UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

MULTIPLICITY PHENOMENON AND MORSE INDEX OF SOLUTIONS FOR SOME ELLIPTIC EQUATIONS

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

WENJING CHEN

PROFESOR GUÍA:
JUAN DIEGO DÁVILA BONCZOS

MIEMBROS DE LA COMISIÓN:
IGNACIO GUERRA BENAVENTE
FETHI MAHMOUDI
SALOMÉ MARTÍNEZ SALAZAR

SANTIAGO DE CHILE
DICIEMBRE 2013

Resumen

Esta tesis contiene cinco capítulos. En el primer capítulo, presentamos algunas motivaciones de los problemas que consideramos en los siguientes cuatro capítulos. En particular, describimos algunos resultados conocidos para el problema Gelfand, ecuación y sistema de Lane-Emden, y el problema clásico de Brézis y Nirenberg, y enunciamos los principales resultados de esta tesis.

En el Capítulo 2, estamos interesados en la estructura de las soluciones al problema de tipo Gelfand

$$\begin{cases} -\Delta u = \lambda(e^u - 1), \quad u > 0 & \text{en } B; \\ u = 0 & \text{en } \partial B, \end{cases}$$

donde B es la bola de radio 1 en $\mathbb{R}^n$, $N \geq 3$ y $\lambda > 0$ es un parámetro. Establecemos multiplicidad infinita de soluciones regulares para $3 \leq N \leq 9$ y un valor particular de λ, y obtenemos una cota para el índice de Morse y el número de soluciones cuando $N \geq 10$.

El Capítulo 3 está dedicado a estudiar soluciones positivas radialmente simétricas estables del sistema de Lane-Emden

$$\begin{cases} -\Delta u = v^p, \ u > 0 & \text{en } \ \mathbb{R}^N, \\ -\Delta v = u^q, \ v > 0 & \text{en } \ \mathbb{R}^N, \end{cases}$$

donde $N \geq 1$ y $p \geq q \geq 1$. Se obtiene una nueva curva crítica que describe de manera óptima la existencia de este tipo de soluciones.

En el Capítulo 4 analizamos la multiplicidad de soluciones para el siguiente problema

$$\begin{cases} -\Delta u = u^p + \lambda u^q, \quad u > 0 & \text{en } \Omega; \\ u = 0 & \text{en } \partial\Omega, \end{cases}$$

donde Ω es un dominio suave y acotado en $\mathbb{R}^3$, $\lambda > 0$, $p = 5 - \varepsilon$, $\varepsilon > 0$ y $1 < q < 3$. En particular, demostrar que si $2 < q < 3$, para $\lambda > 0$ suficientemente grande, $\varepsilon > 0$ pequeño, el problema tiene al menos tres soluciones.

En el último capítulo, utilizando el procedimiento de reducción de Lyapunov-Schmidt, construimos soluciones tipo *torre de burbuja* de la ecuación elíptica ligeramente supercrítica

$$\begin{cases} -\Delta u + u = u^p + \lambda u^q, \quad u > 0 & \text{en } \mathbb{R}^N; \\ u(z) \to 0 \ \text{ cuando } \ |z| \to \infty, \end{cases}$$

donde $p = p^* + \varepsilon$, con $p^* = \frac{N+2}{N-2}$, $1 < q < \frac{N+2}{N-2}$ si $N \geq 4$, $3 < q < 5$ si $N = 3$, $\lambda > 0$ y ε es un parámetro positivo.

Summary

This thesis contains five chapters. In the first chapter, we introduce some motivations for the problems which we consider in the following four chapters. In particular, we mention some known results for the Gelfand problem, Lane-Emden equation and system, the classical Brézis and Nirenberg problem and so on. We also state the main results in this thesis.

In Chapter 2, we are interested in the structure of solutions to the Gelfand-type problem

$$\begin{cases} -\Delta u = \lambda(e^u - 1), \quad u > 0 & \text{in } B; \\ u = 0 & \text{on } \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 3$ and $\lambda > 0$ is a parameter. We establish infinite multiplicity of regular solutions for $3 \leq N \leq 9$ and some λ, and we obtain a bound for the Morse index and the number of solutions when $N \geq 10$.

Chapter 3 is devoted to study stable positive radially symmetric solutions of the Lane-Emden system

$$\begin{cases} -\Delta u = v^p, \ u > 0 & \text{in } \mathbb{R}^N, \\ -\Delta v = u^q, \ v > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $N \geq 1$ and $p \geq q \geq 1$. We obtain a new critical curve that optimally describes existence of such solutions.

In Chapter 4, we are concerned with multiplicity of solutions to the following Dirichlet problem

$$\begin{cases} -\Delta u = u^p + \lambda u^q, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^3$, $\lambda > 0$, $p = 5 - \varepsilon$, $\varepsilon > 0$ and $1 < q < 3$. In particular, we prove that if $2 < q < 3$, for $\lambda > 0$ sufficiently large, $\varepsilon > 0$ small enough, then the problem has at least three solutions.

In the last chapter, using Lyapunov-Schmidt reduction procedure, we construct bubble-tower solutions to slightly supercritical elliptic equation

$$\begin{cases} -\Delta u + u = u^p + \lambda u^q, \quad u > 0 & \text{in } \mathbb{R}^N; \\ u(z) \to 0 \ \text{ as } \ |z| \to \infty, \end{cases}$$

where $p = p^* + \varepsilon$, with $p^* = \frac{N+2}{N-2}$, $1 < q < \frac{N+2}{N-2}$ if $N \geq 4$, $3 < q < 5$ if $N = 3$, $\lambda > 0$, and ε is a positive parameter.

Acknowledgements

I would like to thank my supervisor Professor Juan Dávila, for proposing me the study of these problems in my thesis. I am also deeply grateful for his patient, guidance, continuous encouragement and huge support during my Ph.D study. His knowledge and attitude to mathematics research work are very worth studying for me. Moreover, thank him for giving me many important and useful suggestions during writing my thesis and spending a lot of time in reading this thesis.

Let me thank Prof. Louis Dupaigne (Université de Picardie Jules Verne) and Prof. Marius Ghergu (University College Dublin) for discussing mathematical problem and answering my questions patiently and quickly which I asked. I also thank Profs. Olivier Goubet and Louis Dupaigne for their hospitality during my visit to LAMFA, at the Université de Picardie Jules Verne, France. It is my pleasure to work with them. Thank Prof. Ignacio Guerra (Universidad de Santiago de Chile) for his efforts and contributions to our joint work. Of course, I want to thank Prof. Manuel Del Pino for his interesting seminar, which is very helpful for some topics in my work. Thank Prof. Fethi Mahmoudi for his help and explaining some problems much slower and more times in his classes so that I can understand well. Besides, many thanks to Profs. Ignacio Guerra, Fethi Mahmoudi and Salomé Martínez as committees for my defense of thesis. I also wish to thank Prof. Patricio Felmer and the other professors in partial differential equation group for providing some interesting classes and seminars.

I am thankful for the Department of Mathematical Engineering, University of Chile to create so nice and quiet atmosphere for my Ph.D study. It is very necessary to thank all the staff in CMM and DIM. Especially, I thank postgraduate coordinator Joaquin Fontbona, secretary Silvia Mariano, Oscar Mori, Luis Mella, Eugenio Guerra and Carlos Tapia for providing me a lot of help and giving me much convenience and support for my graduate daily work.

I owe my sincere gratitude to my master advisor Prof. Jianfu Yang for his constant encouragement and good suggestions in my life and study these years.

I also owe my thanks to all friends and my classmates. In particular, I thank Yong Liu, Qiuping Lu, Jinggang Tan, Wei Yao, Natalia, Viviana, Erica, Luis López, Juan Carlos, Clara, Alejandra, Erwin, Miguel, Duver, Paul and César, for their kind help and sharing good time with me.

I express my gratitude to my beloved family. Thank my parents Hongfu Chen and Meixian Yu, brothers and sisters to give me so much love, support and encouragement. Thank my lovely nephews to give me so much funny when I feel boring. Thank my boyfriend Shengbing Deng to help me, show his support and give me positive energy during my hard time.

Finally, I thank the doctoral grant of CONICYT Chile for financial support during my Ph.D study.

Contents

Chapter 1

Introduction

In this thesis, we investigate multiplicity phenomenon and Morse index of solutions for some elliptic equations and a Liouville-type theorem for stable radial solutions of the Lane-Emden system. In Chapter 2 , we are interested in the structure of solutions to a Gelfand-type problem, we establish multiplicity of solutions and analyse the Morse index of solutions. In the third chapter, we obtain a new critical curve that optimally describes existence of radially symmetric stable solutions for the Lane-Emden system in $\mathbb{R}^N$. Using Lyapunov-Schmidt method, we get multiplicity of solutions to elliptic equations with mixed Sobolev growth in the last two chapters. In this chapter, we introduce briefly these problems.

1.1 A Gelfand-type problem

Consider the following elliptic boundary value problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\lambda > 0$ is a parameter and the nonlinearity $f : [0, +\infty) \to \mathbb{R}$ is a C^1, increasing, convex function satisfying

$$f(0) > 0, \tag{1.2}$$

and f is superlinear as $s \to \infty$ in the following sense

$$\lim_{s\to\infty} \frac{f(s)}{s} = \infty. \tag{1.3}$$

Typical examples are $f(u) = e^u$ and $f(u) = (1 + u)^p$ with $p > 1$.

Existence, uniqueness and multiplicity of positive solutions to problem (1.1) in terms of the parameter λ and the domain Ω have brought a lot of attention in the past decades, see for example [13, 15, 34, 51] et al. and references therein.

We note that 0 is a subsolution to problem (1.1) for every $\lambda > 0$. On the other hand, for $\lambda > 0$ small, let ζ solve

$$\begin{cases} -\Delta\zeta = 1 & \text{in } \Omega; \\ \zeta = 0 & \text{on } \partial\Omega, \end{cases}$$

then

$$-\triangle\zeta \geq \lambda f(\zeta)$$

provided $\lambda \leq \inf\limits_{x\in\Omega} \frac{1}{f(\zeta(x))}$. So $\zeta \geq 0$ is a supersolution. Then by the method of sub and supersolutions, we obtain there exists a solution to problem (1.1) for $\lambda > 0$ small.

Moreover, there is no classical solution if $\lambda > 0$ is large. In fact, assume ϕ_1 is the first eigenfunction of $-\triangle$ with Dirichlet boundary condition, i.e. $\phi_1 > 0$ satisfies

$$\begin{cases} -\Delta\phi_1 = \mu_1\phi_1 & \text{in } \Omega; \\ \phi_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ_1 is the first eigenvalue of $-\triangle$. Multiplying (1.1) by ϕ_1 and integrating by parts over Ω, we get

$$\mu_1 \int_\Omega u\phi_1 = \lambda \int_\Omega f(u)\phi_1.$$

By the hypotheses on f, there exists $c > 0$ such that $f(u) \geq cu$ for all $u \geq 0$. Then

$$\mu_1 \int_\Omega u\phi_1 \geq c\lambda \int_\Omega u\phi_1.$$

Thus

$$\lambda \leq \frac{\mu_1}{c}.$$

Define

$$\lambda^* = \sup\{\lambda > 0 : \text{such that (1.1) has a classical solution}\},$$

thus $\lambda^* \in (0, +\infty)$. We recall the following properties for problem (1.1), we refer to see [13, 15, 28, 31, 66, 75].

Proposition 1.1. *Assume $N \geq 1$, then there exists $0 < \lambda^*(N, \Omega, f) < +\infty$ such that*

- *for $0 < \lambda < \lambda^*$, (1.1) has the minimal solution $u_\lambda \in C^2(\bar{\Omega})$;*
- *for $\lambda > \lambda^*$, (1.1) has no solution (even in the weak sense).*

Remark 1.2. *The minimal solution u_λ is in the sense that for any solution u of (1.1), we have $u_\lambda \leq u$.*

In addition, for each $x \in \Omega$, the mapping $\lambda \mapsto u_\lambda$ is increasing in $(0, \lambda^*)$, this allows one define $u^* := \lim_{\lambda\to\lambda^*} u_\lambda$. We call u^* the extremal solution of (1.1) and λ^* the extremal parameter. Furthermore, H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa [13] proved that

Proposition 1.3. *[13] $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is a weak solution of (1.1) for $\lambda = \lambda^*$ in the following sense.*

Definition 1.4. *A weak solution of problem (1.1) is a function $u \in L^1(\Omega)$, $u \geq 0$, such that*

$$f(u)\delta(x) \in L^1(\Omega),$$

where $\delta(x)$ is the distance function with respect to the boundary,

$$\delta(x) = dist(x, \partial\Omega),$$

and

$$-\int_\Omega u\triangle\varphi dx = \lambda\int_\Omega f(u)\varphi dx = 0 \quad \textit{for } \forall\ \varphi \in C_0^2(\bar{\Omega}).$$

It is natural to ask what happens to the solution when $\lambda = \lambda^*$. Before considering this question, we give another characterization of the minimal solution u_λ, i.e. its stability.

Definition 1.5. *Let $f \in C^1(\mathbb{R})$ and $u \in C^2(\Omega)$ be a solution to (1.1),*

(i) We say that u is stable if

$$Q_u(\varphi) := \int_\Omega \left(|\nabla\varphi|^2 - \lambda f'(u)\varphi^2\right) dx \geq 0 \quad \textit{for } \forall\ \varphi \in C_0^\infty(\Omega).$$

(ii) We say that u has Morse index K if $K \geq 1$ is the maximal dimension of a subspace X_K of $C_0^\infty(\Omega)$ such that $Q_u(\varphi) < 0$ for any $\varphi \in X_K\backslash\{0\}$. We write $K = m(u)$.

