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QUASI STATIONARY DISTRIBUTIONS WHEN INFINITY IS AN ENTRANCE BOUNDARY , OPTIMAL CONDITIONS FOR PHASE TRANSITION IN 1 DIMENSIONAL ISING MODEL BY PEIERLS ARGUMENT AND ITS CONSEQUENCES

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA MENCIÓN MODELACIÓN MATEMÁTICA EN COTUTELA CON LA UNIVERSIDAD DE PROVENCE

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Resumen

Este trabajo de tesis contiene dos capítulos principales, donde se estudian dos problemas independientes de Modelación Matemática. En el Captítulo 1 se estudia la existencia y unicidad de distribuciones quasi estacionarias para un movimiento Browniano con drift extinguido en cero, para el caso que infinito es Frontera de Entrada y cero frontera de salida de acuerdo a la clasificación de Feller. El trabajo está relacionado con la publicación pionera [8], donde algunas condiciones suficientes son establecidas para demostrar la existencia y unicidad de QSD en el contexto de una familia de Modelos de Dinámica de Poblaciones y difusiones de Feller. El trabajo generaliza los teoremas más importantes de [8] , ya que no se imponen condiciones extras para obtener los resultados de existencia y unicidad de QSD y la existencia del límite de Yaglom. La parte técnica está basada en la teoría del problema de Sturm Liouville sobre la semirecta positiva. Específicamente, se demuestra que bajo las principales hipótesis existe espectro discreto si y solo si infinito es frontera de entrada y todas las eigenfunciones son simples e integrables respecto a la medida de rapidez del proceso.
En el capítulo 2, se estudia el problema de obtener cotas optimales sobre el Hamiltoniano para el Modelo de Ising de largo alcance, con término de interacción decayendo de acuerdo a $\mathrm{d}^{\alpha-2}$, $\alpha \in [0,1)$. El trabajo está basado en el artículo publicado en 2005 [31], donde cotas optimales son obtenidas para el caso $\alpha \in [0, \frac{log3}{\log 2} - 1)$ en términos de estructuras jerárquicas llamadas triángulos y contornos. Los teoremas principales de este trabajo pueden ser resumidos como (i) No existe una cota optimal para el Hamiltoniano en términos de triángulos para $\alpha \in [\frac{log3}{\log 2} - 1, 1)$. (ii) Existe una cota optimal para el Hamiltoniano en términos de Contornos para $\alpha \in [0,1)$, resultados que son demostrado en los Teoremas 2.16 y 2.24 respectivamente.

Ambos generalizan los resultados existentes, y constituyen la principal contribución de este trabajo. Para demostrar el Teorema 2.16, se construye explícitamente una familia de contraejemplos. La parte técnica está fuertemente basada en la teoría de Fractales sobre Conjuntos Discretos. Para demostrar teorema 2.24 , se usa el argumento se agrupar y sumar sobre contornos con la misma masa. Las demostraciones para ambos resultados son muy técnicas y requieren una gran cantidad de cálculos , los cuales son entregados en detalle. Por otra parte, los teoremas principales tienen importantes implicancias en ésta clase de Modelos. La más importante y directa es la existencia de una fase de transición para bajas temperaturas basada en el argumento de Peierls. Dicha demostraciún, es también entregada en este trabajo.

Abstract

This tesis contains two main Chapters, where we study two independent problems of Mathametical Modelling : In Chapter 1, we study existence and uniqueness of Quasi Stationary Distributions (QSD) for a drifted Browian Motion killed at zero, when $+\infty$ is an entrante Boundary and zero is an exit Boundary accordingly to Feller's classifications. Also, the existence of a Yaglom limit is shown. The work is related to the early paper *"Quasi stationary distributions and diffusion models in population dynamics. Annals of Probability, 37(5):1926-1969"*, where some suficiente conditions are provided to prove the existence and uniqueness of QSD in the context of a family of Dynamic Population Models and Feller's difussions. This work generalizes the most important theorems of this early work, since no extra conditions are imposed to get the existence, uniqueness of QSD and the existence of a Yaglom limit. The technical part is based on the Sturm Liouville theory on the half line. Specifically, we show that under main hypothesis there exists discrete spectrum if and only if $+\infty$ is an entrance boundary and that all the eigenfunctions are simple and integrable respect to the speed measure of the process.

In Chapter 2, we study the problem of getting optimal bounds on the Hamiltonian for the Long Range Ising Model with the interaction term decaying according to $\frac{1}{\mathrm{d}^{2-\alpha}}$, $\alpha \in (0,1)$. This work is based on the piooner paper published in 2005, *"Geometry of contours and Peierls estimates in d=1 Ising models with long range interactions"* where optimal bounds for the Hamiltonian are obtained for $\alpha \in [0, \frac{\log 3}{log 2} - 1)$ in terms of herarchical structures called triangles and Contours . Main theorems of this work can be summarized as : *(1) There is not exists an optimal bound for the Hamiltonian in terms of triangles for $\alpha \in [\frac{\log 2}{\log 3}, 1)$ and (2) There exists an optimal bound for the Hamiltonian in terms of Contours for $\alpha \in [0, 1)$*. Those theorems generalizes existing results, and constitutes the main contribution of this work. For the statement (1) we proceed by building explicitily a family of counterexamples. The thechnical part is strongly based in the Theory of Fractals on Discrete Sets. For stament (2) we use of gropuing and sum over Contours with the mass. Proofs are quite technical and requires lots of detailed computations, which are given in detail.

Main Theorems have some important consecuences into this clases of Models. On the most important and direct consecuence is the existence of a phase transition for low temperatures by giving an explicit proof based on the Peierls argument. This explicit proof is given in this work

Specially dedicated to my mother, for teaching me to persevere and never giving up.

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Contents

Chapter 1

Uniqueness of quasistationary distributions and discrete spectra when ∞ is an entrance boundary and 0 is singular

1.1 History of Quasi Stationary Distributions

The study of the number of survival after a long time started with early work of Kolmogorov in 1938. Later, on 1947 Yaglom [1] showed that the limit behavior of subcritical branching processes conditiond to survival was given by a proper distribution. This work trigged a very important activity in this field.

On 1965, Darroch & Seneta [2] started the study of QSD on finite state irreducible Markov Chain. Many of the most important ideas can be developed by using the Perron-Frobenius theory for finite positive matrices, which gives all the requrided information for finite positive matrices. The exponencial rate of survival time is the Perron-Frobeius eigenvalue and the QSD is its associated eigenmeasure. The chain of trayectories that are never killed is goberned by a process (the h-process), where h is the Perron-Frobenius eigenfunction. In addition, since the dominant eigenvalue is simple, the QSD attracts all the conditioned measures.

For countable state Markov Chains, Seneta and Vere Jones [3] on 1966. Characterization of QSDs as a finite eigenmeasure is due to Pollet . A very important publication was done by Van Doorn in 1991, [4] which states a criteria to determine the existence and uniqueness of QSD for birth and death chains.

For continuous time diffusion process on the half line, the first work is due to Mandl [5], who studied the existence of a QSD on the half line for $+\infty$ being a natural boundary accordingly to Feller's classification. This allows an important development to study the existence of QSD in the context of diffusion process. In subsequent works, many some result of existence of QSD and limit laws for one dimensional difussions killed at 0 are provided (see Ferrari Kesten Martínez Picco (1995) [10], Collet Martínez San Martín [11] (2001) and Martínez San Martín [12] [13] [14] [15].

The existence of QSD on diffusions not satisfying Mandl conditions on $+\infty$ started in 2009, with the recent publication [8], where some necessary and sufficient and conditions are provided to determine the existence and uniqueness of QSD for $+\infty$ being an entrance boundary. On the recently years, Kolb and Steinsaltz in [20] stated necessary conditions for existence of QSD for 0 being a regular boundary.

In this work, we study quasistationary distributions (QSD) for a drifted Brownian motion killed at 0, when $+\infty$ is an entrance boundary according to Feller's classification. The most recent results for the existence and uniqueness of QSDs when 0 is a regular type-boundary are given in [20], and sufficient conditions are stated in [8] when 0 is an exit type boundary.

This work was published in J. Appl. Probab. Volume 49, Number 3 (2012), 719-730 and is related to [8]. We state the existence of a unique QSD when 0 is an exit type boundary and $+\infty$ is an entrance boundary, under the most general conditions. Also, the existence of the Yaglom limit is shown. In Section 1.2 the main hypothesis and some preliminary results are provided. In Section 1.3 the required elements of spectral theory are introduced. Finally, in Section 1.4 we summarize the main theorems on the existence of a unique QSD and the existence of the Yaglom limit.

The most technically difficult result Theorem 1.13, which states the integrability of the eigenfunctions . The key of its proof is an increasing condition property of the eigenfunctions, which is established in (1.4).

1.2 One-dimensional diffusion processes killed at the origin

Let X be a one-dimensional drifted Brownian motion in $(0, \infty)$, i.e.

$$dX_t = dB_t - \alpha(X_t)dt, \qquad X_0 = x > 0 \tag{1.1}$$

where $\alpha \in C^1(0,\infty)$ and $(B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion. We note that the drift α can explode at the origin. A pathwise solution of X in (1.1) exists up to the explosion time τ. We denote by T_y the first time the process hits $y \in (0,\infty)$ (see [6], Chapter VI, Section 3) before the explosion, so $T_y = \inf\{0 \leq t < \tau : X_t = y\}$. We denote by

$$T_\infty = \lim_{n\to\infty} T_n \text{ and } T_0 = \lim_{n\to\infty} T_{1/n}\,.$$

Since α is regular in $(0,\infty)$, then $\tau = T_0 \wedge T_\infty$.

1.2.1 Main Hypothesis

Let $\gamma(x) = 2\int_1^x \alpha(z)\mathrm{d}z$ for $x \in (0,\infty)$. Associated to α, we consider the following functions defined for $x \in (0,\infty)$:

$$\begin{aligned}
\Lambda_1(x) &= \int_1^x \mathrm{e}^{\gamma(z)}\mathrm{d}z \\
\kappa(x) &= \int_x^1 \mathrm{e}^{-\gamma(z)} \int_x^z \mathrm{e}^{\gamma(y)}\mathrm{d}y\mathrm{d}z \\
J(x) &= \int_1^x \mathrm{e}^{-\gamma(z)} \int_1^z \mathrm{e}^{\gamma(y)}\mathrm{d}y\mathrm{d}z\,,
\end{aligned}$$

$\Lambda_1(x)$ is the scale function and $\mu(\mathrm{d}x) = \mathrm{e}^{-\gamma(x)}\mathrm{d}x$ is the speed measure for X.
Observe that under condition $\alpha \in C^1(0,\infty)$, $\gamma(y)$ is finite for all $y > 0$ and both $\int_a^b \mathrm{e}^{\gamma(y)}\mathrm{d}y < \infty$ and $\int_a^b \mathrm{e}^{-\gamma(y)}\mathrm{d}y < \infty$, for all $0 < a < b < \infty$.

Let us to state the following conditions on α.

- $(H1)$ Almost sure absorption at 0: for all $x > 0$ $\mathbb{P}_x(\tau = T_0 < T_\infty) = 1$.
- $(H2)$ $J(+\infty) < \infty$.
- $(H3)$ $\mu((0,\delta)) = \infty$ for all $\delta > 0$.

It is well known (see for example, Chapter VI, Theorem 3.2 [6]) that $(H1)$ holds if and only if: $\Lambda_1(+\infty) = \infty$ and $\kappa(0^+) < \infty$. Also note that $(H1)$ can be written as $\mathbb{P}_x(\lim_{t\to\infty} X_{t\wedge\tau} = 0) = 1$.

As a direct consequence of Hypothesis $(H1)$ and $(H2)$, $+\infty$ is a Entrance Boundary according to Feller classification (see [7] Ch. 15, Table 7.1 for details). Under Hypothesis $(H1)$ and $(H3)$, 0 is an exit boundary according to Feller's classification (see [7] Ch. 15, Table 6.2 for details).

Hypothesis (H) is said to hold when $\alpha \in C^1(0,\infty)$ and $(H1)$, $(H2)$ and $(H3)$ are satisfied. Note that, under (H), the function

$$\Lambda(x) = \int_0^x \mathrm{e}^{\gamma(z)} \mathrm{d}z \quad \text{for all } x \in (0, \infty)$$

is finite. We also obtain some additional properties on the functions already defined, which we summarize in the next lemma.

Lemma 1.1 *Assume that (H) holds. Then the following relations are satisfied:*
(i) $\Lambda_1(0+) > -\infty$.
(ii) $\mu((\delta, \infty)) < \infty \quad$ (for all $\delta > 0$).
(iii) $\int_0^\infty \mathrm{e}^{-\gamma(y)} \Lambda(y) \mathrm{d}y = \int_0^\infty \mathrm{e}^{\gamma(z)} \int_z^\infty \mathrm{e}^{-\gamma(y)} \mathrm{d}y \mathrm{d}z < \infty$.

Proof: (i) For all $0 < \delta < 1$, we have:

$$\left(\int_0^\delta \mathrm{e}^{\gamma(y)} \mathrm{d}y\right) \left(\int_\delta^1 \mathrm{e}^{-\gamma(z)} \mathrm{d}z\right) < \int_0^\delta \mathrm{e}^{\gamma(y)} \left(\int_y^1 \mathrm{e}^{-\gamma(z)} \mathrm{d}z\right) \mathrm{d}y < \kappa(0+) < \infty.$$

Since $0 < \int_\delta^1 \mathrm{e}^{\gamma(z)} \mathrm{d}z < \infty$, the result is shown.

(ii) Since $\int_\delta^M \mathrm{e}^{-\gamma(y)} \mathrm{d}y < \infty$ for all $0 < \delta < M < \infty$, it is enough to show that $\int_M^\infty \mathrm{e}^{-\gamma(y)} \mathrm{d}y < \infty$ for some $M > \delta$. From the condition $\Lambda_1(+\infty) = \infty$, there exists M greater than 1 such that $\Lambda_1(x) > 1$ for all $x > M$, so we have

$$\int_M^\infty \mathrm{e}^{-\gamma(y)} \mathrm{d}y < \Lambda_1(M) \int_M^\infty \mathrm{e}^{-\gamma(y)} \mathrm{d}y < \int_M^\infty \mathrm{e}^{-\gamma(y)} \Lambda_1(y) \mathrm{d}y < J(+\infty) < \infty.$$

(iii) From properties (i) and (ii) we have

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\gamma(y)} \Lambda(y) \mathrm{d}y &= \int_0^1 \mathrm{e}^{-\gamma(z)} \Lambda(y) \mathrm{d}y + \int_1^\infty \mathrm{e}^{-\gamma(y)} \Lambda(y) \mathrm{d}y \\
&< \kappa(0+) + J(+\infty) + \left(\int_1^\infty \mathrm{e}^{-\gamma(y)} \mathrm{d}y\right) \left(\int_0^1 \mathrm{e}^{\gamma(y)} \mathrm{d}y\right) \\
&< \infty.
\end{aligned}$$

□

1.2.2 Preliminary results

Let

$$\lambda^* := \liminf_{t \to \infty} -\frac{\log P_x(T_0 > t)}{t} = \sup\{\lambda : E_x(\mathrm{e}^{\lambda T_0}) < \infty\}. \tag{1.2}$$

In fact, the right-hand equality follows from Fubini's Theorem. Also, by irreducibility, both expressions on the right hand do not depend on x (see claim 1 in the proof of Theorem 1.4 below), so λ^* is well defined.

The next lemma gives some additional information about λ^* when $+\infty$ is a entrance boundary.

Lemma 1.2 *Assume (H) holds. Then:*

$$\lambda^* > \frac{1}{2\int_0^\infty e^{\gamma(z)} \int_z^\infty e^{-\gamma(y)} dy dz} > 0\,.$$

Proof: The proof is analogous to that of Proposition (7.6) of [8]. We have $J_\delta(x) = \int_0^x e^{\gamma(y)} \int_y^\infty e^{-\gamma(z)} dz dy + \delta$, with $\delta > 0$. From $(H2)$, $J_\delta(\infty) < \infty$ and from a straightforward computation we get $\mathcal{L}J_\delta = \frac{1}{2}J_\delta'' - J_\delta' = -\frac{1}{2}$, $\int_\varepsilon^{1/\varepsilon} |J_\delta'(s)|^2 ds < \infty$, for all $\varepsilon > 0$.

By Itô's formula,

$$E_x\left(e^{aT_\varepsilon \wedge T_{1/\varepsilon}\wedge t} J_\delta(X_{T_\varepsilon \wedge T_{1/\varepsilon}\wedge t})\right) = J_\delta(x) + E_x\left(\int_0^{T_\varepsilon \wedge T_{1/\varepsilon}\wedge t} e^{as}(aJ_\delta(X_s) + \mathcal{L}J_\delta(X_s))ds\right).$$

For $0 < a \le 1/2J_\delta(+\infty)$, it is clear that $J_\delta(X_s) + \mathcal{L}J_\delta(X_s) \le 0$, so

$$\delta E_x\left(e^{aT_\varepsilon \wedge T_{1/\varepsilon}\wedge t}\right) \le E_x\left(e^{aT_\varepsilon \wedge T_{1/\varepsilon}\wedge t} J_\delta(X_{T_\varepsilon \wedge T_{1/\varepsilon}\wedge t})\right) \le J_\delta(x)$$

Let $t \to \infty$ and $\varepsilon \to 0$. From the monotone convergence theorem we obtain

$$E_x(e^{aT_0}) \le \frac{J_\delta(x)}{\delta} < \infty \quad \text{for all } a \in \left(0, \frac{1}{2J_\delta(\infty)}\right]$$

Finally, since the result is true for all $\delta > 0$, we conclude the proof by taking $\delta \to 0^+$. □

We recall that a probability measure ν is a QSD if $E_\nu(X_t \in A \,|\, T_0 > t) = \nu(A)$ for all Borel subsets $A \subseteq (0,\infty)$. It is known (see [9], [12]) that if ν is a QSD then $P_\nu(T_0 > t) = e^{-\lambda(\nu)t}$ for some $0 < \lambda(\nu) \le \lambda^*$, where $\lambda(\nu)$ is the survival rate of T_0 starting from ν. Then, each QSD is necessarily associated to a $\lambda \in (0, \lambda^*]$. Theorem 1.4 below shows that, when ∞ is an entrance boundary, any QSD ν satisfies $\lambda(\nu) = \lambda^*$, but before presenting this result we recall the next definition.

Definition 1.3 *X comes down from ∞ if, for some $y > 0$, $\lim_{x\to\infty} P_x(T_y \le t) > 0$.*

Theorem 1.4 *Let hyphotesis (H) hold. Then no QSD is associated to some $\lambda \in (0, \lambda^*)$.*

Proof: We verify this result by proving the following four claims.

Claim 1.5 *If $E_{x_0}(e^{\lambda T_0}) = \infty$ for some $x_0 > 0$, then $E_x(e^{\lambda T_0}) = \infty$ for all $x > 0$.*

Proof: For $x > x_0$, the claim follows straightforwardly since $E_x(\mathrm{e}^{\lambda T_0}) > E_{x_0}(\mathrm{e}^{\lambda T_0})$. For $0 < x < x_0$ we have

$$E_x(\mathrm{e}^{\lambda T_0}) > E_x(\mathrm{e}^{\lambda T_0} 1_{T_0 > T_{x_0}}) = E_x(\mathrm{e}^{\lambda T_{x_0}} 1_{T_0 > T_{x_0}}) E_{x_0}(\mathrm{e}^{\lambda T_0}) \geq P_x(T_{x_0} < T_0) E_{x_0}(\mathrm{e}^{\lambda T_0}) .$$

Since $0 < P_x(T_{x_0} < T_0) = \Lambda(x)/\Lambda(x_0) < 1$ the claim follows.

Claim 1.6 *Let $x_0 > 0$ and $\lambda > 0$ such that $E_{x_0}(\mathrm{e}^{\lambda T_0}) < \infty$. Then X comes down from ∞ if and only if $\sup_{x>0} E_x(\mathrm{e}^{\lambda T_0}) < \infty$.*

Proof: Let us prove the 'if' part. We know that $E_x(\mathrm{e}^{\lambda T_0}) < \infty$ for all $x > 0$. From [8] Proposition (7.6) we know that, for all $\lambda > 0$, there exists y_λ such that $\sup_{x>y_\lambda} E_x(\mathrm{e}^{\lambda T_{y_\lambda}}) < \infty$. By monotonicity and the strong Markov property, $\sup_{x>0} E_x(\mathrm{e}^{\lambda T_0}) < \infty$.

Let us now prove the 'only if' part. Let $y > x > 0$. By using the Markov inequality we have

$$P_x(T_y > t) \leq \mathrm{e}^{-\lambda t} E_x(\mathrm{e}^{\lambda T_y}) \leq \mathrm{e}^{-\lambda t} \sup_{x>0} E_x(\mathrm{e}^{\lambda T_y}) < 1$$

for large enough t. The latter implies that X comes down from ∞ (see Definition 1.3). This completes the proof of the claim.

Claim 1.7 *Assume that (H) holds. If $\lambda > 0$ satisfies $\sup_{x>0} E_x(\mathrm{e}^{\lambda T_0}) < \infty$, then there does not exist a QSD associated to λ.*

Proof: Assume that π_λ is a QSD associated to λ, then $P_{\pi_\lambda}(T_0 > t) = \mathrm{e}^{-\lambda t}$. Now, for all $t > 0$ it is satisfied

$$\begin{aligned} \infty &> \sup_{x>0} E_x(\mathrm{e}^{\lambda T_0}) \\ &\geq E_{\pi_\lambda}(\mathrm{e}^{\lambda T_0}) \\ &= \int_0^\infty E_x(\mathrm{e}^{\lambda T_0}) \pi_\lambda(\mathrm{d}x) \\ &\geq \lambda \int_0^\infty \left(\int_0^t \mathrm{e}^{\lambda s} P_x(T > s) \mathrm{d}s \right) \pi_\lambda(\mathrm{d}x). \end{aligned}$$

By using Fubini's theorem in the expression at the right hand becomes

$$\lambda \int_0^t \left(\int_0^\infty \pi(\mathrm{d}x) P_x(T_0 > s) \right) \mathrm{e}^{\lambda s} \mathrm{d}s = \lambda \int_0^t \mathrm{e}^{-\lambda s} \mathrm{e}^{\lambda s} \mathrm{d}s = \lambda t.$$

The inequality holds for $t > 0$, which is a contradiction, and so the claim is proven.

.

Claim 1.8 *If X comes down from ∞ then, for all $\lambda \in (0, \lambda^*)$ we have* $\sup_{x>0} E(\mathrm{e}^{\lambda T_0}) < \infty$.

Proof: From Claims 2 and 3, it is sufficient to show that, for $\lambda \in (0, \lambda^*)$, $E_x(\mathrm{e}^{\lambda T_0}) < \infty$ holds for some (for all) $x > 0$. From (the lim inf) definition in (1.2), for all $\varepsilon > 0$, there exists t_0 such that for all $t > t_0$ it holds that $\lambda^* > -\log P_x(T > t)/t > \lambda^* - \varepsilon$, or equivalently, $P_x(T_0 > t) < \mathrm{e}^{-(\lambda^*-\varepsilon)t}$ for all $t > t_0$. By choosing $0 < \varepsilon < \lambda^* - \lambda$ we obtain

$$\begin{aligned}
E_x(\mathrm{e}^{\lambda T_0}) &= 1 + \lambda \int_0^\infty \mathrm{e}^{\lambda s} P_x(T_0 > s)\mathrm{d}s \\
&= 1 + \lambda \left(\int_0^{t_0} \mathrm{e}^{\lambda s} P_x(T_0 > s)\mathrm{d}s + \int_{t_0}^\infty \mathrm{e}^{\lambda s} P_x(T_0 > s)\mathrm{d}s \right) \\
&< 1 + \lambda t_0 \mathrm{e}^{\lambda t_0} + \int_{t_0}^\infty \mathrm{e}^{(\lambda - \lambda^* + \varepsilon)s}\mathrm{d}s \\
&< \infty.
\end{aligned}$$

proving the claim.

This completes the proof of Theorem 1.4.

1.3 L^2 and Spectral Theory of the diffusion process

1.3.1 Spectral Theory and the Sturm Liouville problem

As in [8] we will give a L^2 version of the semigroup P_t and its associated generator $\mathcal{L}$. The analysis is based on the theory of the Sturm Liouville problem, which has recently been studied in detail in [19].

Let μ be the speed measure of the process. For $f, g \in L^2(\mu)$, we define

$$(f, g)_\mu = \int_0^\infty f(s)g(s)\mu(\mathrm{d}s).$$

Let $AC_{loc}(0, \infty)$ be the space of locally absolutely continuous functions on $(0, \infty)$, let $D_{\max} = \{f : f, \mathrm{e}^{\gamma}(\mathrm{e}^{-\gamma} f')' \in AC_{loc}(0, \infty) \cap L^2(\mu)\}$ and let $D_0 = \{f \in D_{\max} : f$ has compact support$\}$. Consider the following operators on $L^2(\mu)$: $\mathcal{L}_{\max} f = -\frac{1}{2}\mathrm{e}^{\gamma}(\mathrm{e}^{-\gamma} f')'$ for $f \in D_{\max}$ and $\mathcal{L}_0 f = -\frac{1}{2}\mathrm{e}^{\gamma}(\mathrm{e}^{-\gamma} f')'$ for $f \in D_0$. Note that these expressions are defined μ almost-everywhere in both cases. For f, g in D_0 $(\mathcal{L}_0 f, g)_\mu = (f', g')_\mu$ is a symmetric form on D_0.

Following Lemma (10.3.1) of [19], let $D_{\min}$ and $D_{\max}$ be dense in $L^2(\mu)$, let $\mathcal{L}_0$ be closable, so that its closure, denoted by $\mathcal{L}_{\min}$ is closed, symmetric, densely defined (on D_0) and let $\mathcal{L}_{\min}$ and $\mathcal{L}_{\max}$ satisfy $\mathcal{L}_{\min}^* = \mathcal{L}_{\max}$ and let $\mathcal{L}_{\max}^* = \mathcal{L}_{\min}$. Then, for any self-adjoint extension $\mathcal{L}$ of $\mathcal{L}_{\min}$ we have $\mathcal{L}_{\min} \subseteq \mathcal{L} = \mathcal{L}^* \subseteq \mathcal{L}_{\max}$.

We will see that in our case, $\mathcal{L}_{\min}$ is itself a self-adjoint operator with no proper self-adjoint extensions on $L^2(\mu)$, i.e. if $\mathcal{L}$ is a self-adjoint extension of $\mathcal{L}_{min}$, it satisfies $\mathcal{L}_{min} = \mathcal{L} = \mathcal{L}^* = \mathcal{L}_{max}$. For this purpose, it is necessary to give an appropriate classification of the end points 0 and ∞. We introduce the following notions for the endpoint 0.

Definition 1.9 (i) 0 *is a regular end point if* $\int_0^{\mathrm{d}} \mathrm{e}^{-\gamma(s)}\mathrm{d}s < \infty$ *holds for some (and therefore, for all)* $\mathrm{d} > 0$. *If an end point is not regular, it is called singular.*

(ii) 0 *is a limit circle (LC) end point if all the solutions of the equation* $-\frac{1}{2}\mathrm{e}^{\gamma}(\mathrm{e}^{-\gamma} f')' = \lambda f$, $\lambda \in \mathbb{C}$, *are in* $L^2(\mu, (0, \mathrm{d}))$ *for some* $0 < \mathrm{d} < \infty$. *If* 0 *is not a LC end point, it is called a limit point (LP).*

(iii) 0 *is an oscillatory (O) end point if there exists a non trivial real-valued solution to* $-\frac{1}{2}\mathrm{e}^{\gamma}(\mathrm{e}^{-\gamma} f')' = \lambda f$, $\lambda \in \mathbb{R}$, *with an infinite number of zeros in any neighborhood of the origin. Otherwise, we say that* 0 *is a non-oscillatory (NO) end point.* □

Similar definitions hold for $+\infty$. We recall that the LC/LP classifications are independent of λ, but the O/NO depend on λ in general. It is clear that under hypothesis (H) both end points are singular, in fact

$$\int_0^{\mathrm{d}} \mathrm{e}^{-\gamma(s)}\mathrm{d}s = \infty \text{ and } \int_{\mathrm{d}}^{\infty} \mathrm{e}^{\gamma(s)}\mathrm{d}s = \infty \quad \text{for all } 0 < \mathrm{d} < \infty.$$

In the next result we show that 0 and ∞ are also LP endpoints.

Lemma 1.10 *Assume* (H) *holds. Then* 0 *and* $+\infty$ *are of LP type.*

Proof: Since the classification does not depend on the value of λ, we will use $\lambda = 0$. In this case, the solutions are linear combinations of $f_1(x) = 1$ and $f_2(x) = \Lambda(x)$. The case 0 is trivial since $\int_0^{\mathrm{d}} |f_1(s)|^2 \mathrm{e}^{-\gamma(s)}\mathrm{d}s = \int_0^{\mathrm{d}} \mathrm{e}^{-\gamma(s)}\mathrm{d}s = \infty$, for all $\mathrm{d} > 0$. For the $+\infty$ case, note that $\int_{\mathrm{d}}^{\infty} (\mathrm{e}^{\gamma(s)}/\Lambda(s)^2)\mathrm{d}s = 1/\Lambda(\mathrm{d}) < \infty$ and $0 < \mathrm{e}^{-\gamma(\mathrm{d})}\Lambda^2(\mathrm{d}) < \infty$, for all $0 < \mathrm{d} < \infty$, Then, for $M > 0$ we have

$$\begin{aligned} M &= \int_{\mathrm{d}}^{M+\mathrm{d}} \left(1_{\{\mathrm{e}^{-\gamma(s)}\Lambda^2(s)>1\}} + 1_{\{\mathrm{e}^{-\gamma(s)}\Lambda^2(s)\leq 1\}}\right) \mathrm{d}s \\ &\leq \int_{\mathrm{d}}^{M+\mathrm{d}} \mathrm{e}^{-\gamma(s)}\Lambda^2(s)\mathrm{d}s + \int_{\mathrm{d}}^{M+\mathrm{d}} \frac{\mathrm{e}^{\gamma(s)}}{\Lambda(s)^2}\mathrm{d}s \\ &\leq \int_{\mathrm{d}}^{\infty} \mathrm{e}^{-\gamma(s)}\Lambda^2(s)\mathrm{d}s + \frac{1}{\Lambda(\mathrm{d})}. \end{aligned}$$

Since the inequality holds for all $M > 0$, letting $M \to \infty$ we conclude that the integral on the right hand diverges. □

Theorem 10.4.1 of [19] states that if (and only if) 0 and ∞ are LP end points, then $\mathcal{L}_{\min}$ is itself a self-adjoint operator and has no proper self-adjoint extensions on $L^2(\mu)$ (see above). We conclude that $\mathcal{L}_0$ is a symmetric, closable, densely defined operator on $L^2(\mu)$, whose smallest closure $\mathcal{L}_{\min}$ (denoted by $\mathcal{L}$ in the sequel) is a self-adjoint operator with no proper extensions, and it is also Markovian. Hence, $\mathcal{L}$ is a regular Dirichlet form and possesses the local property (see for example [8] and [16] Theorem 2.1.2). We recall that the approach is the same as that in [8] if D_0 is replaced by $\mathcal{C}_0^\infty(0,\infty)$ (which is included in D_0). Then, Theorem 6.2.2 of [16] applied as in [8] gives the existence of a nonpositive self-adjoint operator $\mathcal{L}$ on $L^2(\mu)$ with domain $D(\mathcal{L}) \supseteq C_0^\infty((0,\infty))$ such that, for $f, g \in C_0^{((0,\infty))}$, $\mathcal{E}(f,g) = -2(f, \mathcal{L}g)_\mu$ (see [16], Theorem 1.3.1). We point out that for $g \in C_0^\infty((0,\infty))$, $\mathcal{L}g = \frac{1}{2}g'' - \alpha g'$. Moreover, $\mathcal{L}$ is the generator of a strongly continuous symmetric semigroup of contractions on $L^2(\mu)$ denoted by $(P_t : t \geq 0)$. This semigroup is subMarkovian, that is, $0 \leq P_t \leq 1$ $\mu-$almost everywhere if $0 \leq f \leq 1$ (see [16] Theorem 1.4.1)).

Following [8] it can be shown that The semigroup P_t and the semigroup induced by the strong Markov process $(X_{t\wedge T})$ coincides on the set of smooth and compactly supported functions. Also, from [8] we know that, when absorption is sure, that is $(H1)$ holds, the semigroup coincides with the semigroup of X killed at 0, that is, $P_t f(x) = \mathbb{E}_x(f(X_t)1_{T_0>t})$.

Now we will show the discreteness of the spectrum which is the main theorem of this section.

Proposition 1.11 *Assume that (H) holds. Then $-\mathcal{L}$ has purely discrete spectrum. The eigenvalues:*

$$-\infty < \lambda_0 < \lambda_1 < ...\lambda_2 < ...$$

are simple, $\lim_{n\to\infty} \lambda_n = +\infty$, and the eigenfunction ψ_n associated to λ_n has exactly n roots belonging to $(0,\infty)$ and a orthonormal basis of $L^2(\mu)$. In particular, we can take ψ_{λ_0} strictly positive.

Proof: Recall that we are analyzing the non nontrivial solutions of the equation

$$(\mathrm{e}^{-\gamma}\psi_\lambda')' = -2\lambda \mathrm{e}^{-\gamma}\psi_\lambda$$

Let $\sigma_{\rm e} = \sigma_{\rm e}(\mathcal{L})$ the essential spectrum and $\sigma_{\rm d} = \sigma_{\rm d}(\mathcal{L})$ the discrete part of the spectrum. From part 8 of Theorem (10.4.1) of [19], we know that if (at least) one end point is LP, then either $\sigma_{\rm e}$ or $\sigma_{\rm d}$ is nonempty. Let $\sigma_0 = \inf \sigma_{\rm e}$, the result will be shown once we prove $\sigma_0 = +\infty$. Since σ_0 is such that all the nontrivial solutions of $\mathcal{L}\psi_\lambda = -\lambda\psi_\lambda$ are NO for $\lambda < \sigma_0$ and O for $\lambda > \sigma_0$, we need to prove that the solutions are NO for each $\lambda > 0$.

