frac{1}{\log\frac{\delta}{\varepsilon_j}})$, $\nabla\bar\eta_2=O(\frac{\varepsilon_j}{\delta})$. Using these facts, estimate (6.113) and that η_2 has zero normal derivative we find

$$\begin{aligned}B(\tilde z_{0j}) &= B(\eta_2)hz_{0j}+\eta_2 B(hz_{0j})\\ &= O(\frac{\varepsilon_j^2 r}{\delta\log\frac{\delta}{\varepsilon_j}})+O(\frac{1}{r^2})+O(\frac{\varepsilon_j}{\log\frac{\delta}{\varepsilon_j}})+O(\frac{\varepsilon_j}{r}) \qquad \frac{\delta}{4\varepsilon_j}<r<\frac{\delta}{3\varepsilon_j}.\end{aligned}$$

From (6.63) we have

$$\tilde W=O(\frac{\varepsilon_j^\alpha}{r}) \quad \frac{\delta}{4\varepsilon_j}<r<\frac{\delta}{\varepsilon_j}.$$

Thus we conclude that for $y\in\partial\Omega_{\varepsilon_j}$, $\frac{\delta}{4\varepsilon_j}<r<\frac{\delta}{3\varepsilon_j}$

$$B(\tilde z_{0j})-\tilde W\tilde z_{0j}=O(\frac{\varepsilon_j^2 r}{\delta\log\frac{\delta}{\varepsilon_j}})+O(\frac{1}{r^2})+O(\frac{\varepsilon_j}{\log\frac{\delta}{\varepsilon_j}})+O(\frac{\varepsilon_j}{r}). \tag{6.115}$$

Estimates (6.111), (6.112), (6.114) and (6.115) give the validity of (6.75).

Proof of (6.83). We shall prove

$$\left\| f'(\tilde U)-\sum_{j=1}^k \varepsilon_j^{-1}e^{w_j}\right\|_{*,\partial\Omega}\le C\lambda.$$

Indeed, we have

$$f'(\tilde U)=\lambda e^{\lambda\tilde U^2}+2\lambda^2\tilde U^2 e^{\lambda\tilde U^2}:=I_a+I_b.$$

For $x\in\partial\Omega$, far away from the points ξ_j, namely for $|x-\xi_j|>\delta$, i.e. $|y-\xi_j'|>\frac{\delta}{\varepsilon_j}$, for all $j=1,2,\dots,k$, a consequence of (6.22) is that

$$I_a=\lambda O(1), \qquad I_b=\lambda^2 O(1).$$

Then we have

$$f'(\tilde U)1_{\mathsf{outer}}=\lambda O(1), \tag{6.116}$$

where 1_{outer} is the characteristic function of the set $\{y : |y-\xi_j'|>\frac{\delta}{\varepsilon_j},\ j=1,\dots,k\}$. Moreover, for $|x-\xi_j|>\delta$, we have

$$\sum_{j=1}^k \varepsilon_j^{-1}e^{w_j}1_{\mathsf{outer}}=O(1)\sum_{j=1}^k\varepsilon_j=O(1)\sum_{j=1}^k 2m_j^2 e^{-\frac{1}{4m_j^2}\frac{1}{\lambda}}=\lambda O(1). \tag{6.117}$$

On the other hand, fix the index j in $\{1,2,\dots,k\}$, for $|x-\xi_j|<\delta$, from (6.23), (6.29) and (6.30) we have

$$\begin{aligned}
I_a &= \lambda e^{\lambda m_j^2\left(w_j(x)+\log\varepsilon_j^{-2}+\beta_j+\theta(x)\right)^2}\\
&= \lambda e^{\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)^2} e^{2\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)w_j} e^{2\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)\theta(x)} e^{\lambda m_j^2(w_j+\theta(x))^2}\\
&= \lambda e^{\frac{1}{2}(\log\frac{1}{\varepsilon_j^2}+\beta_j)} e^{w_j}e^{\theta(x)}e^{\lambda m_j^2(w_j+\theta(x))^2}\\
&= \lambda e^{\beta_j/2}\varepsilon_j^{-1}e^{w_j}e^{\theta(x)}e^{\lambda m_j^2w_j^2}(1+O(\lambda)w_j)\\
&= 2\lambda m_j^2e^{\lambda m_j^2w_j^2}(1+O(\lambda w_j))\varepsilon_j^{-1}e^{w_j},
\end{aligned}$$

and

$$\begin{aligned}
I_b &= 2\lambda^2\left(w_j(x)+\log\varepsilon_j^{-2}+\beta_j+\theta(x)\right)^2 e^{\lambda m_j^2\left(w_j(x)+\log\varepsilon_j^{-2}+\beta_j+\theta(x)\right)^2}\\
&= 2\lambda^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)^2\left(1+(\log\frac{1}{\varepsilon_j^2}+\beta_j)^{-1}(w_j+O(1))\right)^2\\
&\quad\times e^{\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)^2} e^{2\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)w_j} e^{2\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)\theta(x)} e^{\lambda m_j^2(w_j+\theta(x))^2}\\
&= \frac{1}{2m_j^4}\left(1+2\lambda m_j^2(w_j+O(1))\right)^2 e^{\frac{1}{2}(\log\frac{1}{\varepsilon_j^2}+\beta_j)} e^{w_j}e^{\theta(x)}e^{\lambda m_j^2(w_j+\theta(x))^2}\\
&= \frac{1}{2m_j^4}\left(1+2\lambda m_j^2(w_j+O(1))\right)^2 e^{\beta_j/2}\varepsilon_j^{-1}e^{w_j}e^{\theta(x)}e^{\lambda m_j^2w_j^2}(1+O(\lambda)w_j)\\
&= \frac{1}{m_j^2}\left(1+2\lambda m_j^2(w_j+O(1))\right)e^{\lambda m_j^2w_j^2}(1+O(\lambda w_j))\varepsilon_j^{-1}e^{w_j}.
\end{aligned}$$

Then we find

$$I_a 1_{\mathsf{inter}} = \lambda\sum_{j=1}^k\rho_j(x)O(1),\qquad I_b 1_{\mathsf{inter}}-\sum_{j=1}^k\varepsilon_j^{-1}e^{w_j} = \lambda\sum_{j=1}^k\rho_j(x)O(1). \tag{6.118}$$

where 1_{inter} is the characteristic function of the set $\cup_{j=1}^k\{y\ :\ |y-\xi_j'|<\frac{\delta}{\varepsilon_j}\}$. Then from (6.116), (6.117), (6.118) and the definition of $*-$norm, we obtain estimate (6.83).

Proof of (6.84). We shall prove

$$\left\|f''(\tilde U)\right\|_{*,\partial\Omega}\le C.$$

Indeed, we have

$$f''(\tilde U) = 6\lambda^2\tilde U e^{\lambda\tilde U^2}+4\lambda^3\tilde U^3e^{\lambda\tilde U^2} := I_c+I_d.$$

For $x\in\partial\Omega$, far away from the points ξ_j, namely for $|x-\xi_j|>\delta$, i.e. $|y-\xi_j'|>\frac{\delta}{\varepsilon_j}$, for all $j=1,2,\dots,k$, a consequence of (6.22) is that

$$I_c=\lambda^2O(1),\qquad I_d=\lambda^3O(1).$$

Then we have

$$f''(\tilde{U})1_{\mathsf{outer}} = \lambda^2 O(1). \tag{6.119}$$

On the other hand, fix the index j in $\{1,2,\ldots,k\}$, for $|x-\xi_j|<\delta$, from (6.23), (6.29) and (6.30) we have

$$\begin{aligned}
I_c &= 6\lambda^2\left(w_j(x)+\log\varepsilon_j^{-2}+\beta_j+\theta(x)\right)e^{\lambda m_j^2\left(w_j(x)+\log\varepsilon_j^{-2}+\beta_j+\theta(x)\right)^2}\\
&= 6\lambda^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)\left(1+(\log\frac{1}{\varepsilon_j^2}+\beta_j)^{-1}(w_j+O(1))\right)\\
&\quad\times e^{\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)^2}e^{2\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)w_j}e^{2\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)\theta(x)}e^{\lambda m_j^2(w_j+\theta(x))^2}\\
&= \frac{3\lambda}{m_j^2}\left(1+2\lambda m_j^2(w_j+O(1))\right)e^{\frac{1}{2}(\log\frac{1}{\varepsilon_j^2}+\beta_j)}e^{w_j}e^{\theta(x)}e^{\lambda m_j^2(w_j+\theta(x))^2}\\
&= \frac{3\lambda}{m_j^2}\left(1+2\lambda m_j^2(w_j+O(1))\right)e^{\beta_j/2}\varepsilon_j^{-1}e^{w_j}e^{\theta(x)}e^{\lambda m_j^2w_j^2}(1+O(\lambda)w_j)\\
&= 6\lambda\left(1+2\lambda m_j^2(w_j+O(1))\right)e^{\lambda m_j^2w_j^2}(1+O(\lambda w_j))\varepsilon_j^{-1}e^{w_j}\\
&= 6\lambda\left\{\left(1+2\lambda m_j^2(w_j+O(1))\right)e^{\lambda m_j^2w_j^2}(1+O(\lambda w_j))-1\right\}\varepsilon_j^{-1}e^{w_j}+6\lambda\varepsilon_j^{-1}e^{w_j}\\
&= 12\lambda^2m_j^2\underbrace{\frac{1}{2\lambda m_j^2}\left\{\left(1+2\lambda m_j^2(w_j+O(1))\right)e^{\lambda m_j^2w_j^2}(1+O(\lambda w_j))-1\right\}\varepsilon_j^{-1}e^{w_j}}_{=\rho_j(x)}+\underbrace{6\lambda\varepsilon_j^{-1}e^{w_j}}_{=\lambda\rho_j(x)O(1)}\\
&= \lambda\rho_j(x)O(1),
\end{aligned}$$

and

$$\begin{aligned}
I_d &= 4\lambda^3\left(w_j(x)+\log\varepsilon_j^{-2}+\beta_j+\theta(x)\right)^3e^{\lambda m_j^2\left(w_j(x)+\log\varepsilon_j^{-2}+\beta_j+\theta(x)\right)^2}\\
&= 4\lambda^3(\log\frac{1}{\varepsilon_j^2}+\beta_j)^3\left(1+(\log\frac{1}{\varepsilon_j^2}+\beta_j)^{-1}(w_j+O(1))\right)^3\\
&\quad\times e^{\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)^2}e^{2\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)w_j}e^{2\lambda m_j^2(\log\frac{1}{\varepsilon_j^2}+\beta_j)\theta(x)}e^{\lambda m_j^2(w_j+\theta(x))^2}\\
&= \frac{1}{2m_j^6}\left(1+2\lambda m_j^2(w_j+O(1))\right)^3e^{\frac{1}{2}(\log\frac{1}{\varepsilon_j^2}+\beta_j)}e^{w_j}e^{\theta(x)}e^{\lambda m_j^2(w_j+\theta(x))^2}\\
&= \frac{1}{2m_j^6}\left(1+2\lambda m_j^2(w_j+O(1))\right)^3e^{\beta_j/2}\varepsilon_j^{-1}e^{w_j}e^{\theta(x)}e^{\lambda m_j^2w_j^2}(1+O(\lambda)w_j)\\
&= \frac{1}{m_j^4}\left(1+2\lambda m_j^2(w_j+O(1))\right)e^{\lambda m_j^2w_j^2}(1+O(\lambda w_j))\varepsilon_j^{-1}e^{w_j}\\
&= \frac{2\lambda}{m_j^2}\underbrace{\frac{1}{2\lambda m_j^2}\left\{\left(1+2\lambda m_j^2(w_j+O(1))\right)e^{\lambda m_j^2w_j^2}(1+O(\lambda w_j))-1\right\}\varepsilon_j^{-1}e^{w_j}}_{=\rho_j(x)}+\underbrace{\frac{1}{m_j^4}\varepsilon_j^{-1}e^{w_j}}_{=\rho_j(x)O(1)}\\
&= \rho_j(x)O(1).
\end{aligned}$$

Thus we obtain

$$f''(\tilde{U})1_{\mathsf{inter}} = O(1)\sum_{j=1}^{k}\rho_j(x). \tag{6.120}$$

Then from (6.119), (6.120) and the definition of $*-$norm, we obtain estimate (6.84).

Chapter 7

Critical points of the Trudinger-Moser trace funcational

7.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary, and let $H^1(\Omega)$ be the Sobolev space, equipped with the norm

$$\|u\| = \left(\int_\Omega (|\nabla u|^2 + u^2) dx \right)^{\frac{1}{2}}.$$

Let α be a positive number, the Trudinger-Moser trace inequality states that

$$C_\alpha(\Omega) = \sup_{u \in H^1(\Omega),\ \|u\| \le 1} \int_{\partial\Omega} e^{\alpha\,|u|^2} \quad \begin{cases} \le C < +\infty\ , & \text{if}\ \ \alpha \le \pi \\ = +\infty, & \text{if}\ \ \alpha > \pi \end{cases} \tag{7.1}$$

[6, 22, 23, 74, 114]. For (7.1) there is a loss of compactness at the limiting exponent $\alpha = \pi$. Despite of that, it has been proven in [124] that the supremum $C_\pi(\Omega)$ is attained by a function $u \in H^1(\Omega)$ with $\int_\Omega [|\nabla u|^2 + u^2] = 1$, for any bounded domain Ω in $\mathbb{R}^2$, with smooth boundary. Also, for any $\alpha \in (0, \pi)$, the supremum $C_\alpha(\Omega)$ is finite and it is attained. But the exponent $\alpha = \pi$ is critical in the sense that for any $\alpha > \pi$, $C_\alpha(\Omega) = \infty$. See also [24, 72, 73] for generalizations.

The aim of this chapter is to study the existence of critical points of the Trudinger-Moser trace functional

$$E_\alpha(u) = \int_{\partial\Omega} e^{\alpha\, u^2}, \tag{7.2}$$

constrained to functions

$$u \in M = \left\{ u \in H^1(\Omega)\ :\ \|u\|^2 = 1 \right\} \tag{7.3}$$

in the super critical regime

$$\alpha > \pi.$$

In view of the results described above, we will be interested in critical points other than global supremum. As far as we know, no results are known in the literature concerning existence of critical points for the Trudinger-Moser trace constrained problem in the *super critical regime*. Nevertheless, much more is known for the corresponding Trudinger-Moser functional.

Let us recall that the Trudinger-Moser inequality in dimension 2 states that

$$\sup_{u\in H_0^1(\Omega),\, \|\nabla u\|_2\le 1} \int_\Omega e^{\mu\,|u|^2} dx \quad \begin{cases} \le C \ < +\infty\ , & \text{if } \ \mu \le 4\pi \\ = +\infty, & \text{if } \ \mu > 4\pi. \end{cases} \tag{7.4}$$

Here again Ω is a bounded domain of $\mathbb{R}^2$, with smooth boundary. For problem (7.4) there is a loss of compactness at the limiting exponent $\mu = 4\pi$ [79]. Despite of this loss of compactness, the supremum

$$\sup_{u\in H_0^1(\Omega),\, \|\nabla u\|_2\le 1} \int_\Omega e^{4\pi\,|u|^2} dx$$

is attained for any bounded domain $\Omega \subset \mathbb{R}^2$. This was proven first in the seminal work [18] for the ball $\Omega = B_1(0)$ (see also an alternative proof in [30]). In [111] the result was proven for domains Ω which are small perturbation of the ball. The general result in dimension 2 was proven by Flucher in [53], and Lin [80] extended it for the corresponding Trudinger-Moser inequality for general domain of $\mathbb{R}^N$, with $N > 2$.

Concerning the super critical regime for the Trudinger-Moser functional, namely

$$I_\mu(u) = \int_\Omega e^{\mu\,|u|^2} dx\ ,\ u \in H_0^1(\Omega),\ \|\nabla u\|_2^2 = 1, \quad \text{with} \quad \mu > 4\pi, \tag{7.5}$$

some results are known. In the works [111] and [68] it has been proven that a local maxima and saddle point solutions in the supercritical regime $\mu \in (4\pi, \mu_0)$ for the functional (7.5) do exist, for some $\mu_0 > 4\pi$.

Our first result is an extension of the existence of a local maxima for the Trudinger-Moser trace functional in the super critical regime $\alpha \in (\pi, \alpha_0)$. Namely, a local maximizer for Problem (7.2)-(7.3) exists when the value of α is slightly to the right of π.

Theorem 7.1. *Let Ω be a bounded domain in $\mathbb{R}^2$. Then there exists $\alpha_0 > \pi$, such that for any $\alpha \in (0, \alpha_0)$, there exists a function $u_\alpha \in M$ which locally maximizes of E_α on M.*

This result is proved in Section 7.2.

Much more is known for Problem (7.5) and $\mu > 4\pi$. Recently in [40] (see also [39]), the authors obtained several results concerning critical points for Problem (7.5) also in a *very* super critical regime. They found general conditions on the domain Ω under which there is a critical point for $I_\mu(u)$ with $\int_\Omega |\nabla u|^2 dx = 1$ when $\mu \in (4\pi k, \mu_k)$, for any integer $k \geq 1$ and for some μ_k slightly bigger than $4\,\pi\,k$. In particular, for any bounded domain Ω, they found a critical point for $I_\mu(u)$ with $\int_\Omega |\nabla u|^2 dx = 1$ when $\mu \in (4\pi, \mu_1)$, for some $\mu_1 > 4\,\pi$. The L^∞-norm of this solution converges to ∞ as $\mu \to 4\pi$ and its mass is concentrated, in some proper sense, as $\mu \to 4\pi$, around a point in the interior of Ω. On the other hand, if Ω has a hole, namely it is not simply connected, they proved the existence of a critical point for $I_\mu(u)$ with $\int_\Omega |\nabla u|^2 dx = 1$ also in the super critical range $\mu \in (8\pi, \mu_2)$, for some $\mu_2 > 8\pi$. Again in this case, the L^∞-norm of these solutions converges to ∞ as $\mu \to 8\pi$, but now its mass concentrates, as $\mu \to 8\pi$, around two distinct points inside Ω. Furthermore, if Ω is an annulus, taking advantage of the symmetry, a critical point for $I_\mu(u)$ with $\int_\Omega |\nabla u|^2 dx = 1$ and $\mu \in (4\pi k, \mu_k)$ does exist. In this latter case, the L^∞-norm of the solution converges to ∞ as $\mu \to 4\pi k$ and its mass concentrates, as $\mu \to 4\pi k$, around k points distributed along the vertices of a proper regular polygon with k sides lying inside Ω.

The second result of this chapter establishes the counterpart of the above situation for the Trudinger-Moser trace functional in the super critical regime: we will show the existence of critical points for E_α constrained to M, for $\alpha \in (k\,\pi, \alpha_k)$, for any $k \geq 1$ integer and for some α_k slightly to the right of $k\,\pi$. We next describe our result.

Let $G(x, y)$ be the Green's function of the problem

$$\begin{cases} -\Delta_x G(x,y) + G(x,y) = 0 & x \in \Omega; \\ \frac{\partial G(x,y)}{\partial \nu_x} = 2\pi \delta_y(x) & x \in \partial\Omega, \end{cases} \tag{7.6}$$

and H its regular part defined as

$$H(x,y) = G(x,y) - 2\log\frac{1}{|x-y|}. \tag{7.7}$$

Our second result reads as follows.

Theorem 7.2. *Let Ω be any bounded domain in $\mathbb{R}^2$ with smooth boundary. Fix a positive integer $k \geq 1$. Then there exists $\alpha_k > k\pi$ such that for $\alpha \in (k\pi, \alpha_k)$, the functional $E_\alpha(u)$ restricted to M has at least two critical points u^1_α and u^2_α. Furthermore, for any $i = 1, 2$ there exist numbers $m^i_{j,\alpha} > 0$ and points $\xi^i_{j,\alpha} \in \partial\Omega$, for $j = 1, \ldots, k$ such that*

$$\lim_{\alpha \to k\,\pi} m^i_{j,\alpha} = m^i_j \in (0, \infty), \tag{7.8}$$

$$\xi^i_{j,\alpha} \to \xi^i_j \in \partial\Omega, \quad \textit{with} \quad \xi^i_j \neq \xi^i_l \quad \textit{for} \quad j \neq l, \quad \textit{as} \quad \alpha \to k\,\pi \tag{7.9}$$

and

$$u^i_\alpha(x) = \sqrt{\frac{\alpha - k\pi}{\alpha}} \sum_{j=1}^{k} \left[m^i_{j,\alpha} G(x, \xi^i_{j,\alpha}) + o(1) \right], \quad i = 1, 2, \tag{7.10}$$

where $o(1) \to 0$ *uniformly on compact sets of* $\bar{\Omega}\backslash\{\xi_1^i, \dots, \xi_k^i\}$*, as* $\alpha \to k\pi$*. Moreover, for any* $i = 1, 2$*, for any* $\delta > 0$ *small, for any* $j = 1, \dots, k$*,*

$$\sup_{x \in B(\xi_j^i, \delta)} u_\alpha^i(x) \to +\infty, \quad \text{as} \quad \alpha \to k\,\pi. \tag{7.11}$$

There are two important differences between the result stated in Theorem 7.2 and the corresponding result obtained in [40] for the Trudinger-Moser functional (7.5). A first difference is that for Problem (7.2)-(7.3) existence of critical points in the range $\alpha \in (k\,\pi, \alpha_k)$ is guaranteed in *any* bounded domain Ω with smooth boundary, at any integer level k. No further hypothesis on Ω is needed, unlike the Trudinger-Moser case (7.5). The second difference is that, we do find *two* families of critical points for Problem (7.2)-(7.3) when $\alpha \in (k\,\pi, \alpha_k)$, and not only one as in the Trudinger-Moser case (7.5).

7.2 The local maximizer: proof of Theorem 7.1

We set

$$E(u) = \int_{\partial\Omega} e^{u^2}, \tag{7.12}$$

and

$$M_\alpha = \left\{ u \in H^1(\Omega) \ : \ \|u\|^2 = \alpha \right\}. \tag{7.13}$$

We note that by the obvious scaling property, finding critical points of E_α on M (see (7.2) and (7.3)) is equivalent to finding critical points of E on M_α (see (7.12) and (7.13)). In this section, we study the local maximizer for the functional E constrained on the set M_α with α in the right neighborhood of π.

We start with the following Lion's type Lemma. The proof is quite standard, but we reproduce it here for completeness.

Lemma 7.3. *Let* u_m *be a sequence of functions in* $H^1(\Omega)$ *with* $\|u_m\| = 1$*. Suppose that* $u_m \rightharpoonup u_0$ *weakly in* $H^1(\Omega)$*. Then either* $u_0 = 0$*, or there exists* $\alpha > \pi$ *such that the family* $e^{u_m^2}$ *is uniformly bounded in* $L^\alpha(\partial\Omega)$*, and thus we have*

$$\int_{\partial\Omega} e^{\pi u_m^2} \to \int_{\partial\Omega} e^{\pi u_0^2} \quad \text{as } m \to \infty.$$

Proof. Since $\|u_m\| = 1$ and $u_m \rightharpoonup u_0$ weakly in $H^1(\Omega)$, we have

$$\int_\Omega (\nabla u_m \nabla u_0 + u_m u_0) \to \int_\Omega (|\nabla u_0|^2 + u_0^2) \quad \text{as} \quad m \to \infty.$$

Thus we find that

$$\begin{aligned}\lim_{m\to\infty}\|u_m-u_0\|^2 &= \lim_{m\to\infty}\left\{\int_\Omega [|\nabla(u_m-u_0)|^2+(u_m-u_0)^2]\right\}\\ &= \lim_{m\to\infty}\left\{\|u_m\|^2-2\int_\Omega(\nabla u_m\nabla u_0+u_mu_0)+\|u_0\|^2\right\}\\ &= 1-\|u_0\|^2.\end{aligned}$$

Assume $u_0\neq 0$. Take $p\in(1,\frac{1}{1-\|u_0\|^2})$, and choose q_1 and q_2 such that $1<pq_1<\frac{1}{\|u_m-u_0\|^2}$ and $\frac{1}{q_1}+\frac{1}{q_2}=1$. By Hölder inequality we have

$$\begin{aligned}\int_{\partial\Omega}e^{\pi p u_m^2} &= \int_{\partial\Omega}e^{\pi p(u_m-u_0+u_0)^2}=\int_{\partial\Omega}e^{\pi p\left[(u_m-u_0)^2+2(u_m-u_0)u_0+u_0^2\right]}\\ &= \int_{\partial\Omega}e^{\pi p\left[(u_m-u_0)^2+2u_mu_0-u_0^2\right]}\le\int_{\partial\Omega}e^{\pi p\left[(u_m-u_0)^2+2u_mu_0\right]}\\ &= \int_{\partial\Omega}e^{\pi p(u_m-u_0)^2}e^{2\pi p u_mu_0}\le\left(\int_{\partial\Omega}e^{\pi pq_1(u_m-u_0)^2}\right)^{\frac{1}{q_1}}\left(\int_{\partial\Omega}e^{2\pi pq_2u_mu_0}\right)^{\frac{1}{q_2}}.\end{aligned}$$

We now recall that

$$\pi=\sup\left\{\theta\ :\ \sup_{u\in H^1(\Omega),\|u\|\le 1}\int_{\partial\Omega}e^{\theta u^2}d\sigma<\infty\right\}. \tag{7.14}$$

see for instance [6, 22, 23, 74]. Hence, given the choice of p and q_1, we get that there exists a constant C, independent of m, such that

$$\int_{\partial\Omega}e^{\pi pq_1(u_m-u_0)^2}<C.$$

On the other hand, Young's inequality implies that $2|u_mu_0|\le\varepsilon^2u_m^2+\frac{1}{\varepsilon^2}u_0^2$, with $\varepsilon>0$ small. Then from (7.14), we have

$$\int_{\partial\Omega}e^{2\pi pq_2u_mu_0}<\int_{\partial\Omega}e^{\pi pq_2[\varepsilon^2u_m^2+\frac{1}{\varepsilon^2}u_0^2]}=\int_{\partial\Omega}e^{\pi pq_2\varepsilon^2u_m^2}e^{\pi pq_2\frac{1}{\varepsilon^2}u_0^2}<C$$

by choosing ε so that $pq_2\varepsilon^2<1$. Here again C is a constant, independent of m. Thus, we have that there exists $\alpha=p\pi>\pi$ such that the family $e^{u_m^2}$ is uniformly bounded in $L^\alpha(\partial\Omega)$.

We shall now show that

$$\int_{\partial\Omega}e^{\pi u_m^2}\to\int_{\partial\Omega}e^{\pi u_0^2}\quad\text{as } m\to\infty. \tag{7.15}$$

Indeed, let l be a positive number and $p>1$. We have

$$\left|\int_{\partial\Omega}e^{\pi u_m^2}-\int_{\partial\Omega\cap\{|u_m|\le l\}}e^{\pi u_m^2}\right| = \left|\int_{\partial\Omega\cap\{|u_m|>l\}}e^{\pi u_m^2}\right|\le\frac{1}{l^{\frac{2(p-1)}{p}}}\int_{\partial\Omega}e^{\pi u_m^2}u_m^{\frac{2(p-1)}{p}}$$

$$\leq \frac{1}{l^{\frac{2(p-1)}{p}}}\left(\int_{\partial\Omega} e^{\pi p u_m^2}\right)^{\frac{1}{p}}\left(\int_{\partial\Omega} u_m^2\right)^{\frac{p-1}{p}} \leq \frac{C}{l^{\frac{2(p-1)}{p}}}.$$

From the above relation, we conclude that

$$\int_{\partial\Omega} e^{\pi u_m^2} \leq |\partial\Omega| e^{\pi l^2} + \frac{C}{l^{\frac{2(p-1)}{p}}}.$$

Hence dominated convergence Theorem implies (7.15).

Suppose now that $e^{u_m^2}$ is not bounded in $L^\alpha(\partial\Omega)$ for any $\alpha > \pi$. Using Stokes theorem, for $\alpha > \pi$ we have

$$\begin{aligned}\int_{\partial\Omega} e^{\alpha u_m^2} d\sigma &= \int_\Omega div(e^{\alpha u_m^2})dx \leq C\int_\Omega |\nabla u_m||u_m|e^{\alpha u_m^2}dx \\ &\leq C\left(\int_\Omega |\nabla u_m|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |u_m|^q dx\right)^{\frac{1}{q}}\left(\int_\Omega e^{\beta u_m^2}dx\right)^{\frac{\alpha}{\beta}}\end{aligned}$$

where $q > 1$ satisfies $\frac{1}{2} + \frac{1}{q} + \frac{\alpha}{\beta} = 1$ with $\beta > 2\pi$. Then we get that $\int_\Omega e^{\beta u_m^2} dx$ is unbounded for all $\beta > 2\pi$.

Observe now that we can assume that $\int_\Omega u_m dx = 0$, since otherwise we set $\bar{u}_m = u_m - \frac{1}{|\Omega|}\int_\Omega u_m dx$ and obtain $\int_\Omega u_m dx = 0$. We can also assume that $\int_\Omega |\nabla u_m|^2 = 1$. Furthermore, by Poincaré inequality, (u_m) is bounded in $H^1(\Omega)$, and also $(|u_m|)$ is bounded in $H^1(\Omega)$. Hence there exists $u \in H^1(\Omega)$ such that $|u_m| \rightharpoonup u_0$ weakly in $H^1(\Omega)$. We claim that

$$\lim_{m\to\infty}\int_\Omega |\nabla(u_m - \eta)^+|^2 dx = 1 \quad \forall\, \eta > 0. \tag{7.16}$$

By contradiction, assume there exists $\eta > 0$ such that $\lim_{m\to\infty}\int_\Omega |\nabla(u_m-\eta)^+|^2 dx \neq 1$. Define $\gamma = \inf_m \int_\Omega |\nabla(u_m-\eta)^+|^2 dx < 1$ and choose a sufficiently small $\varepsilon > 0$ such that $\alpha' := \frac{2\pi}{\gamma+\varepsilon} > 2\pi$. Let us recall that

$$2\pi = \sup\left\{\theta \;:\; \sup_{u\in H^1(\Omega), \int_\Omega |\nabla u|^2 \leq 1, \int_\Omega u = 0}\int_\Omega e^{\theta u^2}dx < \infty\right\}, \tag{7.17}$$

(see [6, 22, 23, 124]). From (7.17), there exists a positive constant C such that

$$\int_\Omega e^{\alpha'\left[(|u_m|-\eta)^+ - \frac{1}{|\Omega|}\int_\Omega (|u_m|-\eta)^+\right]^2} dx = \int_\Omega e^{2\pi\left[\frac{(|u_m|-\eta)^+ - \frac{1}{|\Omega|}\int_\Omega(|u_m|-\eta)^+}{\sqrt{\gamma+\varepsilon}}\right]^2} dx < C,$$

where we use the fact that $\int_\Omega |\nabla\frac{(u_m-\eta)^+}{\sqrt{\gamma+\varepsilon}}|^2 dx < 1$.

Define $d_m = \frac{1}{|\Omega|}\int_\Omega (|u_m| - \eta)^+$. Choosing $\varepsilon' > 0$ small such that $\tilde{\alpha} := \frac{\alpha'}{1+\varepsilon'} > 2\pi$, and by the Young's inequality,

$$\begin{aligned} u_m^2 &\leq (\eta + d_m)^2 + 2(\eta + d_m)[(|u_m| - \eta)^+ - d_m] + [(|u_m| - \eta)^+ - d_m]^2 \\ &\leq (1+\varepsilon')[(|u_m| - \eta)^+ - d_m]^2 + (\frac{1}{\varepsilon'} + 1)(\eta + d_m)^2. \end{aligned}$$

Thus, since there $d_m = O(1)$ as $m \to \infty$,

$$\int_\Omega e^{\tilde{\alpha} u_m^2} dx = \int_\Omega e^{\frac{\alpha'}{1+\varepsilon'} u_m^2} dx \leq C_1 \int_\Omega e^{\alpha' \left[(|u_m|-\eta)^+ - \frac{1}{|\Omega|}\int_\Omega (|u_m|-\eta)^+\right]^2} dx \leq C_2,$$

for some positive constants C_1 and C_2. This is a contradiction, thus (7.16) holds.

Set $v_m = \min\{|u_m|, \eta\}$, then v_m is bounded in $H^1(\Omega)$ and, up to subsequence, we have that $v_m \rightharpoonup v$. Observe now that $|u_m| = v_m + (|u_m| - \eta)^+$, and

$$1 = \int_\Omega |\nabla u_m|^2 \geq \int_\Omega |\nabla |u_m||^2 dx = \int_\Omega |\nabla v_m|^2 dx + \int_\Omega |\nabla (|u_m| - \eta)^+|^2 dx.$$

Therefore (7.16) implies that that $\int_\Omega |\nabla v_m|^2 dx \to 0$ as $m \to \infty$, so v is constant. On the other hand,

$$\lim_{m\to\infty} \int_\Omega |\nabla v_m|^2 dx = \lim_{m\to\infty} \int_{\Omega \cap \{|u_m| \leq \eta\}} |\nabla |u_m||^2 dx = 0.$$

This implies that $|\{x : |u_m| \geq \eta\}| \to 0$ as $m \to \infty$. By Fatou Lemma,

$$|\{x : u_0 \geq \eta\}| \leq \liminf_{m\to\infty} |\{x : |u_m| \geq \eta\}| = 0,$$

then $|\{x : u_0 \geq \eta\}| = 0$ for any $\eta > 0$. Hence we get $u_0 = 0$. □

We denote $\beta := \sup\limits_{u \in M_\pi} E(u) = \sup\limits_{u\in M} E_\pi(u)$. A direct consequence of the previous Lemma is the following

Proposition 7.4. *Let u_m be a bounded sequence in $H^1(\Omega)$ with $\|u_m\| = 1$. Suppose that $u_m \rightharpoonup u_0$ weakly in $H^1(\Omega)$. Suppose $E_\pi(u_m) \to \beta$ with $\beta > |\partial\Omega|$. Then there exists $\alpha > \pi$ such that the family $e^{u_m^2}$ is uniformly bounded in $L^\alpha(\partial\Omega)$. In particular $E_\pi(u_m) \to E_\pi(u_0)$ and $u_0 \neq 0$.*

Proof. Suppose $e^{u_m^2}$ is unbounded in $L^\alpha(\partial\Omega)$ for all $\alpha > \pi$, and assume the supremum of E_π on M is not attained. Then by Lemma 7.3, we have that $u_0 = 0$, which is impossible because $E_\pi(u_m) \to \beta > |\partial\Omega|$. □

Let K_π be the set defined by

$$K_\pi = \{u \in M \ : \ E_\pi(u) = \beta\}.$$

Lemma 7.5. *The set K_π is compact.*

Proof. Let $\{u_m\} \subset K_\pi$ be such that $u_m \rightharpoonup u_0$ weakly in $H^1(\Omega)$, then by Proposition 7.4,

$$E_\pi(u_m) \to E_\pi(u_0).$$

Moreover, $\|u_0\| \leq \|u_m\| = 1$, then

$$E_\pi(u_0) \leq E_\pi(\frac{u_0}{\|u_0\|}) \leq \sup_{v\in M} E_\pi(v) = \beta.$$

Then we get $E_\pi(u_0) = \beta$, and $\|u_0\| = 1$, hence $u_m \to u_0$ strongly in $H^1(\Omega)$, hence K_π is compact. □

The property of K_π of being compact implies that the family of norm-neighborhoods

$$N_\varepsilon = \{u \in M \mid \exists\, v \in K_\pi : \|u - v\| < \varepsilon\}$$

constitutes a basic neighborhood for K_π in M.

Lemma 7.6. *For sufficiently small $\varepsilon > 0$, one has*

$$\sup_{N_{2\varepsilon}\setminus N_\varepsilon} E_\pi < \beta = \sup_{N_\varepsilon} E_\pi. \tag{7.18}$$

Proof. We argue by contradiction. We suppose that there is a sequence $u_m \in N_{2\varepsilon}\setminus N_\varepsilon$ such that $E_\pi(u_m) \to \beta$. Then we have $u_m \in H^1(\Omega)$ with $\|u_m\|^2 = 1$. Up to subsequence, we can assume that $u_m \rightharpoonup u_0$ weakly in $H^1(\Omega)$. By the definition of $N_{2\varepsilon}$, there is $z_m \in K_\pi$ such that $\|z_m - u_m\| < 2\varepsilon$. By the compactness of K_π, we have that $z_m \to z$ strongly, with $z \in K_\pi$, and z satisfies

$$-\Delta z + z = 0 \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial \nu} = \frac{\pi z e^{z^2}}{\int_{\partial\Omega} z^2 e^{z^2}} \quad \text{on } \partial\Omega.$$

By the maximum principle, we have $z \in L^\infty(\Omega)$.

By the lower-semi continuity, we have $\|z - u_0\| \leq 2\varepsilon$. Then

$$\|z - \frac{u_0}{\|u_0\|}\| \;\leq\; \|z - u_0\| + \|u_0 - \frac{u_0}{\|u_0\|}\| = \|z - u_0\| + 1 - \|u_0\| \leq 4\varepsilon.$$

Thus $\frac{u_0}{\|u_0\|} \in N_{4\varepsilon}$, and so $E_\pi(u_0) \leq E_\pi(\frac{u_0}{\|u_0\|}) \leq \beta$. If $E_\pi(u_0) = \beta$ then $\|u_0\| = 1$, and $u_m \to u_0$. On the other hand, our assumption implies that $u_0 \notin N_\varepsilon$, thus u_0 does not belong to K_π and u_0 can not be relatively maximal. Thus we necessarily get $E_\pi(u_0) < \beta$.

Set $w_m = u_m - z_m + z$, so we have $w_m \rightharpoonup u_0$ weakly in $H^1(\Omega)$. Since

$$e^{\pi|w_m|^2} = e^{\pi|u_m - z_m + z|^2} \leq e^{2\pi|u_m - z_m|^2} e^{2\pi|z|^2} = e^{2\pi\|u_m - z_m\|^2(\frac{u_m - z_m}{\|u_m - z_m\|})^2} e^{2\pi|z|^2} \leq e^{8\pi\varepsilon^2(\frac{u_m - z_m}{\|u_m - z_m\|})^2} e^{2\pi|z|^2}.$$

Choosing ε small such that $16\varepsilon^2 \leq 1$, then from (7.14) we have that $e^{\pi|w_m|^2}$ is uniformly bounded in $L^2(\partial\Omega)$, as $m \to \infty$. Thus $\lim_{m\to\infty} E_\pi(w_m) = E_\pi(u_0)$. On the other hand, we have $w_m - u_m \to 0$ strongly in $H^1(\Omega)$. By uniform local continuity of E_π, and compactness of K_π, we obtain that $E_\pi(w_m) - E_\pi(u_m) \to 0$, and $E_\pi(u_0) = \beta$. This is a contradiction. □

Lemma 7.7. *There exists $\alpha^* > \pi$, $\varepsilon > 0$ such that for all $\alpha \in [\pi, \alpha^*)$, then we have*

(i)

$$\sup_{N_{2\varepsilon}\setminus N_\varepsilon} E_\alpha < \sup_{N_\varepsilon} E_\alpha. \tag{7.19}$$

(ii) $\beta_\alpha := \sup_{N_\varepsilon} E_\alpha$ is achieved in N_ε.

(iii) $K_\alpha = \{u \in N_\varepsilon \mid E_\alpha(u) = \beta_\alpha\}$ is compact.

Proof. (i) Since K_π is compact, there is a neighborhood N of K_π such that, for any $\varsigma > 0$ there exists $\delta' > 0$ such that for all $|\alpha - \pi| < \delta$ then $E_\alpha(u) - E_\pi(u)| \leq \varsigma$, for all $u \in N$. Choose $\varepsilon > 0$ such that (7.18) holds and $N_\varepsilon \subset N$, then (7.19) will be valid for all α in a small neighborhood of π.

(ii) For such α, and let $u_m \in N_\varepsilon$ be a maximizing sequence of E_α, that is, $E_\alpha(u_m) \to \beta_\alpha$ and let $v_m \in K_\pi$ satisfy $\|u_m - v_m\| \leq \varepsilon$.We may assume that $v_m \to v$ strongly in $H^1(\Omega)$ with $v \in L^\infty$, and $u_m \to u$ weakly in $H^1(\Omega)$. Set $w_m = u_m - v_m + v$, as the proof of Lemma 7.6, we obtain that for $\varepsilon > 0$ small, α in a neighborhood of π we have that

$$E_\alpha(w_m) \to E_\alpha(u), \qquad E_\alpha(u_m) - E_\alpha(w_m) \to 0 \quad \text{as } m \to \infty.$$

Then $E_\alpha(u) = \beta_\alpha$. Moreover, by the lower-semi continuity, we have $\|v - u\| \leq \varepsilon$. Then

$$\|v - \frac{u}{\|u\|}\| \;\leq\; \|v - u\| + \|u - \frac{u}{\|u\|}\| = \|v - u\| + 1 - \|u\| \leq 2\varepsilon.$$

We get that $\frac{u}{\|u\|} \in \bar{N}_{2\varepsilon}$ and $E_\alpha(\frac{u}{\|u\|}) \leq \beta_\alpha$. Furthermore, since $\|u\| \leq 1$, we can get $E_\alpha(\frac{u}{\|u\|}) \leq E_\alpha(u)$ and $\|u\| = 1$. It implies that $u \in M$, that is $u \in N_\varepsilon$ and β_α is attained. Moreover, $u_m \to u$ strongly in $H^1(\Omega)$.

(iii) As the proof of (ii), if $u_m \in K_\alpha$, we may assume that $u_m \rightharpoonup u$ weakly in $H^1(\Omega)$, we then get $u \in K_\alpha$, that is K_α is compact. □

Proof of Theorem 7.1: From (7.14), we have that $\sup_{M_\alpha} E$ is achieved for $\alpha < \pi$. Moreover, since $\sup_{u\in M_\pi} E(u) > |\partial\Omega|$, from Lemma 7.7 we have that for α sufficiently close to π, then E has relative maximizers on M_α.

7.3 The proof of Theorem 7.2

In this section, we consider critical points of functional $E(u)$ constrained on the set M_α (which is equivalent to consider critical points of $E_\alpha(u)$ constrained on the set M with $\alpha = k\pi(1+\mu)$, where $\mu > 0$ small). We observe that this problem is equivalent to finding solutions of the following problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = \lambda u e^{u^2} & \text{on } \partial\Omega, \end{cases} \tag{7.20}$$

where

$$\lambda = \frac{\alpha}{\int_{\partial\Omega} u^2 e^{u^2}} = \frac{k\pi(1+\mu)}{\int_{\partial\Omega} u^2 e^{u^2}}. \tag{7.21}$$

In this section we shall prove the existence of solutions to Problem (7.20)-(7.21) with the properties described in Theorem 7.2. In fact, we will construct a solution to (7.20)-(7.21) of the form

$$u = U + \phi, \tag{7.22}$$

where U is the principal part while ϕ represents a lower order correction. In what follows we shall first describe explicitly the function $U(x)$. The definition of this function depends on several parameters: some points ξ on the boundary of Ω and some positive numbers m. Next we find the correction ϕ so that $U + \phi$ solves our Problem in a certain *projected sense* (see Proposition 7.8). Finally we select proper points ξ and numbers m in the definition of U to get an exact solution to Problem (7.20)-(7.21).