Remark 1.6. *If u is stable, we write $m(u) = 0$.*

Many authors are interested in the regularity of the extremal solution u^*, which maybe bounded or singular, depending on the situation. The most well-known cases are exponential and power-type nonlinearities, see for instance [15, 31, 73, 84].

• For $f(u) = e^u$, if $N \leq 9$, then the extremal solution $u^* \in L^\infty(\Omega)$. If $N \geq 10$ and $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^N$, the extremal solution $u^*(x) = -2\log|x|$ is singular.

• For $f(u) = (1+u)^p$ with $p > 1$, if $N < 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$, then u^* is smooth, and when $N \geq 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$, $\Omega = B_1(0)$, $u^*(x) = |x|^{-\frac{2}{p-1}} - 1$ is the extremal solution, which is unbounded.

Let us recall some related results for the exponential and power-type nonlinearities in (1.1). We first study the following classical Gelfand problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with the boundary $\partial\Omega$, and $\lambda > 0$ is a real parameter. When $\Omega = B_1(0)$ is a unit ball in $\mathbb{R}^N$, by the classical result of Gidas-Ni-Nirenberg [67], all smooth solutions of (1.4) are radially symmetric. For $N = 1$, this problem was first considered by Liouville [79] and the author found an explicit solution in 1853. For $N = 2$, Bratu [12] also found an explicit solution to (1.4) in 1914. When $N = 3$, numerical progress for (1.4) was made by Frank-Kamenetshii [62] in his development of thermal explosion theory. Further progress for $N = 3$ was made by Chandrasekhar [23]. Building upon Frank-Kamenetshii's work, in dimension 3, Gelfand [66] used the Emden's transformation to prove the existence of λ for which (1.4) has infinitely many nontrivial solutions. Joseph and Lundgren [73] completely characterized the solution structure of (1.4) for all dimensions via phase plane analysis in 1973. We also refer to see the survey of J. Dávila [34] and the book of L. Dupaigne [51].

Proposition 1.7. *[73] Let Ω be a unit ball in $\mathbb{R}^N$, $N \geq 1$. Then*

(a) If $N = 1, 2$, then there exists $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$, there are exactly two solutions to (1.4), one of them is the minimal solution u_λ. The other one, denote U_λ, has Morse index 1.

(b) If $3 \leq N \leq 9$, then $\lambda^* > 2(N-2)$. For $0 < \lambda < \lambda^*$, $\lambda \neq 2(N-2)$, (1.4) has finitely many solutions; for $\lambda = 2(N-2)$, (1.4) has infinitely many solutions; for λ close to $2(N-2)$, (1.4) has a large number of solutions that converge to $-2\log|x|$.

(c) If $N \geq 10$, then $\lambda^* = 2(N-2)$ and $u^*(x) = -2\log|x|$. Moreover (1.4) has a unique minimal solution u_λ for each $\lambda \in (0, \lambda^*)$.

We summarize these results in Figure 1, which plot the maximum of u against the parameter λ.

Remark 1.8. *Thanks to the following Hardy's inequality, the function $u^*(x) = -2\log|x|$ is a stable weak solution to (1.4) for $\lambda = \lambda^* = 2(N-2)$ if and only if $N \geq 10$.*

Proposition 1.9. *(Hardy's inequality) Let $N \geq 3$. Then for all $\varphi \in C_c^1(\mathbb{R}^N)$,*

$$\frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}dx \leq \int_{\mathbb{R}^N}|\nabla\varphi|^2 dx. \tag{1.5}$$

When nonlinearity $f(u)$ is power-type in (1.1), the problem becomes

$$\begin{cases} -\Delta u = \lambda(1+u)^p, \qquad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with $p > 1$. When the domain Ω is a unit ball in $\mathbb{R}^N$ $(N \geq 3)$, Joseph and Lundgren's results[73] also apply to (1.6). In order to state these results, we introduce the following notations. Denote the critical Sobolev exponent by

$$p_S = \begin{cases} +\infty & \text{if } N \leq 2; \\ \frac{N+2}{N-2} & \text{if } N \geq 3, \end{cases} \tag{1.7}$$

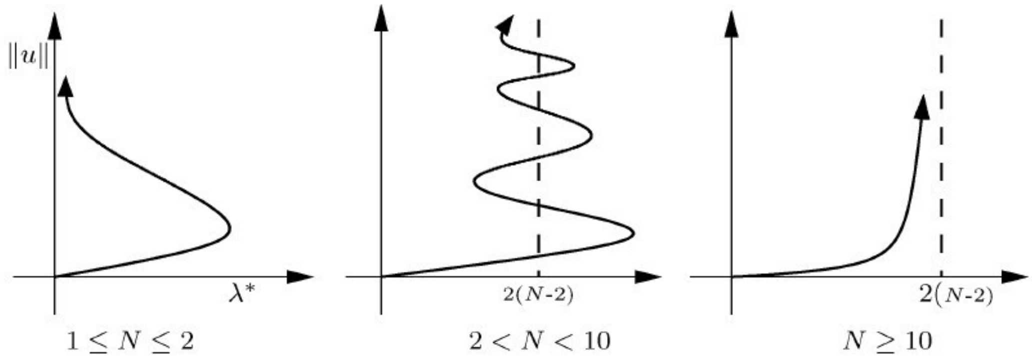

Figure 1: Bifurcation diagrams for positive radial solutions of the Gelfand problem.

we shall refer to the cases $p < p_S$, $p = p_S$, or $p > p_S$ as to Sobolev subcritical, critical, or supercritical respectively.

Define

$$p_{JL} = \begin{cases} \infty & \text{if } 2 \leq N \leq 10; \\ \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N \geq 11, \end{cases} \tag{1.8}$$

which is called Joseph-Lundgren exponent introduced in [73]. Note that the exponent p_{JL} is larger than the classical Sobolev critical exponent p_S.

Proposition 1.10. *[73] Let Ω be a unit ball in $\mathbb{R}^N$, $N \geq 3$, $p > 1$. Then*

(a) If $1 < p \leq p_S$, then there exists $\lambda^* > 0$ such that there are exactly two solutions to (1.6) for any $0 < \lambda < \lambda^*$, while for $\lambda = \lambda^*$ there is a unique solution, which is classical.

(b) If $p_S < p < p_{JL}$, then u^* is bounded and $\lambda^* > \lambda_p$, where $\lambda_p = \frac{2}{p-1}(N - \frac{2p}{p-1})$. For $\lambda = \lambda_p$ there are infinitely many solutions; for λ close to λ_p, there are a large number of solutions.

(c) If $p \geq p_{JL}$, then $\lambda^* = \lambda_p$ and $u^*(x) = |x|^{-\frac{2}{p-1}} - 1$. Moreover (1.6) has a unique minimal solution u_λ for each $\lambda \in (0, \lambda^*)$.

Remark 1.11. *(i) The same bifurcation diagrams as in Figure 1 are true for problem (1.6) when Ω is the unit ball in $\mathbb{R}^N$ and the three cases correspond to $1 < p \leq p_S$, $p_S < p < p_{JL}$ and $p \geq p_{JL}$ respectively.*

(ii) In the supercritical case, the bifurcation diagrams of (1.6) are completely different for $p < p_{JL}$ and $p \geq p_{JL}$.

(iii) Hardy's inequality (1.5) implies that $u^*(x) = |x|^{-\frac{2}{p-1}} - 1$ is a stable weak solution of (1.6) for $\lambda = \lambda_p = \frac{2}{p-1}(N - \frac{2p}{p-1})$ if and only if $p \geq p_{JL}$.

Applying implicit function theorem, one can establish a local solution curve $(\lambda, u) \in [0, \infty) \times C(\Omega)$ to (1.4) and (1.6), which stems from $(0, 0)$. By Propositions 1.7 and 1.10, we note that the exponential and power-type nonlinearities for problem (1.1) in the unit ball of $\mathbb{R}^N$ have similar multiplicity phenomena. A related problem with (1.6) is

$$\begin{cases} -\Delta u = u^p + \lambda u, \quad u > 0 & \text{in } B; \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.9}$$

where $p > 1$ and $\lambda > 0$ is a parameter and B is the unit ball in $\mathbb{R}^N$ with $N \geq 3$. We observe that the nonlinearity $f(0) \equiv 0$ for any $\lambda > 0$, which does not satisfy condition (1.2). According to classical bifurcation theory [32], the point $(\mu_1, 0)$ is a bifurcation point from which emanates an unbounded branch $\mathcal{C}$ of solutions of (1.9), where μ_1 is the first eigenvalue of the negative Laplacian operator under Dirichlet boundary condition in B. Multiplying (1.9) by the first eigenfunction and integrating by parts, we get for any p, (1.9) has no solution for $\lambda \geq \mu_1$, even when B is replaced by a general bounded smooth domain Ω.

For the subcritical case, i.e. $p < p_S$, there is a positive classical solution of (1.9) for $\lambda < \mu_1$ by a standard constrained minimization procedure involving compactness of the Sobolev embedding. More precisely, consider the minimizing of the functional

$$E_\lambda(u) = \int_B (|\nabla u|^2 - \lambda u^2) dx$$

constrained on the manifold

$$M = \left\{ u \in H_0^1(B) \ : \ \int_B |u|^{p+1} dx = 1 \right\}.$$

Using the embedding $H_0^1(B) \hookrightarrow L^{p+1}(B)$ is continuous and compact for $p < p_S (N \geq 3)$, the infimum is achieved.

For the critical case, i.e. $p = p_S$, Brézis and Nirenberg [14] made great contributions to this case. Since the Sobolev embedding $H_0^1(B) \hookrightarrow L^{p+1}(B)$ loses compactness when $p \geq p_S$, problem (1.9) becomes more difficult and delicate. Using the Pohozaev's identity [99], problem (1.9) has no solutions for $\lambda \leq 0$ or $\lambda \geq \mu_1$ whenever $p \geq p_S$. Brézis and Nirenberg [14] established the following results:

- when $N \geq 4$, problem (1.9) has a solution for every $0 < \lambda < \mu_1$;
- when $N = 3$, problem (1.9) has a solution only for $\frac{1}{4}\mu_1 < \lambda < \mu_1$.

For the supercritical case, i.e. $p > p_S$, Budd and Norbury [16] derived formally qualitative properties of the bifurcation branch of solutions to (1.9). In particular, formal asymptotics and numerical computations suggest that before reaching $\lambda = 0$, the curve turns right and then oscillates infinitely many times in the form of an exponentially damped sinusoidal along a line $\lambda = \lambda_*$. Merle and Peletier [81] proved that there is a unique value $\lambda = \lambda_* > 0$ such that there exists a singular solution u_* to (1.9). Moreover,

$$u_*(r) = A(p, N) r^{-\frac{2}{p-1}} [1 - B(p, N) r^2 + o(r^2)] \quad \text{as } r \to 0,$$

where

$$A(p,N) = \left[\frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right)\right]^{\frac{1}{p-1}}, \quad B(p,N) = 4\lambda_*\left(N-1-\frac{3}{p-1}\right)^{-1}.$$

Merle, Peletier and Serrin [82] also studied the asymptotic behavior of the positive solutions (λ_p, u_p) as $p \to \infty$. Dolbeault and Flores [49], using geometric theory of dynamical system, established the numerical computations in [16]. They proved that if

$$N \geq 11 \quad \text{and} \quad p_S < p < p_{JL} \quad \text{or} \quad N \leq 10, \tag{1.10}$$

then there is a unique number $\lambda_* > 0$, such that for λ close to λ_*, a large number of classical radial solutions of (1.9) exist. In particular, there are infinitely many classical radial solutions for $\lambda = \lambda_*$. See the bifurcation diagrams for positive solutions of (1.9) in Figure 2. Moreover, in this paper, the authors also considered problem (1.6) when the power s^p is perturbed by a lower order term. More precisely, they established a similar assertion for the following problem

$$\begin{cases} -\Delta u = \lambda((1+u)^p + (1+u)^q), \quad u > 0 & \text{in } B; \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.11}$$

where $1 < q < p$ and p satisfies (1.10)

Recently, Guo and Wei [71] studied problem (1.9) further. They found the structure of the branch $\mathcal{C}$ changed for

$$p \geq p_{JL} \quad \text{and} \quad p_S < p < p_{JL}.$$

The authors established the following results:

- for $p_S < p < p_{JL}$, $\mathcal{C}$ turns infinitely many times around $\lambda_* \in (0, \mu_1)$;
- for $N \geq 11$ and $p \geq p_{JL}$, all solutions (regular or singular) have finite Morse index;
- for $N \geq 12$ and $p > p_{JL}$ sufficiently large, all solutions (regular or singular) have exactly Morse index one.

Motivated from above results, it is natural to ask: is there similar multiplicity phenomena involving the exponential term in the nonlinearity? The answer is positive.

Chapter 2 is devoted to study the structure of solutions to the following problem

$$\begin{cases} -\Delta u = \lambda(e^u - 1), \quad u > 0 & \text{in } B; \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.12}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 3$ and $\lambda > 0$ is a parameter.