Using the same argument as in Theorem 3.16 of [20], we find that between a local minimum $\underline{x}_{\rm i}$ and a local maximum $\overline{x}_{\rm i} \in (0,\infty)$, there exists exactly one solution to the equation $\psi_\lambda(x) = 0$ and also $\psi_\lambda(\underline{x}_{\rm i}) < 0$ and $\psi_\lambda(\overline{x}_{\rm i}) > 0$. Moreover, for each pair of consecutive $\underline{x}_{\rm i}$ and $\overline{x}_{\rm i}$ local extrema, we have

$$\frac{1}{2\lambda} < \int_{\underline{x}_{\rm i}}^{\overline{x}_{\rm i}} \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s\,.$$

Then the relation

$$\int_0^\infty \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s > \int_\varepsilon^\infty \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s > \sum_{(\underline{x}_{\rm i},\overline{x}_{\rm i})} \int_{\underline{x}_{\rm i}}^{\overline{x}_{\rm i}} \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s \geq \frac{1}{2\lambda}\#(\underline{x}_{\rm i},\overline{x}_{\rm i})$$

is satisfied for all $\varepsilon > 0$, where $\#(\underline{x}_{\rm i},\overline{x}_{\rm i})$ denotes the number of solutions to $\psi_\lambda(x) = 0$ in (ε,∞). So, the number of roots in $[\varepsilon,\infty)$ is bounded uniformly by $2\lambda\int_0^\infty \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s + 1$. The result follows by letting $\varepsilon \to 0^+$. □

A direct consequence of the previous proof is the inequality $\lambda_n \geq \frac{n}{2\int_0^\infty \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s}$. Indeed, it is impled by the fact that ψ_{λ_n} has exactly n roots. Also we can erase the term $+1$ if we already know that ψ_λ has a finite number of zeros.

Theorem 1.12 *Assume (H) holds. Then,*

$$\begin{aligned}(P_t f, g)_\mu &= \sum_{\rm i\geq 0} \mathrm{e}^{-\lambda_{\rm i} t}(\psi_{\lambda_{\rm i}}, f)_\mu(\psi_{\lambda_{\rm i}}, g)_\mu, \quad \text{for all } f, g \in L^2(\mu)\\ P_t f &= \sum_{\rm i\geq 0} \mathrm{e}^{-\lambda_{\rm i} t}(\psi_{\lambda_{\rm i}}, f)_\mu \psi_{\lambda_{\rm i}}(x) \qquad \text{for all } f \in L^2(\mu)\end{aligned}$$

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t}(g, P_t f)_\mu = (\psi_{\lambda_0}, f)_\mu(\psi_{\lambda_0}, g)_\mu \quad \text{for all } f, g \in L^2(\mu), f \geq 0, g \geq 0. \tag{1.3}$$

Proof: It is straightforward from the L^2 version of the process. □

Theorem is similar to Theorem 3.2 of [8], with the main difference being that here we do not impose extra conditions on α to get a discrete spectrum. In fact, we only assume regularity on α to guarantee the existence of a diffusion process, a hypothesis that is often assumed to avoid technical problems.

The next step is to prove the existence (and as we will see below, also uniqueness) of QSDs. The function $\mathrm{e}^{-\gamma}\psi_{\lambda_0}$ is a natural candidate for being a QSD; in fact, for all

$f \in L^2(\mu)$ we have:

$$(\psi_{\lambda_0}, P_t f)_\mu = (\psi_{\lambda_0}, f)_\mu$$

Then, $\mathrm{e}^{-\gamma}\psi_{\lambda_0}$ satisfies the necessary condition on a restricted set of functions. We need to prove the following two facts: (i) $\psi_{\lambda_0} \in L^1(\mu)$ and (ii)$\lambda_0 > 0$. The latter assertion follows from 1.3 . In fact, since ψ_{λ_0} is a positive element in $L^2(\mu)$ we can take $f = g = \psi_{\lambda_0}$ in (1.3) to obtain

$$0 = \lim_{t\to\infty} (P_t\psi_{\lambda_0}, \psi_{\lambda_0})_\mu = \lim_{t\to\infty} \mathrm{e}^{-\lambda_0 t} |\psi_{\lambda_0}|^2_{L^2(\mu)}$$

This implies $\lambda_0 > 0$.

It remains to prove the integrability of ψ_{λ_0}. In Theorem 1.13 we will prove that all the eigenfunctions are in $L_1(\mu)$. First, let us collect some elementary properties of the eigenfunctions.

- $+\infty$ is a LP, and from [19] Lemma 10.4.1 we know that, for all $g \in D_{\max}$ $\lim_{x\to\infty} \mathrm{e}^{-\gamma}[\psi'_{\lambda_k} g - g'\psi_{\lambda_k}](x) = 0$. If we choose $g \in D_{\max}$ such that $g(x) = 1$ for large x (the existence of such a g is guaranteed by $\int_1^\infty 1^2 \mathrm{e}^{-\gamma(s)} < \infty$), we obtain $\lim_{x\to\infty} \mathrm{e}^{-\gamma}\psi'_{\lambda_k}(x) = 0$.
- Since $\psi_{\lambda_0} > 0$, and it satisfies $\left(\mathrm{e}^{-\gamma(x)}\psi'_{\lambda_0}(x)\right)' = -2\lambda_0 \mathrm{e}^{-\gamma(x)}\psi_{\lambda_0}(x)$, $x > 0$, we obtain $\mathrm{e}^{-\gamma(x)}\psi'_{\lambda_0}(x) > \lim_{x\to\infty} \mathrm{e}^{-\gamma}\psi'_{\lambda_0}(x) = 0$. In particular, ψ_{λ_0} is increasing function.
- The last assertion implies the existence of the limit $\psi_{\lambda_0}(0+) = \lim_{x\to 0}\psi_{\lambda_0}(x)$ and also that its value is 0. In fact, if the limit is greater than 0 we obtain $\int_0^1 \psi^2_{\lambda_0}(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s > \psi^2_{\lambda_0}(0+)\int_0^1 \mathrm{e}^{-\gamma(s)}\mathrm{d}s = \infty$, which is a contradiction.

For $\psi_{\lambda_{\mathrm{i}}}$ $\mathrm{i} \geq 1$ we can state similar results; in fact, without loss of generality, we can suppose $\psi_{\lambda_{\mathrm{i}}} > 0$ in $(0, x_{1,\mathrm{i}})$, where $x_{1,\mathrm{i}}$ denotes the smallest positive solution of $\psi_{\lambda_{\mathrm{i}}} = 0$, in which case $\psi'_{\lambda_{\mathrm{i}}}(x_{1,\mathrm{i}}) < 0$, (or equivalently $\mathrm{e}^{-\gamma(x_{1,\mathrm{i}})}\psi'_{\lambda_{\mathrm{i}}}(x_{1,\mathrm{i}}) < 0$) and $\mathrm{e}^{-\gamma}\psi'_{\lambda_{\mathrm{i}}}$ decreases $(0, x_{1,\mathrm{i}})$. Moreover there exists $0 < x^*_{0,\mathrm{i}} < x_{1,\mathrm{i}}$ such that $\mathrm{e}^{-\gamma}\psi'_{\lambda_{\mathrm{i}}}(x^*_{0,\mathrm{i}}) = 0$ (because otherwise we obtain $\psi_{\lambda_{\mathrm{i}}}(0+) > 0$, which is a contradiction). Then, for $x \in (0, x^*_{0,\mathrm{i}})$, $\psi'_{\lambda_{\mathrm{i}}}(x) > \psi'_{\lambda_{\mathrm{i}}}(x^*_{0,\mathrm{i}}) = 0$, so we conclude that $\psi_{\lambda_{\mathrm{i}}}$ is positive and increasing for some neighborhood of 0.

1.3.2 Integrability of the eigenfunctions

First, let us note that for all $\delta > x$ and $k \geq 0$,

$$\begin{aligned}
\int_x^\infty |\psi_{\lambda_{\mathrm{i}}}(s)|\mathrm{e}^{-\gamma(s)}\mathrm{d}s &\leq \sqrt{\int_x^\infty \mathrm{e}^{-\gamma(y)}\mathrm{d}y}\sqrt{\int_x^\infty \psi_{\lambda_{\mathrm{i}}}(y)^2\mathrm{e}^{-\gamma(y)}\mathrm{d}y} \\
&< \sqrt{\int_x^\infty \mathrm{e}^{-\gamma(y)}\mathrm{d}y} \\
&< \infty\,.
\end{aligned}$$

Then, $\int_x^\infty |\psi_{\lambda_{\mathrm{i}}}(s)|\mathrm{e}^{-\gamma(s)}\mathrm{d}s$ converges if and only if $\int_0^x |\psi_{\lambda_{\mathrm{i}}}(s)|\mathrm{e}^{-\gamma(s)}\mathrm{d}s$ for some (and, therefore, for all) $x > 0$. The next theorem shows that, in fact, all the eigenfunctions are absolutely integrable with respect to the measure μ.

Theorem 1.13 *Let Hypothesis (H) holds, then for all $k \geq 0$, $\psi_{\lambda_k} \in L^1(\mu)$.*

Proof: Let us recall the equation

$$\left(\mathrm{e}^{-\gamma(x)}\psi'_{\lambda_\mathrm{i}}(x)\right)' = -2\lambda_\mathrm{i}\mathrm{e}^{-\gamma(x)}\psi_{\lambda_\mathrm{i}}(x)$$

Let $x_{0,0} = \hat{x}^*_{0,0}$, $x_{0,\mathrm{i}} = \min(x^*_{0,\mathrm{i}}, \hat{x}^*_{0,\mathrm{i}})$, where $\hat{x}^*_{0,\mathrm{i}}$ is the solution to the equation $2\lambda_\mathrm{i}\int_0^x \mathrm{e}^{-\gamma(s)}\Lambda(s)\mathrm{d}s = 1$, and $x^*_{0,\mathrm{i}}$ denotes the smallest solution to $\mathrm{e}^{-\gamma}\psi'_{\lambda_\mathrm{i}}(x) = 0$.

Take $g \in D_{\max}$ such that $g(x) = -\Lambda(x)$ for $x \in (0, x^*_{0,\mathrm{i}})$. By integration by parts,

$$\psi_{\lambda_\mathrm{i}}(x) - \mathrm{e}^{-\gamma(x)}\psi'_{\lambda_\mathrm{i}}\Lambda(x) - \left(\mathrm{e}^{-\gamma(\varepsilon)}\psi'_{\lambda_\mathrm{i}}(\varepsilon)\Lambda(\varepsilon) - \psi_{\lambda_\mathrm{i}}(\varepsilon)\right) = 2\lambda_\mathrm{i}\int_\varepsilon^x \Lambda(s)\psi_{\lambda_\mathrm{i}}(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s.$$

Letting $\varepsilon \to 0^+$, since 0 is an LP endpoint, Lemma 10.4.1 of [19] implies that

$$\mathrm{e}^{-\gamma}\left[\psi'_{\lambda_\mathrm{i}}\Lambda - \psi_{\lambda_\mathrm{i}}\Lambda'\right](\varepsilon) \to 0^+.$$

By the Monotone Convergence Theorem,

$$\int_\varepsilon^x \Lambda(s)\psi_{\lambda_\mathrm{i}}(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s \to \int_0^x \Lambda(s)\psi_{\lambda_\mathrm{i}}(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s\,.$$

So, we obtain

$$\psi_{\lambda_\mathrm{i}}(x) - \mathrm{e}^{-\gamma(x)}\psi'_{\lambda_\mathrm{i}}(x)\Lambda(x) = 2\lambda_\mathrm{i}\int_0^x \Lambda(s)\psi_{\lambda_\mathrm{i}}(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s\,.$$

We know that in $(0, x_{0,\mathrm{i}})$, $\psi_{\lambda_\mathrm{i}}$ is positive and increasing, so the following inequality holds

$$\psi_{\lambda_\mathrm{i}}(x) - \mathrm{e}^{-\gamma(x)}\psi'_{\lambda_\mathrm{i}}(x)\Lambda(x) \leq 2\lambda_\mathrm{i}\psi_{\lambda_\mathrm{i}}(x)\int_0^x \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s\,.$$

Using the fact that $\psi_{\lambda_\mathrm{i}}(x) = -\frac{1}{2\lambda_\mathrm{i}}(\mathrm{e}^{-\gamma(x)}\psi'_{\lambda_\mathrm{i}}(x))'\mathrm{e}^{\gamma(x)}$ and by multiplying by $2\lambda_\mathrm{i}\mathrm{e}^{-\gamma(x)}$, we obtain

$$2\lambda_\mathrm{i}\Lambda(x)\mathrm{e}^{-\gamma(x)}\left(\mathrm{e}^{-\gamma(x)}\psi'_{\lambda_\mathrm{i}}(x)\right) + \left(\mathrm{e}^{-\gamma(x)}\psi'_{\lambda_\mathrm{i}}(x)\right)'\left(1 - 2\lambda_\mathrm{i}\int_0^x \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s\right) \geq 0\,.$$

Dividing by $\left(1 - 2\lambda_\mathrm{i}\int_0^x \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s\right)^2$ (which is strictly positive in $(0, x_{0,\mathrm{i}})$), and by noticing that $\left(1 - 2\lambda_\mathrm{i}\int_0^x \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s\right)' = -2\lambda_\mathrm{i}\Lambda(x)\mathrm{e}^{-\gamma(x)}$, we deduce that

$$\left(\frac{\mathrm{e}^{-\gamma(x)}\psi'_{\lambda_\mathrm{i}}(x)}{1 - 2\lambda_\mathrm{i}\int_0^x \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s}\right)' \geq 0 \qquad (1.4)$$

Then, for $0 < x < y < x_{0,\mathrm{i}}$ the following relations is satisfied

$$\mathrm{e}^{-\gamma}\psi'_{\lambda_\mathrm{i}}(0+) < \frac{\mathrm{e}^{-\gamma}\psi'_{\lambda_\mathrm{i}}(x)}{1 - 2\lambda_\mathrm{i}\int_0^x \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s} < \frac{\mathrm{e}^{-\gamma}\psi'_{\lambda_\mathrm{i}}(y)}{1 - 2\lambda_\mathrm{i}\int_0^y \Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s} < \infty$$

The right hand inequality follows from

$$\left|\frac{\mathrm{e}^{-\gamma(x)}\psi'_\lambda(x)}{2\lambda_{\mathrm{i}}}\right| = \left|\int_x^\infty \mathrm{e}^{-\gamma(y)}\psi_{\lambda_{\mathrm{i}}}(y)\mathrm{d}y\right| \leq \int_x^\infty \mathrm{e}^{-\gamma(y)}|\psi_{\lambda_{\mathrm{i}}}(y)|\mathrm{d}y < \infty \text{ for all } x > 0.$$

It guarantees the integrability of $\psi_{\lambda_{\mathrm{i}}}$ because

$$\begin{aligned}
\int_0^{x_{0,\mathrm{i}}/2} |\psi_{\lambda_{\mathrm{i}}}(s)|\mathrm{e}^{-\gamma(s)}\mathrm{d}s &= \int_0^{x_{0,\mathrm{i}}/2} \psi_{\lambda_{\mathrm{i}}}(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s \\
&= \frac{\mathrm{e}^{-\gamma}\psi'_{\lambda_{\mathrm{i}}}(0+) - \mathrm{e}^{-\gamma}\psi'_{\lambda_{\mathrm{i}}}(x_{0,\mathrm{i}}/2)}{2\lambda_{\mathrm{i}}} \\
&\leq \mathrm{e}^{-\gamma}\psi'_{\lambda_{\mathrm{i}}}\left(\frac{x_{0,\mathrm{i}}}{2}\right) \frac{2\lambda_{\mathrm{i}}\int_0^{x_{0,\mathrm{i}}/2}\Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s}{1 - 2\lambda_{\mathrm{i}}\int_0^{x_{0,\mathrm{i}}/2}\Lambda(s)\mathrm{e}^{-\gamma(s)}\mathrm{d}s} \\
&< \infty.
\end{aligned}$$

□

1.4 QSD and the Yaglom Limit

1.4.1 Existence

In the previous section we showed that ψ_{λ_0} is a strictly positive $\mu-$integrable function. Then, by standard methods, for instance as in Theorem 5.2 of [8], the normalized measure defined by $\mathrm{e}^{-\gamma}\psi_{\lambda_0}$ is a QSD. From theorem 1.4, a QSD exists only if it is associated to the value λ^*. Hence, we have proven the following result.

Theorem 1.14 *If Hyphotesis (H) holds then there exists a unique QSD given by*

$$\nu(\mathrm{d}x) = \frac{\psi_{\lambda_0}(x)}{(\mathbf{1}_{(0,\infty)}, \psi_{\lambda_0})_\mu}\mathrm{d}x\,.$$

Moreover $\lambda_0 = \lambda^*$.

In our case, i.e. one-dimensional diffusions killed at 0 verifying hypothesis (H), we will use the same arguments as in the proof of Theorem 5.3 of [8] to show the existence of a Yaglom limit. To achieve this, we need first to study in detail the behavior of the transition density of the diffusion process. We recall Theorem 2.3 of [8], which states that hypothesis $(H1)$ guarantees that, for all $x > 0$ and $t > 0$, there exists a density $r(t, x, y)$ that satisfies

$$\mathbb{E}_x(f(X_t)1_{T_0>t}) = \int_0^\infty r(t,x,y)f(y)\mu(\mathrm{d}y)\,. \tag{1.5}$$

Moreover, we also have the following result on the density $r(t, x, y)$ of (1.5). Under hypothesis (H) the density satisfies

$$r(t,x,y) = \sum_{k\geq 0} \mathrm{e}^{-\lambda_k t}\psi_{\lambda_k}(x)\psi_{\lambda_k}(y) \text{ uniformly on compact sets of } (0,\infty)^3\,. \tag{1.6}$$

The proof of this property is similar to that of Proposition 3.3 of [8], since it only uses the discrete spectrum property. Let us state the last required property on the density $r(t,x,y)$, in order to obtain the Yaglom limit property similarly as was done in Theorem 5.2 of [8].

Proposition 1.15 *Assume that hypothesis (H) holds. Then $r(t,x,y) \in L^2(\mu)$ for all $t>0$ and $x>0$. Moreover, there exists a function $B(t) \geq 0$, $\lim_{t\to\infty} B(t) = 0$, such that*

$$\int_0^\infty r^2(t,x,y)\mu(\mathrm{d}y) < r(t,x,x)B(t) < \infty \qquad \text{for all } t>0\,, x>0\,.$$

Proof: From relation 1.6 and the Cauchy Schwartz inequality, we obtain

$$r^2(t,x,y) \leq \sum_{k\geq 0} \mathrm{e}^{-\lambda_k t}\psi^2_{\lambda_k}(x) \sum_{k\geq 0} \mathrm{e}^{-\lambda_k t}\psi^2_{\lambda_k}(y) = r(t,x,x) \sum_{k\geq 0} \mathrm{e}^{-\lambda_k t}\psi^2_{\lambda_k}(y)\,,$$

where the series are convergent. Moreover, on any compact set K of $\mathbb{R}^+$, we obtain

$$\int_K r^2(t,x,y)\mu(\mathrm{d}y) \leq r(t,x,x) \int_K \sum_{k\geq 0} \mathrm{e}^{-\lambda_k t}\psi^2_{\lambda_k}(y)\mu(\mathrm{d}y)$$

By Tonelli's theorem,

$$\int_K \sum_{k\geq 0} \mathrm{e}^{-\lambda_k t}\psi^2_{\lambda_k}(y)\mu(\mathrm{d}y) = \sum_{k\geq 0} \mathrm{e}^{-\lambda_k t} \int_K \psi^2_{\lambda_k}(y)\mu(\mathrm{d}y) \leq \sum_{k\geq 0} \mathrm{e}^{-\lambda_k t},$$

since $|\psi_{\lambda_k}|_{L^2(\mu)} = 1$. On the other hand, we know that $\frac{k}{2\int_0^\infty \mathrm{e}^{-\gamma(s)}\Lambda(s)\mathrm{d}s} = kJ$ is a lower bound for λ_k, so $\mathrm{e}^{-\lambda_k t} \leq \mathrm{e}^{-kJt}$ and moreover, there exists k_0 such that $\lambda_{k_0} \geq k_0 J \geq \lambda_0$. It follows that

$$\sum_{k\geq 0} \mathrm{e}^{-\lambda_k t} \leq \mathrm{e}^{-\lambda_0 t} + \ldots + \mathrm{e}^{-\lambda_{k_0-1} t} + \sum_{k\geq k_0} \mathrm{e}^{-tkJ} \leq \mathrm{e}^{-\lambda_0 t}\left(k_0 + \frac{1}{1-\mathrm{e}^{-Jt}}\right) = B(t)$$

We obtain, for any compact set K, the inequality

$$\int_K r^2(t,x,y)\mu(\mathrm{d}y) \leq r(t,x,x)B(t)$$

Since the bound on the right hand does not depend on K, letting it to tend to $\mathbb{R}^+$, yields the result. □

Theorem 1.16 *Assume that hypothesis (H) holds. Then, for all $x>0$ and all Borel subsets $A \subseteq (0,\infty)$,*

$$\begin{aligned} \lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} P_x(T_0 > t) &= \psi_{\lambda_0}(x)(\psi_{\lambda_0}, 1)_\mu \\ \lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} P_x(X_t \in A, T_0 > t) &= \nu(A)\psi_{\lambda_0}(x)(\psi_{\lambda_0}, 1)_\mu. \end{aligned}$$

We also have

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} P_x(X_t \in A | T_0 > t) = \nu(A)$$

that is ν is the Yaglom limit distribution. Moreover, any measure ρ with compact support in $(0,\infty)$ satisfies,

$$\begin{aligned}
\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} P_\rho(T_0 > t) &= (\psi_{\lambda_0}, 1)_\mu \int \psi_{\lambda_0} \rho \\
\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} P_x(X_t \in A, T_0 > t) &= \nu(A)(\psi_{\lambda_0}, 1)_\mu \int \psi_{\lambda_0} \rho. \\
\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} P_\rho(X_t \in A | T_0 > t) &= \nu(A).
\end{aligned}$$

Proof: The same proof as in [8] works because $r(t,x,y)$ fulfills all the required properties.

Bibliography

[1] A. M. Yaglom. Certain limit theorems of the theory of branching random processes. Doklady Akad. Nauk SSSR (N.S.), 56:795Ð798, 1947.

[2] J. N. Darroch and E. Seneta. On quasi-stationary distributions in absorbing continuous-time finite Markov chains. J. Appl. Probability, 4:192Ð196, 1967.

[3] E. Seneta and D. Vere-Jones. On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. J. Appl. Probab. 3:403Ð434, 1966.

[4] Van Doorn E. Quasi Stationary Distributions and convergence of quasi-stationary of birth-death processes. Adv. Appl. Prob. 23, 683-700 (1991).

[5] Mandl P. Spectral theory of semigroups connected with diffusion process and its application. Czech. Math. J. 11 558-569. MR0137143(1962).

[6] Ikeda, N. and Watanabe, S. (1989). *Stochastic Differential Equations and Diffusion Processes*. 2nd ed. North-Holland Mathematical Library 24. North-Holland, Amsterdam.

[7] Karlin, S. and Taylor, H. M. (1981). *A second course in Stochastic Processes*. 2nd. edition.

[8] Cattiaux, P., Collet, P., Lambert, A., Martínez, S., Méléard, S., San Martín, J.(2009). Quasi stationary distributions and diffusion models in population dynamics. Annals of Probability, 37(5):1926-1969.

[9] Ferrari P., Martínez S., Picco P. (1992). Existence of non-trivial quasistationary distributions in the birth-death chain. Adv. Appl. Prob. 24, 795-813.

[10] P. A. Ferrari, H. Kesten, S. Martinez, and P. Picco. Existence of Quasi-Stationary Distributions. A Renewal Dynamical Approach . Ann. Probab. Volume 23, Number 2 (1995), 501-521.

[11] Servet Martinez, Pierre Picco, and Jaime San Martin. Domain of attraction of quasi-stationary distributions for the Brownian motion with drift. Adv. in Appl. Probab. Volume 30, Number 2 (1998), 385-408.

[12] Collet P., Martínez S., San Martín J. (1995). Asymptotic laws for one-dimensional diffusions conditioned to nonabsorption, Ann. Probab. 3, no. 3, pp. 1300-1314.

[13] Servet Martinez and Jaime San Martin. Quasi-Stationary Distributions for a Brownian Motion with Drift and Associated Limit Laws. Journal of Applied Probability Vol. 31, No. 4 (Dec., 1994), pp. 911-920

[14] Servet Martínez, Jaime San Martín. Rates of Decay and h-Processes for One Dimensional Diffusions Conditioned on Non-Absorption . Journal of Theoretical Probability January 2001, Volume 14, Issue 1, pp 199-212

[15] Servet Martínez and Jaime San Martín. Classification of Killed One-Dimensional Diffusions. The Annals of Probability Vol. 32, No. 1, A (Jan., 2004), pp. 530-552

[16] Fukushima, M (1980). *Dirichlet Forms and Markov Process*. Kodansha, North-Holland, Amsterdam.

[17] Fukushima, M., Oshima, Y. and Takeda, M. (1994). *Dirichlet forms and Symmetric Markov Process*. Studies in Mathematics 19, de Gruyter, Berlin.

[18] Jacod, J. (1979). *Calcul Stochastique et Problèmes de Martingales*. Lectures Notes in Mathematics 714. Springer, Berlin. MR542115.

[19] Zettl, A. (2005). *Sturm Loiuville Theory*. Vol. 121, Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI.

[20] Kolb, M. and Steinsaltz, D. Quasi limiting behavior for one dimensional diffusions with killing. To appear Annals of Probability.

Chapter 2

Optimal Bounds on the one dimensional Long Range Ising Model and its consequences

2.1 History of the One dimensional Ising model with long range interactions

The history of one dimensional Ising models with long range interactions can be divided in 4 periods:

- 1966-1973

In this period the fundamental works of Dobrushin and Lanford & Ruelle set the framework for mathematical statistical mechanics:

The definition of Gibbs states are linked to the so called DLR equations or DLR states for (Dobrushin-Lanford-Ruelle) that comes from the two articles [1] [2]. Conditions for uniqueness of the DLR state were simultaneously given in [3] for two body potentials and in [4] for multi body potential. They proved that if the interaction between two half lines is bounded then there is only one Gibbs or DLR states at all temperature. For a two body potential decaying as $J(r) = 1/r^{2-\alpha}$ this means $\alpha < 0$. (α must be strictly less than 1 to have stability).

On the other side of the Atlantic Ocean, Mark Kac & Colin Thompson in [6] suggested that for $\alpha \geq 0$ there is a phase transition at low enough temperature.

This was proved for $\alpha > 0$ by Dyson (see below) just after the preprint of Kac & Thompson appears, Dyson mention this as the Kac & Thompson conjecture.

They wrote in the published article that the case $\alpha = 0$ is rather special in the sense that "There is as yet no proof that this is or is not the case" (that a phase transition exists at low temperature)

Dyson proved in [5] that if $0 < \alpha < 1$ then at low enough temperature, there is spontaneous magnetization and therefore at least two Gibbs states by inventing a Hierarchical model and use the Griffiths inequality to compare the true long range model and the hierarchical one. By explicit computation the Hierarchical model has spontaneous magnetization at low enough temperature for the case corresponding to $\alpha > 0$. Concerning the case $\alpha = 0$, Dyson said : "We have no definite opinion concerning the occurrence of a transition for $J(n) = n^{-2}$, and we recommend this case as an interesting object for further study".

An historical remark seems important to the two citations mentioned above: the preprint of Kac & Thompson is quoted by Dyson in his 1969 paper. In the published version, Kac & Thompson quoted the Dyson result as a preprint.

Moreover Dyson, interested by this model, he called it Anderson model, built a Hierarchical model where the magnetization exhibit a jump at the critical temperature, he called this fact a Thouless effect. this was done in [7]. The hierarchical model introduced by Dyson was then studied by Bleher & Sinai with interest in critical behaviors in two articles [8] where gaussian limit theorem with non usual normalization is proved at the critical temperature in [9]. See also the Collet & Eckmann monograph [10]

On the other side of the Atlantic Ocean (EAST) in a rather fundamental work Dobrushin [11] where he proved that in the case $\alpha < 0$ the Free energy is an analytic function of the parameters of the interaction. The proof is given for two body potential however Dobrushin wrote "The proof is not simple; we give a complete proof here only for a bounded two body potential. The proof in the general case, and some additional results, will be published elsewhere" which is never did.

- the period 1979-1988

In 1979 a paper by Khanin & Sinai proved that for long range system with random mean zero interactions the infinite volume free energy exists even in the regime $1 \leq \alpha < 3/2$. In ferromagnetic models, this is not true since the energy grows faster than the volume. This is the so called long range Edwards and Anderson model for spin glasses [12]. In 1980 Khanin extend the uniqueness result of Dobrushin 1968 to the case of the Edwards and Anderson model with long range interactions with $0 \leq \alpha \leq 1/2$ [13] . This is a strong uniqueness results since it holds for all the boundary conditions that could depend on the realization of the disorder. In 1981 in a fundamental article Cassandro & Olivieri proved the case left by Dobrushin for analyticity in [14]. This paper is more important that just giving a complete proof of analyticity in the multi body case since it will open the way to a sequence of articles dedicated to uniqueness, infinite differentiability or analyticity of the free energy with respect to the interaction parameter in one dimension. The first one concerns the Edrwards and Anderson model with $0 \leq \alpha < 1/2$ where strong uniqueness occurs as proved bt Khanin [15], the second concerns the case of unbounded spin system when $\alpha < 0$ [16]. The third one concerns the Edwards and Anderson model in the case $1/2 \leq \alpha < 1$ which is a lot more difficult than the case $0 \leq \alpha < 1/2$ since only weak uniqueness is proved in [17] while the infinite differentiability is proved in the fourth paper [18].

On the other side of Atlantic Ocean, the ferromagnetic case $\alpha = 0$ was intensively studied : In a fondamental article Fröhlich and Spencer solved what they call the Dyson conjecture even if Dyson-Kac-Thompson conjecture is more appropriate [19] and J. Imbrie showed optimal bound on the correlation functions in [20]. These two papers are difficult to read. On the other hand, starting from a fundamental article on the long range percolation in one dimension, [21] [22]. Then Imbrie and Newman proved the existence of an intermediate phase in an article that can be considered as a milestone in mathematical physic : [23]. In the same period, but on the Pacific Ocean coast, Rogers & Thompson, in [24] give sufficient condition for uniqueness at all temperature that is weaker to the one of Dyson.

Concerning the long range Edwards and Anderson model, among the few rigorous results note that Fröhlich & Zegarlinski proved that in the large temperature phase there is a weak uniqueness when $1 \leq \alpha < 3/2$, see [25].

• The period 1990-2000

A highly non trivial model is the antiferromagnetic Ising model with long range interactions, it was first studied at the level of ground states by Burkov & Sinai [26]. Uniqueness for all temperature is delicate when $0 \leq \alpha \leq 1$ owing to a very large number of ground states,

This was proved first at very low temperature which is the most complicated part by Kerimov [27] and later at all temperature in [28]. Concerning the long range Edwards and Anderson model, among the few rigorous results one has [29]. For these models (antiferromagnetic and Edwards and Anderson model) one expect phase transition at small temperature only for $1 \leq \alpha < 3/2$. However proof of an existence of phase transition is missing.

• The period 2000-2013

In this period, the study of phase transition for one dimensional long range model changed of the side of the Atlantic Ocean and start with a re–reading of the Fröhlich & Spencer article. This was done by Cassandro Ferrari Merola and Presutti where there extend the Fröhlich & Spencer proof of a phase transition by using a Peierls argument [31]. Important results were proved using their method in particular : For low enough temperature, the existence of a phase transition for the random field long range Ising model was proved by Cassandro, Orlandi & Picco when $1/2 < \alpha < (\log 3)/(\log 2) - 1$ [30].

When $0 \leq \alpha < 1/2$, that is is the regime of uniqueness (as it follows from the Aizenman & Wher result, a description of typical configurations was given by Cassandro, Orlandi & Picco (Comm. Math. Phys.). The extension of the results of Imbrie about optimal bounds on the decay of correlation functions and rather unexpect behavior of the localization of the point separation of phases when $\alpha = 0$ were shown in a work by Cassandro, Merola, Picco & Rozikov [32] (submitted Comm. Math. Phys). They proved that when $\alpha = 0$ the localization of the point separation of phases has macroscopic fluctuations while when $\alpha > 0$ these fluctuations are microscopic, and the point of separation is located at zero when one take Dobrushin type boundary conditions on volume that are symetric with respect to the origin. This fact seems to have escaped to all the people working in this subject. Some other works in preparation include coexistence of phases in the sense of Minlos & Sinai. One important question is the limitation of the Cassandro, Ferrari, Merola & Presutti method to the regime $0 \leq \alpha < (\log 3)/(\log 2) - 1$ and is the subject of this part of the thesis.

2.2 Model Description

In the one dimensional case, each $\mathrm{i} \in \mathbb{Z}$ is associated with a value, we say $+1$, -1 which describes the spin orientation of the particle stated in the lattice i (just two states for each particle are allowed) . Each $\sigma \subset \{-1, +1\}^{\mathbb{Z}}$ is called a spin configuration and σ_{i} describes the state of the particle i. The corresponding Hamiltonian associated with the model is

$$H(\sigma) = \frac{1}{2} \sum_{\mathrm{i},j \in \mathbb{Z}} J(x,y) 1_{\sigma_x \neq \sigma_y} \tag{2.1}$$

where

$$J(n) = \begin{cases} J+1 & n = 1 \\ \frac{1}{n^{2-\alpha}} & n > 1 \end{cases} \tag{2.2}$$

and $0 \leq \alpha < 1$. In this work, homogeneous boundary conditions are considered only, so either $\sigma_x = +1$, $\sigma_x = -1$ $\forall |x| \geq L$, for some $L \geq 1$. By symmetry, we can restrict

without loss of generality to the case $\sigma_x = +1$, $|x| \geq L$.
For our purposes, it is more convenient to introduce the dual lattice $\mathbb{Z}^* \sim \mathbb{Z} + 1/2$ of nearest neighbor bonds. Each feasible configuration σ specifies a subset $\sigma^* \subset \mathbb{Z}^*$ which is the set of spin flip points, i.e. for any $x \in \mathbb{Z}$

$$\sigma_x \neq \sigma_{x+1} \Leftrightarrow x + 1/2 \in \sigma^* \tag{2.3}$$

We note that when homogeneous boundary conditions are considered, the cardinality of the spin flips points associated to any Z^* is an even number and determines uniquely a configuration of spin states with homogeneous boundary conditions.