To define the function U, first we introduce the following limit problem

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^2_+; \\ \frac{\partial w}{\partial \nu} = e^w & \text{on } \partial\mathbb{R}^2_+; \\ \int_{\partial\mathbb{R}^2_+} e^w < \infty. \end{cases} \tag{7.23}$$

A family solutions to (7.23) is given by

$$w_{t,\mu}(x) = w_{t,\mu}(x_1, x_2) = \log \frac{2\mu}{(x_1 - t)^2 + (x_2 + \mu)^2}, \tag{7.24}$$

where $t \in \mathbb{R}$ and $\mu > 0$ are parameters. See [76, 99, 126]. Set

$$w_\mu(x) := w_{0,\mu}(x) = \log \frac{2\mu}{x_1^2 + (x_2 + \mu)^2}. \tag{7.25}$$

Let $\xi_1, \ldots, \xi_k$ be k distinct points on the boundary and $m_1, \ldots, m_k$ be k positive numbers. We assume there exists a sufficiently small but fixed number $\delta > 0$ such that

$$|\xi_i - \xi_j| > \delta \quad \text{for } i \neq j, \qquad \delta < m_j < \frac{1}{\delta}. \tag{7.26}$$

For notational convenience through out the paper we will use the notation

$$(\xi, m) = (\xi_1, \ldots, \xi_k, m_1, \ldots, m_k).$$

For any $j = 1, \ldots, k$, we define ε_j to be the positive numbers given by the relation

$$2\lambda m_j^2 \left(\log \frac{1}{\varepsilon_j^2} + 2\log(2m_j^2) \right) = 1. \tag{7.27}$$

Since the parameters m_j satisfy assumption (7.26), it follows that $\lim_{\lambda \to 0} \varepsilon_j = 0$. Define moreover μ_j to be the positive constants given by

$$\log(2\mu_j) = -2\log(2m_j^2) + H(\xi_j, \xi_j) + \sum_{i \neq j} m_i m_j^{-1} G(\xi_i, \xi_j). \tag{7.28}$$

Using once more assumption (7.26), we get that there exists two positive constants c and C, such that $c \leq \mu_j \leq C$, as $\lambda \to 0$.

We define the function U in (7.22) to be given by

$$U(x) = \sqrt{\lambda} \sum_{j=1}^{k} m_j \left[u_j(x) + H_j(x) \right], \tag{7.29}$$

where

$$u_j(x) = \log \frac{1}{|x - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2}, \tag{7.30}$$

$\nu(\xi_j)$ denoting the unitary outer normal to $\partial\Omega$ at the point ξ_j, and where H_j is a correction term given as the solution of

$$\begin{cases} -\Delta H_j + H_j = -u_j & \text{in } \Omega; \\ \frac{\partial H_j}{\partial \nu} = 2\varepsilon_j \mu_j e^{u_j} - \frac{\partial u_j}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{7.31}$$

The maximum principle allows a precise asymptotic description of the functions H_j, namely we have that

$$H_j(x) = H(x, \xi_j) + O(\varepsilon_j^\sigma) \quad \text{for } 0 < \sigma < 1 \tag{7.32}$$

uniformly in Ω, as $\lambda \to 0$. Recall that H is the regular part of the Green's function, as defined in (7.6). Therefore, the function U can be described as follows

$$U(x) = \sqrt{\lambda} \sum_{j=1}^{k} m_j \left[G(x, \xi_j) + O(\varepsilon_j^\sigma) \right] \tag{7.33}$$

uniformly on compact sets of $\bar{\Omega} \setminus \{\xi_1, \ldots, \xi_k\}$, as $\lambda \to 0$. On the other hand, if we consider a region close to ξ_j, for some j fixed, say for $|x - \xi_j| < \delta$, with sufficiently small but fixed δ, we can rewrite

$$U(x) \;=\; \sqrt{\lambda} m_j \left(w_j(x) + \log \varepsilon_j^{-2} + \beta_j + \theta(x) \right), \tag{7.34}$$

where

$$w_j(x) = w_{\mu_j}(\frac{x - \xi_j}{\varepsilon_j}) = \log \frac{2\mu_j}{|y - \xi_j' - \mu_j \nu(\xi_j')|^2}, \quad y = \frac{x}{\varepsilon_j}, \quad \xi_j' = \frac{\xi_j}{\varepsilon_j}, \tag{7.35}$$

and

$$\beta_j = -\log(2\mu_j) + H(\xi_j, \xi_j) + \sum_{i \neq j} m_j^{-1} m_i G(\xi_j, \xi_i), \qquad \theta(x) = O(|x - \xi_j|) + \sum_{j=1}^{k} O(\varepsilon_j^{\alpha}).$$

Define on the boundary $\partial\Omega$ the error of approximation

$$R := f(U) - \frac{\partial U}{\partial \nu}. \tag{7.36}$$

Here and in what follows f denotes the nonlinearity

$$f(\tilde{u}) = \lambda \tilde{u} e^{\tilde{u}^2}.$$

The choice we made of μ_j in (7.28) and of ε_j in (7.27) gives that in the region $|x - \xi_j| < \delta$, the error of approximation can be described as follows

$$R = m_j \sqrt{\lambda} \left\{ \left(1 + 2\lambda m_j^2 (w_j + O(1))\right) e^{\lambda m_j^2 w_j^2} (1 + O(\lambda w_j)) - 1 \right\} \varepsilon_j^{-1} e^{w_j}, \tag{7.37}$$

where w_j is defined in (7.35). Indeed, for $x \in \partial\Omega$ with $|x - \xi_j| < \delta$, we have that

$$\begin{aligned}
\lambda^{-\frac{1}{2}} f(U) &= \lambda \left[m_j \left(w_j(x) + \log \varepsilon_j^{-2} + \beta_j + \theta(x) \right) \right] e^{\lambda \left[m_j \left(w_j(x) + \log \varepsilon_j^{-2} + \beta_j + \theta(x) \right) \right]^2} \\
&= \left(\lambda m_j (\log \frac{1}{\varepsilon_j^2} + \beta_j) + \lambda m_j (w_j + O(1)) \right) \\
&\quad \times e^{\lambda m_j^2 (\log \frac{1}{\varepsilon_j^2} + \beta_j)^2} e^{2\lambda m_j^2 (\log \frac{1}{\varepsilon_j^2} + \beta_j) w_j} e^{2\lambda m_j^2 (\log \frac{1}{\varepsilon_j^2} + \beta_j)\theta(x)} e^{\lambda m_j^2 (w_j + \theta(x))^2} \\
&= \lambda m_j (\log \frac{1}{\varepsilon_j^2} + \beta_j) \left(1 + (\log \frac{1}{\varepsilon_j^2} + \beta_j)^{-1} (w_j + O(1)) \right) \\
&\quad \times e^{\lambda m_j^2 (\log \frac{1}{\varepsilon_j^2} + \beta_j)^2} e^{2\lambda m_j^2 (\log \frac{1}{\varepsilon_j^2} + \beta_j) w_j} e^{2\lambda m_j^2 (\log \frac{1}{\varepsilon_j^2} + \beta_j)\theta(x)} e^{\lambda m_j^2 (w_j + \theta(x))^2} \\
&= \frac{1}{2m_j} \left(1 + 2\lambda m_j^2 (w_j + O(1)) \right) e^{\frac{1}{2}(\log \frac{1}{\varepsilon_j^2} + \beta_j)} e^{w_j} e^{\theta(x)} e^{\lambda m_j^2 (w_j + \theta(x))^2} \\
&= \frac{1}{2m_j} \varepsilon_j^{-1} e^{\beta_j/2} \left(1 + 2\lambda m_j^2 (w_j + O(1)) \right) e^{w_j} e^{\theta(x)} e^{\lambda m_j^2 w_j^2} (1 + O(\lambda) w_j)
\end{aligned}$$

thanks to the definition of ε_j in (7.27). On the other hand, in the same region, we have

$$\lambda^{-\frac{1}{2}}\frac{\partial U}{\partial \nu} = \frac{\partial}{\partial \nu}\left[m_j\left(w_j(x) + \log \varepsilon_j^{-2} + \beta_j + \theta(x)\right)\right] = m_j\varepsilon_j^{-1}e^{w_j} + \sum_{j=1}^{k} O(\varepsilon_j^2), \quad \text{as} \quad \lambda \to 0.$$

The definition of μ_j in (7.28) allows to match at main order the two terms $\frac{\partial \tilde{U}}{\partial \nu}$ and $f(\tilde{U})$ in the region under consideration, since we , we easily get that

$$\lambda^{-\frac{1}{2}}f(\tilde{U}) \;\;=\;\; m_j\left(1 + 2\lambda m_j^2(w_j + O(1))\right)\varepsilon_j^{-1}e^{w_j}e^{\lambda m_j^2 w_j^2}(1 + O(\lambda w_j)).$$

These facts imply the validity of expansion (7.37). Let us now observe that a direct computation shows that $R(x) \sim \lambda^{\frac{3}{2}}\varepsilon_j^{-1}e^{w_j(x)}$ in the region $|x - \xi_j| = O(\lambda)$; while, in the region $|x - \xi_j| > \delta$ for all j, we have that $|R(x)| \leq C\lambda^{\frac{3}{2}}$, for some positive constant C. We thus conclude that the error of approximation satisfies the global bound

$$|R| \leq C\lambda^{\frac{3}{2}}\rho(x),$$

where

$$\rho(x) := \sum_{j=1}^{k} \rho_j(x)\chi_{B_\delta(\xi_j)}(x) + 1.$$

Here $\chi_{B_\delta(\xi_j)}$ is the characteristic function on $B_\delta(\xi_j)\bigcap \partial\Omega$ and

$$\rho_j(x) := \frac{1}{2\lambda m_j^2}\left\{\left(1 + 2\lambda m_j^2(w_j + O(1))\right)e^{\lambda m_j^2 w_j^2}(1 + O(\lambda w_j)) - 1\right\}\varepsilon_j^{-1}e^{w_j}$$

From now on, let us write

$$\rho_j(x) = c\gamma_j\left\{\left(1 + \frac{1}{\gamma_j}(w_j + 1)\right)\left(1 + \frac{1}{\gamma_j}(1 + |w_j|)\right)e^{\frac{w_j^2}{2\gamma_j}} - 1\right\}\varepsilon_j^{-1}e^{w_j}, \tag{7.38}$$

where $\gamma_j = \log \varepsilon_j^{-2}$. We define the $L^\infty-$weight norm

$$\|h\|_{*,\partial\Omega} = \sup_{x\in\partial\Omega} \rho(x)^{-1}|h(x)|. \tag{7.39}$$

We thus have the validity of the following key estimate for the error term R

$$\|R\|_{*,\partial\Omega} \leq C\lambda^{\frac{3}{2}}. \tag{7.40}$$

Up to this point, we have defined a function U, whose expression depends of $\xi_1, \ldots, \xi_k$ points on $\partial\Omega$, and depends of $m_1, \ldots, m_k$ positive numbers. These points and numbers satisfy the bounds (7.26). We next describe the problem that the function ϕ in (7.22) solves.

Define in $\mathbb{R}^2_+ = \{(x_1, x_2) : x_2 > 0\}$ the functions

$$z_{0j}(x_1, x_2) = \frac{1}{\mu_j} - 2\frac{x_2 + \mu_j}{x_1^2 + (x_2 + \mu_j)^2}, \qquad z_{1j}(x_1, x_2) = -2\frac{x_1}{x_1^2 + (x_2 + \mu_j)^2}.$$

It has been shown in [27] that these functions are all the bounded solutions to the linearized equation around w_{μ_j} (7.25) associated to Problem (7.23), that is they are the only bounded solutions to

$$\Delta\psi = 0 \quad \text{in} \quad \mathbb{R}^2_+, \qquad -\frac{\partial\psi}{\partial x_2} = e^{w_{\mu_j}}\psi \quad \text{on} \quad \partial\mathbb{R}^2_+. \tag{7.41}$$

For $\xi_j \in \partial\Omega$, we define $F_j : B_\delta(\xi_j) \to \mathcal{O}$ to be a diffeomorphism, where $\mathcal{O}$ is an open neighborhood of the origin in $\mathbb{R}^2_+$ such that $F_j(\Omega \cap B_\delta(\xi_j)) = \mathbb{R}^2_+ \cap \mathcal{O}$, $F_j(\partial\Omega \cap B_\delta(\xi_j)) = \partial\mathbb{R}^2_+ \cap \mathcal{O}$. We can select F_j so that it preserves area. Define

$$Z_{ij}(x) = z_{ij}\left(\varepsilon_j^{-1} F_j(x)\right), \qquad i = 0, 1, \quad j = 1, \ldots, k. \tag{7.42}$$

Next, let us consider a large but fixed number $R_0 > 0$ and a nonnegative radial and smooth cut-off function χ with $\chi(r) = 1$ if $r < R_0$ and $\chi(r) = 0$ if $r > R_0 + 1$, $0 \le \chi \le 1$. Then set

$$\chi_j(x) = \varepsilon_j^{-1}\chi\left(\varepsilon_j^{-1} F_j(x)\right). \tag{7.43}$$

The problem we solve is the following: given $\xi_1, \ldots, \xi_k$ and $m_1, \ldots, m_k$ satisfying the bounds (7.26), find a function ϕ and numbers c_{ij} such that

$$\begin{cases} -\Delta(U + \phi) + (U + \phi) = 0 & \text{in } \Omega; \\ \frac{\partial(U+\phi)}{\partial\nu} = \lambda(U + \phi)e^{(U+\phi)^2} + \sqrt{\lambda}\sum_{i=0,1}\sum_{j=1}^{k} c_{ij}\chi_j Z_{ij} & \text{on } \partial\Omega; \\ \int_\Omega \chi_j Z_{ij}\phi = 0 \qquad \text{for} \ \ i = 0, 1, \quad j = 1, \ldots, k. \end{cases} \tag{7.44}$$

Consider the norm

$$\|\phi\|_\infty = \sup_{x\in\Omega} |\phi(x)|.$$

We have the following result.

Proposition 7.8. *Let $\delta > 0$ be a small but fixed number and assume points the $\xi_1, \ldots, \xi_k \in \partial\Omega$ and the numbers $m_1, \ldots, m_k$ satisfy (7.26). Furthermore we assume that ε_j and μ_j are given by (7.27) and (7.28). Then there exist positive numbers λ_0 and C, such that for any $0 < \lambda < \lambda_0$, there is a unique solution $\phi = \phi(\lambda, \xi, m)$, $c_{ij} = c_{ij}(\lambda, \xi, m)$ to (7.44). Moreover,*

$$\|\phi\|_\infty \le C\lambda^{\frac{3}{2}}, \qquad |c_{ij}| \le C\lambda. \tag{7.45}$$

Furthermore, function ϕ and constant c_{ij} are C^1 with respect to (ξ, m), and we have

$$\|D_{\xi,m}\phi\|_\infty \le C\lambda^{\frac{3}{2}}, \qquad |D_{\xi,m}c_{ij}| \le C\lambda. \tag{7.46}$$

We will sketch the proof in Section 7.4.

Assuming for the moment the validity of the statement in the above Proposition, we observe that $U+\phi$ is an exact solution to Problem (7.20), if there exists a proper choice of λ, of the points ξ_j and the parameters m_j, such that

$$\lambda = \frac{k\pi(1+\mu)}{\int_{\partial\Omega}(U+\phi)^2 e^{(U+\phi)^2}} \quad \text{and} \quad c_{ij}=0, \quad \text{for all } i,j, \tag{7.47}$$

or equivalently

$$\int_{\Omega}\left[|\nabla(U+\phi)|^2+(U+\phi)^2\right]dx = k\pi(1+\mu) \quad \text{and} \quad c_{ij}=0, \quad \text{for all } i,j. \tag{7.48}$$

In order to solve (7.48), we are in the need of understanding the asymptotic expansion, as $\lambda\to 0$, of $\int_\Omega[|\nabla(U+\phi)|^2+(U+\phi)^2]\,dx$ in terms of the localization of the points ξ and the values of the parameters m. Next Proposition contains this result, together with the asymptotic expansion of $\int_{\partial\Omega}e^{(U+\phi)^2}$, as $\lambda\to 0$, again in terms of in terms of ξ and m.

Proposition 7.9. *Under the conditions of Proposition 7.8, Assume that ε_j and μ_j are given by (7.27) and (7.28). Furthermore, we assume that λ is a free parameter. Then, as $\lambda\to 0$, we have*

$$\int_{\Omega}\left[|\nabla(U+\phi)|^2+(U+\phi)^2\right]dx = k\pi\left\{1+\lambda f_k(\xi,m)+\lambda^2\Theta_\lambda(\xi,m)\right\} \tag{7.49}$$

where

$$f_k(\xi,m)=\frac{2}{k}\left[2\sum_{j=1}^{k}m_j^2\log(2m_j^2)-\sum_{j=1}^{k}m_j^2H(\xi_j,\xi_j)-\sum_{i\neq j}m_im_jG(\xi_i,\xi_j)\right]. \tag{7.50}$$

Moreover, as $\lambda\to 0$,

$$\int_{\partial\Omega}e^{(U+\phi)^2}=|\partial\Omega|+4\pi\sum_{j=1}^{k}m_j^2+\lambda\sum_{j=1}^{k}m_j^2\left[\tilde c+\int_{\partial\Omega}G^2(x,\xi_j)\right]+\lambda^2\Theta_\lambda(\xi,m), \tag{7.51}$$

where $\tilde c$ is a positive constant. In (7.50) and (7.51) the function $\Theta_\lambda(\xi,m)(x)$ denotes a generic smooth function, uniformly bounded together with its derivatives, as $\lambda\to 0$, for (ξ,m) satisfying (7.26). In (7.50) and (7.51), G is the Green function defined in (7.6) and H its regular part, as defined in (7.7).

Next Proposition will suggest how to solve Problem in (7.48).

Proposition 7.10. *Under the conditions of Proposition 7.8, let R be the set of points (ξ, m) satisfy (7.26). then there exist $\mu_0 > 0$ and a subregion R' of R such that for all $0 < \mu < \mu_0$ and for all $(\xi, m) \in R'$, there exists a function $\lambda = \lambda(\mu, \xi, m)$ such that*

$$\int_\Omega \left[|\nabla(U+\phi)|^2 + (U+\phi)^2\right] dx = k\pi(1+\mu) \quad \text{for all } \mu > 0, \quad \mu \to 0. \tag{7.52}$$

Moreover, λ is a smooth function of the free parameter μ, of the points $\xi_1, \ldots, \xi_k$ and of the parameters $m_1, \ldots, m_k$. Furthermore, $\lambda \to 0$ as $\mu \to 0$ for points $\xi_1, \ldots, \xi_k$ and parameters $m_1, \ldots, m_k$ belonging to R'. With this definition of λ, we have that the function ϕ and the constants c_{ij} are C^1 with respect to (ξ, m). We finally have that

$$D_{\xi,m} E(U+\phi) = 0 \implies c_{ij} = 0 \quad \text{for all } i, j. \tag{7.53}$$

See (7.12) for the definition of E.

The proofs of Proposition 7.9 and of Proposition 7.10 are postponed to Section 7.5.

Given the choice of λ defined through formula (7.52), for all $\mu > 0$ small, Proposition 7.10 gives that $U + \phi$ is a solution to problem (7.20)-(7.21) if we can find (ξ, m) to be a critical point of the function

$$\mathcal{I}(\xi, m) := E(U+\phi). \tag{7.54}$$

We have now all the elements to give the

Proof of Theorem 7.2: Let $\mathcal{D}$ be the open set such that

$$\bar{\mathcal{D}} \subset \left\{(\xi, m) \in (\partial\Omega)^k \times \mathbb{R}_+^k : \ \xi_i \neq \xi_j, \ \forall \, i \neq j\right\}$$

Let $U(x)$ be defined as in (7.29), and $\phi(x)$ be the solution of problem (7.44), whose existence and properties are stated in Proposition 7.8. Proposition 7.10 gives that

$$u(x) = U(x) + \phi(x)$$

is a solution to problem (7.20)-(7.21) if we can find (ξ, m) to be a critical point of the function

$$\mathcal{I}(\xi, m) := E(U+\phi).$$

From (7.52) and (7.49), we have

$$\lambda f_k(\xi, m) + \lambda^2 \Theta_\lambda(\xi, m) = \mu \tag{7.55}$$

where

$$f_k(\xi, m) = \frac{2}{k}\left[2\sum_{j=1}^k m_j^2 \log(2m_j^2) - \sum_{j=1}^k m_j^2 H(\xi_j, \xi_j) - \sum_{i \neq j} m_i m_j G\xi_i, \xi_j)\right].$$

In (7.55), $\Theta_\lambda(\xi, m)(x)$ denotes a smooth function, uniformly bounded together with its derivatives, as $\lambda \to 0$, for (ξ, m) satisfying (7.26). Make the change of variables $s_j = m_j^2$. So we write, with abuse of notation,

$$f_k(\xi, s) = \frac{2}{k}\left[2\sum_{j=1}^{k} s_j \log(2s_j) - \sum_{j=1}^{k} s_j H(\xi_j, \xi_j) - \sum_{i\neq j} \sqrt{s_i s_j} G\xi_i, \xi_j)\right]$$

Fix ξ. Observe that the function $s \to f_k(\xi, s)$ has a unique zero, namely there exists a unique $\bar{s} = (\bar{s}_1(\xi), \ldots, \bar{s}_k(\xi)) \in \mathbb{R}_+^k$ satisfying $f_k(\xi, \bar{s}) = 0$. We have the following properties:

(i) $\bar{s}_j$ is a C^1 function with respect to ξ defined in $(\partial\Omega)^k$;

(ii) There is a positive constant c_0, independent of the points ξ, such that $\bar{s}_j \geq c_0$ for each $j = 1, \ldots, k$;

(iii) $\bar{s}_j \to +\infty$ as $|\xi_i - \xi_j| \to 0$ for some $i \neq j$;

(iv) Define

$$M^+ = \{(\xi, s) \in (\partial\Omega)^k \times \mathbb{R}_+^k \ : \ s_1 s_2 \ldots s_k \neq 0, f_k(\xi, s) > 0\}.$$

Then $(\xi, (1+r)\bar{s}) \in M^+$ for $r > 0$ small.

Proof of (i). Since $f(\xi, \bar{s}) = 0$, and for j fixed,

$$\partial_{s_j} f_k(\xi, s)\big|_{s=\bar{s}} = \frac{2}{k}\left\{2\log(2\bar{s}_j) + 2 - \left[H(\xi_j, \xi_j) - \frac{1}{2}\sum_{i\neq j}\sqrt{\bar{s}_i/\bar{s}_j}\, G(\xi_i, \xi_j)\right]\right\}.$$

Then

$$\nabla_s f_k(\xi, \bar{s}) \cdot \bar{s} = \partial_{s_1} f_k(\xi, \bar{s})\bar{s}_1 + \ldots + \partial_{s_k} f_k(\xi, \bar{s})\bar{s}_k \quad = \quad \frac{4}{k}\sum_{j=1}^{k} \bar{s}_j > 0 \tag{7.56}$$

Thus we get $\nabla_s f_k(\xi, s)\big|_{s=\bar{s}} \neq 0$. The implicit function theorem implies the validity of (i).

Proof of (ii). According to the definition of $\bar{s}$, we know that

$$\frac{2}{k}\sum_{j=1}^{k} \bar{s}_j \left[2\log(2\bar{s}_j) - H(\xi_j, \xi_j) - \sum_{i\neq j}\sqrt{\frac{\bar{s}_i}{\bar{s}_j}} G\xi_i, \xi_j)\right] = 0.$$

It yields that

$$2\log(2\bar{s}_j) - H(\xi_j, \xi_j) = \sum_{i\neq j}\sqrt{\frac{\bar{s}_i}{\bar{s}_j}} G\xi_i, \xi_j) > 0.$$

So

$$\bar{s}_j > \frac{1}{2} e^{\frac{H(\xi_j, \xi_j)}{2}}$$

Then we get (ii).

Proof of (iii). Since $G(\xi_i,\xi_j)\to+\infty$ if $|\xi_i-\xi_j|\to 0$, for some $i\neq j$, if we suppose that $\bar{s}_l$ is bounded, for some l, then the relation $f_k(\xi,\bar{s})=0$ would provide a contradiction. This proves (iii).

Proof of (iv). For $r>0$ small, by the Taylor expansion, from (7.56) we have

$$\begin{aligned} f_k(\xi,(1+r)\bar{s}) &= f_k(\xi,\bar{s})+\left[\partial_{s_1}f_k(\xi,\bar{s})\bar{s}_1+\ldots+\partial_{s_k}f_k(\xi,\bar{s})\bar{s}_k\right]r+o(r)\\ &= \frac{4}{k}r\sum_{j=1}^{k}\bar{s}_j+o(r)>0. \end{aligned} \tag{7.57}$$

Making the change of variable, define $s=(1+r)\bar{s}$ with $r>0$ small, we have $(\xi,(1+r)\bar{s})\in M^+$.

Thanks to the above properties, we conclude that relation (7.55) defines λ as a function of the free parameter μ and (ξ,s). More precisely,

$$\lambda=\frac{\mu}{f_k(\xi,(1+r)\bar{s})}+\frac{\mu^2}{f_k(\xi,(1+r)\bar{s})^3}\Theta_\lambda(\xi,s) \tag{7.58}$$

where $\Theta_\lambda(\xi,s)$ is a smooth function, uniformly bounded together with its derivatives, as $\lambda\to 0$.

Taking (7.58) into (7.51), we get that

$$\begin{aligned} &\mathcal{I}(\xi,(1+r)\bar{s})\\ =\;& |\partial\Omega|+4(1+r)\pi\sum_{j=1}^{k}\bar{s}_j+\mu\frac{\sum_{j=1}^{k}\bar{s}_j\left[\tilde{c}+\int_{\partial\Omega}G^2(x,\xi_j)\right]}{f_k(\xi,(1+r)\bar{s})}\\ &+\left(\frac{\mu}{f_k(\xi,(1+r)\bar{s})}\right)^2\Theta_\mu(\xi,s)\\ =\;& |\partial\Omega|+4(1+r)\pi\sum_{j=1}^{k}\bar{s}_j+\mu\frac{\sum_{j=1}^{k}\bar{s}_j\left[\tilde{c}+\int_{\partial\Omega}G^2(x,\xi_j)\right]}{\frac{4}{k}r\sum_{j=1}^{k}\bar{s}_j}+\mu\Theta_\mu(\xi,s), \end{aligned} \tag{7.59}$$

where $\Theta_\mu(\xi,s)$ is a smooth function, uniformly bounded together with its derivatives, as $\mu\to 0$.

We claim that, given $\delta>0$, for all $\mu>0$ small enough, the function

$$\varphi_\mu(\xi,\bar{s},r):=|\partial\Omega|+4\pi\sum_{j=1}^{k}\bar{s}_j+4r\pi\sum_{j=1}^{k}\bar{s}_j+\mu\frac{\sum_{j=1}^{k}\bar{s}_j\left[\tilde{c}+\int_{\partial\Omega}G^2(x,\xi_j)\right]}{\frac{4}{k}r\sum_{j=1}^{k}\bar{s}_j}$$

has a critical point in the region $|\xi_i - \xi_j| > \delta$ for $i \neq j$, $\xi_j \in \partial\Omega$, and $\delta\sqrt{\mu} < r < \delta^{-1}\sqrt{\mu}$, with value $|\partial\Omega| + 4\pi \sum_{j=1}^{k} \bar{s}_j + O(\sqrt{\mu})$, as $\mu \to 0$, in the region considered. By construction, the critical point situation is stable under proper small C^1 perturbation of φ_μ: to be more precise, any function ψ such that $\|\psi - \varphi_\mu\|_\infty + \|\nabla\psi - \nabla\varphi_\mu\|_\infty \leq C\mu$ in the region considered, also has a critical point. This fact will conclude the proof of Theorem 7.2.

Observe that the function

$$r \mapsto \varphi_\mu(\xi, \bar{s}, r) := |\partial\Omega| + 4\pi \sum_{j=1}^{k} \bar{s}_j + 4r\pi \sum_{j=1}^{k} \bar{s}_j + \mu \frac{\sum_{j=1}^{k} \bar{s}_j \left[\tilde{c} + \int_{\partial\Omega} G^2(x, \xi_j)\right]}{\frac{4}{k} r \sum_{j=1}^{k} \bar{s}_j}$$

has a critical point $\bar{r}$ given by

$$\bar{r} = \frac{\sqrt{\sum_{j=1}^{k} \bar{s}_j \left[\tilde{c} + \int_{\partial\Omega} G^2(x, \xi_j)\right]}}{4\frac{\sqrt{\pi}}{\sqrt{k}} \sum_{j=1}^{k} \bar{s}_j} \sqrt{\mu},$$

which is a non-degenerate mimimum, since

$$\partial_{rr}^2 \varphi_\mu(\xi, \bar{s}, r) = \mu \frac{\sum_{j=1}^{k} \bar{s}_j \left[\tilde{c} + \int_{\partial\Omega} G^2(x, \xi_j)\right]}{\frac{2}{k} \sum_{j=1}^{k} \bar{s}_j} \frac{1}{r^3} > 0.$$

Inserting the value of $\bar{r}$ in φ_μ, in the new variables $\xi \in (\partial\Omega)^k$, we get

$$\begin{aligned} \Phi(\xi) &:= \mathcal{I}(\xi, (1+\bar{r})\bar{s}) \\ &= |\partial\Omega| + 4\pi \sum_{j=1}^{k} \bar{s}_j + 2\sqrt{k\pi} \sqrt{\sum_{j=1}^{k} \bar{s}_j \left[\tilde{c} + \int_{\partial\Omega} G^2(x, \xi_j)\right]} \sqrt{\mu} + \mu\Theta_\mu(\xi, s) \\ &= |\partial\Omega| + 4\pi \sum_{j=1}^{k} \bar{s}_j + O(\sqrt{\mu}) \quad \text{as } \mu \to 0 \end{aligned}$$

for $\xi \in \hat{\Omega}_k = \{(\xi_1, \ldots, \xi_k) \in (\partial\Omega)^k \ : \ \xi_i \neq \xi_j \text{ if } i \neq j\}$.

Next we show that functional $\Phi(\xi)$ has at least two critical points. Let $\mathcal{C}_0$ be a component of $\partial\Omega$. Let $\Lambda : S^1 \to \mathcal{C}_0$ be a continuous bijective function that parametrizes $\mathcal{C}_0$. Set

$$\tilde{\Omega}_k = \{(\xi_1, \ldots, \xi_k) \in \mathcal{C}_0^k \ : \ |\xi_i - \xi_j| > \delta \text{ for } i \neq j\}.$$

The function Φ is C^1, bounded from below in $\tilde{\Omega}_k$, and from (iii) we have

$$\Phi(\xi) = \Phi(\xi_1, \dots, \xi_k) \to +\infty \quad \text{as} \quad |\xi_i - \xi_j| \to 0 \quad \text{for some } i \neq j.$$

Hence, since δ is arbitrarily small, Φ has an absolute minimum c_m in $\tilde{\Omega}_k$.

On the other hand, using the Ljusternik-Schnirelmann theory, we get that Φ has at least two distinct points in $\tilde{\Omega}_k$. Let $cat(\tilde{\Omega}_k)$ be the Ljusternik-Schnirelmann category of $\tilde{\Omega}_k$ relative to $\tilde{\Omega}_k$, which is the minimum number of closed and contractible sets in $\tilde{\Omega}_k$ whose union covers $\tilde{\Omega}_k$. We will estimate the number of critical points for Φ by $cat(\tilde{\Omega}_k)$.

Claim: $cat(\tilde{\Omega}_k) > 1$.

Indeed, by contradiction, suppose that $cat(\tilde{\Omega}_k) = 1$. This means that $\tilde{\Omega}_k$ is contractible in itself, namely there exist a point $\xi^0 \in \tilde{\Omega}_k$ and a continuous function $\Gamma : [0,1] \times \tilde{\Omega}_k \to \tilde{\Omega}_k$, such that, for all $\xi \in \tilde{\Omega}_k$,

$$\Gamma(0,\xi) = \xi, \quad \Gamma(1,\xi) = \xi_0.$$

Define $f : S^1 \to \tilde{\Omega}_k$ to be the continuous function given by

$$f(\bar{\xi}) = \left(\Lambda(\bar{\xi}), \Lambda(e^{2\pi i \frac{1}{k}}\bar{\xi}), \dots, \Lambda(e^{2\pi i \frac{k-1}{k}}\bar{\xi})\right).$$

Let $\eta : [0,1] \times S^1 \to S^1$ be the well defined continuous map given by

$$\eta(t,\bar{\xi}) = \Lambda^{-1} \circ \pi_1 \circ \Gamma(t, f(\bar{\xi})),$$

where π_1 is the projection on the first component. The function η is a contraction of S^1 to a point and this gives a contradiction, then claim follows.

Therefore we have that $cat(\tilde{\Omega}_k) \geq 2$ for any $k \geq 1$. Define

$$c = \sup_{C \in \Xi} \inf_{\xi \in C} \Phi(\xi)$$

where

$$\Xi = \{C \subset \tilde{\Omega}_k \ : \ C \text{ closed and } cat(C) \geq 2\}.$$

Then by Ljusternik-Schnirelmann theory we obtain that c is a critical level.

If $c \neq c_m$, we conclude that Φ has at least two distinct critical points in $\tilde{\Omega}_k$. If $c = c_m$, there is at least one set C such that $cat(C) \geq 2$, where the function Φ reaches its absolute minimum. In this case we conclude that there are infinitely many critical points for Φ in $\tilde{\Omega}_k$.

Thus we obtain that the function Φ has at least two distinct critical points in $\tilde{\Omega}_k$, denoted say by ξ^1, ξ^2. Hence, for μ sufficiently small, the function $\mathcal{I}(\xi, s)$ has two distinct points $\left(\xi^1_\mu, s^1_\mu\right)$ and $\left(\xi^2_\mu, s^2_\mu\right)$ close respectively to $(\xi^1, (1+\bar{r}(\xi^1))\bar{s}(\xi^1))$ and to $(\xi^2, (1+\bar{r}(\xi^2))\bar{s}(\xi^2))$. This implies the existence of a solution to our Problem of the form $U + \phi$. Finally, let us remark that (7.10) holds as a direct consequence of the construction of U and of the fact that ϕ is a smaller perturbation. This ends the proof of the Theorem.

7.4 Proof of Proposition 7.8

The proof of Proposition 7.8 is based on a fixed point argument and the invertibility property of the following linear Problem: Given $h \in L^\infty(\partial\Omega)$, find a function ϕ and constants c_{ij} such that

$$\begin{cases} -\Delta\phi + \phi = 0 & \text{in } \Omega; \\ L(\phi) = h + \sum_{i=0,1} \sum_{j=1}^{k} c_{ij}\chi_j Z_{ij} & \text{on } \partial\Omega; \\ \int_\Omega \chi_j Z_{ij}\phi = 0 & \text{for } \; i = 0, 1, \quad j = 1, \dots, k. \end{cases} \tag{7.60}$$

In chapter six, we have proven the following result, see Proposition 6.3 and Lemma 6.8.

Proposition 7.11. *Let $\delta > 0$ be a small but fixed number and assume we have $\xi_1, \dots, \xi_k \in \partial\Omega$ and $m_1, \dots, m_k$ with*

$$|\xi_i - \xi_j| \geq \delta, \quad \forall\; i \neq j, \qquad \delta < m_j < \frac{1}{\delta}. \tag{7.61}$$

Then there exist positive numbers λ_0 and C such that, for any $0 < \lambda < \lambda_0$ and any $h \in L^\infty(\partial\Omega)$, there is a unique solution $\phi \equiv T_\lambda(h)$, and $c_{ij} \in \mathbb{R}$ to (7.60). Moreover,

$$\|\phi\|_\infty \leq C\|h\|_{*,\partial\Omega}. \tag{7.62}$$

Moreover, the operator T_λ is differentiable with respect to the variable $\xi_1, \dots, \xi_k$ on $\partial\Omega$, and $m_1, \dots, m_k$, one has the estimate

$$\|D_\xi T_\lambda(h)\|_\infty \leq C\|h\|_{*,\partial\Omega}, \qquad \|D_m T_\lambda(h)\|_\infty \leq C\|h\|_{*,\partial\Omega}. \tag{7.63}$$

for a given positive C, independent of λ, and for all λ small enough.

We are now in the position to prove Proposition 7.8.

Proof of Proposition 7.8 In terms of the operator T_λ defined in Proposition 7.11, problem (7.44) becomes

$$\phi = T_\lambda\left(R + N(\phi)\right) := A(\phi), \tag{7.64}$$

where R is defined in (7.36). For a given number $\gamma > 0$, let us consider the region

$$\mathcal{F}_\gamma := \left\{\phi \in C(\bar\Omega) \;:\; \|\phi\|_\infty \leq \gamma\lambda^{\frac{3}{2}}\right\}.$$

From Proposition 7.11, we get

$$\|A(\phi)\|_\infty \leq C\left[\|R\|_{*,\partial\Omega} + \|N(\phi)\|_{*,\partial\Omega}\right].$$

An involved but direct computation shows that, see the proof of (6.83) and (6.84),

$$\left\| f'(\tilde{U}) - \sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j} \right\|_{*,\partial\Omega} \leq C\lambda^{\frac{3}{2}}. \tag{7.65}$$

and

$$\left\| f''(\tilde{U}) \right\|_{*,\partial\Omega} \leq C. \tag{7.66}$$

From (7.40), (7.65) and (7.66), from the definition of $N(\phi)$ in (7.64), namely

$$N(\phi) := f(\tilde{U}+\phi) - f(\tilde{U}) - f'(\tilde{U})\phi + \left[f'(\tilde{U}) - \sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j} \right] \phi, \tag{7.67}$$

it follows that

$$\|A(\phi)\|_\infty \ \leq \ C\left(\lambda^{\frac{3}{2}} + \|\phi\|_\infty^2 + \lambda\|\phi\|_\infty\right).$$

We then get that $A(\mathcal{F}_\gamma) \subset \mathcal{F}_\gamma$ for a sufficiently large but fixed γ and all small λ. Moreover, for any $\phi_1, \phi_2 \in \mathcal{F}_\gamma$, one has

$$\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega} \leq C\left[\left(\max_{i=1,2} \|\phi_i\|_\infty\right) + \lambda\right] \|\phi_1 - \phi_2\|_\infty,$$

In fact, using directly (7.67),

$$\begin{aligned}
& N(\phi_1) - N(\phi_2) \\
= \ & f(\tilde{U}+\phi_1) - f(\tilde{U}+\phi_2) - f'(\tilde{U})(\phi_1-\phi_2) + \left[f'(\tilde{U}) - \sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j} \right] (\phi_1 - \phi_2) \\
= \ & \int_0^1 \left(\frac{d}{dt} f(\tilde{U}+\phi_2+t(\phi_1-\phi_2)) \right) dt - f'(\tilde{U})(\phi_1-\phi_2) + \left[f'(\tilde{U}) - \sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j} \right] (\phi_1 - \phi_2) \\
= \ & \int_0^1 \left(f'(\tilde{U}+\phi_2+t(\phi_1-\phi_2)) - f'(\tilde{U}) \right) dt\, (\phi_1-\phi_2) + \left[f'(\tilde{U}) - \sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j} \right] (\phi_1 - \phi_2)
\end{aligned}$$

Thus, for a certain $t^* \in (0,1)$, and $s \in (0,1)$

$$\begin{aligned}
& |N(\phi_1) - N(\phi_2)| \\
\leq \ & C\left[|f'(\tilde{U}+\phi_2+t^*(\phi_1-\phi_2)) - f'(\tilde{U})| + \left(f'(\tilde{U}) - \sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j} \right) \right] \|\phi_1 - \phi_2\|_\infty \\
\leq \ & C\Big[|f''(\tilde{U}+s\phi_2+t^*(\phi_1-\phi_2))| \left(\|\phi_1\|_{L^\infty(\Omega)} + \|\phi_2\|_\infty \right)
\end{aligned}$$

$$+[f'(\tilde{U}) - \sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j}]\Bigg] \|\phi_1 - \phi_2\|_\infty.$$

Thanks to (7.65), (7.66) and the fact that $\|\phi_1\|_\infty$, $\|\phi_2\|_\infty \to 0$ as $\lambda \to 0$, we conclude that

$$\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega} \;\; \leq \;\; C\left[\|\phi_1\|_\infty + \|\phi_2\|_\infty + \lambda\right] \|\phi_1 - \phi_2\|_\infty.$$

Then we have

$$\|A(\phi_1) - A(\phi_2)\|_\infty \leq C\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega} \leq C\left[\max_{i=1,2} \|\phi_i\|_\infty + \lambda\right] \|\phi_1 - \phi_2\|_\infty.$$

Thus the operator A has a small Lipschitz constant in $\mathcal{F}_\gamma$ for all small λ, and therefore a unique fixed point of A exists in this region.

We shall next analyze the differentiability of the map $(\xi, m) = (\xi_1, \dots, \xi_k, m_1, \dots, m_k) \mapsto \phi$. Assume for instance that the partial derivative $\partial_{\xi_{sl}}\phi$ exists, for $s = 1, \dots, k$, $l = 1, 2$. Since $\phi = T_\lambda\left(N(\phi) + R\right)$, formally we have that

$$\partial_{\xi_{sl}}\phi = \left(\partial_{\xi_{sl}} T_\lambda\right)\left(N(\phi) + R\right) + T_\lambda\left(\partial_{\xi_{sl}} N(\phi) + \partial_{\xi_{sl}} R\right).$$

From (7.63), we have

$$\|\partial_{\xi_{sl}} T_\lambda\left(N(\phi) + R\right)\|_\infty \;\; \leq \;\; C\|N(\phi) + R\|_{*,\partial\Omega} \leq C\lambda^{\frac{3}{2}}.$$

On the other hand,

$$\begin{aligned} \partial_{\xi_{sl}} N(\phi) \;\; &= \;\; [f'(\tilde{U} + \phi) - f'(\tilde{U}) - f''(\tilde{U})\phi]\partial_{\xi_{sl}}\tilde{U} + \partial_{\xi_{sl}}\left(\frac{\partial Z_{ij}}{\partial \nu} - [\sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_j}]\right)\phi \\ &\quad + [f'(\tilde{U} + \phi) - f'(\tilde{U})]\partial_{\xi_{sl}}\phi + \left(f'(\tilde{U}) - [\sum_{j=1}^{k} \varepsilon_j^{-1} e^{w_{\mu_j}}]\right)\partial_{\xi_{sl}}\phi. \end{aligned}$$

Then,

$$\|\partial_{\xi_{sl}} N(\phi)\|_{*,\partial\Omega} \leq C\left\{\|\phi\|_\infty^2 + \lambda\|\phi\|_\infty + \|\phi\|_\infty\|\partial_{\xi_{sl}}\phi\|_\infty + \lambda\|\partial_{\xi_{sl}}\phi\|_\infty\right\}.$$

Since $\|\partial_{\xi_{sl}} R\|_{*,\partial\Omega} \leq \lambda^{\frac{3}{2}}$, Proposition 7.11 guarantees that

$$\|\partial_{\xi_{sl}}\phi\|_\infty \leq C\lambda^{\frac{3}{2}}$$

for all s, l. Analogous computation holds true if we differentiate with respect to m_j. Then, the regularity of the map $(\xi, m) \mapsto \phi$ can be proved by standard arguments involving the implicit function theorem and the fixed point representation (7.64). This concludes proof of the Proposition.