Smooth solutions to (1.12) are radially symmetric and decreasing by the classical result of Gidas, Ni and Nirenberg [67]. We observe that $f(0) = 0$, which does not satisfy condition (1.2). Note that $u = 0$ is a trivial solution to (1.12) for any $\lambda > 0$. According to classical

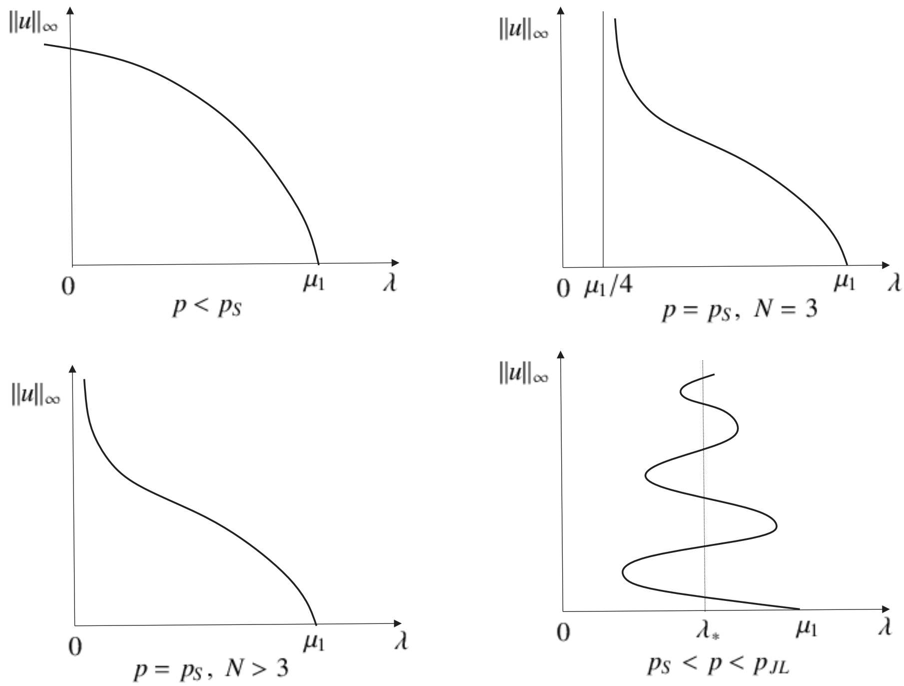

Figure 2: Bifurcation diagrams for positive solutions of (1.9) in the unit ball of $\mathbb{R}^N$.

bifurcation theory [32], the point $(\mu_1, 0)$ is a bifurcation point from which emanates an unbounded branch $\mathcal{C}$ of solutions of (1.12), where μ_1 is the first eigenvalue of the negative Laplacian operator under Dirichlet boundary condition in B.

We get multiplicity of regular radial solutions to problem (1.12) for $3 \leq N \leq 9$.

Theorem 1.12. *If $3 \leq N \leq 9$, then there exists a unique λ_* such that problem (1.12) has infinitely many regular radial solutions for $\lambda = \lambda_*$. Moreover $\lambda \neq \lambda_*$ but close to λ_*, there is a large number of regular radial solutions for (1.12).*

Multiplicity results were obtained by using geometric theory of dynamical systems in three-dimensional phase space, which was applied by Bamon, del Pino, and Flores [8] to study the following problem,

$$\begin{cases} -\triangle u = u^p + u^q & \text{in } \mathbb{R}^N; \\ 0 < u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.13}$$

where p and q are subcritical and supercritical respectively, namely

$$1 < p < p_S < q. \tag{1.14}$$

By the result of Zou [119], all the ground states to (1.13) are radially symmetric around some point for p and q satisfying (1.14). Thus it can be written as an ODE equation

$$\begin{cases} -u'' - \frac{N-1}{r} u' = u_+^p + u_+^q & r > 0; \\ u'(0) = 0, \quad 0 < u(r) \to 0 & \text{as } r \to \infty, \end{cases} \tag{1.15}$$

where $u_+ = \max\{u, 0\}$. A positive solution $u(r)$ of (1.15) in $(0, \infty)$ is said to have *slow decay* if

$$u(r) = Cr^{-\frac{2}{p-1}} + o(r^{-\frac{2}{p-1}}) \quad \text{as } r \to \infty,$$

for some positive constant C. $u(r)$ is said to have *fast decay* if

$$u(r) = O(r^{-(N-2)}) \quad \text{as } r \to \infty.$$

Thus $u(r)$ is said to be a *radial ground state* of (1.13) if it is finite up to $r = 0$ with $u'(0) = 0$. The first result of existence of radial ground states of (1.13) was given by Lin and Ni in [78]. They found if p and q satisfy (1.14) and $q = 2p - 1$, then there is an explicit solution of the form $u(r) = (\frac{A}{B+r^2})^{\frac{1}{p-1}}$, where A, B are positive constants depending on p and N. It is a ground state of slow decay.

Problem (1.15) is equivalent to a three dimensional autonomous first order system after the classical Emden-Fowler transformation, then a ground state with fast decay corresponds to a heteroclinic orbit connecting two stationary points of the system with a two-dimensional unstable manifold and a two-dimensional stable manifold respectively. Using phase-space analysis, Bamon, del Pino, and Flores [8] proved that for $q > p_S$ is fixed and p approaches p_S from below, then problem (1.13) has a large number of radial ground states with fast decay. A similar fact holds for $\frac{N}{N-2} < p < p_S$ fixed and q approaches p_S. Moreover, if q is fixed and p close enough to $\frac{N}{N-2}$, then no solutions exist.

It is also worth mentioning the case $q = 2p - 1$ and the range of p is further restricted to

$$\frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}} < p. \tag{1.16}$$

Flores [59] showed that not only Lin and Ni's solution exists, but also infinitely many solutions with fast decay. In addition, if $\frac{N}{N-2} < p < p_S < q$, p satisfies (1.16), and there is a slow decay radial ground state of (1.13), then there are infinitely many radial ground states with fast decay.

This method was subsequently applied in [49, 59, 60]. There are some analogies between the results and techniques of this work and [4, 5, 38, 40, 41] on fourth order problems involving the exponential nonlinearity.

Although the question of multiplicity of solutions to (1.13) under restriction (1.14) has been studied in the nearly sub-supercritical case with the help of geometric dynamical systems tools, Campus [21], using Lyapunov-Schmidt procedure, proved that there exist a large finite number of ground states of (1.13) with fast decay when $\frac{N}{N-2} < p < p_S$ is fixed with $N \geq 3$ and q lies above but close enough to the critical exponent p_S, these solutions behave like a superposition of "bubbles" of different blow-up orders centered at the origin. In the last chapter, we are interested in multiplicity of solutions of the following problem

$$\begin{cases} -\Delta u + u = u^p + \lambda u^q, \quad u > 0 \qquad \text{in } \mathbb{R}^N; \\ u(z) \to 0 \quad \text{as} \quad |z| \to \infty, \end{cases} \tag{1.17}$$

where p and q are in some ranges. λ is a positive parameter. We will introduce this problem at the end of this chapter.

Let us come back to problem (1.12). Another interesting question is: what does happen in high dimensions? Inspired by the result of Guo and Wei [71], we estimate the Morse index of solution to (1.12) for $N \geq 10$.

Theorem 1.13. *Assume $N \geq 10$. Then there exists $K < \infty$ such that the Morse index of any radial solution (λ, u_λ) of (1.12) (regular or singular) is bounded by K. The number of intersections of any regular solution and the radial singular solution is uniformly bounded by $2K+1$. Moreover, for each $\lambda \in (\lambda_0, \mu_1)$, the number of regular solutions to (1.12) is bounded by $(K+1)^2$.*

Remark 1.14. *By Pohozaev's identity, there exists $\lambda_0 > 0$ such that classical solutions of (1.12) can exist only for $\lambda \in (\lambda_0, \mu_1)$, where μ_1 is the first eigenvalue of the negative Laplacian operator under Dirichlet boundary condition in B.*

1.2 Lane-Emden system

Consider the Lane-Emden system

$$\begin{cases} -\Delta u = v^p, \ u > 0 & \text{in } \mathbb{R}^N, \\ -\Delta v = u^q, \ v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.18}$$

where $N \geq 1$ and $p \geq q \geq 1$. This system arises in chemical, biological and physical studies, and has been investigated by many authors, see for example, de Figueiredo-Felmer [42], Mitidieri [85], Serrin and Zou [106] and Van der Vorst [114].

System (1.18) is a natural extension of the celebrated Lane-Emden equation

$$-\Delta u = u^p \quad x \in \mathbb{R}^N, \quad u > 0, \quad N \geq 3, \ p > 1. \tag{1.19}$$

Problem (1.19) has been studied extensively. There has been much work done on existence and nonexistence of positive classical solutions of (1.19), see for instance [19, 25, 69, 67]. B. Gidas and J. Spruck [69] obtained the following beautiful result: the Lane-Emden equation (1.19) has no positive solution if

$$1 < p < p_S = \frac{N+2}{N-2}.$$

L. Caffarelli, B. Gidas and J. Spruck [19] established that if $p = p_S$, up to rescaling and translation, the positive solution is unique. It is known that the Sobolev exponent

$$p_S = \frac{N+2}{N-2},$$

which is the dividing number for existence and non-existence of solutions of (1.19), that is, equation (1.19) admits non-negative, non-trivial solutions if and only if $p \geq p_S$, see [69].

Moreover, Farina [55] proved Liouville-type results for C^2 solutions of (1.19) belonging to one of the following classes: stable solutions, finite Morse index solutions, solutions which are stable outside a compact set, radial solutions and non-negative solutions. The author got existence of a new critical exponent p_{JL}. This new critical exponent is larger than the classical Sobolev critical exponent. We state here one of results in [55]. The author obtained that no nontrivial stable solution (also nonradial) exists if

$$N \leq 10 \quad \text{or} \quad N \geq 11 \quad \text{and} \quad 1 < p < p_{JL},$$

where p_{JL} is the Joseph-Lundgren exponent. On the other hand, for

$$N \geq 11 \quad \text{and} \quad \text{every} \quad p \geq p_{JL}$$

(1.19) admits a positive smooth stable radial solution.

For the Lane-Emden system, concerning the question of existence and nonexistence of entire solutions, it is expected that the role of the Sobolev exponent p_S should be played by the so-called Sobolev hyperbola. It has been conjectured, see for example De Figueiredo and Felmer [42], that the hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}, \qquad p,\ q > 0,$$

is the dividing curve between existence and nonexistence of solutions to (1.18). That is, there is no positive classical solution of (1.18) if and only if

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \tag{1.20}$$

This conjecture is supported by the results that there are no radial positive solutions to (1.18) provided that p, q satisfy (1.20), see Mitidieri [86] for $p, q > 1$ and Serrin and Zou [107] for $p, q > 0$. Moreover, system (1.18) admits positive radial classical solutions provided that

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2}{N}, \tag{1.21}$$

see Serrin and Zou [107]. This conjecture was proved for the radial case in all dimensions. For non-radial solutions, in dimension $N \leq 2$, the conjecture is a consequence of a result of Mitidieri and Pohozaev [87]. Poláčik, Quittner and Souplet [100] proved that the conjecture is true for $N = 3$. The conjecture was proved by Souplet [110] for $N = 4$. Moreover, some partial results were also established for $N \geq 5$, see for example [17, 26, 110].

Recently, Cowan [30] proved various Liouville-type theorems for positive stable solutions of the Lane-Emden system and the fourth scalar equation. For example, the author showed that the nonexistence of positive classical stable solutions (not necessary radial) to (1.18) for

$1 \le N \le 10$ and $p \ge q \ge 2$. The author also examined nonexistence of positive classical stable solutions of the fourth order equation, i.e. the case $q = 1$ in (1.18)

Question: is there a new dividing curve in the $pq-$plane for existence and nonexistence of stable radially symmetric positive solution to the Lane-Emden system?

In Chapter 3, we characterize the stability of radially symmetric solutions of the Lane-Emden system (1.18). This gives a positive answer for above question. In order to state our result, we introduce the definition of stable solution for system (1.18) and some notations.

Definition 1.15. *A solution (u, v) to (1.18) is stable if there exists a positive supersolution of the linearized system i.e. if there exists $(\phi, \psi) \in C^2(\mathbb{R}^N)^2$ such that*

$$\begin{cases} -\Delta\phi \ge pv^{p-1}\psi & \text{in } \mathbb{R}^N, \\ -\Delta\psi \ge qu^{q-1}\phi & \text{in } \mathbb{R}^N, \\ \ \ \phi, \psi > 0 & \text{in } \mathbb{R}^N. \end{cases}$$

Let us also note that if (1.21) holds, then

$$(u_s, v_s) = (a|x|^{-\alpha}, b|x|^{-\beta}), \quad x \in \mathbb{R}^N \setminus \{0\} \tag{1.22}$$

is a weak solution of (1.18) provided

$$\alpha = \frac{2(p+1)}{pq-1}\ , \quad \beta = \frac{2(q+1)}{pq-1} \tag{1.23}$$

and $a = (ST^p)^{\frac{1}{pq-1}}$, $b = (S^qT)^{\frac{1}{pq-1}}$, $S = \alpha(N-2-\alpha)$, $T = \beta(N-2-\beta)$.

Theorem 1.16. *Assume $p \ge q \ge 1$.*

(i) If $N \ge 11$ and (p, q) lies on or above the Joseph-Lundgren critical curve i.e.

$$\Big[\frac{(N-2)^2 - (\alpha-\beta)^2}{4}\Big]^2 \ge pq\alpha\beta(N-2-\alpha)(N-2-\beta), \tag{1.24}$$

then any radially symmetric solution (u, v) of (1.18) is stable and satisfies

$$u < u_s \quad \text{and} \quad v < v_s \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

where (u_s, v_s) is the singular solution given by (1.22) and α, β are the scaling exponents given by (1.23).

(ii) If $N \le 10$ or if $N \ge 11$ and (1.24) fails, then there is no stable radially symmetric solution of (1.18).