In particular, given a configuration of spins with boundary conditions Λ^+ whose spin flip points are σ^* any subset $\sigma_0^* \in \sigma^*$ with even cardinality determines uniquely a spin configuration with boundary condition Λ^+

2.3 Triangles and Contours

2.3.1 Triangles

In this section we start recalling the content of [31][32]. For each $\mathrm{i}^* \in \Lambda^*$, we consider an interval $[\mathrm{i}^* - \frac{1}{100}, \mathrm{i}^* + \frac{1}{100}] \subset \mathbb{R}$ and choose one point in each interval, say $r_\mathrm{i} \in \mathbb{R}$ in such a way that for any four distinct points r_i, $\mathrm{i} = 1, ..., 4$ $|r_1 - r_2| \neq |r_3 - r_4|$.

We next embbed $\mathbb{R} \in \mathbb{R}^2$ where the line containing the r_i^* represents the states at $t = 0$, and the orthogonal axis represent the envolving time of a process of growing "$\vee - lines$": each point r_i^* branches into two twin lines growing in the positive half plane, in the directions respectively of angles $\pi/4$ and $3\pi/4$, until one of the two meets another line coming from a diferent r_j^*. At the instant when two branches of different $\vee - lines$ meet, they are frozen and stop their growth, at the same their twin lines dissapear, while all the other $\vee - lines$ meet associate to the other points are undisturbed and keep growing. The collision of two lines is represented graphically in the (r,t) plane by a triangle whose basis is the interval between the two points r_i^*, r_j^* roots of the two lines that collided, ad the third vertex is the point representing the collision in the plane (r, t). The construction is a way to construct a pairing of spin flips with a criterium of minimal distance. Our choice of r_i^*, makes the definition of triangles non ambiguous.

For any finite L, in the case of homogeneous b.c, the process stops at the finite time $T \leq L + 1$ giving rise to a configuration of triangles. In addition, since the cardinality of the spin flip points is even, each point r_i is the basis of some triangle.

The triangles will be denoted by T and by x_-, x_+ respectively the left and the right root of the associated $\vee - lines$. We also write

$$\begin{aligned} \Delta(T) &= [x_-, x_+] \cap \mathbb{Z} \qquad \text{The basis of the triangle T} && (2.4)\\ |T| &= \#\{\Delta(T)\} \qquad \text{The mass of the triangle T} && (2.5)\\ sf^*(T) &= \{\inf \Delta(T) - 1/2, \sup \Delta(T) + 1/2\} && (2.6) \end{aligned}$$

where $\mathbb{Z}$ is equipped with its natural order

$$\mathrm{d}ist(T, T^{'}) = \mathrm{d}ist(sf(T), sf(T^{'})) \tag{2.7}$$

From our construction it follows that for all triangles $T_{\rm i} \neq T_j$,

$$\mathrm{d}ist(T_{\rm i}, T_j) \geq \min(|T_{\rm i}|, |T_j|) \tag{2.8}$$

We denote $\mathcal{T}_{\Lambda^+}$ the set of configuration of triangles $\underline{T} = \{T_1, ..., T_n\}$ that satisfy (2.8) and such that $\Delta(T_{\rm i}) \subset \Lambda$ for $= 1, ..., n$. Note that the above construction uniquely groups the spin flip points into dipoles satisfying the distance rule given by (2.8).

2.3.2 Contours

In order to get local energy estimates, those used in Peierls argument, we collect the triangles in subsets suitably separated. As in [31] section 3.1 [32], section 4.2 respectively, the triangles of a configuration $\underline{\mathrm{T}}$ are clustered into more sophisticated structures, giving rise to a configuration $\underline{\Gamma}$ of contours. A contour Γ is a collection of "connected triangles", where the network of connections is defined hierarchically and depends on a parameter c and on a distance rule.

Formally, we will call contour to a family of triangles $\Gamma \equiv \{T : T \in \Gamma\}$ that satisfy the definition given next.

Definition 2.1 *Given a configuration of triangles $\underline{T} \in \mathcal{T}_{\Lambda^+}$, a configuration of contours $\Gamma = \Gamma(\underline{T})$ is a partition of $\underline{T}$ whose atoms, called contours are determined by the following properties*

P.0 *Let $\mathcal{R} \equiv (\Gamma_1, ..., \Gamma_N)$, $\Gamma_{\rm i} = \{T_{j,{\rm i}}, 1 \leq j \leq k_{\rm i}\}$, then $\underline{T} = \{T_{j,{\rm i}}, 1 \leq j \leq k_{\rm i}, 1 \leq j \leq n\}$.*

P.1 *Contours are well separated from each other. Any pair $\Gamma \neq \Gamma^{'}$ in $\mathcal{R}(\underline{T})$ verifies one of the following two alternatives.*

- $\Delta(\Gamma_{\rm i}) \cap \Delta(\Gamma_j) = \varnothing$.

- $\Delta(\Gamma_{\rm i}) \subseteq \Delta(\Gamma_j)$ *or* $\Delta(\Gamma_j) \subseteq \Delta(\Gamma_{\rm i})$. *If we suposse the first case,* $\forall T_{m,j} \in \Gamma_j$, *we have one of the next*

  1. $\Delta(\Gamma_{\rm i}) \subset T_{m,j}$
  2. $\Delta(\Gamma_{\rm i}) \cap T_{m,j} = \varnothing$

  where $\Delta(\Gamma) \equiv \bigcup_{T\in\Gamma} \Delta(T)$. *In both cases*

$$\mathrm{dist}(\Gamma, \Gamma') \equiv \inf_{T\in\Gamma, T'\in\Gamma'} \mathrm{dist}(T, T') > c|\Gamma_{\rm i}|^3 \tag{2.9}$$

  where $c < 1$ *is a constant that we will be chosen enough large.*

P.2 *Independence. Let* $\{\underline{T}^{(1)}..., \underline{T}^{(k)}\}$, *be* $k > 1$ *configurations of triangles;* $\mathcal{R}(\underline{T}^{(\rm i)}) = \{\Gamma_j^{(\rm i)}, j = 1, ..., n_{\rm i}\}$ *the contours of the configuration* $\underline{T}^{(\rm i)}$. *Then, for any distinct pair* $\Gamma_j^{(\rm i)}$ *and* $\Gamma_{j'}^{(\rm i')}$ *satisfie s P.1*

$$\mathcal{R}(\underline{T}^{(1)}, \underline{T}^{(2)}, ..., \underline{T}^{(k)}) = \left\{\Gamma_j^{(\rm i)}, j = 1, ..., n_{\rm i} : {\rm i} = 1, ..., k\right\} \tag{2.10}$$

For any single contour, we introduce the next definitions

$$\begin{aligned} supp(\Gamma) &\equiv \mathrm{sf}^*(\Gamma) \cap \mathbb{Z} && (2.11) \\ \mathrm{sf}^*(\Gamma) &\equiv \bigcup_{T\in\Gamma} \mathrm{sf}^*(T) && (2.12) \end{aligned}$$

A very detailed proof of existence and uniqueness of an algorithm $\mathcal{R}(\underline{T})$ satisfying definition 2.1 can be founded in Theorem 3.1 from [31]. Since we are interested into the consequences of this construction, we avoid to rewrite the proof.

We emphasize that we have a bijection between spin configurations in $\mathcal{S}_\Lambda$ and triangles in $\mathcal{T}_\Lambda$ and another between $\mathcal{T}_\Lambda$ and its image by $\mathcal{R}$, in particular there is a one to one correspondence between spin and contours configurations.

2.3.3 Internal and external Triangles

In a typical compatible triangle configuration, we could find triangles whose basis is contained into the basis of a bigger triangle or conversely, its basis could contain the basis of another triangle. In the following sections, it will be very important to distinguish those cases, so we will give next some useful definitions.

Definition 2.2 *Given a compatible configuration of triangles $\underline{T}$, a triangle $T \in \underline{T}$ is external respect to $\underline{T}$ if* $\Delta(\Gamma)$ *if its basis is not contained into the basis of another triangle that belongs to $\underline{T}$, i.e.*

$$\Delta(T) \not\subset \bigcup_{T' \in \underline{T}, T' \neq T} \Delta(T^{'}) \tag{2.13}$$

A triangle which is not external is called internal.

We emphasize that the notion of internal and external depends strongly on $\underline{T}$. For example, if $\underline{T} = \{T\}$ for an arbitrary triangle T, it is always external accordingly to the definition. Nevertheless, it could be internal respect to another configuration containing more then one triangle.

Remark 2.3 *For an arbitrary compatible configuration of triangles, we can decompose it uniquely as follows*

$$\underline{T} = \cup_{j=1}^{N_{\mathrm{ex}}(T)} (T_{j,\mathrm{ex}} \cup \underline{T}_{\mathrm{in}}(T_{j,\mathrm{ex}})) \tag{2.14}$$

where $N_{\mathrm{ex}}(T)$ *the number of external contours of* $\underline{T}$ *and* $T_{\mathrm{i,ex}}$, $\mathrm{i} = 1, ..., N_{\mathrm{ex}}(T)$ *are external contours of* $\underline{T}$. *For all* $\mathrm{i} = 1, ..., N_{\mathrm{ex}}(T)$, *we denote*

$$\underline{T}_{\mathrm{in}}(T_{j,\mathrm{ex}}) \equiv \{T \in \Gamma : \Delta(T) \subset T_{j,\mathrm{ex}}\} \tag{2.15}$$

the subset of internal contours contained into the basis of $T_{j,\mathrm{ex}}$.

In this work, we are interested mainly into the decomposition of internal and external triangles for a single contours, nevertheless the definition is valid for an arbitrary compatible configuration.

2.3.4 Internal and External Contours

In a very similar way to the triangles, and accordingly to the properties of contours, it can be possible that the basis of a typical contour can either be contained into the basis of another contour or not, so it is necessary to distinguish between those class of contours. The notion of internal and external contours is given next.

Definition 2.4 *Given a compatible configuration of contours $\underline{\Gamma}$, a contour $\Gamma \in \underline{\Gamma}$ is* *external respect to $\underline{\Gamma}$ if the basis of $\Delta(\Gamma)$ is not contained into the basis of another contour that belongs to $\underline{\Gamma}$.*

$$\forall T \in T, \Delta(T) \not\subset \bigcup_{\Gamma' \in \underline{\Gamma}, \Gamma' \neq \Gamma} \Delta(\Gamma') \tag{2.16}$$

A contour which is not external is called internal.

Similar to triangles, for a single contour, the condition of internal and external depends strongly on the configuration $\underline{\Gamma}$ and in general a contour can be external respect to $\underline{\Gamma}$ but internal respect to another compatible configuration $\underline{\Gamma}'$.

Remark 2.5 *Note that a collection of contours $\underline{\Gamma}$ can be uniquely decomposed at follows:*

$$\underline{\Gamma} = \cup_{j=1}^{N_{ex}(\underline{\Gamma})} (\Gamma_{j,ex} \cup \underline{\Gamma}_{in}(\Gamma_{j,ex})) \equiv \cup_{\Gamma_{ex} \in \underline{\Gamma}} (\Gamma_{ex} \cup \underline{\Gamma}_{in}(\Gamma_{ex})) \tag{2.17}$$

where $N_{ex}(\Gamma)$ the number of external contours of $\underline{\Gamma}$ and $\Gamma_{i,ex}$, $i = 1, ..., N_{ex}(\Gamma)$ is the set of external contours of $\underline{\Gamma}$. For all $j = 1, ..., N_{ex}(\Gamma)$, we denote $\underline{\Gamma}_{in}(\Gamma_{j,ex}) \equiv \{\Gamma \in \underline{\Gamma} : supp(\Gamma) \subset \Gamma_{j,ex}\}$.

For any single contour we set,

$$|\Gamma| \equiv \sum_{T \in \Gamma} |T| \tag{2.18}$$

$$||\Gamma||_\alpha \equiv \sum_{T \in \Gamma} |T|^\alpha \quad \alpha > 0 \tag{2.19}$$

$$||\Gamma||_0 \equiv \sum_{T \in \Gamma} (\log(|T|) + 4) \tag{2.20}$$

Note that the partition function can be written in term of contours

$$Z = \sum_{\sigma_\Lambda} e^{-\beta H_\Lambda(\sigma_\Lambda)} = \sum_{\underline{\Gamma}: \mathcal{R}(\underline{T}) = \underline{\Gamma}} e^{-\beta H_\Lambda(\underline{\Gamma})} \tag{2.21}$$

2.4 Contours and the Peierls argument

The main idea behind building contours is to distinguish between triangles that are close enough and those that are far away. Intituively, the interaction between distant triangles is weak, so it should not have a significant contribution to the Hamiltonian. As we will see it makes sense, but we need to be more precise with which means exactly "close" and "far" triangles.

In order to introduce the main concerns of this work, we revisit the results given in [31] associated to optimal bounds for the Hamiltonian and the connection with the Peierls argument.

2.4.1 Some existing Results

Theorem 2.6 *(Theorem 3.2 [31]) Let* $\underline{\Gamma} \cup \Gamma_0$ *be a compatible contour configuration. For all* $0 < \alpha < \frac{\log 3}{\log 2} - 1$ *and* c *enough large*

$$H[\underline{\Gamma} \cup \Gamma_0] - H[\underline{\Gamma}] \geq \frac{\zeta}{2} \sum_{T \in \Gamma_0} |T|^\alpha \tag{2.22}$$

where $\zeta = 2\frac{3-2^{\alpha+1}}{\alpha(1-\alpha)}$. *For* $\alpha = 0$, $|T|^\alpha$ *is replaced by* $\log |T| + 4$.

Theorem 2.7 *(Theorem 4.2 [31]) For any* b *enough large and for any* $m \geq 1$

$$\sum_{\Gamma : |\Gamma| = m, 0 \in \Gamma} \mathrm{e}^{-\frac{\beta\zeta}{2} \sum_{T \in \Gamma} |T|^\alpha} \leq 2m \mathrm{e}^{-bm^\alpha} \tag{2.23}$$

Phase Transition

In this section, we revisit the proof of existence of a phase transition by using a Peierls argument for the one dimensional mode, which is given in section 3.3 from [31]. Let Λ an interval containing the origin and μ_{Λ^+} the Gibbs measure in Λ^+ with boundary conditions. Then

$$\mu_\Lambda^+(\sigma_0 = -1) \leq \mu_\Lambda^+(\{\sigma_0 \in \Gamma\}) \tag{2.24}$$

where $\{0 \in \Gamma\}$ denotes the event that there is a contour Γ which has a triangle T which contains the origin. Then

$$\mu_\Lambda^+(\{\sigma_0 \in \Gamma\}) = \frac{1}{Z_\Lambda^+} \sum_{0\Gamma} \sum_{\underline{\Gamma} : \Gamma \in \underline{\Gamma}} \mathrm{e}^{-\beta H[\underline{\Gamma}]} \tag{2.25}$$

From Theorem (2.6)

$$\mathrm{e}^{-\beta H[\underline{\Gamma}]} \leq \mathrm{e}^{-\beta H[\underline{\Gamma} \setminus \Gamma]} \mathrm{e}^{-\frac{\beta\zeta}{2} \sum_{T \in \Gamma} |T|^\alpha} \tag{2.26}$$

then

$$\mu_\Lambda^+(\{\sigma_0 \in \Gamma\}) \leq \sum_{0\Gamma} \mathrm{e}^{-\frac{\beta\zeta}{2}\sum_{T\in\Gamma}|T|^\alpha} = \sum_m \sum_{\Gamma:|\Gamma|=m, 0\in\Gamma} \mathrm{e}^{-\frac{\beta\zeta}{2}\sum_{T\in\Gamma}|T|^\alpha} \tag{2.27}$$

From Theorem (2.7), we get for β enough large

$$\mu_\Lambda^+(\{0 \in \Gamma\}) \leq 2\sum_m m\mathrm{e}^{-\frac{\zeta\beta m^\alpha}{2}} \tag{2.28}$$

We observe that the Peierls argument depends strongly on Theorems (2.6), (2.7) where some optimal bounds are provided for $0 \leq \alpha < \frac{\log 3}{\log 2} - 1$. The main concern of this work is to find equivalent versions of those Theorems for $\frac{\log 3}{\log 2} - 1 \leq \alpha < 1$ in order to generalize the Peierls argument to the complete interval $0 \leq \alpha < 1$. So, the main conjectures that motivates this work are

- Given an arbitrary compatible configuration of contours $\underline{\Gamma} \cup \Gamma_0$, there exists $\delta_\alpha > 0$ such that
$$H[\underline{\Gamma} \cup \Gamma_0] - H[\underline{\Gamma}] \geq \delta_\alpha H[\underline{\Gamma}] \tag{2.29}$$

- Given any single contour, there exists $\omega_\alpha > 0$ such that
$$H[\Gamma] \geq \omega_\alpha \sum_{T\in\Gamma} |T|^\alpha \tag{2.30}$$

2.4.2 Optimal Bounds on the Hamiltonian

Before studying properties on triangles and contours, we will give some preliminary bounds for the Hamiltonian when a single contour is considered. This preliminary results we will show some facts of the behavior of the Hamiltonian, specifically the dependence on some physical quantities. Let us to start by providing the next definition.

Definition 2.8 *Let $\underline{\Gamma}$ be a compatible contour configuration such that $\underline{\Gamma} = \{\Gamma_0\}$. We denote by $Q(\Gamma_0)$ the cardinality of the subset of negative spins, i.e.*

$$Q(\Gamma_0) = \sum_{x\in\Lambda} \mathbf{1}_{\sigma_x=-1} \tag{2.31}$$

The quantity $Q(\Gamma_0)$ represents the total amount of "negative charge" present in the model, so have a very definite physical interpretation. Next proposition states lower and upper bounds for the Hamiltonian, in terms of Q.

Proposition 2.9 *Let be $\underline{\Gamma}$ a compatible contour configuration with boundary condition Λ^+ such that $\underline{\Gamma} = \{\Gamma_0\}$. Then* *(i) The following inequalities are satisfied*

$$H[\Gamma_0] \geq 2\sum_{x=1}^{Q}\sum_{y=Q+1}^{\infty} J(x,y) \tag{2.32}$$

$$H[\Gamma_0] < 2Q(J+\xi(\alpha-2)) \tag{2.33}$$

where $\xi(\alpha-2) = \sum_{k\geq 1}\frac{1}{k^{2-\alpha}}$ *is the Riemann zeta function.*

(ii) As a consequence, there exists constants $\omega_1(\alpha)$, $\omega_2(\alpha)$ *does not depending on* Γ_0, *such that for* $0<\alpha<1$.

$$\omega_1(\alpha)Q^{\alpha} < H[\Gamma_0] \leq \omega_2(\alpha)Q \tag{2.34}$$

for $\alpha = 0$, Q^{α} *is replaced by* $\log Q + 4$.

Proof:

Part (ii) of the proof is a direct consequence from (i) and that for $Q \geq 1$, $0<\alpha<1$

$$\sum_{x=1}^{Q}\sum_{y=Q+1}^{\infty} J(x,y) \geq \omega_1(\alpha)Q^{\alpha} \tag{2.35}$$

for some $\omega_1(\alpha) > 0$. When $\alpha = 0$, Q^{α} is replaced by $\log Q$. Inequality (2.35) is obtained from the well known argument on replace sums by integrals and its left to the reader. To prove part (i), let Γ_0 be a compatible configuration with boundary condition Λ^+ containing a single contour. Note that if $x_1 < x_2 < ... < x_Q$ are those sites such that $\sigma_{x_{\rm i}} = -1$, i $= 1, ..., Q$, the Hamiltonian can be written as

$$H[\Gamma_0] = \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} \tag{2.36}$$

where $\sigma_x(\Gamma_0)$ denotes the value of the spin state at site x when the configuration $\underline{\Gamma} = \{\Gamma_0\}$ is considered. Let us to write $\sigma_x(\Gamma_0)$. Note that the above expression is equivalent to

$$
\begin{aligned}
H[\Gamma_0] &= \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} && (2.37)\\
&= \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)=-1}\mathbb{1}_{\sigma_y(\Gamma_0)=+1} + \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)=+1}\mathbb{1}_{\sigma_y(\Gamma_0)=-1} && (2.38)\\
&= \sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)=-1}\mathbb{1}_{\sigma_y(\Gamma_0)=+1} && (2.39)\\
&= \sum_{x\in\mathbb{Z}}\sum_{\substack{y\in\mathbb{Z}\\ y\neq x}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)=-1}\mathbb{1}_{\sigma_y(\Gamma_0)=+1} && (2.40)
\end{aligned}
$$

It is certain $\sigma_x(\Gamma_0) = \sigma_x(\Gamma_0)$ and $\mathbb{1}_{\sigma_x(\Gamma_0)=-1} = 1 - \mathbb{1}_{\sigma_x(\Gamma_0)=+1}$, so we have

$$
\begin{aligned}
H[\Gamma_0] &= \sum_{x\in\mathbb{Z}}\sum_{\substack{y\in\mathbb{Z}\\ y\neq x}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)=-1}\mathbb{1}_{\sigma_y(\Gamma_0)=+1} && (2.41)\\
&= \sum_{x\in\mathbb{Z}}\sum_{\substack{y\in\mathbb{Z}\\ y\neq x}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)=-1}(1-\mathbb{1}_{\sigma_y(\Gamma_0)=-1}) && (2.42)\\
&= \sum_{x\in\mathbb{Z}}\sum_{\substack{y\in\mathbb{Z}\\ y\neq x}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)=-1} - \sum_{x\in\mathbb{Z}}\sum_{\substack{y\in\mathbb{Z}\\ y\neq x}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)=-1}\mathbb{1}_{\sigma_y(\Gamma_0)=-1} && (2.43)
\end{aligned}
$$

We recall that if $x_1 < x_2 < ... < x_Q$ are those sites such that $\sigma_{x_\mathrm{i}}(\Gamma_0) = -1$, $\mathrm{i} = 1, ..., Q$. By using this, we get for the Hamiltonian

$$
\begin{aligned}
H[\Gamma_0] &= \sum_{x\in\mathbb{Z}}\sum_{\substack{y\in\mathbb{Z}\\ y\neq x_\mathrm{i}}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)=-1} - \sum_{x\in\mathbb{Z}}\sum_{\substack{y\in\mathbb{Z}\\ y\neq x}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)=-1}\mathbb{1}_{\sigma_y(\Gamma_0)=-1} && (2.44)\\
&= \sum_{\mathrm{i}=1}^{Q}\sum_{\substack{y\in\mathbb{Z}\\ y\neq x_\mathrm{i}}} J(x_\mathrm{i},y) - \sum_{\mathrm{i}=1}^{Q}\sum_{\substack{j=1\\ j\neq \mathrm{i}}}^{Q} J(x_\mathrm{i},x_j) && (2.45)\\
&= \sum_{\mathrm{i}=1}^{Q}\sum_{\substack{y\in\mathbb{Z}\\ y\neq x_\mathrm{i}}} J(x_\mathrm{i},y) - 2\sum_{\mathrm{i}=1}^{Q}\sum_{j=\mathrm{i}+1}^{Q} J(x_\mathrm{i},x_j) && (2.46)
\end{aligned}
$$

We note first that for each x_i, $\mathrm{i} = 1, ..., Q$ fixed, $\sum_{\substack{y\in\mathbb{Z}\\ y\neq x_\mathrm{i}}} J(x_\mathrm{i},y) = 2(J + \xi(2-\alpha))$ does not depend on the site x_i, so the next equality follows

$$\sum_{\mathrm{i}=1}^{Q}\sum_{y\neq x_{\mathrm{i}}} J(x_{\mathrm{i}},y) = 2Q(J+\xi(2-\alpha)) \tag{2.47}$$

In addition, since $J(x_{\mathrm{i}},x_j)$ depends only on $x_1,...x_Q$, specifically on its distances, we write for $\mathrm{i}=1,...,Q-1$

$$x_{\mathrm{i}+1} = x_{\mathrm{i}} + 1 + [\mathrm{d}(x_{\mathrm{i}+1},x_{\mathrm{i}})-1] \tag{2.48}$$

An equivalent expression by using the notation $\mathrm{d}_{\mathrm{i},\mathrm{i}+1} = [\mathrm{d}(x_{\mathrm{i}+1},x_{\mathrm{i}})-1]$ is

$$x_{\mathrm{i}+1} - x_{\mathrm{i}} = 1 + \mathrm{d}_{\mathrm{i},\mathrm{i}+1} \tag{2.49}$$

where $\mathrm{d}_{\mathrm{i},\mathrm{i}+1} \geq 0$. Also, we check that

- $\mathrm{d}(x_{\mathrm{i}+1},x_{\mathrm{i}})-1=0$ if and only if $x_{\mathrm{i}+1}=x_{\mathrm{i}}+1$ and $x_{\mathrm{i}}+1/2$ is not a spin flip point.
- $\mathrm{d}(x_{\mathrm{i}+1},x_{\mathrm{i}})-1\neq 0$ if and only if both $x_{\mathrm{i}+1/2}$ and $x_{\mathrm{i}+1}-1/2$ are spin flip points.

For any pair x_j, x_{i}, $j>\mathrm{i}$ we get

$$\begin{aligned} x_j - x_{\mathrm{i}} &= j-\mathrm{i}+\sum_{m=\mathrm{i}}^{j-1}\mathrm{d}_{m,m+1} \\ &:= j-\mathrm{i}+D_{\mathrm{i},j} \end{aligned} \tag{2.50}$$

where $D_{\mathrm{i},j} = \sum_{m=\mathrm{i}}^{j-1}\mathrm{d}_{m,m+1}$. In the same way as (2.49), $D_{\mathrm{i},j}=0$ implies $\mathrm{d}_{m,m+1}=0$, $m=\mathrm{i},...,j-1$ and by consequence $x_{\mathrm{i}}+1/2,...,x_{j-1}+1/2$ are not spin flip points. By using (2.46), (2.47) and (2.50) and recalling that $J(x_{\mathrm{i}},x_j)=J(|x_j-x_{\mathrm{i}}|)$, we deduce

$$H[\Gamma_0] = 2Q(J+\xi(\alpha-2)) - 2\sum_{\mathrm{i}=1}^{Q-1}\sum_{j=\mathrm{i}+1}^{Q} J(|j-\mathrm{i}+D_{\mathrm{i},j}|) \tag{2.51}$$

$$< 2Q(J+\xi(\alpha-2)) \tag{2.52}$$

We remark that $D_{\mathrm{i},j}$ $\mathrm{i}\neq j$ and by consequence the Hamiltonian depends only on $\mathrm{d}_{l,l+1}$, $\mathrm{i}=\mathrm{i},...,j-1$. In addition, since $D_{\mathrm{i},j}\geq 0$ for all $1\leq \mathrm{i},j\leq Q$ and $J(|\mathrm{d}|)$ is a decreasing function on the distance, we get $J(|j-\mathrm{i}+D_{\mathrm{i},j}|) \leq J(|j-\mathrm{i}|)$ for any compatible configuration , and it implies

$$H[\Gamma_0] \geq 2Q(J+\xi(\alpha-2)) - 2\sum_{\mathrm{i}=1}^{Q-1}\sum_{j=\mathrm{i}+1}^{Q} J(|j-\mathrm{i}|) \tag{2.53}$$

To complete the proof, we proceed by direct computation. We recall the identity

$$2Q(J+\xi(\alpha-2)) = 2\sum_{\mathrm{i}=1}^{Q}\sum_{j=\mathrm{i}+1}^{\infty} J(|j-\mathrm{i}|) \tag{2.54}$$

By replacing (2.54) in (2.53)

$$\begin{aligned}
H[\Gamma_0] &> 2\sum_{\mathrm{i}=1}^{Q}\sum_{j=\mathrm{i}+1}^{\infty} J(|j-\mathrm{i}|) - 2\sum_{\mathrm{i}=1}^{Q-1}\sum_{j=\mathrm{i}+1}^{Q} J(|j-\mathrm{i}|) && (2.55)\\
&= 2\left(\sum_{\mathrm{i}=1}^{Q-1}\sum_{j=\mathrm{i}+1}^{\infty} J(|j-\mathrm{i}|) - \sum_{\mathrm{i}=1}^{Q-1}\sum_{j=\mathrm{i}+1}^{Q} J(|j-\mathrm{i}|\right) + 2\sum_{j=Q+1}^{\infty} J(|j-Q|) && (2.56)\\
&= 2\sum_{\mathrm{i}=1}^{Q-1}\sum_{j=Q+1}^{\infty} J(|j-\mathrm{i}|) + 2\sum_{j=Q+1}^{\infty} J(|j-Q|) && (2.57)\\
&= 2\sum_{\mathrm{i}=1}^{Q}\sum_{j=Q+1}^{\infty} J(|j-\mathrm{i}|) && (2.58)
\end{aligned}$$

concluding the proof.

2.5 Optimal bounds for a single contour

In this section we are concerned primarily on finding optimal bounds for the Hamiltonian of a single contour, specifically we are intersted in accepting or refusing the conjecture

$$H[\Gamma] \geq \omega_\alpha \sum_{T\in\Gamma} |T|^\alpha \tag{2.59}$$

Let us to explain shortly the importance of thisFor any single triangle T we have

$$H[T] \sim |T|^\alpha \tag{2.60}$$

So, the idea behind the conjecture is that, for any contour

$$H[\Gamma] \geq \omega_\alpha \sum_{T\in\Gamma} |T|^\alpha \tag{2.61}$$

for some ω_α. Intuitively, given a fixed compatible configuration, for α close to zero the interaction between two sites is weaker than in the case α close to zero, so it is more difficult that the conjecture could be certain for α close to 1. The strategy to get the results shown in this section are basically based in exploting the hierarchical and self similar structure of contours. It is well known that the contour structure exhibes some self similar behavior. In [31] was shown that a single contour configuration can be decomposed into some primaries structures called pre-contours, which are compatible contours by itself. We recommend to read sections (4.1) (4.2) from [31], where a very proof was made. So, self similarity plays an important role in a appropriate description of the model, including some optimal bounds for the Hamiltonian. In fact, we will see in Theorem 2.16, by building some discrete type fractal counterexamples that the conjecture (2.59) is not certain, for $\alpha \geq \frac{\log 2}{\log 3}$, so it is not possible to get an optimal bound in terms of $||\Gamma||_\alpha$, which is the optimal bound for the Hamiltonian in the interval $0 < \alpha < \frac{\log 3}{\log 2} - 1$. The intermediate case $\frac{\log 3}{\log 2} - 1 \leq \alpha < \frac{\log 2}{\log 3}$ remains as an open problem.

2.5.1 Preliminaries

Let us to start by setting up some notation.

Definition 2.10 *Let be $A \subseteq \mathbb{Z}$, the following sets are defined*

$$\begin{aligned} tA &= \{tx : x \in A\} \\ A + y &= \{y + x : x \in A\} \\ tA + y &= \{tx + y : x \in A\} \end{aligned} \tag{2.62}$$

(2.63)

Definition 2.11 *Let be $A, B \subset \mathbb{Z}$. We say that $A < B$ if*

$$x < y \text{ for all x} \in \text{A}, \text{ y} \in \text{B} \tag{2.64}$$

Definition 2.12 *Let $A \subset \mathbb{N}$, we define the upper counting measure as*

$$\overline{\mathcal{M}}(A) = \limsup_n \frac{\log |A \cap \{1, ..., n\}|}{\log n} \tag{2.65}$$

In [37], Theorem (2.1) was shown that an equivalent formulation of (2.65) is the zeta dimension defined as

$$\dim_{\zeta}(A) = \inf\{s : \zeta(A) < \infty\} \tag{2.66}$$

where $\zeta(A) = \sum_{n\in A} n^{-s}$. For any arbitrary finite discrete set

$$\begin{aligned}
\overline{\mathcal{M}}(A) &= \limsup_n \frac{\log|A\cap\{1,...,n\}|}{\log n} && (2.67)\\
&\leq \limsup_n \frac{\log|A|}{\log n} && (2.68)\\
&= \log|A| \limsup_n \frac{1}{\log n} && (2.69)\\
&= 0. && (2.70)
\end{aligned}$$

i.e. the discrete fractal dimension is zero. In a typical spin configuration, the number of negative spins is a finite number, and the fractal dimension is zero. Nevertheless it will not be the case for some interesting limiting examples. Those limiting examples are studied next.

2.5.2 Discrete Fractals

There exists a close connection between fractal sets and in some limiting cases of spin configurations, which plays an important role in the studying of the behavior of the Long Range Ising Model. We start by giving an explicit building to the discrete analogue of the Cantor Set.

The Discrete Cantor Set

Let be C_{d}^{n}, $n \geq 1$ a sequence of subsets of $\mathbb{Z}$ defined recursively

P.0 $C_{\mathrm{d}}^{(0)} = \{1\}$.

P.1 $C_{\mathrm{d}}^{(1)} = \{1, 3\}$.

P.2 $C_{\mathrm{d}}^{(2)} = \{1, 3\} \cup \{7, 9\}$.

P.3 $C_{\mathrm{d}}^{(3)} = \{1, 3, 7, 9\} \cup \{19, 21, 25, 27\}$.

P.N Given a configuration $C_{\mathrm{d}}^{(n-1)}$ at step $n-1$, we set

$$C_{\mathrm{d}}^{(n)} = C_{\mathrm{d}}^{(n-1)} \bigcup \{2 \times 3^{n-1} + C_{\mathrm{d}}^{(n-1)}\} \tag{2.71}$$

The procedure continues for all $n \in \mathbb{N}$. We note that $C_{\rm d}^n \subseteq C_{\rm d}^m$ when $n < m$. The discrete Cantor set is defined as

$$C_{\rm d} = \bigcup_{n \geq 1} C_{\rm d}^{(n)} \tag{2.72}$$

Now, let us to show the connection with the classical example. We consider the family of continuous intervals on the form $[\mathrm{i}, j]$, $\mathrm{i}, j \in \mathbb{Z}$ and numerable union of them, which will be denoted by $\mathcal{I}_{\mathbb{Z}}$ on $\mathbb{R}$. In addition, we define the mapping

$$\begin{aligned} f \;\; : \;\; & \mathbb{Z} \longrightarrow \mathcal{I}_{\mathbb{Z}} \\ & x \longrightarrow [x-1, x]. \end{aligned}$$

Note that for any subset $A \subseteq \mathbb{Z}$

$$f(A) = \bigcup_{j \in A} [j-1, j] \tag{2.73}$$

and more generally

$$f\left(\bigcup_{\mathrm{i}=1}^{n} A\right) = \bigcup_{\mathrm{i}=1}^{n} f(A_{\mathrm{i}}) \tag{2.74}$$

By consequence, we get that for any $n \geq 1$

$$\frac{f(C_{\rm d}^{(n)})}{3^n} = \bigcup_{\mathrm{i} \in C_{\rm d}^{(n)}} \left[\frac{\mathrm{i}-1}{3^n}, \frac{\mathrm{i}}{3^n}\right] \tag{2.75}$$

coincides with the classical Cantor set $C^{(n)}$ at step n. Since $f(C_{\rm d}^{(0)}) = [0,1]$ we get the identity

$$\bigcap_{n \geq 0} \frac{f(C_{\rm d}^{(n)})}{3^n} = C \tag{2.76}$$

where C denotes the classical Cantor set.