7.5 Proofs of Proposition 7.9 and of Proposition 7.10

7.5.1 Proof of Proposition 7.9

Let us write

$$U(x)=\sum_{j=1}^{k}U_j(x),\qquad \text{with } \ U_j(x)=\sqrt{\lambda}m_j[u_j(x)+H_j(x)]$$

where u_j and H_j are given by (7.30) and (7.31). We observe that U_j satisfies

$$\begin{cases}-\Delta U_j(x)+U_j(x)=0 & \text{in } \ \Omega;\\ \frac{\partial U_j(x)}{\partial \nu}=2\sqrt{\lambda}m_j\varepsilon_j\mu_j e^{u_j(x)} & \text{on } \ \partial\Omega.\end{cases} \tag{7.68}$$

We have

$$\begin{aligned}&\int_\Omega \left[|\nabla(U+\phi)|^2+(U+\phi)^2\right]\\ =\ &\int_\Omega \left(|\nabla U|^2+U^2\right)+\int_\Omega \left[2\left(\nabla U\nabla\phi+U\phi\right)+\left(|\nabla\phi|^2+\phi^2\right)\right]:=I_a+I_b.\end{aligned} \tag{7.69}$$

For I_a, we have

$$I_a=\sum_{j=1}^{k}\int_\Omega \left(|\nabla U_j|^2+U_j^2\right)+\sum_{i\neq j}\int_\Omega \left(\nabla U_i\nabla U_j+U_iU_j\right):=I_{a,1}+I_{a,2}. \tag{7.70}$$

Multiplying (7.68) by U_j and integrating on Ω, by (7.32) we find

$$\begin{aligned}I_{a,1}\ &=\ \sum_{j=1}^{k}2\sqrt{\lambda}m_j\varepsilon_j\mu_j\int_{\partial\Omega}e^{u_j(x)}U_j(x)=\sum_{j=1}^{k}2\lambda m_j^2\varepsilon_j\mu_j\int_{\partial\Omega}e^{u_j}(u_j+H_j)\\ &=\ \sum_{j=1}^{k}2\lambda m_j^2\int_{\partial\Omega}\frac{\varepsilon_j\mu_j}{|x-\xi_j-\varepsilon_j\mu_j\nu(\xi_j)|^2}\left(\log\frac{1}{|x-\xi_j-\varepsilon_j\mu_j\nu(\xi_j)|^2}+H(x,\xi_j)+O(\varepsilon_j^\sigma)\right)\\ &=\ \sum_{j=1}^{k}2\lambda m_j^2\int_{\partial\Omega_{\varepsilon_j\mu_j}}\frac{1}{|y-\nu(0)|^2}\left[\log\frac{1}{|y-\nu(0)|^2}+H(\xi_j,\xi_j)-2\log(\varepsilon_j\mu_j)+O(\varepsilon_j^\sigma)\right]\end{aligned}$$

where $\Omega_{\varepsilon_j\mu_j}=\frac{\Omega-\xi_j}{\varepsilon_j\mu_j}$. Using the following facts

$$\int_{\partial\Omega_{\varepsilon_j\mu_j}}\frac{1}{|y-\nu(0)|^2}=\pi+O(\varepsilon_j^\sigma),$$

$$\int_{\partial\Omega_{\varepsilon_j\mu_j}}\frac{1}{|y-\nu(0)|^2}\log\frac{1}{|y-\nu(0)|^2}=-2\pi\log 2+O(\varepsilon_j^\sigma),$$

and the definition of ε_j given in (7.27), we obtain

$$\begin{aligned} I_{a,1} &= \sum_{j=1}^{k} 2\lambda m_j^2 \left[-2\pi\log 2 + \pi H(\xi_j,\xi_j) - 2\pi\log(\varepsilon_j\mu_j) + O(\varepsilon_j^\sigma)\right] \\ &= k\pi + 2\pi\lambda\sum_{j=1}^{k} m_j^2\left[H(\xi_j,\xi_j) - 2\log(2m_j^2) - 2\log(2\mu_j) + O(\varepsilon_j^\alpha)\right]. \end{aligned} \tag{7.71}$$

Multiplying (7.68) by U_i and integrating on Ω, we find

$$\begin{aligned} I_{a,2} &= \sum_{i\neq j}\int_{\partial\Omega} 2\sqrt{\lambda}m_j\varepsilon_j\mu_j e^{u_j(x)}U_i(x) = 2\sum_{i\neq j}\lambda m_i m_j\varepsilon_j\mu_j\int_{\partial\Omega} e^{u_j}(u_i+H_i) \\ &= 2\sum_{i\neq j}\lambda m_i m_j\int_{\partial\Omega_{\varepsilon_j\mu_j}} \frac{1}{|y-\nu(0)|^2}\left[\log\frac{1}{|\xi_j-\xi_i+\varepsilon_j\mu_j y-\varepsilon_i\mu_i\nu(\xi_i)|^2} + H_i(\varepsilon_j\mu_j y+\xi_j)\right] \\ &= 2\pi\lambda\sum_{i\neq j} m_i m_j\left[G(\xi_i,\xi_j) + O\left(\varepsilon_i\log\frac{1}{\varepsilon_i} + \varepsilon_j\log\frac{1}{\varepsilon_j}\right) + O(\varepsilon_i^\sigma+\varepsilon_j^\sigma)\right]. \end{aligned} \tag{7.72}$$

Thus from (7.70), (7.71), (7.72) and the definition of μ_j given in (7.28) we get

$$\int_\Omega \left(|\nabla U|^2 + U^2\right) = k\pi\left\{1 + \lambda f_k(\xi,m) + \sum_{j=1}^{k}\varepsilon_j\log\frac{1}{\varepsilon_j}\Theta_\lambda(\xi,m)\right\} \tag{7.73}$$

where f_k is the function defined in (7.50) and $\Theta_\lambda(\xi,m)$ is a smooth function, uniformly bounded as $\lambda\to 0$, in the region for (ξ,m) satisfying (7.26). This is a estimate in the C^0-sense. For C^1-closeness, the derivatives in ξ and in m, by the same argument of C^0-estimate, we have

$$D_\xi\left(\int_\Omega\left(|\nabla U|^2+U^2\right)\right) = k\pi\lambda D_\xi\left(f_k(\xi,m)\right) + \sum_{j=1}^{k}\varepsilon_j\log\frac{1}{\varepsilon_j}\Theta_\lambda(\xi,m), \tag{7.74}$$

$$D_m\left(\int_\Omega\left(|\nabla U|^2+U^2\right)\right) = k\pi\lambda D_m\left(f_k(\xi,m)\right) + \sum_{j=1}^{k}\varepsilon_j\log\frac{1}{\varepsilon_j}\Theta_\lambda(\xi,m), \tag{7.75}$$

where $\Theta(\xi,m)$ is uniformly bounded, as $\lambda\to 0$, in the region for (ξ,m) satisfying (7.26). From the choice of ε_j in (7.27), we note that $\varepsilon_j\log\frac{1}{\varepsilon_j} = o(\lambda^3)$.

On the other hand, for I_b given in (7.69). We have

$$I_b \leq 2\left|\int_\Omega\left[\nabla(U+\phi)\nabla\phi + (U+\phi)\phi\right]\right|$$

Multiplying (7.44) by ϕ and integrating on Ω, we find

$$\int_\Omega\left[\nabla(U+\phi)\nabla\phi + (U+\phi)\phi\right] = \lambda\int_{\partial\Omega}(U+\phi)e^{(U+\phi)^2}\phi.$$

By (7.45) we have $\|\phi\|_\infty \le C\lambda^{\frac{3}{2}}$ for some fixed constant C independent of λ, and using a Taylor expansion, we find

$$\lambda \int_{\partial\Omega} (U+\phi)e^{(U+\phi)^2}\phi \le \lambda\|\phi\|_\infty \left|\int_{\partial\Omega}(U+\phi)e^{(U+\phi)^2}\right| \le C\lambda^{\frac{5}{2}}\left|\int_{\partial\Omega} Ue^{U^2}\right| + C\lambda^4.$$

Since, for some $\delta > 0$ small, we write

$$\int_{\partial\Omega} Ue^{U^2} = \sum_{j=1}^{k} \int\limits_{\partial\Omega\cap B(\xi_j,\delta\sqrt{\varepsilon_j})} Ue^{U^2} + \int\limits_{\partial\Omega\setminus\bigcup\limits_{j=1}^{k} B(\xi_j,\delta\sqrt{\varepsilon_j})} Ue^{U^2} := I_c + I_d,$$

where

$$\begin{aligned}\int\limits_{\partial\Omega\cap B(\xi_j,\delta\sqrt{\varepsilon_j})} Ue^{U^2} &= \int\limits_{\partial\Omega\cap B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} Ue^{U^2} + \int\limits_{\partial\Omega\cap(B(\xi_j,\delta\sqrt{\varepsilon_j})\setminus B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|))} Ue^{U^2}\\ &:= I_{c,1} + I_{c,2}.\end{aligned}$$

From (7.27) and (7.34), for x close to point ξ_j, we have $U = \sqrt{\lambda}m_j\left(w_j + \frac{1}{2\lambda m_j^2} + O(1)\right)$ and $e^{U^2} = 2m_j^2\varepsilon_j^{-1}e^{w_j}(1+O(\lambda))$, where w_j is defined in (7.35). Hence,

$$\begin{aligned}I_{c,1} &= 2\sqrt{\lambda}m_j^3\varepsilon_j^{-1} \int\limits_{\partial\Omega\cap B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} \left(w_j + \frac{1}{2\lambda m_j^2} + O(1)\right) e^{w_j}(1+O(\lambda))\\ &= 2\sqrt{\lambda}m_j^3 \int\limits_{\frac{\partial\Omega-\xi_j}{\varepsilon_j\mu_j}\cap B(0,\frac{\delta|\log\varepsilon_j|}{\mu_j})} \left(\log\frac{2\mu_j^{-1}}{|y-\nu(0)|^2} + \frac{1}{2\lambda m_j^2} + O(1)\right)\frac{2}{|y-\nu(0)|^2}(1+O(\lambda)).\end{aligned}$$

Moreover,

$$\begin{aligned}|I_{c,2}| &\le C\sqrt{\lambda}\int\limits_{\delta|\log\varepsilon_j|}^{\delta\varepsilon_j^{-\frac{1}{2}}} \frac{1}{r^2}e^{\frac{\log^2 r}{\gamma_j^2}}\, r\, dr\\ &= C\sqrt{\lambda}\int\limits_{R_1+\log\gamma_j^2}^{R_2+\frac{\gamma_j^2}{4}} e^{-2t+\frac{4t^2}{\gamma_j^2}}dt \le C\sqrt{\lambda}\int\limits_{R_1+\log\gamma_j^2}^{R_2+\frac{\gamma_j^2}{4}} e^{-t}dt = O(\lambda^{\frac{3}{2}}).\end{aligned}$$

For I_d, since in the region $\partial\Omega\setminus\bigcup\limits_{j=1}^{k} B(\xi_j,\delta\sqrt{\varepsilon_j})$, the function $U(x)$ satisfies $U(x) = \sqrt{\lambda}[\sum_{j=1}^{k} m_jG(x,\xi_j) + o(1)]$, with $o(1)\to 0$ as $\lambda\to 0$, we then have

$$I_d = \int_{\partial\Omega\setminus\bigcup\limits_{j=1}^{k} B(\xi_j,\delta\sqrt{\varepsilon_j})} Ue^{U^2}$$

$$
\begin{aligned}
&= \sqrt{\lambda}\sum_{j=1}^{k} m_j \int_{\partial\Omega} G(x,\xi_j)\left[1+\lambda\left(\sum_{j=1}^{k} m_j G(x,\xi_j)\right)^2\right](1+o(1)) \\
&= \sqrt{\lambda}\sum_{j=1}^{k} m_j \int_{\partial\Omega} G(x,\xi_j)(1+o(1)).
\end{aligned}
$$

Thanks to above facts, we obtain

$$
I_b = \lambda^3 \Theta_\lambda(m,\xi) \tag{7.76}
$$

with $\Theta_\lambda(m,\xi)$ is a function, uniformly bounded, in the region for (ξ,m) satisfying (7.26), as $\lambda \to 0$. Therefore, from (7.69), (7.73) and (7.76) we obtain that estimate (7.49) holds in the C^0 sense.

Next let us show the C^1−closeness in estimate (7.49). From (7.44) and (7.46) we have

$$
\begin{aligned}
D_\xi\left(\int_\Omega \left(|\nabla(U+\phi)|^2+(U+\phi)^2\right)\right) &= 2\int_\Omega [\nabla(U+\phi)\nabla(\partial_\xi U+\partial_\xi \phi)+(U+\phi)(\partial_\xi U+\partial_\xi\phi)] \\
&= 2\int_{\partial\Omega}\frac{\partial(U+\phi)}{\partial\nu}(\partial_\xi U+\partial_\xi\phi) = 2\int_{\partial\Omega}\frac{\partial U}{\partial \nu}\partial_\xi U+\lambda^2\Theta_\lambda(m,\xi)
\end{aligned} \tag{7.77}
$$

where $\Theta_\lambda(m,\xi)$ is a function, uniformly bounded, in the region for (ξ,m) satisfying (7.26), as $\lambda \to 0$, here we use the facts $\|\partial_\xi \phi\|_\infty \le C\lambda^{\frac{3}{2}}$ and $\int_{\partial\Omega}\frac{\partial U}{\partial\nu} \le C\sqrt{\lambda}$. On the other hand, we note that $-\Delta U + U = 0$ in Ω, hence

$$
D_\xi\left(\int_\Omega \left(|\nabla U|^2+U^2\right)\right) = 2\int_\Omega [\nabla U\nabla\partial_\xi U+U\partial_\xi U] = 2\int_{\partial\Omega}\frac{\partial U}{\partial\nu}\partial_\xi U. \tag{7.78}
$$

From (7.74), (7.77) and (7.78), we obtain the C^1−closeness in estimate (7.49)

$$
D_\xi\left(\int_\Omega \left(|\nabla(U+\phi)|^2+(U+\phi)^2\right)\right) = k\pi\lambda D_\xi\left(f_k(\xi,m)\right)+\lambda^2\Theta_\lambda(\xi,m), \tag{7.79}
$$

and by the same argument, we have

$$
D_m\left(\int_\Omega \left(|\nabla(U+\phi)|^2+(U+\phi)^2\right)\right) = k\pi\lambda D_m\left(f_k(\xi,m)\right)+\lambda^2\Theta_\lambda(\xi,m), \tag{7.80}
$$

where $\Theta_\lambda(m,\xi)$ is a function, uniformly bounded, in the region for (ξ,m) satisfying (7.26), as $\lambda \to 0$.

Finally, let us evaluate $\int_{\partial\Omega} e^{(U+\phi)^2}$. By a Taylor expansion, we find

$$
\int_{\partial\Omega} e^{(U+\phi)^2} = \int_{\partial\Omega} e^{U^2} + \lambda^2\Theta_\lambda(m,\xi). \tag{7.81}
$$

We write

$$\int_{\partial\Omega} e^{U^2} = \sum_{j=1}^{k} \int\limits_{\partial\Omega\cap B(\xi_j,\delta\sqrt{\varepsilon_j})} e^{U^2(x)} + \int\limits_{\partial\Omega\setminus\bigcup\limits_{j=1}^{k} B(\xi_j,\delta\sqrt{\varepsilon_j})} e^{U^2(x)} := I_e + I_f. \tag{7.82}$$

Since

$$\int\limits_{\partial\Omega\cap B(\xi_j,\delta\sqrt{\varepsilon_j})} e^{U^2(x)} = \int\limits_{\partial\Omega\cap B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} e^{U^2(x)} + \int\limits_{\partial\Omega\cap(B(\xi_j,\delta\sqrt{\varepsilon_j})\setminus B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|))} e^{U^2(x)} := I_{e,1} + I_{e,2}.$$

From (7.27), (7.28), (7.34) and definition of β_j, we have

$$\begin{aligned} I_{e,1} &= \int\limits_{\partial\Omega\cap B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} e^{U^2(x)} = \varepsilon_j^{-1} e^{\frac{\beta_j}{2}} \int\limits_{\partial\Omega\cap B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} e^{w_j} e^{\theta(x)} e^{\lambda m_j^2[w_j^2+2w_j\theta(x)+\theta^2(x)]} \\ &= 2m_j^2 \int_{\frac{\partial\Omega-\xi_j}{\varepsilon_j\mu_j}\cap B(0,\frac{\delta|\log\varepsilon_j|}{\mu_j})} \frac{2}{|y-\nu(0)|^2}\left(1+O(\lambda)\right) = 4\pi m_j^2\left(1+O(\lambda)\right), \end{aligned} \tag{7.83}$$

with $\Theta_\lambda(m,\xi)$ a function, uniformly bounded, in the region for (ξ,m) satisfying (7.26), as $\lambda\to 0$. Moreover,

$$|I_{e,2}| \le C\int_{\delta|\log\varepsilon_j|}^{\delta\varepsilon_j^{-\frac{1}{2}}} \frac{1}{r^2} e^{\frac{\log^2 r}{\gamma_j^2}}\, r\, dr = C\int_{R_1+\log\gamma_j^2}^{R_2+\frac{\gamma_j^2}{4}} e^{-2t+\frac{4t^2}{\gamma_j^2}}\, dt \le C\int_{R_1+\log\gamma_j^2}^{R_2+\frac{\gamma_j^2}{4}} e^{-t} dt = O(\lambda). \tag{7.84}$$

Furthermore, we have

$$\begin{aligned} I_f &= \int_{\partial\Omega\setminus\bigcup\limits_{j=1}^{k} B(\xi_j,\delta\sqrt{\varepsilon_j})} e^{U^2} = \int_{\partial\Omega\setminus\bigcup\limits_{j=1}^{k} B(\xi_j,\delta\sqrt{\varepsilon_j})} \left[1+\lambda\sum_{j=1}^{k} m_j^2 G^2(x,\xi_j)\right](1+o(1)) \\ &= |\partial\Omega| + \lambda\sum_{j=1}^{k} m_j^2 \int_{\partial\Omega} G^2(x,\xi_j) + \lambda^2\Theta_\lambda(m,\xi) \end{aligned} \tag{7.85}$$

with $|\partial\Omega|$ denotes the measure of domain $\partial\Omega$, and $\Theta_\lambda(m,\xi)$ is a function, uniformly bounded, in the region for (ξ,m) satisfying (7.26), as $\lambda\to 0$. Then from (7.81)-(7.85) we get that estimate (7.51) hold true in C^0-sense.

On the other hand, by a Taylor expansion and the facts $\|\phi\|_\infty \le C\lambda^{\frac{3}{2}}$ and $\int_{\partial\Omega} U \le C\sqrt{\lambda}$, we have

$$D_\xi\left(\int_{\partial\Omega} e^{(U+\phi)^2}\right) = 2\int_{\partial\Omega} e^{U^2} U\partial_\xi U + \lambda^2\Theta_\lambda(m,\xi) = D_\xi\left(\int_{\partial\Omega} e^{U^2}\right) + \lambda^2\Theta_\lambda(m,\xi),$$

and

$$D_m\left(\int_{\partial\Omega} e^{(U+\phi)^2}\right) = D_m\left(\int_{\partial\Omega} e^{U^2}\right) + \lambda^2\Theta_\lambda(m,\xi)$$

with $\Theta_\lambda(m,\xi)$ is a function, uniformly bounded, in the region for (ξ,m) satisfying (7.26), as $\lambda\to 0$. Then we obtain that the C^1-closeness in (7.51) by the same way as in the proof of C^1-closeness in (7.49).

7.5.2 Proof of Proposition 7.10

Define the set

$$R' = \{(\xi,m)\in R : f_k(\xi,m)\neq 0\}.$$

From Proposition 7.9, replacing expansion (7.49) into (7.52), we see that (7.52) gives

$$\lambda f_k(\xi,m) + \lambda^2\Theta_\lambda(\xi,m) = \mu. \tag{7.86}$$

In R', (7.86) defines λ as a function of μ,ξ and m, which is smooth in (ξ,m) in the region R'. Furthermore, as $\mu\to 0$,

$$\lambda = \frac{\mu}{f_k(\xi,m)} + \frac{\mu^2}{f_k^3(\xi,m)}\Theta_\mu(\xi,m)$$

with $\Theta_\mu(m,\xi)$ is a function, uniformly bounded with its derivatives, as $\mu\to 0$.

Assume now (7.52), we shall prove (7.53). Let us denote ∂ by the partial derivative with respect to m_j for any $j=1,\dots,k$, or the partial derivative with respect to ξ_{j1} for $j=1,\dots,k$. By a direct computation we have

$$J'(U+\phi)\left[\partial(U+\phi)\right] = \frac{1}{2}\partial\left(\int_\Omega (|\nabla(U+\phi)|^2 + (U+\phi)^2)\right) - \frac{\lambda}{2}\partial\left(\int_{\partial\Omega} e^{(U+\phi)^2}\right).$$

From (7.52) we have that $\partial\left(\int_\Omega(|\nabla(U+\phi)|^2+(U+\phi)^2)\right)=0$. Thus $\partial\left(\int_{\partial\Omega} e^{(U+\phi)^2}\right)=0$ if and only if $J'(U+\phi)\left[\partial(U+\phi)\right]=0$. Let us now rewrite

$$\frac{1}{\sqrt{\lambda}}(U+\phi)(\xi,m)(x) \;=\; m_l v_l\left(\frac{x-\xi_l}{\varepsilon_l}\right) + \frac{1}{2\lambda m_l}$$

for some $l=1,\dots,k$, with

$$v_l(y) := w_{\mu_l}(y) + \sum_{j=1}^k \left(O(|\varepsilon_l y + \xi_l - \xi_j|) + O(\varepsilon_j^2)\right) \quad \text{for } \; |y|\le \frac{\delta}{\varepsilon_l}.$$

Since $U+\phi$ is the solution of (7.44), then v_l satisfies

$$\begin{cases} -\Delta v_l + \varepsilon_l^2\left(v_l + \frac{1}{2\lambda m_l^2}\right) = 0 & \text{in } \Omega_l; \\ \frac{\partial v_l}{\partial \nu} - (1+2\lambda m_l^2 v_l)e^{v_l}e^{\lambda m_l^2 v_l^2} = m_l^{-1}\varepsilon_l \sum_{i=0,1}\sum_{j=1}^{k} c_{ij}\varepsilon_j^{-1}\chi\left(\frac{F_j(\varepsilon_l y+\xi_l-\xi_j)}{\varepsilon_j}\right) z_{ij}\left(\frac{F_j(\varepsilon_l y+\xi_l-\xi_j)}{\varepsilon_j}\right) & \text{on } \partial\Omega_l, \end{cases}$$

where $\Omega_l = \frac{\Omega-\xi_l}{\varepsilon_l}$. For any l, we define

$$I_l(v_l) = \frac{1}{2}\int_{\Omega_l}\left[|\nabla v_l|^2 + \varepsilon_l^2\left(v_l + \frac{1}{2\lambda m_l^2}\right)^2\right] - \int_{\partial\Omega_l} e^{v_l}e^{\lambda m_l^2 v_l^2}.$$

We observe that

$$J'(U+\phi)\left[\partial(U+\phi)\right] = \lambda m_l^2 I_l'(v_l)[\partial v_l].$$

and

$$\begin{aligned} & \lambda m_l^2 I_l'(v_l)[\partial v_l] \\ = \; & \lambda m_l\varepsilon_l \sum_{i=0,1}\sum_{j=1}^{k}\left(\int_{\partial\Omega_l}\varepsilon_j^{-1}\chi\left(\frac{F_j(\varepsilon_l y+\xi_l-\xi_j)}{\varepsilon_j}\right) z_{ij}\left(\frac{F_j(\varepsilon_l y+\xi_l-\xi_j)}{\varepsilon_j}\right)\partial v_l\; dy\right) c_{ij}. \end{aligned}$$

Now, fix i and j, we compute the coefficient in front of c_{ij}, we choose $l=j$, $\partial v_l = D_{m_s}v_l(y)$, and obtain

$$\begin{aligned} & \int_{\partial\Omega_l}\varepsilon_j^{-1}\chi\left(\frac{F_j(\varepsilon_l y+\xi_l-\xi_j)}{\varepsilon_j}\right) z_{ij}\left(\frac{F_j(\varepsilon_l y+\xi_l-\xi_j)}{\varepsilon_j}\right) D_{m_s}v_l(y)\; dy \\ = \; & \int_{\partial\Omega_l}\varepsilon_j^{-1}\chi\left(y\right) z_{ij}\left(y\right) D_{m_s}\left[w_{\mu_j}(y) + \sum_{j=1}^{k}\left(O(|\varepsilon_j y|) + O(\varepsilon_j^2)\right)\right]\; dy \\ = \; & \frac{\partial\mu_j}{\partial m_s}\int_{\partial\mathbb{R}^2_+} z_{0j}^2(y)\; dy(1+o(1)). \end{aligned}$$

Thus we concludes that for any $s = 1,2,\cdots,k$, we have

$$J'(U+\phi)\left[\partial_{m_s}(U+\phi)\right] = \lambda m_l\varepsilon_l\sum_{j=1}^{k}\frac{\partial\mu_j}{\partial m_s}\int_{\partial\mathbb{R}^2_+} z_{0j}^2(y)\; dy c_{0j}(1+o(1)).$$

Similarly, we get that for all s,l

$$\begin{aligned} & J'(U+\phi)\left[\partial_{\xi_{s1}}(U+\phi)\right] \\ = \; & \lambda m_l\varepsilon_l\left[\sum_{j=1}^{k}\left(\frac{\partial\mu_j}{\partial\xi_{s1}}\int_{\partial\mathbb{R}^2_+} z_{0j}^2(y)dy\right)c_{0j} + \left(\int_{\partial\mathbb{R}^2_+} z_{1s}^2(y)dy\right)c_{1s}\right](1+o(1)). \end{aligned}$$

Thus, we can conclude that $J'(U+\phi)\left[\partial(U+\phi)\right]=0$, that is $D_{\xi,m}E(U+\phi)=0$ then we have the following system

$$\left[\sum_{j=1}^{k}\frac{\partial\mu_j}{\partial m_s}c_{0j}\right](1+o(1))=0,\quad s=1,2,\cdots,k, \tag{7.87}$$

$$\left[A\sum_{j=1}^{k}\frac{\partial\mu_j}{\partial\xi_{s1}}c_{0j}+c_{1s}\right](1+o(1))=0,\quad \text{for all } s, \tag{7.88}$$

for some fixed constant A, with $o(1)$ small in the sense of the L^∞ norm as $\lambda\to 0$. Then (7.53) follows if we show that the matrix $\frac{\partial\mu_j}{\partial m_s}$ of dimension $k\times k$ is invertible in the region for (ξ,m) satisfying (7.26). Indeed, this fact implies unique solvability of (8.99). Inserting this in (8.100) we get unique solvability of (8.100).

Consider the definition of the μ_j, in terms of m_j's and points ξ_j given in (7.26). These relations correspond to the gradient $D_mF(m,\xi)$ of the function $F(m,\xi)$ defined as follows

$$F(m,\xi)=\frac{1}{2}\sum_{j=1}^{k}m_j^2\left[-2\log\left(2m_j^2\right)-\log(2\mu_j)+2+H(\xi_j,\xi_j)\right]+\sum_{i\neq j}m_im_jG(\xi_i,\xi_j).$$

We set $s_j=m_j^2$, then the above function can be written as follows

$$F(s,\xi)\;=\;\frac{1}{2}\sum_{j=1}^{k}s_j\left[-2\log(2s_j)-\log(2\mu_j)+2+H(\xi_j,\xi_j)\right]+\sum_{i\neq j}\sqrt{s_is_j}G(\xi_i,\xi_j).$$

This function is strictly convex function of the parameters s_j, for parameters s_j uniformly bounded and uniformly bounded away from 0 and for points ξ_j in Ω uniformly far away from each other and from the boundary. For this reason, the matrix $(\frac{\partial^2F}{\partial s_i\partial s_j})$ is invertible in the range of parameters and points we are considering. Thus, by the implicit function theorem, relation (7.28) defines a diffeomorphism between μ_j and m_j. This fact gives the invertibility of $(\frac{\partial\mu_j}{\partial m_s})$. Thus we finish the proof of Proposition 7.10.

Chapter 8

Multipeak solutions for asymptotically critical elliptic equations on Riemannian manifold

[1]

8.1 Introduction

Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $n \geq 3$, where g denotes the metric tensor. We are interested in the following asymptotically critical elliptic equation

$$\Delta_g u + a(x)u = u^{2^*-1-\varepsilon}, \quad u > 0 \text{ in } \mathcal{M}, \tag{8.1}$$

where $\Delta_g = -div_g(\nabla)$ is the Laplace-Beltrami operator on $\mathcal{M}$, $a(x)$ is a $\mathcal{C}^1$ function on $\mathcal{M}$, $2^* = \frac{2n}{n-2}$ denotes the Sobolev critical exponent, ε is a small real parameter such that $\varepsilon \to 0$. The equation with $\varepsilon > 0$ is subcritical, and the equation with $\varepsilon < 0$ is supercritical.

Recently, nonlinear elliptic equations on compact Riemannian manifold have been brought much attention. Consider the following problem

$$\varepsilon^2 \Delta_g u + u = |u|^{p-2}u \quad \text{in } \mathcal{M}, \tag{8.2}$$

where $(\mathcal{M}, g)$ is a compact, connected, Riemannian manifold of class C^∞ with Riemannian metric g, $\dim \mathcal{M} = n \geq 3$, $2 < p < 2^*$ and ε is a positive parameter. In [10], the authors proved that the problem (8.2) has a mountain pass solution u_ε which exhibits a spike layer.

[1]The main result of this chapter was published in *Nonlinear Analysis: Theory, Methods and Applications*,74(3)(2011), 859-881.

In particular, they proved that the maximum point of u_ε converges to a maximum point of the scalar curvature Scal_g as ε goes to zero. Multiple solutions were obtained in [12] for the problem (8.2), the authors showed that multiplicity of solutions to (8.2) depends on the topological properties of the manifold $\mathcal{M}$. More precisely, they showed that problem (8.2) has at least $cat(\mathcal{M})+1$ nontrivial solutions provided ε is small enough. Here $cat(\mathcal{M})$ denotes the Lusternik-Schnirelmann category of $\mathcal{M}$. While for zero mass case, similar result was obtained in [117]. And in [65] the author constructed an interesting example of two manifolds having the same topology, for which the number of solutions to the problem (8.2) is different.

In [87] the authors showed that for any stable critical point of the scalar curvature it is possible to construct a single peak solution, whose peak approaches such a point as ε goes to zero. In [26] the authors proved that for any fixed positive integer k, problem (8.2) has a $k-$peak solution, whose peaks collapse, as ε goes to zero, to an isolated local minimum point of the scalar curvature. Recently in [89] the authors proved that the existence of positive or sign changing multi-peak solutions of (8.2), whose both positive and negative peaks approach different stable critical points of the scalar curvature as ε goes to zero.

The asymptotically critical case on Riemannian manifold in [90] the authors proved problem (8.1) exists blowing-up families of positive solutions provide the graph of $a(x)$ is distinct at some point from the graph of $\frac{n-2}{4(n-1)}\text{Scal}_g$.

If $a \equiv \frac{n-2}{4(n-1)}\text{Scal}_g$, problem (8.9) is the intensively studied Yamabe problem

$$\Delta_g u + \frac{n-2}{4(n-1)}\text{Scal}_g u = u^{2^*-1-\varepsilon} \quad \text{in } \mathcal{M} \quad u > 0 \text{ in } \mathcal{M}, \tag{8.3}$$

is just the so called prescribed scalar curvature problem with $\varepsilon = 0$. The existence of a conformal metric with constant scalar curvature on compact Riemannian manifolds was studied by Yamabe [116], Trudinger [115], Aubin [8] and Schoen [108]. If u is a solution, then $\frac{4(n-1)}{n-2}$ is the scalar curvature of the conformal metric $\tilde{g} = u^{\frac{1}{n-2}}g$. On the compact manifold $(\mathcal{M}, g)$, the coercivity of the operator $\Delta_g + a$ is a necessary condition for the existence of a solution to problem (8.3). In [43] the author consider (8.1) with $\varepsilon \geq 0$, for any smooth, compact Riemannian manifold of dimensional $n \geq 3$ and any smooth function a on $\mathcal{M}$ such that $\Delta_g + a$ is coercive and $a(\xi) < \frac{n-1}{4(n-2)}\text{Scal}_g(\xi)$, then (8.1) exists a solution.

In order to state our main result, it is useful to recall some definitions and results. First, Let us introduce the definition of $\mathcal{C}^1$ stable critical set.

Definition 8.1. *([69]) Let $f \in \mathcal{C}^1(\mathcal{M}, \mathbb{R})$, for any given integer $k \geq 2$, set $\bar{\xi} = (\xi_1, \xi_2, \cdots, \xi_k)$, let $C_1, C_2, \cdots, C_k \subset \mathcal{M}$ be k mutually disjoint closed subsets of critical points of f, we say that $(C_1, C_2, \cdots, C_k) \subset \mathcal{M}^k$ is a $\mathcal{C}^1$ stable critical set of function $F(\bar{\xi}) := \sum_{j=1}^{k} f(\xi_j)$, if for any $\sigma > 0$ there exists $\gamma > 0$ such that if $\Phi \in \mathcal{C}^1(\mathcal{M}^k, \mathbb{R})$ with*

$$\max_{d_g(\xi_j, C_j) \leq \sigma, 1 \leq j \leq k} \left(|F(\bar{\xi}) - \Phi(\bar{\xi})| + |\nabla_g F(\bar{\xi}) - \nabla_g \Phi(\bar{\xi})|\right) \leq \gamma,$$

then Φ *has at least one critical point* $\overline{\xi}$ *in* $\mathcal{M}^k$ *with* $d_g(\xi_j, C_j) \leq \sigma$*.*

Remark 8.2. *([69])* $(C_1, C_2, \cdots, C_k) \subset \mathcal{M}^k$ *is a* $\mathcal{C}^1$ *stable critical set of function* F *if one of the following condition is satisfied:*

(i) Each C_j *is a strict local minimum set of* f*.*

(ii) Each C_j *is a strict local maximum set of* f*.*

(iii) Each $C_j = (\{\xi_j^0\})$ *is an isolated critical point of* f *with* $\deg\left(\nabla_g f, B_g(\xi_j^0, \varrho), 0\right) \neq 0$ *for some* $\varrho > 0$*, where* deg *denotes the Brouwer degree.*

Next, we introduce the following equation which correspond to limiting equation to problem (8.1).

$$\Delta U = U^{2^*-1} \quad \text{in} \quad \mathbb{R}^n, \tag{8.4}$$

where $\Delta = -div(\nabla)$ is the Laplace-Beltrami operator associated with the Euclidean metric. It is known that [8, 115] the functions $\lambda^{(2-n)/2}U(\lambda^{-1}z)$ satisfy equation (8.4), where

$$U(z) = U(|z|) = \left(\frac{\sqrt{n(n-2)}}{1+|z|^2}\right)^{(n-2)/2}. \tag{8.5}$$

Let us define a smooth cut-off function χ_r satisfies

$$\chi_r(z) := \begin{cases} 1 & \text{if} \quad z \in B(0, \frac{r}{2}); \\ \in (0,1) & \text{if} \quad z \in B(0,r) \backslash B(0, \frac{r}{2}); \\ 0 & \text{if} \quad z \in \mathbb{R}^n \backslash B(0,r), \end{cases} \tag{8.6}$$

and $|\nabla \chi_r(z)| \leq \frac{2}{r}$, $|\nabla^2 \chi_r(z)| \leq \frac{2}{r^2}$. For any point ξ in $\mathcal{M}$ and for any positive real number λ, we define the function $W_{\lambda,\xi}$ on $\mathcal{M}$ by

$$W_{\lambda,\xi}(x) := \begin{cases} \chi_r\left(\exp_\xi^{-1}(x)\right) \lambda^{\frac{2-n}{2}} U\left(\lambda^{-1}\exp_\xi^{-1}(x)\right) & \text{if } x \in B_g(\xi, r); \\ 0 & \text{otherwise.} \end{cases} \tag{8.7}$$

We assume in this chapter that the operator $\Delta_g + a$ is coercive, we can provide the Hilbert space $H_g^1(\mathcal{M})$ with the inner product

$$\langle u, v \rangle_a = \int_{\mathcal{M}} \left(\langle \nabla u, \nabla v\rangle_g + a(x)uv\right) \, d\mu_g,$$

which induces the norm

$$\|u\|_a^2 = \int_{\mathcal{M}} \left(|\nabla_g u|^2 + a(x)u^2\right) \, d\mu_g.$$

Let

$$\varphi(\xi) = a(\xi) - \frac{n-1}{4(n-2)}\mathrm{Scal}_g(\xi). \tag{8.8}$$

In this chapter we construct a family of solutions of equation (8.1), whose peaks approach different stable critical points of $\varphi(\xi)$ with ε small enough, which blow-up and concentrate at some points in $\mathcal{M}$, in the sense of the following definition.

Definition 8.3. *For $k \geq 2$ be a positive integer, let u_ε be a family of solution of (8.1), we say that u_ε blow-up and concentrates at point $\bar{\xi}^0 = (\xi_1^0, \cdots, \xi_k^0) \in \mathcal{M}^k$ if there exist $\bar{\xi}^\varepsilon = (\xi_1^\varepsilon, \cdots, \xi_k^\varepsilon) \in \mathcal{M}^k$ and $(\lambda_1(\varepsilon), \cdots, \lambda_k(\varepsilon)) \in (\mathbb{R}^+)^k$ with $\lambda_j(\varepsilon) > 0$ such that*

$$\xi_j^\varepsilon \to \xi_j^0, \quad \lambda_j(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{for} \quad j = 1, 2, \cdots, k.$$

and

$$\left\| u_\varepsilon - \sum_{j=1}^{k} W_{\lambda_j(\varepsilon), \xi_j^\varepsilon} \right\|_a \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Our main result is the following.

Theorem 8.4. *Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $n \geq 6$, let $a(x)$ be a $\mathcal{C}^1$ positive function on $\mathcal{M}$ such that the operator $\Delta_g + a$ is coercive, and for any given integer $k \geq 2$, set $\bar{\xi}^0 = (\xi_1^0, \cdots, \xi_k^0)$, let ξ_j^0 be an isolated critical point of $\varphi(\xi_j)$ with $\deg(\nabla_g \varphi, B_g(\xi_j^0, \varrho), 0) \neq 0$ for some $\varrho > 0$ and $j = 1, \cdots, k$, we have*

(i) If $\varphi(\xi_j^0) > 0$ and ε is small enough, there exists a family of solutions of the subcritical problem, which blow-up and concentrates at $\bar{\xi}^0$.

(ii) If $\varphi(\xi_j^0) < 0$ and ε is small enough, there exists a family of solutions of the supercritical problem, which blow-up and concentrates at $\bar{\xi}^0$.

When $\mathcal{M}$ is a flat domain of $\mathbb{R}^n$, problems like (8.1) have been widely investigated. In the bounded domain, with the Neumann boundary condition, the following problem

$$-\Delta u + \mu u = u^{q-1} \quad u > 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{8.9}$$

arises in several branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in biological pattern formation [96] or of parabolic equations in chemotaxis, such as Keller-Segel model [81]. When q is subcritical, that is $q < \frac{n+2}{n-2}$, Lin, Ni, and Takagi in [81] proved that the only solution, for small μ, is the constant one, whereas nonconstant solutions appear for large μ, the solution blow up at one or several points as μ goes to infinity. In [97, 98], the authors proved that the least energy solution blows up at a boundary point which maximizes the mean curvature of the boundary. Higher energy solutions exist which blow up at one or several points, located on the boundary [57, 59], and in the interior of the domain [56]. In the critical case, as in the subcritical case the least energy solution blows up, as μ goes to infinity, at a unique point which maximizes the mean curvature of the boundary [98]. The higher energy solutions have also been exhibited, blow up at one [118] or several points [119, 120]. In the asymptotically critical case, in [105, 106], the authors considered the problem (8.9) for fixed μ, when the exponent q is close to the critical one, i.e. $q = \frac{2n}{n-2} + \varepsilon$ and ε is a small nonzero number, they proved that a single interior or boundary peak solution exist for finite μ, provided that q is close enough to the critical exponent. In super-critical case, del Pino-Musso-Pistoia in [38] proved that the existence of solutions with blow-up points located on the boundary and determined by the mean curvature of $\partial\Omega$, see also [34, 35, 37]. In the unbounded case, Micheletti-Pistoia in [86] constructed a family of positive solutions for both the slightly subcritical and slightly supercritical equation

$$-\Delta u + V(x)u = n(n-2)(u^+)^{2^*-1-\varepsilon}, \quad \text{in } \mathbb{R}^n,$$

with ε is small, the solutions blow-up and concentrate at a single point as ε goes to 0 under certain conditions on the potential V.

This chapter is organized as follows. In Section 2, we introduce some framework and preliminary results. The proof of the main result is given in Section 3. Section 4 is devoted to perform the finite dimensional reduction. Section 5 contains the asymptotic expansion of the reduced energy. Some technical estimates are given in Section 6.

8.2 The framework and preliminary results

Let $\mathcal{M}$ be a compact Riemannian manifold of class $\mathcal{C}^\infty$. On the tangent bundle of $\mathcal{M}$ it is defined the exponential map $\exp : T\mathcal{M} \to \mathcal{M}$ which has the following properties:

(i) exp is of class C^∞;

(ii) there exists a constant $r > 0$ such that $\exp_\xi\big|_{B(0,r)} : B(0,r) \to B_g(\xi, r)$ is a diffeomorphism for all $\xi \in \mathcal{M}$.
where $B(0,r)$ denotes the ball in $\mathbb{R}^n$ centered at 0 with radius r and $B_g(\xi, r)$ denotes the ball in $\mathcal{M}$ centered at ξ with radius r with respect to the distance induced by the metric g.