The above result states that the stability of a radial solution of the Lane-Emden system is determined by the position of the exponents (p, q) with respect to a new critical curve, which we call "Joseph and Lundgren", since the exponent introduced by these authors in [73] is the intersection of the curve with the diagonal $p = q$.

1.3 Bubbling solutions for some elliptic equations

In Chapters 4 and 5, we use Lyapunov-Schmidt Reduction method to consider existence and multiplicity of bubbling solutions to some asymptotic critical elliptic equations. First we state this method, see also the book [24]. Then we introduce our main problems and results.

1.3.1 Lyapunov-Schmidt Reduction

Let X, Y be Banach space, and let Λ be a topological space. Assume that $F : \mathcal{O} \times \Lambda \to Y$ is continuous, where $\mathcal{O} \subset X$ is a neighborhood of x_0. We assume that $F_x(x_0, \lambda_0)$ is a Fredholm operator, i.e.

(a) $ImF_x(x_0, \lambda_0)$ is closed in Y,

(b) $d = dimkerF_x(x_0, \lambda_0) < \infty$,

(c) $d^* = codimImF_x(x_0, \lambda_0) < \infty$.

Set

$$X_1 = kerF_x(x_0, \lambda_0), \qquad Y_1 = ImF_x(x_0, \lambda_0).$$

Since both $dimX_1$ and $codimY_1$ are finite, we have the direct sum decompositions:

$$X = X_1 \oplus X_2, \qquad Y = Y_1 \oplus Y_2,$$

and the projection operator $P : Y \to Y_1$, for every $x \in X$, there exists a unique decomposition:

$$x = x_1 + x_2, \quad x_i \in X_i, \ i = 1, 2.$$

Thus

$$F(x, \lambda) = 0 \iff \begin{cases} PF(x_1 + x_2, \lambda) = 0, \\ (Id - P)F(x_1 + x_2, \lambda) = 0. \end{cases}$$

Now, $PF_x(x_0, \lambda_0) : X_2 \to Y_1$ is a surjection as well as an injection. According to the Banach theorem, it has a bounded inverse. If we already have $F(x_0, \lambda_0) = 0$, then from the implicit function theorem, we have a unique solution

$$u : V_1 \times V \to V_2$$

satisfying

$$PF(x_1 + u(x_1, \lambda), \lambda) = 0,$$

where V_1 is a neighborhood of x_1 in $U \cap X_1$, V_2 is a neighborhood of 0 in $U \cap X_2$, and V is a neighborhood of λ_0.

It remains to solve the equation:

$$(Id - P)F(x_1 + u(x_1, \lambda), \lambda) = 0$$

on $V_1 \times V$. This is a nonlinear system of d variables and d^* equations.

Above procedure is called Lyapunov-Schmidt reduction which reduces an infinite-dimensional problem to a finite-dimensional system. This method has been used broadly by many mathematicians to construct bubbling solutions to elliptic equations, which was first developed by Bahri and Coron in [7]. We refer to see the nice survey of del Pino and Musso [47], also see[35, 43, 44, 45, 46, 48, 58, 64, 83, 89, 90, 91, 98, 104, 105, 116] et al. and references therein. By bubbles we mean the functions

$$w_\mu(z) = \alpha_N \frac{\mu^{\frac{N-2}{2}}}{(\mu^2 + |z|^2)^{\frac{N-2}{2}}}, \quad \text{with} \quad \alpha_N = (N(N-2))^{\frac{N-2}{4}},$$

where $\mu > 0$, which are the unique positive solutions (except translations) of

$$-\Delta w = w^{p^*} \quad \text{in } \mathbb{R}^N.$$

1.3.2 Multiplicity of solutions to asymptotic critical elliptic equations

In this subsection, first we are interested in the following semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = u^p + \lambda u^q, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.25}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^3$, $\lambda > 0$ and $p > q$.

Existence and multiplicity of solutions to (1.25) have been studied in many works for the exponents p and q in different ranges. Ambrosetti, Brézis and Cerami [2], using the method of sub and super solutions, established that for $0 < q < 1$ and $p > 1$ arbitrary, there exists $\Lambda > 0$ such that problem (1.25) has a minimal solution u_λ for $\lambda \in (0, \Lambda)$, and u_λ is increasing with respect to λ; for $\lambda = \Lambda$, problem (1.25) has at least one weak soltuion; for all $\lambda > \Lambda$, problem (1.25) has no solution. Moreover, using variational tools, the authors [2] also showed that if $0 < q < 1 < p \le 5$, for all $\lambda \in (0, \Lambda)$, problem (1.25) has a second solution.

Let us mention some related results of (1.25) for $q = 1$. Namely, (1.25) reduces to

$$\begin{cases} -\Delta u = u^p + \lambda u, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.26}$$

In Section 1.1, we state some results of (1.26) when Ω is a until ball in $\mathbb{R}^N$ with $N \ge 3$. Especially, we recall here some results for $N = 3$.

If $1 < p < 5$, for $0 < \lambda < \mu_1$, where μ_1 is the first eigenvalue of $-\Delta$ under Dirichlet boundary condition, a solution can be found by the standard constrained minimization procedure thanks to compactness of Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$.

If $p \geq 5$, this case is more delicate, since for $p = 5$ the embedding loses compactness while for $p > 5$ Sobolev embedding fails. Pohozaev [99] proved that if Ω is strictly star-shaped, then there is no solution of (1.26) if $\lambda \leq 0$ and $p \geq 5$. For $p = 5$, the great contribution to this case was the pioneering work of Brézis and Nirenberg [14]. They obtained that (1.26) has a solution if and only if $\lambda \in (\frac{1}{4}\mu_1, \mu_1)$ when Ω is a ball, where μ_1 denotes the first eigenvalue of $-\Delta$ under Dirichlet boundary condition on a ball. Moreover, the authors considered the case $q > 1$: if $1 < q \leq 3$, there exists a solution if and only if $\lambda > 0$ is large enough. If $3 < q < 5$, (1.25) has a solution for every $\lambda > 0$. In addition, based on numerical computations, they gave the following conjecture when Ω is a ball.

(a) If $q = 3$, there is some $\tilde{\lambda}$ such that

- (i) for $\lambda > \tilde{\lambda}$, there is a unique solution of (1.25);
- (ii) for $\lambda \leq \tilde{\lambda}$, there is no solution of (1.25).

(b) If $1 < q < 3$, there is some $\tilde{\lambda}$ such that

- (i) for $\lambda > \tilde{\lambda}$, there are two solutions of (1.25);
- (ii) for $\lambda = \tilde{\lambda}$, there is a unique solution of (1.25);
- (iii) for $\lambda < \tilde{\lambda}$, there is no solution of (1.25).

Afterwards, Atkinson and Peletier [6] proved the nonuniqueness of solutions to (1.25) conjectured by Brézis and Nirenberg for $N = 3$, $p = 5$ and $1 < q < 3$. Not restricting to integer values of N, they established for $2 < N < 4$, $p = \frac{N+2}{N-2}$ and $1 < q < \frac{6-N}{N-2}$, then there exists some $\tilde{\lambda} > 0$ such that (1.25) has at least two solutions for any $\lambda > \tilde{\lambda}$, and it has no solution for $\lambda < \tilde{\lambda}$. Rey [103] provided another partial answer to above conjecture. He obtained that for $p = 5$ and $2 < q < 3$, $\lambda > 0$ large enough, problem (1.25) has at least $Cat(\Omega) + 1$ solutions, where Ω is any smooth and bounded domain in $\mathbb{R}^3$ and $Cat(\Omega)$ denotes Ljusternik-Schnirelman (L-S, for short) category of Ω, see [3] for the definition of L-S category. We put the bifurcation diagrams of positive solutions to problem (1.25) in the unit ball of $\mathbb{R}^3$ in Figure 3.

Next, we are also interested in the elliptic equation

$$\begin{cases} -\Delta u + u = u^p + \lambda u^q, \qquad u > 0 \qquad \text{in } \mathbb{R}^N, \\ u(z) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{1.27}$$

where $N \geq 3$, $\lambda > 0$ and $1 < q < p$. This problem arises in the study of standing waves of a nonlinear Schrödinger equation with two power type nonlinearities, see for example Tao, Visan and Zhang [113].

If $p = q$, equation (1.27) reduces to

$$\begin{cases} -\Delta u + u = u^p, \qquad u > 0 \qquad \text{in } \mathbb{R}^N, \\ u(z) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{1.28}$$

after a suitable scaling.

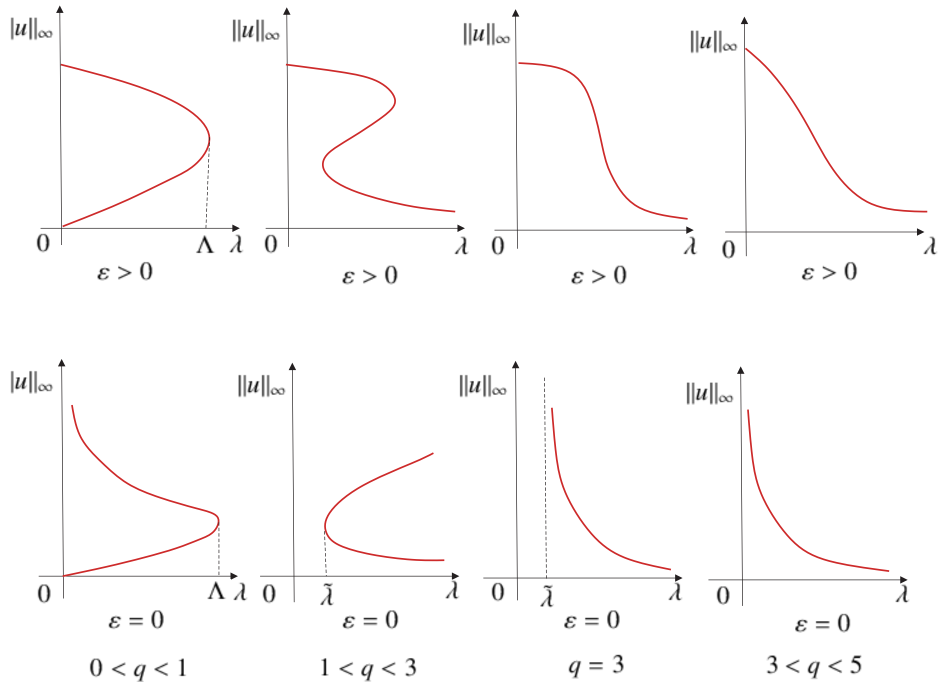

Figure 3: Bifurcation diagrams of positive solutions to problem (1.25) when $p = 5 - \varepsilon$ and Ω is the unit ball in $\mathbb{R}^3$. The case $q = 1$ is given in Figure 2

Thanks to the classical result of Gidas, Ni and Nirenberg [68], solutions of (1.27) and (1.28) are radially symmetric about some point, which we will assume is always the origin.

It is well known that problem (1.28) has a solution if and only if $1 < p < \frac{N+2}{N-2}$. Existence was proved by Berestycki and Lions [10], while non-existence from the Pohozaev identity [99]. Uniqueness also holds and was fully settled by Kwong [76], after a series of contributions [22, 80, 93, 94, 96, 97]. See also Felmer, Quaas, Tang and Yu [57] for further properties.

Concerning (1.27), the work of Berestycki and Lions [10] is still applicable if $1 < q < p < \frac{N+2}{N-2}$, and one obtains existence of a solution. If $p, q \geq \frac{N+2}{N-2}$ there is no solution, again from the Pohozaev identity.

Recently, Dávila, del Pino and Guerra [35] proved that uniqueness does not hold in general for (1.27), if $1 < q < p < \frac{N+2}{N-2}$. More precisely if $N = 3$, the authors obtained at least three solutions to problem (1.27) if $1 < q < 3$, $\lambda > 0$ is sufficiently large and fixed, and $p < 5$ is close enough to 5.

Let us next mention some contributions to the question of existence for (1.27) when one exponent is subcritical and other is critical or supercritical. If $1 < q < p = \frac{N+2}{N-2}$ in (1.27), using variational methods, Alves, de Morais Filho and Souto [1] proved:

- when $N \geq 4$, there exists a nontrivial classical solution for all $\lambda > 0$ and $1 < q < \frac{N+2}{N-2}$;
- when $N = 3$, there exists a nontrivial classical solution for all $\lambda > 0$ and $3 < q < 5$;

• when $N = 3$, there exists a nontrivial classical solution for $\lambda > 0$ large enough and $1 < q \leq 3$.

Moreover, Ferrero and Gazzola [56] proved that for $q < \frac{N+2}{N-2} \leq p$, there exists $\bar{\lambda} > 0$, such that if $\lambda > \bar{\lambda}$, then (1.27) has at least one solution, while for $q < \frac{N+2}{N-2} < p$, there exists $0 < \underline{\lambda} < \bar{\lambda}$ such that if $\lambda < \underline{\lambda}$, then there is no solution.

An interesting problem is bubble-tower phenomena for a slightly supercritical Brézis and Nirenberg problem. In the work of del Pino, Dolbeault and Musso [43], the authors found for $\lambda = o(1)$, depending on ε, a new phenomena happened: the presence of towers constituted by superposition of bubbles of different blow-up orders for (1.26) in a ball when $p = \frac{N-2}{N+2} + \varepsilon$ with $\varepsilon > 0$, $N \geq 4$. After that, these authors [44] established bubble-tower solutions to (1.26) in a general bounded and smooth domain in $\mathbb{R}^3$. J. Campos [21] considered the existence of bubble-tower solutions to a problem related to (1.27):

$$\begin{cases} -\Delta u = u^{p^* \pm \varepsilon} + u^q, \quad u > 0 \quad \text{in } \mathbb{R}^N; \\ u(z) \to 0 \ \text{ as } \ |z| \to \infty, \end{cases} \tag{1.29}$$

with $\frac{N}{N-2} < q < p^* = \frac{N+2}{N-2}$, $N \geq 3$. These solutions were obtained by Lyapunov-Schmidt reduction procedure. We refer to see [21, 44, 46, 48, 64, 65, 83, 89, 91, 98] for bubble-tower phenomena.