A generalized discrete Cantor set

We will build a generalization of the discrete Cantor set defined above, which is the discrete analog to the generalized continuous Cantor set. Let $\lambda > 1 \in \mathbb{N}$, the sequence of sets $C^{(n)}_{\mathrm{d},\lambda}$, $n \geq 1$ is defined recursively

$$\begin{aligned} C^{(0)}_{\lambda,\mathrm{d}} &= \{1\} \\ C^{(0)}_{\lambda,\mathrm{d}} &= \{1, \lambda\} \qquad (2.77) \\ C^{(n)}_{\lambda,\mathrm{d}} &= C^{(n-1)}_{\lambda,\mathrm{d}} \cup \left\{(\lambda-1)\lambda^{n-1} + C^{(n-1)}_{\lambda,\mathrm{d}}\right\}, \; n \geq 2. \qquad (2.78) \end{aligned}$$

The generalized Cantor Set is defined as

$$C_{\lambda,\mathrm{d}} = \bigcup_{n\geq 1} C^{(n)}_{\lambda,\mathrm{d}} \qquad (2.79)$$

For $\lambda = 3$, we recover the discrete Cantor set. On the other hand, analougsly to (2.75), we get that $\frac{f(C^{(n)}_{\lambda})}{\lambda^n}$, satisfies the recursive formula

$$\frac{f(C^{(n)}_{\lambda})}{\lambda^n} = \frac{1}{\lambda}\frac{f(C^{(n-1)}_{\lambda})}{\lambda^{n-1}} \cup \left\{\frac{\lambda-1}{\lambda} + \frac{1}{\lambda}\frac{f(C^{(n-1)}_{\lambda})}{\lambda^{n-1}}\right\} \qquad (2.80)$$

where $\frac{f(C^{(0)}_{\lambda})}{\lambda^0} = [0,1]$. We deduce that

$$C_{\lambda} = \bigcap_{n\geq 1} \frac{f(C^{(n)}_{\lambda})}{\lambda^n} \qquad (2.81)$$

is the generalized Cantor set defined.

Proposition 2.13 *Let $C_{\lambda,\mathrm{d}}$ defined as (2.79). Then,*

$$\dim_{\zeta}(C_{\lambda,\mathrm{d}}) = \frac{\log 2}{\log \lambda} \qquad (2.82)$$

in particular, for $\lambda = 3$, we get that the zeta dimension of the discrete Cantor set is $\frac{\log 2}{\log 3}$.

Proof: The proof is given in 2.7

2.5.3 Spin Configurations associated to discrete fractal sets

The Discrete Cantor Set Spin Configuration

Let $\sigma_C^{(n)}$, $n \geq 1$ be a sequence of spin flips configuration with boundary condition Λ^+, defined by

$$\sigma_{C,x}^{(n)} = \begin{cases} -1 & \text{if } x \in C_{\mathrm{d}}^{(n)} \\ +1 & \text{otherwise} \end{cases} \tag{2.83}$$

Note that the spin flip configurations are given by the next procedure

P.0 $\mathrm{sf}^*(\sigma_{C,x}^{(0)}) = \{\frac{1}{2}, \frac{3}{2}\}$.

P.1 $\mathrm{sf}^*(\sigma_{C,x}^{(1)}) = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}\}$.

P.2 $\mathrm{sf}^*(\sigma_{C,x}^{(2)}) = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}\} \cup \{\frac{13}{2}, \frac{15}{2}, \frac{17}{2}, \frac{19}{2}\}$.

P.N Given a configuration $\sigma_{C,x}^{(n-1)}$ at step $n-1$, we set

$$\mathrm{sf}^*(\sigma_{C,x}^{(n)}) = \mathrm{sf}^*(C_{\mathrm{d}}^{(n-1)}) \cup \left\{2 \times 3^{n-1} + \mathrm{sf}^*(C_{\mathrm{d}}^{(n-1)})\right\} \tag{2.84}$$

Triangle Structure of the Discrete Cantor Spin Configuration

We recall that in order to get a consistent construction of triangles, each spin flip point $x^* \in \mathbb{Z}^*$ is associated with a number $r_x \in [x^* - \frac{1}{100}, x^* + \frac{1}{100}]$, which satisfies $|r_{x_1^*} - r_{x_2^*}| \neq |r_{x_3^*} - r_{x_4^*}|$, for any arbitrary selection of four spin flip points. This property guarantees a consistent construction of the triangles. Nevertheless, the contour norm $||\Gamma||_\alpha$ can dramatically change in some examples, depending on r_x^*'s choose. Let us to review in detail the Cantor discrete case, specifically when the points r_x^* according to the next procedure

P.0 For $\mathrm{sf}^*(\sigma_{C,x}^{(0)})$, we choose

$$\begin{aligned} r_{1/2} &= \frac{1}{2} + \delta_1 \\ r_{3/2} &= \frac{3}{2} + \delta_2 \end{aligned} \tag{2.85}$$

where $|\delta_1| \vee |\delta_2| < \frac{1}{100}$. At first step we have just one option, which is a triangle whose mass is 1.

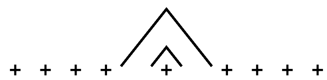
P.1 For $\mathrm{sf}^*(\sigma_C^{(1)})$ we set r_1, r_2 as (2.86) and

$$\begin{aligned} r_{5/2} &= \frac{5}{2} + \delta_3 \\ r_{7/2} &= \frac{7}{2} + \delta_3 + \delta_2 - \delta_1 \end{aligned} \tag{2.86}$$

where $|\delta_3| < \frac{1}{100}$, and $\delta_3 - \delta_2 < \delta_2 - \delta_1$. Then

$$\begin{aligned} r_{5/2} - r_{3/2} &= 1 + \delta_3 - \delta_2 \\ &< 1 + \delta_2 - \delta_1 \\ &= r_{3/2} - r_{1/2} := r_{7/2} - r_{5/2} \end{aligned}$$

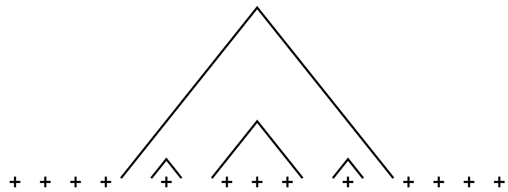
we have two triangles: an external triangle whose mass is 3 and an internal triangle whose mass is 1.



P.N Given a collection of points $r(\mathrm{sf}^*(\sigma_C^{(n-1)}))$ at step $n-1$, we set

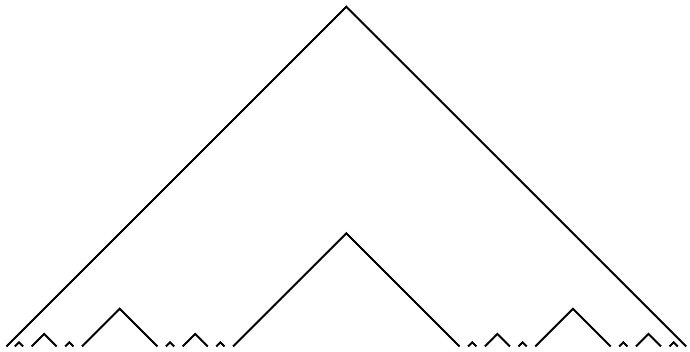
$$r(\mathrm{sf}^*(\sigma_{C,x}^{(n)})) = r(\mathrm{sf}^*(\sigma_{C,x}^{(n-1)})) \cup \left\{ r(\mathrm{sf}^*(\sigma_C^{(n-1)})) + 2 \times 3^{n-1} + \delta_n \right\} \tag{2.87}$$

where $|\delta_n| < \frac{1}{100}$ is chosen in a way such that $r_{2\times 3^{n-1}+\frac{1}{2}} - r_{3^{n-1}+\frac{1}{2}} < r_{3^{n-1}+\frac{1}{2}} - r_{\frac{1}{2}}$.



Contour Structure

We use the notation $\underline{T}_{C_\mathrm{d}}^{(n)}$ and $\underline{\Gamma}_{C_\mathrm{d}}^{(n)}$ to denote respectively the triangle and contour configuration associated $\sigma_{C_\mathrm{d}}^{(n)}$. We claim that $\underline{\Gamma}_{C_\mathrm{d}}^{(n)} = \{\Gamma_{C_\mathrm{d}}\}$ is a single contour, for all $n \geq 1$. To prove that, we first note that by construction $\underline{T}_{C_\mathrm{d}}^{(n)}$ contains exactly one maximal triangle, we say T and $\underline{T}_{\mathrm{in}}(T) = \{T_1, T_2, ..., T_{2^n-1}\}$ contains exactly $2^n - 1$ internal triangles,



the above picture shows the Cantor discrete set for $n = 4$. We observe that for any pair $T_{\rm i}, T_{\rm i+1} \in \underline{T}^{(n)}_{C_{\rm d}}$, $\rm i = 1, ...2^n - 2$, $\mathrm{dist}(T_{\rm i}, T_{\rm i+1}) = 1 < c|T_{\rm i}|^3 \wedge |T_{\rm i+1}|^3$ and by consequence the configuration is a single contour, i.e.

$$\underline{\Gamma}^{(n)}_{C_{\rm d}} = \{\Gamma^{(n)}_{C_{\rm d}}\} \tag{2.88}$$

where $\Gamma^{(n)}_{C_{\rm d}} = T \cup_{\rm i=1}^{2^n-1} T_{\rm i}$.

Contour mass

We recall equation (2.75)

$$\frac{f(C^{(n)}_{\rm d})}{3^n} = \bigcup_{{\rm i} \in C^{(n)}_{\rm d}} \left[\frac{\rm i - 1}{3^n}, \frac{\rm i}{3^n}\right]$$

From the building procedure, we can check that, up to scale, the internal triangles are the "erased" third middle intervals of the Classical Cantor set, whereas the negative spins is its complement. The mass contour is given by the expression

$$\begin{aligned} |\Gamma^{(n)}_{C_{\rm d}}| &= 3^n + 3^{n-1} + 2 \times 3^{n-2} + ... + 2^{n-1} \times 3^0 \\ &= 3^n + \sum_{j=0}^{n} 2^{j-1} 3^{n-j} \end{aligned} \tag{2.89}$$

Analougsly,

$$\begin{aligned} ||\Gamma^{(n)}_{C_{\rm d}}||_\alpha &= 3^{n\alpha} + 3^{(n-1)\alpha} + 2 \times 3^{(n-2)\alpha} + ... + 2^{n-1} \times 3^{0\alpha} \\ &= 3^{n\alpha} + \sum_{j=0}^{n} 2^{j-1} 3^{(n-j)\alpha} \end{aligned} \tag{2.90}$$

We note that for $0 \le \alpha \le 1$, $||\Gamma^{(n)}_{C_\mathrm{d}}||_\alpha$ satisfies the recursive equation

$$||\Gamma^{(n)}_{C_\mathrm{d}}||_\alpha = 2||\Gamma^{(n-1)}_{C_\mathrm{d}}||_\alpha + (3^\alpha - 1)\, 3^{(n-1)\alpha} \tag{2.91}$$

Either, by equations (2.89), (2.90) or solving the recursive equation (2.91), we get

$$||\Gamma^{(n)}_{C_\mathrm{d}}||_\alpha = \begin{cases} 3^{n\alpha} + \frac{3^{n\alpha} - 3^{n\alpha_0}}{3^\alpha - 3^{\alpha_0}} & \alpha \neq \alpha_0 \\ 2^n \left(1 + \frac{n}{2}\right) & \alpha = \alpha_0 \end{cases} \tag{2.92}$$

where $\alpha_0 = \frac{\log 2}{\log 3}$.

Hamiltonian

For each $n \ge 1$, the configuration $\Gamma^{(n)}_{C_\mathrm{d}}$ contains 2^n negative spins, distributed in 2^n intervals whose length is 1. It implies that the configuration contains 2^{n+1} spin flips, so a lower bound for the Hamiltonian is

$$H[\Gamma^{(n)}_{C_\mathrm{d}}] > 2^{n+1}\,(J+1) \tag{2.93}$$

from Proposition 2.9 we know that

$$H[\Gamma^{(n)}_{C_\mathrm{d}}] < 2^{n+1}\,(J + \xi(\alpha - 2)) \tag{2.94}$$

where ξ is the Riemann zeta function. It means

$$H[\Gamma^{(n)}_{C_\mathrm{d}}] \sim 2^n \tag{2.95}$$

Limit Behaviour

Proposition 2.14 *For all $\alpha \ge \alpha_0$*

$$\lim_{n\to\infty} \frac{H[\Gamma^{(n)}_{C_\mathrm{d}}]}{||\Gamma^{(n)}_{C_\mathrm{d}}||_\alpha} = 0 \tag{2.96}$$

Proof: It is a direct consequence from equations (2.92), (2.93), (2.94). For all $n \ge 1$, we have

$$H[\Gamma^{(n)}_{C_{\mathrm{d}}}] < \begin{cases} \frac{2^{n+1}(J+\xi(\alpha-2))}{3^{n\alpha}+\frac{3^{n\alpha}-3^{n\alpha_0}}{3^{\alpha}-3^{\alpha_0}}} & \alpha > \alpha_0 \\ \frac{2^{n+1}}{2^n\left(1+\frac{n}{2}\right)} & \alpha = \alpha_0 \end{cases} \tag{2.97}$$

since $3^{\alpha_0} = 2$, we have rewriting the above expresions

$$\frac{H[\Gamma^{(n)}_{C_{\mathrm{d}}}]}{||\Gamma^{(n)}_{C_{\mathrm{d}}}||_{\alpha}} < \begin{cases} \frac{2(J+\xi(\alpha-2))}{3^{n(\alpha-\alpha_0)}+\frac{3^{n(\alpha-\alpha_0)}-1}{3^{\alpha}-3^{\alpha_0}}} & \alpha > \alpha_0 \\ \frac{2(J+\xi(\alpha-2))}{\left(1+\frac{n}{2}\right)} & \alpha = \alpha_0 \end{cases} \tag{2.98}$$

For $\alpha > \alpha_0$, $\lim_{n\to\infty} 3^{n(\alpha-\alpha_0)} = \infty$, similarily, for $\alpha = \alpha_0$ $\lim_{n\to\infty} 1 + \frac{n}{2} = \infty$ and in both cases the limit is zero.

2.5.4 A generalized Cantor type Spin configuration

Let $\lambda \in (0, 1)$ and $x \in \mathbb{Z}$ such that $\lambda x \geq 1$. We define the following sequence of sub sets $\in \mathbb{Z}$ accordingly to the recursive formula

$$\begin{aligned} C^{(0)}_{\lambda,\mathrm{d}} &= \{1, ..., x\} \\ C^{(n)}_{\lambda,\mathrm{d}} &= C^{(n-1)}_{\lambda,\mathrm{d}} \cup \left\{[\lambda l_{n-1}] + C^{(n-1)}_{\lambda,\mathrm{d}}\right\} \end{aligned} \tag{2.99}$$

where $[\lambda l_{n-1}]$ denotes the entire part of λl_{n-1} and $l_{n-1} = \mathrm{diam}(C^{(n-1)}_{\lambda,\mathrm{d}}) + 1$. Associated to (2.99), we define the following sequence of spin configurations with boundary condition Λ^+

$$\sigma^{(n)}_{C_{\lambda,\mathrm{d}},x} = \begin{cases} -1 & \text{if } x \in C^{(n)}_{\lambda,\mathrm{d}} \\ +1 & \text{otherwise} \end{cases} \tag{2.100}$$

Triangle and Contour Configuration

Since $\lambda < 1$ we have $[\lambda l_{n-1}] < l_{n-1}$, so there is not confusion about the triangle configuration. Similary to the discrete Cantor configuration, we get for all $n \geq 1$, $\sigma^{(n)}_{C_{\lambda,\mathrm{d}}}$ has one maximal triangle and $2^n - 1$ internal triangles that satisfies $T_1 < T_2 < .... < T_{2^n-1}$. Moreover, for $\lambda \geq \left(\frac{2}{c}\right)^{\frac{1}{3}}$

$$\begin{aligned}
\mathrm{d}ist(T_{\mathrm{i}}, T_{\mathrm{i}+1}) &= x \\
&= \frac{(\lambda x)^3}{\lambda^3 x^2} \\
&\leq \frac{1}{\lambda^3}(\lambda x)^3 \\
&\leq \frac{2}{\lambda^3}[\lambda x]^3 \\
&\leq c|T_{\mathrm{i}}|^3 \wedge |T_{\mathrm{i}+1}|^3
\end{aligned} \tag{2.101}$$

In addition, $||\Gamma^{(n)}_{\lambda,\mathrm{d}}||_\alpha$ satisfies the recursive equation

$$||\Gamma^{(n)}_{\lambda,\mathrm{d}}||_\alpha = 2||\Gamma^{(n-1)}_{\lambda,\mathrm{d}}||_\alpha - 2l^\alpha_{n-1} + [\lambda l_{n-1}]^\alpha + l^\alpha_n \tag{2.102}$$

$$l^\alpha_n = (2l_{n-1} + [\lambda l_{n-1}])^\alpha \tag{2.103}$$

The argument to get that equations is the very same to the used in the Cantor case. Actually, for $\lambda \to 1$, we recover equation (2.91).

Limit Behaviour

Proposition 2.15 *For all* $\alpha \geq \frac{\log 2}{\log(2+\lambda)}$

$$\lim_{n\to\infty} \frac{H[\Gamma^{(n)}_{C_{\lambda,\mathrm{d}}}]}{||\Gamma^{(n)}_{C_{\lambda,\mathrm{d}}}||_\alpha} = 0 \tag{2.104}$$

Proof:

For all $n \geq 1$, we get from equation (2.103), for $\alpha = 1$

$$\begin{aligned}
l_n &= (2l_{n-1} + [\lambda l_{n-1}]) \\
&\geq (2+\lambda)l_{n-1} - 1 \\
&\geq (2+\lambda)^2 l_{n-2} - (2+\lambda) - 1 \\
&\geq (2+\lambda)^3 l_{n-3} - (2+\lambda)^2 - (2+\lambda) - 1 \\
&\vdots \\
&\geq (2+\lambda)^n l_0 - \sum_{j=0}^{n-1}(2+\lambda)^j \\
&\geq (2+\lambda)^n l_0 - \frac{(2+\lambda)^n - 1}{1+\lambda}
\end{aligned} \tag{2.105}$$

since $l_0 = x$ and $\frac{1}{1+\lambda} > 0$, we have

$$l_n \geq (2+\lambda)^n (x-1) \tag{2.106}$$

$$l_n \geq (2+\lambda) l_{n-1} - 1 \Rightarrow l_n^\alpha \geq (2+\lambda)^\alpha l_{n-1}^\alpha - 1 \tag{2.107}$$

$$[\lambda l_{n-1}]^\alpha \geq \lambda l_{n-1}^\alpha - 1 \tag{2.108}$$

so replacing in (2.102)

$$\begin{aligned} ||\Gamma_{\lambda,\mathrm{d}}^{(n)}||_\alpha &= 2||\Gamma_{\lambda,\mathrm{d}}^{(n-1)}||_\alpha - 2l_{n-1}^\alpha + [\lambda l_{n-1}]^\alpha + l_n^\alpha \\ &\geq 2||\Gamma_{\lambda,\mathrm{d}}^{(n-1)}||_\alpha - 2 + \left((2+\lambda)^\alpha - 2 + \lambda^\alpha\right) l_{n-1}^\alpha \\ &\geq \left((2+\lambda)^\alpha - 2 + \lambda^\alpha\right) l_{n-1}^\alpha \end{aligned}$$

By using equation (2.106)

$$||\Gamma_{\lambda,\mathrm{d}}^{(n)}||_\alpha \geq (2+\lambda)^{n\alpha} \frac{(x-1)^\alpha}{(2+\lambda)^\alpha} \left((2+\lambda)^\alpha - 2 + \lambda^\alpha\right) \tag{2.109}$$

On the other hand, for all $n \geq 1$, the configuration $\Gamma_{\lambda,\mathrm{d}}^{(n)}$ contains 2^n intervals of negative spins whose length is 1. So, by using proposition 2.9, an upper bound of the Hamiltonian

$$H[\Gamma_{C_{\lambda,\mathrm{d}}}^{(n)}] < 2^{n+1} \sum_{\mathrm{i}=1}^{x} \sum_{j=x+1}^{\infty} J(\mathrm{i}, j) \tag{2.110}$$

So, from equations (2.109), (2.110) we get for all $n \geq 1$

$$\frac{H[\Gamma_{C_{\lambda,\mathrm{d}}}^{(n)}]}{||\Gamma_{C_{\lambda,\mathrm{d}}}^{(n)}||_\alpha} \leq \left(\frac{2}{(2+\lambda)^\alpha}\right)^n \frac{(2+\lambda)^\alpha 2 \sum_{\mathrm{i}=1}^{x} \sum_{j=x+1}^{\infty} J(\mathrm{i}, j)}{\left((2+\lambda)^\alpha - 2 + \lambda^\alpha\right)(x-1)^\alpha} \tag{2.111}$$

For $\alpha > \frac{\log 2}{\log(2+\lambda)}$ we have $\frac{2}{(2+\lambda)^\alpha} < 1$, so we deduce the result directly from (2.111) and letting $n \to \infty$.

We have builded explicitly some sequences $\sigma^{(n)}$ of configurations such that the limit satisfies

$$\lim_{n\to\infty} \frac{H[\Gamma_\sigma^{(n)}]}{||\Gamma_\sigma^{(n)}||_\alpha} = 0 \tag{2.112}$$

The next theorem generalizes the above equations.

Theorem 2.16 *Let $\mathcal{T}_{\Lambda_L^+}$ be the set of compatible configurations with boundary condition $\sigma_x = +1$ for $|x| \geq L$. For all $\alpha \geq \alpha_0$, given any constant δ, there exists a family $\mathcal{T}_{\delta,\Lambda_L^+} \subset \cup_{L\geq 1}\mathcal{T}_{\Lambda_L^+}$ of compatible triangle configuration such that*

$$\sup_{\underline{T}\in\mathcal{T}_{\delta,\Lambda_L^+}} \frac{H[\underline{T}]}{||\underline{T}||_\alpha} \leq \delta \tag{2.113}$$

Proof: It is a direct consequence from equations (2.104), (2.96). Given $\delta > 0$, there exists $n_0(\delta)$ such that

$$\sup_{n\geq n_0} \frac{H[\Gamma_{C_{\lambda,\mathrm{d}}}^{(n)}]}{||\Gamma_{C_{\lambda,\mathrm{d}}}^{(n)}||_\alpha} \leq \delta \tag{2.114}$$

and

$$\sup_{n\geq n_0} \frac{H[\Gamma_{C_\mathrm{d}}^{(n)}]}{||\Gamma_{C_\mathrm{d}}^{(n)}||_\alpha} \leq \delta \tag{2.115}$$

It suffices to take

$$\mathcal{T}_\delta = \bigcup_{n\geq n_0} \Gamma_{C_{\lambda,\mathrm{d}}}^{(n)} \tag{2.116}$$

for $\lambda = 2^{1/\alpha} - 1$ to deduce the theorem.

2.6 An upper bound for the contour mass

An equivalent form to state Theorem (2.16) is that, given $\alpha \geq \alpha_0$ and a constant $\delta > 0$ there exists a contour Γ_δ such that

$$H[\underline{\Gamma}] \leq \delta \sum_{T\in\Gamma} H[T] \tag{2.117}$$

the above equation implies that the Hamiltonian is not quasi additive in term of triangles. In order to get some general bounds for the Hamiltonian in similarly as Proposition

2.9, we study in this section an upper bound for the contour mass and for $||\underline{\Gamma}||_\alpha$, $\alpha \in (0,1)$ in terms of $Q(\underline{T}) = \sum_{x\in supp(\underline{T})} \mathbb{1}_{\sigma_x=-1}$. To do this, we work directly on a maximization problem over a single contour. So, we define for $\alpha \in (0,1)$

$$f(q,\alpha) := \max_{\underline{T}\in\mathcal{T}^q_{\Lambda^+}} ||\underline{T}||_\alpha \qquad (\mathcal{P})$$

where

$$\mathcal{T}^q_{\Lambda^+} = \{\underline{T} \in \mathcal{T}_{\Lambda^+} : Q(\underline{T}) = q\} \tag{2.118}$$

denotes the set of compatible configurations whose number of negative spins is q. In addition, we define

$$f_k(q,\alpha) := \max_{\underline{T}\in\mathcal{T}^q_{\Lambda^+,k}} ||\underline{T}||_\alpha \qquad (\mathcal{P}_{q,k})$$

$$\mathcal{T}^q_{\Lambda^+,k} = \{\underline{T} \in \mathcal{T}^q_{\Lambda^+} : |sf^*(\underline{T})| = k\} \tag{2.119}$$

we observe that $\mathcal{T}^q_{\Lambda,k}$ is the subset of compatible configurations of $\mathcal{T}^q_\Lambda$ whose number of spin flips is k. We recall that under boundary conditions Λ^+, k is an even number, and moreover

$$\mathcal{T}^q_{\Lambda^+} = \bigcup_{\substack{k=2\\ k\ even}}^{2q} \mathcal{T}^q_{\Lambda^+,k} \tag{2.120}$$

We state next the main theorem of this section

Theorem 2.17 *Let $\mathcal{T}^q_{\Lambda^+}$ be the set of compatible configurations defined in (2.118). For all $\alpha \in (0,1)$ we have*

(i) For all $n \geq 0$, $f(2^n,\alpha)$ gets the maximum at the discrete Cantor set configuration $C^{(n)}_{\mathrm{d}}$.

(ii) Given an arbitrary q, there exists constants $0 < K_1(\alpha) < K_2(\alpha) < \infty$ such that for all $q \geq 1$

$a \neq \alpha_0$

$$K_1(\alpha) q^{\frac{\alpha\vee\alpha_0}{\alpha_0}} < f(q,\alpha) < K_2(\alpha) q^{\frac{\alpha\vee\alpha_0}{\alpha_0}} \tag{2.121}$$

$a = \alpha_0$

$$K_1(\alpha_0) q \log q < f(q,\alpha_0) < K_2(\alpha_0) q \log q \tag{2.122}$$

2.6.1 Proof of Theorem (2.17)

Proof of (ii)

We show first part (ii). It is a direct consequence from part (i) and the following Lemma

Lemma 2.18 *For all $\alpha \in [0,1)$ fixed, and $1 \leq q_1 < q_2$, we have*

$$f(q_1,\alpha) < f(q_1+1,\alpha). \tag{2.123}$$

and by consequence $f(q_1,\alpha) < f(q_2,\alpha)$ for any pair $q_1 < q_2$.

Proof: Suposse that $\underline{T}$ is a configuration satisfying $f(q_1,\alpha) = ||\underline{T}||_\alpha$. The configuration

$$\underline{T}^* = \underline{T} \cup T_1 \tag{2.124}$$

where T_1 is a triangle with mass is 1 located at the right hand of $\underline{T}$ and $dist(\underline{T}, T_1) = 1$. We get that $\underline{T} \cup T_1$ is compatible configuration and satisfies

$$\begin{aligned} \underline{T}^* &\in \mathcal{T}^{q_1+1}_{\Lambda^+} \\ ||\underline{T}^*||_\alpha &= f(q_1,\alpha)+1 \end{aligned}$$

So, we have the inequality

$$f(q_1,\alpha) < ||\underline{T}^*||_\alpha \leq f(q_1+1,\alpha) \tag{2.125}$$

concluding the proof.

Now, by using part (i), we have that for $q = 2^n$, $n \geq 1$

$$f(q,\alpha) \leq \begin{cases} \frac{q^{\frac{\alpha}{\alpha_0}} - q}{3^\alpha - 2} + q^{\frac{\alpha}{\alpha_0}} & \text{if } \alpha \neq \alpha_0 \\ q\left(\frac{\log(q)}{\log(2)} + 1\right) & \text{if } \alpha = \alpha_0 \end{cases} \tag{2.126}$$

For q arbitrary, we have that $m = \left[\frac{\log(q)}{\log(2)}\right]$ satisfies $2^m \leq q \leq 2^{m+1}$. So, by using lemma (2.18)

$$f(2^m,\alpha) < f(q,\alpha) < f(2^{m+1},\alpha) \tag{2.127}$$

and from a straightforward computation (see Appendix 2.6.3)

$$\begin{aligned} 1/2 &< \tfrac{f(q,\alpha)}{q} < \frac{2}{2-3^\alpha} & \alpha < \alpha_0 \\ \frac{1}{8\log(2)} &< \tfrac{f(q,\alpha)}{q\log(q)} < \frac{6}{\log(2)} & \alpha = \alpha_0 \\ 3^{-\alpha} &< \tfrac{f(q,\alpha)}{q^{\frac{\alpha}{\alpha_0}}} < \frac{2}{3}3^\alpha & \alpha > \alpha_0 \end{aligned} \tag{2.128}$$

□

Proof of part (i)

The proof is based in giving first some necessary conditions to get the maximum of $(\mathcal{P})$. We start by giving some preliminaries definitions.

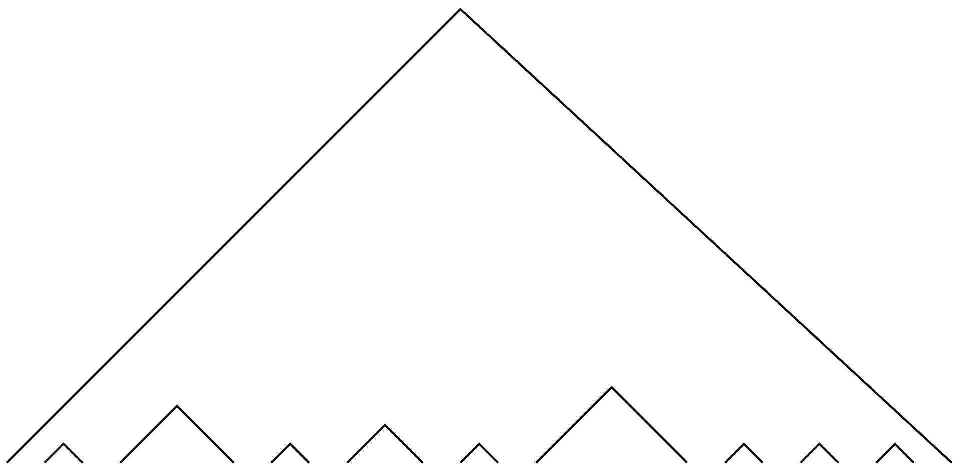
Definition 2.19 *Let be $\mathcal{T}_\Lambda^q$ as (2.118). We say that $\underline{T} \in \mathcal{T}_{\Lambda+}^q$ is a candidate configuration if satisfies the following conditions*

(C1) *The number of maximal triangles is 1*

$$\underline{T} = T \cup \underline{T}_{\text{in}}(T) \tag{2.129}$$

(C2) *The set of internal triangles satisfies $\Delta(T_{\text{i}}) \cap \Delta(T_j) = \varnothing$, $\text{i} \neq j$ or equivalently the set of internal triangles of $\underline{T}$ satisfies $T_1 < T_2 < ... < T_n$.*

(C3) *The number of spin flips is $2q$ or equivalently, the length of each interval of negative spin is 1.*



The subset of candidate triangle configuration of $\mathcal{T}_{\Lambda+,k}^q$ will be denoted as $\mathcal{C}_{\Lambda+,k}^q$

The above picture shows an example of a candidate type configuration. The fractal type examples including the Cantor discrete configuration satisfies $(C1)$, $(C2)$, $(C3)$. The next proposition states a necessary condition for the configuration where the problem $(\mathcal{P})$ gets the maximum.

Proposition 2.20 *For each $q \geq 2$, $\alpha \in (0,1)$ if $\underline{T} \in \mathcal{T}_{\Lambda+}^q$ is not a candidate , there exists a configuration $\underline{T}^* \in \mathcal{T}_{\Lambda+}^q$ satisfiyng $||\underline{T}^*||_\alpha > ||\underline{T}||_\alpha$.*

Proof: The proof requires some technical and long computation, so it is given in Appendix 2.6.2, where we show separately that (C1), (C2) and (C3) are necessary conditions.

Main part of the Proof

Note the configuration that maximize $\mathcal{P}$ is a candidate type, so we can write

$$\begin{aligned} f(q,\alpha) &:= \max_{\underline{T}\in\mathcal{T}^q_{\Lambda+,k}} ||\underline{T}||_\alpha & (\mathcal{P}) \\ &:= \max_{\underline{T}\in\mathcal{C}^q_{\Lambda+,k}} ||\underline{T}||_\alpha & (2.130) \end{aligned}$$

i.e. , we only need to search the maximum over candidate configurations. To prove that σ^0_C maximizes $\mathcal{P}$ for $Q = 2^m$, $m \geq 0$ we proceed by induction over m.

m=0 We have just one option, a single triangle with mass 1, and the spin configuration coincides with σ^0_C.

m=1 For $m = 1$ there exists three possible type of configuration depending on the location of the negative spins. Let d = di$st(x_1, x_2) - 1$ where x_1, x_2 are the negative spins , we have

- If d = 0 the configuration only contains a single external triangle whose length is 2.
- If d = 1 the configuration contains an external triangle whose length 3 and an internal triangle whose length is 1.
- If d $\geq$ 2 the configuration contains two external triangles whose length is 1.