Fix such an r in this paper with $r < i_g/2$, where i_g denotes the injectivity radius of $(\mathcal{M}, g)$. Let $\mathfrak{C}$ be the atlas on $\mathcal{M}$ whose charts are given by the exponential map and $\mathcal{P} = \{\psi_\omega\}_{\omega\in\mathfrak{C}}$ be a partition of unity subordinate to the atlas $\mathfrak{C}$. For $u \in H^1_g(\mathcal{M})$, we have

$$\int_{\mathcal{M}} |\nabla_g u|^2 \, d\mu_g = \sum_{\omega\in\mathfrak{C}} \int_\omega \psi_\omega(x) |\nabla_g u|^2 \, d\mu_g,$$

where $d\mu_g = \sqrt{\det g}\, dz$ denotes the volume form on $\mathcal{M}$ associated to the metric g. Moreover, if u has support inside one chart $\omega = B_g(\xi, r)$, then

$$\int_\omega |\nabla_g u|^2 \, d\mu_g = \int_{B(0,r)} \left(\sum_{a,b=1}^n g_\xi^{ab}(z) \frac{\partial u(\exp_\xi(z))}{\partial z_a} \frac{\partial u(\exp_\xi(z))}{\partial z_b} \right) |g_\xi(z)|^{\frac{1}{2}} \, dz, \tag{8.10}$$

where g_ξ denotes the Riemannian metric reading in $B(0,r)$ through the normal coordinates defined by the exponential map $\exp_\xi$ at ξ. We denote $|g_\xi(z)| := \det(g_\xi(z))$ and $(g_\xi^{ab})(z)$ is the inverse matrix of $g_\xi(z)$. In particular, it holds

$$g_\xi^{ab}(0) = \delta_{ab}, \qquad g_\xi(0) = Id, \tag{8.11}$$

where δ_{ab} is the Kronecker symbol and

$$\frac{\partial g_\xi^{ab}}{\partial z_c}(0) = 0 \quad \text{for any} \quad a, b, c. \tag{8.12}$$

Since $\mathcal{M}$ is compact, there are two strictly positive constants C and $\tilde{C}$ such that

$$\forall\, \xi \in \mathcal{M}, \quad \forall\, v \in T_\xi \mathcal{M}, \quad C\|v\|^2 \le g_\xi(v,v) \le \tilde{C}\|v\|^2.$$

Hence, we have

$$\forall\, \xi \in \mathcal{M}, \quad C^n \le |g_\xi| \le \tilde{C}^n.$$

Let L^q be the Banach space $L^q(\mathcal{M})$ with the norm

$$|u|_q = \left(\int_{\mathcal{M}} |u|^q \, d\mu_g \right)^{1/q}.$$

It is clear that the embedding $i : H_g^1(\mathcal{M}) \hookrightarrow L^{2^*}(\mathcal{M})$ is a continuous map. We let $i^* : L^{2n/(n+2)}(\mathcal{M}) \hookrightarrow H_g^1(\mathcal{M})$ be the adjoint operator of the embedding i, the embedding i^* is a continuous map such that for any w in $L^{2n/(n+2)}(\mathcal{M})$, the function $u = i^*(w)$ in $H_g^1(\mathcal{M})$ is the unique solution of the equation $\Delta_g u + au = w$ in $\mathcal{M}$. By the continuity of the embedding $H_g^1(\mathcal{M})$ into $L^{2^*}(\mathcal{M})$, we have

$$\|i^*(w)\|_a \le C|w|_{2n/(n+2)} \tag{8.13}$$

for some positive constant C independent of w.

In order to study the supercritical, by the standard elliptic estimates (see [55]), given a real number $s > 2n/(n-2)$, that is $ns/(n+2s) > 2n/(n+2)$, for any w in $L^{ns/(n+2s)}(\mathcal{M})$, the function $i^*(w)$ belongs to $L^s(\mathcal{M})$ and satisfies

$$|i^*(w)|_s \le C|w|_{ns/(n+2s)} \tag{8.14}$$

for some positive constant C independent of w. For ε small, we set

$$s_\varepsilon := \begin{cases} 2^* - \frac{n}{2}\varepsilon & \text{if} \quad \varepsilon < 0; \\ 2^* & \text{if} \quad \varepsilon > 0, \end{cases}$$

and set $\mathcal{H}_\varepsilon = H_g^1(\mathcal{M}) \cap L^{s_\varepsilon}(\mathcal{M})$ be the Banach space provided with the norm

$$\|u\|_{a,s_\varepsilon} = \|u\|_a + |u|_{s_\varepsilon}.$$

If $\varepsilon > 0$, the subcritical case, the space $\mathcal{H}_\varepsilon$ is the Sobolev space $H_g^1(\mathcal{M})$, and the norm $\|\cdot\|_{a,s_\varepsilon}$ is equivalent to the norm $\|\cdot\|_a$. And we can compute that there holds

$$\frac{ns_\varepsilon}{n+2s_\varepsilon} = \begin{cases} \frac{s_\varepsilon}{2^*-1-\varepsilon} & \text{if} \quad \varepsilon < 0; \\ \frac{2n}{n+2} & \text{if} \quad \varepsilon > 0, \end{cases} \tag{8.15}$$

and by (8.13) (or (8.14) in the supercritical case), we can write problem (8.1) as

$$u = i^*\left(f_\varepsilon(u)\right), \qquad u \in \mathcal{H}_\varepsilon, \tag{8.16}$$

where $f_\varepsilon(u) = u_+^{2^*-1-\varepsilon}$ and $u_+ = \max\{u, 0\}$.

It is known that [13] every solution of the linear equation

$$\Delta v = (2^*-1)U^{2^*-2}v, \qquad v \in \mathcal{D}_0^{1,2}(\mathbb{R}^n) \tag{8.17}$$

is a linear combination of the functions

$$V_0(z) = \frac{d\left(\lambda^{(2-n)/2}U(\lambda^{-1}z)\right)}{d\lambda}\Big|_{\lambda=1} = \frac{1}{2}n^{\frac{n-2}{4}}(n-2)^{\frac{n+2}{4}}\frac{|z|^2-1}{(1+|z|^2)^{n/2}}, \tag{8.18}$$

and

$$V_i(z) = -\frac{\partial U}{\partial z_i}(z) = n^{\frac{n-2}{4}}(n-2)^{\frac{n+2}{4}}\frac{z_i}{(1+|z|^2)^{n/2}} \quad \text{for} \;\; i = 1, 2, \cdots, n. \tag{8.19}$$

Let us define on $\mathcal{M}$ the functions

$$Z^i_{\lambda,\xi}(x) := \begin{cases} \chi_r\left(\exp_\xi^{-1}(x)\right)\lambda^{\frac{2-n}{2}}V_i\left(\lambda^{-1}\exp_\xi^{-1}(x)\right) & \text{if } x \in B_g(\xi, r); \\ 0 & \text{otherwise}, \end{cases} \tag{8.20}$$

for $i = 0, 1, 2, \cdots, n$. We set $\Lambda_\varepsilon(\bar{d}) = (\lambda_1, \lambda_2. \cdots, \lambda_k) \in (\mathbb{R}^+)^k$, $\bar{d} = (d_1, d_2, \cdots, d_k) \in (\mathbb{R}^+)^k$ with

$$\lambda_j = \sqrt{|\varepsilon| d_j}, \quad \eta < d_j < \frac{1}{\eta}, \tag{8.21}$$

for fixed small $\eta > 0$, and we denote $\bar{\xi} = (\xi_1, \xi_2, \cdots, \xi_k) \in \mathcal{M}^k$. For $\rho > 0$, we define

$$\mathcal{O}_{\eta,\rho} := \left\{(\Lambda_\varepsilon(\bar{d}), \bar{\xi}) \in (\mathbb{R}^+)^k \times M^k \;:\; d_g(\xi_j, \xi_l) \geq \rho > 2r \;\text{ for } j,\; l = 1, 2, \cdots, k,\; j \neq l\right\} \tag{8.22}$$

Let

$$K_{\Lambda_\varepsilon(\bar d),\bar\xi} = \mathrm{Span}\left\{Z^i_{\lambda_j,\xi_j}:\ i=0,1,2,\cdots,n;\ j=1,2,\cdots,k\right\},$$

and

$$K^\perp_{\Lambda_\varepsilon(\bar d),\bar\xi} = \left\{\phi\in\mathcal{H}_\varepsilon:\ \langle\phi, Z^i_{\lambda_j,\xi_j}\rangle_a = 0,\ \forall\ i=0,1,2,\cdots,n;\ j=1,2,\cdots,k\right\}.$$

We will look for a solution to (8.16), or equivalently to (8.1), of the form

$$u_\varepsilon = V_{\Lambda_\varepsilon(\bar d),\bar\xi} + \phi_{\varepsilon,\bar d,\bar\xi} \quad \text{with} \quad V_{\Lambda_\varepsilon(\bar d),\bar\xi} = \sum_{j=1}^k W_{\lambda_j,\xi_j} \tag{8.23}$$

for $(\Lambda_\varepsilon(\bar d),\bar\xi)\in\mathcal{O}_{\eta,\rho}$, where the rest term $\phi_{\varepsilon,\bar d,\bar\xi}$ belongs to the space $K^\perp_{\Lambda_\varepsilon(\bar d),\bar\xi}$ and the functions W_{λ_j,ξ_j} are defined in (8.7) with $r<\rho/2$, so that W_{λ_j,ξ_j} and W_{λ_l,ξ_l} have disjoint supports if $j\neq l$.

Let $\Pi_{\Lambda_\varepsilon(\bar d),\bar\xi}:\mathcal{H}_\varepsilon\to K_{\Lambda_\varepsilon(\bar d),\bar\xi}$ and $\Pi^\perp_{\Lambda_\varepsilon(\bar d),\bar\xi}:\mathcal{H}_\varepsilon\to K^\perp_{\Lambda_\varepsilon(\bar d),\bar\xi}$ be the orthogonal projections. In order to solve problem (8.16) we will solve the system

$$\Pi^\perp_{\Lambda_\varepsilon(\bar d),\bar\xi}\left\{V_{\Lambda_\varepsilon(\bar d),\bar\xi}+\phi - i^*\left[f_\varepsilon\left(V_{\Lambda_\varepsilon(\bar d),\bar\xi}+\phi\right)\right]\right\}=0, \tag{8.24}$$

$$\Pi_{\Lambda_\varepsilon(\bar d),\bar\xi}\left\{V_{\Lambda_\varepsilon(\bar d),\bar\xi}+\phi - i^*\left[f_\varepsilon\left(V_{\Lambda_\varepsilon(\bar d),\bar\xi}+\phi\right)\right]\right\}=0. \tag{8.25}$$

8.3 The existence result

We first give the result whose proof is postponed until Section 4 to solve equation (8.24).

Proposition 8.5. *If $n\geq 6$, for any $\eta,\rho>0$, $(\Lambda_\varepsilon(\bar d),\bar\xi)\in\mathcal{O}_{\eta,\rho}$, if ε is small enough, there exists a unique $\phi_{\varepsilon,\bar d,\bar\xi}=\phi(\varepsilon,\bar d,\bar\xi)$ which solves equation (8.24), which is continuously differential with respect to $\bar\xi$ and $\bar d$, moreover,*

$$\|\phi_{\varepsilon,\bar d,\bar\xi}\|_{a,s_\varepsilon}\leq C\begin{cases} |\varepsilon|\,|\ln|\varepsilon|\,|^{2/3} & \text{if}\quad n=6\quad\text{and}\quad \varepsilon>0;\\ |\varepsilon|\,|\ln|\varepsilon|\,| & \text{otherwise,}\end{cases} \tag{8.26}$$

where C is a positive constant.

We introduce the functional $J_\varepsilon: H^1_g(\mathcal{M})\to\mathbb{R}$ defined by

$$J_\varepsilon(u)=\frac12\int_{\mathcal{M}}|\nabla_g u|^2\,d\mu_g+\frac12\int_{\mathcal{M}}a(x)u^2\,d\mu_g-\frac{1}{2^*-\varepsilon}\int_{\mathcal{M}}u_+^{2^*-\varepsilon}\,d\mu_g,$$

It is well known that any critical point of J_ε is solution to problem (8.1). We also define the functional $\widetilde{J}_\varepsilon : (\mathbb{R}^+)^k \times \mathcal{M}^k \to \mathbb{R}$ by

$$\widetilde{J}_\varepsilon(\bar{d}, \bar{\xi}) = J_\varepsilon\left(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}}\right), \tag{8.27}$$

where $V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}$ is as (8.23) and $\phi_{\varepsilon,\bar{d},\bar{\xi}}$ is given by Proposition 8.5.

The next result, whose proof is postponed until Section 5, allows to solve equation (8.25), by reducing the problem to a finite dimensional one. We denote K_n the sharp constant for the embedding of $\mathcal{D}^{1,2}(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$, that is, $K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{1/n}}}$ with ω_n is the volume of the unite $n-$sphere.

Proposition 8.6. *(i) For ε small, if $(\bar{d}, \bar{\xi})$ is a critical point of the functional $\widetilde{J}_\varepsilon$, then $V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}}$ is a solution of (8.16), or equivalently of problem (8.1).*

(ii) If $n \geq 6$, for $(\Lambda_\varepsilon(\bar{d}), \bar{\xi}) \in \mathcal{O}_{\eta,\rho}$, there holds

$$J_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) = \frac{k}{n}K_n^{-n} - \frac{k}{n}K_n^{-n}\alpha_n\varepsilon + \frac{1}{n}K_n^{-n}\Psi_k(\bar{d}, \bar{\xi}) + o(\varepsilon), \tag{8.28}$$

as $\varepsilon \to 0$, $\mathcal{C}^1-$uniformly with respect to $\bar{\xi}$ in $\mathcal{M}^k$ and to $\bar{d}$ in compact subsets of $(\mathbb{R}^+)^k$, where

$$\alpha_n = 2^{n-3}(n-2)^2\frac{\omega_{n-1}}{\omega_n}\int_0^{+\infty} \frac{t^{(n-2)/2}\ln(1+t)}{(1+t)^n}\,dt + K_n^{-n}\frac{(n-2)^2}{4n}\left(1 - n\ln\sqrt{n(n-2)}\right) \tag{8.29}$$

$$\Psi_k(\bar{d}, \bar{\xi}) = \sum_{j=1}^{k}\left[-c_1\varepsilon\ln\left(|\varepsilon|d_j\right) + c_2 d_j|\varepsilon|\varphi(\xi_j)\right], \tag{8.30}$$

with $c_1 = \frac{(n-2)^2}{8}$, $c_2 = \frac{2(n-1)}{(n-2)(n-4)}$.

(iii) If $n \geq 6$, for $(\Lambda_\varepsilon(\bar{d}), \bar{\xi}) \in \mathcal{O}_{\eta,\rho}$, there holds

$$\tilde{J}_\varepsilon(\bar{d}, \bar{\xi}) = J_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}}) = J_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) + o(\varepsilon)$$

as $\varepsilon \to 0$, $\mathcal{C}^1$ uniformly with respect to $\bar{\xi} \in \mathcal{M}^k$ and to $\bar{d}$ in compact subsets of $(\mathbb{R}^+)^k$.

Proof of Theorem 8.4:

(i) We first prove part (i) of Theorem 8.4.

We define the functional $\tilde{J} : (\mathbb{R}^+)^k \times \mathcal{M}^k \to \mathbb{R}$ by

$$\tilde{J}(\bar{d}, \bar{\xi}) = \sum_{j=1}^{k} f(d_j, \xi_j), \quad \text{with} \quad f(d_j, \xi_j) = -c_1\ln d_j + c_2 d_j\varphi(\xi_j). \tag{8.31}$$

Since ξ_j^0 be an isolated critical point of $\varphi(\xi_j)$ with $\varphi(\xi_j^0) > 0$ for $\varepsilon > 0$, and set $d_j^0 = \frac{c_1}{c_2\varphi(\xi_j^0)}$, we have $d_j^0 > 0$ and (d_j^0, ξ_j^0) is an isolated critical point of $f(d_j, \xi_j)$. Since $\deg(\nabla_g \varphi, B_g(\xi_j^0, \varrho), 0) \neq 0$ for some $\varrho > 0$, then $\deg(\nabla_g f, B_g(\xi_j^0, \varrho), 0) \neq 0$, by the continuity of the Brouwer degree via homotopy considering the function $H : [0,1] \times (\mathbb{R}^+)^k \times \mathcal{M}^k \to \mathbb{R}^{k\times(n+1)}$ defined by

$$\begin{aligned} H(\tau, \bar{d}, \bar{\xi}) &= \tau \begin{pmatrix} \frac{\partial \tilde{J}(\bar{d},\bar{\xi})}{\partial d_1} & \left(\frac{\partial f(d_1,\xi_1(y^1))}{\partial y_1^1}\right)_{|y=0} & \cdots & \left(\frac{\partial f(d_1,\xi_1(y^1))}{\partial y_n^1}\right)_{|y=0} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \tilde{J}(\bar{d},\bar{\xi})}{\partial d_k} & \left(\frac{\partial f(d_k,\xi_k(y^k))}{\partial y_1^k}\right)_{|y=0} & \cdots & \left(\frac{\partial f(d_k,\xi_k(y^k))}{\partial y_n^k}\right)_{|y=0} \end{pmatrix} \\ &+ (1-\tau) \begin{pmatrix} d_1 - d_1^0 & \left(\frac{\partial(\varphi\circ\xi_1(y^1))}{\partial y_1^1}\right)_{|y=0} & \cdots & \left(\frac{\partial(\varphi\circ\xi_1(y^1))}{\partial y_n^1}\right)_{|y=0} \\ \vdots & \vdots & \ddots & \vdots \\ d_k - d_k^0 & \left(\frac{\partial(\varphi\circ\xi_k(y^k))}{\partial y_1^k}\right)_{|y=0} & \cdots & \left(\frac{\partial(\varphi\circ\xi_k(y^k))}{\partial y_n^k}\right)_{|y=0} \end{pmatrix} \end{aligned}$$

We get that $(\bar{d}^0, \bar{\xi}^0)$ is an isolated critical point of $\tilde{J}(\bar{d}, \bar{\xi})$, where $\bar{d}^0 = (d_1^0, d_2^0, \cdots, d_k^0)$, $\bar{\xi}^0 = (\xi_1^0, \xi_2^0, \cdots, \xi_k^0)$, such that

$$\deg(\nabla_g \tilde{J}, B_g(\xi_j^0, \varrho), 0) \neq 0,$$

thus by Remark 8.2, we have that $(\bar{d}^0, \bar{\xi}^0)$ is a $\mathcal{C}^1$ stable critical set of $\tilde{J}(\bar{d}, \bar{\xi})$. By Proposition 3.2, we have $\left[\left|\partial_{\bar{d}}\left(\frac{1}{\varepsilon}\tilde{J}_\varepsilon - \tilde{J}\right)\right| + \left|\partial_{\bar{\xi}}\left(\frac{1}{\varepsilon}\tilde{J}_\varepsilon - \tilde{J}\right)\right| \to 0 \quad \text{for} \quad j = 1, 2, \cdots, k,\right] as \varepsilon \to 0$, uniformly with respect to $\bar{\xi}$ in $\mathcal{M}^k$ and to $\bar{d}$ in compact subsets of $(\mathbb{R}^+)^k$. By the properties of the Brouwer degree, it follows that there exists a family of critical points $(\bar{d}^\varepsilon, \bar{\xi}^\varepsilon)$ of $\tilde{J}_\varepsilon$ converging to $(\bar{d}^0, \bar{\xi}^0)$ as $\varepsilon \to 0$. Then, from Proposition 8.6, we get that the function $u_\varepsilon = V_{\lambda_\varepsilon(\bar{d}^\varepsilon), \bar{\xi}^\varepsilon} + \phi_{\varepsilon, \bar{d}^\varepsilon, \bar{\xi}^\varepsilon}$ is a solution of equation (8.16), and it is a solution of problem (8.1) for ε small enough. Moreover, by Definition 8.3, We get that the $u_\varepsilon' s$ blow up at $\bar{\xi}^0$ as $\varepsilon \to 0$.

(ii) For supercritical case when $\varepsilon < 0$, we introduce the function f on $\mathbb{R}^+ \times \mathcal{M}$ by $f(d_j, \xi_j) = -c_1 \ln d_j - c_2 d_j \varphi(\xi_j)$ replace in (8.31), and then we proceed in a similar way as in proof of part (i).

8.4 The finite dimensional reduction

This section is devoted to the proof of Proposition 8.5. Let us introduce the linear operator $L_{\varepsilon,\bar{d},\bar{\xi}} : \mathcal{H}_\varepsilon \cap K^\perp_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \to K^\perp_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}$ defined by

$$L_{\varepsilon,\bar{d},\bar{\xi}}(\phi) := \Pi^\perp_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \left\{\phi - i^*\left[f_\varepsilon'(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\phi\right]\right\}.$$

This operator is well defined by using (8.13) and (8.14). Therefore equation (8.24) is equivalent to

$$L_{\varepsilon,\bar{d},\bar{\xi}}(\phi) = N_{\varepsilon,\bar{d},\bar{\xi}}(\phi) + R_{\varepsilon,\bar{d},\bar{\xi}} \tag{8.32}$$

where

$$N_{\varepsilon,\bar{d},\bar{\xi}}(\phi) = \Pi^{\perp}_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \left\{ i^* \left[f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi) - f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) - f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\phi \right] \right\}, \tag{8.33}$$

and

$$R_{\varepsilon,\bar{d},\bar{\xi}} = \Pi^{\perp}_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \left\{ i^* \left(f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) \right) - V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \right\}. \tag{8.34}$$

As a first step, we want to study the invertibility of $L_{\varepsilon,\bar{d},\bar{\xi}}$.

Lemma 8.7. *If $n \geq 6$ and for any $\eta, \rho > 0$, for any $(\Lambda_\varepsilon(\bar{d}), \bar{\xi}) \in \mathcal{O}_{\eta,\rho}$, and for any $\phi \in \mathcal{H}_\varepsilon \cap K^{\perp}_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}$, if ε is small enough, there holds*

$$\|L_{\varepsilon,\bar{d},\bar{\xi}}(\phi)\|_{a,s_\varepsilon} \geq C\|\phi\|_{a,s_\varepsilon}, \tag{8.35}$$

where C is a positive constant.

Proof. We argue by contradiction. Assume there exist $\eta, \rho > 0$ and a sequences $\varepsilon_\alpha \to 0$, $(\Lambda_{\varepsilon_\alpha}(\bar{d}_\alpha), \bar{\xi}_\alpha) \in \mathcal{O}_{\eta,\rho}$, with $\bar{\xi}_\alpha = (\xi_{1\alpha}, \xi_{2\alpha}, \cdots, \xi_{k\alpha}) \in \mathcal{M}^k$, and a sequences of functions $\phi_\alpha \in \mathcal{H}_{\varepsilon_\alpha} \cap K^{\perp}_{\Lambda_{\varepsilon_\alpha}(\bar{d}_\alpha),\bar{\xi}_\alpha}$ such that

$$L_{\varepsilon_\alpha,\bar{d}_\alpha,\bar{\xi}_\alpha}(\phi_\alpha) = \psi_\alpha, \quad \|\phi_\alpha\|_{a,s_{\varepsilon_\alpha}} = 1 \quad \text{and} \quad \|\psi_\alpha\|_{a,s_{\varepsilon_\alpha}} \to 0. \tag{8.36}$$

For any α, for notation's convenience we will write $\Lambda_\alpha = \Lambda_{\varepsilon_\alpha}(\bar{d}_\alpha)$. From (8.36) we get there exists $\zeta_\alpha \in \mathcal{H}_{\varepsilon_\alpha} \cap K_{\Lambda_\alpha,\bar{\xi}_\alpha}$ such that

$$\phi_\alpha - i^* \left[f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})\phi_\alpha \right] = \psi_\alpha + \zeta_\alpha. \tag{8.37}$$

Step 1, we claim that

$$\|\zeta_\alpha\|_{a,s_{\varepsilon_\alpha}} \to 0 \quad \text{as} \quad \alpha \to \infty. \tag{8.38}$$

Let $\zeta_\alpha := \sum_{i=0,\cdots,n} \sum_{j=1,\cdots,k} \lambda^{ij}_\alpha Z^i_{\lambda_{j\alpha},\xi_{j\alpha}}$. For any $h = 0, 1, \cdots, n$ and $l = 1, \cdots, k$, we multiply (8.37) by $Z^h_{\lambda_{l\alpha},\xi_{l\alpha}}$, and taking into account that $\phi_\alpha, \psi_\alpha \in K^{\perp}_{\Lambda_\alpha,\bar{\xi}_\alpha}$, we get

$$\sum_{i=0,\cdots,n} \sum_{j=1,\cdots,k} \lambda^{ij}_\alpha \left\langle Z^i_{\lambda_{j\alpha},\xi_{j\alpha}}, Z^h_{\lambda_{l\alpha},\xi_{l\alpha}} \right\rangle_a = -\left\langle i^* \left[f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})\phi_\alpha \right], Z^h_{\lambda_{l\alpha},\xi_{l\alpha}} \right\rangle_a \tag{8.39}$$

Since $d_g(\xi_{j\alpha}, \xi_{l\alpha}) \geq \rho > 2r$, from the definition of χ_r, and by properties of the exponential map, we get, $\left\langle Z^i_{\lambda_{j\alpha},\xi_{j\alpha}}, Z^h_{\lambda_{l\alpha},\xi_{l\alpha}} \right\rangle_a = 0$ if $j_\alpha \neq l_\alpha$. Therefore, by changing of variable $x = \exp_{\xi_{j\alpha}}(\lambda_{j\alpha} z)$, for $i, h = 0, 1, \cdots, n$ and for any α, we have

$$\left\langle Z^i_{\lambda_{j\alpha},\xi_{j\alpha}}, Z^h_{\lambda_{l\alpha},\xi_{l\alpha}} \right\rangle_a$$

$$
\begin{aligned}
&= \int_{\mathcal{M}} \left\langle \nabla Z^i_{\lambda_{j\alpha},\xi_{j\alpha}}, \nabla Z^h_{\lambda_{l\alpha},\xi_{l\alpha}} \right\rangle_g \, d\mu_g + \int_{\mathcal{M}} a(x) Z^i_{\lambda_{j\alpha},\xi_{j\alpha}} Z^h_{\lambda_{l\alpha},\xi_{l\alpha}} \, d\mu_g \\
&= \delta_{j_\alpha l_\alpha} \left\{ \int_{\mathcal{M}} \left\langle \nabla Z^i_{\lambda_{j\alpha},\xi_{j\alpha}}, \nabla Z^h_{\lambda_{l\alpha},\xi_{l\alpha}} \right\rangle_g \, d\mu_g + \int_{\mathcal{M}} a(x) Z^i_{\lambda_{j\alpha},\xi_{j\alpha}} Z^h_{\lambda_{l\alpha},\xi_{l\alpha}} \, d\mu_g \right\} \\
&= \delta_{j_\alpha l_\alpha} \lambda_{j\alpha}^{2-n} \int\limits_{B_g(\xi_{j\alpha},r)} \nabla_g \left(\chi_r(\exp^{-1}_{\xi_{j\alpha}}(x)) V_i(\lambda_{j\alpha}^{-1} \exp^{-1}_{\xi_{j\alpha}}(x)) \right) \\
&\qquad \times \nabla_g \left(\chi_r(\exp^{-1}_{\xi_{j\alpha}}(x)) V_h(\lambda_{j\alpha}^{-1} \exp^{-1}_{\xi_{j\alpha}}(x)) \right) \, d\mu_g \\
&\quad + \delta_{j_\alpha l_\alpha} \lambda_{j\alpha}^{2-n} \int\limits_{B_g(\xi_{j\alpha},r)} a(x) \chi_r^2(\exp^{-1}_{\xi_{j\alpha}}(x)) V_i(\lambda_{j\alpha}^{-1} \exp^{-1}_{\xi_{j\alpha}}(x)) V_h(\lambda_{j\alpha}^{-1} \exp^{-1}_{\xi_{j\alpha}}(x)) \, d\mu_g \\
&= \delta_{j_\alpha l_\alpha} \lambda_{j\alpha}^{2} \int\limits_{B(0,r/\lambda_{j\alpha})} \sum_{a,b=1}^n g^{ab}_{\xi_{j\alpha}}(\lambda_{j\alpha} z) \left[\frac{1}{\lambda_{j\alpha}} \frac{\partial V_i(z)}{\partial z_a} \chi_r(\lambda_{j\alpha} z) + \frac{\partial \chi_r(\lambda_{j\alpha} z)}{\partial z_a} V_i(z) \right] \\
&\qquad \times \left[\frac{1}{\lambda_{j\alpha}} \frac{\partial V_h(z)}{\partial z_b} \chi_r(\lambda_{j\alpha} z) + \frac{\partial \chi_r(\lambda_{j\alpha} z)}{\partial z_b} V_h(z) \right] \left| g_{\xi_{j\alpha}}(\lambda_{j\alpha} z) \right|^{\frac{1}{2}} \, dz \\
&\quad + \delta_{j_\alpha l_\alpha} \lambda_{j\alpha}^{2} \int\limits_{B(0,r/\lambda_{j\alpha})} a\left(\exp_{\xi_{j\alpha}}(\lambda_{j\alpha} z) \right) \chi_r(\lambda_{j\alpha} z) V_i(z) V_h(z) \left| g_{\xi_{j\alpha}}(\lambda_{j\alpha} z) \right|^{\frac{1}{2}} \, dz \\
&:= I_1 + I_2. \qquad (8.40)
\end{aligned}
$$

By the Taylor's expansion, from (8.12), we have

$$
g^{ab}_{\xi_{j\alpha}}(\lambda_{j\alpha} z) = \delta_{ab} + O(\lambda_{j\alpha}^2 |z|^2) = \delta_{ab} + O(|\varepsilon_\alpha| d_{j\alpha} |z|^2); \qquad (8.41)
$$

and

$$
\left| g_{\xi_{j\alpha}}(\lambda_{j\alpha} z) \right|^{\frac{1}{2}} = 1 + O(\lambda_{j\alpha}^2 |z|^2) = 1 + O(|\varepsilon_\alpha| d_{j\alpha} |z|^2). \qquad (8.42)
$$

Moreover, for $i, h = 1, 2, \cdots, n$

$$
\begin{aligned}
\int\limits_{B(0,r/\lambda_{j\alpha}) - B(0,r/2\lambda_{j\alpha})} V_i V_h \, dz &= n^{\frac{n-2}{2}} (n-2)^{\frac{n+2}{2}} \int\limits_{B(0,r/\lambda_{j\alpha}) - B(0,r/2\lambda_{j\alpha})} \frac{z_i z_h}{(1+|z|^2)^n} \, dz \\
&\le C \int_{r/2\lambda_{j\alpha}}^{r/\lambda_{j\alpha}} t^{-n+1} dt = O(1). \qquad (8.43)
\end{aligned}
$$

and by the similar way we have

$$
\begin{aligned}
&\int\limits_{B(0,r/\lambda_{j\alpha}) - B(0,r/2\lambda_{j\alpha})} \frac{\partial V_i}{\partial z_a} V_h \, dz = O(1), \\
&\int\limits_{B(0,r/\lambda_{j\alpha}) - B(0,r/2\lambda_{j\alpha})} V_0 V_h \, dz = O(1), \qquad (8.44)
\end{aligned}
$$

$$\int_{B(0,r/\lambda_{j\alpha})-B(0,r/2\lambda_{j\alpha})} \frac{\partial V_0}{\partial z_a} V_h \, dz = O(1).$$

From (8.41)-(8.44), by the property of function χ_r, we have

$$I_1 \to \delta_{j_\alpha l_\alpha} \delta_{ih} \|V_i\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}, \quad \text{as} \quad \alpha \to \infty. \tag{8.45}$$

Moreover,

$$I_2 = |\varepsilon_\alpha| d_{l\alpha} \int_{\mathbb{R}^n} a\left(\exp_{\xi_{j\alpha}}(\lambda_{j\alpha} z)\right) \chi_r^2(\lambda_{l\alpha} z) V_i V_h \left|g_{\xi_{j\alpha}}(\lambda_{j\alpha} z)\right|^{\frac{1}{2}} \, dz \to 0, \tag{8.46}$$

as $\alpha \to \infty$. Thus, by (8.40), (8.45) and (8.46), we get

$$\left\langle Z^i_{\lambda_{j\alpha},\xi_{j\alpha}}, Z^h_{\lambda_{l\alpha},\xi_{l\alpha}} \right\rangle_a = \delta_{j_\alpha l_\alpha} \delta_{ih} \|V_i\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} + o(1), \quad \text{as} \quad \alpha \to \infty. \tag{8.47}$$

Now, set

$$\widetilde{\phi}_\alpha(z) := \begin{cases} \lambda_{l\alpha}^{(n-2)/2} \phi_\alpha\left(\exp_{\xi_{l\alpha}}(\lambda_{l\alpha} z)\right) & \text{if } z \in B(0, r/\lambda_{l\alpha}); \\ 0 & \text{otherwise.} \end{cases}$$

By (8.36) and by an change of variable, we get that the sequence $\{\widetilde{\phi}_\alpha\}_\alpha$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Passing if necessary to a subsequence, we may assume that $\{\widetilde{\phi}_\alpha\}_\alpha$ converges weakly to a function $\tilde{\phi}$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, and thus in $L^{2^*}(\mathbb{R}^n)$ by the continuity of the embedding of $\mathcal{D}^{1,2}(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$.

Since, for any α, the function $\phi_\alpha \in K^\perp_{\Lambda_\alpha,\bar{\xi}_\alpha}$, by the same change of variable for $x = \exp_{\xi_{l\alpha}}(\lambda_{l\alpha} z)$, we have

$$\begin{aligned} 0 \;=\; \left\langle Z^h_{\lambda_{l\alpha},\xi_{l\alpha}}, \phi_\alpha \right\rangle_a &= \int_{\mathbb{R}^n} \left\langle \nabla\left(\chi_r(\lambda_{l\alpha} z) V_h\right), \nabla \widetilde{\phi}_\alpha \right\rangle_{g_\alpha} \, d\mu_{g_\alpha} \\ &+ \lambda_{l\alpha}^2 \int_{\mathbb{R}^n} a\left(\exp_{\xi_{l\alpha}}(\lambda_{l\alpha} z)\right) \chi_r(\lambda_{l\alpha} z) V_h \widetilde{\phi}_\alpha \, d\mu_{g_\alpha}, \end{aligned} \tag{8.48}$$

where $g_\alpha(z) = \exp^*_{\xi_{l\alpha}} g(\lambda_{l\alpha} z)$ with $d\mu_{g_\alpha} = |g_{\xi_{l\alpha}}(\lambda_{l\alpha} z)|^{\frac{1}{2}} \, dz$. Then, passing the limit in (8.48), we get

$$\int_{\mathbb{R}^n} \langle \nabla V_h, \nabla \tilde{\phi} \rangle \, dz = 0.$$

Since the function V_h is a solution of equation (8.17), it yields that

$$\int_{\mathbb{R}^n} \langle \nabla V_h, \nabla \tilde{\phi} \rangle \, dz = (2^* - 1) \int_{\mathbb{R}^n} U^{2^*-2} V_h \tilde{\phi} \, dz = 0. \tag{8.49}$$

Moreover,

$$\int_{\mathcal{M}} f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha}) Z^h_{\lambda_{l\alpha},\xi_{l\alpha}} \phi_\alpha d\mu_g$$

$$
\begin{aligned}
&= \int_{\mathcal{M}} f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})\chi_r(\exp^{-1}_{\xi_{l\alpha}}(x))\lambda_{l\alpha}^{(2-n)/2}V_h(\lambda_{l\alpha}^{-1}\exp^{-1}_{\xi_{l\alpha}}(x))\phi_\alpha d\mu_g \\
&= \lambda_{l\alpha}^2 \int\limits_{B(0,r/\lambda_{l\alpha})} f'_{\varepsilon_\alpha}\left(\sum_{j=1}^{k}\chi_r(\lambda_{l\alpha}z)\lambda_{j\alpha}^{(2-n)/2}U(z)\right)\chi_r(\lambda_{l\alpha}z)V_h(z)\widetilde{\phi}_\alpha(z)d\mu_{g_\alpha} \\
&= (2^*-1-\varepsilon_\alpha)\lambda_{l\alpha}^{\frac{(n-2)\varepsilon_\alpha}{2}} \int\limits_{B(0,r/\lambda_{l\alpha})} \left(\chi_r(\lambda_{l\alpha}z)U(z)\right)^{2^*-2-\varepsilon_\alpha}\chi_r(\lambda_{l\alpha}z)V_h(z)\widetilde{\phi}_\alpha(z)d\mu_{g_\alpha}. \quad (8.50)
\end{aligned}
$$

Since there holds

$$\lambda_{l\alpha}^{\frac{(n-2)\varepsilon_\alpha}{2}} = (|\varepsilon_\alpha|d_{l\alpha})^{\frac{(n-2)\varepsilon_\alpha}{4}} \to 1, \quad \text{as} \quad \alpha\to\infty. \tag{8.51}$$

And the sequence $\{\widetilde{\phi}_\alpha\}_\alpha$ converges weakly to $\tilde{\phi}$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, then taking the limit into (8.50) yields

$$\int_{\mathcal{M}} f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})Z^h_{\lambda_{l\alpha},\xi_{l\alpha}}\phi_\alpha\, d\mu_g \to (2^*-1)\int_{\mathbb{R}^n} U(z)^{2^*-2}V_h(z)\tilde{\phi}(z)\, dz = 0, \tag{8.52}$$

as $\alpha\to\infty$, because (8.49) holds.

It follows from (8.39), (8.47) and (8.52) that for any $i=0,1,\cdots,n$ and for any $j=1,2,\cdots,k$, there holds $\lambda_\alpha^{ij}\to 0$ as $\alpha\to\infty$, therefore our claim (8.38) is proved.

Step 2: We prove that

$$\liminf_{\alpha\to\infty}\int_{\mathcal{M}} f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})u_\alpha^2\, d\mu_g \to c>0. \tag{8.53}$$

where

$$u_\alpha = \phi_\alpha - \psi_\alpha - \zeta_\alpha, \quad \text{with} \quad \|u_\alpha\|_{a,s_{\varepsilon_\alpha}} \to 1. \tag{8.54}$$

Let us write equation (8.37) as

$$\Delta_g u_\alpha + a(x)u_\alpha = f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})u_\alpha + f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})(\psi_\alpha+\zeta_\alpha), \tag{8.55}$$

We first prove that

$$\liminf_{\alpha\to\infty}\|u_\alpha\|_a = c>0. \tag{8.56}$$

In fact, by (8.55) we deduce

$$u_\alpha = i^*\left\{f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})u_\alpha + f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})(\psi_\alpha+\zeta_\alpha)\right\}, \tag{8.57}$$

and by (8.14), (8.36), (8.38) and (8.54), from (i) and (ii) in Lemma 8.15, use the Hölder inequality, we get

$$\begin{aligned}
|u_\alpha|_{s_{\varepsilon_\alpha}} &\leq C\left[\left|f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})u_\alpha\right|_{\frac{ns_{\varepsilon_\alpha}}{n+2s_{\varepsilon_\alpha}}} + \left|f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})(\psi_\alpha+\zeta_\alpha)\right|_{\frac{ns_{\varepsilon_\alpha}}{n+2s_{\varepsilon_\alpha}}}\right] \\
&\leq C\left[\left|f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})\right|_{\frac{2ns_{\varepsilon_\alpha}}{2n-(n-6)s_{\varepsilon_\alpha}}}|u_\alpha|_{2^*} + \left|f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})\right|_{\frac{n}{2}}|\psi_\alpha+\zeta_\alpha|_{s_{\varepsilon_\alpha}}\right] \\
&\leq C\left|f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})\right|_{\frac{2ns_{\varepsilon_\alpha}}{2n-(n-6)s_{\varepsilon_\alpha}}}|u_\alpha|_{2^*} + C\left|f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})\right|_{\frac{n}{2}}\left(\|\psi_\alpha\|_{a,s_{\varepsilon_\alpha}} + \|\zeta_\alpha\|_{a,s_{\varepsilon_\alpha}}\right) \\
&\leq C\left|f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})\right|_{\frac{2ns_{\varepsilon_\alpha}}{2n-(n-6)s_{\varepsilon_\alpha}}}|u_\alpha|_{2^*} + o(1) \\
&\leq C\|u_\alpha\|_a + o(1),
\end{aligned} \tag{8.58}$$

as $\alpha \to \infty$. Then, if $\|u_\alpha\|_a \to 0$, by (8.58) we deduce that also $|u_\alpha|_{s_{\varepsilon_\alpha}} \to 0$, this is not impossible because of (8.54). This gives the validity of (8.56).

We multiply (8.55) by u_α we deduce that

$$\|u_\alpha\|_a^2 = \int_{\mathcal{M}} f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})u_\alpha^2\, d\mu_g + \int_{\mathcal{M}} f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})(\psi_\alpha+\zeta_\alpha)u_\alpha\, d\mu_g. \tag{8.59}$$

By Hölder inequality, from (8.36), (8.38) and (i) of Lemma 8.15, we have

$$\begin{aligned}
\left|\int_{\mathcal{M}} f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})(\psi_\alpha+\zeta_\alpha)u_\alpha\, d\mu_g\right| &\leq \left|f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})\right|_{\frac{n}{2}}|\psi_\alpha+\zeta_\alpha|_{\frac{2n}{n-2}}|u_\alpha|_{\frac{2n}{n-2}} \\
&\leq C\|\psi_\alpha+\zeta_\alpha\|_{a,s_{\varepsilon_\alpha}}\|u_\alpha\|_{a,s_{\varepsilon_\alpha}} = o(1).
\end{aligned} \tag{8.60}$$

Then, (8.53) follows by (8.56), (8.59) and (8.60).

Step 3: Let us prove that a contradiction arises, by showing that

$$\liminf_{\alpha\to\infty}\int_{\mathcal{M}} f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})u_\alpha^2\, d\mu_g \to 0. \tag{8.61}$$

In fact, for $l \in \{1, 2, \cdots, k\}$, set

$$\tilde{u}_\alpha(z) := \begin{cases} \lambda_{l\alpha}^{(n-2)/2}u_\alpha\left(\exp_{\xi_{l\alpha}}(\lambda_{l\alpha}z)\right) & \text{if } z \in B(0, r/\lambda_{l\alpha}); \\ 0 & \text{otherwise.} \end{cases} \tag{8.62}$$

We will show that

$$\tilde{u}_\alpha \to 0 \quad \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^n) \text{ and strongly in } L^q_{loc}(\mathbb{R}^n), \tag{8.63}$$

for any $q \in [2, 2^*)$ if $n \geq 3$ or $q \geq 2$ if $n = 2$. That fact implies that

$$\int_{\mathcal{M}} f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})u_\alpha^2\, d\mu_g = \sum_{l=1}^{k}\int_{B_g(\xi_{l\alpha},r)} f'_{\varepsilon_\alpha}(W_{\lambda_{l\alpha},\xi_{l\alpha}})u_\alpha^2\, d\mu_g$$

$$
\begin{aligned}
&= \sum_{l=1}^{k} \int_{B_g(\xi_{l\alpha},r)} f'_{\varepsilon_\alpha}\left(\chi_r(\exp^{-1}_{\xi_{l\alpha}}(x))\lambda_{l\alpha}^{\frac{2-n}{2}} U(\lambda_{l\alpha}^{-1}\exp^{-1}_{\xi_{l\alpha}}(x))\right) u_\alpha^2 \, d\mu_g \\
&= \sum_{l=1}^{k} \lambda_{l\alpha}^{\frac{n-2}{2}\varepsilon_\alpha} \int_{B(0,r/\lambda_{l\alpha})} f'_{\varepsilon_\alpha}\left(\chi_r(\lambda_{l\alpha}z)U(z)\right) \tilde{u}_\alpha^2 \, |g_{\xi_{l\alpha}}(\lambda_{l\alpha}z)|^{\frac{1}{2}} \, dz \\
&\leq C\sum_{l=1}^{k} \lambda_{l\alpha}^{\frac{n-2}{2}\varepsilon_\alpha} \left|f'_{\varepsilon_\alpha}(U(z))\right|_{L^{n/2}(\mathbb{R}^n)} |\tilde{u}_\alpha|_{L^{2^*}(\mathbb{R}^n)} = o(1),
\end{aligned}
\tag{8.64}
$$

for ε small enough, because (8.51) and $\left|f'_{\varepsilon_\alpha}(U(z))\right|_{L^{n/2}(\mathbb{R}^n)} = O(1)$ hold.