Motivated from above, the left question is whether there exist multiplicity of solutions to problems (1.25) and (1.27). We will answer it in the last two chapters.

In Chapter 4, we will establish multiplicity of solutions to subcritical problem

$$\begin{cases} -\Delta u = u^{5-\varepsilon} + \lambda u^q, \quad u > 0 \quad & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.30}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $1 < q < 3$, $\lambda > 0$ and $\varepsilon > 0$ small enough.

In Chapter 5, we are concerned with multiplicity of solutions of (1.27), and for this we take an asymptotic approach, that is, we consider

$$\begin{cases} -\Delta u + u = u^p + \lambda u^q, \quad u > 0 \quad \text{in } \mathbb{R}^N; \\ u(z) \to 0 \ \text{ as } \ |z| \to \infty, \end{cases} \tag{1.31}$$

where $p = p^* + \varepsilon$, with $p^* = \frac{N+2}{N-2}$, $\lambda > 0$ and $\varepsilon > 0$ are parameters, and q satisfies

$$1 < q < \frac{N+2}{N-2} \quad \text{if } N \geq 4; \qquad 3 < q < 5 \ \text{ if } N = 3. \tag{1.32}$$

The main results in Chapters 4 and 5 are as follows.

Theorem 1.17. *Let $1 < q < 3$, there exists $\lambda_0 > 0$, depending on Ω, q, and $\varepsilon_0 > 0$, such that for any $\lambda \geq \lambda_0$, $\varepsilon \in (0, \varepsilon_0)$, problem (1.30) has at least two solutions.*

Theorem 1.18. *Assume that* $2 < q < 3$. *Then there exist* $\hat{\lambda} \geq \lambda_0$ *and* $\delta_0 > 0$, *such that for any* $\lambda \geq \hat{\lambda}$ *satisfying*

$$0 < \varepsilon \lambda^{\frac{2}{3-q}} \log \lambda < \delta_0, \tag{1.33}$$

then for all sufficiently small $\varepsilon > 0$, *problem (1.30) has at least three solutions.*

Theorem 1.19. *Let* $\lambda > 0$ *and let* q *satisfy (1.32). Given an integer* $k \geq 1$, *then there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$, *there is a solution* $u_\varepsilon(z)$ *of problem (1.31) of the form*

$$u_\varepsilon(z) = (N(N-2))^{\frac{N-2}{4}} \sum_{j=1}^{k} \frac{\varepsilon^{-[(j-1)+\frac{2}{p^*-q}]}(\Lambda_j^*)^{-\frac{N-2}{2}}}{\left(1 + \varepsilon^{-\frac{4}{N-2}[(j-1)+\frac{1}{p^*-q}]}(\Lambda_j^*)^{-2}|z|^2\right)^{\frac{N-2}{2}}}(1 + o(1)), \tag{1.34}$$

where the constants $\Lambda_j^* > 0$, $j = 1, 2, \ldots, k$, *can be computed explicitly and depend on* k, N, q.

The first solution in Theorem 1.17 is obtained by mountain pass theorem [102, Theorem 2.2]. Regarding $\varepsilon > 0$ as a small parameter, we use Lyapunov-Schmidt reduction procedure to construct the second solution.

Basing on Theorem 1.17 which provides a mountain pass solution and a bubble solution as $\varepsilon > 0$ is a small parameter. In order to prove Theorem 1.18, it is sufficient to show that if (1.33) holds, then (1.30) has a third solution. This solution is also constructed by Lyapunov-Schmidt reduction procedure by regarding $\lambda > 0$ as a large parameter. In the case $1 < q \leq 2$, it is also possible to find a third solution but the proof is more delicate and will be addressed in future work.

The proof of Theorem 1.19 starts with a variation of the so-called Emden-Fowler transformation, which reduces the problem of finding k-bubble solution to the problem of finding a k-bump solution of a second-order ordinary differential equation in $\mathbb{R}$. After a Lyapunov-Schmidt reduction procedure, see for example [58, 83, 21], the problem becomes to find a critical point of some functional depending on k real parameters.

Chapter 2

Resonance phenomenon for a Gelfand-type problem

2.1 Introduction

In this chapter, we consider the structure of the solution set of the boundary value problem

$$\begin{cases} -\Delta u = \lambda(e^u - 1), \quad u > 0 & \text{in } B; \\ u = 0 & \text{on } \partial B, \end{cases} \tag{2.1}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 3$ and $\lambda > 0$ is a parameter. Smooth solutions to (2.1) are radially symmetric and decreasing by the classical result of Gidas, Ni and Nirenberg [67].

Problem (2.1) is related to the following *Gelfand* problem:

$$\begin{cases} -\Delta u = \lambda e^u, & \text{in } B; \\ u = 0 & \text{on } \partial B. \end{cases} \tag{2.2}$$

Barenblatt [66] and Joseph and Lundgren [73], using phase-plane analysis, gave a complete description of the classical solutions to (2.2), which are again radially symmetric [67], see Proposition 1.7 in Chapter 1.

Nagasaki and Suzuki [92] classified the solutions of (2.2) according to their Morse index. In a few words, the family of regular solutions of (2.2) can be described as a curve $(u(s), \lambda(s))$ with $s \in [0, \infty)$, such that $(u(s), \lambda(s)) \to (0, 0)$ as $s \to 0$ and $(u(s), \lambda(s)) \to (u_\sigma, \lambda_\sigma)$ as $s \to \infty$, where $u_\sigma(r) = -2\log(r)$, $\lambda_\sigma = 2(N-2)$ is a singular solution of (2.2). In dimensions $3 \leq N \leq 9$, $\lambda(s)$ oscillates around $2(N-2)$ as $s \to \infty$ and the Morse index of $u(s)$ increases by one in each oscillation. In dimensions $N \geq 10$, $\lambda(s)$ is monotone, $u(s)$ is monotone and is stable for each s.

A problem analogous to (2.1) is

$$\begin{cases} -\Delta u = u^p + \lambda u, \quad u > 0 & \text{in } B; \\ u = 0 & \text{on } \partial B. \end{cases} \tag{2.3}$$

where $p > 1$ and $\lambda > 0$ is a parameter. According to classical bifurcation theory [32], the point $(\mu_1, 0)$ is a bifurcation point from which emanates an unbounded branch $\mathcal{C}$ of solutions of (2.3), where μ_1 is the first eigenvalue of the negative Laplacian operator under Dirichlet boundary condition in B.

• If $p < \frac{N+2}{N-2} (N \geq 3)$, for $\lambda < \mu_1$, there is a positive solution of (2.3) by a standard constrained minimization procedure involving compactness of the Sobolev embedding. Moreover, by Pohozaev's identity [99], problem (2.3) has no solutions for $\lambda \leq 0$ whenever $p \geq \frac{N+2}{N-2}$.

• If $p = \frac{N+2}{N-2}$, which is the classical Brezis-Nirenberg problem [14], problem (2.3) has a solution for $0 < \lambda < \mu_1$ if $N \geq 4$, and for $\frac{1}{4}\mu_1 < \lambda < \mu_1$ if $N = 3$.

• If $p > \frac{N+2}{N-2}$, Dolbeault and Flores found that if $p > \frac{N+2}{N-2}$, and $p < p_{JL}$ or $N \leq 10$, then there is a unique number $\lambda_* > 0$, such that for λ close to λ_*, a large number of classical solutions of (2.3) exist. In particular, there are infinitely many classical solutions for $\lambda = \lambda_*$. Recently, Guo and Wei in [71] showed that the structure of the branch $\mathcal{C}$ changes for

$$p \geq p_{JL} \quad \text{and} \quad \frac{N+2}{N-2} < p < p_{JL}$$

where p_{JL} is defined as in (1.8). Moreover, they established that for $\frac{N+2}{N-2} < p < p_{JL}$, $\mathcal{C}$ turns infinitely many times around $\lambda_* \in (0, \mu_1)$. For $p \geq p_{JL}$, all solutions have a finite Morse index, and for $N \geq 12$ and $p > p_{JL}$ sufficiently large all solutions have exactly Morse index one.

The aim of this chapter is to study the structure of solutions to problem (2.1). We start with some general remarks. First, classical solutions of (2.1) can exist only for λ in some interval.

Proposition 2.1. *Let μ_1 be the first eigenvalue of the $-\Delta$ under Dirichlet boundary condition in B. Then there exists $\lambda_0 > 0$, such that a necessary condition for existence of classical solutions to problem (2.1) is $\lambda \in (\lambda_0, \mu_1)$.*

See a proof in the Appendix. By classical bifurcation theory [24, 32] we have that $(\mu_1, 0)$ is a bifurcation point of solutions to (2.1). Both observations are also valid if we replace the ball by a bounded smooth domain (star shaped in the case of Proposition 2.1).

We are interested also in weak solutions, allowing for possible singularities.

Definition 2.2. *We say that $u \in H_0^1(B)$ is a weak solution of (2.1) if $e^u \in L^1(B)$ and*

$$\int_B \nabla u \nabla \varphi = \lambda \int_B (e^u - 1)\varphi \quad \textit{for all } \varphi \in C_0^\infty(B). \tag{2.4}$$

We say that a weak solution u of (2.1) is regular (resp., singular) if $u \in L^\infty(B)$ (resp., $u \notin L^\infty(B)$).
We say that a radial weak solution u of (2.1) is weakly singular solution if it is singular and $\lim_{r\to 0} ru'(r)$ exists.

We first study singular solutions to (2.1).

Theorem 2.3. *Assume $N \geq 3$. Let $\lambda > 0$ and suppose that $u \in C^2(B \setminus \{0\})$, $u \geq 0$ is a radial solution of*

$$-\Delta u = \lambda(e^u - 1) \quad \text{in } B\backslash\{0\}. \tag{2.5}$$

Then either

a) u can be extended as a function in $C^\infty(B)$ and (2.5) holds in B,

or

b) u is singular at $r = 0$ and satisfies

$$\lim_{r\to 0}(u(r) + 2\log r) = \log \frac{2(N-2)}{\lambda},$$
$$\lim_{r\to 0} ru'(r) = -2.$$

As a consequence, u is a radial singular weak solution to (2.1) if and only if u is a weakly singular solution.

Theorem 2.4. *For $N \geq 3$, there exists a unique $\lambda_* > 0$, such that (2.1) admits a radial singular solution for $\lambda = \lambda_*$, and the radial singular solution is unique.*

By Theorem 2.3 the singular solution is weakly singular.

Next, we consider the question of multiplicity of solutions to (2.1).

Theorem 2.5. *If $3 \leq N \leq 9$, then problem (2.1) has infinitely many regular radial solutions for $\lambda = \lambda_*$. For $\lambda \neq \lambda_*$ but close to λ_*, there is a large number of regular radial solutions for (2.1).*

Let us recall the definition of the Morse index of solution to (2.1), see Definition 1.5. Namely, for a weak solution (λ, u) of (2.1), we define the Morse index of u as the largest dimension k of a subspace $Y \subset C_c^\infty(B)$ such that

$$Q_u(\varphi) = \int_B |\nabla\varphi|^2 - \lambda e^u \varphi^2 < 0 \quad \forall\, \varphi \in Y \setminus \{0\}.$$

If u is a regular solution this is the number of negative eigenvalues, counting multiplicity, of the operator $-\Delta - \lambda e^u$. By Theorem 3 of Dancer and Farina [33], if $3 \leq N \leq 9$, for a sequence of solutions $(\lambda_n, u_{\lambda_n})$ to (2.1) with $\|u_n\|_{L^\infty(B)} \to \infty$ as $n \to \infty$, then the Morse index of u_{λ_n} goes to infinity as $n \to \infty$.

Theorem 2.6. *Assume $N \geq 10$. Then there exists $K < \infty$ such that the Morse index of any radial solution (λ, u_λ) of (2.1)(regular or singular) is bounded by K. The number of intersections of any regular solution and the radial singular solution is uniformly bounded by $2K+1$. Moreover, for each $\lambda \in (\lambda_0, \mu_1)$, the number of regular solutions to (2.1) is bounded by $(K+1)^2$.*

A natural conjecture for $N \geq 10$, which is observed in numerical calculations, is that the Morse index of any radial solution of (2.1) (regular or singular) is 1, the number of intersections of any regular solution and the radial singular solution is 1, and that for each $\lambda \in (\lambda_*, \mu_1)$ there is a unique solution.

We use geometric theory of dynamical systems in three-dimensional phase space, which was applied in [8], and subsequently in [49, 59, 60], to obtain multiplicity of solutions to problem (2.1). There are some analogies between the results and techniques of this work and [4, 5, 38, 40, 41] on fourth order problems involving the exponential nonlinearity.

In Section 2.2 we give some preliminaries. In Section 2.3 we prove Theorem 2.3, namely that radial solutions either are regular or weakly singular. Theorem 2.4, which is about the existence and uniqueness of a singular solution is proved in Section 2.4. In Section 2.5 we prove Theorem 2.5 on multiplicity of solutions in dimensions $3 \leq N \leq 9$. In Section 2.6 we analyze the Morse index of solutions to problem (2.1), give the structure of the branch of solutions to (2.1), and prove Theorem 2.6. Finally, we give the proof of Proposition 2.1 in the Appendix.