We notice that we have only one candidate configuration, consisting on an external triangle with mass 3, and an internal triangle with mass 1, whose spin configuration is exactly σ^1_C. From proposition 2.20 we get that maximizes ($\mathcal{P}$) for Q=2.
To prove $m \Rightarrow m+1$, from condition $(C3)$ of Definition 2.19 there exists an internal triangle, we say T_c such that for any $x_0 \in \Delta(T_c)$ (the basis of T_c)

$$Q_{left}(\underline{T}, x) = Q_{right}(\underline{T}, x) = 2^m \tag{2.131}$$

where

$$Q_{left}(\underline{T}, x_0) = \sum_{x<x_0} \mathbb{1}_{\sigma_x=-1} \tag{2.132}$$
$$Q_{right}(\underline{T}, x_0) = \sum_{x>x_0} \mathbb{1}_{\sigma_x=-1} \tag{2.133}$$
$$\tag{2.134}$$

denotes respectively the number of negative spins at the left and right hand of x_0. On the other hand, for any $\underline{T} \in \mathcal{C}^q_{\Lambda^+,k}$ we get

$$||\underline{T}||_\alpha = \sum_{T\in\underline{T}} |T|_\alpha \tag{2.135}$$
$$||\underline{T}||_\alpha = \sum_{T<T_c} |T|^\alpha + \sum_{T>T_c} |T|^\alpha + |T_c|^\alpha + |T_{\rm ex}|^\alpha \tag{2.136}$$

Now, let $\{x^*_{1,c} < x^*_{2,c}\}$ and $\{x^*_{1,{\rm ex}} < x^*_{2,{\rm e}x}\}$ be the spin flip points of T_c and T_{ex} respectively. We denote

$$l_1 = |x^*_{1,c} - x^*_{1,{\rm e}x}| \tag{2.137}$$
$$l_2 = |x^*_{2,c} - x^*_{2,{\rm e}x}| \tag{2.138}$$

The mass of the external triangle can be written as

$$|T_{{\rm e}x}| = |T_c| + l_1 + l_2 \tag{2.139}$$

In addition, from the distance rule we get the next inequality for $|T_c|$

$$|T_c| \leq l_1 \wedge l_2 \tag{2.140}$$

so, we write equation (2.135) in the form

$$||\underline{T}||_\alpha = \sum_{T<T_c} |T|^\alpha + l_1^\alpha - l_1^\alpha \tag{2.141}$$
$$+ \sum_{T>T_c} |T|^\alpha + l_2^\alpha - l_2^\alpha \tag{2.142}$$
$$+ |T_c|^\alpha + |T_{{\rm e}x}|^\alpha \tag{2.143}$$

Note that

$$\sum_{T<T_c} |T|^\alpha + l_1^\alpha = \sum_{T \in \underline{T}'} |T|^\alpha \tag{2.144}$$

for some $\underline{T}' \in \mathcal{C}_\Lambda^{2^m}$. The same follows for $\sum_{T>T_c} |T|^\alpha + l_2^\alpha$. It implies the following inequalities,

$$\sum_{T<T_c} |T|^\alpha + l_1^\alpha \leq f_\alpha(2^m) \tag{2.145}$$

$$\sum_{T>T_c} |T|^\alpha + l_2^\alpha \leq f_\alpha(2^m) \tag{2.146}$$

$$\tag{2.147}$$

and by consequence, for each $\underline{T} \in \mathcal{C}_\Lambda^{2^{m+1}}$

$$||\underline{T}||_\alpha \leq 2f_\alpha(2^m) - l_1^\alpha - l_2^\alpha + |T_c|^\alpha + |T_c + l_1 + l_2|^\alpha \tag{2.148}$$

We claim

$$|T_c|^\alpha + |T_c + l_1 + l_2|^\alpha - l_1^\alpha - l_2^\alpha \leq (3^\alpha - 1)\max\{l_1^\alpha, l_2^\alpha\} \tag{2.149}$$

If equation (2.149) is true, we get

$$||\underline{T}||_\alpha \leq 2f_\alpha(2^m) + (3^\alpha - 1)\max\{l_1^\alpha, l_2^\alpha\} \tag{2.150}$$

$$\leq 2f_\alpha(2^m) + (3^\alpha - 1)3^{m\alpha} \tag{2.151}$$

The inequality does not depend on $||\underline{T}|| \in \mathcal{C}_{\Lambda^+,k}^{2^m}$, then

$$f_\alpha(2^{m+1}) \leq 2f_\alpha(2^m) + (3^\alpha - 1)3^{m\alpha} \tag{2.152}$$

$$\tag{2.153}$$

From the induction Hypothesis , $f_\alpha(2^m) = ||\Gamma_{C^{2^m}}||_\alpha$, so

$$
\begin{aligned}
f_\alpha(2^{m+1}) &\leq 2||\Gamma_{C^{2^m}}||_\alpha + (3^\alpha - 1)3^{m\alpha} && (2.154)\\
&= ||\Gamma_{C^{2^{m+1}}}||_\alpha && (2.155)
\end{aligned}
$$

concluding the proof. We left to prove equation (2.149), note that we have already shown the claim for $m = 0, 1$ (see 2.6.1) so we will proceed by induction on m. To prove $m \Rightarrow m+1$, we observe

$$
|T_{\mathrm{e}x}| = |T_c| + l_1 + l_2 \qquad (2.156)
$$

Since $l_1 = |T_{\mathrm{e}x}|$ for some $T_{ex} \in \underline{T}_a$, $\underline{T}_a \subseteq \mathcal{C}^{2^m}_{\Lambda,k}$ and from the Induction hyphotesis, we get $l_1 \leq 3^m$ and we use the very same argument to prove $l_2 \leq 3^m$. In addition, $|T_c| \leq l_1 \wedge l_2 \leq 3^m$ and it implies

$$
\begin{aligned}
|T_{\mathrm{e}x}| &\leq 3^m + 3^m + 3^m && (2.157)\\
&= 3^{m+1} && (2.158)
\end{aligned}
$$

□

2.6.2 Appendix

Proof of Proposition 2.20

Lemma 2.21 *(Proof of Condition (C1)). Let $\underline{T}$ be a compatible triangle configuration such that $\mathcal{R}(\underline{T}) = \Gamma$ is a single contour and the number of negative spins is $Q(\underline{T}) = q$. If $N_{\mathrm{e}x}(\underline{T}) > 1$, then there exists a compatible triangle configuration $\underline{T}^{'}$ satisfying*

$$
\begin{aligned}
\mathcal{R}(\underline{T}^{'}) &= \{\Gamma^{'}\} && (2.159)\\
N_{\mathrm{e}x}(\underline{T}^{'}) &< N_{\mathrm{e}x}(\underline{T}) && (2.160)\\
Q(\underline{T}^{'}) &= q && (2.161)\\
||\underline{T}^{'}||_\alpha &> ||\underline{T}||_\alpha && (2.162)
\end{aligned}
$$

Proof: We recall the notation

$$
\underline{T} = \bigcup_{\mathrm{i}=1}^{N_{\mathrm{e}x}(\underline{T})} T_{\mathrm{e}x,\mathrm{i}}(\underline{T}) \cup \underline{T}_{\mathrm{i}n}(T_{\mathrm{e}x,\mathrm{i}}) \qquad (2.163)
$$

where $T_{ex,1} < T_{ex,1} < \cdots T_{N_{ex}(\underline{T})}$ are the external triangles of $\underline{T}$ and its internal triangles are denoted by $\underline{T}_{in}(T_{ex,i})$. By writing $\underline{T}_i^* = T_{ex,i}(\underline{T}) \cup \underline{T}_{in}(T_{ex,i})$, $i = 1, \cdots, N_{ex}(\underline{T})$ we get an equivalent expression for (2.163)

$$\underline{T} = \bigcup_{i=1}^{N_{ex}(\underline{T})} \underline{T}_i^* \tag{2.164}$$

Also, we denote the spin flip points of $\underline{T}$ as

$$\mathrm{sf}^*(\underline{T}) = \bigcup_{i=1}^{N_{ex}(\underline{T})} \mathrm{sf}^*(\underline{T}_i^*) \tag{2.165}$$

$$\mathrm{sf}^*(\underline{T}_i^*) = \bigcup_{T\in\underline{T}_i^*} \mathrm{sf}^*(T) \tag{2.166}$$

in a very similar way, we denote the middle points $r^{'}$s

$$r(\underline{T}) = \bigcup_{i=1}^{N_{ex}(\underline{T})} r(\underline{T}_i^*) \tag{2.167}$$

$$r(\underline{T}_i^*) = \bigcup_{T\in\underline{T}_i^*} r(T) \tag{2.168}$$

We will build explicitly $\underline{T}^{'}$. To do that, we first set the constants

$$K := K_{T_{ex,1}(\underline{T}),T_{ex,2}(\underline{T})} = |T_{ex,1}(\underline{T})| \wedge |T_{ex,2}(\underline{T})| - \mathrm{di}st(T_{ex,1}(\underline{T}), T_{ex,2}(\underline{T})) \tag{2.169}$$

If $N_{ex}(\underline{T}) \geq 3$ we define additionally

$$M := M_{T_{ex,2}(\underline{T}),T_{ex,3}(\underline{T})} = K(T_{ex,1}(\underline{T}), T_{ex,2}(\underline{T})) - \mathrm{di}st(T_{ex,2}(\underline{T}), T_{ex,3}(\underline{T})) \tag{2.170}$$

We consider the following spin flip configurations

- If $N_{ex}(\underline{T}) = 2$

$$\mathrm{sf}^*(T_{ex,1}^*) \bigcup \{\mathrm{sf}^*(T_{ex,2}^*) + K\} \tag{2.171}$$

- If $N_{ex}(\underline{T}) \geq 3$

$$\mathrm{sf}^*(T_{ex,1}^*) \bigcup \{\mathrm{sf}^*(T_{ex,2}^*) + K\} \bigcup_{i=3}^{N_{ex}(\underline{T})} \{\mathrm{sf}^*(T_{ex,i}^*) + M\} \tag{2.172}$$

The middle points r_{i}'s of the configuration are given by

- If $N_{\mathrm{ex}}(\underline{T}) = 2$

$$r(T^*_{\mathrm{ex},1}) \bigcup \{r(T^*_{\mathrm{ex},2}) + K\} \tag{2.173}$$

- If $N_{\mathrm{ex}}(\underline{T}) \geq 3$

$$r(T^*_{\mathrm{ex},1}) \bigcup \{r(T^*_{\mathrm{ex},2}) + K - \delta_1\} \bigcup_{\mathrm{i}=1}^{N_{\mathrm{ex}}(\underline{T})} \{r(T^*_{\mathrm{ex},\mathrm{i}}) + M - \delta_2\} \tag{2.174}$$

where δ_1, δ_2 enough small satisfying

$$\begin{aligned} \mathrm{d}ist(r(T^*_{\mathrm{ex},1}), r(T^*_{\mathrm{ex},2}) + K - \delta_1) &< |T_{\mathrm{ex},1}(\underline{T})| \wedge |T_{\mathrm{ex},2}(\underline{T})| & (2.175) \\ \mathrm{d}ist(r(T^*_{\mathrm{ex},2}) + K - \delta_1, r(T^*_{\mathrm{ex},3}) + M - \delta_2) &> \mathrm{d}iam(r(T^*_{\mathrm{ex},1}) \cup \{r(T^*_{\mathrm{ex},2}) + K - \delta_1\}) & (2.176) \end{aligned}$$

We call $\underline{T}^{'}$ to the compatible triangle configuration defined uniquely by (2.173). We compare next this new configuration with $\underline{T}$.

- From condition (2.175), we get that both $T_{\mathrm{ex},1}(\underline{T})$ and $T_{\mathrm{ex},2}(\underline{T})$ disappears, obtaining a new external triangle, we say $\widehat{T}_{\mathrm{ex},1}$ whose mass is $|T_{\mathrm{ex},1}(\underline{T})| + |T_{\mathrm{ex},2}(\underline{T})| + |T_{\mathrm{ex},1}(\underline{T})| \wedge |T_{\mathrm{ex},2}(\underline{T})|$ and a new internal triangle, we say $\widehat{T}$ whose mass is $|T_{\mathrm{ex},1}(\underline{T})| \wedge |T_{\mathrm{ex},2}(\underline{T})|$.
- If $N_{\mathrm{ex}}(\underline{T}) \geq 3$, condition (2.176) guarantees that for $\mathrm{i} \geq 3$, $T^*_{\mathrm{ex},\mathrm{i}}(\underline{T})$ are preserved up to translation.

Note that $\underline{T}^{'}$ can be written as

$$\underline{T}^{'} = \bigcup_{\mathrm{i}=3}^{N_{\mathrm{ex}}(\underline{T})} \widehat{T}^*_{\mathrm{i}} \bigcup \widehat{T}^*_1 \tag{2.177}$$

where

$$\widehat{T}^*_1 = \widehat{T}_1 \cup \widehat{\underline{T}}_{in}(T_{1,\mathrm{ex}}) \cup \widehat{\underline{T}}_{in}(T_{\mathrm{ex},2}) \cup \widehat{T}_{in} \tag{2.178}$$

$$\begin{aligned} \widehat{T}^*_{\mathrm{i}} &= \{\underline{T}^*_{\mathrm{i}} + M\} & \mathrm{i} \geq 3 \quad (2.179) \\ \widehat{\underline{T}}_{in}(T_{\mathrm{ex},\mathrm{i}}) &= \{\underline{T}_{in}(T_{\mathrm{ex},\mathrm{i}}) + K\} & \mathrm{i} = 1,2 \quad (2.180) \end{aligned}$$

We compute next $||\underline{T}^{'}||_{\alpha}$

$$
\begin{aligned}
||\underline{T}^{'}||_\alpha &= ||\widehat{\underline{T}}^*_1||_\alpha + \sum_{\mathrm{i}=3}^{N_{\mathrm{ex}}(\underline{T}^{'})} ||\widehat{\underline{T}}^*_{\mathrm{i}}||_\alpha && (2.181)\\
&= ||\widehat{\underline{T}}^*_1||_\alpha + \sum_{j=3}^{N_{\mathrm{ex}}(\underline{T})} ||\underline{T}^*_{\mathrm{i}}||_\alpha && (2.182)\\
&= \sum_{j=1}^{N_{\mathrm{ex}}(\underline{T})} ||\underline{T}^*_{\mathrm{i}}||_\alpha - ||\underline{T}^*_1||_\alpha - ||\underline{T}^*_2||_\alpha + ||\widehat{\underline{T}}^*_1||_\alpha && (2.183)\\
&= ||\underline{T}||_\alpha - ||\underline{T}^*_1||_\alpha - ||\underline{T}^*_2||_\alpha + ||\widehat{\underline{T}}^*_1||_\alpha && (2.184)\\
&> ||\underline{T}||_\alpha && (2.185)
\end{aligned}
$$

The last inequality follows from

$$
||\widehat{T}^*_1||_\alpha = ||T^*_1||_\alpha + ||T^*_2||_\alpha + ||\widehat{T}_{\mathrm{ex},1}||_\alpha + ||T_{\mathrm{ex},1} \wedge T_{\mathrm{ex},2}||_\alpha - ||T_{\mathrm{ex},1}||_\alpha - ||_{\mathrm{ex},2}||_\alpha \quad (2.186)
$$

and

$$
||\widehat{T}^*_{\mathrm{ex},1}||_\alpha > ||T_{\mathrm{ex},1}||_\alpha \vee ||T_{\mathrm{ex},2}||_\alpha \quad (2.187)
$$

□

Finally, we prove $Q(\underline{T}) = Q(\underline{T}^{'})$. In fact

$$
\begin{aligned}
Q(\underline{T}^{'}) &= \sum_{\mathrm{i}=1}^{N_{\mathrm{ex}}(\underline{T}^{'})} Q(\widehat{T}^*_{\mathrm{ex,i}}) && (2.188)\\
&= \sum_{\mathrm{i}=3}^{N_{\mathrm{ex}}(\underline{T}^{'})} Q(\widehat{T}^*_{\mathrm{ex,i}}) + Q(\widehat{T}^*_{\mathrm{ex},1}) + Q(\widehat{T}^*_{\mathrm{ex},2}) && (2.189)\\
&= \sum_{\mathrm{i}=1}^{N_{\mathrm{ex}}(\underline{T}^{'})} Q(T^*_{\mathrm{ex,i}}) && (2.190)\\
&= Q && (2.191)
\end{aligned}
$$

The main argument used in the above equations is based on the fact that there is not exist negative spins outside external triangles and , in addition, that the translation of $\mathrm{sf}(\underline{T}^*_{\mathrm{i}})$, $\mathrm{i} = 1, ..., N_{\mathrm{ex}}(\underline{T})$ does not affect the number of negative spins.

Corollary 2.22 *(Proof of Condition (C2)). Let* $\underline{T}$ *be a triangle configuration such that* $\mathcal{R}(\underline{T}) = \{\underline{\Gamma}\}$ *and* $N_{\rm ex}(\underline{T}) = 1$. *Suposse that some non external triangle* $T \in \underline{T}$ *satisfies* $\underline{T}_{\rm in}(T) \neq \varnothing$, *where*

$$\mathcal{D}(T) = \{T_a \in \underline{T},\ T_a \neq T : \quad \Delta(T_a) \subset \Delta(T)\} \tag{2.192}$$

then there exists a compatible triangle configuration $\underline{T}^{'}$ *such that*

$$\begin{aligned} \mathcal{R}(\underline{T}^{'}) &= \{\underline{\Gamma}^{'}\} && (2.193)\\ Q(\underline{T}^{'}) &= q && (2.194)\\ ||\underline{T}^{'}||_\alpha &> ||\underline{T}||_\alpha && (2.195) \end{aligned}$$

Proof: If $\mathcal{D}(T) \neq \varnothing$ for some internal triangle of $\underline{T}$, necessarily

$$\mathcal{D}(T_a) \neq \varnothing \tag{2.196}$$

for some $T_a \in \underline{T}$ which is external respect to the configuration $\underline{T} \backslash T_{ex}$. By taking the spin configuration whose spin flip points are

$$\{\mathrm{sf}^*(\underline{T}) \backslash \mathrm{sf}^*(\underline{T}_{\rm in}(T_a))\} \cup \{\mathrm{sf}^*(\underline{T}_{\rm in}(T_a)) + M\} \tag{2.197}$$

where $M > 0$ is chosen such that

$$\mathrm{sf}^*(T_{\rm ex}) < \{\mathrm{sf}^*(\underline{T}_{\rm in}(T_a)) + M\} \tag{2.198}$$

$$\mathrm{d}ist(T_{\rm ex}, \{\mathrm{sf}^*(\underline{T}_{\rm in}(T_a)) + M\}) > |T_a| \tag{2.199}$$

The associated middle points are

$$\{r(\underline{T}) \backslash r(\underline{T}_{\rm in}(T_a))\} \cup \{r(\underline{T}_{\rm in}(T_a)) + M\} \tag{2.200}$$

The above configuration consists on the original triangle configuration, where the spin flips of the triangle T_a and its internal triangles are removed and placed at the right hand of T_{ex}.

The triangle configuration $\underline{T}$ associated to the spin flip points given by (2.197) and middle points given by (2.200) satisfies

(a) $Q(\underline{T}^{'}) = Q(\underline{T})$
(b) $N_{\text{ex}}(\underline{T}^{'}) > 1$
(c) $||\underline{T}^{'}||_\alpha = ||\underline{T}||_\alpha$

Equation (2.199) guarantees that there exists more than one external triangle. The number of external triangles is greater than 1, so the result follows directly from proposition 2.21. □

We left to check (a), (c). To prove (a) we take

$$supp(\underline{T}_{\text{in}}(T_a)) \cap \mathbb{Z} = [x^-(\underline{T}_{\text{in}}(T_a)), x^+(\underline{T}_{\text{in}}(T_a))] \cap \mathbb{Z} \tag{2.201}$$

the minimal interval that contains all the sites of $\underline{T}_{\text{in}}(T_a)$. We also define

$$\begin{aligned} I_1 &= supp(T_{\text{ex}}) \cap \Lambda && (2.202)\\ I_2 &= supp(T_{\text{ex}}) \backslash supp(T_a) \cap \Lambda && (2.203)\\ I_3 &= supp(T_a) + M && (2.204) \end{aligned}$$

We have

$$\sigma_x(\underline{T}^{'}) = \begin{cases} +1 & \text{if } x \in I_1 \\ \sigma_x(\underline{T}) & \text{if } x \in I_2 \\ \sigma_{x-M}(\underline{T}) & \text{if } x \in I_3 \end{cases} \tag{2.205}$$

Then

$$\begin{aligned} Q(\underline{T}^{'}) &= \sum_{x\in I_1} \mathbb{1}_{\sigma_x(\underline{T}^{'})=-1} + \sum_{x\in I_2} \mathbb{1}_{\sigma_x(\underline{T}^{'})=-1} + \sum_{x\in I_3} \mathbb{1}_{\sigma_x(\underline{T}^{'})=-1} && (2.206)\\ &= \sum_{x\in I_2} \mathbb{1}_{\sigma_x(\underline{T}^{'})=-1} + \sum_{x\in I_3} \mathbb{1}_{\sigma_x(\underline{T}^{'})=-1} && (2.207)\\ &= \sum_{x\in supp(T_{\text{ex}})\backslash supp(T_a)} \mathbb{1}_{\sigma_x(\underline{T}^{'})=-1} + \sum_{x\in supp(T_a)+M} \mathbb{1}_{\sigma_x(\underline{T}^{'})=-1} && (2.208)\\ &= \sum_{x\in supp(T_{\text{ex}})\backslash supp(T_a)} \mathbb{1}_{\sigma_x(\underline{T})=-1} + \sum_{x\in supp(T_a)+M} \mathbb{1}_{\sigma_{x-M}(\underline{T})=-1} && (2.209)\\ &= \sum_{x\in supp(T_{\text{ex}})\backslash supp(T_a)} \mathbb{1}_{\sigma_x(\underline{T})=-1} + \sum_{x\in supp(T_a)} \mathbb{1}_{\sigma_x(\underline{T})=-1} && (2.210)\\ &= \sum_{x\in supp(T_{\text{ex}})} \mathbb{1}_{\sigma_x(\underline{T})=-1} && (2.211)\\ &= Q(\underline{T}) && (2.212) \end{aligned}$$

In addition

$$\begin{aligned} ||\underline{T}^{'}||_\alpha &= ||\underline{T}\backslash\underline{T}_{\text{in}}(T_a)||_\alpha + ||\underline{T}_{\text{in}}(T_a)||_\alpha && (2.213)\\ &= ||\underline{T}||_\alpha - ||\underline{T}_{\text{in}}(T_a)||_\alpha + ||\underline{T}_{\text{in}}(T_a)||_\alpha && (2.214)\\ &= ||\underline{T}||_\alpha && (2.215)\end{aligned}$$

Proposition 2.23 *(Proof of Condition (C3)). If $\underline{T}$ is a triangle configuration satisfying $\mathcal{R}(\underline{T}) = \{\Gamma\}$ and $|\text{sf}^*(\underline{T})| < 2Q$, then there exists a triangle configuration $\underline{T}^{'}$ such that*

(a) $|\text{sf}^*(\underline{T})| < |\text{sf}^*(\underline{T}^{'})|$.
(b) $Q(\underline{T}) = Q(\underline{T}^{'})$.
(c) $||\underline{T}^{'}||_\alpha > ||\underline{T}||_\alpha$.

Proof: We suppose that $\underline{T}$ contains one external triangle and it internal triangles satisfies $T_1 < T_2 < T_{N(\underline{T})-1}$, otherwise the result is a direct consequence from 2.21 and 2.22. We observe that if $|\text{sf}^*(\underline{T})| = 2k < 2Q$. there exists a site $x \in \Lambda$ such that $\sigma_x = \sigma_{x+1} = -1$. In addition, from the condition on the internal triangles $T_1 < T_2 < T_{N(\underline{T})-1}$ we deduce that the distance between two consecutive internal triangles is the length of some interval of negative spins. Then, there exists a pair of triangles T_a, T_b such that

$$\text{d}ist(T_a, T_b) = \text{d} > 1 \tag{2.216}$$

For each $x \in \mathbb{Z}$ fixed we denote the set of spin flip points of $\underline{T}$ at the right hand of x

$$\text{sf}^*_{left}(\underline{T}, x) = \{x^* \in \text{sf}^*(\underline{T}) : x^* < x\} \tag{2.217}$$

Similarily, we denote the spin flip points of $\underline{T}$ at the right hand of x

$$\text{sf}^*_{right}(\underline{T}, x) = \{x^* \in \text{sf}^*(\underline{T}) : x^* > x\} \tag{2.218}$$

Now, let $\{x_0, ..., x_{\text{d}-1}\}$ be those sites of negative spins and $\underline{T}^{'}$ the configuration whose spin flip points are

$$\text{sf}^*_{left}(\underline{T}, x_0) \cup \{x_0 + \frac{1}{2}, x_0 + \frac{3}{2}\} \cup \{\text{sf}^*_{right}(\underline{T}, x_0) + 1\} \tag{2.219}$$

Let us to explain shortly the action of the above procedure

(i) Add a pair of spin flips at points $\{x_0 + \frac{1}{2}, x_0 + \frac{3}{2}\}$.
(ii) The set of spin $\{\text{sf}^*_{right}(\underline{T}, x_0) + 1\}$ are translated one site at the right hand.

The associated middle points are

$$r_{left}(\underline{T}, x_0) \cup \{x_0 + \frac{1}{2} + \delta, x_0 + \frac{3}{2} - \delta\} \cup r_{right}\{(\underline{T}, x_0) + 1\} \tag{2.220}$$

where

$$r_{left}(\underline{T}, x) = \{x^* \in r(\underline{T}) : x^* < x\} \tag{2.221}$$

$$r_{right}(\underline{T}, x) = \{x^* \in r(\underline{T}) : x^* > x\} \tag{2.222}$$

and $|\delta| < 1/100$ is chosen such that

$$1 - 2\delta < \mathrm{d}is t(x_0 + 1/2 - \delta, r_{left}(\underline{T}, x)) \wedge \mathrm{d}is t(x_0 + 3/2 - \delta, r_{right}(\underline{T}, x)) \tag{2.223}$$

Note that the new configuration contains $2(k+1)$ spin flip points and from equation (2.223) we get that the effect of adding the spin flip points $\{x_0 + \frac{1}{2} + \delta, x_0 + \frac{3}{2} - \delta\}$ is that a new internal triangle whose length is 1 appears. Moreover, $Q(\underline{T}) = Q(\underline{T}^{'})$ and

$$\begin{aligned} ||\underline{T}^{'}||_\alpha &= ||\underline{T}||_\alpha + 1 &(2.224)\\ &> ||\underline{T}||_\alpha &(2.225)\end{aligned}$$

concluding the proof.

2.6.3 Technical Results

Proof of 2.128

We recall that for $q = 2^n$, $n \geq 1$

$$f(q, \alpha) = \begin{cases} \frac{q^{\frac{\alpha}{\alpha_0}} - q}{3^\alpha - 2} + q^{\frac{\alpha}{\alpha_0}} & \text{if } \alpha \neq \alpha_0 \\ q\left(\frac{\log(q)}{\log(2)} + 1\right) & \text{if } \alpha = \alpha_0 \end{cases} \tag{2.226}$$

For an arbitrary q, $m = \left[\frac{\log(q)}{\log(2)}\right]$ satisfies $2^m \leq q \leq 2^{m+1}$.

$\alpha = \alpha_0$

$$\begin{aligned} f(2^{m+1}, \alpha) &= 2^{m+1}\left(\frac{\log(2^{m+1})}{\log(2)} + 1\right) \\ &= 2^{m+1}(m+2) \\ &\leq 6m2^m \\ &\leq \frac{6}{\log 2} q \log q \end{aligned} \tag{2.227}$$

whereas

$$\begin{aligned} f(2^m, \alpha) &= 2^m (m+1) \\ &= \frac{1}{2} 2^{m+1}(m+1) \\ &\geq \frac{1}{2\log 2} q \log q \end{aligned} \tag{2.228}$$

So, by using lemma (2.18)

$$\frac{1}{2\log 2} q \log q < f(q, \alpha) < \frac{6}{\log 2} q \log q \tag{2.229}$$

$\alpha < \alpha_0$

$$\begin{aligned} f(2^{m+1}, \alpha) &= \frac{2^{m+1} - 2^{(m+1)\frac{\alpha}{\alpha_0}}}{2 - 3^\alpha} + 2^{(m+1)\frac{\alpha}{\alpha_0}} \\ &\leq \frac{2^{m+1}}{2 - 3^\alpha} + 2^{m+1} \\ &= \frac{3 - 3^\alpha}{2 - 3^\alpha} 2^{m+1} \\ &\leq \frac{4}{2 - 3^\alpha} 2^m \\ &\leq \frac{4}{2 - 3^\alpha} q \end{aligned} \tag{2.230}$$

whereas

$$\begin{aligned}
f(2^{m+1},\alpha) &= \frac{2^m - 2^{(m)\frac{\alpha}{\alpha_0}}}{2-3^\alpha} + 2^{(m)\frac{\alpha}{\alpha_0}} \\
&= \frac{1}{2-3^\alpha}2^m + \frac{1-3^\alpha}{2-3^\alpha}2^{(m)\frac{\alpha}{\alpha_0}} \\
&\geq \frac{1}{2-3^\alpha}2^m + \frac{1-3^\alpha}{2-3^\alpha}2^m \\
&= 2^m \\
&= \frac{1}{2}2^{m+1} \\
&\geq \frac{1}{2}q
\end{aligned} \tag{2.231}$$

So, by using lemma (2.18)

$$\frac{1}{2}q < f(q,\alpha) < \frac{4}{2-3^\alpha}q \tag{2.232}$$

$\alpha > \alpha_0$

$$\begin{aligned}
f(2^{m+1},\alpha) &= \frac{2^{m+1} - 2^{(m+1)\frac{\alpha}{\alpha_0}}}{2-3^\alpha} + 2^{(m+1)\frac{\alpha}{\alpha_0}} \\
&\leq \frac{2^{(m+1)\frac{\alpha}{\alpha_0}}}{3^\alpha - 2} + 2^{(m+1)\frac{\alpha}{\alpha_0}} \\
&= \frac{3^\alpha - 1}{3^\alpha - 2}2^{(m+1)\frac{\alpha}{\alpha_0}} \\
&= 2^{\frac{\alpha}{\alpha_0}}\frac{3^\alpha - 1}{3^\alpha - 2}2^{m\frac{\alpha}{\alpha_0}} \\
&\leq \frac{6}{3^\alpha - 2}q^{\frac{\alpha}{\alpha_0}}
\end{aligned} \tag{2.233}$$

whereas

$$\begin{aligned}
f(2^m,\alpha) &= \frac{2^m - 2^{m\frac{\alpha}{\alpha_0}}}{2-3^\alpha} + 2^{m\frac{\alpha}{\alpha_0}} \\
&\geq 2^{m\frac{\alpha}{\alpha_0}} \\
&= \frac{1}{2^{\frac{\alpha}{\alpha_0}}}2^{(m+1)\frac{\alpha}{\alpha_0}} \\
&\geq \frac{1}{2^{\frac{\alpha}{\alpha_0}}}q^{\frac{\alpha}{\alpha_0}}
\end{aligned} \tag{2.234}$$

So, by using lemma (2.18)

$$\frac{1}{2^{\frac{\alpha}{\alpha_0}}}q^{\frac{\alpha}{\alpha_0}} < f(q,\alpha) < \frac{6}{2-3^\alpha}q^{\frac{\alpha}{\alpha_0}} \tag{2.235}$$

2.7 Proof of Proposition 2.13

Proof:

We first claim that for all $k \geq 1$, $\mathrm{di}am(C^{(k)}_{\lambda,\mathrm{d}}) = \lambda^k - 1$. To prove this, we proceed by induction. For $k = 0$ we have $C^{(0)}_{\lambda,\mathrm{d}} = \{1\}$ and $\mathrm{di}am(C^{(k)}_{\lambda,\mathrm{d}}) = 0 = \lambda^0 - 1$, so the hyphotesis is satisfied. To prove $n-1 \Rightarrow n$, we first note that from construction, we get for $n-1$

$$C^{(n)}_{\lambda,\mathrm{d}} = C^{(n-1)}_{\lambda,\mathrm{d}} \cup \left\{\lambda(\lambda-1)^{n-1} + C^{(n-1)}_{\lambda,\mathrm{d}}\right\} \tag{2.236}$$

and by definition of $\mathrm{di}am(C^{(n)}_{\lambda,\mathrm{d}})$

$$\begin{aligned}
\mathrm{di}am(C^{(n)}_{\lambda,\mathrm{d}}) &= \max\left\{|x-y| : x \in C^{(n-1)}_{\lambda,\mathrm{d}},\ y \in \lambda(\lambda-1)^{n-1} + C^{(n-1)}_{\lambda,\mathrm{d}}\right\} \\
&= \max\{|x-y| : x \in C^{(n-1)}_{\lambda,\mathrm{d}},\ y \in C^{(n-1)}_{\lambda,\mathrm{d}}\} + (\lambda-1)\lambda^{n-1} \\
&= \mathrm{di}am(C^{(n-1)}_{\lambda,\mathrm{d}}) + (\lambda-1)\lambda^{n-1} \\
&= \lambda^{n-1} - 1 + (\lambda-1)\lambda^{n-1} \\
&= \lambda^n - 1
\end{aligned} \tag{2.237}$$

then, for all $k \geq 1$

$$|C_{\lambda,\mathrm{d}} \cap \{1, ..., \lambda^k\}| = 2|C_{\lambda,\mathrm{d}} \cap \{1, ..., \lambda^{k-1}\}| \tag{2.238}$$

by using the above equation recursively

$$|C_{\lambda,\mathrm{d}} \cap \{1, ..., \lambda^k\}| = 2^k \tag{2.239}$$

On the other hand, $k = \left[\frac{\log n}{\log \lambda}\right]$, satisfies

$$\lambda^k < n < \lambda^{k+1} \tag{2.240}$$

or equivalently

$$k \log \lambda < \log n < (k+1) \log \lambda \tag{2.241}$$

It implies

$$|C_{\lambda,\mathrm{d}} \cap \{1, ..., \lambda^k\}| \leq |C_{\lambda,\mathrm{d}} \cap \{1, ..., n\}| \leq |C_{\lambda,\mathrm{d}} \cap \{1, ..., \lambda^{k+1}\}| \tag{2.242}$$

from equations (2.239), (2.240), (2.242)

$$k \log 2 \leq \log |C_{\lambda,\mathrm{d}} \cap \{1, ..., n\}| \leq (k+1) \log 2 \tag{2.243}$$

So

$$\frac{k}{k+1} \frac{\log 2}{\log \lambda} \leq \frac{\log |C_{\lambda,\mathrm{d}} \cap \{1, ..., n\}|}{\log n} \leq \frac{k+1}{k} \frac{\log 2}{\log \lambda} \tag{2.244}$$

Finally, since $k = \left[\frac{\log n}{\log \lambda}\right]$ tends to infinity when $n \to \infty$, we conclude

$$\lim_{n \to \infty} \frac{\log |C_{\lambda,\mathrm{d}} \cap \{1, ..., n\}|}{\log n} = \frac{\log 2}{\log \lambda} \tag{2.245}$$

2.8 Optimal bounds on Contours

In this section, we are concentrated in finding some sufficient conditions on the building of contour configurations, in order to get sort of quasi additivity on the Hamiltonian. We remark that in the classical One Dimensional Ising Model , for any $\underline{\Gamma} = \Gamma \cup \Gamma_0$

$$H[\Gamma_0 \cup \underline{\Gamma}] = H[\Gamma_0] + H[\underline{\Gamma}] \tag{2.246}$$

This is not our case, because of Long Range interactions. As we will see below, Main theorem of the section states

$$H[\Gamma_0 \cup \underline{\Gamma}] \sim H[\Gamma_0] + H[\underline{\Gamma}] \tag{2.247}$$

for a suitable choose of the constant c of the contour structure. The proof is quite technical and we will be the main concern of this section. Consequently, we revisit the Peierls argument by using the bounds obtained in this work.