Finally, we prove (8.63). By (8.55) we get

$$
\begin{aligned}
& \int_{\mathcal{M}} |\nabla_g u_\alpha|_g \, d\mu_g + \int_{\mathcal{M}} a(x) u_\alpha^2 \, d\mu_g \\
&= \int_{\mathcal{M}} f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha}) u_\alpha^2 \, d\mu_g + \int_{\mathcal{M}} f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha})(\psi_\alpha + \zeta_\alpha) u_\alpha \, d\mu_g \\
&= \int_{\mathcal{M}} f'_{\varepsilon_\alpha}(V_{\Lambda_\alpha,\bar{\xi}_\alpha}) u_\alpha^2 \, d\mu_g + o(1),
\end{aligned}
\tag{8.65}
$$

because (8.60) holds.

By an change of variable $x = \exp_{\xi_{l\alpha}}(\lambda_{l\alpha}z)$ in (8.65), we get

$$
\begin{aligned}
& \int_{\mathbb{R}^n} |\nabla_g \tilde{u}_\alpha|_{g_\alpha} \, d\mu_\alpha + \lambda_{l\alpha}^2 \int_{\mathbb{R}^n} a\left(\exp_{\xi_{l\alpha}}(\lambda_{l\alpha}z)\right) \tilde{u}_\alpha^2 \, d\mu_{g_\alpha} \\
&= \lambda_{l\alpha}^{\frac{n-2}{2}\varepsilon_\alpha} \int_{\mathbb{R}^n} f'_{\varepsilon_\alpha}\left(\chi_r(\lambda_{l\alpha}z)U(z)\right) \tilde{u}_\alpha^2 \, d\mu_{g_\alpha} + o(1).
\end{aligned}
\tag{8.66}
$$

Moreover, we observe that $\|\tilde{u}_\alpha\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} \leq c\|u_\alpha\|_{a,s_{\varepsilon_\alpha}} \leq c$, that is the sequence $\{\tilde{u}_\alpha\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, then there exists $\tilde{u}$ such that $\tilde{u}_\alpha(z) \to \tilde{u}$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and strongly in $L^q(\mathbb{R}^n)$ for any $q \in [2, 2^*)$ if $n \geq 3$ or $q \geq 2$ if $n = 2$. Then we deduce that $\tilde{u}$ solve the problem

$$
\Delta \tilde{u} = (2^* - 1)U^{2^*-2}\tilde{u} \quad \text{in} \quad \mathbb{R}^n, \tag{8.67}
$$

by (8.49), we get that the function $\tilde{u}$ is identically zero, then (8.63) holds.

Therefore from the contradiction (8.53) with (8.61), we end proof of Lemma 8.7.

□

Next, we want to study the estimate the term of $R_{\varepsilon,\bar{d},\bar{\xi}}$.

Lemma 8.8. *If $n \geq 6$ and for any $\eta, \rho > 0, (\Lambda_\varepsilon(\bar{d}), \bar{\xi}) \in \mathcal{O}_{\eta,\rho}$, if ε is small enough, there holds*

$$\|R_{\varepsilon,\bar{d},\bar{\xi}}\|_{a,s_\varepsilon} \leq C \begin{cases} |\varepsilon|\,|\ln|\varepsilon||^{2/3} & \text{if } n = 6 \text{ and } \varepsilon > 0; \\ |\varepsilon|\,|\ln|\varepsilon|| & \text{otherwise,} \end{cases} \tag{8.68}$$

where C is a positive constant.

Proof. Let us introduce the function $Z_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}$ defined by $V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} := i^*(Z_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})$, that is,

$$\Delta_g V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + a(x)V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} = Z_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \quad \text{on} \quad \mathcal{M}.$$

We remark that $V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}(x) = 0$ if $x \notin B_g(\xi_1, r) \cup \cdots \cup B_g(\xi_k, r)$, $V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}(x) = W_{\lambda_l,\xi_l}$ if $x \in B_g(\xi_l, r)$. Therefore, we have $Z_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}(x) = 0$, if $x \notin B_g(\xi_1, r) \cup \cdots \cup B_g(\xi_k, r)$ and

$$Z_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} = \Delta_g W_{\lambda_l,\xi_l} + a(x)W_{\lambda_l,\xi_l}, \quad \text{if} \quad x \in B_g(\xi_l, r).$$

We have

$$\begin{aligned} &\left\| i^*\left(f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\right) - V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \right\|_{a,s_\varepsilon} = \left\| i^*\left(f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\right) - i^*(Z_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) \right\|_{a,s_\varepsilon} \\ \leq\ & C\left| f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) - Z_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \right|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}} = C\left(\int_{\mathcal{M}} \left| f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) - Z_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}} d\mu_g \right)^{\frac{n+2s_\varepsilon}{ns_\varepsilon}} \\ =\ & C\sum_{l=1}^{k} \left(\int\limits_{B_g(\xi_l,r)} \left| f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) - Z_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \right|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}} d\mu_g \right)^{\frac{n+2s_\varepsilon}{ns_\varepsilon}} \\ =\ & C\sum_{l=1}^{k} \left(\int\limits_{B_g(\xi_l,r)} |f_\varepsilon(W_{\lambda_l,\xi_l}) - (\Delta_g W_{\lambda_l,\xi_l} + a(x)W_{\lambda_l,\xi_l})|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}} d\mu_g \right)^{\frac{n+2s_\varepsilon}{ns_\varepsilon}} \end{aligned} \tag{8.69}$$

By Lemma 3.2 in [90], for any $l = 1, 2, \cdots, k$, we have

$$\begin{aligned} &\left(\int\limits_{B_g(\xi_l,r)} |f_\varepsilon(W_{\lambda_l,\xi_l}) - (\Delta_g W_{\lambda_l,\xi_l} + a(x)W_{\lambda_l,\xi_l})|^{\frac{ns_\varepsilon}{n+2s_\varepsilon}} d\mu_g \right)^{\frac{n+2s_\varepsilon}{ns_\varepsilon}} \\ &\leq C \begin{cases} |\varepsilon|\,|\ln|\varepsilon||^{2/3} & \text{if } n = 6 \text{ and } \varepsilon > 0; \\ |\varepsilon|\,|\ln|\varepsilon|| & \text{otherwise.} \end{cases} \end{aligned} \tag{8.70}$$

Then (8.68) holds from (8.69) and (8.70), that concludes the proof of Lemma 8.8.

□

Proof of Proposition 8.5: In order to solve (8.24) or equivalently equation (8.32), we need to find a fixed point for the operator $T_{\varepsilon,\bar{d},\bar{\xi}} : \mathcal{H}_\varepsilon \cap K^{\perp}_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \to \mathcal{H}_\varepsilon \cap K^{\perp}_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}$ defined

$$T_{\varepsilon,\bar{d},\bar{\xi}}(\phi) = L^{-1}_{\varepsilon,\bar{d},\bar{\xi}}\left(N_{\varepsilon,\bar{d},\bar{\xi}}(\phi) + R_{\varepsilon,\bar{d},\bar{\xi}}\right),$$

for ε small and for any $(\Lambda_\varepsilon(\bar{d}), \bar{\xi}) \in \mathcal{O}_{\eta,\rho}$. We also let

$$\mathcal{B}(\beta) = \left\{\phi \in \mathcal{H}_\varepsilon \cap K^{\perp}_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \ : \ \|\phi\|_{a,s_\varepsilon} \le \beta \|R_{\varepsilon,\bar{d},\bar{\xi}}\|_{a,s_\varepsilon}\right\},$$

where β is a positive constant to be chosen later on.

By Lemma 8.7, we deduce

$$\left\|T_{\varepsilon,\bar{d},\bar{\xi}}(\phi)\right\|_{a,s_\varepsilon} \le C\left(\left\|N_{\varepsilon,\bar{d},\bar{\xi}}(\phi)\right\|_{a,s_\varepsilon} + \left\|R_{\varepsilon,\bar{d},\bar{\xi}}\right\|_{a,s_\varepsilon}\right), \tag{8.71}$$

and

$$\left\|T_{\varepsilon,\bar{d},\bar{\xi}}(\phi_1) - T_{\varepsilon,\bar{d},\bar{\xi}}(\phi_2)\right\|_{a,s_\varepsilon} \le C\left(\left\|N_{\varepsilon,\bar{d},\bar{\xi}}(\phi_1) - N_{\varepsilon,\bar{d},\bar{\xi}}(\phi_2)\right\|_{a,s_\varepsilon}\right). \tag{8.72}$$

By (8.13) and (8.14), we deduce

$$\begin{aligned}\left\|N_{\varepsilon,\bar{d},\bar{\xi}}(\phi)\right\|_{a,s_\varepsilon} \ &\le \ C\left|f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi) - f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) - f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\phi\right|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}} \\ &\quad + \left|f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi) - f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) - f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\phi\right|_{\frac{2n}{n+2}},\end{aligned} \tag{8.73}$$

and

$$\begin{aligned}&\left\|N_{\varepsilon,\bar{d},\bar{\xi}}(\phi_1) - N_{\varepsilon,\bar{d},\bar{\xi}}(\phi_2)\right\|_{a,s_\varepsilon} \\ \le \ & C\left|f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_1) - f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_2) - f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})(\phi_1 - \phi_2)\right|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}} \\ & + \left|f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_1) - f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_2) - f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})(\phi_1 - \phi_2)\right|_{\frac{2n}{n+2}}.\end{aligned} \tag{8.74}$$

Then by the mean value theorem and the Hölder inequality, by Lemma 8.16, it follows that if $n = 6$ and $\varepsilon > 0$, for any $\tau \in (0, 1)$,

$$\begin{aligned}&\left|f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_1) - f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_2) - f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})(\phi_1 - \phi_2)\right|_{\frac{2n}{n+2}} \\ = \ & \left|\left(f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_2 + \tau(\phi_1 - \phi_2)) - f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\right)(\phi_1 - \phi_2)\right|_{\frac{2n}{n+2}} \\ \le \ & C\left(|\phi_1|_{s_\varepsilon}^{\frac{2s_\varepsilon}{n}} + |\phi_2|_{s_\varepsilon}^{\frac{2s_\varepsilon}{n}}\right)|\phi_1 - \phi_2|_{\frac{2n}{n-2}} \le C\left(\|\phi_1\|^{1-\varepsilon}_{a,s_\varepsilon} + \|\phi_2\|^{1-\varepsilon}_{a,s_\varepsilon}\right)\|\phi_1 - \phi_2\|_{a,s_\varepsilon}.\end{aligned} \tag{8.75}$$

We note that by (8.15) we have $\frac{ns_\varepsilon}{n+2s_\varepsilon} = \frac{2n}{n+2}$ for $\varepsilon > 0$.

If $n \geq 7$ or $\varepsilon < 0$, there holds

$$\begin{aligned}
& \left| f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_1) - f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_2) - f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})(\phi_1 - \phi_2) \right|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}} \\
= & \left| \left(f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \tau\phi_2 + (1-\tau)\phi_1) - f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) \right) (\phi_1 - \phi_2) \right|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}} \\
\leq & \; C \left| \left(|V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}|^{2^*-3-\varepsilon} |\tau\phi_2 + (1-\tau)\phi_1| + |\tau\phi_2 + (1-\tau)\phi_1|^{2^*-2-\varepsilon} \right) (\phi_1 - \phi_2) \right|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}} \\
= & \; C \left(|V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}|_{s_\varepsilon} + |\phi_1|_{s_\varepsilon} + |\phi_2|_{s_\varepsilon} \right)^{2^*-3-\varepsilon} \left(|\phi_1|_{s_\varepsilon} + |\phi_2|_{s_\varepsilon} \right) |\phi_1 - \phi_2|_{s_\varepsilon} \\
\leq & \; C \left(|V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}|_{s_\varepsilon} + \|\phi_1\|_{a,s_\varepsilon} + \|\phi_2\|_{a,s_\varepsilon} \right)^{2^*-3-\varepsilon} \left(\|\phi_1\|_{a,s_\varepsilon} + \|\phi_2\|_{a,s_\varepsilon} \right) \|\phi_1 - \phi_2\|_{a,s_\varepsilon} \quad (8.76)
\end{aligned}$$

Since the problem is supercritical if $\varepsilon < 0$, $s > \frac{2n}{n-2}$, i.e., $\frac{ns_\varepsilon}{n+2s_\varepsilon} > \frac{2n}{n+2}$, by the embedding $L^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}(\mathcal{M}) \hookrightarrow L^{\frac{2n}{n+2}}(\mathcal{M})$, we get

$$\begin{aligned}
& \left| f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_1) - f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_2) - f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})(\phi_1 - \phi_2) \right|_{\frac{2n}{n+2}} \\
= & \; C \left(\left| V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \right|_{s_\varepsilon} + \|\phi_1\|_{a,s_\varepsilon} + \|\phi_2\|_{a,s_\varepsilon} \right)^{2^*-3-\varepsilon} \left(\|\phi_1\|_{a,s_\varepsilon} + \|\phi_2\|_{a,s_\varepsilon} \right) \|\phi_1 - \phi_2\|_{a,s_\varepsilon}.
\end{aligned} \tag{8.77}$$

Moreover, if $n \geq 7$ and $\varepsilon > 0$, from (8.15), we have $\frac{ns_\varepsilon}{n+2s_\varepsilon} = \frac{2n}{n+2}$.

Taking $\phi_1 = \phi, \phi_2 = 0$ into (8.75) or (8.76) and (8.77), from (8.73), we can get

$$\|N_{\varepsilon,\bar{d},\bar{\xi}}(\phi)\|_{a,s_\varepsilon} \leq \begin{cases} C\|\phi\|^{2-\varepsilon}_{a,s_\varepsilon} & \text{if} \quad n = 6 \text{ and } \varepsilon > 0; \\ C\left(|V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}|^{2^*-3-\varepsilon}_{s_\varepsilon} \|\phi\|^2_{a,s_\varepsilon} + \|\phi\|^{2^*-1-\varepsilon}_{a,s_\varepsilon} \right) & \text{if} \quad n \geq 7 \text{ or } \varepsilon < 0. \end{cases} \tag{8.78}$$

By Lemma 8.15 (iii), we have $|V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}|_{s_\varepsilon} = O(1)$ for ε small. By the definition of $\mathcal{B}(\beta)$, from (8.68), (8.71) and (8.78), we can get that there exists $\beta > 0$ such that

$$\phi \in \mathcal{B}(\beta) \quad \Longrightarrow \quad T_{\varepsilon,\bar{d},\bar{\xi}}(\phi) \in \mathcal{B}(\beta), \tag{8.79}$$

provided that ε is sufficiently small. Next we will show that the map $T_{\varepsilon,\bar{d},\bar{\xi}}$ is a contraction map for any ε small enough.

If $n = 6$ and $\varepsilon > 0$, by (8.72), (8.74) and (8.75), we deduce that there exists $\vartheta \in (0,1)$ such that

$$\begin{aligned}
& \|\phi_1\|_{a,s_\varepsilon}, \|\phi_2\|_{a,s_\varepsilon} \leq |\varepsilon| \left| \ln |\varepsilon| \right|^{2/3} \\
\Longrightarrow \quad & \|T_{\varepsilon,\bar{d},\bar{\xi}}(\phi_1) - T_{\varepsilon,\bar{d},\bar{\xi}}(\phi_2)\|_{a,s_\varepsilon} \leq \vartheta \|\phi_1 - \phi_2\|_{a,s_\varepsilon}.
\end{aligned} \tag{8.80}$$

If $n \geq 7$ or $\varepsilon < 0$, by (8.72), (8.74),(8.76) and (8.77), we can deduce that there exists $\vartheta \in (0,1)$ such that

$$\|\phi_1\|_{a,s_\varepsilon}, \|\phi_2\|_{a,s_\varepsilon} \leq |\varepsilon| \left| \ln |\varepsilon| \right|$$

$$\implies \quad \|T_{\varepsilon,\bar{d},\bar{\xi}}(\phi_1) - T_{\varepsilon,\bar{d},\bar{\xi}}(\phi_2)\|_{a,s_\varepsilon} \leq \vartheta\|\phi_1 - \phi_2\|_{a,s_\varepsilon}. \tag{8.81}$$

By (8.79) and (8.80) or (8.81), we deduce that $T_{\varepsilon,\bar{d},\bar{\xi}}$ is a contraction mapping from $\mathcal{B}(\beta)$ into $\mathcal{B}(\beta)$ for ε small enough, so it has a fixed point $\phi_{\varepsilon,\bar{d},\bar{\xi}}$ which satisfies (8.24), and (8.26) holds from (8.68).

In order to prove that the map $(\bar{d},\bar{\xi}) \to \phi_{\varepsilon,\bar{d},\bar{\xi}}$ is a $\mathcal{C}^1$ map, we apply the Implicit Function Theorem to the function $G(\bar{d},\bar{\xi},u) : (\mathbb{R}^+)^k \times \mathcal{M}^k \times \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$ defined by

$$G(\bar{d},\bar{\xi},u) = u - L^{-1}_{\varepsilon,\bar{d},\bar{\xi}}\left(N_{\varepsilon,\bar{d},\bar{\xi}}(u) + R_{\varepsilon,\bar{d},\bar{\xi}}\right).$$

Indeed, $\phi_{\varepsilon,\bar{d},\bar{\xi}}$ satisfies

$$G(\bar{d},\bar{\xi},\phi_{\varepsilon,\bar{d},\bar{\xi}}) = 0. \tag{8.82}$$

We have

$$\partial_u G(\bar{d},\bar{\xi},u)[v] = v - L^{-1}_{\varepsilon,\bar{d},\bar{\xi}}\left((\partial_u N_{\varepsilon,\bar{d},\bar{\xi}})(u)v\right). \tag{8.83}$$

Moreover, by the mean value theorem we have

$$N_{\varepsilon,\bar{d},\bar{\xi}}(\phi_1) - N_{\varepsilon,\bar{d},\bar{\xi}}(\phi_2) = (\partial_u N_{\varepsilon,\bar{d},\bar{\xi}})(u)(\phi_1 - \phi_2),$$

for some $u = \tau\phi_1 + (1-\tau)\phi_2, \tau \in [0,1]$. Then from (8.26), (8.74) and (8.75), there exists a positive constant c such that

$$\|\partial_u N_{\varepsilon,\bar{d},\bar{\xi}}\|_{a,s_\varepsilon} \leq c|\varepsilon| \quad \text{for} \quad n = 6 \quad \text{or} \quad \varepsilon > 0. \tag{8.84}$$

In the similarly that if $n > 6$ or $\varepsilon < 0$, from (8.26), (8.74), (8.76) and (8.77) , it holds that

$$\|\partial_u N_{\varepsilon,\bar{d},\bar{\xi}}\|_{a,s_\varepsilon} \leq c|\varepsilon| \quad \text{for} \quad n \geq 7 \quad \text{or} \quad \varepsilon < 0. \tag{8.85}$$

Consequently, using Lemma 8.7, (8.83), (8.84) or (8.85) we obtain that $\partial_u G(\bar{d},\bar{\xi},\phi_{\varepsilon,\bar{d},\bar{\xi}})$ is invertible with uniformly bounded inverse. Then, the fact that $(\bar{d},\bar{\xi}) \mapsto \phi_{\varepsilon,\bar{d},\bar{\xi}}$ is $\mathcal{C}^1$ follows from the fact that $(\bar{d},\bar{\xi},\phi_{\varepsilon,\bar{d},\bar{\xi}}) \mapsto L^{-1}_{\varepsilon,\bar{d},\bar{\xi}}\left(N_{\varepsilon,\bar{d},\bar{\xi}}(u)\right)$ is $\mathcal{C}^1$ and the implicit functions theorem.

8.5 Expansion of the energy

This section is devoted to the proof of Proposition 8.6. At the first step, we have

Lemma 8.9. *For ε small, if $(\bar{d},\bar{\xi})$ is a critical point of the functional $\widetilde{J}_\varepsilon$, then $V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}}$ is a solution of (8.16), or equivalently of problem (8.1).*

Proof. Let $(\bar{d}, \overline{\xi})$ be a critical point of $\tilde{J}_\varepsilon$, where $\bar{d} = (d_1, d_2, \cdots, d_k) \in (\mathbb{R}^+)^k$ and $\overline{\xi} = (\xi_1, \xi_2, \cdots, \xi_k) \in \mathcal{M}^k$. Let $\xi_j = \xi_j(y^j) = \exp_{\xi_j}(y^j)$, $y^j \in B(0, r)$ and

$$\overline{\xi} = \overline{\xi}(y) = \left(\exp_{\xi_1}(y^1), \cdots, \exp_{\xi_k}(y^k)\right), \quad y := (y^1, \ldots, y^k) \in B(0,r) \times \cdots \times B(0,r).$$

We remark that $\overline{\xi}(0) = \overline{\xi}$, since $(\bar{d}, \overline{\xi})$ is a critical point of $\tilde{J}_\varepsilon$, for $c = 1, \cdots, n$, $j = 1, \cdots, k$, there holds

$$J'_\varepsilon\left(V_{\Lambda_\varepsilon(\bar{d}),\overline{\xi}} + \phi_{\varepsilon,\bar{d},\overline{\xi}}\right)\left[\frac{\partial}{\partial d_j} V_{\Lambda_\varepsilon(\bar{d}),\overline{\xi}} + \frac{\partial}{\partial d_j}\phi_{\varepsilon,\bar{d},\overline{\xi}}\right] = 0, \tag{8.86}$$

and

$$J'_\varepsilon\left(V_{\Lambda_\varepsilon(\bar{d}),\overline{\xi}} + \phi_{\varepsilon,\bar{d}_\varepsilon,\overline{\xi}}\right)\left[\frac{\partial}{\partial y^j_c} V_{\Lambda_\varepsilon(\bar{d}),\overline{\xi}} + \frac{\partial}{\partial y^j_c}\phi_{\varepsilon,\bar{d},\overline{\xi}}\right] = 0. \tag{8.87}$$

Let ∂_m denotes ∂_{d_h} or $\partial_{y^h_c}$ for $h = 1, 2, \cdots, k$ and $c = 1, \cdots, n$. By (8.24) we get

$$\begin{aligned}
0 &= \partial_m \tilde{J}_\varepsilon(\bar{d}, \overline{\xi}) = J'_\varepsilon\left(V_{\Lambda_\varepsilon(\bar{d}),\overline{\xi}} + \phi_{\varepsilon,\bar{d},\overline{\xi}}\right)\left[\partial_m V_{\Lambda_\varepsilon(\bar{d}),\overline{\xi}} + \partial_m \phi_{\varepsilon,\bar{d},\overline{\xi}}\right] \\
&= \left\langle V_{\Lambda_\varepsilon(\bar{d}),\overline{\xi}} + \phi_{\varepsilon,\bar{d},\overline{\xi}} - i^*\left[f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\overline{\xi}} + \phi_{\varepsilon,\bar{d},\overline{\xi}})\right], \partial_m V_{\Lambda_\varepsilon(\bar{d}),\overline{\xi}} + \partial_m \phi_{\varepsilon,\bar{d},\overline{\xi}}\right\rangle_a \\
&= \sum_{i=0}^{n}\sum_{j=1}^{k} c^{ij}_\varepsilon \left\langle Z^i_{\lambda_j,\xi_j(y^j)}, \partial_m V_{\Lambda_\varepsilon(\bar{d}),\overline{\xi}} + \partial_m \phi_{\varepsilon,\bar{d},\overline{\xi}}\right\rangle_a,
\end{aligned} \tag{8.88}$$

for some $c^{ij}_\varepsilon \in \mathbb{R}$. We have to prove that if we compute (8.88) at $y = 0$ then, provided ε is small enough, it holds

$$c^{ij}_\varepsilon = 0 \quad \text{for any} \quad i = 0, 1, \cdots n; \quad j = 1, 2, \cdots, k. \tag{8.89}$$

First of all, from (8.126) and (8.127) in Lemma 8.14 we have

$$\begin{aligned}
&\sum_{i=0}^{n}\sum_{j=1}^{k} c^{ij}_\varepsilon \left\langle Z^i_{\lambda_j,\xi_j(y^j)}, \left(\frac{\partial}{\partial y^h_c} W_{\lambda_h,\xi_h(y^h)}\right)\Big|_{y=0}\right\rangle_a \\
= \;& \sum_{i=0}^{n}\sum_{j=1}^{k} \frac{1}{\lambda_j} c^{ij}_\varepsilon \delta_{jh}\delta_{ic}\|V_i\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + \sum_{i=0}^{n}\sum_{j=1}^{k} o(1) c^{ij}_\varepsilon \delta_{jh}\delta_{ic},
\end{aligned} \tag{8.90}$$

and

$$\begin{aligned}
&\sum_{i=0}^{n}\sum_{j=1}^{k} c^{ij}_\varepsilon \left\langle Z^i_{\lambda_j,\xi_j(y^j)}, \frac{\partial}{\partial d_h} V_{\Lambda_\varepsilon(\bar{d}),\overline{\xi}}\right\rangle_a \\
= \;& \frac{1}{2d_j}\sum_{i=0}^{n}\sum_{j=1}^{k} c^{ij}_\varepsilon \delta_{jh}\delta_{i0}\|V_0\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + \sum_{i=0}^{n}\sum_{j=1}^{k} c^{ij}_\varepsilon \delta_{jh}\delta_{i0}\, o\left(\left\|Z^0_{\lambda_j,\xi_j(y^j)}\right\|_a\right).
\end{aligned} \tag{8.91}$$

Moreover, since $\phi_{\varepsilon,\bar d,\bar\xi}\in K^{\perp}_{\Lambda_\varepsilon(\bar d),\bar\xi}$, it holds $\left\langle Z^i_{\lambda_j,\xi_j(y^j)},\phi_{\varepsilon,\bar d,\bar\xi}\right\rangle_a=0$, which implies

$$\begin{aligned}\left\langle Z^i_{\lambda_j,\xi_j(y^j)},\partial_m\phi_{\varepsilon,\bar d,\bar\xi}\right\rangle_a &= -\left\langle \partial_m Z^i_{\lambda_j,\xi_j(y^j)},\phi_{\varepsilon,\bar d,\bar\xi}\right\rangle_a\\ &= O\left(\left\|\partial_m Z^i_{\lambda_j,\xi_j(y^j)}\right\|_a\left\|\phi_{\varepsilon,\bar d,\bar\xi}\right\|_a\right).\end{aligned} \tag{8.92}$$

From Lemma 8.13, there hold

$$\left\|\partial_{d_j} Z^i_{\lambda_j,\xi_j(y^j)}\right\|_a=O(|\varepsilon|^{1/2}),\qquad \left\|\left(\frac{\partial}{\partial y^j_c}Z^i_{\lambda_j,\xi_j(y^j)}\right)_{|y=0}\right\|_a=O(|\varepsilon|^{-1/2}). \tag{8.93}$$

By Proposition 3.1, from (8.92) and (8.93), for any $\kappa\in(0,1)$, we get

$$\left\langle Z^i_{\lambda_j,\xi_j(y^j)},\partial_m\phi_{\varepsilon,\bar d,\bar\xi}\right\rangle_a=o\left(|\varepsilon|^{\kappa}\right). \tag{8.94}$$

Therefore, by (8.90), (8.91), (8.92) and (8.94) we deduce that the linear system in (8.88) has only the trivial solution provide ε small. That concludes the proof of the part (i) of Proposition 8.6.

□

In the next Lemma, we give the expansion of $J_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})$ as $\varepsilon\to0$ for $(\Lambda_\varepsilon(\bar d),\bar\xi)\in\mathcal{O}_{\eta,\rho}$.

Lemma 8.10. *If $n\geq5$ and $(\Lambda_\varepsilon(\bar d),\bar\xi)\in\mathcal{O}_{\eta,\rho}$ satisfies (8.22), then there holds*

$$J_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})=\frac{k}{n}K_n^{-n}-\frac{k}{n}K_n^{-n}\alpha_n\varepsilon+\frac{1}{n}K_n^{-n}\Psi_k(\bar d,\bar\xi)+o(\varepsilon), \tag{8.95}$$

as $\varepsilon\to0$, $\mathcal{C}^1-$uniformly with respect to $\bar\xi$ in $\mathcal{M}^k$ and to $\bar d$ in compact subsets of $(\mathbb{R}^+)^k$, where α_n and $\Psi_k(\bar d,\bar\xi)$ defined in (8.29) and (8.30).

Proof. We have

$$\begin{aligned} J_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi}) &= J_\varepsilon\left(\sum_{j=1}^k W_{\lambda_j,\xi_j}\right)\\ &= \frac12\int_{\mathcal{M}}\left|\nabla_g\left(\sum_{j=1}^k W_{\lambda_j,\xi_j}\right)\right|^2 d\mu_g+\frac12\int_{\mathcal{M}}a(x)\left(\sum_{j=1}^k W_{\lambda_j,\xi_j}\right)^2 d\mu_g\\ &\quad -\frac{1}{2^*-\varepsilon}\int_{\mathcal{M}}\left(\sum_{j=1}^k W_{\lambda_j,\xi_j}\right)_+^{2^*-\varepsilon} d\mu_g\\ &= \sum_{j=1}^k\left[\frac12\int_{\mathcal{M}}\left|\nabla_g W_{\lambda_j,\xi_j}\right|^2 d\mu_g+\frac12\int_{\mathcal{M}}a(x)W^2_{\lambda_j,\xi_j}\,d\mu_g-\frac{1}{2^*-\varepsilon}\int_{\mathcal{M}}\left(W^{2^*-\varepsilon}_{\lambda_j,\xi_j}\right)_+ d\mu_g\right]\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j,l=1;j\neq l}^{k}\frac{1}{2}\int_{\mathcal{M}}\nabla_g W_{\lambda_j,\xi_j}\nabla_g W_{\lambda_l,\xi_l}\,d\mu_g+\frac{1}{2}\sum_{j,l=1;j\neq l}^{k}\int_{\mathcal{M}}a(x)W_{\lambda_j,\xi_j}W_{\lambda_l,\xi_l}\,d\mu_g\\
&-\frac{1}{2^*-\varepsilon}\int_{\mathcal{M}}\left\{\left|\sum_{j=1}^{k}W_{\lambda_j,\xi_j}\right|^{2^*-\varepsilon}-\sum_{j=1}^{k}\left|W_{\lambda_j,\xi_j}\right|^{2^*-\varepsilon}\right\}\,d\mu_g. \qquad (8.96)
\end{aligned}
$$

Since W_{λ_j,ξ_j} and W_{λ_l,ξ_l} have disjoint supports, we get

$$
\int_{\mathcal{M}}\nabla_g W_{\lambda_j,\xi_j}\nabla_g W_{\lambda_l,\xi_l}\,d\mu_g=\int_{\mathcal{M}}a(x)W_{\lambda_j,\xi_j}W_{\lambda_l,\xi_l}\,d\mu_g=0,\quad \text{for}\ \ j\neq l,
$$

and

$$
\int_{\mathcal{M}}\left\{\left|\sum_{j=1}^{k}W_{\lambda_j,\xi_j}\right|^{2^*-\varepsilon}-\sum_{j=1}^{k}\left|W_{\lambda_j,\xi_j}\right|^{2^*-\varepsilon}\right\}\,d\mu_g=0.
$$

Moreover, by Lemma 4.1 in [90], we get that for any $j=1,2,\cdots,k$

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathcal{M}}\left|\nabla_g W_{\lambda_j,\xi_j}\right|^2\,d\mu_g+\frac{1}{2}\int_{\mathcal{M}}a(x)W_{\lambda_j,\xi_j}^2\,d\mu_g-\frac{1}{2^*-\varepsilon}\int_{\mathcal{M}}\left(W_{\lambda_j,\xi_j}^{2^*-\varepsilon}\right)_+\,d\mu_g\\
=\ &\frac{K_n^{-n}}{n}\left(1-\frac{(n-2)^2}{8}\varepsilon\ln(|\varepsilon|d_j)-\alpha_n\varepsilon\right)\\
&+\frac{K_n^{-n}}{n}\frac{2(n-1)}{(n-2)(n-4)}|\varepsilon|d_j\left(a(\xi_j)-\frac{n-2}{4(n-1)}\mathrm{Scal}_g(\xi_j)\right)+o(\varepsilon). \qquad (8.97)
\end{aligned}
$$

Thus, (8.95) follows from (8.96) and (8.97). □

Lemma 8.11. *If $n\geq 6$ and $\Lambda_\varepsilon(\bar{d})=(\lambda_1,\lambda_2,\cdots,\lambda_k)$ satisfies (8.21), then there holds*

$$
\tilde{J}_\varepsilon(\bar{d},\bar{\xi})=J_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}+\phi_{\varepsilon,\bar{d},\bar{\xi}})=J_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})+o(\varepsilon) \qquad (8.98)
$$

as $\varepsilon\to 0$, $\mathcal{C}^0$ uniformly with respect to $\bar{\xi}\in\mathcal{M}^k$ and to $\bar{d}$ in compact subsets of $(\mathbb{R}^+)^k$.

Proof. We argue as Lemma 4.2 in [90]. From the equation (8.24), it holds

$$
\begin{aligned}
&\tilde{J}_\varepsilon(\bar{d},\bar{\xi})-J_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})=J_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}+\phi_{\varepsilon,\bar{d},\bar{\xi}})-J_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\\
=\ &\frac{1}{2}\left\|\phi_{\varepsilon,\bar{d},\bar{\xi}})\right\|_a^2+\int_{\mathcal{M}}\left[\nabla_g V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}\nabla_g\phi_{\varepsilon,\bar{d},\bar{\xi}}+a(x)V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}\phi_{\varepsilon,\bar{d},\bar{\xi}}-f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\phi_{\varepsilon,\bar{d},\bar{\xi}}\right]\,d\mu_g\\
&-\int_{\mathcal{M}}\left[F_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}+\phi_{\varepsilon,\bar{d},\bar{\xi}})-F_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})-f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\phi_{\varepsilon,\bar{d},\bar{\xi}}\right]\,d\mu_g\\
=\ &\frac{1}{2}\left\|\phi_{\varepsilon,\bar{d},\bar{\xi}})\right\|_a^2+\int_{\mathcal{M}}\left[f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}+\phi_{\varepsilon,\bar{d},\bar{\xi}})-f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\right]\phi_{\varepsilon,\bar{d},\bar{\xi}}\,d\mu_g\\
&-\int_{\mathcal{M}}\left[F_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}+\phi_{\varepsilon,\bar{d},\bar{\xi}})-F_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})-f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\phi_{\varepsilon,\bar{d},\bar{\xi}}\right]\,d\mu_g. \qquad (8.99)
\end{aligned}
$$

We prove that the right hand side of (8.99) is $o(\varepsilon)$. In fact, the first term in the right side of (8.99) is $o(\varepsilon)$ because of (8.26). Moreover, by the mean value theorem, for some $\tau_1, \tau_2 \in [0,1]$, we have

$$\begin{aligned} &\int_{\mathcal{M}} \left[f(V_{\Lambda_\varepsilon(\bar d),\bar\xi} + \phi_{\varepsilon,\bar d,\bar\xi}) - f_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi}) \right] \phi_{\varepsilon,\bar d,\bar\xi}\, d\mu_g \\ = &\int_{\mathcal{M}} f'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi} + \tau_1 \phi_{\varepsilon,\bar d,\bar\xi}) \phi^2_{\varepsilon,d,\bar\xi}\, d\mu_g, \end{aligned} \tag{8.100}$$

and

$$\begin{aligned} &\int_{\mathcal{M}} \left[F_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi} + \phi_{\varepsilon,\bar d,\bar\xi}) - F_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi}) - f_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})\phi_{\varepsilon,\bar d,\bar\xi} \right]\, d\mu_g \\ = &\int_{\mathcal{M}} f'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi} + \tau_2 \phi_{\varepsilon,\bar d,\bar\xi}) \phi^2_{\varepsilon,d,\bar\xi}\, d\mu_g. \end{aligned} \tag{8.101}$$

By the Hölder inequality and (8.26), we have for any $\tau \in [0,1]$

$$\begin{aligned} &\int_{\mathcal{M}} \left| f'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi} + \tau \phi_{\varepsilon,\bar d,\bar\xi}) \right| \phi^2_{\varepsilon,d,\bar\xi}\, d\mu_g \\ \le\ & C \int_{\mathcal{M}} V^{2^*-2-\varepsilon}_{\Lambda_\varepsilon(\bar d),\bar\xi} \phi^2_{\varepsilon,d,\bar\xi}\, d\mu_g + C \int_{\mathcal{M}} \phi^{2^*-\varepsilon}_{\varepsilon,d,\bar\xi}\, d\mu_g \\ \le\ & C \left| V^{2^*-2-\varepsilon}_{\Lambda_\varepsilon(\bar d),\bar\xi} \right|_{\frac{n}{2}} \left| \phi_{\varepsilon,\bar d,\bar\xi} \right|_{\frac{2n}{n-2}} + C |\phi_{\varepsilon,\bar d,\bar\xi}|^{2^*-\varepsilon}_{s_\varepsilon} \\ \le\ & C \sum_{j=1}^{k} \left| W^{2^*-2-\varepsilon}_{\lambda_j,\xi_j} \right|_{\frac{n}{2}} \left\| \phi_{\varepsilon,\bar d,\bar\xi} \right\|_{a,s_\varepsilon} + C \|\phi_{\varepsilon,\bar d,\bar\xi}\|^{2^*-\varepsilon}_{a,s_\varepsilon} \\ =\ & o(\varepsilon), \end{aligned} \tag{8.102}$$

because of $\left| W^{2^*-2-\varepsilon}_{\lambda_j,\xi_j} \right|_{\frac{n}{2}} = O(1)$ by (i) in Lemma 8.15. Therefore (8.98) follows from (8.99), (8.100), (8.101) and (8.102). By Lemma 8.10, we deduced that (8.28) holds.

□

Next, we estimate the gradient of the reduced energy.

Lemma 8.12. *For any $\eta, \rho > 0$ in (8.22), if ε is small enough, for $j = 1, 2, \cdots, k$, it holds*

$$\partial_{d_j} \left[J_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi} + \phi_{\varepsilon,\bar d,\bar\xi}) \right] = \partial_{d_j} \Psi_k(\bar d, \bar\xi) + o(\varepsilon), \tag{8.103}$$

and set $\bar\xi = \bar\xi(y) = \left(\exp_{\xi_1}(y^1), \cdots, \exp_{\xi_k}(y^k)\right)$, $y := (y^1, \ldots, y^k) \in B(0,r) \times \cdots \times B(0,r)$, for any $c = 1, 2, \cdots, n$ and $j = 1, \cdots, k$, it holds that

$$\left(\frac{\partial}{\partial y^j_c} J_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi} + \phi_{\varepsilon,\bar d,\bar\xi}) \right)_{|y=0} = \left(\frac{\partial}{\partial y^j_c} \Psi_k(\bar d, \bar\xi) \right)_{|y=0} + o(\varepsilon), \tag{8.104}$$

$\mathcal{C}^0$ uniformly with respect to $\bar\xi$ in $\mathcal{M}^k$ and $\bar d$ in $(\mathbb{R}^+)^k$, the function Ψ_k is defined in (8.30).

Proof. It holds

$$\begin{aligned}
&\partial_{d_j}\left[J_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi}+\phi_{\varepsilon,\bar d,\bar\xi})\right]-\partial_{d_j}\Psi(\bar d,\bar\xi)\\
=\;&\left(J'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi}+\phi_{\varepsilon,\bar d,\bar\xi})-J'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})\right)\left[\partial_{d_j}V_{\Lambda_\varepsilon(\bar d),\bar\xi}\right]\\
&+J'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi}+\phi_{\varepsilon,\bar d,\bar\xi})\left[\partial_{d_j}\phi_{\varepsilon,\bar d,\bar\xi}\right]+\left(\partial_{d_j}J_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})-\partial_{d_j}\Psi(\bar d,\bar\xi)\right)\\
:=\;&I_3+I_4+I_5. \qquad (8.105)
\end{aligned}$$

First, because the parameters $\bar d$ is bounded and bounded from zero, using Lemma 8.10 the expansion of $J_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})$, we have $I_5=o(\varepsilon)$. Next we show that $I_3, I_4=o(\varepsilon)$.