2.2 Preliminary results

Let u satisfy (2.1) and make the change of variables

$$v(t) = u(r) \quad \text{with } r = e^t, \quad \text{for} \quad t \in (-\infty, 0). \tag{2.6}$$

Then problem (2.1) becomes

$$\begin{cases} -v''(t) + (2-N)v'(t) = \lambda e^{2t}(e^{v(t)} - 1), & t \in (-\infty, 0), \\ v(0) = 0, \quad \lim_{t \to -\infty} e^{-t} v'(t) = 0. \end{cases} \tag{2.7}$$

Define

$$\begin{cases} v_1(t) = \frac{\lambda}{2(N-2)} e^{v(t)+2t}, \\ v_2(t) = v'(t), \\ v_3(t) = \lambda e^{2t}. \end{cases} \tag{2.8}$$

We find that (v_1, v_2, v_3) satisfies the following differential system

$$\begin{cases} v_1' = v_1(v_2 + 2), \\ v_2' = -2(N-2)v_1 - (N-2)v_2 + v_3, \\ v_3' = 2v_3, \end{cases} \tag{2.9}$$

with the condition

$$v_3(0) = 2(N-2)v_1(0). \tag{2.10}$$

System (2.9) has two stationary points

$$P_1 = (0,0,0) \quad \text{and} \quad P_2 = (1,-2,0).$$

The linearization of (2.9) around P_1 is given by $X' = M_1 X$, with

$$M_1 = \begin{bmatrix} 2 & 0 & 0 \\ -2(N-2) & 2-N & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

The eigenvalues of M_1 are

$$\tilde{\nu}_1 = \tilde{\nu}_2 = 2, \quad \tilde{\nu}_3 = 2-N.$$

Thus for $N \geq 3$, $P_1 = (0,0,0)$ is a hyperbolic point, which has a 2-dimensional unstable manifold $W^u(P_1)$ and a 1-dimensional stable manifold $W^s(P_1)$.

The linearization of (2.9) around P_2 is given by $X' = M_2 X$, with

$$M_2 = \begin{bmatrix} 0 & 1 & 0 \\ -2(N-2) & 2-N & 1 \\ 0 & 0 & 2 \end{bmatrix}. \tag{2.11}$$

The eigenvalues of M_2 are given by

$$\nu_1 = 2, \quad \nu_{2,3} = \frac{(2-N) \pm \sqrt{(N-2)(N-10)}}{2}. \tag{2.12}$$

For $3 \leq N \leq 9$, ν_2 and ν_3 are complex conjugate and $Re(\nu_2) = Re(\nu_3) = \frac{2-N}{2} < 0$. For $N \geq 10$, all the eigenvalues are real and $\nu_1 > 0$, $\nu_2 < 0$, $\nu_3 < 0$. Thus for all $N \geq 3$, $P_2 = (1,-2,0)$ is a hyperbolic point, which has a 1-dimensional unstable manifold $W^u(P_2)$ and a 2-dimensional stable manifold $W^s(P_2)$. Actually $W^s(P_2)$ is contained in the plane $\{v_3 = 0\}$, which is invariant for (2.9).

Also we note that solutions of system (2.9) restricted to $\{v_3 = 0\}$ are related to radial solutions of the equation

$$-\Delta u = \lambda e^u \tag{2.13}$$

by exactly the same change of variables (2.6) and the first two equations in (2.8). This yields immediately a heteroclinic connection from P_1 to P_2, which is associated to the unique radial solution of (2.13) with $\lambda = 2(N-2)$ and initial condition $u(0) = u'(0) = 0$.

Proposition 2.7. *For $N \geq 3$, system (2.9) has a heteroclinic orbit from P_1 to P_2, which is contained in the plane $\{v_3 = 0\}$.*

Thanks to a result of Belickiĭ [9], we have the following Lemma.

Lemma 2.8. *System (2.9) is C^1-conjugate to its linearization around $P_2 = (1, -2, 0)$.*

Proof. We just need to check that none of the following relations

$$Re(\nu_i) = Re(\nu_j) + Re(\nu_k), \tag{2.14}$$

holds for different indices $i, j, k \in \{1, 2, 3\}$ such that $Re(\nu_j) < 0$ and $Re(\nu_k) > 0$, where ν_1, ν_2, ν_3 are corresponding eigenvalues of M_2. It is easy to check this by calculation for $N \geq 3$. □

Lemma 2.9. *Let $v^{(1)}$, $v^{(2)}$, $v^{(3)}$ be the eigenvectors of M_2 associated to ν_1, ν_2, ν_3. Then $v^{(k)} = (1, \nu_k, \nu_k(\nu_k - (2 - N)) + 2(N - 2))$ and $v^{(1)}$ is always real; for $3 \leq N \leq 9$, $v^{(2)}$, $v^{(3)}$ are complex conjugates. In particular the components of $v^{(1)} = (1, 2, 4(N - 1))$ are positive.*

Proof. By direct calculations, $v^{(k)} = (1, \nu_k, \nu_k(\nu_k - (2 - N)) + 2(N - 2))$ is an eigenvector associated to ν_k. □

2.3 Characterization of weakly singular solutions

In this section our aim is to prove Theorem 2.3. We assume that $u \in C^2(0, 1)$, $u \geq 0$ satisfies

$$-\Delta u = 2(N - 2)(e^u - 1) \quad \text{in } (0, 1), \tag{2.15}$$

where we assume, by using a scaling, that $\lambda = 2(N - 2)$. The scaling changes the length of the interval where the solution is defined, but this is not relevant for the next arguments, so we assume that the interval is (0, 1).

Define $v(t) = u(e^t)$, $w(t) = v(t) + 2t$ for $t \leq 0$. Then w satisfies

$$-w''(t) + (2 - N)w'(t) = 2(N - 2)\left(e^{w(t)} - e^{2t} - 1\right) \quad \text{for all } t \leq 0. \tag{2.16}$$

We also let v_1, v_2, v_3 be defined in (2.8).

By similar arguments as in [40], we have the following results.

Lemma 2.10. *One has*

$$\liminf_{t \to -\infty} w(t) \leq 0. \tag{2.17}$$

Proof. We follow [87]. Let $L := \liminf_{t \to -\infty} w(t)$ and suppose by contradiction that $L > 0$. Then there exists $T_0 > 0$, such that $w(t) \geq L/2$ for all $t \leq -T_0$. Let ϕ be a smooth cut-off function in $\mathbb{R}$ such that $0 \leq \phi(t) \leq 1$, $\phi(t) = 0$ for $t \leq -(T_0 + 3)$ and $t \geq -T_0$; $\phi(t) = 1$ for $t \in [-(T_0 + 2), -(T_0 + 1)]$, and for $i = 1, 2$

$$\int_{-(T_0+3)}^{-T_0} \frac{(\phi^{(i)})^2}{\phi}\, dt := c_i < +\infty.$$

Let $\tau > 1$ and $\phi_\tau(t) = \phi(\frac{t}{\tau})$. Multiplying (2.16) by ϕ_τ and integrating, we get

$$\int_{-(T_0+3)\tau}^{-T_0\tau} (e^{w(t)} - 1)\phi_\tau dt = \sum_{i=1}^{2} a_i \int_{-(T_0+3)\tau}^{-T_0\tau} w\phi_\tau^{(i)}\ dt + \int_{-(T_0+3)\tau}^{-T_0\tau} e^{2t}\phi_\tau dt, \tag{2.18}$$

where $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{2(N-2)}$. Using Young's inequality with $\varepsilon_1 > 0$ to be fixed later on, we have

$$\begin{aligned}\Big| \int_{-(T_0+3)\tau}^{-T_0\tau} w\phi_\tau^{(i)}\ dt\Big| &\leq \varepsilon_1 \int_{-(T_0+3)\tau}^{-T_0\tau} w^2\phi_\tau\ dt + C_{\varepsilon_1} \int_{-(T_0+3)\tau}^{-T_0\tau} \frac{(\phi_\tau^{(i)})^2}{\phi_\tau}\ dt \\ &\leq \varepsilon_1 \int_{-(T_0+3)\tau}^{-T_0\tau} w^2\phi_\tau\ dt + C_{\varepsilon_1} c_i \tau^{1-2i}. \end{aligned} \tag{2.19}$$

We also have

$$\int_{-(T_0+3)\tau}^{-T_0\tau} e^{2t}\phi_\tau\ dt \leq \frac{1}{2} e^{-2T_0\tau}. \tag{2.20}$$

From (2.18), (2.19), (2.20) we get

$$\int_{-(T_0+3)\tau}^{-T_0\tau} \left[e^{w(t)} - 1 - \varepsilon_1 K w(t)^2\right] \phi_\tau\ dt \leq C_{\varepsilon_1} K \max_{i=1,2} c_i \tau^{1-2i} + \frac{1}{2} e^{-2T_0\tau}$$

with $K = |a_1| + |a_2|$. Since $w(t) \geq L/2 > 0$ for all $t \leq -T_0$, we can choose $\varepsilon_1 > 0$ small, such that $e^{w(t)} - 1 - \varepsilon_1 K w(t)^2 \geq \varrho$ for $t \leq -T_0$, where $\varrho > 0$ is fixed. Then

$$\varrho\tau \leq \int_{-(T_0+3)\tau}^{-T_0\tau} \left[e^{w(t)} - 1 - \varepsilon_1 K w(t)^2\right] \phi_\tau\ dt \leq C_{\varepsilon_1} K \max_{i=1,2} c_i \tau^{1-2i} + \frac{1}{2} e^{-2T_0\tau},$$

which is impossible for $\tau > 1$ large. □

Lemma 2.11. *We have*

$$\limsup_{t\to-\infty} w(t) < +\infty.$$

Proof. Assume by contradiction that $\limsup_{t\to-\infty} w(t) = +\infty$. Then there is a sequence $t_k \to -\infty$ such that $w(t_k) \to +\infty$. Furthermore we can assume that for all $k \geq 1$ we have $t_{k+1} + \log 2 < t_k$, $w(t_{k+1}) \geq w(t_k)$.

Set $M_k = w(t_k)$, $r_k = e^{t_k}$ and $\rho_k = \frac{r_{k+1}}{r_k}$. Note that $0 < \rho_k < \frac{1}{2}$. Let $\eta_k(r) = \frac{N-2}{N} r_k^2 (1 - r^2)$ so that it satisfies

$$-\Delta\eta_k = 2(N-2)r_k^2 \quad \text{in } B, \qquad \eta_k = 0 \quad \text{on } \partial B.$$

Define

$$u_k(r) = u(rr_k) - M_k + 2\log(r_k) + \eta_k(r).$$

Then we have

$$-\Delta u_k(r) = 2(N-2)r_k^2 e^{u(r_k r)} = 2(N-2)e^{M_k-\eta_k(r)}e^{u_k(r)}, \quad \text{for } 0<r<r_k^{-1}.$$

Since η_k is bounded from above,

$$-\Delta u_k \geq C_0 e^{M_k} e^{u_k} \quad \forall 0<r<r_k^{-1}, \tag{2.21}$$

for some $C_0>0$ independent of k. Also note that

$$\begin{aligned} u_k(1) &= u(r_k) - M_k + 2t_k = 0,\\ u_k(\rho_k) &= M_{k+1} - M_k + 2(t_k - t_{k+1}) + \eta_k(\rho_k) \geq 0. \end{aligned}$$

Let $\lambda_{1,k}$ be the first eigenvalue for $-\Delta$ with Dirichlet boundary condition in the annulus $B\backslash B_{\rho_k}$ and $\phi_k>0$ be the corresponding eigenfunction, that is,

$$\begin{cases} -\Delta\phi_k = \lambda_{1,k}\phi_k, \quad \phi_k>0 & \text{in } B\backslash B_{\rho_k};\\ \phi_k = 0; & \text{on } \partial\left(B\backslash B_{\rho_k}\right), \end{cases}$$

normalized so that $\|\phi_k\|_{L^\infty(B)} = 1$. Multiplying (2.21) by ϕ_k and integrating in $B\setminus B_{\rho_k}$, we get

$$C_0 e^{M_k}\int_{B\backslash B_{\rho_k}} e^{u_k}\phi_k\,dx \leq \int_{\partial(B\backslash B_{\rho_k})}\frac{\partial\phi_k}{\partial\nu}u_k\,d\sigma + \lambda_{1,k}\int_{B\backslash B_{\rho_k}} u_k\phi_k\,dx.$$

But $u_k\geq 0$ and $\frac{\partial\phi_k}{\partial\nu}\leq 0$ on $\partial(B\backslash B_{\rho_k})$ so that

$$C_0 e^{M_k}\int_{B\backslash B_{\rho_k}} e^{u_k}\phi_k\,dx \leq \lambda_{1,k}\int_{B\backslash B_{\rho_k}} u_k\phi_k\,dx.$$

Now using the inequality $e^u\geq u$, it yields that

$$C_0 e^{M_k}\leq \lambda_{1,k}.$$

However, since the annulus $B\backslash B_{\rho_k}$ has a width that does not converge to zero, $\lambda_{1,k}$ remains uniformly bounded. It follows that M_k is bounded as $k\to\infty$, which is a contradiction. □

Lemma 2.12. *For $i=0,1,2$, we have*

$$|w^{(i)}(t)| \leq C(1+|t|) \quad \text{for all } \ t\leq 0, \tag{2.22}$$

and for all $i=1,2,3$

$$|v_i(t)| \leq C(1+|t|) \quad \text{for all } \ t\leq 0. \tag{2.23}$$

Proof. Since $u \geq 0$ and w is bounded above, we have $|w(t)| \leq C(1+|t|)$. Moreover, by equation (2.16), and interpolation inequalities such as in Chapter 6 of [70], we get that for any $t \leq -1$ and $i = 1, 2$

$$\begin{aligned} |w^{(i)}(t)| &\leq C \sup_{[t-1,t+1]} \left(|w| + 2(N-2)|e^w - e^{2t} - 1|\right) \\ &\leq C \sup_{[t-1,t+1]} \left(|w| + 2(N-2)|e^w - 1|\right). \end{aligned}$$

Since w is bounded above, the second term in the supremum is bounded. Then (2.22) and (2.23) follow from the bound of w. □

Lemma 2.13. *For $i = 1, 2, 3$*

$$|v_i(t)| \leq C \quad \text{for all} \quad t \leq 0, \tag{2.24}$$

for $i = 1, 2$

$$|w^{(i)}(t)| \leq C \quad \text{for all} \quad t \leq 0. \tag{2.25}$$

Proof. It is direct that v_3 is bounded for all $t \leq 0$. Since $v_1(t) = e^{w(t)}$ (recall the change of variables (2.8) and that we assume $\lambda = 2(N-2)$) and w is bounded above, we have $v_1(t)$ is bounded as $t \to -\infty$. Next we prove that v_2 is bounded for all $t \leq 0$.