Theorem 2.24 *Let $\Gamma_0, ..., \Gamma_n$ be an arbitrary configuration of compatible contours. Then, for all $0 \leq \alpha < 1$*

(i) The next inequalities are satisfied

$$0 > H[\Gamma_0, ..., \Gamma_n] - H[\Gamma_1, ..., \Gamma_n] - H[\Gamma_0] \geq -\varphi_c(\alpha) H[\Gamma_0] \tag{2.248}$$

or equivalently

$$H[\Gamma_0] > H[\Gamma_0, ..., \Gamma_n] - H[\Gamma_1, ..., \Gamma_n] \geq (1 - \varphi_c(\alpha)) H[\Gamma_0] \tag{2.249}$$

where c is the constant used in the contours construction and

$$\varphi_c(\alpha) = \frac{(2-\alpha)}{c^{1-\alpha}} + \frac{\pi^2}{6c} \tag{2.250}$$

(ii) By using the formula

$$H[\Gamma_0, ..., \Gamma_n] = H[\Gamma_n] + \sum_{\mathrm{i}=0}^{n-1} H[\Gamma_{\mathrm{i}}, ..., \Gamma_n] - H[\Gamma_{\mathrm{i}+1}, ..., \Gamma_n] \tag{2.251}$$

and equation (2.249) recursively, we get

$$\sum_{\mathrm{i}=0}^{n} H[\Gamma_{\mathrm{i}}] > H[\Gamma_0, ..., \Gamma_n] \geq (1 - \varphi_c(\alpha)) \sum_{\mathrm{i}=0}^{n} H[\Gamma_{\mathrm{i}}] \tag{2.252}$$

(iii) For $0 \leq \alpha < \alpha^+ = \frac{\log 3}{\log 2} - 1$, there exists functions $0 < \xi_1(\alpha) < \xi_2(\alpha)$ such that for any single contour

$$\xi_2(\alpha) ||\Gamma||_\alpha > H[\Gamma] > \xi_1(\alpha) ||\Gamma||_\alpha \tag{2.253}$$

Then, equation (2.252) can be rewritten

$$\xi_2(\alpha) \sum_{\mathrm{i}=0}^{n} ||\Gamma_{\mathrm{i}}||_\alpha > H[\Gamma_0, ..., \Gamma_n] \geq \xi_1(\alpha)(1 - \varphi_c(\alpha)) \sum_{\mathrm{i}=0}^{n} ||\Gamma_{\mathrm{i}}||_\alpha \tag{2.254}$$

Theorem 2.24 proves the quasi additivity of contours for all $0 \leq \alpha < 1$ and could be extended to "reasonable" non increasing interaction. We recall that in the classical Ising Model, any family of compatible contours is additive, i.e

$$H[\Gamma_0, ..., \Gamma_n] = \sum_{\mathrm{i}=0}^{n} H[\Gamma_{\mathrm{i}}] \tag{2.255}$$

We emphasize that the case $0 \leq \alpha < \alpha^+$, which is extensively studied in [31] [32] is also considered in our theorem, specifically in (iii), where a lower a bound in terms of $|| \, ||_\alpha$ is provided. Also, equation (2.253) states the quasi adittivity not only of contours but also of triangles, which is not true for any α in (0,1) (We have already shown that for $\alpha \geq \log(2)/\log(3)$ there exists an infinity amount of fractal type contours which does not satisfies an inequality similar to equation (2.253).

A especial case of interest is when only mutually external contours are considered, i.e. we make a partition of $\underline{\Gamma} = \cup_{j=1}^{N^{ex}(\underline{\Gamma})}(\Gamma_j^{\text{ex}} \cup \underline{\Gamma}^{\text{in}}(\Gamma_j^{\text{ex}}))$. The Hamiltonian can be written as

$$H[\underline{\Gamma}] = \sum_{\text{i}=1}^{N^{\text{ex}}(\underline{\Gamma})} H[\Gamma_{\text{i}}^*] + 2\sum_{\Gamma \neq \Gamma'} \mathcal{K}[\underline{\Gamma}_{\text{i}}^*, \underline{\Gamma}_j^*] \tag{2.256}$$

where $\underline{\Gamma}_{\text{i}}^* = \Gamma_{\text{i}}^{\text{ex}} \cup \underline{\Gamma}^{\text{in}}(\Gamma_{\text{i}}^{\text{ex}})$ and

$$\begin{aligned} \mathcal{K}[\underline{\Gamma}_{\text{i}}^*, \underline{\Gamma}_j^*] &= \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} J(x,y) \left\{ 1_{\sigma_x(\underline{\Gamma}_{\text{i}}^*, \underline{\Gamma}_j^*) \neq \sigma_y(\underline{\Gamma}_{\text{i}}^*, \underline{\Gamma}_j^*)} - 1_{\sigma_x(\underline{\Gamma}_{\text{i}}^*)=-1} - 1_{\sigma_y(\underline{\Gamma}_j^*)=-1} \right\} \\ &= -2\sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} J(x,y) 1_{\sigma_x(\underline{\Gamma}_{\text{i}}^*, \underline{\Gamma}_j^*)=-1, \sigma_y(\underline{\Gamma}_{\text{i}}^*, \underline{\Gamma}_j^*)=-1} \end{aligned} \tag{2.257}$$

the notation $(\sigma_x(\underline{\Gamma}_{\text{i}}^*, \underline{\Gamma}_j^*), x \in \Lambda)$ represents the spin configuration when only the family of contours $\underline{\Gamma}_{\text{i}}^*, \underline{\Gamma}_j^*$ is present, $\sigma_x(\underline{\Gamma}_{\text{i}}^*)$ denotes the spin configuration where only the family of contours $\underline{\Gamma}_{\text{i}}^*$ is present. The same follows for $\sigma_x(\underline{\Gamma}_{\text{i}}^*)$

In this case, the Hamiltonian can be written a sum of two body interaction terms. The importance of this type of interpretation has important consequences, specifically in studying a cluster type expansion. We emphasize that the equation (2.248) does not depend on the choose of Γ_0, the "erased" contour. It makes possible to state a very similar result for mutually external contours and in general for an arbitrary partition of $\underline{\Gamma}$.

Corollary 2.25 *Let $\underline{\Gamma}$ be an arbitrary configuration of compatible contours, and P a partition whose atoms are $\{\underline{\Gamma}_1, ..., \underline{\Gamma}_l\}$. For c enough large, it is satisfied*

$$H[\underline{\Gamma}] \geq (1 - \varphi_c(\alpha)) \sum_{\text{i}=1}^{l} H[\underline{\Gamma}_{\text{i}}] \tag{2.258}$$

In particular, if the partition whose atoms are $\underline{\Gamma}_{\text{i}}^* = \left\{ \Gamma_{\text{i}}^{\text{ex}} \cup \underline{\Gamma}^{\text{in}}(\Gamma_{\text{i}}^{\text{ex}}) \right\}$, $\text{i} = 1, ..., n_{\text{ex}}(\underline{\Gamma})$

is considered, we get

$$H[\underline{\Gamma}] \geq (1-\varphi_c(\alpha)) \sum_{\mathrm{i}=1}^{n^{ex}(\underline{\Gamma})} H[\underline{\Gamma}_{\mathrm{i}}^*] \tag{2.259}$$

Proof: From part (i) of Theorem 2.24 we get

$$\begin{aligned} H[\Gamma_0, \Gamma_1, ..., \Gamma_n] &\geq (1-\varphi_c(\alpha)) \sum_{\mathrm{i}=0}^{n} H[\Gamma_{\mathrm{i}}] && (2.260)\\ &= (1-\varphi_c(\alpha)) \sum_{\mathrm{i}=0}^{l} \sum_{\Gamma \in \underline{\Gamma}_{\mathrm{i}}} H[\Gamma] && (2.261)\\ &\geq (1-\varphi_c(\alpha)) \sum_{\mathrm{i}=0}^{l} H[\underline{\Gamma}_{\mathrm{i}}] && (2.262) \end{aligned}$$

2.8.1 Revisiting the Peierls argument

In this section, we revisit the Peierls argument used to show the existence of a Phase Transition for $\alpha \geq \frac{\log 3}{\log 2} - 1$.

$$\begin{aligned} \mu_\Lambda^+(\sigma_0 = -1) &\leq \mu_\Lambda^+(\{\sigma_0 \in \Gamma\}) && (2.263)\\ &= \frac{1}{Z_\Lambda^+} \sum_{0\Gamma} \sum_{\underline{\Gamma}:\Gamma\in\underline{\Gamma}} \mathrm{e}^{-\beta H[\underline{\Gamma}]} && (2.264) \end{aligned}$$

From Theorem (2.24)

$$\mathrm{e}^{-\beta H[\underline{\Gamma}]} \leq \mathrm{e}^{-\beta H[\underline{\Gamma}\setminus\Gamma]} \mathrm{e}^{-\beta(1-\varphi_c(\alpha))H[\Gamma]} \tag{2.265}$$

then

$$\mu_\Lambda^+(\{\sigma_0 \in \Gamma\}) \leq \sum_{0\Gamma} \mathrm{e}^{-\beta(1-\varphi_c(\alpha))H[\Gamma]} = \sum_{m} \sum_{\Gamma:|\Gamma|=m, 0\in\Gamma} \mathrm{e}^{-\beta(1-\varphi_c(\alpha))H[\Gamma]} \tag{2.266}$$

In the following, given $\delta > 0$we differentiate between those "problematic" contours such that

$$H[\Gamma] < \delta ||\Gamma||_\alpha \tag{2.267}$$

from those contours satisfying

$$H[\Gamma] \geq \delta ||\Gamma||_\alpha \tag{2.268}$$

For $\delta > 0$ fixed close to zero, a necessary condition for a contour satisfying (2.267) is $|\Gamma| > M_0$ for some M_0 enough large. Then, the right hand of 2.266 can be decomposed as

$$\sum_m \sum_{\Gamma:|\Gamma|=m, 0\in\Gamma} \mathrm{e}^{-\beta(1-\varphi_c(\alpha))H[\Gamma]} \mathbb{1}_{m>m_0} + \sum_m \sum_{\Gamma:|\Gamma|=m, 0\in\Gamma} \mathrm{e}^{-\beta(1-\varphi_c(\alpha))H[\Gamma]} \mathbb{1}_{m\leq m_0} \tag{2.269}$$

For $m \leq m_0$, we get from equation (2.268) for β enough large

$$\begin{aligned}
\sum_m \sum_{\Gamma:|\Gamma|=m, 0\in\Gamma} \mathrm{e}^{-\beta(1-\varphi_c(\alpha))H[\Gamma]} \mathbb{1}_{m\leq m_0} &\leq \sum_m \sum_{\Gamma:|\Gamma|=m, 0\in\Gamma} \mathrm{e}^{-\beta(1-\varphi_c(\alpha))\delta||\Gamma||_\alpha} \mathbb{1}_{m\leq m_0} && (2.270)\\
&\leq \sum_m \sum_{\Gamma:|\Gamma|=m, 0\in\Gamma} \mathrm{e}^{-\beta\delta(1-\varphi_c(\alpha))||\Gamma||_\alpha} && (2.271)\\
&\leq \sum_{m\geq 1} \mathrm{e}^{-\beta\delta(1-\varphi_c(\alpha))m^\alpha} && (2.272)
\end{aligned}$$

Analougsly, for $m > 0$ it could happen (2.267) and we can proceed directly as before. However, given a fixed configuration, the Hamiltonian is a increasing function of α we can use the equality

$$\begin{aligned}
H_\alpha[\Gamma] &= \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} \left(|x-y|^{\alpha-2} + J\mathbb{1}_{|x-y|=1}\right) \mathbb{1}_{\sigma_x(\Gamma)\neq\sigma_y(\Gamma)} && (2.273)\\
&> \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} \left(|x-y|^{\alpha^*-2} + J\mathbb{1}_{|x-y|=1}\right) \mathbb{1}_{\sigma_x(\Gamma)\neq\sigma_y(\Gamma)} && (2.274)\\
&= H_{\alpha^*}[\Gamma] && (2.275)
\end{aligned}$$

where $\alpha^* < \frac{\log 3}{\log 2} - 1 < \alpha < 1$. We have deduced that is possible to use a lower value of α to estimate the Hamiltonian

$$H_\alpha[\Gamma] > H_{\alpha^*}[\Gamma] > \xi(\alpha^*)||\Gamma||_{\alpha^*} \tag{2.276}$$

Replacing in equation 2.269

$$\begin{aligned}
\sum_m \sum_{\Gamma:|\Gamma|=m, 0\in\Gamma} \mathrm{e}^{-\beta(1-\varphi_c(\alpha))H[\Gamma]} \mathbb{1}_{m>m_0} &\leq \sum_m \sum_{\Gamma:|\Gamma|=m, 0\in\Gamma} \mathrm{e}^{-\beta(1-\varphi_c(\alpha))\delta\xi(\alpha^*)||\Gamma||_{\alpha^*}} \mathbb{1}_{m>m_0} && (2.277)\\
&\leq \sum_{m>m_0} \mathrm{e}^{-\beta(1-\varphi_c(\alpha))\delta\xi(\alpha^*)||\Gamma||_{\alpha^*}} && (2.278)
\end{aligned}$$

We have gotten

$$\mu_\Lambda^+(\sigma_0=-1)\leq\sum_{m>m_0}\mathrm{e}^{-\beta(1-\varphi_c(\alpha))\delta\xi(\alpha^*)||\Gamma||_{\alpha^*}}+\sum_{m\geq1}\mathrm{e}^{-\beta\delta(1-\varphi_c(\alpha))m^\alpha} \tag{2.279}$$

For m_0 enough large, the term $\sum_{m>m_0}\mathrm{e}^{-\beta(1-\varphi_c(\alpha))\delta\xi(\alpha^*)||\Gamma||_{\alpha^*}}$ decrease faster than the another one. Intuitively, contour with great mass and by consequence fractal type contours are unlikely.

2.8.2 Proof of Theorem 2.24

We recall the expression for the Hamiltonian

$$H[\Gamma_0,\underline{\Gamma}]=\frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}}J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})\neq\sigma_y(\Gamma_0,\underline{\Gamma})} \tag{2.280}$$

where $\sigma_x(\Gamma_0,\underline{\Gamma})$ denote the spin state at site x when the contours $\underline{\Gamma}$ and Γ_0 are present. Analougsly, we write

$$H[\Gamma_0] = \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}}J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} \tag{2.281}$$

$$H[\underline{\Gamma}] = \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}}J(x,y)\mathbb{1}_{\sigma_x(\underline{\Gamma})\neq\sigma_y(\underline{\Gamma})} \tag{2.282}$$

where $\sigma_x(\underline{\Gamma}_0)$ denotes the spin state at site x when only Γ_0 is present. Similarily, $\sigma_x(\underline{\Gamma})$ denotes the spin state at site x when only $\underline{\Gamma}$ is present.

We rewrite the Hamiltonian in a more convenient form

$$H[\Gamma_0,\underline{\Gamma}]=H[\Gamma_0,\underline{\Gamma}]-H[\underline{\Gamma}]-H[\Gamma_0]+H[\underline{\Gamma}]+H[\Gamma_0] \tag{2.283}$$

So

$$H[\Gamma_0,\underline{\Gamma}]=H[\underline{\Gamma}]+H[\Gamma_0]-\mathcal{K}[\Gamma_0,\underline{\Gamma}] \tag{2.284}$$

where

$$\mathcal{K}[\Gamma_0,\underline{\Gamma}]=H[\underline{\Gamma}]+H[\Gamma_0]-H[\Gamma_0,\underline{\Gamma}] \tag{2.285}$$

can be interpreted as an interaction term between Γ_0 and the remaining contours.

We expand explicitly this interaction term

$$\mathcal{K}[\Gamma_0, \underline{\Gamma}] = \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} \quad (2.286)$$

$$+ \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} J(x,y)\mathbb{1}_{\sigma_x(\underline{\Gamma})\neq\sigma_y(\underline{\Gamma})} \quad (2.287)$$

$$- \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})\neq\sigma_y(\Gamma_0,\underline{\Gamma})} \quad (2.288)$$

The above expression is the same as

$$\mathcal{K}[\Gamma_0, \underline{\Gamma}] = \frac{1}{2}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} J(x,y)\theta_{(\Gamma_0,\underline{\Gamma})}(x,y) \quad (2.289)$$

where

$$\theta_{(\Gamma_0,\underline{\Gamma})}(x,y) = \mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} + \mathbb{1}_{\sigma_x(\underline{\Gamma})\neq\sigma_y(\underline{\Gamma})} - \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})\neq\sigma_y(\Gamma_0,\underline{\Gamma})} \quad (2.290)$$

Let us to define

$$l_{\Gamma_0}(x) = \sum_{\mathrm{i}=-\infty}^{x} \mathbb{1}_{\sigma_{\mathrm{i}-1}(\Gamma_0)\neq\sigma_{\mathrm{i}}(\Gamma_0)} \quad (2.291)$$

the number of spin flips at the left hand of Γ_0. We remark that $x^* \in \mathbb{Z}^*$ is an spin flip point of Γ_0 if $\sigma_{x^*-1/2}(\Gamma_0) \neq \sigma_{x^*+1/2}(\Gamma_0)$. We also emphasize that $l_{\Gamma_0}(x)$ depends only on the spin flips of Γ_0 and that $l_{\Gamma_0}(x)$ is well defined when boundary conditions Λ^+ are considered, since for any compatible configuration the number of negative spins is a finite number

$$l_{\Gamma_0}(x) = \sum_{\mathrm{i}=-\infty}^{x} \mathbb{1}_{\sigma_{\mathrm{i}-1}(\Gamma_0)\neq\sigma_{\mathrm{i}}(\Gamma_0)} \quad (2.292)$$

$$= \sum_{\mathrm{i}\in\Lambda,\mathrm{i}\leq x} \mathbb{1}_{\sigma_{\mathrm{i}-1}(\Gamma_0)\neq\sigma_{\mathrm{i}}(\Gamma_0)} \quad (2.293)$$

$$\leq |\Lambda| \quad (2.294)$$

$$< \infty \quad (2.295)$$

We claim

$$\frac{\theta_{(\Gamma_0,\underline{\Gamma})}(x,y)}{2} = \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} \times \mathbb{1}_{l_{\Gamma_0}(x)+l_{\Gamma_0}(y)\ \text{odd}} \tag{2.296}$$

2.8.3 An explicit characterization for the difference

We will check claim 2.296 explicitly. We observe first

$$\sigma_x(\Gamma_0) = (-1)^{l_{\Gamma_0}(x)} \qquad \forall x \in \mathbb{Z} \tag{2.297}$$

$$\sigma_x(\underline{\Gamma}) = (-1)^{l_{\Gamma_0}(x)} \cdot \sigma_x(\Gamma_0,\underline{\Gamma}) \qquad \forall x \in \mathbb{Z} \tag{2.298}$$

For any pair $x, y \in \mathbb{Z}$ we necessarily have one of the next cases

$$l_{\Gamma_0}(x) + l_{\Gamma_0}(y) \quad \text{odd} \tag{2.299}$$

$$l_{\Gamma_0}(x) + l_{\Gamma_0}(y) \quad \text{even} \tag{2.300}$$

If $l_{\Gamma_0}(x) + l_{\Gamma_0}(y)$ even

(i) Since $(-1)^{l_{\Gamma_0}(x)} = (-1)^{l_{\Gamma_0}(y)}$, we deduce

$$\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} \equiv \mathbb{1}_{(-1)^{l_{\Gamma_0}(x)}\neq(-1)^{l_{\Gamma_0}(y)}} \tag{2.301}$$

$$\equiv \mathbb{1}_{(-1)^{l_{\Gamma_0}(x)}\neq(-1)^{l_{\Gamma_0}(x)}} \tag{2.302}$$

$$\equiv 0 \tag{2.303}$$

(ii) In addition $\sigma_x(\underline{\Gamma}) = (-1)^{l_{\Gamma_0}(x)}\sigma_x(\underline{\Gamma},\Gamma_0)$, $\sigma_y(\underline{\Gamma}) = (-1)^{l_{\Gamma_0}(y)}\sigma_y(\underline{\Gamma},\Gamma_0)$, so we deduce

$$\mathbb{1}_{\sigma_x(\underline{\Gamma})\neq\sigma_y(\underline{\Gamma})} \equiv \mathbb{1}_{(-1)^{l_{\Gamma_0}(x)}\sigma_x(\Gamma_0,\underline{\Gamma})\neq(-1)^{l_{\Gamma_0}(y)}\sigma_y(\Gamma_0,\underline{\Gamma})} \tag{2.304}$$

$$\equiv \mathbb{1}_{(-1)^{l_{\Gamma_0}(x)}\sigma_x(\Gamma_0,\underline{\Gamma})\neq(-1)^{l_{\Gamma_0}(x)}\sigma_y(\Gamma_0,\underline{\Gamma})} \tag{2.305}$$

$$\equiv \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})\neq\sigma_y(\Gamma_0,\underline{\Gamma})}. \tag{2.306}$$

From (i), (ii) we get

$$\frac{\theta_{(\Gamma_0,\underline{\Gamma})}(x,y)}{2} = \mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} + \mathbb{1}_{\sigma_x(\underline{\Gamma})\neq\sigma_y(\underline{\Gamma})} - \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})\neq\sigma_y(\Gamma_0,\underline{\Gamma})} \tag{2.307}$$

$$= 0 + \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})\neq\sigma_y(\Gamma_0,\underline{\Gamma})} - \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})\neq\sigma_y(\Gamma_0,\underline{\Gamma})} \tag{2.308}$$

$$= 0 \tag{2.309}$$

If $l_{\Gamma_0}(x) + l_{\Gamma_0}(y)$ is odd

(i) Since $(-1)^{l_{\Gamma_0}(x)} = -(-1)^{l_{\Gamma_0}(y)}$

$$\begin{aligned} \mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} &\equiv \mathbb{1}_{(-1)^{l_{\Gamma_0}(x)}\neq(-1)^{l_{\Gamma_0}(y)}} && (2.310)\\ &\equiv \mathbb{1}_{(-1)^{l_{\Gamma_0}(x)}\neq-(-1)^{l_{\Gamma_0}(x)}} && (2.311)\\ &\equiv 1 && (2.312)\end{aligned}$$

(ii) In adittion

$$\sigma_x(\underline{\Gamma}) = (-1)^{l_{\Gamma_0}(x)}\sigma_x(\underline{\Gamma}, \Gamma_0) \quad (2.313)$$

$$\begin{aligned} \sigma_y(\underline{\Gamma}) &= (-1)^{l_{\Gamma_0}(y)}\sigma_y(\underline{\Gamma}, \Gamma_0) && (2.314)\\ &= -(-1)^{l_{\Gamma_0}(x)}\sigma_y(\underline{\Gamma}, \Gamma_0) && (2.315)\end{aligned}$$

Implies

$$\begin{aligned} \mathbb{1}_{\sigma_x(\underline{\Gamma})\neq\sigma_y(\underline{\Gamma})} &\equiv \mathbb{1}_{(-1)^{l_{\Gamma_0}(x)}\sigma_x(\Gamma_0,\underline{\Gamma})\neq(-1)^{l_{\Gamma_0}(y)}\sigma_y(\Gamma_0,\underline{\Gamma})} && (2.316)\\ &\equiv \mathbb{1}_{(-1)^{l_{\Gamma_0}(x)}\sigma_x(\Gamma_0,\underline{\Gamma})\neq-(-1)^{l_{\Gamma_0}(x)}\sigma_y(\Gamma_0,\underline{\Gamma})} && (2.317)\\ &\equiv \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})}. && (2.318)\end{aligned}$$

From (i), (ii) we get

$$\begin{aligned} \theta_{(\Gamma_0,\underline{\Gamma})}(x, y) &= \mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} + \mathbb{1}_{\sigma_x(\underline{\Gamma})\neq\sigma_y(\underline{\Gamma})} - \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})\neq\sigma_y(\Gamma_0,\underline{\Gamma})} && (2.319)\\ &= \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} + 1 - \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})\neq\sigma_y(\Gamma_0,\underline{\Gamma})} && (2.320)\\ &= 2\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} && (2.321)\end{aligned}$$

The last equality follows directly from the identity

$$\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})\neq\sigma_y(\Gamma_0,\underline{\Gamma})} + \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} = 1 \quad (2.322)$$

We write 2.296 as

$$\theta_{(\Gamma_0,\underline{\Gamma})}(x, y) = \begin{cases} 0 & l_{\Gamma_0}(x) + l_{\Gamma_0}(y) \text{ even} \\ 2\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} & l_{\Gamma_0}(x) + l_{\Gamma_0}(y) \text{ odd} \end{cases} \quad (2.323)$$

2.8.4 A characterization of the sites in terms of the spin flip points of Γ_0

We start by introducing the collection of subsets

$$I_{\Gamma_0}(l) = \{x \in \mathbb{Z} : l_{\Gamma_0}(x) = l\} \tag{2.324}$$

which denotes an interval of $\mathbb{Z}$ of sites that has the same number of spin flips at its left hand. Note that if $x_1^* < x_2^* < \cdots < x_{2n}^* \subset \mathbb{Z}^*$ are the spin flip points of Γ_0 then

$$I_{\Gamma_0}(l) = \begin{cases} [x_l^*, x_{l+1}^*] \cap \mathbb{Z} & 1 \leq l \leq 2n-1 \\ (-\infty, x_1^*] \cap \mathbb{Z} & l = 0 \\ [x_{2n}^*, \infty) \cap \mathbb{Z} & l = 2n \\ \varnothing & \text{otherwise} \end{cases} \tag{2.325}$$

Also, we define

$$\underline{\Delta}_{\Gamma_0,in} = \cup_{l=1}^{2n-1} I_{\Gamma_0}(l) \tag{2.326}$$

$$\underline{\Delta}_{\Gamma_0,ex} = I(0) \cup I(2n) \tag{2.327}$$

Since $\mathbb{Z} = \cup_{l=0}^{2n} I_{\Gamma_0}(l)$, or equivalently $\mathbb{Z} = \underline{\Delta}_{\Gamma_0,in} \cup \underline{\Delta}_{\Gamma_0,ex}$ we can decompose the interaction as

$$\mathcal{K}[\Gamma_0, \underline{\Gamma}] = \frac{1}{2} \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} J(x,y) \theta_{(\Gamma_0, \underline{\Gamma})}(x,y) \tag{2.328}$$

$$= \frac{1}{2} \sum_{x \in \underline{\Delta}_{\Gamma_0,in}} \left(\sum_{y \in \underline{\Delta}_{\Gamma_0,in}} + \sum_{y \in \underline{\Delta}_{\Gamma_0,ex}} \right) J(x,y) \theta_{(\Gamma_0, \underline{\Gamma})}(x,y) \tag{2.329}$$

$$+ \frac{1}{2} \sum_{x \in \underline{\Delta}_{\Gamma_0,ex}} \left(\sum_{y \in \underline{\Delta}_{\Gamma_0,in}} + \sum_{y \in \underline{\Delta}_{\Gamma_0,ex}} \right) J(x,y) \theta_{(\Gamma_0, \underline{\Gamma})}(x,y) \tag{2.330}$$

From the above expression and from the symmetry of $J(x,y)$ we get

$$\mathcal{K}[\Gamma_0, \underline{\Gamma}] = \frac{1}{2} \sum_{x \in \underline{\Delta}_{\Gamma_0,in}} \sum_{y \in \underline{\Delta}_{\Gamma_0,in}} J(x,y) \theta_{(\Gamma_0, \underline{\Gamma})}(x,y) \tag{2.331}$$

$$+ 2 \times \frac{1}{2} \sum_{x \in \underline{\Delta}_{\Gamma_0,in}} \sum_{y \in \underline{\Delta}_{\Gamma_0,ex}} J(x,y) \theta_{(\Gamma_0, \underline{\Gamma})}(x,y)$$

$$+ \frac{1}{2} \sum_{x \in \underline{\Delta}_{\Gamma_0,ex}} \sum_{y \in \underline{\Delta}_{\Gamma_0,ex}} J(x,y) \theta_{(\Gamma_0, \underline{\Gamma})}(x,y) \tag{2.332}$$

For an arbitrary pair of sets $A, B \subset \mathbb{Z}$ we define

$$\mathcal{K}_{A,B}[\Gamma_0, \underline{\Gamma}] = \frac{1}{2} \sum_{x \in A} \sum_{y \in B} J(x,y) \theta_{(\Gamma_0, \underline{\Gamma})}(x,y) \tag{2.333}$$

By using this notation, equation (2.331) can be written as

$$\mathcal{K}[\Gamma_0, \underline{\Gamma}] = \mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}_{\Gamma_0,in}}[\Gamma_0, \underline{\Gamma}] + 2\mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}_{\Gamma_0,ex}}[\Gamma_0, \underline{\Gamma}] + \mathcal{K}_{\underline{\Delta}_{\Gamma_0,ex}, \underline{\Delta}_{\Gamma_0,ex}}[\Gamma_0, \underline{\Gamma}] \tag{2.334}$$

We need next to discriminate those points that are close to Γ_0 from those that are far away. To do that, we introduce the following partition of $\underline{\Delta}_{\Gamma_0,ex}$

$$\underline{\Delta}^{(1)}_{\Gamma_0,ex} = \{x \in \underline{\Delta}_{\Gamma_0,ex} : \mathrm{d}(x, \Gamma_0) > c|\Gamma_0|^3\} \tag{2.335}$$

$$\underline{\Delta}^{(2)}_{\Gamma_0,ex} = \{x \in \underline{\Delta}_{\Gamma_0,ex} : \mathrm{d}(x, \Gamma_0) \le c|\Gamma_0|^3\} \tag{2.336}$$

Since

$$\mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}_{\Gamma_0,ex}}[\Gamma_0, \underline{\Gamma}] = \mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}^{(1)}_{\Gamma_0,ex}}[\Gamma_0, \underline{\Gamma}] + \mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}^{(2)}_{\Gamma_0,ex}}[\Gamma_0, \underline{\Gamma}] \tag{2.337}$$

The next expression is valid for the interaction term

$$\mathcal{K}[\Gamma_0, \underline{\Gamma}] = \mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}_{\Gamma_0,in}}[\Gamma_0, \underline{\Gamma}] + 2\mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}^{(1)}_{\Gamma_0,ex}}[\Gamma_0, \underline{\Gamma}] + 2\mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}^{(2)}_{\Gamma_0,ex}}[\Gamma_0, \underline{\Gamma}] + \mathcal{K}_{\underline{\Delta}_{\Gamma_0,ex}, \underline{\Delta}_{\Gamma_0,ex}}[\Gamma_0, \underline{\Gamma}] \tag{2.338}$$

The next proposition states that the terms at the right hand of equation (2.338) are well bounded.