From (8.130) in Lemma 8.14 and the function $\phi_{\varepsilon,\bar d,\bar\xi}$ belongs to $K^\perp_{\Lambda_\varepsilon(\bar d),\bar\xi}$, we have,

$$\begin{aligned}
I_3\;=\;&\left(J'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi}+\phi_{\varepsilon,\bar d,\bar\xi})-J'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})\right)\left[\partial_{d_j}V_{\Lambda_\varepsilon(\bar d),\bar\xi}\right]\\
=\;&\int_{\mathcal M}\left(\nabla_g\phi_{\varepsilon,\bar d,\bar\xi}\nabla_g\left(\partial_{d_j}V_{\Lambda_\varepsilon(\bar d),\bar\xi}\right)+a(x)\phi_{\varepsilon,\bar d,\bar\xi}\partial_{d_j}V_{\Lambda_\varepsilon(\bar d),\bar\xi}\right)\,d\mu_g\\
&-\int_{\mathcal M}\left(f_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi}+\phi_{\varepsilon,\bar d,\bar\xi})-f_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})\right)\partial_{d_j}V_{\Lambda_\varepsilon(\bar d),\bar\xi}\,d\mu_g\\
=\;&\frac{1}{2d_j}\int_{\mathcal M}\left(\nabla_g\phi_{\varepsilon,\bar d,\bar\xi}\nabla_g Z^0_{\lambda_j,\xi_j}+a(x)\phi_{\varepsilon,\bar d,\bar\xi}Z^0_{\lambda_j,\xi_j}\right)\,d\mu_g\\
&-\frac{1}{2d_j}\int_{\mathcal M}\left(f_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi}+\phi_{\varepsilon,\bar d,\bar\xi})-f_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})\right)Z^0_{\lambda_j,\xi_j}\,d\mu_g\\
=\;&-\frac{1}{2d_j}\int_{\mathcal M}f'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})\phi_{\varepsilon,\bar d,\bar\xi}Z^0_{\lambda_j,\xi_j}\,d\mu_g-\frac{1}{2d_j}\int_{\mathcal M}\Big\{f_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi}+\phi_{\varepsilon,\bar d,\bar\xi})\\
&-f_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})-f'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})\phi_{\varepsilon,\bar d,\bar\xi}\Big\}Z^0_{\lambda_j,\xi_j}d\mu_g. \qquad (8.106)
\end{aligned}$$

From (8.26), by the boundary of d_j, using the similarly proof of (8.60), we have

$$\left|\frac{1}{2d_j}\int_{\mathcal M}f'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})\phi_{\varepsilon,\bar d,\bar\xi}Z^0_{\lambda_j,\xi_j}\,d\mu_g\right|=o(|\varepsilon|). \qquad (8.107)$$

Moreover,

$$\begin{aligned}
&\left|\int_{\mathcal M}\left(f_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi}+\phi_{\varepsilon,\bar d,\bar\xi})-f_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})-f'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})\phi_{\varepsilon,\bar d,\bar\xi}\right)Z^0_{\lambda_j,\xi_j}\,d\mu_g\right|\\
\le\;&\int_{\mathcal M}\left|f'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi}+t\phi_{\varepsilon,\bar d,\bar\xi})-f'_\varepsilon(V_{\Lambda_\varepsilon(\bar d),\bar\xi})\right|\phi_{\varepsilon,\bar d,\bar\xi}Z^0_{\lambda_j,\xi_j}\,d\mu_g\\
\le\;&\begin{cases}C\int_{\mathcal M}\phi^{2^*-1-\varepsilon}_{\varepsilon,\bar d,\bar\xi}Z^0_{\lambda_j,\xi_j}\,d\mu_g & \text{if } n=6 \text{ and } \varepsilon>0;\\ C\int_{\mathcal M}\left(V^{2^*-3-\varepsilon}_{\Lambda_\varepsilon(\bar d),\bar\xi}\phi^2_{\varepsilon,\bar d,\bar\xi}Z^0_{\lambda_j,\xi_j}+\phi^{2^*-1-\varepsilon}_{\varepsilon,\bar d,\bar\xi}Z^0_{\lambda_j,\xi_j}\right)d\mu_g & \text{otherwise}\end{cases}\\
\le\;&\begin{cases}C\left|\phi_{\varepsilon,\bar d,\bar\xi}\right|^{2^*-1-\varepsilon}_{s_\varepsilon}\left|Z^0_{\lambda_j,\xi_j}\right|_{\frac{ns_\varepsilon}{ns_\varepsilon-n-2s_\varepsilon}} & \text{if } n=6 \text{ and } \varepsilon>0;\\ C\Big(\left|V^{2^*-3-\varepsilon}_{\Lambda_\varepsilon(\bar d),\bar\xi}\right|_{\frac{s_\varepsilon}{s_\varepsilon-3}}\left|\phi_{\varepsilon,\bar d,\bar\xi}\right|^2_{s_\varepsilon}\left|Z^0_{\lambda_j,\xi_j}\right|_{s_\varepsilon}\\ \qquad+\left|\phi_{\varepsilon,\bar d,\bar\xi}\right|^{2^*-1-\varepsilon}_{s_\varepsilon}\left|Z^0_{\lambda_j,\xi_j}\right|_{\frac{ns_\varepsilon}{ns_\varepsilon-n-2s_\varepsilon}}\Big) & \text{otherwise.}\end{cases} \qquad (8.108)
\end{aligned}$$

From (8.26), (8.106), (8.107), (8.108) and (iv) in Lemma 8.15, we get $I_3 = o(|\varepsilon|)$ for ε small enough.

Now, for ε small, for any $(\Lambda_\varepsilon(\bar{d}), \bar{\xi}) \in \mathcal{O}_{\eta,\rho}$, since (8.24), we have

$$\begin{aligned} I_4 &= J'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}})\left[\partial_{d_j}\phi_{\varepsilon,\bar{d},\bar{\xi}}\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{k} c^{ij}\left\langle Z^i_{\lambda_j,\xi_j}, \partial_{d_j}\phi_{\varepsilon,\bar{d},\bar{\xi}}\right\rangle_a \\ &= -\sum_{i=1}^{n}\sum_{j=1}^{k} c^{ij}\left\langle \frac{\partial}{\partial d_j} Z^i_{\lambda_j,\xi_j}, \phi_{\varepsilon,\bar{d},\bar{\xi}}\right\rangle_a. \end{aligned} \tag{8.109}$$

We argue in Lemma 4.2 in [90], we have that for any $\kappa \in (0,1)$, there holds

$$\sum_{i=1}^{n}\sum_{j=1}^{k} |c^{ij}| = O(|\varepsilon|^\kappa). \tag{8.110}$$

Then, from (8.26), (8.110) and (8.120) in Lemma 8.13, we have $I_4 = o(|\varepsilon|)$, therefore, the estimate (8.103) holds.

Next, we prove (8.104) holds. From Lemma 8.10, we have

$$\begin{aligned} &\frac{\partial}{\partial y_c^j} J_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}}) - \frac{\partial}{\partial y_c^j}\Psi_k(\bar{d},\bar{\xi}) \\ = &\frac{\partial}{\partial y_i^j} J_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}}) - \frac{\partial}{\partial y_c^j} J_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) + \frac{\partial}{\partial y_c^j}\left[J_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) - \Psi_k(\bar{d},\bar{\xi})\right] \\ = &J'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}})\left[\frac{\partial}{\partial y_c^j}V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \frac{\partial}{\partial y_c^j}\phi_{\varepsilon,\bar{d},\bar{\xi}}\right] - J'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\left[\frac{\partial}{\partial y_c^j}V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}\right] + o(\varepsilon) \\ = &\left[J'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}}) - J'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\right]\left[\frac{\partial}{\partial y_c^j}V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}\right] \\ &+ J'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}})\left[\frac{\partial}{\partial y_c^j}\phi_{\varepsilon,\bar{d},\bar{\xi}}\right] + o(\varepsilon). \end{aligned} \tag{8.111}$$

We will prove that

$$J'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}})\left[\frac{\partial}{\partial y_c^j}\phi_{\varepsilon,\bar{d},\bar{\xi}}\right] = o(\varepsilon) \tag{8.112}$$

and

$$\left[J'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}}) - J'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\right]\left[\frac{\partial}{\partial y_c^j}V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}\right] = o(\varepsilon) \tag{8.113}$$

Then (8.104) will follow from (8.112) and (8.113).

First, we prove (8.112). Since (8.24) holds, and we take into account that $\phi_{\varepsilon,\bar{d},\bar{\xi}} \in K^{\perp}_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}$, we have

$$\begin{aligned} J'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}}) \left[\frac{\partial}{\partial y^j_c} \phi_{\varepsilon,\bar{d},\bar{\xi}} \right] &= \sum_{l=1}^{k} \sum_{h=0}^{n} c_\varepsilon^{lh} \left\langle Z^h_{\lambda_l,\xi_l}, \frac{\partial}{\partial y^j_c} \phi_{\varepsilon,\bar{d},\bar{\xi}} \right\rangle_a \\ &= -\sum_{l=1}^{k} \sum_{h=0}^{n} c_\varepsilon^{lh} \left\langle \frac{\partial}{\partial y^j_c} Z^h_{\lambda_l,\xi_l}, \phi_{\varepsilon,\bar{d},\bar{\xi}} \right\rangle_a . \end{aligned} \tag{8.114}$$

Then, from (8.26), (8.110) and (8.120) in Lemma 8.13, we have (8.112) holds.

Finally, by the mean value theorem for some $\tau \in [0,1]$, from (8.26) and (8.128), it holds

$$\begin{aligned} & \left[J'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}}) - J'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) \right] \left[\frac{\partial}{\partial y^j_c} V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \right] \\ = & \int_{\mathcal{M}} \left(\nabla_g \phi_{\varepsilon,\bar{d},\bar{\xi}} \nabla_g \left(\frac{\partial}{\partial y^j_c} V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \right) + a(x) \phi_{\varepsilon,\bar{d},\bar{\xi}} \frac{\partial}{\partial y^j_c} V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \right) \, d\mu_g \\ & - \int_{\mathcal{M}} \left(f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}}) - f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) \right) \frac{\partial}{\partial y^j_c} V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} \, d\mu_g \\ = & \left\langle \phi_{\varepsilon,\bar{d},\bar{\xi}}, \frac{1}{\lambda_j} Z^i_{\lambda_j,\xi_j} + o(|\varepsilon|) \right\rangle_a \\ & - \int_{\mathcal{M}} \left(f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi_{\varepsilon,\bar{d},\bar{\xi}}) - f_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}) \right) \left(\frac{1}{\lambda_j} Z^i_{\lambda_j,\xi_j} + o(|\varepsilon|) \right) \, d\mu_g \\ = & - \int_{\mathcal{M}} f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \tau \phi_{\varepsilon,\bar{d},\bar{\xi}}) \phi_{\varepsilon,\bar{d},\bar{\xi}} \left(\frac{1}{\lambda_j} Z^i_{\lambda_j,\xi_j} + o(|\varepsilon|) \right) \, d\mu_g + o(\varepsilon) \\ = & \, o(\varepsilon). \end{aligned}$$

□

The proof of the part (ii) of Proposition 8.6 follows from Lemma 8.10, 8.11, 8.12.

8.6 Appendix

Let $\bar{\xi}^0 = (\xi^0_1, \cdots, \xi^0_k) \in \mathcal{M}^k$ be fixed. Let $\xi_j = \xi_j(y^j) = \exp_{\xi^0_j}(y^j)$, $y^j \in B(0,r) \subset \mathbb{R}^n$ and set

$$\bar{\xi} = \bar{\xi}(y) = \left(\exp_{\xi^0_1}(y^1), \cdots, \exp_{\xi^0_k}(y^k) \right), \qquad y = (y^1, \cdots, y^k) \in B(0,r) \times \cdots \times B(0,r).$$

We remark that $\bar{\xi}(0) = \bar{\xi}^0$. Let us introduce the function $\mathcal{E}$ defined by

$$\mathcal{E}(y,x) = \left(\mathcal{E}^1(y^1,x), \cdots, \mathcal{E}^k(y^k,x) \right), \quad x \in \mathcal{M},$$

where $\mathcal{E}^j(y^j,x) = \exp^{-1}_{\xi_j(y^j)}(x) = \exp^{-1}_{\exp_{\xi_j^0}(y^j)}(x) \in B(0,r)$.

Now, by (8.7), we can write $\left[W_{\lambda_j,\xi_j(y^j)}(x) = \lambda_j^{\frac{2-n}{2}} \chi_r\left(\mathcal{E}^j(y^j,x)\right) U\left(\lambda_j^{-1}\mathcal{E}^j(y^j,x)\right),\right]$ setx$=\exp_{\xi_j(y^j)}(\lambda_j z) = \exp_{\exp_{\xi_j^0}(y^j)}(\lambda_j z)$, for $c = 1,2,\cdots,n$ we have

$$\begin{aligned}&\left(\frac{\partial}{\partial y_c^j} W_{\lambda_j,\xi_j(y^j)}\right)(x)\\ =\ & \sum_{a=1}^n \lambda_j^{\frac{2-n}{2}} \left[\frac{1}{\lambda_j}\frac{\partial U(z)}{\partial z_a}\chi_r(\lambda_j z) + \frac{\partial \chi_r(\lambda_j z)}{\partial z_a} U(z)\right] \frac{\partial \mathcal{E}_a^j}{\partial y_c^j}\left(y^j, \exp_{\exp_{\xi_j^0}(y^j)}(\lambda_j z)\right).\end{aligned} \tag{8.115}$$

In particular,

$$\begin{aligned}&\left(\frac{\partial}{\partial y_c^j} W_{\lambda_j,\xi_j(y^j)}\right)\Big|_{y=0}\\ =\ & \sum_{a=1}^n \lambda_j^{\frac{2-n}{2}} \left[\frac{1}{\lambda_j}\frac{\partial U(z)}{\partial z_a}\chi_r(\lambda_j z) + \frac{\partial \chi_r(\lambda_j z)}{\partial z_a} U(z)\right] \frac{\partial \mathcal{E}_a^j}{\partial y_c^j}\left(0, \exp_{\xi_j^0}(\lambda_j z)\right).\end{aligned} \tag{8.116}$$

In a similar way, for $c = 1,2,\cdots,n$,

$$\begin{aligned}&\left(\frac{\partial}{\partial y_c^j} Z^i_{\lambda_j,\xi_j(y^j)}\right)(x)\\ =\ & \sum_{a=1}^n \lambda_j^{\frac{2-n}{2}} \left[\frac{1}{\lambda_j}\frac{\partial V_i(z)}{\partial z_a}\chi_r(\lambda_j z) + \frac{\partial \chi_r(\lambda_j z)}{\partial z_a} V_i(z)\right] \frac{\partial \mathcal{E}_a^j}{\partial y_c^j}\left(y^j, \exp_{\exp_{\xi_j^0}(y^j)}(\lambda_j z)\right),\end{aligned} \tag{8.117}$$

and,

$$\begin{aligned}&\left(\frac{\partial}{\partial y_c^j} Z^i_{\lambda_j,\xi_j(y^j)}\right)\Big|_{y=0}\\ =\ & \sum_{a=1}^n \lambda_j^{\frac{2-n}{2}} \left[\frac{1}{\lambda_j}\frac{\partial V_i(z)}{\partial z_a}\chi_r(\lambda_j z) + \frac{\partial \chi_r(\lambda_j z)}{\partial z_a} V_i(z)\right] \frac{\partial \mathcal{E}_a^j}{\partial y_c^j}\left(0, \exp_{\xi_j^0}(\lambda_j z)\right).\end{aligned} \tag{8.118}$$

From Lemma 6.4 in [87], we deduce the Taylor's expansion

$$\frac{\partial \mathcal{E}_a^j}{\partial y_c^j}\left(0, \exp_{\xi_j^0}(\lambda_j z)\right) = \frac{\partial \mathcal{E}_a^j}{\partial y_c^j}\left(0, \exp_{\xi_j^0}(0)\right) + O(\lambda_j^2|z|^2) = \delta_{ac} + O(d_j|\varepsilon||z|^2). \tag{8.119}$$

Lemma 8.13. *Let ∂_m denote ∂_{d_j} or $\partial_{y_c^j}$ for $j = 1,2,\cdots,k$ and $c = 1,\cdots,n$, it holds*

$$\left\|\partial_m Z^i_{\lambda_j,\xi_j(y^j)}\right\|_a = \begin{cases} \left\|\frac{\partial}{\partial y_c^j} Z^i_{\lambda_j,\xi_j(y^j)}\right\|_a = O\left(|\varepsilon|^{-\frac{1}{2}}\right) & \text{if } c = 1,2,\cdots,n; \quad \text{for } \ y=0;\\ \left\|\frac{\partial}{\partial d_l} Z^i_{\lambda_j,\xi_j(y^j)}\right\|_a = 0 & \text{if } l = 1,2,\cdots,k,\ l \neq j;\\ \left\|\frac{\partial}{\partial d_j} Z^i_{\lambda_j,\xi_j(y^j)}\right\|_a = O\left(|\varepsilon|^{\frac{1}{2}}\right). \end{cases} \tag{8.120}$$

Proof. By (8.118) and (8.119), we have

$$\begin{aligned}
&\int_{\mathcal{M}} \left| \nabla_g \left(\frac{\partial}{\partial y_c^j} Z^i_{\lambda_j,\xi_j(y^j)} \right)_{|y=0} \right|^2 d\mu_g = \int\limits_{B(0,\frac{r}{\lambda_j})} \left| g_{\xi_j(y^j)}(\lambda_j z) \right|^{\frac{1}{2}} \sum_{a,b=1}^{n} g^{ab}_{\xi_j(y^j)}(\lambda_j z) \\
&\times \frac{\partial}{\partial z_a} \left\{ \sum_{d=1}^{n} \left[\frac{1}{\lambda_j} \frac{\partial V_i(z)}{\partial z_d} \chi_r(\lambda_j z) + \frac{\partial \chi_r(\lambda_j z)}{\partial z_d} V_i(z) \right] \frac{\partial \mathcal{E}^j_d}{\partial y^j_c} \left(y^j, \exp_{\exp_{\xi_j^0}(y^j)}(\lambda_j z) \right) \right\} \\
&\times \frac{\partial}{\partial z_b} \left\{ \sum_{d=1}^{n} \left[\frac{1}{\lambda_j} \frac{\partial V_i(z)}{\partial z_d} \chi_r(\lambda_j z) + \frac{\partial \chi_r(\lambda_j z)}{\partial z_d} V_i(z) \right] \frac{\partial \mathcal{E}^j_d}{\partial y^j_c} \left(y^j, \exp_{\exp_{\xi_j^0}(y^j)}(\lambda_j z) \right) \right\} dz \\
= \; & O\left(\frac{1}{\lambda_j^2} \right) = O\left(\frac{1}{|\varepsilon|} \right). \qquad (8.121)
\end{aligned}$$

And

$$\begin{aligned}
&\int_{\mathcal{M}} a(x) \left(\frac{\partial}{\partial y_c^j} Z^i_{\lambda_j,\xi_j(y^j)} \right)^2_{\big|_{y=0}} d\mu_g \\
= \; & \int\limits_{B(0,\frac{r}{\lambda_j})} \left| g_{\xi_j(y^j)}(\lambda_j z) \right|^{\frac{1}{2}} a \left(\exp_{\exp_{\xi_j^0}(y^j)}(\lambda_j z) \right) \\
&\times \left\{ \sum_{a=1}^{n} \left[\frac{1}{\lambda_j} \frac{\partial V_i(z)}{\partial z_a} \chi_r(\lambda_j z) + \frac{\partial \chi_r(\lambda_j z)}{\partial z_a} V_i(z) \right] \frac{\partial \mathcal{E}^j_a}{\partial y^j_c} \left(y^j, \exp_{\exp_{\xi_j^0}(y^j)}(\lambda_j z) \right) \right\}^2 dz \\
= \; & O\left(\frac{1}{\lambda_j^2} \right) = O\left(\frac{1}{|\varepsilon|} \right). \qquad (8.122)
\end{aligned}$$

By (8.121) and (8.122) we can get the first estimate holds.

From (8.20), we have $\left\| \frac{\partial}{\partial d_l} Z^i_{\lambda_j,\xi_j(y^j)} \right\|_a = 0$ for $l, j = 1, 2, \cdots, k$, $l \neq j$. Moreover

$$\begin{aligned}
\frac{\partial}{\partial d_j} Z^i_{\lambda_j,\xi_j(y^j)} &= \frac{\partial}{\partial \lambda_j} \left\{ \chi_r \left(\exp^{-1}_{\xi_j}(x) \right) \lambda_j^{\frac{2-n}{2}} V_i \left(\lambda_j^{-1} \exp^{-1}_{\xi_j}(x) \right) \right\} \frac{\partial \lambda_j}{\partial d_j} \\
&= \frac{|\varepsilon|}{2\lambda_j} \chi_r \left(\exp^{-1}_{\xi_j}(x) \right) \left\{ \frac{2-n}{2} \lambda_j^{\frac{-n}{2}} V_i \left(\lambda_j^{-1} \exp^{-1}_{\xi_j}(x) \right) \right. \\
&\quad \left. + \lambda_j^{\frac{2-n}{2}} \frac{\partial}{\partial \lambda_j} V_i \left(\lambda_j^{-1} \exp^{-1}_{\xi_j}(x) \right) \right\}. \qquad (8.123)
\end{aligned}$$

It yields that

$$\left\| \frac{\partial}{\partial d_j} Z^i_{\lambda_j,\xi_j(y^j)} \right\|^2_a$$

$$
\begin{aligned}
= & \int\limits_{B(0,r/\lambda_j)} \left|g_{\xi_j(y^j)}(\lambda_j z)\right|^{\frac{1}{2}} \sum_{a,b=1}^{n} g^{ab}_{\xi_j(y^j)}(\lambda_j z) \\
& \quad \times \frac{\partial}{\partial z_a}\left\{\frac{|\varepsilon|}{2\lambda_j}\chi_r(\lambda_j z)\left[\frac{2-n}{2}V_i(z)+\frac{\partial}{\partial \lambda_j}V_i(z)\right]\right\} \\
& \quad \times \frac{\partial}{\partial z_b}\left\{\frac{|\varepsilon|}{2\lambda_j}\chi_r(\lambda_j z)\left[\frac{2-n}{2}V_i(z)+\frac{\partial}{\partial \lambda_j}V_i(z)\right]\right\} dz \\
& + \int\limits_{B(0,r/\lambda_j)} \left|g_{\xi_j(y^j)}(\lambda_j z)\right|^{\frac{1}{2}} a\left(\exp_{\exp_{\xi_j^0}(y^j)}(\lambda_j z)\right) \\
& \quad \times \left\{\frac{|\varepsilon|}{2\lambda_j}\chi_r(\lambda_j z)\left[\frac{2-n}{2}V_i(z)+\frac{\partial}{\partial \lambda_j}V_i(z)\right]\right\}^2 dz \\
= & \left(\frac{|\varepsilon|}{2\lambda_j}\right)^2 \int\limits_{B(0,r/\lambda_j)} \left|g_{\xi_j(y^j)}(\lambda_j z)\right|^{\frac{1}{2}} \sum_{a,b=1}^{n} g^{ab}_{\xi_j(y^j)}(\lambda_j z) \\
& \times\Big\{\frac{\partial}{\partial z_a}\chi_r(\lambda_j z)\left[\frac{2-n}{2}V_i(z)+\frac{\partial}{\partial \lambda_j}V_i(z)\right] \\
& \quad +\chi_r(\lambda_j z)\frac{\partial}{\partial z_a}\left[\frac{2-n}{2}V_i(z)+\frac{\partial}{\partial \lambda_j}V_i(z)\right]\Big\} \\
& \times\Big\{\frac{\partial}{\partial z_b}\chi_r(\lambda_j z)\left[\frac{2-n}{2}V_i(z)+\frac{\partial}{\partial \lambda_j}V_i(z)\right] \\
& \quad +\chi_r(\lambda_j z)\frac{\partial}{\partial z_b}\left[\frac{2-n}{2}V_i(z)+\frac{\partial}{\partial \lambda_j}V_i(z)\right]\Big\} dz \\
& + \left(\frac{|\varepsilon|}{2\lambda_j}\right)^2 \int\limits_{B(0,r/\lambda_j)} \left|g_{\xi_j(y^j)}(\lambda_j z)\right|^{\frac{1}{2}} a\left(\exp_{\xi_j(y^j)}(\lambda_j z)\right) \\
& \quad \times\left\{\chi_r(\lambda_j z)\left[\frac{2-n}{2}V_i(z)+\frac{\partial}{\partial \lambda_j}V_i(z)\right]\right\}^2 dz \\
\leq & \; C\left(\frac{2-n}{2}\right)^2\left(\frac{|\varepsilon|}{2\lambda_j}\right)^2 \int\limits_{B(0,r/\lambda_j)} \sum_{a,b=1}^{n}\left(V_i(z)+\frac{\partial}{\partial z_a}V_i(z)\right)\left(V_i(z)+\frac{\partial}{\partial z_b}V_i(z)\right) dz \\
& + C\left(\frac{2-n}{2}\right)^2\left(\frac{|\varepsilon|}{2\lambda_j}\right)^2 \int\limits_{B(0,r/\lambda_j)} a\left(\exp_{\xi_j(y^j)}(\lambda_j z)\right) V_i^2 \, dz \\
= & \; O(|\varepsilon|).
\end{aligned} \tag{8.124}
$$

□

Lemma 8.14. *For $i,h=0,1,\cdots,n$ and $j,l=1,2,\cdots k$, it holds*

$$
\left\langle Z^i_{\lambda_j,\xi_j(y^j)}, Z^h_{\lambda_l,\xi_l(y^l)}\right\rangle_a = \delta_{jl}\delta_{ih}\left\|V_i\right\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)} + o(1), \quad \text{as} \;\; \varepsilon\to 0. \tag{8.125}
$$

Moreover for $c = 1, 2, \cdots, n$, if ε is small, we have

$$\left\langle Z^i_{\lambda_j,\xi_j(y^j)}, \left(\frac{\partial}{\partial y^l_c}\sum_{l=1}^k W_{\lambda_l,\xi_l(y^l)}\right)_{|y=0}\right\rangle_a = \frac{1}{\lambda_j}\delta_{jl}\delta_{ic}\left\|V_i\right\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)} + o(1), \tag{8.126}$$

and

$$\left\langle Z^i_{\lambda_j,\xi_j(y^j)}, \frac{\partial}{\partial d_j}V_{\Lambda_\varepsilon(\bar{d}_\varepsilon),\bar{\xi}}\right\rangle_a = \frac{1}{2d_j}\delta_{i0}\left\|V_0\right\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)} + o\left(\left\|Z^0_{\lambda_j,\xi_j(y^j)}\right\|_a\right). \tag{8.127}$$

Proof. By the similar way as proof of (8.47), we can get (8.125) holds.

Next, we prove (8.126). From (8.116) and (8.119), set $x = \exp_{\xi_l(y^l)}(\lambda_l z)$, we have

$$\begin{aligned}
\left(\frac{\partial}{\partial y^l_c}W_{\lambda_l,\xi_l(y^l)}\right)_{|y=0} &= \lambda_l^{\frac{2-n}{2}}\sum_{a=1}^n\left\{\frac{1}{\lambda_l}\frac{\partial U(z)}{\partial z_a}\chi_r(\lambda_l z) + \frac{\partial \chi_r(\lambda_l z)}{\partial z_a}U(z)\right\}\left(\delta_{ac} + O(\lambda_l^2|z|^2)\right) \\
&= \frac{1}{\lambda_l}\lambda_l^{\frac{2-n}{2}}\chi_r(\lambda_l z)\frac{\partial U(z)}{\partial z_c} + \lambda_l^{\frac{2-n}{2}}\frac{\partial \chi_r(\lambda_l z)}{\partial z_c}U(z) + O(\lambda_l^2|z|^2) \\
&= \frac{1}{\lambda_l}Z^c_{\lambda_l,\xi_l} + o(|\lambda_l|^2) = \frac{1}{\lambda_l}Z^c_{\lambda_l,\xi_l} + o(|\varepsilon|).
\end{aligned} \tag{8.128}$$

Then, from (8.125) we have

$$\begin{aligned}
&\left\langle Z^i_{\lambda_j,\xi_j(y^j)}, \left(\frac{\partial}{\partial y^l_c}\sum_{l=1}^k W_{\lambda_l,\xi_l(y^l)}\right)_{|y=0}\right\rangle_a = \left\langle Z^i_{\lambda_j,\xi_j(y^j)}, \sum_{l=1}^k\frac{1}{\lambda_l}Z^c_{\lambda_l,\xi_l} + o(|\varepsilon|)\right\rangle_a \\
= \;& \frac{1}{\lambda_l}\left\langle Z^i_{\lambda_j,\xi_j(y^j)}, Z^c_{\lambda_l,\xi_l}\right\rangle_a + o(|\varepsilon|)\|Z^i_{\lambda_j,\xi_j(y^j)}\|_a = \frac{1}{\lambda_j}\delta_{jl}\delta_{ic}\left\|V_i\right\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)} + o(1)
\end{aligned} \tag{8.129}$$

for $\varepsilon \to 0$.

Finally, we prove (8.127) holds. From (8.23), a straightforward computation establishes that

$$\frac{\partial}{\partial d_j}V_{\Lambda_\varepsilon(\bar{d}_\varepsilon),\bar{\xi}} = \frac{1}{2d_j}Z^0_{\lambda_j,\xi_j}. \tag{8.130}$$

Then, by (8.125) and (8.130), for ε small enough, we have

$$\begin{aligned}
&\left\langle Z^i_{\lambda_j,\xi_j(y^j)}, \frac{\partial}{\partial d_j}V_{\Lambda_\varepsilon(\bar{d}_\varepsilon),\bar{\xi}}\right\rangle_a = \left\langle Z^i_{\lambda_j,\xi_j(y^j)}, \frac{1}{2d_j}Z^0_{\lambda_j,\xi_j(y^j)}\right\rangle_a \\
= \;& \frac{1}{2d_j}\left\langle Z^i_{\lambda_j,\xi_j(y^j)}, Z^0_{\lambda_j,\xi_j(y^j)}\right\rangle_a = \frac{1}{2d_j}\delta_{i0}\left\|V_0\right\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)} + o\left(\left\|Z^0_{\lambda_j,\xi_j(y^j)}\right\|_a\right).
\end{aligned}$$

□

Lemma 8.15. *For $(\Lambda_\varepsilon(\bar{d}), \bar{\xi}) \in \mathcal{O}_{\eta,\rho}$, for ε is small enough, we have*

(i) $\left|f'_\varepsilon\left(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}\right)\right|_{\frac{n}{2}} = O(1)$.

(ii) $\left|f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\right|_{\frac{2ns_\varepsilon}{2n-(n-6)s_\varepsilon}} = O(1)$.

(iii) $\left|V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}\right|_{s_\varepsilon} = O(1)$, for $n \geq 7$ or $\varepsilon < 0$.

(iv) $\left|f''_\varepsilon\left(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}\right)\right|_{\frac{s_\varepsilon}{s_\varepsilon-3}} = O(1)$ for $n \geq 7$ or $\varepsilon < 0$.

Proof. (i)

$$
\begin{aligned}
&\left|f'_\varepsilon\left(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}\right)\right|_{\frac{n}{2}}^{\frac{n}{2}} = \int_{\mathcal{M}} \left|f'_\varepsilon\left(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}\right)\right|^{\frac{n}{2}} \, d\mu_g \\
= \;& (2^*-1-\varepsilon)^{\frac{n}{2}} \int_{\mathcal{M}} \left|\sum_{j=1}^k W_{\lambda_j,\xi_j}\right|^{(2^*-2-\varepsilon)\frac{n}{2}} d\mu_g \\
= \;& (2^*-1-\varepsilon)^{\frac{n}{2}} \sum_{j=1}^k \lambda_j^n \int\limits_{B(0,r/\lambda_j)} \left|\chi_r(\lambda_j z)\lambda_j^{\frac{2-n}{2}} U(z)\right|^{(2^*-2-\varepsilon)\frac{n}{2}} \left|g_{\xi_j}(\lambda_j z)\right|^{\frac{1}{2}} \, dz \\
\leq \;& C(2^*-1-\varepsilon)^{\frac{n}{2}} \sum_{j=1}^k \lambda_j^{\frac{n(2-n)}{4}(2^*-2-\varepsilon)+n} \int\limits_{B(0,r/\lambda_j)} |U(z)|^{(2^*-2-\varepsilon)\frac{n}{2}} \, dz \\
\leq \;& C(2^*-1-\varepsilon)^{\frac{n}{2}} \sum_{j=1}^k \lambda_j^{\frac{n(n-2)}{4}\varepsilon} (n(n-2))^{\frac{n}{2}-\frac{n(n-2)}{8}\varepsilon} \omega_n \int_0^{r/\lambda_j} t^{-1+\frac{n(n-2)}{4}\varepsilon} \, dt \\
= \;& O(1).
\end{aligned}
$$

(ii) Since

$$
\begin{aligned}
&\left|f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\right|_{\frac{2ns_\varepsilon}{2n-(n-6)s_\varepsilon}} = \left|f'_\varepsilon\left(\sum_{j=1}^k W_{\lambda_j,\xi_j}\right)\right|_{\frac{2ns_\varepsilon}{2n-(n-6)s_\varepsilon}} \\
= \;& \left|f'_\varepsilon\left(\sum_{j=1}^k \chi_r(\exp_{\xi_j}^{-1}(x))\lambda_j^{\frac{2-n}{2}} U(\lambda_j^{-1}\exp_{\xi_j}^{-1}(x))\right)\right|_{\frac{2ns_\varepsilon}{2n-(n-6)s_\varepsilon}} \\
\leq \;& C\lambda_{j\alpha}^{\frac{2-n}{2}(2^*-2-\varepsilon)+n} \left(\int_0^{r\lambda_{j\alpha}^{-1}} |U(z)|^{(2^*-2-\varepsilon)\frac{2ns_\varepsilon}{2n-(n-6)s_\varepsilon}} \, dz\right)^{\frac{2n-(n-6)s_\varepsilon}{2ns_\varepsilon}}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\lambda_{j\alpha}^{(n-2)(1+\frac{\varepsilon}{2})}\left(\int_0^{r\lambda_{j\alpha}^{-1}} t^{(-4+(n-2)\varepsilon)\theta+n-1}\,dz\right)^{\frac{1}{\theta}}\\
&= O(1) \quad \text{as} \quad \varepsilon\to 0,
\end{aligned}
$$

because $(-4+(n-2)\varepsilon)\,\theta+n-1<0$, where $\theta=\frac{2ns_\varepsilon}{2n-(n-6)s_\varepsilon}$.

(iii) We have

$$
\begin{aligned}
\left|V_{\Lambda_\varepsilon(\bar d),\bar\xi}\right|_{s_\varepsilon}^{s_\varepsilon} &= \int_{\mathcal{M}}\left|V_{\Lambda_\varepsilon(\bar d),\bar\xi}\right|^{s_\varepsilon}\,d\mu_g=\int_{\mathcal{M}}\left|\sum_{j=1}^k W_{\lambda_j,\xi_j}\right|^{s_\varepsilon}\,d\mu_g\\
&= \sum_{j=1}^k\int_{B_g(\xi_j,r)}\left|\chi_r(\exp_{\xi_j}^{-1}(x))\lambda_j^{\frac{2-n}{2}}U(\lambda_j^{-1}\exp_{\xi_j}^{-1}(x))\right|^{s_\varepsilon}\,d\mu_g\\
&= \sum_{j=1}^k\lambda_j^{\frac{2-n}{2}s_\varepsilon+n}\int\limits_{B(0,r/\lambda_j)}|\chi_r(\lambda_j z)U(z)|^{s_\varepsilon}\left|g_{\xi_j}(\lambda_j z)\right|^{\frac12}\,dz\\
&\le \sum_{j=1}^k\lambda_j^{\frac{2-n}{2}s_\varepsilon+n}\int\limits_{B(0,r/\lambda_j)}|U(z)|^{s_\varepsilon}\,dz\\
&\le (n(n-2))^{\frac{n-2}{4}s_\varepsilon}\sum_{j=1}^k\lambda_j^{\frac{n(n-2)}{4}\varepsilon}\int_0^{\lambda_j^{-1}r}t^{-n-1+\frac{n(n-2)\varepsilon}{2}}\,dt\\
&= O(1).
\end{aligned}
$$

(iv) We have

$$
\begin{aligned}
&\left|f_\varepsilon''\left(V_{\Lambda_\varepsilon(\bar d),\bar\xi}\right)\right|_{\frac{s_\varepsilon}{s_\varepsilon-3}}^{\frac{s_\varepsilon}{s_\varepsilon-3}}\\
&= [(2^*-1-\varepsilon)(2^*-2-\varepsilon)]^{\frac{s_\varepsilon}{s_\varepsilon-3}}\int_{\mathcal{M}}\left|V_{\Lambda_\varepsilon(\bar d),\bar\xi}\right|^{(2^*-3-\varepsilon)\frac{s_\varepsilon}{s_\varepsilon-3}}\,d\mu_g\\
&\le [(2^*-1-\varepsilon)(2^*-2-\varepsilon)]^{\frac{s_\varepsilon}{s_\varepsilon-3}}\sum_{j=1}^k\int\limits_{B_g(\xi_j,r)}\left|W_{\lambda_j,\xi_j}\right|^{(2^*-3-\varepsilon)\frac{s_\varepsilon}{s_\varepsilon-3}}\,d\mu_g\\
&\le C\sum_{j=1}^k\lambda_j^{\frac{2-n}{2}(2^*-3-\varepsilon)\frac{s_\varepsilon}{s_\varepsilon-3}+n}\int\limits_{B(0,r/\lambda_j)}|U(z)|^{(2^*-3-\varepsilon)\frac{s_\varepsilon}{s_\varepsilon-3}}\left|g_{\xi_j}(\lambda_j z)\right|^{\frac12}\,dz\\
&\le C\int_0^{r/\lambda_j}t^{n-1-(n-2)(2^*-3-\varepsilon)\frac{s_\varepsilon}{s_\varepsilon-3}}\,dt=C\int_0^{r/\lambda_j}t^{\frac{-(n-6)(n+1)+n(n-2)(3n-13)}{n(n-2)\varepsilon+2(n-6)}}\,dt\\
&= O(1).
\end{aligned}
$$

□

Lemma 8.16. *([69]) For any $a > 0$, b real, we have*

$$\left||a+b|^\beta - a^\beta\right| \le \begin{cases} C(\beta)\min\{|b|^\beta, a^{\beta-1}|b|\} & \text{if } 0 < \beta < 1; \\ C(\beta)\left(a^{\beta-1}|b| + |b|^\beta\right) & \text{if } \beta \ge 1. \end{cases} \tag{8.131}$$

In particular, we get for any $\phi \in \mathcal{H}_\varepsilon$, we have

$$\left|f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}} + \phi) - f'_\varepsilon(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}})\right| \le \begin{cases} C|\phi|^{2^*-2-\varepsilon} & \text{if } n = 6 \text{ and } \varepsilon > 0; \\ C\left(V_{\Lambda_\varepsilon(\bar{d}),\bar{\xi}}^{2^*-3-\varepsilon}|\phi| + |\phi|^{2^*-2-\varepsilon}\right) & \text{otherwise.} \end{cases} \tag{8.132}$$

Chapter 9

Blow-up Solutions for Paneitz-Branson type equations with critical growth

[1]

9.1 Introduction

In 1983 Paneitz [102] introduced a conformally fourth order operator defined on 4-dimensional Riemannian manifolds. Branson [14] generalized the definition to $n-$dimensional Riemannian manifolds.

We let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$. We also let $H_2^2(M)$ be the Sobolev space consisting of functions in $L^2(M)$ with two derivatives in $L^2(M)$. We consider the geometric Paneitz equation

$$P_g^n u = |u|^{2^\sharp - 2} u \quad \text{in } M. \tag{9.1}$$

Here $2^\sharp = \frac{2n}{n-4}$ is the critical exponent for the Sobolev embedding, P_g^n is the Paneitz-Branson operator which is given by

$$P_g^n u = \Delta_g^2 u - \operatorname{div}_g (A_g du) + \frac{n-4}{2} Q_g u \tag{9.2}$$

where $\Delta_g = -\operatorname{div}_g \nabla$ is the Laplace-Beltrami operator, Q_g is the $Q-$curvature of g, A_g is the

[1]The main result of this chapter was worked with Angela Pistoia, was published in *Asymptotic Analysis*, Volume 73(4), 2011, 225-248.

smooth symmetrical $(2,0)-$tensor field

$$A_g = \frac{(n-2)^2+4}{2(n-1)(n-2)} S_g g - \frac{4}{n-2} Rc_g, \tag{9.3}$$

where Rc_g and S_g are respectively the Ricci curvature and the Scalar curvature of g.

The Paneitz operator is conformally invariant in the sense that if $\tilde{g} = \phi^{\frac{4}{n-2}} g$ is conformal to g then $P^n_{\tilde{g}} u = \phi^{-\frac{n+4}{n-4}} P^n_g(\phi u)$ for any $u \in C^\infty(M)$. From the viewpoint of conformal geometry equation (9.1) turns out to be the natural fourth order analogue of the second order Yamabe problem. That is why we are led to study extensions to this operator of some classical problems.

Using a terminology introduced by Hebey, we refer to a *Paneitz-Branson type operator with general coefficients* as an operator of the form

$$P_g u = \Delta_g^2 u - \operatorname{div}_g(A du) + au \tag{9.4}$$

where $A \in \Lambda^\infty_{(2,0)}(M)$ is a smooth symmetric $(2,0)-$tensor field and $a \in C^\infty(M)$ and we refer to *Paneitz-Branson type operator with constant coefficients* as an operator of the form

$$P_g u = \Delta_g^2 u + b\Delta_g u + cu \tag{9.5}$$

where b and c are real numbers.

The Paneitz-Branson operator (9.2) is as in (9.4) whatever (M,g) is. In particular, when (M,g) is Einstein, i.e. $Rc_g = \lambda g$ for some $\lambda \in \mathbb{R}$, the Paneitz-Branson operator (9.2) has constant coefficients as in (9.5) with $b = \frac{n^2-2n-4}{2(n-1)}\lambda$ and $c = \frac{n(n-4)(n^2-4)}{16(n-1)^2}\lambda^2$.

Equation

$$P_g u = \Delta_g^2 u + b\Delta_g u + cu = |u|^{2^\sharp-2} u \quad \text{in } M, \tag{9.6}$$

when P_g is a Paneitz-Branson type operator with constant coefficients as in (9.5), was widely studied. Examples of compact manifolds including locally conformally flat manifold for which equations (9.6) have non constant solutions are in [42, 51]. Compactness of problem (9.6) was studied in [61, 62, 63, 64]. Recently, in [63] Hebey and Robert also studied the stability of problem (9.6). They introduce the following definition of stability. Equation (9.6) is said to be *stable* if for any sequences $(b_\alpha)_\alpha$ and $(c_\alpha)_\alpha$ of real numbers converging to b and c and for any sequence $(u_\alpha)_\alpha$ of solutions to

$$\Delta_g^2 u_\alpha + b_\alpha \Delta_g u_\alpha + c_\alpha u_\alpha = |u_\alpha|^{2^\sharp-2} u_\alpha \quad \text{in } M,$$

bounded in $H_2^2(M)$, there holds that, up to a subsequence, $u_\alpha \to u$ in $C^4(M)$ where u is a smooth solution of (9.6). In other words, problem (9.6) is stable if arbitrary bounded sequences in $H_2^2(M)$ of solutions of equations close to (9.6) do not blow up in one or more points of the manifold. In particular, they prove that if (M,g) is locally conformally flat and the Paneitz-Branson type operator is coercive then problem (9.6) is stable provided $b \neq \frac{1}{n} Tr_g A_g$ if $n \geq 9$ or $n = 7$ and $b < \frac{1}{8} Tr_g A_g$ if $n = 8$. Here and in what follows, if A denotes a smooth $(2,0)-$tensor field, we let $Tr_g A = g^{ij} A_{ij}$ be the trace of A with respect to g. It is easily seen that if A_g is defined in (9.3) then

$$Tr_g A_g = \frac{n^2 - 2n - 4}{2(n-1)} S_g. \tag{9.7}$$

As far as we know, a few results are known about problem

$$P_g u = \Delta_g^2 u - \operatorname{div}_g (A du) + au = |u|^{2^\sharp - 2} u \quad \text{in } M, \tag{9.8}$$

when P_g is a Paneitz-Branson type operator with general coefficients as in (9.4). In [50] among other existence results, Esposito and Robert proved that problem (9.8) when $n \geq 8$ has a non constant solution provided $\min_M Tr_g (A - A_g) < 0$. In [108] Sandeep proved that problem (9.8) is stable provided $A - A_g$ is either positive or negative definite. We would like to point out that in the quoted results the quantity $Tr_g A$ plays a crucial role in studying existence of solutions and stability of problems (9.6) and (9.8).