Integrating the following equation

$$\frac{d}{ds}\left(v_2(s)e^{(N-2)s}\right) = \left[-2(N-2)v_1(s) + v_3(s)\right]e^{(N-2)s}$$

in $[t, t_0]$ with $t \leq t_0 \leq 0$, we get

$$\begin{aligned} v_2(t) = e^{-(N-2)t}\Big(v_2(t_0)e^{(N-2)t_0} + 2(N-2)\int_t^{t_0} e^{(N-2)s}v_1(s)\,ds \\ - \frac{2(N-2)}{N}(e^{Nt_0} - e^{Nt})\Big). \end{aligned}$$

Since v_1 is bounded, the integral $\int_{-\infty}^{t_0} e^{(N-2)s}v_1(s)\,ds$ exists. If

$$\frac{2(N-2)}{N}e^{Nt_0} - 2(N-2)\int_{-\infty}^{t_0} e^{(N-2)s}v_1(s)\,ds \neq v_2(t_0)e^{(N-2)t_0},$$

we deduce that $|v_2(t)|$ grows exponentially as $t \to -\infty$, which contradicts (2.23). Therefore we get

$$v_2(t_0) = -2(N-2)e^{-(N-2)t_0}\int_{-\infty}^{t_0} e^{(N-2)s}v_1(s)\,ds + \frac{2(N-2)}{N}e^{2t_0} \quad \forall\, t_0 \leq 0, \tag{2.26}$$

It follows that $|v_2(t)| \leq C$ for all $t \leq 0$, because v_1 is bounded.

Finally, the relations

$$w'(t) = v_2 + 2, \qquad w''(t) = -2(N-2)v_1 + (2-N)v_2 + v_3,$$

imply (2.25). □

Proof of Theorem 2.3. The statements in the theorem are consequence of the following properties, that we will prove next:

(i) If $\liminf_{t\to-\infty} w(t) = -\infty$, then $w(t) \to -\infty$, $v_i(t) \to 0$ as $t \to -\infty$ for $i = 1, 2, 3$, and u is a regular solution.

(ii) If $\liminf_{t\to-\infty} w(t) > -\infty$, then $w(t) \to 0$, $(v_1, v_2, v_3) \to P_2$ as $t \to -\infty$, and u is a weakly singular solution.

To prove these claims it is useful to define

$$E(t) = \frac{1}{2}(w'(t))^2 + 2(N-2)(e^{w(t)} - w(t)) - (N-2)C_1 e^{2t},$$

where $C_1 > 0$ is a constant such that $|w'(t)| \leq C_1$ for all $t \leq 0$. This constant exists thanks to Lemma 2.13. Let us compute

$$E'(t) = (w(t)'' + 2(N-2)(e^{w(t)} - 1))w(t)' - 2(N-2)C_1 e^{2t}$$

for $t \leq 0$. Using equation (2.16) we get

$$E'(t) = -(N-2)w'(t)^2 + 2(N-2)e^{2t}(w'(t) - C_1) \leq 0. \tag{2.27}$$

Let us prove (i) and so we assume $\liminf_{t\to-\infty} w(t) = -\infty$. First, we show that $w(t) \to -\infty$ as $t \to -\infty$. By contradiction, we assume that $w(t)$ does not tend to $-\infty$ as $t \to -\infty$. Then we can find sequences $s_k \to -\infty, \tau_k \to -\infty$, such that $s_k > \tau_k$,

$$w(s_k) \to -\infty, \quad w(\tau_k) \quad \text{is bounded.}$$

But then $E(\tau_k)$ is bounded and $E(s_k) \to \infty$ as $k \to \infty$. However, by (2.27), $E(s_k) \leq E(\tau_k)$, which is a contradiction.

Now, since $w(t) \to -\infty$ as $t \to -\infty$, we can easily deduce $v_1(t) \to 0$ as $t \to -\infty$. Using formula (2.26), we obtain $v_2(t) \to 0$ as $t \to -\infty$. Therefore $\lim_{t\to-\infty} V(t) = P_1$.

Since $v_2(t) \to 0$ as $t \to -\infty$, we have $\lim_{r\to 0} ru'(r) = 0$. Then for any $\epsilon > 0$, there exists $r_0 > 0$ such that for any $0 < r < r_0$, we have $|ru'(r)| < \epsilon$. Integrating from r to r_0 in this inequality, for any $0 < r < r_0$ we obtain

$$0 \leq u(r) \leq -\epsilon \ln r + C, \quad e^{u(r)} \leq Cr^{-\epsilon}, \tag{2.28}$$

for some $C > 0$.

We can then get that $u'(r)$ is bounded for $r > 0$ small enough. In fact, equation (2.1) can be written as

$$-\left(s^{N-1}u'(s)\right)' = \lambda s^{N-1}(e^{u(s)} - 1).$$

Integrating the above equation from δ to r with $(\delta, r) \subset (0, r_0)$ and using (2.28), letting $\delta \to 0$, we have

$$|u'(r)| \leq Cr^{1-N} \int_0^r s^{N-1}\left(s^{-\epsilon} - 1\right)\, ds \leq C$$

for $0 < r < r_0$. From the boundedness of u' near $r = 0$ we also get that u is bounded near $r = 0$. This shows that u is regular.

We prove now (ii), so we assume that $\liminf_{t\to-\infty} w(t) > -\infty$. Since w is bounded above by Lemma 2.11, we have w is bounded. By Lemma 2.13, the derivatives of w are bounded, then we get that $E(t)$ is bounded as $t \to -\infty$. From the boundedness of E together with the boundedness of the derivatives of w and (2.27), we deduce that

$$\int_{-\infty}^{0} w'(t)^2 \, dt < +\infty. \tag{2.29}$$

Set $\psi_T(t) = w'(t + T)$, then we get that

$$\psi_T \to 0 \quad \text{in } L^2(0,1) \quad \text{as } T \to -\infty.$$

Moreover, ψ_T satisfies the equation

$$-\psi_T''(t) + (2 - N)\psi_T'(t) = 2(N - 2)e^{w(T+t)}\psi_T(t) - 4(N - 2)e^{2(T+t)}.$$

Using regularity theory, we have $\psi_T(\frac{1}{2}) \to 0$ and $\psi_T'(\frac{1}{2}) \to 0$ as $T \to -\infty$. Thus we obtain that $w'(t) \to 0$ as $t \to -\infty$ and similarly $w''(t) \to 0$ as $\to -\infty$. This implies that $\lim_{t\to-\infty} v'(t) = -2$. Since $v'(t) = u'(e^t)e^t$ we see that u is a weakly singular solution by the definition. We get in addition that $(v_1, v_2, v_3) \to (1, -2, 0)$ as $t \to -\infty$. That is, $\lim_{t\to-\infty} V(t) = P_2$. □

A direct corollary of the proof of Theorem 2.3 is the following.

Corollary 2.14. *Let u be a radial singular solution to (2.1) and let $V(t) = (v_1(t), v_2(t), v_3(t))$ be the corresponding trajectory to (2.9). Then $\lim\limits_{t\to-\infty} V(t) = P_2 = (1, -2, 0)$.*

As a consequence of Theorem 2.3 and Corollary 2.14, we have the following.

Corollary 2.15. *For u a radial solution of (2.1) we have:*

(a) u is regular if and only if $\lim\limits_{t\to-\infty} V(t) = P_1$;

(b) u is singular if and only if $\lim\limits_{t\to-\infty} V(t) = P_2$.

2.4 The unstable manifold at P_2

In this section, we study the unstable manifold of P_2 and prove Theorem 2.4. First we have the following result.

Proposition 2.16. *Let $V(t) = (v_1(t), v_2(t), v_3(t)) : (-\infty, T) \to \mathbb{R}^3$ be the trajectory in $W^u(P_2)$ such that $v_3'(t) > 0$ as $t \to -\infty$, where T is the maximal time of existence. Then there exists some $t < T$ such that $v_3(t) \geq 2(N - 2)v_1(t)$.*

Proof. First we observe that this trajectory satisfies

$$v_1'(t) > 0, \quad v_2'(t) > 0, \quad v_3'(t) > 0$$

for t close to $-\infty$ since the tangent vector to this trajectory becomes parallel to $(1, 2, 4(N-1))$ as it approaches P_2.

Let $z(t) = v_3(t) - 2(N-2)v_1(t)$ and by contradiction we assume that

$$z(t) < 0 \quad \text{for} \quad \forall t \in (-\infty, T). \tag{2.30}$$

First, we remark that

$$v_2(t) < 0 \quad \text{for} \quad \forall t \in (-\infty, T). \tag{2.31}$$

To prove this, let us suppose it fails, and so there is the first time $t_0 \in (-\infty, T)$, such that $v_2(t_0) = 0$. Since $\lim_{t\to-\infty} v_2(t) = -2$ we must have $v_2'(t_0) \geq 0$. But writing the second equation in (2.9) as

$$v_2'(t) = z(t) - (N-2)v_2(t)$$

we would get $z(t_0) \geq 0$, a contradiction with (2.30).

Using (2.9) and $v_2(t) < 0$ for all $t < T$ we can assert that the solution is defined for all t, that is $T = +\infty$. Indeed, the first equation in (2.9) yields

$$v_1(t) = v_1(t_0)e^{\int_{t_0}^{t}(2+v_2(s))\,ds} \tag{2.32}$$

Since $v_2(t) < 0$ we see that $v_1(t)$ cannot blow up as $t \to T$, if T were finite. Also v_3 cannot blow up. This and the linearity of the second equation in (2.9) yield that $T = +\infty$.

Now, let us establish that

$$v_1(t) > 0 \quad \text{for} \quad \forall t \in (-\infty, +\infty). \tag{2.33}$$

In fact, this is valid for t near $-\infty$ since $v_1(t) \to 1$ as $t \to -\infty$. If inequality (2.33) does not hold, then $v_1(t_0) = 0$ for some t_0, and it follows from (2.32) that $v_1(t) = 0$ for all t, a contradiction.

Next, we prove that

$$\limsup_{t\to+\infty} v_2(t) = 0. \tag{2.34}$$

Indeed, suppose not, we assume that there is a small number $\delta > 0$ such that $v_2(t) < -\delta < 0$ for all t. From the first equation in (2.9), we then get $v_1'(t) < (2-\delta)v_1(t)$, so we have $v_1(t) < v_1(0)e^{(2-\delta)t}$ for all $t > 0$. But by the third equation in (2.9), we have $v_3(t) = v_3(0)e^{2t}$. Hence $z(t) = v_3(0)e^{2t} - 2(N-2)v_1(0)e^{(2-\delta)t} \geq 0$ for some $t > 0$, which contradicts assumption (2.30).