Proposition 2.26 *For any compatible configuration* $\Gamma_0, \underline{\Gamma}$ *and for all* $\alpha \in (0,1)$ *it holds*

$$\mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}_{\Gamma_0,in}}[\Gamma_0, \underline{\Gamma}] = 0 \tag{2.339}$$

$$2\mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}^{(1)}_{\Gamma_0,ex}}[\Gamma_0, \underline{\Gamma}] \le \frac{(2-\alpha)}{c^{1-\alpha}} H[\Gamma_0] \tag{2.340}$$

$$2\mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}^{(2)}_{\Gamma_0,ex}}[\Gamma_0, \underline{\Gamma}] \le \frac{2\pi^2}{6c} \sum_{x \in \underline{\Delta}_{\Gamma_0,in}} \sum_{y \in \underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y) \mathbb{1}_{\sigma_x(\Gamma_0) \ne \sigma_y(\Gamma_0)} \tag{2.341}$$

$$\mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}_{\Gamma_0,in}}[\Gamma_0, \underline{\Gamma}] \le \frac{2\pi^2}{6c} \sum_{x \in \underline{\Delta}_{\Gamma_0,in}} \sum_{y \in \underline{\Delta}_{\Gamma_0,in}} J(x,y) \mathbb{1}_{\sigma_x(\Gamma_0) \ne \sigma_y(\Gamma_0)} \tag{2.342}$$

And by consequence

$$\mathcal{K}[\Gamma_0, \underline{\Gamma}] \leq \left(\frac{(2-\alpha)}{c^{1-\alpha}} + 2\frac{\pi^2}{6c} \right) H[\Gamma_0] \tag{2.343}$$

Proof of 2.339

From the definition of $\underline{\Delta}_{\Gamma_0,ex}$, for each $x \in \underline{\Delta}_{\Gamma_0,ex}$,either $l_{\Gamma_0}(x) = 0$ or $l_{\Gamma_0}(x) = 2n$. Then , for $x, y \in \underline{\Delta}_{\Gamma_0,ex}$ it is certain that $l_{\Gamma_0}(x) + l_{\Gamma_0}(y)$ is an even number and by consequence

$$\theta_{(\Gamma_0,\underline{\Gamma})}(x,y) = 0 \qquad x, y \in \underline{\Delta}_{\Gamma_0,ex} \tag{2.344}$$

Equation 2.339 follows directly from (2.344)

Proof of 2.340

For each $y \in \underline{\Delta}_{\Gamma_0,ex}$, we have that $l_{\Gamma_0}(y)$ is an even number, so for $x \in \underline{\Delta}_{\Gamma_0,in}$, $y \in \underline{\Delta}_{\Gamma_0,ex}$

$$\begin{aligned} \theta_{(\Gamma_0,\underline{\Gamma})}(x,y) \neq 0 \;\; &\Rightarrow \;\; l_{\Gamma_0}(x) \quad \text{is an odd number.} &(2.345)\\ &\Rightarrow \;\; x \in I_{\Gamma_0}(l) \;\; \text{for some l odd.} &(2.346)\end{aligned}$$

From the above equation, we get

$$\begin{aligned} \mathcal{K}_{\underline{\Delta}_{\Gamma_0,in}, \underline{\Delta}^{(1)}_{\Gamma_0,ex}}[\Gamma_0, \underline{\Gamma}] &= \frac{1}{2} \sum_{x \in \underline{\Delta}_{\Gamma_0,in}} \sum_{y \in \underline{\Delta}^{(1)}_{\Gamma_0,ex}} J(x,y)\theta_{(\Gamma_0,\underline{\Gamma})}(x,y) &(2.347)\\ &= \sum_{x \in \underline{\Delta}_{\Gamma_0,in}} \sum_{y \in \underline{\Delta}^{(1)}_{\Gamma_0,ex}} J(x,y) \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma}) = \sigma_y(\Gamma_0,\underline{\Gamma})} \mathbb{1}_{l_{\Gamma_0}(x) + l_{\Gamma_0}(y) \;\; odd} &(2.348)\\ &= \sum_{l \text{ odd}} \sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in \underline{\Delta}^{(1)}_{\Gamma_0,ex}} J(x,y) \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma}) = \sigma_y(\Gamma_0,\underline{\Gamma})} &(2.349)\\ &= \sum_{l \text{ odd}} \mathcal{K}_{I_{\Gamma_0}(l), \underline{\Delta}^{(1)}_{\Gamma_0,ex}}[\Gamma_0, \underline{\Gamma}] &(2.350)\end{aligned}$$

From the inequality $\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} \leq 1$, we get for each l odd fixed

$$\begin{aligned} \mathcal{K}_{I_{\Gamma_0}(l),\underline{\Delta}^{(1)}_{\Gamma_0,ex}}[\Gamma_0,\underline{\Gamma}] &= \sum_{x\in I_{\Gamma_0}(l)} \sum_{y\in\underline{\Delta}^{(1)}_{\Gamma_0,ex}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} && (2.351)\\ &\leq \sum_{x\in I_{\Gamma_0}(l)} \sum_{y\in\underline{\Delta}^{(1)}_{\Gamma_0,ex}} J(x,y) && (2.352)\end{aligned}$$

In addition, for any pair $x \in I_{\Gamma_0}(l)$, l odd, $y \in \underline{\Delta}^{(1)}_{\Gamma_0,ex}$ we have

$$\sigma_x(\Gamma_0) = -1 \qquad \sigma_y(\Gamma_0) = +1 \tag{2.353}$$

$$\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} = 1 \qquad x \in I_{\Gamma_0}(l) \text{ l odd}, \; y \in \underline{\Delta}_{\Gamma_0,ex} \tag{2.354}$$

Then

$$\sum_{x\in I_{\Gamma_0}(l)} \sum_{y\in\underline{\Delta}^{(1)}_{\Gamma_0,ex}} J(x,y) = \sum_{x\in I_{\Gamma_0}(l)} \sum_{y\in\underline{\Delta}^{(1)}_{\Gamma_0,ex}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} \tag{2.355}$$

And by consequence

$$\begin{aligned} \mathcal{K}_{\underline{\Delta}_{\Gamma_0,in},\underline{\Delta}^{(1)}_{\Gamma_0,ex}}[\Gamma_0,\underline{\Gamma}] &= \sum_{l\text{ odd}} \mathcal{K}_{I_{\Gamma_0}(l),\underline{\Delta}^{(1)}_{\Gamma_0,ex}}[\Gamma_0,\underline{\Gamma}] && (2.356)\\ &\leq \sum_{l\text{ odd}} \sum_{x\in I_{\Gamma_0}(l)} \sum_{y\in\underline{\Delta}^{(1)}_{\Gamma_0,ex}} J(x,y) && (2.357)\\ &= \sum_{x\in\underline{\Delta}_{\Gamma_0,in}} \sum_{y\in\underline{\Delta}^{(1)}_{\Gamma_0,ex}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} && (2.358)\end{aligned}$$

We remark that the last equality holds since $\sigma_x(\Gamma_0) = -1$ if and only if $x \in I_{\Gamma_0}(l)$, l odd.

We recall the definition

$$Q(\Gamma_0) = \sum_{x\in\Lambda} \mathbb{1}_{\sigma_x(\Gamma_0)=-1} \tag{2.359}$$

the number of negative spins when only Γ_0 is present. Since $J(x,y)$ is decreasing as a function of $|x-y|$ and $|\Gamma_0| > Q(\Gamma_0)$ we get

$$\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}^{(1)}_{\Gamma_0,ex}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} \leq 2\sum_{x=1}^{Q}\sum_{y=cQ^3+Q+1}^{\infty} J(x,y) \tag{2.360}$$

From the inequality $cQ^3+Q>cQ$ and from Propostion (2.27) for $k=c$

$$\sum_{x=1}^{Q}\sum_{y=cQ^3+Q+1}^{\infty} J(x,y) \leq 2\frac{(2-\alpha)}{c^{1-\alpha}}\sum_{x=1}^{Q}\sum_{y=Q+1}^{\infty} J(x,y) \tag{2.361}$$

We emphasize that $2\sum_{x=1}^{Q}\sum_{y=Q+1}^{\infty} J(x,y)$ denotes the Hamiltonian of a single contours that contains just one triangle of mass Q.

We have already proven in Proposition (2.9) that for any single contour

$$2\sum_{x=1}^{Q}\sum_{y=Q+1}^{\infty} J(x,y) \leq H[\Gamma_0] \tag{2.362}$$

We conclude from equations (2.356), (2.360), (2.361), (2.362)

$$\mathcal{K}_{\underline{\Delta}_{\Gamma_0,in},\underline{\Delta}^{(1)}_{\Gamma_0,ex}}[\Gamma_0,\underline{\Gamma}] \leq \frac{(2-\alpha)}{c^{1-\alpha}}H[\Gamma_0] \tag{2.363}$$

Proof of 2.341

We remark that for $y\in\underline{\Delta}_{\Gamma_0,ex}$, $l_{\Gamma_0}(y)$ is an even number, whereas for $x\in\underline{\Delta}_{\Gamma_0,in}$, $l_{\Gamma_0}(x)$ is an odd number if and only if $x\in I_{\Gamma_0}(l)$ for l $o$dd. Then

$$\begin{aligned}\frac{1}{2}\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y)\theta_{(\Gamma_0,\underline{\Gamma})}(x,y) &= \sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})}\mathbb{1}_{l_{\Gamma_0}(x)+l_{\Gamma_0}(y)\text{ odd}}\\ &= \sum_{l\text{ odd}}\sum_{x\in I_{\Gamma_0}(l)}\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})}\end{aligned} \tag{2.364}$$

We recall that σ_0 denotes the spin state at the leftmost point of Γ_0 when $\Gamma_0,\underline{\Gamma}$ are present

$$\sigma_0=\sigma_{x^-(\Gamma_0)}(\Gamma_0,\underline{\Gamma}) \tag{2.365}$$

From Proposition (2.31), we get

$$\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_0}\mathbb{1}_{\sigma_y(\Gamma_0,\underline{\Gamma})=\sigma_0} \leq \frac{\pi^2}{6c}\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y) \quad (2.366)$$

$$\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=-\sigma_0}\mathbb{1}_{\sigma_y(\Gamma_0,\underline{\Gamma})=-\sigma_0} \leq \frac{\pi^2}{6c}\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y) \quad (2.367)$$

and by using the identity

$$\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} = \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_0}\mathbb{1}_{\sigma_y(\Gamma_0,\underline{\Gamma})=\sigma_0} + \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=-\sigma_0}\mathbb{1}_{\sigma_y(\Gamma_0,\underline{\Gamma})=-\sigma_0} \quad (2.368)$$

We get for each l odd fixed

$$\sum_{x\in I_{\Gamma_0}(l)}\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} \leq 2\frac{\pi^2}{6c}\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y) \quad (2.369)$$

Since

$$\sigma_x(\Gamma_0) = -1 \quad \Leftrightarrow \quad x \in I_{\Gamma_0}(l) \text{ l odd} \quad (2.370)$$

$$\sigma_y(\Gamma_0) = +1 \qquad y \in \underline{\Delta}_{\Gamma_0,ex} \quad (2.371)$$

We get $\sigma_x(\Gamma_0) = -1$ for $x \in I_{\Gamma_0}(l)$ l odd and $\sigma_y(\Gamma_0) = +1$ for $y \in \underline{\Delta}_{\Gamma_0,ex}$. It implies

$$\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} = 1 \quad (2.372)$$

From equations (2.369), (2.372) we deduce

$$2\frac{\pi^2}{6c}\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y) \leq 2\frac{\pi^2}{6c}\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} \quad (2.373)$$

which is exactly equation (2.341)

Proof of 2.342

$$\frac{1}{2}\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}_{\Gamma_0,in}} J(x,y)\theta_{(\Gamma_0,\underline{\Gamma})}(x,y) = \frac{1}{2}\sum_{l=1}^{2n-1}\sum_{m=1}^{2n-1}\sum_{x\in I_{\Gamma_0}(l)}\sum_{y\in I_{\Gamma_0}(m)} J(x,y)\theta_{(\Gamma_0,\underline{\Gamma})}(x,y) \quad (2.374)$$

$$= \frac{1}{2}\sum_{l=1}^{2n-1}\sum_{m=1}^{2n-1}\sum_{x\in I_{\Gamma_0}(l)}\sum_{y\in I_{\Gamma_0}(m)} J(x,y)\theta_{(\Gamma_0,\underline{\Gamma})}(x,y)\mathbb{1}_{l+m \text{ odd}}$$

$$+ \frac{1}{2}\sum_{l=1}^{2n-1}\sum_{m=1}^{2n-1}\sum_{x\in I_{\Gamma_0}(l)}\sum_{y\in I_{\Gamma_0}(m)} J(x,y)\theta_{(\Gamma_0,\underline{\Gamma})}(x,y)\mathbb{1}_{l+m \text{ even}}$$

If $l+m$ is even, we have that $l_{\Gamma_0}(x)+l_{\Gamma_0}(y)$ is an even number, so $\theta_{(\Gamma_0,\underline{\Gamma})}(x,y)=0$. It means

$$\frac{1}{2}\sum_{l=1}^{2n-1}\sum_{m=1}^{2n-1}\sum_{x\in I_{\Gamma_0}(l)}\sum_{y\in I_{\Gamma_0}(m)} J(x,y)\theta_{(\Gamma_0,\underline{\Gamma})}(x,y)\mathbb{1}_{l+m \text{ even}} = 0$$

On the other hand, if $l+m$ is an odd number

$$\theta_{(\Gamma_0,\underline{\Gamma})}(x,y) = 2\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} \quad (2.375)$$

From equations (2.374), (2.375), (2.375) we deduce

$$\frac{1}{2}\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}_{\Gamma_0,in}} J(x,y)\theta_{(\Gamma_0,\underline{\Gamma})}(x,y) = \sum_{l=1}^{2n-1}\sum_{m=1}^{2n-1}\sum_{x\in I_{\Gamma_0}(l)}\sum_{y\in I_{\Gamma_0}(m)} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})}\mathbb{1}_{l+m \text{ odd}}$$

Since

$$\mathbb{1}_{l+m \text{ odd}} = \mathbb{1}_{l \text{ odd}}\mathbb{1}_{m \text{ even}} + \mathbb{1}_{m \text{ odd}}\mathbb{1}_{l \text{ even}} \quad (2.376)$$

By symmetry we get

$$\frac{1}{2}\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}_{\Gamma_0,in}} J(x,y)\theta_{(\Gamma_0,\underline{\Gamma})}(x,y) = 2\frac{1}{2}\sum_{\substack{l=1:\\ l \text{ odd}}}^{2n-1}\sum_{\substack{m=1:\\ m \text{ even}}}^{2n-1}\sum_{x\in I_{\Gamma_0}(l)}\sum_{y\in I_{\Gamma_0}(m)} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} \quad (2.377)$$

We fix $1 \leq l, m \leq 2n-1$ such that $l+m$ is an odd number, so we will estimate

$$\sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in I_{\Gamma_0}(m)} J(x,y) \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} \tag{2.378}$$

It can be checked

$$\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} = \mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=\sigma_0} \mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=\sigma_0} + \mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=-\sigma_0} \mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=-\sigma_0} \tag{2.379}$$

By replacing the above identity, we get

$$\begin{aligned} \sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in I_{\Gamma_0}(m)} J(x,y) \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} &= \sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in I_{\Gamma_0}(m)} J(x,y) \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_0} \mathbb{1}_{\sigma_y(\Gamma_0,\underline{\Gamma})=\sigma_0} \\ &+ \sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in I_{\Gamma_0}(m)} J(x,y) \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=-\sigma_0} \mathbb{1}_{\sigma_y(\Gamma_0,\underline{\Gamma})=-\sigma_0} \end{aligned}$$

By using Proposition (2.32)

$$\sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in I_{\Gamma_0}(m)} J(x,y) \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_0} \mathbb{1}_{\sigma_y(\Gamma_0,\underline{\Gamma})=\sigma_0} \leq \frac{\pi^2}{6c} \sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in I_{\Gamma_0}(m)} J(x,y) \tag{2.380}$$

$$\sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in I_{\Gamma_0}(m)} J(x,y) \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=-\sigma_0} \mathbb{1}_{\sigma_y(\Gamma_0,\underline{\Gamma})=-\sigma_0} \leq \frac{\pi^2}{6c} \sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in I_{\Gamma_0}(m)} J(x,y) \tag{2.381}$$

And consequently

$$\sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in I_{\Gamma_0}(m)} J(x,y) \mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} \leq 2\frac{\pi^2}{6c} \sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in I_{\Gamma_0}(m)} J(x,y) \tag{2.382}$$

We remark that $\sigma_x(\Gamma_0,\underline{\Gamma}) \neq \sigma_y(\Gamma_0,\underline{\Gamma})$ if and only if $l_{\Gamma_0}(x) + l_{\Gamma_0}(y)$ is an odd number. For each fixed pair $x \in I_{\Gamma_0}(l)$, $y \in I_{\Gamma_0}(m)$ we have $\mathbb{1}_{\sigma_x(\Gamma_0) \neq \sigma_y(\Gamma_0)} = 1$ and it implies

$$2\frac{\pi^2}{6c} \sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in I_{\Gamma_0}(m)} J(x,y) = 2\frac{\pi^2}{6c} \sum_{x \in I_{\Gamma_0}(l)} \sum_{y \in I_{\Gamma_0}(m)} J(x,y) \mathbb{1}_{\sigma_x(\Gamma_0) \neq \sigma_y(\Gamma_0)} \tag{2.383}$$

Finally, by taking all the possible values of l odd, m even

$$
\begin{aligned}
2\sum_{\substack{l=1:\\ l \text{ odd}}}^{2n-1}\sum_{\substack{m=1:\\ m \text{ even}}}^{2n-1}\sum_{x\in I_{\Gamma_0}(l)}\sum_{y\in I_{\Gamma_0}(m)} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0,\underline{\Gamma})=\sigma_y(\Gamma_0,\underline{\Gamma})} &\leq 4\frac{\pi^2}{6c}\sum_{\substack{l=1:\\ l \text{ odd}}}^{2n-1}\sum_{\substack{m=1:\\ m \text{ even}}}^{2n-1}\sum_{x\in I_{\Gamma_0}(l)}\sum_{y\in I_{\Gamma_0}(m)} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y} \\
&= 2\frac{\pi^2}{6c}\sum_{l=1}^{2n-1}\sum_{m=1}^{2n-1}\sum_{x\in I_{\Gamma_0}(l)}\sum_{y\in I_{\Gamma_0}(m)} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma} \\
&= 2\frac{\pi^2}{6c}\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}_{\Gamma_0,in}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)}
\end{aligned}
$$

We rewrite the above inequality

$$
\frac{1}{2}\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}_{\Gamma_0,in}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} \leq 2\frac{\pi^2}{6c}\sum_{x\in\underline{\Delta}_{\Gamma_0,in}}\sum_{y\in\underline{\Delta}_{\Gamma_0,in}} J(x,y)\mathbb{1}_{\sigma_x(\Gamma_0)\neq\sigma_y(\Gamma_0)} \qquad (2.385)
$$

which is exactly equation 2.342.

2.8.5 Technical Results

The most technical part of the Proof of (2.24) is given in this section.

Proposition 2.27 *Let*

$$
\begin{aligned}
S(N,\mathrm{d}) &= \sum_{x=1}^{N}\sum_{y=N+\mathrm{d}+1}^{\infty} J_{|y-x|} \\
&= \sum_{x=1}^{N}\sum_{y'=1}^{\infty} J_{|y'+N+\mathrm{d}-x|} \qquad y' = N-\mathrm{d} \\
&= \sum_{x'=1}^{N}\sum_{y'=1}^{\infty} J_{|y'+x'+\mathrm{d}-1|} \qquad x' = N-x+1
\end{aligned}
$$

then , for all $\mathrm{d} > kN$, $N \geq 1$

$$
\frac{S(x,\mathrm{d})}{S(x,0)} = \frac{\sum_{x'=1}^{N}\sum_{y'=1}^{\infty} J_{|y'+x'+\mathrm{d}-1|}}{\sum_{x'=1}^{N}\sum_{y'=1}^{\infty} J_{|y'+x'-1|}} \leq (2-\alpha)k^{\alpha-1} \qquad (2.386)
$$

Proof: We use the approach of replacing sums by integrals. From the property that $J(x)$ is a decreasing function, we get for all $\mathrm{d} > 0$

$$0 \leq \sum_{y=1}^{\infty} J_{|x+y+\mathrm{d}-1|} - \int_1^{\infty} J_{|x+s+\mathrm{d}-1|}\mathrm{d}s \leq J_{|x+\mathrm{d}|} \tag{2.387}$$

So

$$\begin{aligned}
\frac{\sum_{y=1}^{\infty} J_{|x+y+\mathrm{d}-1|}}{\sum_{y=1}^{\infty} J_{|x+y-1|}} &\leq \frac{J_{|x+\mathrm{d}|} + \int_1^{\infty} J_{|x+s+\mathrm{d}-1|}\mathrm{d}s}{\int_1^{\infty} J_{|x+s-1|}\mathrm{d}s} \\
&\leq \max_{1\leq x\leq N} \frac{J_{|x+\mathrm{d}|} + \int_1^{\infty} J_{|x+s+\mathrm{d}-1|}\mathrm{d}s}{\int_1^{\infty} J_{|x+s-1|}\mathrm{d}s}
\end{aligned} \tag{2.388}$$

For $J(n) = n^{\alpha-2} + J\mathbb{1}_{n=1}$ we get

$$\begin{aligned}
\max_{1\leq x\leq N} \frac{J_{|x+\mathrm{d}|} + \int_1^{\infty} J_{|x+s+\mathrm{d}-1|}\mathrm{d}s}{\int_1^{\infty} J_{|x+s-1|}\mathrm{d}s} &= \max_{1\leq x\leq N} \frac{|x+\mathrm{d}|^{\alpha-2} + \frac{|x+\mathrm{d}|^{\alpha-1}}{1-\alpha}}{J\mathbb{1}_{x=1} + \frac{|x|^{\alpha-1}}{1-\alpha}} \\
&= \max_{1\leq x\leq N} \left[\frac{x+\mathrm{d}}{x}\right]^{\alpha-1} \left(1 + \frac{1-\alpha}{|x+\mathrm{d}|}\right) \\
&\leq \left[\frac{N+\mathrm{d}}{N}\right]^{\alpha-1} \left(1 + \frac{1-\alpha}{|1+\mathrm{d}|}\right)
\end{aligned} \tag{2.389}$$

The last inequality is a consequence from that as a function of x, $\left[\frac{x+kN}{x}\right]^{\alpha-1}$ is increasing where as $\frac{1}{x+\mathrm{d}}$ is decreasing. Finally, since $\mathrm{d} > kN > 1$, we deduce

$$\max_{1\leq x\leq N} \frac{J_{|x+\mathrm{d}|} + \int_1^{\infty} J_{|x+s+\mathrm{d}-1|}\mathrm{d}s}{\int_1^{\infty} J_{|x+s-1|}\mathrm{d}s} \leq (k+1)^{\alpha-1}\left(1+\frac{1-\alpha}{1+k}\right) \tag{2.390}$$

$$\leq k^{\alpha-1}(1+1-\alpha) \tag{2.391}$$

$$= (2-\alpha)k^{\alpha-1} \tag{2.392}$$

We have proven

$$\begin{aligned}
\sum_{x=1}^{N}\sum_{y=1}^{\infty} J_{|x+y+\mathrm{d}-1|} &= \sum_{x=1}^{N} \frac{\sum_{y=1}^{\infty} J_{|x+y+\mathrm{d}-1|}}{\sum_{y=1}^{\infty} J_{|x+y-1|}} \sum_{y=1}^{\infty} J_{|x+y-1|} \\
&\leq \frac{(2-\alpha)}{k^{1-\alpha}} \sum_{x=1}^{N}\sum_{y=1}^{\infty} J_{|x+y-1|}
\end{aligned} \tag{2.393}$$

which is exactly equation 2.386.

Propositions 2.31 and 2.32

We left to prove Propositions 2.31 2.32, which are based on the argument of grouping contours with the same mass in order to bound "far interaction". Before we provide the proof, we need to introduce a tree expansion of contours.

A tree expansion of contours

In this section, we give a tree representation of a typical contour configuration. To achieve this, we start the section by setting up some definitions for the contour configuration.

Definition 2.28 *Let* $\underline{\Gamma}$ *be an arbitrary configuration of compatible contours. We say that* Γ *is a "white" contour if* $\sigma_{x^-(\Gamma)}(\underline{\Gamma})$ *is equal to* -1*. Otherwise, we say that* Γ *is a "black" contour*

We recall that $\sigma_{x^-(\Gamma)}(\underline{\Gamma})$ represents the spin state at the leftmost point of Γ when the family of contours $\underline{\Gamma}$ is present. Every one can be identified either as a white or black, depending of its colour. Complementarily, we introduce the next definition to determine such an order notion in $\underline{\Gamma}$.

Definition 2.29 *Let* $\underline{\Gamma}$ *be an arbitrary configuration of compatible contours. For any pair* $\Gamma, \Gamma^{'} \in \underline{\Gamma}$ *we say that* $\Gamma \hookrightarrow \Gamma^{'}$ *if* $supp(\Gamma) \subseteq supp(\Gamma^{'})$.

Definition 2.30 *Let* $\underline{\Gamma}$ *be an arbitrary configuration of compatible contours. We say that* Γ *is a maximal contour respect to* $\underline{\Gamma}$ *if* $\Gamma \hookrightarrow \Gamma^{'}$ *implies* $\Gamma^{'} = \Gamma$*, i.e any* $\Gamma^{'} \neq \Gamma$,$supp(\Gamma) \nsubseteq supp(\Gamma^{'})$.

Note that the definition of maximal contour is the very same we give at the very beginning of this work. For any $\Gamma_a \in \underline{\Gamma}$ we introduce the subset

$$\mathcal{D}(\Gamma_a) = \{\Gamma \in \underline{\Gamma} : \Gamma \hookrightarrow \Gamma_a\} \tag{2.394}$$

which denotes the family of such contours whose support is contained in $supp(\Gamma_a)$. The next properties can be deduced

- If $\Gamma \hookrightarrow \Gamma^{'}$ and $\Gamma^{'} \hookrightarrow \Gamma^{''}$, then $\Gamma \hookrightarrow \Gamma^{''}$.

- If $\Gamma \hookrightarrow \Gamma'$ and $\Gamma' \hookrightarrow \Gamma$, then $\Gamma \equiv \Gamma'$

- $\Gamma_a \in \mathcal{D}(\Gamma_a)$ for all $\Gamma_a \in \underline{\Gamma}$.

- $\Gamma_a \hookrightarrow \Gamma_b$ implies $\mathcal{D}(\Gamma_a) \subseteq \mathcal{D}(\Gamma_b)$

- $\Gamma_a \hookrightarrow \Gamma_b$ and $\Gamma_b \hookrightarrow \Gamma_c$ implies $\Gamma_a \hookrightarrow \Gamma_c$ and by consequence $\mathcal{D}(\Gamma_a) \subseteq \mathcal{D}(\Gamma_b) \subseteq \mathcal{D}(\Gamma_c)$

The next algorithm shows the main rules to build the tree expansion

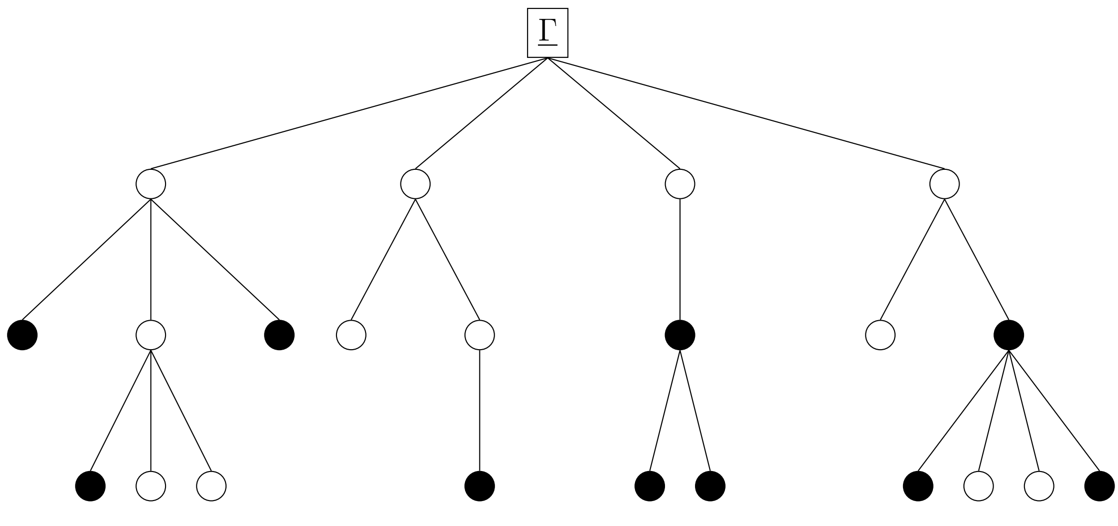
Lev. 0 Represents $\underline{\Gamma}$, is a single node.

Lev. 1 The nodes at the first level are given by the family of maximal contours of $\underline{\Gamma}$. If $\Gamma_{1,1}, ..., \Gamma_{1,n_1}$ are those contours, we define $\mathcal{A}(\Gamma_{1,\mathrm{i}}) = \mathcal{D}(\Gamma_{1,\mathrm{i}})\backslash\{\Gamma_{1,\mathrm{i}}\}$, $1 \leq \mathrm{i} \leq n_1$. At the first level, the nodes are white uniquely.

Lev. 2 At the second level, branches of $\Gamma_{1,\mathrm{i}}$, $\mathrm{i} = 1, ..., n_1$ is the family maximal contours respect to $\mathcal{A}(\Gamma_{1,\mathrm{i}})$. We define $\mathcal{A}_{2,\mathrm{i}} = \mathcal{D}(\Gamma_{2,\mathrm{i}})\backslash\{\Gamma_{2,\mathrm{i}}\}$, $1 \leq \mathrm{i} \leq n_2$, where $\Gamma_{2,\mathrm{i}}$ $1 \leq \mathrm{i} \leq n_2$ is the number of nodes at the second level.

Lev. t Given the nodes $\Gamma_{t-1,1}, ..., \Gamma_{t-1,n_{t-1}}$ of $\Gamma_{t-1,m}$ at level t-1, the branches of $\Gamma_{t-1,m}$ at level t are given by the family of maximal contours respect to $\mathcal{A}(\Gamma_{t-1,m}) = \mathcal{D}(\Gamma_{t-1,m})\backslash\{\Gamma_{t-1,m}\}$, $1 \leq m \leq n_{t-1}$

The algorithm continues while there exist some contour such that the subset of its branches is not empty. Note that the maximum number of levels is the number of contours of $\underline{\Gamma}$, so the procedure stops in a finite time.



We recall the definitions

$$\begin{aligned}
\underline{\Delta}^{(2)}_{\Gamma_0,ex} &= \{y \in \underline{\Delta}^{(2)}_{\Gamma_0,ex} : \mathrm{di}st(y,\Gamma_0) \le c|\Gamma_0|^3\} \\
l_{\Gamma_0}(x) &= \sum_{x\in\Lambda} \mathbb{1}_{\sigma_x(\Gamma_0)=-1} \\
I(l) &= \{x \in \mathbb{Z} : l(x) = l\} \\
\sigma_0 &= \sigma_{x^-(\Gamma_0)}
\end{aligned}$$

Also, we introduce the following subsets of contours

$$\begin{aligned}
\underline{\Gamma}_{in,l} &:= \{\Gamma \in \underline{\Gamma} : supp(\Gamma) \subset I(l)\} \\
\underline{\Gamma}_{ex,2} &:= \{\Gamma \in \underline{\Gamma} : supp(\Gamma) \subset \underline{\Delta}^{(2)}_{\Gamma_0,ex}\}
\end{aligned} \tag{2.395}$$

Note that for $l = 1, ..., 2n-1$, $\underline{\Gamma}_{in,l}$ denotes the set of internal contours of Γ_0 whose support is contained in $[x^*_l, x^*_{l+1}]$. Since those contours are internal, we get

$$\mathrm{di}st(\Gamma,\Gamma_0) > c|\Gamma|^3 \tag{2.396}$$

for all $\Gamma \in \underline{\Gamma}_{in,l}$, $l = 1, ..., 2n-1$. Similarily, for $\Gamma \in \underline{\Gamma}_{ex,2}$, $\mathrm{di}st(\Gamma_0, \underline{\Delta}^{(2)}_{\Gamma_0,ex}) \le c|\Gamma_0|^3$ implies $|\Gamma| < |\Gamma_0|$ and $\mathrm{di}st(\Gamma,\Gamma_0) > c|\Gamma|^3$

Associated to the above subsets we define

$$\begin{aligned}
\mathcal{E}_0^{-\sigma_0}(\underline{\Gamma}_{ex,2}) &= \{\Gamma \in \underline{\Gamma}_{ex,2} : \sigma_{x^-(\Gamma)}(\underline{\Gamma},\Gamma_0) = -\sigma_0\} \\
\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{ex,2}) &= \bigcup_{\Gamma \in \mathcal{E}_0^{-\sigma_0}(\underline{\Gamma}_{ex,2})} \mathcal{D}(\Gamma)
\end{aligned} \tag{2.397}$$

$$\begin{aligned}
\mathcal{E}_0^{-(-1)^l\sigma_0}(\underline{\Gamma}_{in,l}) &= \{\Gamma \in \underline{\Gamma}_{in,l} : \sigma_{x^-(\Gamma)}(\underline{\Gamma},\Gamma_0) = -(-1)^l\sigma_0\} \\
\mathcal{E}^{-(-1)^l\sigma_0}(\underline{\Gamma}_{in,l}) &= \bigcup_{\Gamma \in \mathcal{E}_0^{-(-1)^l\sigma_0}(\underline{\Gamma}_{in,l})} \mathcal{D}(\Gamma)
\end{aligned} \tag{2.398}$$

Let us to make some comments about the subsets defined in (2.398) and (2.397).

- If $\sigma_0 := \sigma_{x^-(\widetilde{\underline{\Gamma}})}(\underline{\Gamma}) = -1$, then $\mathcal{E}_0^{-\sigma_0}(\underline{\Gamma}_{ex,2})$ contains the black contours of $\underline{\Gamma}_{ex,2}$ and $\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{ex,2})$ is the union of $\mathcal{E}^{-\sigma_0}(\widetilde{\underline{\Gamma}})$ and its descendants accordingly to the tree expansion, so $\underline{\Gamma}\backslash\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{ex,2})$ is a set of compatible contours.
- In the same way if $\sigma_0 = +1$, $\mathcal{E}_0^{-\sigma_0}(\underline{\Gamma}_{ex,2})$ contains the black contours of $\underline{\Gamma}_{ex,2}$ and $\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{ex,2})$ is the union of $\mathcal{E}^{-\sigma_0}(\widetilde{\underline{\Gamma}})$ and its descendants.

The very same argument can be used to show that $\underline{\Gamma}\backslash\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{in,l})$ is a set of compatible contours.