The aim of the present paper is to show how stability of the problem (9.8) actually depends on the trace of A_g. In particular, by building blowing-up solutions of the slightly subcritical problem (9.9), we will show that problem (9.8) is not stable if $\max_M Tr_g (A - A_g) > 0$ and $n \geq 8$ or if $\min_M Tr_g (A - A_g) > 0$ and $n \geq 7$.

More precisely, we consider the following Paneitz-Branson type equation with slightly subcritical growth

$$\Delta_g^2 u - \operatorname{div}_g ((A_g + B) du) + au = |u|^{2^\sharp - 2 - \varepsilon} u, \text{ in } M, \tag{9.9}$$

where A_g is given in (9.3), $B \in \Lambda^\infty_{(2,0)}(M)$ is a smooth symmetric $(2,0)-$tensor field, $a \in C^\infty(M)$ and ε is a small positive parameter.

Let $P_{g,B}(u) := \Delta_g^2 u - \operatorname{div}_g ((A_g + B) u) + au$. We will assume that $P_{g,B}$ is coercive, i.e. there exists $c > 0$ such that

$$\int_M (P_{g,B} u) u d\mu_g \geq c \int_M u^2 d\mu_g \quad \text{for any } u \in H_2^2(M).$$

Coercivity was studied in [61].

Given a C^1-function φ on M, we say that a critical point ξ_0 of φ is $\mathcal{C}^1-$stable if there exists an open neighborhood Ω of ξ_0 such that for any point $\xi \in \overline{\Omega}$ there holds $\nabla \varphi(\xi) = 0$ if and only if $\xi = \xi_0$ and such that the Brouwer degree

$$\deg (\nabla_g \varphi, \Omega, 0) \neq 0.$$

If ξ_0 is a strict local minimum point or a strict local maximum point of φ then ξ_0 is a $\mathcal{C}^1-$stable critical point of φ. Moreover, if φ is a $\mathcal{C}^2-$function on M, then any non degenerate critical point of φ is $\mathcal{C}^1-$stable.

Theorem 9.1. *Assume*

- $n \geq 8$ *and* ξ_0 *is a* $\mathcal{C}^1-$*stable critical point of* $Tr_g B$ *with* $Tr_g B(\xi_0) > 0$,
- $n \geq 7$, $Tr_g B$ *is not constant and* $\min\limits_M Tr_g B > 0$.

Then there exists $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *equation (8.1) admits a solution* u_ε *such that the family* $(u_\varepsilon)_\varepsilon$ *is bounded in* $H_2^2(M)$ *and the* u_ε'*s blow up at the point* ξ_0 *if* $n \geq 8$ *or at a global minimum point of* $Tr_g B$ *if* $n = 7$, *as* ε *goes to zero.*

In particular, as far as it concerns the stability of equation (9.8), we can extend the definition given by Hebey and Robert in [63]. We say that equation (9.8) is *stable* if for any sequences $(\varepsilon_\alpha)_\alpha$ of positive real numbers converging to zero and for any sequence $(u_\alpha)_\alpha$ of solutions to

$$\Delta_g^2 u_\alpha - \mathrm{div}_g\left((A_g + B)u_\alpha\right) + a u_\alpha = |u_\alpha|^{2^\sharp - 2 - \varepsilon_\alpha} u_\alpha \quad \text{in } M,$$

bounded in $H_2^2(M)$, there holds that, up to a subsequence, $u_\alpha \to u$ in $C^4(M)$ where u is a smooth solution of (9.9) with $\epsilon = 0$.

Therefore, Theorem 9.1 immediately implies the following stability result.

Corollary 9.2. *Assume*

- $n \geq 8$ *and* ξ_0 *is a* $\mathcal{C}^1-$*stable critical point of* $Tr_g B$ *with* $Tr_g B(\xi_0) > 0$,
- $n \geq 7$, $Tr_g B$ *is not constant and* $\min\limits_M Tr_g B > 0$.

Then (9.9) *with* $\epsilon = 0$ *is not stable.*

The proof of our result relies on a very well known Liapunov-Schmidt reduction procedure, introduced in [9, 52]. We use Liapunov-Schmidt reduction method to reduce the problem to a finite dimensional one, with some reduced energy. Then, the solutions in Theorems 9.1 turn out to be generated by critical points of the reduced energy functionals. In particular, we follow some ideas recently developed in [90], where the authors studied the Yamabe type equation with slightly subcritical growth

$$\Delta_g u + \left(\frac{n-2}{4(n-1)} S_g + h\right) u = u^{2^* - 1 - \varepsilon}, \; u > 0, \text{ in } M,$$

where $2^* = \frac{2n}{n-2}$ is the critical exponent for the Sobolev embedding, $h \in C^\infty(M)$ and ε is a small positive parameter.

This chapter is organized as follows. In Section 9.2, we describe the proof of the main result. Section 9.3 is devoted to perform the finite dimensional reduction. Section 9.4 contains the asymptotic expansion of the reduced energy.

9.2 The existence result

Let $H_2^2(M)$ be the standard Sobolev space defined as the completion of $C^\infty(M)$ with respect to the norm

$$\|u\|_{H_2^2(M)} = \left(\int_M (\Delta_g u)^2 \, d\mu_g + \int_M |\nabla_g u|^2 \, d\mu_g + \int_M u^2 \, d\mu_g \right)^{1/2}.$$

Let $L^q(M)$ be the Banach space equipped with the standard norm

$$|u|_q = \left(\int_M |u|^q \, d\mu_g \right)^{1/q},$$

The Sobolev embedding theorem asserts that $H_2^2(M)$ is continuously embedded in $L^q(M)$ for $1 < q \leq 2^\sharp$, and this embedding is compact when $q < 2^\sharp$.

Since $P_{g,B}$ is assumed to be coercive, we can provide the Hilbert space $H_2^2(M)$ with the inner product

$$\langle u, v \rangle = \int_M P_{g,B}(u) v \, d\mu_g = \int_M \left[\Delta_g u \Delta_g v + (A_g + B)(\nabla_g u, \nabla_g v) + auv \right] \, d\mu_g,$$

which induces the norm equivalent to the standard one

$$\|u\| = \left(\int_M \left[(\Delta_g u)^2 + (A_g + B) |\nabla_g u|^2 + a u^2 \right] \, d\mu_g \right)^{1/2}.$$

It will be useful to rewrite equation (9.9) in a differential setting, we introduce the following operator.

Definition 9.3. *Let $i^* : L^{\frac{2n}{n+4}}(M) \to H_2^2(M)$ be the adjoint operator of the embedding $i : H_2^2(M) \to L^{2^\sharp}(M)$, namely*

$$i^*(w) = u \Leftrightarrow \langle u, v \rangle = \int_M uv \, d\mu_g, \ \forall v \in H_2^2(M) \Leftrightarrow P_{g,B}(u) = w \ \text{ on } M, \quad u \in H_2^2(M). \tag{9.10}$$

By the continuity of the embedding $H_2^2(M)$ into $L^{2^\sharp}(M)$, we get

$$\|i^*(w)\| \leq C |w|_{2n/(n+4)} \quad \text{for any } w \in L^{\frac{2n}{n+4}}(M), \tag{9.11}$$

where C is a positive constant independent of w.

We can rewrite equation (9.9) in the equivalent way

$$u = i^*\left(f_\varepsilon(u) \right), \quad u \in H_2^2(M), \tag{9.12}$$

where $f_\varepsilon(s) := |s|^{2^\sharp-2-\varepsilon}s$.

Let δ be a positive number less or equal than the injectivity radius of M and χ be a smooth cut-off function such that $0 \le \chi \le 1$ in $\mathbb{R}^n$, $\chi \equiv 1$ in $B(0,\delta)$ and $\chi \equiv 0$ out of $B(0,2\delta)$, $|\nabla\chi(z)| \le \frac{2}{\delta}$ and $|\nabla^2\chi(z)| \le \frac{2}{\delta^2}$.

For any point ξ in M and for any positive real number λ, we define the function $W_{\lambda,\xi}$ on M by

$$W_{\lambda,\xi}(x) := \begin{cases} \chi\left(\exp_\xi^{-1}(x)\right)\lambda^{\frac{4-n}{2}}U\left(\lambda^{-1}\exp_\xi^{-1}(x)\right) & \text{if } x \in B_g(\xi,2\delta); \\ 0 & \text{otherwise,} \end{cases} \tag{9.13}$$

where

$$U(z) = \alpha_n\left(\frac{1}{1+|z|^2}\right)^{\frac{n-4}{2}}, \quad \text{with } \alpha_n = \big(n(n-4)(n^2-4)\big)^{\frac{n-4}{8}}. \tag{9.14}$$

In particular, the functions $\lambda^{\frac{4-n}{2}}U(\lambda^{-1}z)$ satisfy the following equation (see [80])

$$\Delta^2 U = U^{2^\sharp-1} \text{ in } \mathbb{R}^n, \quad u \in D^{2,2}(\mathbb{R}^n)$$

where $\Delta = -\mathrm{div}(\nabla)$ is the Laplace-Beltrami operator in $\mathbb{R}^n$ associated with the Euclidean metric.

We will look for a solution to (9.12), or equivalently to (9.9) of the form

$$u_\varepsilon = W_{\lambda_\varepsilon(t),\xi} + \phi_{\varepsilon,\lambda,\xi} \qquad \text{with } \lambda_\varepsilon(t) = \sqrt{\varepsilon t},\ t>0, \text{and } \xi \in M \tag{9.15}$$

where the functions $W_{\lambda_\varepsilon(t),\xi}$ are defined in (9.13), and the rest term $\phi_{\varepsilon,\lambda,\xi}$ belongs to the space $K^\perp_{\lambda_\varepsilon(t),\xi}$ defined as follows.

It is known that (see [82]) every solution of the linear equation

$$\Delta^2 v = (2^\sharp-1)U^{2^\sharp-2}v \text{ in } \mathbb{R}^n, \quad v \in D^{2,2}(\mathbb{R}^n) \tag{9.16}$$

is a linear combination of the functions

$$V_0(z) = \frac{d\left(\lambda^{(4-n)/2}U(\lambda^{-1}z)\right)}{d\lambda}\Big|_{\lambda=1} = \alpha_n\frac{n-4}{2}\frac{|z|^2-1}{(1+|z|^2)^{(n-2)/2}}, \tag{9.17}$$

and

$$V_i(z) = -\frac{\partial U}{\partial z_i}(z) = \alpha_n(n-4)\frac{z_i}{(1+|z|^2)^{(n-2)/2}} \qquad \text{for } i = 1,2,\cdots,n. \tag{9.18}$$

Let us define on M the functions

$$Z^i_{\lambda,\xi}(x) := \begin{cases} \chi\left(\exp_\xi^{-1}(x)\right)\lambda^{\frac{4-n}{2}}V_i\left(\lambda^{-1}\exp_\xi^{-1}(x)\right) & \text{if } x \in B_g(\xi,2\delta); \\ 0 & \text{otherwise,} \end{cases} \tag{9.19}$$

for $i = 0, 1, 2, \cdots, n$. We also define the projections $\Pi_{\lambda,\xi}$ and $\Pi^{\perp}_{\lambda,\xi}$ of the Sobolev space $H^2_2(M)$ onto the respective subspaces $K_{\lambda,\xi} = \text{Span}\left\{Z^i_{\lambda,\xi} :\ i = 0, 1, 2, \cdots, n\right\}$ and $K^{\perp}_{\lambda,\xi} = \left\{\phi \in H^2_2(M) :\ \langle \phi, Z^i_{\lambda,\xi}\rangle = 0,\ \forall\, i = 0, 1, 2, \cdots, n\right\}$.

Finally, in order to solve problem (9.12) we will solve the system

$$\Pi^{\perp}_{\lambda_\varepsilon(t),\xi}\left\{W_{\lambda_\varepsilon(t),\xi} + \phi - i^*\left[f_\varepsilon\left(W_{\lambda_\varepsilon(t),\xi} + \phi\right)\right]\right\} = 0, \tag{9.20}$$

$$\Pi_{\lambda_\varepsilon(t),\xi}\left\{W_{\lambda_\varepsilon(t),\xi} + \phi - i^*\left[f_\varepsilon\left(W_{\lambda_\varepsilon(t),\xi} + \phi\right)\right]\right\} = 0. \tag{9.21}$$

Equation (9.20) is solved in the following Proposition, whose proof is postponed to Section 3.

Proposition 9.4. *If $n \geq 7$ and $\lambda_\varepsilon(t)$ is as in (9.15), then for any real numbers c_1 and c_2 satisfying $0 < c_1 < c_2$, such that for ε small, for any point ξ in M, and for any real number t in $[c_1, c_2]$, equation (9.20) admits a unique solution $\phi_{\varepsilon,\lambda,\xi}$ in $K^{\perp}_{\lambda_\varepsilon(t),\xi}$, which is continuously differential with respect to ξ and t, moreover,*

$$\|\phi_{\varepsilon,\lambda,\xi}\| \leq C\begin{cases} \varepsilon^{\frac{3}{4}} & \text{if} \quad n = 7; \\ \varepsilon|\ln\varepsilon| & \text{if} \quad n \geq 8. \end{cases} \tag{9.22}$$

where C is a positive constant dependent on c_1 and c_2.

Then, we introduce the functional $J_\varepsilon : H^2_2(M) \to \mathbb{R}$ defined by

$$J_\varepsilon(u) = \frac{1}{2}\int_M P_{g,B}(u)u\, d\mu_g - \frac{1}{2^\sharp - \varepsilon}\int_M u^{2^\sharp-\varepsilon}\, d\mu_g,$$

whose critical points are solutions to equation (9.9). We also define the functional $\widetilde{J}_\varepsilon : \mathbb{R}^*_+ \times M \to \mathbb{R}$ by

$$\widetilde{J}_\varepsilon(t, \xi) = J_\varepsilon\left(W_{\lambda_\varepsilon(t),\xi} + \phi_{\varepsilon,\lambda,\xi}\right), \tag{9.23}$$

where $W_{\lambda_\varepsilon(t),\xi}$ is defined as (9.13) and $\phi_{\varepsilon,\lambda,\xi}$ is given by Proposition 9.4.

The next result, whose proof is postponed until Section 4, allows to solve equation (9.21), by reducing the problem to a finite dimensional one.

Proposition 9.5. *(i) If $n \geq 7$ and $\lambda_\varepsilon(t)$ is as in (9.15), there holds*

$$J_\varepsilon(W_{\lambda_\varepsilon(t),\xi}) = \frac{2}{n}K_n^{-\frac{n}{4}}\left\{1 - C_n\varepsilon - \frac{(n-4)^2}{16}\varepsilon\ln(\varepsilon t) + \frac{(n-1)}{(n-6)(n^2-4)}Tr_g B(\xi)\varepsilon t + o(\varepsilon)\right\} \tag{9.24}$$

as $\varepsilon \to 0$, $\mathcal{C}^1-$uniformly with respect to ξ in M and t in compact subsets of $\mathbb{R}^*_+$. Here

$$C_n = 2^{n-4}(n-4)^2\frac{\omega_{n-1}}{\omega_n}\int_0^{+\infty}\frac{r^{\frac{n-2}{2}}\ln(1+r)}{(1+r)^n}dr + \frac{(n-4)^2}{8(n-2)}\left(1 - n\ln\sqrt{n(n-4)}\right).$$

(ii) If $n \geq 8$ and $\lambda_\varepsilon(t)$ is as in (9.15), there holds

$$\tilde{J}_\varepsilon(t,\xi) = J_\varepsilon(W_{\lambda_\varepsilon(t),\xi}) + o(\varepsilon) \tag{9.25}$$

as $\varepsilon \to 0$, C^1-uniformly with respect to ξ in M and t in compact subsets of $\mathbb{R}^*_+$. If $n = 7$ estimate (9.25) holds only C^0-uniformly with respect to ξ in M and t in compact subsets of $\mathbb{R}^*_+$.

(iii) For ε small, if (t,ξ) is a critical point of the functional $\widetilde{J}_\varepsilon$, then $W_{\lambda_\varepsilon(t),\xi} + \phi_{\varepsilon,\lambda,\xi}$ is a solution of (9.12), or equivalently of equation (9.9).

Proof of Theorem 9.1: By (i) and (ii) of Proposition 9.5, we have

$$\tilde{J}_\varepsilon(t,\xi) = C_1 - C_2\varepsilon - C_3\varepsilon \ln(\varepsilon t) + C_4 Tr_g B(\xi)\varepsilon t + o(\varepsilon) \text{ as } \varepsilon \to 0.$$

Here C_1, C_2, C_3, C_4 are positive constants which only depend on n. We define the functional $\tilde{J} : \mathbb{R}^*_+ \times M \to \mathbb{R}$ by

$$\tilde{J}(t,\xi) = -C_3 \ln t + C_4 Tr_g B(\xi) t.$$

If $n \geq 8$ we argue exactly as in the proof of Theorem 1.1 in [90]. If $n \geq 7$ and $Tr_g B$ has a strict global minimum point with $\min_M Tr_g B > 0$, then the function $\tilde{J}$ has a global minimum point which is stable under C^0-perturbation, which easily implies the existence for ε small enough of a critical point $(t_\varepsilon, \xi_\varepsilon)$ of the function $\tilde{J}_\varepsilon$ such that ξ_ε approaches the minimum set of $Tr_g B$ as ε goes to zero. The claim follows by (iii) of Proposition 9.5.

9.3 The finite dimensional reduction

This section is devoted to the proof of Proposition 9.4. Let us introduce the linear operator $L_{\varepsilon,\lambda,\xi} : K^\perp_{\lambda,\xi} \to K^\perp_{\lambda,\xi}$ defined by

$$L_{\varepsilon,\lambda,\xi}(\phi) := \Pi^\perp_{\lambda_\varepsilon(t),\xi}\left\{\phi - i^*\left[f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\phi\right]\right\}.$$

This operator is well defined because of (9.11). Therefore equation (9.20) turns out to be equivalent to

$$L_{\varepsilon,\lambda,\xi}(\phi) = N_{\varepsilon,\lambda,\xi}(\phi) + R_{\varepsilon,\lambda,\xi}, \tag{9.26}$$

where

$$N_{\varepsilon,\lambda,\xi}(\phi) = \Pi^\perp_{\lambda_\varepsilon(t),\xi}\left\{i^*\left[f_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi}) - f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\phi\right]\right\}, \tag{9.27}$$

and

$$R_{\varepsilon,\lambda,\xi} = \Pi^\perp_{\lambda_\varepsilon(t),\xi}\left\{i^*\left(f_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\right) - W_{\lambda_\varepsilon(t),\xi}\right\}. \tag{9.28}$$

As a first step, we want to study the invertibility of $L_{\varepsilon,\lambda,\xi}$.

Lemma 9.6. *If $\lambda_\varepsilon(t)$ is as in (9.15), then for any real numbers c_1 and c_2 satisfying $0 < c_1 < c_2$, there exists a positive constant C dependent on c_1 and c_2 such that for ε small, for any point ξ in M, any real number t in $[c_1, c_2]$, and any function $\phi \in K^{\perp}_{\lambda_\varepsilon(t),\xi}$, there holds*

$$\|L_{\varepsilon,\lambda,\xi}(\phi)\| \geq C\|\phi\|. \tag{9.29}$$

Proof. We argue by contradiction. Assume there exist a sequences of $(\varepsilon_\alpha)_\alpha$ converging to 0, a sequence of points $(\xi_\alpha)_\alpha$ in M, a sequence of real numbers $(t_\alpha)_\alpha$ in $[c_1, c_2]$, and a sequence of functions $(\phi_\alpha)_\alpha \in K^{\perp}_{\lambda_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$ satisfying

$$L_{\varepsilon_\alpha,\lambda_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}(\phi_\alpha) = \psi_\alpha, \quad \|\phi_\alpha\| = 1 \quad \text{and} \quad \|\psi_\alpha\| \to 0. \tag{9.30}$$

For any α, for notation's convenience we will write $\lambda_\alpha = \lambda_{\varepsilon_\alpha}(t_\alpha)$. From (9.30) we get there exists $\zeta_\alpha \in K_{\lambda_\alpha,\xi_\alpha}$ such that

$$\phi_\alpha - i^*\left[f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha})\phi_\alpha\right] = \psi_\alpha + \zeta_\alpha. \tag{9.31}$$

We set $g_\alpha(z) = \exp^*_{\xi_\alpha} g(\lambda_\alpha z)$.

Step 1. We claim that

$$\|\zeta_\alpha\| \to 0 \quad \text{as } \alpha \to \infty. \tag{9.32}$$

Let $\zeta_\alpha := \sum_{i=0}^{n} C^i_\alpha Z^i_{\lambda_\alpha,\xi_\alpha}$. For any $j = 0, 1, \cdots, n$, we multiply (9.31) by $Z^j_{\lambda_\alpha,\xi_\alpha}$, and taking into account that $\phi_\alpha, \psi_\alpha \in K^{\perp}_{\lambda_\alpha,\xi_\alpha}$, we get

$$\sum_{i=0}^{n} C^i_\alpha \left\langle Z^i_{\lambda_\alpha,\xi_\alpha}, Z^j_{\lambda_\alpha,\xi_\alpha}\right\rangle = -\left\langle i^*\left[f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha})\phi_\alpha\right], Z^j_{\lambda_\alpha,\xi_\alpha}\right\rangle. \tag{9.33}$$

For $i, j = 0, 1, \cdots, n$ and any α, we have

$$\begin{aligned}
\left\langle Z^i_{\lambda_\alpha,\xi_\alpha}, Z^j_{\lambda_\alpha,\xi_\alpha}\right\rangle &= \int_M P_{g,B}(Z^i_{\lambda_\alpha,\xi_\alpha}) Z^j_{\lambda_\alpha,\xi_\alpha}\, d\mu_g \\
&= \int_M \Delta_g Z^i_{\lambda_\alpha,\xi_\alpha} \Delta_g Z^j_{\lambda_\alpha,\xi_\alpha}\, d\mu_g \\
&\quad + \int_M (A_g + B)\left(\nabla_g Z^i_{\lambda_\alpha,\xi_\alpha}, \nabla_g Z^j_{\lambda_\alpha,\xi_\alpha}\right)\, d\mu_g \\
&\quad + \int_M a Z^i_{\lambda_\alpha,\xi_\alpha} Z^j_{\lambda_\alpha,\xi_\alpha}\, d\mu_g \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{9.34}$$

By (9.55) we have

$$I_1 = \int_M \Delta_g Z^i_{\lambda_\alpha,\xi_\alpha} \Delta_g Z^j_{\lambda_\alpha,\xi_\alpha}\, d\mu_g$$

$$\begin{aligned}
&= \lambda_\alpha^{4-n}\int_{B_g(\xi_\alpha,2\delta)} \Delta_g\left(\chi\left(\exp_{\xi_\alpha}^{-1}(x)\right)V_i\left(\lambda_\alpha^{-1}\exp_{\xi_\alpha}^{-1}(x)\right)\right)\\
&\qquad \times\Delta_g\left(\chi\left(\exp_{\xi_\alpha}^{-1}(x)\right)V_j\left(\lambda_\alpha^{-1}\exp_{\xi_\alpha}^{-1}(x)\right)\right)\, d\mu_g\\
&= \int_{B(0,2\lambda_\alpha^{-1}\delta)}\left(\sum_{a,b,c=1}^n\left(g_\alpha^{ab}(z)\frac{\partial^2\left(\chi(\lambda_\alpha z)V_i(z)\right)}{\partial z_a\partial z_b}-\lambda_\alpha\Gamma_{ab}^c(\lambda_\alpha z)\frac{\partial\left(\chi(\lambda_\alpha z)V_i(z)\right)}{\partial z_c}\right)\right)\\
&\quad\times\left(\sum_{a,b,c=1}^n\left(g_\alpha^{ab}(z)\frac{\partial^2\left(\chi(\lambda_\alpha z)V_j(z)\right)}{\partial z_a\partial z_b}-\lambda_\alpha\Gamma_{ab}^c(\lambda_\alpha z)\frac{\partial\left(\chi(\lambda_\alpha z)V_j(z)\right)}{\partial z_c}\right)\right)|g_\alpha(z)|^{\frac12}\, dz,\\
&\to \begin{cases}\int_{\mathbb{R}^n}\Delta^2 V_i\, dz & \text{if } \ i=j;\\ 0 & \text{if } \ i\neq j\end{cases} \quad \text{as } \alpha\to+\infty.
\end{aligned} \tag{9.35}$$

Moreover, setting $(A_g+B)_\alpha(z)=(A_g+B)(\exp_{\xi_\alpha}(\lambda_\alpha z))$, we have

$$\begin{aligned}
I_2 &= \int_M (A_g+B)\left(\nabla_g Z^i_{\lambda_\alpha,\xi_\alpha},\nabla_g Z^j_{\lambda_\alpha,\xi_\alpha}\right)\, d\mu_g\\
&= \lambda_\alpha^{4-n}\int_{B_g(\xi_\alpha,2\delta)}(A_g+B)\left(\nabla_g\left(\chi\left(\exp_{\xi_\alpha}^{-1}(x)\right)V_i\left(\lambda_\alpha^{-1}\exp_{\xi_\alpha}^{-1}(x)\right)\right),\right.\\
&\qquad\left.\nabla_g\left(\chi\left(\exp_{\xi_\alpha}^{-1}(x)\right)V_j\left(\lambda_\alpha^{-1}\exp_{\xi_\alpha}^{-1}(x)\right)\right)\right)\, d\mu_g\\
&= \lambda_\alpha^4\int_{B(0,2\lambda_\alpha^{-1}\delta)}\sum_{a,b=1}^n\left((A_g+B)_\alpha(z)\right)_{st}g_\alpha^{sa}g_\alpha^{tb}(z)\left(\frac{1}{\lambda_\alpha}\frac{\partial V_i(z)}{\partial z_a}\chi(\lambda_\alpha z)+\frac{\partial\chi(\lambda_\alpha z)}{\partial z_a}V_i(z)\right)\\
&\qquad\times\left(\frac{1}{\lambda_\alpha}\frac{\partial V_j(z)}{\partial z_b}\chi(\lambda_\alpha z)+\frac{\partial\chi(\lambda_\alpha z)}{\partial z_b}V_j(z)\right)|g_\alpha(z)|^{\frac12}\, dz\\
&\to 0 \text{ as } \alpha\to+\infty,
\end{aligned} \tag{9.36}$$

and setting $a_\alpha(z)=a(\exp_{\xi_\alpha}(\lambda_\alpha z))$ we also have

$$\begin{aligned}
I_3 &= \int_M aZ^i_{\lambda_\alpha,\xi_\alpha}Z^j_{\lambda_\alpha,\xi_\alpha}\, d\mu_g\\
&= \lambda_\alpha^{4-n}\int_{B_g(\xi_\alpha,2\delta)}a\chi^2\left(\exp_{\xi_\alpha}^{-1}(x)\right)V_i\left(\lambda_\alpha^{-1}\exp_{\xi_\alpha}^{-1}(x)\right)V_j\left(\lambda_\alpha^{-1}\exp_{\xi_\alpha}^{-1}(x)\right)\, d\mu_g\\
&= \lambda_\alpha^4\int_{B(0,2\lambda_\alpha^{-1}\delta)}a_\alpha\chi^2(\lambda_\alpha z)V_i(z)V_j(z)\,|g_\alpha(z)|^{\frac12}\, dz\\
&\to 0 \text{ as } \alpha\to+\infty.
\end{aligned} \tag{9.37}$$

Then from (9.34)- (9.37) we have

$$\left\langle Z^i_{\lambda_\alpha,\xi_\alpha},Z^j_{\lambda_\alpha,\xi_\alpha}\right\rangle \to \begin{cases}\int_{\mathbb{R}^n}\Delta^2V_i\, dz & \text{if } \ i=j;\\ 0 & \text{if } \ i\neq j.\end{cases} \tag{9.38}$$

Now, set

$$\widetilde{\phi}_\alpha(z):=\begin{cases}\lambda_\alpha^{(n-4)/2}\phi_\alpha\left(\exp_{\xi_\alpha}(\lambda_\alpha z)\right) & \text{if } z\in B(0,2\lambda_\alpha^{-1}\delta),\\ 0 & \text{otherwise.}\end{cases}$$

By (9.30), we deduce that the sequence $\{\widetilde{\phi}_\alpha\}_\alpha$ is bounded in $\mathcal{D}^{2,2}(\mathbb{R}^n)$, where $\mathcal{D}^{2,2}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\|_{\mathcal{D}^{2,2}(\mathbb{R}^n)} = \|\Delta u\|_{L^2(\mathbb{R}^n)}$. Passing to a subsequence, we may assume that $\{\widetilde{\phi}_\alpha\}_\alpha$ converges weakly to a function $\tilde{\phi}$ in $\mathcal{D}^{2,2}(\mathbb{R}^n)$, and thus in $L^{2^\sharp}(\mathbb{R}^n)$ by the continuity of the embedding of $\mathcal{D}^{2,2}(\mathbb{R}^n)$ into $L^{2^\sharp}(\mathbb{R}^n)$.

Since, for any α, the function $\phi_\alpha \in K^\perp_{\lambda_\alpha,\xi_\alpha}$, we have

$$\begin{aligned} 0 &= \left\langle Z^j_{\lambda_\alpha,\xi_\alpha}, \phi_\alpha \right\rangle = \int_M P_{g,B}(Z^j_{\lambda_\alpha,\xi_\alpha})\phi_\alpha \, d\mu_g \\ &= \int_M \Delta_g Z^j_{\lambda_\alpha,\xi_\alpha} \Delta_g \phi_\alpha + (A_g + B)\left(\nabla_g Z^j_{\lambda_\alpha,\xi_\alpha}, \nabla_g \phi_\alpha\right) \, d\mu_g \\ &\quad + \int_M a Z^j_{\lambda_\alpha,\xi_\alpha} \phi_\alpha \, d\mu_g \\ &= \int_{\mathbb{R}^n} \Delta V_j \Delta \tilde{\phi} \, dz + o(1) \quad \text{as } \alpha \to \infty, \end{aligned} \tag{9.39}$$

Since the function V_j solves (9.16), it yields that

$$\int_{\mathbb{R}^n} \Delta V_j \Delta \tilde{\phi} \, dz = (2^\sharp - 1) \int_{\mathbb{R}^n} U^{2^\sharp - 2} V_j \tilde{\phi} \, dz = 0. \tag{9.40}$$

Moreover, we have

$$\begin{aligned} &\left\langle i^* \left[f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha})\phi_\alpha \right], Z^j_{\lambda_\alpha,\xi_\alpha} \right\rangle \\ = &\int_M f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha}) Z^j_{\lambda_\alpha,\xi_\alpha} \phi_\alpha d\mu_g \\ = &\int_M f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha}) \chi(\exp^{-1}_{\xi_\alpha}(x)) \lambda_\alpha^{\frac{4-n}{2}} V_j(\lambda_\alpha^{-1} \exp^{-1}_{\xi_\alpha}(x)) \phi_\alpha d\mu_g \\ = &\lambda_\alpha^4 \int_{B(0,2\lambda_\alpha^{-1}\delta)} f'_{\varepsilon_\alpha}\left(\chi(\lambda_\alpha z)\lambda_\alpha^{\frac{4-n}{2}} U(z)\right) \chi(\lambda_\alpha z) V_j(z) \widetilde{\phi}_\alpha(z) \, d\mu_{g_\alpha} \\ = &(2^\sharp - 1 - \varepsilon_\alpha)\lambda_\alpha^{\frac{(n-4)\varepsilon_\alpha}{2}} \int_{B(0,2\lambda_\alpha^{-1}\delta)} [\chi(\lambda_\alpha z) U(z)]^{2^\sharp - 2 - \varepsilon_\alpha} \chi(\lambda_\alpha z) V_h(z) \widetilde{\phi}_\alpha(z) d\mu_{g_\alpha} \\ \to &(2^\sharp - 1) \int_{\mathbb{R}^n} U(z)^{2^\sharp - 2} V_j(z) \tilde{\phi}(z) \, dz = 0 \text{ as } \alpha \to +\infty, \end{aligned} \tag{9.41}$$

because $\lambda_\alpha^{\frac{(n-4)\varepsilon_\alpha}{2}} = (\varepsilon_\alpha t)^{\frac{(n-4)\varepsilon_\alpha}{4}} \to 1$, the sequence $\{\widetilde{\phi}_\alpha\}_\alpha$ converges weakly to $\tilde{\phi}$ in $\mathcal{D}^{2,2}(\mathbb{R}^n)$ and (9.40) holds. It follows from (9.33), (9.38) and (9.41) that for any $i = 0, 1, \cdots, n$ $C^i_\alpha \to 0$ as $\alpha \to \infty$ and, so (9.32) is proved.

Step 2. We prove that

$$\liminf_{\alpha\to\infty} \int_M f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha}) u_\alpha^2 \, d\mu_g \to 1, \tag{9.42}$$

where

$$u_\alpha = \phi_\alpha - \psi_\alpha - \zeta_\alpha, \quad \text{with} \quad \|u_\alpha\| \to 1. \tag{9.43}$$

Let us write equation (9.31) as

$$P_{g,B}(u_\alpha) = f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha})u_\alpha + f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha})(\psi_\alpha + \zeta_\alpha). \tag{9.44}$$

If we multiply (9.44) by u_α, we get

$$\|u_\alpha\|^2 = \int_M f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha})u_\alpha^2 \, d\mu_g + \int_M f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha})(\psi_\alpha + \zeta_\alpha)u_\alpha \, d\mu_g \tag{9.45}$$

By Hölder inequality, from (9.11), (9.30) and (9.32), then we have

$$\begin{aligned} & \left| \int_M f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha})(\psi_\alpha + \zeta_\alpha)u_\alpha \, d\mu_g \right| \\ \leq \; & \left| f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha}) \right|_{n/4} |\psi_\alpha + \zeta_\alpha|_{2n/n-4} |u_\alpha|_{2n/n-4} = o(1). \end{aligned} \tag{9.46}$$

Therefore, from (9.43), (9.45) and (9.46) (9.42) follows.

Step 3. Set

$$\tilde{u}_\alpha(z) := \begin{cases} \lambda_\alpha^{(n-4)/2} u_\alpha\left(\exp_{\xi_\alpha}(\lambda_\alpha z)\right) & \text{if } z \in B(0, 2\lambda_\alpha^{-1}\delta); \\ 0 & \text{otherwise.} \end{cases} \tag{9.47}$$

We claim that

$$\tilde{u}_\alpha \rightharpoonup 0 \text{ weakly in } \mathcal{D}^{2,2}(\mathbb{R}^n) \text{ and strongly in } L^q(\mathbb{R}^n) \text{ for any } q \in [1, \tfrac{2n}{n-4}). \tag{9.48}$$

In fact, by (9.43) we get that $(\tilde{u}_\alpha)_\alpha$ are bounded in $\mathcal{D}^{2,2}(\mathbb{R}^n)$. Then, up to a subsequence, $\tilde{u}_\alpha \rightharpoonup \tilde{u}$ weakly in $\mathcal{D}^{2,2}(\mathbb{R}^n)$ and strongly in $L^q(\mathbb{R}^n)$ for any $q \in [1, \frac{2n}{n-4})$. By (9.44) we easily deduce that $\tilde{u}$ solves the linearized problem (9.16) and by (9.40) we also deduce that the function $\tilde{u}$ is identically zero and (9.48) holds.

Therefore, we have that

$$\liminf_{\alpha\to\infty} \int_M f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha})u_\alpha^2 \, d\mu_g \to 0. \tag{9.49}$$

In fact, by (9.48) we deduce that the sequence $\left(f'_{\varepsilon_\alpha}(W_{\lambda_\alpha,\xi_\alpha})\right)_\alpha$ converges strongly to $f'_0(U)$ in $L^{\frac{n}{4}}(\mathbb{R}^n)$ and by the fact that the functions $\tilde{u}_\alpha^2$ are uniformly bounded in $L^{\frac{n}{n-4}}(\mathbb{R}^n)$ and converge almost everywhere to zero in $\mathbb{R}^n$ we deduce that they converge weakly to zero in $L^{\frac{n}{n-4}}(\mathbb{R}^n)$.

Finally a contradiction arises, because of (9.42) and (9.49).

That concludes the proof. □

Next, we want to study the estimate the term of $R_{\varepsilon,\lambda,\xi}$.

Lemma 9.7. *If $\lambda_\varepsilon(t)$ is as in (9.15), then for any real numbers c_1 and c_2 satisfying $0 < c_1 < c_2$, there exists a positive constant C dependent on c_1 and c_2 such that for ε small, for any point ξ in M, any real number t in $[c_1, c_2]$, there holds*

$$\|R_{\varepsilon,\lambda,\xi}\| \leq C \begin{cases} \varepsilon^{\frac{3}{4}} & \text{if } \; n = 7; \\ \varepsilon|\ln \varepsilon| & \text{if } \; n \geq 8. \end{cases} \tag{9.50}$$

Proof. Let us introduce the function $Z_{\lambda,\xi}$ defined by $W_{\lambda,\xi} := i^*(Z_{\lambda,\xi})$, i.e. $P_{g,B}(W_{\lambda,\xi}) = Z_{\lambda,\xi}$ on M.

By (9.28) and (9.11), we get there exists a positive constant C such that for ε small, for any point ξ in M and any positive real number $t \in [c_1, c_2]$, there holds,

$$\|R_{\varepsilon,\lambda,\xi}\| \leq C \left| f_\varepsilon(W_{\lambda,\xi}) - P_{g,B}(W_{\lambda,\xi}) \right|_{\frac{2n}{n+4}} .$$

The claim will follow once we prove that

$$\left| f_\varepsilon(W_{\lambda,\xi}) - P_{g,B}(W_{\lambda,\xi}) \right|_{\frac{2n}{n+4}} \leq C \begin{cases} \varepsilon^{\frac{3}{4}} & \text{if } \; n = 7; \\ \varepsilon|\ln \varepsilon| & \text{if } \; n \geq 8. \end{cases} \tag{9.51}$$

We have

$$\begin{aligned} \left| f_\varepsilon(W_{\lambda,\xi}) - P_{g,B}(W_{\lambda,\xi}) \right|_{\frac{2n}{n+4}} \leq & \left| f_\varepsilon(W_{\lambda,\xi}) - f_0(W_{\lambda,\xi}) \right|_{\frac{2n}{n+4}} \\ & + \left| f_0(W_{\lambda,\xi}) - P_{g,B}(W_{\lambda,\xi}) \right|_{\frac{2n}{n+4}} . \end{aligned} \tag{9.52}$$

Let us estimate the first term of the right hand side of (9.52). A change of variable yields (setting $\chi_{\varepsilon,\lambda} = \chi(\lambda_\varepsilon(t) z)$)

$$\begin{aligned} & \left| f_\varepsilon(W_{\lambda,\xi}) - f_0(W_{\lambda,\xi}) \right|_{\frac{2n}{n+4}} \\ \leq \; & C \lambda_\varepsilon(t)^{\frac{n-4}{2}\varepsilon} \left\| \chi_{\varepsilon,\lambda}^{2^\sharp-1-\varepsilon} \left(U^{2^\sharp-1-\varepsilon} - U^{2^\sharp-1} \right) \right\|_{L^{\frac{2n}{n+4}}(\mathbb{R}^n)} \\ & + C \left\| \left(\lambda_\varepsilon(t)^{\frac{n-4}{2}\varepsilon} \chi_{\varepsilon,\lambda}^{2^\sharp-1-\varepsilon} - \chi_{\varepsilon,\lambda}^{2^\sharp-1} \right) U^{2^\sharp-1} \right\|_{L^{\frac{2n}{n+4}}(\mathbb{R}^n)} \\ = \; & O\left(|\varepsilon \ln \varepsilon| \right), \end{aligned} \tag{9.53}$$

because

$$\begin{aligned} & \int_{\mathbb{R}^n} \left| \chi_{\varepsilon,\lambda}^{2^\sharp-1-\varepsilon} \left(U^{2^\sharp-1-\varepsilon} - U^{2^\sharp-1} \right) \right|^{\frac{2n}{n+4}} \, dz \\ = \; & O\left(\int_{B(0, 2\lambda_\varepsilon(t)^{-1}\delta)} \left| U^{2^\sharp-1-\varepsilon} - U^{2^\sharp-1} \right|^{\frac{2n}{n+4}} \, dz \right) \end{aligned}$$

$$
\begin{aligned}
&= O\left(\int_0^{\frac{2\delta}{\lambda_\varepsilon(t)}} \frac{s^{n-1}}{(1+s^2)^n} \left|(1+s^2)^{\frac{n-4}{2}\varepsilon}-1\right|^{\frac{2n}{n+4}} dz\right)\\
&= O\left(\int_0^{\frac{2\delta}{\lambda_\varepsilon(t)}} \frac{s^{n-1}}{(1+s^2)^n} \left|\frac{n-4}{2}\varepsilon\ln(1+s^2)\right|^{\frac{2n}{n+4}} dz\right)\\
&= O\left(\varepsilon^{\frac{2n}{n+4}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\left|\left(\lambda_\varepsilon(t)^{\frac{n-4}{2}\varepsilon}\chi_{\varepsilon,\lambda}^{2^\sharp-1-\varepsilon}-\chi_{\varepsilon,\lambda}^{2^\sharp-1}\right)U^{2^\sharp-1}\right|^{\frac{2n}{n+4}} dz\\
= \; & O\left(|\varepsilon\ln\lambda_\varepsilon(t)|^{\frac{2n}{n+4}}\int_0^{\frac{\delta}{2\lambda_\varepsilon(t)}}\frac{s^{n-1}}{(1+s^2)^n}dz+\int_{\frac{\delta}{2\lambda_\varepsilon(t)}}^{+\infty}\frac{s^{n-1}}{(1+s^2)^n}dz\right)\\
= \; & O\left(|\varepsilon\ln\lambda_\varepsilon(t)|^{\frac{2n}{n+4}}+\lambda_\varepsilon(t)^n\right)\\
= \; & O\left(|\varepsilon\ln\varepsilon|^{\frac{2n}{n+4}}\right).
\end{aligned}
$$

Let us estimate the second term of the right hand side of (9.52).