Proposition 2.31 *Let $\underline{\Gamma}, \Gamma_0$ be an arbitrary set of compatible contours , and $1 \le l \le 2n-1$ an odd number, the next inequalities hold*

(I) For each fixed $x_0 \in I(l)$

$$\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x_0,y)\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=\sigma_0} \le \sum_{y\in supp(\underline{\Gamma}_{ex,2})} J(x_0,y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0\backslash\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{ex,2}))=\sigma_0} \tag{2.399}$$

$$\le \frac{\pi^2}{6c}\sum_{y\in supp(\underline{\Delta}^{(2)}_{\Gamma_0,ex})} J(x_0,y) \tag{2.400}$$

(II) For each fixed $y_0 \in \underline{\Delta}^{(2)}_{\Gamma_0,ex}$,

$$\sum_{x\in\underline{\Delta}_{\Gamma_0,in,l}} J(x,y_0)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=-\sigma_0} \le \sum_{x\in supp(\underline{\Gamma}_{in,l})} J(x_0,y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0\backslash\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{in,l}))=-\sigma_0} \tag{2.401}$$

$$\le \frac{\pi^2}{6c}\sum_{x\in I(l)} J(x,y_0) \tag{2.402}$$

From equations (2.400), (2.402) we get

$$\sum_{x\in I_{\Gamma_0}(l)}\left(\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y)\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=\sigma_0}\right) \le \frac{\pi^2}{6c}\sum_{x\in I_{\Gamma_0}(l)}\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}} J(x,y) \tag{2.403}$$

$$\sum_{y\in\underline{\Delta}^{(2)}_{\Gamma_0,ex}}\left(\sum_{x\in I_{\Gamma_0}(l)} J(x,y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=-\sigma_0}\right) \le \frac{\pi^2}{6c}\sum_{y\in\underline{\Delta}_{\Gamma_0,in,l}}\sum_{x\in I_{\Gamma_0}(l)} J(x,y) \tag{2.404}$$

Proof: We suppose $\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2})$ is not empty, otherwise it is direct. We claim

$$\{x \in supp(\underline{\Gamma}_{\mathrm{e}x,2}) : \sigma_x(\underline{\Gamma}, \Gamma_0) = \sigma_0\} \subseteq \{x \in supp(\underline{\Gamma}_{\mathrm{e}x,2}) : \sigma_x(\underline{\Gamma}, \Gamma_0 \backslash \mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2}) = \sigma_0\} \tag{2.405}$$

If (2.405) is true, it follows

$$\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=\sigma_0} \leq \mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0\backslash\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2})=\sigma_0} \tag{2.406}$$

and (2.399) is a direct consequence. To prove the claim (2.405), we notice that

$$x \notin \bigcup_{\Gamma\in\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2})} supp(\Gamma) \ \Rightarrow \ \sigma_x(\underline{\Gamma}, \Gamma_0) = \sigma_x(\underline{\Gamma}, \Gamma_0\backslash\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2})) \tag{2.407}$$

$$x \in \bigcup_{\Gamma\in\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2})} supp(\Gamma) \ \Rightarrow \ \sigma_x(\underline{\Gamma}, \Gamma_0\backslash\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2})) = \sigma_0 \tag{2.408}$$

but not necessarily $\sigma_0 = \sigma_x(\underline{\Gamma}, \Gamma_0)$ for $x \in \bigcup_{\Gamma\in\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2})} supp(\Gamma)$. In fact, it is certain for $x = x^-(\Gamma_a)$, $\Gamma_a \subset \mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2})$

$$\begin{aligned} \sigma_x(\underline{\Gamma}, \Gamma_0) &= -\sigma_0 \\ \sigma_x(\underline{\Gamma}, \Gamma_0\backslash\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2})) &= \sigma_0 \end{aligned} \tag{2.409}$$

□

Proof of (2.400). We will use the argument of grouping into contours with the same mass, based on the techniques used in [31] [32]. Let x_0 be a fixed point

$$\sum_{y\in supp(\underline{\Gamma}_{\mathrm{e}x,2})} J(x_0,y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0\backslash\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2}))=\sigma_0} \leq \sum_{\Gamma\in\widehat{\underline{\Gamma}}_{\mathrm{e}x,2}} \sum_{y\in supp(\Gamma)} J(x_0,y)\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0\backslash\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2}))=\sigma_0} \tag{2.410}$$

$$= \sum_{M\geq 1} \sum_{\substack{\Gamma\in\widehat{\underline{\Gamma}}_{\mathrm{e}x,2} \\ |\Gamma|=M}} \sum_{y\in supp(\Gamma)} J(x_0,y)\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0\backslash\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2}))=\sigma_0} \tag{2.411}$$

where

$$\widehat{\underline{\Gamma}}_{\mathrm{e}x,2} = \underline{\Gamma}_{\mathrm{e}x,2}\backslash\mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{\mathrm{e}x,2})$$

Equation (2.411) depends strongly on the fact that those contours whose state at the leftmost point is $-\sigma_0$ were erased, so the interaction can be decomposed into a sum of terms where the contours belonging to $\underline{\Gamma}, \Gamma_0 \backslash \mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{ex,2})$. Now, by fixing M and denoting by $\Gamma_1, ..., \Gamma_{n_M} \subset \underline{\Gamma}_{ex,2}$ the subset of contours with mass is M and by using some of the main properties of contours, we deduce

- There exists at most M points in $supp(\Gamma_i)$, $1 \leq i \leq n_M$ whose state is σ_0.
- Contours with the same mass are mutually external, so we can denote $\Gamma_1 < \Gamma_2 ..., < \Gamma_{n_{1,M}}$ for contours at the right hand of x_0 and $\Gamma_1^{'} > \Gamma_2^{'} ..., > \Gamma_{n_{2,M}}^{'}$ for contours at the left hand of x_0.
- If $y_k \subset supp(\Gamma_k)$ is the nearest point to x_0, then

$$\begin{aligned} dist(y_k, x_0) &\geq [(k-1)M + ckM^3] \\ dist(y_k, y_{k+1}) &> [M + cM^3] \end{aligned}$$

  The same follows for contours at the left hand of x_0. If $y_k^{'} \subset supp(\Gamma_k^{'})$ is the nearest point to x_0, then

$$\begin{aligned} dist(y_k^{'}, x_0) &\geq [(k-1)M + ckM^3] \\ dist(y_k^{'}, y_{k+1}^{'}) &> [M + cM^3] \end{aligned}$$

The interaction is a decreasing function of the distance, so

$$\begin{aligned} \sum_{\substack{\Gamma \in \widehat{\underline{\Gamma}}_{ex,2} \\ |\Gamma| = M}} \sum_{y \in supp(\Gamma)} J(x,y) \mathbb{1}_{\sigma_y(\underline{\Gamma}, \Gamma_0 \backslash \mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{ex,2})) = \sigma_0} &\leq M \left(\sum_{k=1}^{n_{1,M}} J(y_k, x_0) + \sum_{k=1}^{n_{2,M}} J(y_k^{'}, x_0) \right) \\ &\leq \frac{M}{cM^3} \left(\sum_{\substack{y \in \underline{\Delta}^{(2)}_{\Gamma_0, ex} \\ y < x_0}} J(y, x_0) + \sum_{\substack{y \in \underline{\Delta}^{(2)}_{\Gamma_0, ex} \\ y > x_0}} J(y, x_0) \right) \quad (2.412) \\ &= \frac{1}{cM^2} \sum_{y \in \underline{\Delta}^{(2)}_{\Gamma_0, ex}} J(x, y_0) \quad (2.413) \end{aligned}$$

The above inequality implies

$$\sum_{M \geq 1} \sum_{\substack{\Gamma \in \widehat{\underline{\Gamma}}_{ex,2} \\ |\Gamma| = M}} \sum_{x \in supp(\Gamma)} J(x,y) \mathbb{1}_{\sigma_x(\underline{\Gamma}, \Gamma_0 \backslash \mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{ex,2})) = \sigma_0} \leq \sum_{M \geq 1} \frac{1}{cM^2} \sum_{x \in supp(\underline{\Gamma}_{ex,2})} J(x, y_0) \quad (2.414)$$

Finally, since $\sum_{M \geq 1} \frac{1}{M^2} = \frac{\pi^2}{6}$, we conclude the result.

□

Proof of (2.401) The argument used is the very same as (2.401). Nevertheless we will reproduce it again. Suppose $\mathcal{E}^{-(-1)^l\sigma_0}(\underline{\Gamma}_{\text{in},l})$ is not empty, otherwise it is direct. We claim

$$\{x \in supp(\underline{\Gamma}_{\text{in},l}) : \sigma_x(\underline{\Gamma}, \Gamma_0) = -\sigma_0\} \subseteq \{x \in supp(\underline{\Gamma}_{\text{in},l}) : \sigma_x(\underline{\Gamma}, \Gamma_0 \backslash \mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{in},l}) = -\sigma_0\} \tag{2.415}$$

It implies

$$\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=\sigma_0} \leq \mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0\backslash\mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{e}x,2})=\sigma_0} \tag{2.416}$$

and (2.401) is a direct consequence from this. To prove (2.415), we check

$$x \notin \bigcup_{\Gamma\in\mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{in},l})} supp(\Gamma) \quad : \quad \sigma_x(\underline{\Gamma}, \Gamma_0) = \sigma_x(\underline{\Gamma}, \Gamma_0 \backslash \mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{in},l})) \tag{2.417}$$

$$x \in \bigcup_{\Gamma\in\mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{in},l})} supp(\Gamma) \;\Rightarrow\; \sigma_x(\underline{\Gamma}, \Gamma_0 \backslash \mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{in},l})) = -\sigma_0 \tag{2.418}$$

but not necessarily $\sigma_0 = -\sigma_x(\underline{\Gamma}, \Gamma_0)$ for $x \in \bigcup_{\Gamma\in\mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{in},l})} supp(\Gamma)$. In fact, it is certain for $x = x^-(\Gamma_b)$, $\Gamma_b \subset \mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{in},l})$.

$$\begin{aligned} \sigma_x(\underline{\Gamma}, \Gamma_0) &= \sigma_0 \\ \sigma_x(\underline{\Gamma}, \Gamma_0 \backslash \mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{e}x,2})) &= -\sigma_0 \end{aligned} \tag{2.419}$$

□

Proof of (2.402). We use the very same argument as (2.400). Let y_0 be a fixed point

$$\sum_{x\in supp(\underline{\Gamma}_{\text{in},l})} J(x, y_0)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0\backslash\mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{in},l}))=-\sigma_0} \leq \sum_{\Gamma\in\widehat{\underline{\Gamma}}_{\text{in},l}} \sum_{x\in supp(\Gamma)} J(x_0, y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0\backslash\mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{in},l}))=-\sigma_0} \tag{2.420}$$

$$= \sum_{M\geq 1} \sum_{\substack{\Gamma\in\widehat{\underline{\Gamma}}_{\text{in},l} \\ |\Gamma|=M}} \sum_{x\in supp(\Gamma)} J(x_0, y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0\backslash\mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{in},l}))=-\sigma_0} \tag{2.421}$$

where

$$\widehat{\underline{\Gamma}}_{\text{in},l} = \underline{\Gamma}_{\text{in},l} \backslash \mathcal{E}^{\sigma_0}(\underline{\Gamma}_{\text{in},l}) \tag{2.422}$$

Equation (2.421) depends strongly on the fact that those contours whose state at the leftmost point is σ_0 were erased, so the interaction can be decomposed into a sum over contours belonging to $\underline{\Gamma}, \Gamma_0 \backslash \mathcal{E}^{\sigma_0}(\underline{\Gamma}_{ex,2})$. Now, by fixing M and denoting $\Gamma_1, ..., \Gamma_{n_M} \subset \underline{\Gamma}_{ex,2}$ to those contours with mass is M and using some of the main properties of contours, we can deduce

- There exists at most M points in $supp(\Gamma_k)$, $1 \leq \text{i} \leq n_M$ whose state is $-\sigma_0$.
- Contours with the same mass are mutually external, so we can denote $\Gamma_1 < \Gamma_2 ..., < \Gamma_{n_{1,M}}$ for contours at the right hand of y_0 and $\Gamma_1^{'} > \Gamma_2^{'} ..., > \Gamma_{n_{2,M}}^{'}$ for contours at the left hand of y_0.
- If $x_k \subset supp(\Gamma_k)$ is the nearest point to y_0, then

$$\begin{aligned} dist(x_k, x_0) &\geq [(k-1)M + ckM^3] \\ dist(x_k, x_{k+1}) &> [M + cM^3] \end{aligned}$$

  The same follows for those contours at the left hand of y_0. If $x_k^{'} \subset supp(\Gamma_k^{'})$ is the nearest point to y_0, then

$$\begin{aligned} dist(x_k^{'}, y_0) &\geq [(k-1)M + ckM^3] \\ dist(x_k^{'}, x_{k+1}^{'}) &> [M + cM^3] \end{aligned}$$

since the interaction is not increasing as a function on the distance, we get

$$\begin{aligned} \sum_{\substack{\Gamma \in \widehat{\underline{\Gamma}}_{in,l} \\ |\Gamma| = M}} \sum_{x \in supp(\Gamma)} J(x, y_0) \mathbb{1}_{\sigma_x(\underline{\Gamma}, \Gamma_0 \backslash \mathcal{E}^{\sigma_0}(\underline{\Gamma}_{in,l})) = -\sigma_0} &\leq M \left(\sum_{k=1}^{n_{1,M}} J(y_k, x_0) + \sum_{k=1}^{n_{2,M}} J(y_k^{'}, x_0) \right) \\ &\leq \frac{M}{cM^3} \left(\sum_{\substack{x \in I_{\Gamma_0}(l) \\ x < y_0}} J(y_0, x) + \sum_{\substack{x \in I_{\Gamma_0}(l) \\ x > y_0}} J(y_0, x) \right) \\ &= \frac{1}{cM^2} \sum_{x \in I_{\Gamma_0}(l)} J(x, y_0) \end{aligned} \tag{2.423}$$

It implies

$$\sum_{M \geq 1} \sum_{\substack{\Gamma \in \widehat{\underline{\Gamma}}_{in,l} \\ |\Gamma| = M}} \sum_{x \in supp(\Gamma)} J(x, y) \mathbb{1}_{\sigma_x(\underline{\Gamma}, \Gamma_0 \backslash \mathcal{E}^{-\sigma_0}(\underline{\Gamma}_{in,l})) = \sigma_0} \leq \sum_{M \geq 1} \frac{1}{cM^2} \sum_{x \in supp(\underline{\Gamma}_{in,l})} J(x, y_0) \tag{2.424}$$

Finally, since $\sum_{M \geq 1} \frac{1}{M^2} = \frac{\pi^2}{6}$, we conclude the result.

Proposition 2.32 *Let l, m be a pair of numbers such that l is odd and m is even, then*

$$\sum_{x\in I(l)}\sum_{y\in I(m)} J(x,y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=\sigma_0}\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=\sigma_0} \leq \frac{\pi^2}{6c}\sum_{x\in I(l)}\sum_{y\in I(m)} J(x,y) \quad (2.425)$$

$$\sum_{x\in I(l)}\sum_{y\in I(m)} J(x,y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=-\sigma_0}\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=-\sigma_0} \leq \frac{\pi^2}{6c}\sum_{x\in I(l)}\sum_{y\in I(m)} J(x,y)$$

Proof:

We observe for $x \in I(l)$ fixed

$$\sum_{y\in I(m)} J(x,y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=\sigma_0}\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=\sigma_0} \leq \sum_{y\in I(m)} J(x,y)\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=\sigma_0} \quad (2.426)$$

It can be shown for each $x \in I(l)$ that

$$\begin{aligned}\sum_{y\in I(m)} J(x,y)\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=\sigma_0} &= \sum_{y\in supp(\underline{\Gamma}_{\mathrm{in},m})} J(x,y)\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=\sigma_0} \\ &\leq \frac{\pi^2}{6c}\sum_{y\in I(m)} J(x,y) \quad (2.427)\end{aligned}$$

see equation (2.402) from Proposition 2.31. Note that equation (2.427) implies

$$\begin{aligned}\sum_{x\in I(l)}\sum_{y\in I(m)} J(x,y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=\sigma_0}\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=\sigma_0} &\leq \sum_{x\in I(l)}\sum_{y\in I(m)} J(x,y)\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=\sigma_0} \quad (2.428) \\ &\leq \frac{\pi^2}{6c}\sum_{x\in I(l)}\sum_{y\in I(m)} J(x,y) \quad (2.429) \\ & \quad (2.430)\end{aligned}$$

Analougsly, for $y \in I(m)$ fixed

$$\sum_{x\in I(l)} J(x,y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=-\sigma_0}\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=-\sigma_0} \leq \sum_{x\in I(l)} J(x,y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=-\sigma_0} \quad (2.431)$$

From equation (2.402) of Proposition 2.31

$$\begin{aligned}
\sum_{x\in I(l)}\sum_{y\in I(m)} J(x,y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=-\sigma_0}\mathbb{1}_{\sigma_y(\underline{\Gamma},\Gamma_0)=-\sigma_0} &\le \sum_{x\in I(l)}\sum_{y\in I(m)} J(x,y)\mathbb{1}_{\sigma_x(\underline{\Gamma},\Gamma_0)=-\sigma_0} &(2.432)\\
&\le \frac{\pi^2}{6c}\sum_{x\in I(l)}\sum_{y\in I(m)} J(x,y) &(2.433)
\end{aligned}$$

.1 Some general notes on Fractals

.1.1 The Cantor set

The most famous fractal set is the Cantor set on $[0,1]$, so we start by revisiting this classical example. The procedure to build this set by "erasing" the middle third part is summarized in the next algorithm

P.0 $C^{(0)} = [0,1]$.

P.1 $C^{(1)} = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$.

P.2 $C^{(2)} = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{6}{9},\frac{7}{9}] \cup [\frac{8}{9},1]$.

P.3 $C^{(3)} = [0,\frac{1}{27}] \cup [\frac{2}{27},\frac{1}{9}] \cup [\frac{6}{27},\frac{7}{27}] \cup [\frac{8}{27},\frac{9}{27}] \cup [\frac{2}{3},\frac{19}{27}] \cup [\frac{21}{27},\frac{22}{27}] \cup [\frac{24}{27},\frac{25}{27}] \cup [\frac{26}{27},1]$.

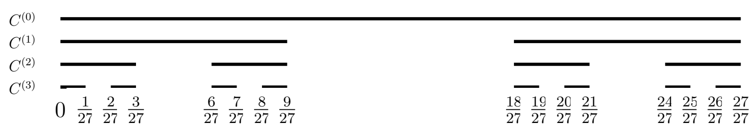
P.N Given a configuration $C^{(n-1)}$ at step $n-1$, we set

$$C^{(n)} \;=\; \frac{C^{(n-1)}}{3} \cup \left\{\frac{2}{3} + \frac{C^{(n-1)}}{3}\right\} \tag{434}$$

Note that for $n < m$, we have $C^m \subset C^n$. The Cantor set can be defined as

$$C = \cap_{n\ge 1} C^{(n)} \tag{435}$$

The above picture shows the procedure for the first three steps



note that C satisfies

$$\frac{1}{3}C \cup \left\{\frac{2}{3} + \frac{1}{3}C\right\} \quad = \quad C \tag{436}$$

and

$$\frac{1}{3}C \cap \left\{\frac{2}{3} + \frac{1}{3}C\right\} \quad = \quad \varnothing \tag{437}$$

Equations (436), (437) plays an important role , which will be formalized next.

.1.2 Self Similarity

Definition .33 *We say that a mapping*

$$\mathcal{S} : \mathbb{R} \longrightarrow \mathbb{R} \tag{438}$$

is a similitude if there exists $0 < r < 1$ *such that*

$$|\mathcal{S}(x) - \mathcal{S}(y)| = r|x - y|, \qquad x, y \in \mathbb{R} \tag{439}$$

where r *is called the contraction ratio.*

Definition .34 *Suppose* $\mathcal{S} = \{\mathcal{S}_1, ..., \mathcal{S}_N\}$, $N \geq 2$ *is a finite sequence of similitudes with contraction ratios* $r_1, ..., r_N$. *We say that a non empty compact set* K *is invariant under* S *if*

$$K = \bigcup_{\mathrm{i}=1}^{N} \mathcal{S}_{\mathrm{i}} K \tag{440}$$

We remark that for any finite collection of similarities $\mathcal{S}$ there exists a unique invariant compact set that satisfies (440) . A formal definition of a self similar set is given next

Definition .35 *We say that* $\mathcal{S} = \{\mathcal{S}_1, ..., \mathcal{S}_N\}$ *satisfies the open condition if there exists an open set* $\mathcal{O}$ *such that*

$$\cup_{\mathrm{i}=1}^{N} \mathcal{S}_{\mathrm{i}}(O) \subseteq \mathcal{O} \qquad \mathcal{S}_{\mathrm{i}}(\mathcal{O}) \cap \mathcal{S}_j(\mathcal{O}) = \varnothing \quad , \mathrm{i} \neq j \tag{441}$$

From equation (436) we know that the Cantor can be written as

$$C = \mathcal{S}_1(C) \cup \mathcal{S}_2(C) \tag{442}$$

where

$$\begin{aligned} \mathcal{S}_1(x) &= \frac{1}{3}C \\ \mathcal{S}_2(x) &= \frac{2}{3} + \frac{1}{3}C \end{aligned}$$

we note that C satisfies definition (.33) with $r_1 = r_2 = 1/3$. We will see in the next lemma that , the Cantor set also satisfies the Open Condition set defined in (.35).

Lemma .36 *Let be the open set*

$$B(C,\varepsilon) = \{x \in \mathbb{R} : \mathrm{d}(x,C) < \varepsilon\} \tag{443}$$

Then, for $0 < \varepsilon < 1/12$ we have

(i)

$$B(C,\varepsilon) \subset \frac{B(C,\varepsilon)}{3} \cup \left\{\frac{B(C,\varepsilon)}{3} + \frac{2}{3}\right\} \tag{444}$$

(ii)

$$\frac{B(C,\varepsilon)}{3} \cap \left\{\frac{B(C,\varepsilon)}{3} + \frac{2}{3}\right\} = \varnothing \tag{445}$$

in particular, the Cantor set satisfies the Open condition

Proof:

We have that $x \in B(C,\varepsilon)$ if and only if there exists $y \in K$ such that $|x-y| < \varepsilon$. On the other hand, from equation (443) we get for $\mathrm{i} = 1,2$

$$|\mathcal{S}_\mathrm{i}(x) - \mathcal{S}_\mathrm{i}(y)| = \frac{|x-y|}{3} < \frac{\varepsilon}{3} < \varepsilon \tag{446}$$

since $\mathcal{S}(y) \in C$, we deduce that $S(x) \in B(C,\varepsilon)$ for all $x \in B(C,\varepsilon)$, and by consequence

$$\mathcal{S}_\mathrm{i}(B(C,\varepsilon)) \subset B(C,\varepsilon) \tag{447}$$

for $\mathrm{i} = 1,2$. On the other hand, for $\varepsilon < 1/12$, we get

$$\mathrm{dis}t(\mathcal{S}_1(B(C,\varepsilon)),\mathcal{S}_2(B(C,\varepsilon))) > |\frac{1}{3} - 2\varepsilon| > \frac{1}{6} \tag{448}$$

then $\mathcal{S}_1(B(C,\varepsilon)) \cap \mathcal{S}_2(B(C,\varepsilon)) = \varnothing$. We conclude that C satisfies the open condition.

.1.3 Hausdorff Dimension

Let X be a separable metric space and $0 \leq s < \infty$. For any $E \in X$, we denote de s-dimensional Hausdorff measure by

$$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) \tag{449}$$

where

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum \mathrm{d}(E_\mathrm{i})^s : A \subset \cup_{\mathrm{i}\geq 1} E_\mathrm{i},\ E_\mathrm{i} \text{ measurable and } \mathrm{d}(E_\mathrm{i}) \leq \delta\right\} \tag{450}$$

where $\mathrm{d}(E) = \max\{|x-y| : x, y \in E\}$ is the diameter of E, with the convention $0^0 = 1$ and $\mathrm{d}(\varnothing) = 0$. The Hausdorff measure have nice properties under translation and dilation transformations on $\mathbb{R}$. For $A \in \mathbb{R}$, $y \in \mathbb{R}$, $0 < t < \infty$

$$\begin{aligned} \mathcal{H}^s(A+y) &= \mathcal{H}^s(A) \\ \mathcal{H}^s(tA+y) &= t^s\mathcal{H}^s(A) \end{aligned} \tag{451}$$

Definition .37 *The Hausdorff dimension on a set $A \in \mathbb{R}$ is*

$$\begin{aligned} \dim_H(A) &= \sup\{s : \mathcal{H}_s(A) > 0\} \\ &= \sup\{s : \mathcal{H}_s(A) = \infty\} \end{aligned} \tag{452}$$

or equivalently

$$\begin{aligned} \dim_H(A) &= \inf\{s : \mathcal{H}_s(A) < \infty\} \\ &= \inf\{s : \mathcal{H}_s(A) = 0\} \end{aligned} \tag{453}$$

The Hausdorff dimension has some desirable properties

$$\begin{aligned} \dim_H(A) &\leq \dim_H(B) \quad if\ A \subseteq B \\ \dim_H(\cup_{i\geq 1} A_i) &= \sup_i \dim_H(A_i) \\ \dim_H(tA+y) &= \dim_H(A) \end{aligned} \tag{454}$$

The Hausdorff dimension is the unique number such that

$$\begin{aligned} s &< \dim_H(A) \Rightarrow \mathcal{H}^s(A) = \infty \\ s &> \dim_H(A) \Rightarrow \mathcal{H}^s(A) = 0 \end{aligned}$$

At the border case $t = \dim_H(A)$ there is not exist a general non trivial behavior, and the three cases $\mathcal{H}^s(A) = \infty$, $0 < \mathcal{H}^s(A) < \infty$, $\mathcal{H}^s(A) = 0$ are possible.
In general, a computation of the Hausdorff dimension could be very hard, so it is desirable to get a simple criteria to the most important cases. We state next a simple criteria to compute the Hausdorff dimension, which applies for a general subset of fractal sets

Theorem .38 *Let $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_N\}$ a similitude with contraction ratios $r_1, ..., r_N$. and K the unique compact set that satisfies $S(K) = \cup_{i=1}^N \mathcal{S}_i(K)$. Then the Hausdorff dimension of K is given for the unique real solution to the equation*

$$\sum_{i=1}^{N} r_i^s = 1 \tag{455}$$

Proof: See Theorem 4.14 from [36]

Corollary .39 *The Hausdorff Dimension of the Cantor set is $s = \frac{\log 2}{\log 3}$*

Proof: From the identity $C = \frac{1}{3}C \cup \left\{\frac{2}{3} + \frac{1}{3}C\right\}$, we have that the contraction ratios are $r_1 = r_2 = 1/3$, so we get directly from theorem (.38) that the Hausdorff dimension is given by the solution to the equation

$$\frac{1}{3^s} + \frac{1}{3^s} = 1 \Leftrightarrow 3^s = 2 \Leftrightarrow s = \frac{\log 2}{\log 3} \tag{456}$$

.1.4 A generalized Cantor Set

We will build in the following a more general case, accordingly to the next procedure. Let $\lambda \in (0, 1)$, we set recursively

$$
\begin{aligned}
C_\lambda^{(0)} &= [0,1] \\
C_\lambda^{(n)} &= \lambda C_\lambda^{(n-1)} \cup \left\{(1-\lambda) + \lambda C^{(n-1)}\right\} \;,\; n \geq 1.
\end{aligned} \tag{457}
$$

Note that for $n < m$, we have $C_\lambda^m \subset C_\lambda^n$. The generalized Cantor set is defined by

$$
C_\lambda = \cap_{n\geq 1} C_\lambda^{(n)} \tag{458}
$$

We remark that for $\lambda = 1/3$, the Classical Cantor Set is recovered. This generalized set satisfies the identity

$$
C_\lambda = \lambda C_\lambda \cup \{(1-\lambda) + \lambda C_\lambda\}
$$

In addition, for ε close to zero

$$
\begin{aligned}
B(C_\lambda, \varepsilon) &\subset \lambda B(C_\lambda, \varepsilon) \cup \{(1-\lambda) + \lambda B(C_\lambda, \varepsilon)\} \\
\varnothing &= \lambda B(C_\lambda, \varepsilon) \cap \{(1-\lambda) + \lambda B(C_\lambda, \varepsilon)\}
\end{aligned} \tag{459}
$$

where $B(C_\lambda, \varepsilon) = \{x \in \mathbb{R} : \mathrm{d}(x, C) < \varepsilon\}$ (the argument is the very same as the used in Lemma (.36)). We deduce that the compact set C_λ is self similar and satisfies the open condition, so the Hausdorff dimension is given by the solution to the equation

$$
\lambda^s + \lambda^s = 1 \Leftrightarrow s = -\frac{\log 2}{\log \lambda} \tag{460}
$$

Bibliography

[1] Dobrushin, R. L.: Description of a random field by means of conditional probabilities and regularity conditions. Probability theory and applications (in Russian) 13 (2) 201–222 (1968).

[2] Lanford,O.E., Ruelle,D.: Observables at infinity and states with short-range correlations in statistical mechanics, Commun. math. Phys., 13, (3) 194–215 (1968).

[3] Dobrushin, R. L.: Problem of uniqueness of a Gibbs random field and phase transitions. Functional Analysis and applications (in Russian) 2 (4) 44–57 (1968).

[4] Ruelle,D.: Statistical mechanics of one-dimensional lattice gas, Commun. math. Phys., 9, (4) 267—278 (1968).

[5] Freeman J. Dyson Existence of a Phase-Transition in a One-Dimensional Ising Ferromagnet Commun. math. Phys. 12, 91–107 (1969)

[6] Mark Kac and Colin J. Thompson. Critical Behavior of Several Lattice Models with Long Range Interaction J. Math. Phys. 10 , 1373 (1969)

[7] Freeman J. Dyson. An Ising Ferromagnet with Discontinuous Long-Range Order (Comm. Math. Phys 21, 269–283 (1971)

[8] P. M. Bleher, Ja. G. Sinai. Investigation of the critical point in models of the type of Dyson's hierarchical models. Comm. Math. Phys. 33 (1973), no. 1, 23–42.

[9] P. M. Bleher, Ya. G. Sinai. Critical indices for Dyson's asymptotically-hierarchical models. Comm. Math. Phys. 45 (1975), no. 3, 247–278.

[10] T. C. Dorlas. A renormalization group analysis of the hierarchical model in statistical mechanics. Lecture Notes in Physics, Vol. 74. Springer-Verlag, Berlin-New York, 1978.

[11] R. L. Dobrushin Analyticity of Correlation Functions in One-Dimensional Classical Systems with Slowly Decreasing Potentials Commun. math. Phys.32, 269–289 (1973)

[12] Khanin, K.M., Sinai, Ya.G.: Existence of free energy for models with long range random Hamiltonians. J. Stat. Phys. 20, 573-584 (1979).

[13] Khanin, K.M.: Absence of phase transitions in one-dimensional long-range spin systems with random Hamiltonian. Theor. Math. Phys. 43, 445-449 (1980).

[14] M. Cassandro and E. Olivieri Renormalization Group and Analyticity in one Dimension: A Proof of Dobrushin's Theorem Commun. Math. Phys. 80, 255-269 (1981).

[15] Cassandro, M., Olivieri, E., Tirozzi, B.: Infinite differentiability for one-dimensional spin systems with long range random interaction. Commun. Math. Phys. 87, 229-252 (1982).

[16] Campanino, M., Capocaccia, D., Olivieri, E.: Analyticity for one-dimensional systems with long range superstable interactions. J. Stat. Phys. 33, 437-476 (1983)

[17] M. Campanino, E. Olivieri , and A. C. D. van Enter One Dimensional Spin Glasses with Potential Decay $1/r^{1+\varepsilon}$. Absence of Phase Transitions and Cluster Properties Commun. Math. Phys. 108, 241-255 (1987)

[18] M. Campanino, E. Olivieri One-dimensional random Ising systems with interaction decay $r^{-(1+\varepsilon)}$: a convergent cluster expansion Comm. Math. Phys. Volume 111, Number 4 (1987), 555-577.

[19] Fröhlich, J., and Spencer, T., The phase transition in the one-dimensional Ising model with $1/r^2$ interaction energy,î Commun. Math. Phys. 84, 87–101 1982

[20] Imbrie, J. Z., Decay of correlations in one dimensional Ising model with $Jij = |\mathrm{i} - j|^{-2}$,î Commun. Math. Phys. 85, 491–515 (1982).

[21] Aizenman, M., and Newman, C. M., Discontinuity of the percolation density in one-dimensional $1/|x-y|^2$ percolation models,î Commun. Math. Phys. 107, 611–647 (1986).

[22] Aizenman, M., Chayes, J. T., Chayes, L., and Newman, C. M., Discontinuity of the magnetization in one-dimensional $1/|x-y|^2$ percolation, Ising and Potts models,î J. Stat. Phys. 50, 1–40 (1988).

[23] Imbrie, J. Z., and Newman, C. M., An intermediate phase with slow decay of correlations in one-dimensional $1/|x-y|^2$ percolation, Ising and Potts models,î Commun. Math. Phys. 118, 303–336 (1988).

[24] Jeffrey B. Rogers & Colin J. Thompson, Absence of long range order in one-dimensional spin sysstems, J. Stat. Phys, 25 669–678 (1981)

[25] J. Fröhlich and B. Zegarlinski : The High-Temperature Phase of Long-Range Spin Glasses., Comm. Math. Phys 110, 121–155 (1987)). Weak uniqueness means that allowed boundary conditions are the one that are independent of the random coupling.

[26] S. E. Burkov and Ya. G. Sinai, Phase diagrams of one-dimensional lattice models with long-range antiferromagnetic interaction. Uspekhi. Mat. Nauk 38:205–225 (1983).

[27] Kerimov, Azer Absence of phase transitions in one-dimensional antiferromagnetic models with long-range interactions. J. Statist. Phys. 72 (1993), no. 3-4, 571–620.

[28] Kerimov, Azer Uniqueness of Gibbs states in one-dimensional antiferromagnetic model with long-range interaction. J. Math. Phys. 40 (1999), no. 10, 4956–4974).

[29] Gandolfi, A.; Newman, C. M.; Stein, D. L. Exotic states in long-range spin glasses. Comm. Math. Phys. 157 (1993), no. 2, 371?387.

[30] Cassandro, Orlandi & Picco Typical Gibbs Configurations for the 1d Random Field Ising Model with Long Range Interaction. Comm. Math. Phys. Volume 309, Issue 1, pp 229-253

[31] Marzio Cassandro , Pablo Augusto Ferrari , Immacolata Merola and Errico Presutti (2005) *Geometry of contours and Peierls estimates in d=1 Ising models with long range interactions.* Journal of Mathematical Physics 46-5.

[32] Marzio Cassandro, Immacolata Merola, Pierre Picco, Utkir Rozikov (2013). *One Dimensional Ising Models With Long Range Interactions: Cluster expansion, Phase-separating point* (Work in Progress).

[33] F.J. Dyson (1969). Existence of a phase transition in a one dimensional Ising ferrmagnet Commun. Math. Phys. 12 , 91-107.

[34] F.J. Dyson (1971). *Non-Existence of Spontaneous Magnetization in a One-Dimensional Ising Ferromagnet Comm. Math. Phys.,12, 212-215.*

[35] M T Barlow and S J Taylor (1989)*J. Phys. A: Math. Gen 22, 2621-2626.*

[36] Michel L. Lapidus, Michel van FRANKENHUISEN Geometry Fractal, Complex Dimension and Zeta Functions, 2006.

[37] David Doty, Xiaoyang Gu, Jack H. Lutz ,Elvira Mayordomo ,Philippe Moser (2005)*Zeta-Dimension* http://arxiv.org/abs/cs/0503052.

[38] Jurg Frohlich and Thomas Spencer (1982) *The Phase Transition in the One-Dimensional Ising Model with* $1/r^2$ *Interaction Energy* Commun Math Phys. 84, 87-101.