We claim that

$$
P_{g,B}u = \Delta^2 u + \mathcal{R}u, \quad \mathcal{R}u = O\left(|u| + |\partial_k u| + |\partial^2_{ik}u| + |x||\partial^3_{ijk}u| + |x|^2|\partial^4_{ijlk}u|\right). \tag{9.54}
$$

In fact, by standard properties of the exponential map, in geodesic normal coordinates, there hold

$$
\Delta_g u = -\Delta u + a^{ij}\partial^2_{ij}u + b^k\partial_k u, \tag{9.55}
$$

$$
a^{ij}(x) := -\left[g^{ij}(x) - \delta^{ij}(x)\right] = \frac{1}{3}R_{i\alpha\beta j}(\xi)x^\alpha x^\beta + O(|x|^3), \tag{9.56}
$$

and

$$
b^k(x) := g^{ij}(x)\Gamma^k_{ij}(x) = \partial_l\Gamma^k_{ii}(\xi)x^l + O(|x|^2). \tag{9.57}
$$

By (9.55) we can compute

$$
\begin{aligned}
&\Delta_g\left[-\Delta u + a^{rs}\partial^2_{rs}u + b^h\partial_h u\right]\\
&= -\Delta\left[-\Delta u + a^{rs}\partial^2_{rs}u + b^h\partial_h u\right] + a^{ij}\partial^2_{ij}\left[-\Delta u + a^{rs}\partial^2_{rs}u + b^h\partial_h u\right]\\
&+b^k\partial_k\left[-\Delta u + a^{rs}\partial^2_{rs}u + b^h\partial_h u\right]
\end{aligned}
$$

and by (9.56) and (9.57), using the definition of $P_{g,B}$, estimate (9.54) follows.

We perform a change of variable $x = \exp_\xi^{-1}(z)$ and set $U_\lambda(x) := \lambda^{-\frac{n-4}{2}} U(x/\lambda)$. By (9.54), taking into account that $\Delta^2 U_\lambda = f_0(U_\lambda)$ we deduce that

$$P_{g,B}(W_{\lambda,\xi}) - f_0(W_{\lambda,\xi}) = \mathcal{R} U_\lambda + r, \tag{9.58}$$

where $r := P_{g,B} g\left[(\chi - 1)(W_{\lambda,\xi})\right] - (\chi^p - \chi) f_0(W_{\lambda,\xi})$.

We also point out that for some positive constant c we have

$$|\partial_k U| \le c \frac{1}{(1+|x|^2)^{\frac{n-3}{2}}}, \quad |\partial^2_{ik} U| + |x||\partial^3_{ijk} U| + |x|^2 |\partial^4_{ijlk} U| \le c \frac{1}{(1+|x|^2)^{\frac{n-2}{2}}}.$$

Therefore, by (9.54) we deduce

$$\begin{aligned} \|\mathcal{R} U_\lambda\|_{\frac{2n}{n+4}} &= O\left(\left\| |U_\lambda| + |\partial_k U_\lambda| + |\partial^2_{ik} U_\lambda| + |x||\partial^3_{ijk} U_\lambda| + |x|^2 |\partial^4_{ijlk} U_\lambda| \right\|_{\frac{2n}{n+4}} \right) \\ &= \begin{cases} O\left(\lambda^2\right) & \text{if } n \ge 9, \\ O\left(\lambda^2 |\ln \lambda|\right) & \text{if } n = 8, \\ O\left(\lambda^{\frac{n-4}{2}}\right) & \text{if } 5 \le n \le 7, \end{cases} \end{aligned} \tag{9.59}$$

because

$$\|U_\lambda\|_{\frac{2n}{n+4}} = \begin{cases} O\left(\lambda^4\right) & \text{if } n \ge 13, \\ O\left(\lambda^4 |\ln \lambda|\right) & \text{if } n = 12, \\ O\left(\lambda^{\frac{n-4}{2}}\right) & \text{if } 5 \le n \le 11, \end{cases}$$

$$\|\partial_k U_\lambda\|_{\frac{2n}{n+4}} = \begin{cases} O\left(\lambda^3\right) & \text{if } n \ge 11, \\ O\left(\lambda^3 |\ln \lambda|\right) & \text{if } n = 10, \\ O\left(\lambda^{\frac{n-4}{2}}\right) & \text{if } 5 \le n \le 9 \end{cases}$$

and

$$\left\| |\partial^2_{ik} U_\lambda| + |x||\partial^3_{ijk} U_\lambda| + |x|^2 |\partial^4_{ijlk} U_\lambda| \right\|_{\frac{2n}{n+4}} = \begin{cases} O\left(\lambda^2\right) & \text{if } n \ge 9, \\ O\left(\lambda^2 |\ln \lambda|\right) & \text{if } n = 8, \\ O\left(\lambda^{\frac{n-4}{2}}\right) & \text{if } 5 \le n \le 7. \end{cases}$$

Is it easy to check that the term

$$\|r\|_{\frac{2n}{n+4}} = o\left(\|\mathcal{R} U_\lambda\|_{\frac{2n}{n+4}} \right). \tag{9.60}$$

Finally, by (9.58), (9.59) and (9.60) we get

$$|f_0(W_{\lambda,\xi}) - P_{g,B}(W_{\lambda,\xi})|_{\frac{2n}{n+4}} = \begin{cases} O(\varepsilon) & \text{if } n \ge 9, \\ O(\varepsilon |\ln \epsilon|) & \text{if } n = 8, \\ O\left(\varepsilon^{\frac{3}{4}}\right) & \text{if } n = 7. \end{cases} \tag{9.61}$$

Finally, by (9.52), (9.53) and (9.61), estimate (9.51) follows. □

Proof of Proposition 9.4: For ε small, for any point ξ in M and any positive real number $t \in [c_1, c_2]$, we let $T_{\varepsilon,\lambda,\xi} : K^{\perp}_{\lambda,\xi} \to K^{\perp}_{\lambda,\xi}$ be defined by

$$T_{\varepsilon,\lambda,\xi}(\phi) = L^{-1}_{\varepsilon,\lambda,\xi}\left(N_{\varepsilon,\lambda,\xi}(\phi) + R_{\varepsilon,\lambda,\xi}\right),$$

where $N_{\varepsilon,\lambda,\xi}(\phi)$ and $R_{\varepsilon,\lambda,\xi}$ are as (9.27) and (9.28). We also set

$$\mathcal{B}_{\varepsilon,\lambda,\xi}(\gamma) = \left\{\phi \in K^{\perp}_{\lambda,\xi} \;\middle|\; \|\phi\| \le \gamma \|R_{\varepsilon,\lambda,\xi}\|\right\},$$

where γ is a positive constant to be chosen large later on. We take $\lambda = \lambda_\varepsilon(t)$ for some real number t in $[c_1, c_2]$. In order to solve (9.20) or equivalently equation (9.26), it suffices to show that the map $T_{\varepsilon,\lambda_\varepsilon(t),\xi}$ admits a fixed point $\phi_{\varepsilon,\lambda,\xi}$.

By Lemma 9.6, we deduce that

$$\left\|T_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi)\right\| \le C\left(\left\|N_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi)\right\| + \left\|R_{\varepsilon,\lambda_\varepsilon(t),\xi}\right\|\right), \tag{9.62}$$

and

$$\left\|T_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi_1) - T_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi_2)\right\| \le C\left(\left\|N_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi_1) - N_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi_2)\right\|\right). \tag{9.63}$$

By (9.11) and (9.27), we deduce

$$\left\|N_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi)\right\| \le C\left|f_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi}) - f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\phi\right|_{\frac{2n}{n+4}}, \tag{9.64}$$

and

$$\begin{aligned} &\left\|N_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi_1) - N_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi_2)\right\| \\ \le\;& C\left|f_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_1) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_2) - f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})(\phi_1 - \phi_2)\right|_{\frac{2n}{n+4}}. \end{aligned} \tag{9.65}$$

Then by the mean value theorem and Hölder and Sobolev inequalities, it follows that, for any $\tau \in (0,1)$, we have

$$\begin{aligned} &\left|f_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_1) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_2) - f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})(\phi_1 - \phi_2)\right|_{\frac{2n}{n+4}} \\ =\;& \left|\left[f'_\varepsilon\left(W_{\lambda_\varepsilon(t),\xi} + \tau\phi_2 + (1-\tau)\phi_1\right) - f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\right](\phi_1 - \phi_2)\right|_{\frac{2n}{n+4}} \\ \le\;& \left|f'_\varepsilon\left(W_{\lambda_\varepsilon(t),\xi} + \tau\phi_2 + (1-\tau)\phi_1\right) - f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\right|_{\frac{n}{4}} |\phi_1 - \phi_2|_{2^\sharp} \\ \le\;& C\left|f'_\varepsilon\left(W_{\lambda_\varepsilon(t),\xi} + \tau\phi_2 + (1-\tau)\phi_1\right) - f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\right|_{\frac{n}{4}} \|\phi_1 - \phi_2\|. \end{aligned}$$

By Lemma 8.16, we deduce that

$$\left|f_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_1) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_2) - f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})(\phi_1 - \phi_2)\right|_{\frac{2n}{n+4}}$$

$$\leq \begin{cases} C\left(|\phi_1|_{2^\sharp}^{2^\sharp-2-\varepsilon}+|\phi_2|_{2^\sharp}^{2^\sharp-2-\varepsilon}\right)\|\phi_1-\phi_2\| & \text{if } n\geq 12;\\ C\left(|W_{\lambda_\varepsilon(t),\xi}|_{2^\sharp}+|\phi_1|_{2^\sharp}+|\phi_2|_{2^\sharp}\right)^{2^\sharp-3-\varepsilon}\left(|\phi_1|_{2^\sharp}+|\phi_2|_{2^\sharp}\right)\|\phi_1-\phi_2\| & \text{if } 5\leq n<12,\end{cases}$$

$$\leq \begin{cases} C\left(\|\phi_1\|^{2^\sharp-2-\varepsilon}+\|\phi_2\|^{2^\sharp-2-\varepsilon}\right)\|\phi_1-\phi_2\| & \text{if } n\geq 12;\\ C\left(|W_{\lambda_\varepsilon(t),\xi}|_{2^\sharp}+\|\phi_1\|+\|\phi_2\|\right)^{2^\sharp-3-\varepsilon}\left(\|\phi_1\|+\|\phi_2\|\right)\|\phi_1-\phi_2\| & \text{if } 5\leq n<12.\end{cases} \tag{9.66}$$

Taking $\phi_1=\phi, \phi_2=0$ into (9.66), from (9.64) we have

$$\left\|N_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi)\right\| \leq \begin{cases} C\|\phi\|^{2^\sharp-1-\varepsilon} & \text{if } n\geq 12;\\ C\left(|W_{\lambda_\varepsilon(t),\xi}|_{2^\sharp}^{2^\sharp-3-\varepsilon}\|\phi\|^2+\|\phi\|^{2^\sharp-1-\varepsilon}\right) & \text{if } 5\leq n<12.\end{cases} \tag{9.67}$$

Since

$$|W_{\lambda_\varepsilon(t),\xi}|_{2^\sharp}^{2^\sharp} = \int_M |W_{\lambda_\varepsilon(t),\xi}|^{2^\sharp}\, d\mu_g = O\left(\int_{B(0,\frac{2\delta}{\lambda_\varepsilon(t)})}\frac{s^{n-1}}{(1+s^2)^n}\, ds\right) = O(1), \tag{9.68}$$

then we have $|W_{\lambda_\varepsilon(t),\xi}|_{2^\sharp}^{2^\sharp-3-\varepsilon}=O(1)$. From (9.62), (9.63), (9.65), (9.66) and (9.67), for any functions ϕ, ϕ_1 and ϕ_2 in $\mathcal{B}_{\varepsilon,\lambda_\varepsilon(t),\xi}(\gamma)$ and for ε small, we have

$$\left\|T_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi)\right\| \leq \begin{cases} C\left(\gamma^{2^\sharp-1-\varepsilon}\|R_{\varepsilon,\lambda_\varepsilon(t),\xi}\|^{2^\sharp-1-\varepsilon}+\left\|R_{\varepsilon,\lambda_\varepsilon(t),\xi}\right\|\right) & \text{if } n\geq 12;\\ C\left(\gamma^2\|R_{\varepsilon,\lambda_\varepsilon(t),\xi}\|^2+\gamma^{2^\sharp-1-\varepsilon}\|R_{\varepsilon,\lambda_\varepsilon(t),\xi}\|^{2^\sharp-1-\varepsilon}+\left\|R_{\varepsilon,\lambda_\varepsilon(t),\xi}\right\|\right) & \text{if } 5\leq n<12,\end{cases}$$

and

$$\left\|T_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi_1)-T_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi_2)\right\| \leq C\gamma^{2^\sharp-2-\varepsilon}\|R_{\varepsilon,\lambda_\varepsilon(t),\xi}\|^{2^\sharp-2-\varepsilon}\|\phi_1-\phi_2\|,$$

where C is a positive constant independent of $\gamma, \varepsilon, \xi, t, \phi, \phi_1$ and ϕ_2. By Lemma 9.7, it follows that if γ is fixed large enough, then for ε small, for any point ξ in M, and any real number t in $[c_1,c_2]$, $T_{\varepsilon,\lambda_\varepsilon(t),\xi}$ is a contraction mapping on $\mathcal{B}_{\varepsilon,\lambda_\varepsilon(t),\xi}(\gamma)$ and satisfies

$$\phi\in\mathcal{B}_{\varepsilon,\lambda_\varepsilon(t),\xi}(\gamma) \implies T_{\varepsilon,\lambda_\varepsilon(t),\xi}(\phi)\in\mathcal{B}_{\varepsilon,\lambda_\varepsilon(t),\xi}(\gamma),$$

therefore $T_{\varepsilon,\lambda_\varepsilon(t),\xi}$ has a fixed point $\phi_{\varepsilon,\lambda,\xi}$ which satisfies (9.20), and (9.22) holds from (9.50). The regularity of the map $(t,\xi)\to\phi_{\varepsilon,\lambda,\xi}$ can be proved by standard arguments involving the implicit function theorem.

9.4 The reduced problem: proof of Proposition 9.5

Let K_n be the sharp constant for the embedding of $\mathcal{D}^{2,2}(\mathbb{R}^n)$ into $L^{2^\sharp}(\mathbb{R}^n)$, i.e.

$$\frac{1}{K_n}=\frac{n(n^2-4)(n-4)\omega_n^{\frac{4}{n}}}{16},$$

where ω_n is the volume of $\mathbb{S}^n$.

Proof of (i) of Proposition 9.5:

Let us prove the C^0-estimate. The C^1-estimate can be proved using the same argument as in [87, 90].

In Section 4 of [50] it was proved that for $n \geq 7$

$$\begin{aligned}
& \frac{1}{2}\int_M P_{g,B}\left(W_{\lambda_\varepsilon(t),\xi}\right) W_{\lambda_\varepsilon(t),\xi}\, d\mu_g \\
= \ & \frac{1}{2}K_n^{-\frac{n}{4}}\left\{1+\left[\frac{4(n-1)}{n(n-6)(n^2-4)}\left(Tr_g A_g(\xi)+Tr_g B(\xi)\right)-\frac{(n^2+4n-20)}{6(n-6)(n^2-4)}\mathrm{Scal}_g(\xi)\right]\lambda_\varepsilon(t)^2\right\} \\
& +o\left(\lambda_\varepsilon(t)^2\right) \\
= \ & \frac{1}{2}K_n^{-\frac{n}{4}}\left\{1+\frac{4(n-1)}{n(n-6)(n^2-4)}\left(\left(Tr_g A_g(\xi)+Tr_g B(\xi)\right)-\frac{n(n^2+4n-20)}{24(n-1)}\mathrm{Scal}_g(\xi)\right)\varepsilon t\right\} \\
& +o\left(\varepsilon\right), \qquad (9.69)
\end{aligned}$$

as $\varepsilon \to 0$, $\mathcal{C}^0-$uniformly with respect to ξ in M and t in compact subsets of $\mathbb{R}_+^*$.

Now, let us estimate the term $\frac{1}{2^\sharp-\varepsilon}\int_M W_{\lambda_\varepsilon(t),\xi}^{2^\sharp-\varepsilon}\, d\mu_g$. It is useful to introduce some notations.

For any positive real numbers p and q satisfying $p-q>1$, we set

$$I_p^q=\int_0^{+\infty}\frac{s^q}{(1+s)^p}\, ds \text{ and } \tilde{I}_p^q=\int_0^{+\infty}\frac{s^q\ln(1+s)}{(1+s)^p}\, ds.$$

As is easily checked (see [8]) there hold

$$I_{p+1}^{q+1}=\frac{q+1}{p-q-1}I_{p+1}^q \text{ and } I_{p+1}^q=\frac{p-q-1}{p}I_p^q.$$

Moreover, we have

$$I_n^{n/2}=\frac{n\omega_n}{2^{n-1}(n-2)\omega_{n-1}}=\frac{2K_n^{-n/4}}{\alpha_n^2(n-2)(n-4)(n^2-4)\omega_{n-1}}.$$

We also recall the Cartan expansion of the metric

$$\sqrt{|g|}(z)=1-\frac{1}{6}R_{ij}z^iz^j+O(|z|^3),$$

where the R_{ij} is the component of the Ricci tensor in the exponential chart and $|g|$ is the determinant of the components of the metric g in geodesic normal coordinates. Moreover, we point out that

$$I_{n-\frac{n-4}{2}\varepsilon}^{\frac{n-2}{2}}=I_n^{\frac{n-2}{2}}+\frac{n-4}{2}\tilde{I}_n^{\frac{n-2}{2}}\varepsilon+O(\varepsilon^2), \text{ and } I_{n-\frac{n-4}{2}\varepsilon}^{\frac{n}{2}}=I_n^{\frac{n}{2}}+\frac{n}{2}\tilde{I}_n^{\frac{n-2}{2}}\varepsilon+O(\varepsilon^2).$$

Finally, using both the previous facts, we compute

$$
\begin{aligned}
& \frac{1}{2^\sharp-\varepsilon}\int_M W_{\lambda_\varepsilon(t),\xi}^{2^\sharp-\varepsilon}\,d\mu_g \\
= & \frac{1}{2^\sharp-\varepsilon}\int_M \left(\chi\left(\exp_\xi^{-1}(x)\right)\lambda_\varepsilon(t)^{\frac{4-n}{2}}U\left(\lambda_\varepsilon(t)^{-1}\exp_\xi^{-1}(x)\right)\right)^{2^\sharp-\varepsilon}\,d\mu_g \\
= & \frac{1}{2^\sharp-\varepsilon}\lambda_\varepsilon(t)^{\frac{n-4}{2}\varepsilon}\int\limits_{B(0,2\delta/\lambda_\varepsilon(t))}(\chi_{\varepsilon,t}(z)U(z))^{2^\sharp-\varepsilon}\sqrt{|g_{\varepsilon,\xi}|}(z)\,dz \\
= & \frac{1}{2^\sharp-\varepsilon}\lambda_\varepsilon(t)^{\frac{n-4}{2}\varepsilon}\int\limits_{B(0,\delta/\lambda_\varepsilon(t))}U(z)^{2^\sharp-\varepsilon}\sqrt{|g_{\varepsilon,\xi}|}(z)\,dz \\
= & \frac{1}{2^\sharp-\varepsilon}\lambda_\varepsilon(t)^{\frac{n-4}{2}\varepsilon}\int\limits_{B(0,\delta/\lambda_\varepsilon(t))}\left(\alpha_n\left(\frac{1}{1+|z|^2}\right)^{\frac{n-4}{2}}\right)^{2^\sharp-\varepsilon}\sqrt{|g_{\varepsilon,\xi}|}(z)\,dz \\
= & \frac{\alpha_n^{2^\sharp-\varepsilon}}{2^\sharp-\varepsilon}\lambda_\varepsilon(t)^{\frac{n-4}{2}\varepsilon}\int\limits_{B(0,\delta/\lambda_\varepsilon(t))}\left(\frac{1}{1+|z|^2}\right)^{n-\frac{n-4}{2}\varepsilon}\sqrt{|g_{\varepsilon,\xi}|}(z)\,dz \\
= & \frac{\alpha_n^{2^\sharp-\varepsilon}}{2^\sharp-\varepsilon}\lambda_\varepsilon(t)^{\frac{n-4}{2}\varepsilon}\frac{\omega_{n-1}}{2}\left(I_{n-\frac{n-4}{2}\varepsilon}^{\frac{n-2}{2}}-\frac{1}{6n}\mathrm{Scal}_g(\xi)I_{n-\frac{n-4}{2}\varepsilon}^{\frac{n}{2}}\lambda_\varepsilon(t)^2+o(\lambda_\varepsilon(t)^2)\right) \\
= & \lambda_\varepsilon(t)^{\frac{n-4}{2}\varepsilon}\frac{\alpha_n^{2^\sharp-\varepsilon}}{2^\sharp-\varepsilon}\frac{\omega_{n-1}}{2}\left(I_n^{\frac{n-2}{2}}+\frac{n-4}{2}\tilde I_n^{\frac{n-2}{2}}\varepsilon-\frac{1}{6n}\mathrm{Scal}_g(\xi)I_n^{\frac{n}{2}}\lambda_\varepsilon(t)^2+o(\lambda_\varepsilon(t)^2)\right) \\
= & \lambda_\varepsilon(t)^{\frac{n-4}{2}\varepsilon}\frac{(n(n-4)(n^2-4))^{\frac{n}{4}-\frac{n-4}{8}\varepsilon}}{2^\sharp-\varepsilon}\frac{\omega_{n-1}}{2}\left(I_n^{\frac{n-2}{2}}+\frac{n-4}{2}\tilde I_n^{\frac{n-2}{2}}\varepsilon-\frac{1}{6n}\mathrm{Scal}_g(\xi)I_n^{\frac{n}{2}}\lambda_\varepsilon(t)^2+o(\lambda_\varepsilon(t)^2)\right) \\
= & \frac{(n(n-4)(n^2-4))^{\frac{n}{4}}\,\omega_{n-1}}{2}\frac{n-4}{2n}\Big\{I_n^{\frac{n-2}{2}}+\frac{n-4}{2}I_n^{\frac{n-2}{2}}\varepsilon\ln(\lambda_\varepsilon(t)) \\
& +\frac{n-4}{2n}\left(n\tilde I_n^{\frac{n-2}{2}}+(1-n\ln\sqrt{n(n-4)})I_n^{\frac{n}{2}}\right)\varepsilon-\frac{1}{6n}\mathrm{Scal}_g(\xi)I_n^{\frac{n}{2}}\lambda_\varepsilon(t)^2+o(\lambda_\varepsilon(t)^2)\Big\} \\
= & \frac{n-4}{2n}\frac{(n(n-4)(n^2-4))^{\frac{n}{4}}\,\omega_{n-1}}{2}I_n^{\frac{n-2}{2}}\Big\{1+\frac{n-4}{2}\varepsilon\ln(\lambda_\varepsilon(t)) \\
& +\frac{1}{I_n^{\frac{n-2}{2}}}\frac{n-4}{2n}\left(n\tilde I_n^{\frac{n-2}{2}}+(1-n\ln\sqrt{n(n-4)})I_n^{\frac{n}{2}}\right)\varepsilon-\frac{1}{6(n-2)}\mathrm{Scal}_g(\xi)\lambda_\varepsilon(t)^2+o(\lambda_\varepsilon(t)^2)\Big\} \\
= & \frac{n-4}{2n}K_n^{-\frac{n}{4}}\Big\{1+\frac{n-4}{4}\varepsilon\ln(\varepsilon t)+\frac{n-4}{2n}\left(n\tilde I_n^{\frac{n-2}{2}}/I_n^{\frac{n-2}{2}}+\frac{n\left(1-n\ln\sqrt{n(n-4)}\right)}{n-2}\right)\varepsilon \\
& -\frac{1}{6(n-2)}\mathrm{Scal}_g(\xi)\varepsilon t+o(\varepsilon)\Big\},
\end{aligned}
\tag{9.70}
$$

as $\varepsilon\to 0$, $\mathcal{C}^0-$uniformly with respect to ξ in M and t in compact subsets of $\mathbb{R}_+^*$. Thus, (9.24) follows by (9.7), (9.69) and (9.70).

Proof of (ii) of Proposition 9.5:

We argue as Lemma 4.2 in [90]. Let us write:

$$\begin{aligned}&\tilde{J}_\varepsilon(t,\xi) - J_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\\ =\;& \int_M \left(P_{g,B}(W_{\lambda_\varepsilon(t),\xi}) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\right)\phi_{\varepsilon,\lambda,\xi}\, d\mu_g\\ &- \int_M \left(F_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_{\varepsilon,\lambda,\xi}) - F_\varepsilon(W_{\lambda_\varepsilon(t),\xi}) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\phi_{\varepsilon,\lambda,\xi}\right)\, d\mu_g,\end{aligned} \tag{9.71}$$

where $F_\varepsilon(u) = \int_0^u f_\varepsilon(v)\, dv$. We firstly estimate the first term in the right hand side of (9.71). By Hölder and Sobolev inequalities, we have

$$\begin{aligned}&\left|\int_M \left(P_{g,B}(W_{\lambda_\varepsilon(t),\xi}) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\right)\phi_{\varepsilon,\lambda,\xi}\, d\mu_g\right|\\ \le\;& \left|P_{g,B}(W_{\lambda_\varepsilon(t),\xi}) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\right|_{\frac{2n}{n+4}} |\phi_{\varepsilon,\lambda,\xi}|_{2^\sharp}\\ \le\;& C\left|P_{g,B}(W_{\lambda_\varepsilon(t),\xi}) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\right|_{\frac{2n}{n+4}} \|\phi_{\varepsilon,\lambda,\xi}\| = o(\varepsilon),\end{aligned} \tag{9.72}$$

because of estimates (9.51) and (9.22). Next, we estimate the second term in the right hand side of (9.71). By the mean value theorem and Hölder inequality, it holds that

$$\begin{aligned}&\left|\int_M \left(F_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_{\varepsilon,\lambda,\xi}) - F_\varepsilon(W_{\lambda_\varepsilon(t),\xi}) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\phi_{\varepsilon,\lambda,\xi}\right)\, d\mu_g\right|\\ \le\;& \int_M W_{\lambda_\varepsilon(t),\xi}^{2^\sharp-2-\varepsilon}\phi_{\varepsilon,\lambda,\xi}^2\, d\mu_g + \int_M \phi_{\varepsilon,\lambda,\xi}^{2^\sharp-\varepsilon}\, d\mu_g\\ \le\;& C|\phi_{\varepsilon,\lambda,\xi}|_{2^\sharp}^2\left(|W_{\lambda_\varepsilon(t),\xi}|_{\frac{n}{4}}^{2^\sharp-2-\varepsilon} + |\phi_{\varepsilon,\lambda,\xi}|_{2^\sharp}^{2^\sharp-2-\varepsilon}\right)\\ \le\;& C\|\phi_{\varepsilon,\lambda,\xi}\|^2\left(|W_{\lambda_\varepsilon(t),\xi}|_{\frac{n}{4}}^{2^\sharp-2-\varepsilon} + \|\phi_{\varepsilon,\lambda,\xi}\|^{2^\sharp-2-\varepsilon}\right) = o(\varepsilon),\end{aligned} \tag{9.73}$$

because of (9.22). The $\mathcal{C}^0-$uniform estimate (9.25) then follows from (9.72) and (9.73).

Now, let us show the $\mathcal{C}^1-$uniform estimate (9.25). From Proposition 9.4, for ε small, for any point ξ in M and any positive real number t, there holds

$$DJ_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_{\varepsilon,\lambda,\xi}) = \sum_{i=0}^n C^i_{\lambda_\varepsilon(t),\xi}\langle Z^i_{\lambda_\varepsilon(t),\xi}, \cdot\rangle \tag{9.74}$$

for some real numbers $C^0_{\lambda_\varepsilon(t),\xi}, \cdots, C^n_{\lambda_\varepsilon(t),\xi}$, where the functions $Z^i_{\lambda_\varepsilon(t),\xi}$ are as in (9.19). First, we claim that

$$\sum_{i=0}^n |C^i_{\lambda_\varepsilon(t),\xi}| = O\left(\varepsilon^{3/4} \text{ if } n = 7,\ \varepsilon|\ln\varepsilon| \text{ if } n \ge 8\right). \tag{9.75}$$

Indeed, for $i, j = 0, 1, 2, \cdots, n$ we have

$$\left\langle Z^i_{\lambda_\varepsilon(t),\xi}, Z^j_{\lambda_\varepsilon(t),\xi} \right\rangle \;\to\; \begin{cases} \|V_i\|^2_{\mathcal{D}^{2,2}(\mathbb{R}^n)} & \text{if } \; i = j; \\ 0 & \text{if } \; i \neq j. \end{cases} \tag{9.76}$$

On the other hand, from (9.74) and (9.76), for $i = 0, 1, \cdots, n$, there holds

$$DJ_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_{\varepsilon,\lambda,\xi})[Z^i_{\lambda_\varepsilon(t),\xi}] = C^i_{\lambda_\varepsilon(t),\xi} \|V_i\|^2_{\mathcal{D}^{2,2}(\mathbb{R}^n)} + o\left(\sum_{j=0}^{n} |C^j_{\lambda_\varepsilon(t),\xi}| \right). \tag{9.77}$$

Let us prove that

$$DJ_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_{\varepsilon,\lambda,\xi})[Z^i_{\lambda_\varepsilon(t),\xi}] = O\left(\varepsilon^{3/4} \text{ if } n = 7, \;\; \varepsilon|\ln \varepsilon| \text{ if } n \geq 8\right). \tag{9.78}$$

Estimate (9.75) will follow by (9.77) and (9.78).

Let us prove (9.78). Since, for ε small, the function $\phi_{\varepsilon,\lambda,\xi}$ belongs to $K^\perp_{\varepsilon,\lambda,\xi}$, by Hölder inequality and Lemma 8.16 we have

$$\begin{aligned}
& DJ_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_{\varepsilon,\lambda,\xi})[Z^i_{\lambda_\varepsilon(t),\xi}] \\
= \; & \int_M \left(P_{g,B}(W_{\lambda_\varepsilon(t),\xi}) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi}) \right) Z^i_{\lambda_\varepsilon(t),\xi} \, d\mu_g \\
& - \int_M \left(f_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_{\varepsilon,\lambda,\xi}) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi}) \right) Z^i_{\lambda_\varepsilon(t),\xi} \, d\mu_g \\
& = O\left(\left| P_{g,B}(W_{\lambda_\varepsilon(t),\xi}) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi}) \right|_{\frac{2n}{n+4}} \left| Z^i_{\lambda_\varepsilon(t),\xi} \right|_{2^\sharp} \right) \\
& + O\left(|\phi_{\varepsilon,\lambda,\xi}|_{2^\sharp} |Z^i_{\lambda_\varepsilon(t),\xi}|_{2^\sharp} \left(|W_{\lambda_\varepsilon(t),\xi}|^{2^\sharp - 2 - \varepsilon}_{\frac{n}{4}} + |\phi_{\varepsilon,\lambda,\xi}|^{2^\sharp - 2 - \varepsilon}_{\frac{n}{4}} \right) \right) \\
= \; & O\left(\varepsilon^{3/4} \text{ if } n = 7, \;\; \varepsilon |\ln \varepsilon| \text{ if } n \geq 8 \right),
\end{aligned} \tag{9.79}$$

because of (9.51) and (9.22).

Finally, let us compute the derivative of $\tilde{J}_\varepsilon(t, \xi)$. Firstly, we remark that

$$\frac{d(W_{\lambda_\varepsilon(t),\xi})}{dt} = \frac{1}{2t} Z^0_{\lambda_\varepsilon(t),\xi}, \tag{9.80}$$

which implies

$$\begin{aligned}
& \frac{d\tilde{J}_\varepsilon(t,\xi)}{dt} - \frac{dJ_\varepsilon(W_{\lambda_\varepsilon(t),\xi})}{dt} \\
= \; & \frac{1}{2t} \Big\{ \int_M \left(P_{g,B}(Z^0_{\lambda_\varepsilon(t),\xi}) - f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi}) Z^0_{\lambda_\varepsilon(t),\xi} \right) \phi_{\varepsilon,\lambda,\xi} \, d\mu_g \\
& - \int_M \left(f_\varepsilon(W_{\lambda_\varepsilon(t),\xi} + \phi_{\varepsilon,\lambda,\xi}) - f_\varepsilon(W_{\lambda_\varepsilon(t),\xi}) - f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi}) \phi_{\varepsilon,\lambda,\xi} \right) Z^0_{\lambda_\varepsilon(t),\xi} \; \mu_g \Big\}
\end{aligned}$$

$$+DJ_\varepsilon(W_{\lambda_\varepsilon(t),\xi}+\phi_{\varepsilon,\lambda,\xi})\left[\frac{d(\phi_{\varepsilon,\lambda,\xi})}{dt}\right]. \tag{9.81}$$

Secondly, if $\xi=\xi(y)=\exp_\xi(y)$, $y\in B(0,r)$, with $\xi(0)=\xi$, we have for any $i=1,2,\cdots,n$ (see estimate (6.13) in [87])

$$\frac{\partial(W_{\lambda_\varepsilon(t),\exp_\xi(y)})}{\partial y_i}\Big|_{y=0}=\frac{1}{\lambda_\varepsilon(t)}Z^i_{\lambda_\varepsilon(t),\xi}+R_{\lambda_\varepsilon(t),\xi},\quad \|R_{\lambda_\varepsilon(t),\xi}\|=O(\lambda_\varepsilon(t)^2), \tag{9.82}$$

which easily implies

$$\begin{aligned}
&\frac{\partial\tilde J_\varepsilon(t,\exp_\xi(y))}{\partial y_i}\Big|_{y=0}-\frac{\partial J_\varepsilon(W_{\lambda_\varepsilon(t),\exp_\xi(y)})}{\partial y_i}\Big|_{y=0}\\
=\;&\frac{1}{\lambda_\varepsilon(t)}\Big\{\int_M\left(P_{g,B}(Z^i_{\lambda_\varepsilon(t),\xi})-f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})Z^i_{\lambda_\varepsilon(t),\xi}\right)\phi_{\varepsilon,\lambda,\xi}\,d\mu_g\\
&-\int_M\left(f_\varepsilon(W_{\lambda_\varepsilon(t),\xi}+\phi_{\varepsilon,\lambda,\xi})-f_\varepsilon(W_{\lambda_\varepsilon(t),\xi})-f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\phi_{\varepsilon,\lambda,\xi}\right)Z^i_{\lambda_\varepsilon(t),\xi}\,\mu_g\Big\}\\
&+DJ_\varepsilon(W_{\lambda_\varepsilon(t),\xi}+\phi_{\varepsilon,\lambda,\xi})\left[\frac{\partial(\phi_{\varepsilon,\lambda,\exp_\xi(y)})}{\partial y_i}\Big|_{y=0}\right]+O\left(\|\phi_{\varepsilon,\lambda,\xi}\|\,\|R_{\lambda_\varepsilon(t),\xi}\|\right).
\end{aligned} \tag{9.83}$$

We will estimate each term of the right hand of (9.81) and (9.83). By Hölder inequality and (9.22), for $i=0,1,\cdots,n$, we have

$$\begin{aligned}
&\left|\int_M\left(P_{g,B}(Z^i_{\lambda_\varepsilon(t),\xi})-f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})Z^i_{\lambda_\varepsilon(t),\xi}\right)\phi_{\varepsilon,\lambda,\xi}\,d\mu_g\right|\\
\le\;&\left|P_{g,B}(Z^i_{\lambda_\varepsilon(t),\xi})-f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})Z^i_{\lambda_\varepsilon(t),\xi}\right|_{\frac{2n}{n+4}}|\phi_{\varepsilon,\lambda,\xi}|_{2^\sharp}\\
=\;&O\left(\varepsilon^{3/2}\text{ if }n=7,\ \varepsilon^2|\ln\varepsilon|^2\text{ if }n\ge 8\right),
\end{aligned} \tag{9.84}$$

because arguing exactly as in the proof of Lemma 9.7, we can show that for any $i=0,1,\cdots,n$

$$\left|P_{g,B}(Z^i_{\lambda_\varepsilon(t),\xi})-f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})Z^i_{\lambda_\varepsilon(t),\xi}\right|_{\frac{2n}{n+4}}=O\left(\varepsilon^{\frac34}\text{ if }n=7,\ \varepsilon|\ln\varepsilon|\text{ if }n\ge 8\right) \tag{9.85}$$

From Lemma 9.7, by Hölder inequality and (9.22), we have

$$\begin{aligned}
&\left|\int_M\left(f_\varepsilon(W_{\lambda_\varepsilon(t),\xi}+\phi_{\varepsilon,\lambda,\xi})-f_\varepsilon(W_{\lambda_\varepsilon(t),\xi})-f'_\varepsilon(W_{\lambda_\varepsilon(t),\xi})\phi_{\varepsilon,\lambda,\xi}\right)Z^i_{\lambda_\varepsilon(t),\xi}\,d\mu_g\right|\\
\le\;&\begin{cases} C\int_M W^{2^\sharp-3-\varepsilon}_{\lambda_\varepsilon(t),\xi}\phi^2_{\varepsilon,\lambda,\xi}Z^i_{\lambda_\varepsilon(t),\xi}d\mu_g & \text{if } n\ge 12;\\ C\int_M\left[W^{2^\sharp-3-\varepsilon}_{\lambda_\varepsilon(t),\xi}\phi^2_{\varepsilon,\lambda,\xi}+\phi^{2^\sharp-1-\varepsilon}_{\varepsilon,\lambda,\xi}\right]Z^i_{\lambda_\varepsilon(t),\xi}d\mu_g & \text{if } 5\le n<12,\end{cases}\\
\le\;&\begin{cases} C|\phi_{\varepsilon,\lambda,\xi}|^2_{\frac{2n}{n-4}}|W^{2^\sharp-3-\varepsilon}_{\lambda_\varepsilon(t),\xi}Z^i_{\lambda_\varepsilon(t),\xi}|_{\frac n4} & \text{if } n\ge 12;\\ C\left(|\phi_{\varepsilon,\lambda,\xi}|^2_{\frac{2n}{n-4}}|W^{2^\sharp-3-\varepsilon}_{\lambda_\varepsilon(t),\xi}Z^i_{\lambda_\varepsilon(t),\xi}|_{\frac n4}+|\phi_{\varepsilon,\lambda,\xi}|^{2^\sharp-1-\varepsilon}_{\frac{2n}{n-4}}|Z^i_{\lambda_\varepsilon(t),\xi}|_{\frac{2n}{n-4}}\right) & \text{if } 5\le n<12.\end{cases}
\end{aligned}$$

$$= O\left(\varepsilon^2\right), \tag{9.86}$$

because $|W^{2^\sharp-3-\varepsilon}_{\lambda_\varepsilon(t),\xi} Z^i_{\lambda_\varepsilon(t),\xi}|_{\frac{n}{4}} = O(1)$.

Finally, since the function $\phi_{\varepsilon,\lambda,\xi}$ belongs to $K^\perp_{\varepsilon,\lambda,\xi}$, from (9.74) we have

$$\begin{aligned} DJ_\varepsilon(W_{\lambda_\varepsilon(t),\xi}+\phi_{\varepsilon,\lambda,\xi})\left[\frac{d(\phi_{\varepsilon,\lambda,\xi})}{dt}\right] &= \sum_{j=0}^n C^j_{\lambda_\varepsilon(t),\xi}\left\langle Z^j_{\lambda_\varepsilon(t),\xi}, \frac{d(\phi_{\varepsilon,\lambda,\xi})}{dt}\right\rangle \\ &= -\sum_{j=0}^n C^j_{\lambda_\varepsilon(t),\xi}\left\langle \frac{d(Z^j_{\lambda_\varepsilon(t),\xi})}{dt}, \phi_{\varepsilon,\lambda,\xi}\right\rangle, \end{aligned} \tag{9.87}$$

and for $i = 1, 2, \cdots, n$

$$\begin{aligned} DJ_\varepsilon(W_{\lambda_\varepsilon(t),\xi}+\phi_{\varepsilon,\lambda,\xi})\left[\frac{\partial(\phi_{\varepsilon,\lambda,\exp_\xi(y)})}{\partial y_i}\Big|_{y=0}\right] &= \sum_{j=0}^n C^j_{\lambda_\varepsilon(t),\xi}\left\langle Z^j_{\lambda_\varepsilon(t),\xi}, \frac{\partial(\phi_{\varepsilon,\lambda,\exp_\xi(y)})}{\partial y_i}\Big|_{y=0}\right\rangle \\ &= -\sum_{j=0}^n C^j_{\lambda_\varepsilon(t),\xi}\left\langle \frac{\partial(Z^j_{\lambda_\varepsilon(t),\xi})}{\partial y_i}\Big|_{y=0}, \phi_{\varepsilon,\lambda,\xi}\right\rangle \end{aligned} \tag{9.88}$$

It is easy to check that

$$\left\|\frac{d(Z^j_{\lambda_\varepsilon(t),\xi})}{dt}\right\| \to \frac{1}{2t}\left\|\frac{d\left(\lambda^{\frac{4-n}{4}}V_j(\lambda^{-1}y)\right)}{d\lambda}\Big|_{\lambda=1}\right\| = O(1), \tag{9.89}$$

and

$$\left\|\frac{\partial(Z^j_{\lambda_\varepsilon(t),\xi})}{\partial y_i}\Big|_{y=0}\right\| \to \frac{1}{\lambda_\varepsilon(t)}\left\|\frac{\partial V_j}{\partial y_i}\right\| = O\left(\frac{1}{\lambda_\varepsilon(t)}\right) \tag{9.90}$$

By (9.87) (9.89), (9.22) and (9.75), we get

$$\begin{aligned} DJ_\varepsilon(W_{\lambda_\varepsilon(t),\xi}+\phi_{\varepsilon,\lambda,\xi})\left[\frac{d(\phi_{\varepsilon,\lambda,\xi})}{dt}\right] &= O\left(\|\phi_{\varepsilon,\lambda,\xi}\|\sum_{j=0}^n |C^j_{\lambda_\varepsilon(t),\xi}|\right) \\ &= O\left(\varepsilon^{3/2} \text{ if } n=7,\ \varepsilon^2|\ln\varepsilon|^2 \text{ if } n\geq 8\right), \end{aligned} \tag{9.91}$$

and by (9.88),(9.90), (9.22) and (9.75) we get

$$DJ_\varepsilon(W_{\lambda_\varepsilon(t),\xi}+\phi_{\varepsilon,\lambda,\xi})\left[\frac{\partial(\phi_{\varepsilon,\lambda,\exp_\xi(y)})}{\partial y_i}\Big|_{y=0}\right] = O\left(\frac{\|\phi_{\varepsilon,\lambda,\xi}\|\sum_{j=0}^n |C^j_{\lambda_\varepsilon(t),\xi}|}{\lambda_\varepsilon(t)}\right)$$

$$= \quad O\left(\varepsilon \text{ if } n=7,\ \varepsilon^{5/2}|\ln\varepsilon|^2 \text{ if } n\geq 8\right). \tag{9.92}$$

By (9.22) and (9.82) we get

$$O\left(\|\phi_{\varepsilon,\lambda,\xi}\|\ \|R_{\lambda_\varepsilon(t),\xi}\|\right) \quad = \quad o\left(\varepsilon\right). \tag{9.93}$$

Finally, collecting all the estimates (9.81), (9.83), (9.84), (9.86), (9.91), (9.92) and (9.93) we get the $\mathcal{C}^1-$uniform estimate (9.25).

That concludes the proof.

Proof of (iii) of Proposition 9.5: We argue exactly as in [90], see also the proof of Lemma 8.9 in chapter eight.

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