From (2.31) and (2.34), there exists a sequence (t_k) with $t_k \to +\infty$ as $k \to +\infty$, such that

$$v_2'(t_k) > 0, \quad \text{and} \quad v_2(t_k) \to 0 \quad \text{as } k \to +\infty.$$

Moreover, by the second equation in (2.9) we have $0 > z(t_k) = v_2'(t_k) + (N-2)v_2(t_k) > (N-2)v_2(t_k)$. Therefore,

$$z(t_k) \to 0 \quad \text{as } k \to +\infty. \tag{2.35}$$

From (2.9), we have $z'(t) - 2z(t) = -2(N-2)v_1(t)v_2(t)$. Multiplying by e^{-2t} and integrating from t to t_k, we get

$$z(t_k) = e^{2(t_k - t)} \left(z(t) - 2(N-2)e^{2t} \int_t^{t_k} e^{-2s} v_1(s) v_2(s) ds \right) \tag{2.36}$$

From (2.31), (2.33), (2.35) and (2.36) we have that

$$\int_t^{+\infty} e^{-2s} v_1(s)|v_2(s)| ds < +\infty \quad \text{for any } t < +\infty. \tag{2.37}$$

Note that $v_1(t) = \frac{v_3(t) - z(t)}{2(N-2)}$ and hence

$$z'(t) - 2z(t) = (z(t) - v_3(t))v_2(t).$$

Multiplying by e^{-2t} and integrating from 0 to t_k, we find

$$z(t_k) = e^{2t_k} \left(z(0) + \int_0^{t_k} e^{-2s} z(s) v_2(s) ds - \int_0^{t_k} e^{-2s} v_2(s) v_3(s) ds \right).$$

Since $z(0) < 0$, $\int_0^{t_k} e^{-2s} z(s) v_2(s) ds$ and $-\int_0^{t_k} e^{-2s} v_2(s) v_3(s) ds$ are positive, we get

$$\int_0^{+\infty} e^{-2s} |v_2(s)| v_3(s) ds < +\infty. \tag{2.38}$$

Since $v_3(t) = v_3(0)e^{2t}$, (2.38) implies that

$$\int_0^{+\infty} |v_2(s)| ds < +\infty. \tag{2.39}$$

Since $z(t) < 0$ by assumption, we have $v_2(s) \le v_2(0)e^{-(N-2)s}$ for $s \ge 0$. Then for $t \ge 0$,

$$\begin{aligned} \int_t^{+\infty} e^{-2s} v_1(s)|v_2(s)| ds &= -\int_t^{+\infty} e^{-2s} v_1(s) v_2(s) ds \\ &\ge -v_2(0) \int_t^{+\infty} e^{-Ns} v_1(s) ds. \end{aligned} \tag{2.40}$$

Integrating by parts and using (2.9) we get

$$\int_t^\infty e^{-Ns} v_1(s)\,ds = \frac{1}{N}e^{-Nt}v_1(t) + \frac{1}{N}\int_t^\infty e^{-Ns}v_1'(s)\,ds$$
$$= \frac{1}{N}e^{-Nt}v_1(t) + \frac{2}{N}\int_t^\infty e^{-Ns}v_1(s)\,ds + \frac{1}{N}\int_t^\infty e^{-Ns}v_1(s)v_2(s)\,ds$$

and we deduce

$$\int_t^\infty e^{-Ns}v_1(s) = \frac{1}{N-2}e^{-Nt}v_1(t) + \frac{1}{N-2}\int_t^\infty e^{-Ns}v_1(s)v_2(s)\,ds.$$

Hence for $t > 0$, and since $v_2(s) < 0$

$$\int_t^\infty e^{-Ns}v_1(s) \geq \frac{1}{N-2}e^{-Nt}v_1(t) + \frac{1}{N-2}\int_t^\infty e^{-2s}v_1(s)v_2(s)\,ds. \tag{2.41}$$

From (2.40) and (2.41) we have

$$\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds \geq -\frac{v_2(0)}{N-2}v_1(t)e^{-Nt} + \frac{v_2(0)}{N-2}\int_t^{+\infty} v_1(s)|v_2(s)|e^{-2s}ds,$$

which implies that

$$\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds \geq \frac{-v_2(0)}{N-2-v_2(0)}v_1(t)e^{-Nt}. \tag{2.42}$$

Now, from (2.35) and (2.36) we have

$$-z(t) = 2(N-2)e^{2t}\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds. \tag{2.43}$$

From(2.43) and (2.42), we observe that

$$-z(t) \geq \frac{-2(N-2)v_2(0)}{N-2-v_2(0)}v_1(t)e^{(-N+2)t}. \tag{2.44}$$

Moreover, using (2.39)

$$v_1(t) = v_1(0)e^{2t}e^{\int_0^t v_2(s)ds} = v_1(0)e^{2t}e^{-\int_0^t |v_2(s)|ds} \geq v_1(0)e^{-C}e^{2t} \tag{2.45}$$

for some constant $C > 0$. Hence,

$$-z(t) \geq \frac{-2(N-2)v_1(0)v_2(0)}{N-2-v_2(0)}e^{-C}e^{(4-N)t} := C_1e^{(4-N)t}, \tag{2.46}$$

for $C_1 > 0$, which is a contradiction with (2.35) for $N = 3, 4$.

From now on we assume $N > 4$. By the second equation in (2.9) and $z(t) = v_3(t) - 2(N-2)v_1(t)$, we get that

$$-v_2(t) = -v_2(0)e^{(2-N)t} + e^{(2-N)t}\int_0^t (-z(s))e^{(N-2)s}ds.$$

By (2.46) we have

$$\begin{aligned}|v_2(t)| = -v_2(t) &\geq -v_2(0)e^{(2-N)t} + C_1 e^{(2-N)t}\int_0^t e^{2s}ds \\ &\geq \frac{C_1}{2}e^{(2-N)t}(e^{2t}-1) \geq C_2 e^{(4-N)t},\end{aligned}$$

for $t > 1$ where C_2 is a positive constant. Therefore,

$$\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds \geq C_2\int_t^{+\infty} e^{(2-N)s}v_1(s)ds, \tag{2.47}$$

while, for $N > 4$ and $t > 0$

$$\begin{aligned}&\int_t^{+\infty} e^{(2-N)s}v_1(s)ds \\ =&\frac{1}{N-2}v_1(t)e^{(2-N)t} - \frac{1}{N-2}\int_t^{+\infty} e^{(2-N)s}v_1(s)|v_2(s)|ds \\ &\quad + \frac{2}{N-2}\int_t^{+\infty} e^{(2-N)s}v_1(s)ds \\ \geq&\frac{1}{N-2}v_1(t)e^{(2-N)t} - \frac{1}{N-2}\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds \\ &\quad + \frac{2}{N-2}\int_t^{+\infty} e^{(2-N)s}v_1(s)ds.\end{aligned}$$

So,

$$\int_t^{+\infty} e^{(2-N)s}v_1(s)ds \geq \frac{1}{N-4}v_1(t)e^{(2-N)t} - \frac{1}{N-4}\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds. \tag{2.48}$$

Combining (2.47) and (2.48), we get

$$\int_t^{+\infty} e^{-2s}v_1(s)|v_2(s)|ds \geq \frac{C_2}{N-4+C_2}v_1(t)e^{(2-N)t}. \tag{2.49}$$

Then, from (2.43), (2.45) and (2.49) we obtain that

$$-z(t) \geq \frac{2(N-2)C_2v_1(0)e^{-C}}{N-4+C_2}e^{(6-N)t} := C_3e^{(6-N)t}, \tag{2.50}$$

for $C_3 > 0$, which is a contradiction with (2.35) for $N = 5, 6$.

Starting with (2.50) we can do the same process and obtain a contradiction for all $N \geq 3$. This ends the proof of the proposition. □

Proposition 2.17. *At any point of $W^u(P_2) \cap \{v_3 = 2(N-2)v_1\}$ the intersection is transversal.*

Proof. Let $V(t) = (v_1, v_2, v_3)$ be a trajectory in $W^u(P_2)$ with t in some interval $(-\infty, T)$ and $\lim_{t\to-\infty} V(t) = P_2$. Suppose that t_1 is such that $v_3(t_1) = 2(N-2)v_1(t_1)$. By contradiction, assume that $V'(t_1)$ is not transversal to the plane $\{v_3(t) = 2(N-2)v_1(t)\}$, that is,

$$V'(t_1) \in \{v_3 = 2(N-2)v_1\}.$$

Then, $v_3(t_1) = 2(N-2)v_1(t_1)$, $v_3'(t_1) = 2(N-2)v_1'(t_1)$. From (2.9) we get $v_2(t_1) = 0$. Let $z(t) = v_3(t) - 2(N-2)v_1(t)$. The ODE (2.9) implies that

$$v_2' = z - (N-2)v_2, \quad z' = 2z - 2(N-2)v_1v_2.$$

Treating v_1 as a given function, we see that v_2, z satisfy a first order non-autonomous linear ODE and the initial condition $v_2(t_1) = 0$, $z(t_1) = 0$. Since $v_2 = z = 0$ is a solution of the ODE with the same initial condition, by uniqueness we deduce $v_2(t) = 0$ for all t where it is defined. This contradicts $\lim_{t\to-\infty} v_2(t) = -2$. □

Proof of Theorem 2.4. The existence of some $\lambda_* > 0$ such that (2.1) has a singular solution is a consequence of Proposition 2.16. Indeed, let $V(t) = (v_1(t), v_2(t), v_3(t)) : (-\infty, T) \to \mathbb{R}^3$ be the trajectory in $W^u(P_2)$ such that $v_3'(t) > 0$ as $t \to -\infty$, where T is the maximal time of existence. Then there exists some $t < T$ such that $v_3(t) \geq 2(N-2)v_1(t)$. Let t_1 be the first time such that $v_3(t_1) = 2(N-2)v_1(t_1)$. Because the system (2.9) is autonomous, by shifting time, we can assume $t_1 = 0$. Let $P^* = V(0)$ be the point of intersection, and write $P^* = (P_1^*, P_2^*, P_3^*)$. Then

$$u(r) = -2\log(r) + \log\left(\frac{2(N-2)v_1(\log(r))}{\lambda_*}\right)$$

is a singular solution of (2.1) for $\lambda_* = P_3^*$.

The uniqueness of λ_* such that a singular solution of (2.1) exists is a consequence of Corollary 2.15, which says that singular solutions must be associated to trajectories in $W^u(P_2)$, and the trajectory in $W^u(P_2)$ with tangent vector close $(1, 2, 4(N-1))$ as it approaches P_2 is unique except a shift in time. This also yields the uniqueness of the singular solution. □

2.5 Multiplicity result: proof of Theorem 2.5

In this section, we assume that $3 \leq N \leq 9$ and prove multiplicity of solutions to problem (2.1). Let $P_1 = (0, 0, 0)$ and $P_2 = (1, -2, 0)$ be the stationary points of (2.9). We recall that P_1 has a 2-dimensional unstable manifold $W^u(P_1)$ and 1-dimensional stable manifold $W^s(P_1)$, while P_2 has a 1-dimensional unstable manifold $W^u(P_2)$ and a 2-dimensional stable manifold $W^s(P_2)$.

From Corollary 2.15 it follows that each regular radial solution of (2.1) corresponds to exactly one point in $W^u(P_1) \cap \{v_3 = 2(N-2)v_1\}$. By Proposition 2.17, we define λ_* to be the height $v_3 = \lambda_*$ where $W^u(P_2)$ first intersects the plane $\{v_3 = 2(N-2)v_1\}$, and we denote this intersection point by

$$P^* = (P_1^*, P_2^*, P_3^*) = (\frac{\lambda_*}{2(N-2)}, P_2^*, \lambda_*). \tag{2.51}$$

Let $V_0 : \mathbb{R} \to \mathbb{R}^3$ be the heteroclinic connection from P_1 to P_2 contained in $\{v_3 = 0\}$ as stated in Proposition 2.7 and let $\hat{V}_0 = V_0(-\infty, +\infty)$. Then $\hat{V}_0$ is contained in both $W^u(P_1)$ and $W^s(P_2)$.

Lemma 2.18. *$W^u(P_1)$ and $W^s(P_2)$ intersect transversally on points of $\hat{V}_0$. More precisely, for points $Q \in \hat{V}_0$ sufficiently close to P_2, there are directions in the tangent plane to $W^u(P_1)$ which are almost parallel to $v^{(1)}$, the tangent vector to $W^u(P_2)$ at P_2.*

Proof. Let u_β be the solution of the following initial value problem

$$\begin{cases} -\Delta u_\beta(r) = 2(N-2)e^{u_\beta(r)} - \beta & \text{for } 0 < r < R(\beta), \\ u_\beta(0) = 0, \quad u'_\beta(0) = 0, \end{cases} \tag{2.52}$$

where $\beta \in \mathbb{R}$ is a parameter and $R(\beta) > 0$ is the maximal time of existence. We claim that $R(\beta) = +\infty$. Indeed, assume $R(\beta) < +\infty$ and fix $r_0 < R(\beta)$. Then for $r \in [r_0, R(\beta))$, from equation (2.52) we get

$$u'_\beta(r) = r_0^{N-1}u'_\beta(r_0)r^{1-N} - r^{1-N}\int_{r_0}^{r} t^{N-1}\left(2(N-2)e^{u_\beta(t)} - \beta\right)dt, \tag{2.53}$$

and this implies

$$u'_\beta(r) \le r_0^{N-1}u'_\beta(r_0)r^{1-N} + \frac{|\beta|}{N}(r - r^{1-N}r_0^N) \quad \text{for } r_0 \le r < R(\beta).$$

Integrating we see that

$$\limsup_{r\to R(\beta)} u_\beta(r) < +\infty.$$

Since u_β is bounded above in $[r_0, R(\beta))$, using again (2.53) we obtain

$$r_0^{N-1}u'_\beta(r_0)r^{1-N} - C(r - r^{1-N}r_0^N) \le u'_\beta(r) \quad \text{for } r_0 \le r < R(\beta),$$

and this shows that

$$\liminf_{r\to R(\beta)} u_\beta(r) > -\infty.$$

Control of u_β as $r \to R(\beta)$ also yields control of u'_β by (2.53) and this contradicts that $R(\beta)$ is the maximal time of existence. Therefore the solution $u_\beta(r)$ of (2.52) is defined for all $r > 0$.

Let $v_\beta(t) = u_\beta(r)$ with $r = e^t$ for $t \in (-\infty, +\infty)$ and set

$$v_{1,\beta}(t) = e^{v_\beta(t)+2t}, \quad v_{2,\beta} = v'_\beta(t), \quad v_{3,\beta}(t) = \beta e^{2t}.$$

Then $v_{1,\beta}$, $v_{2,\beta}$, $v_{3,\beta}$ satisfies system (2.9). Let $V_\beta = (v_{1,\beta}, v_{2,\beta}, v_{3,\beta})$. We have created in this way a family of trajectories in $W^u(P_1)$ with β as a parameter. Note that for $\beta = 0$, V_0 is just the heteroclinic connection of system (2.9) from P_1 to P_2 contained in the plane $\{v_3 = 0\}$ described in Proposition 2.7.

Define $X = \frac{\partial V}{\partial \beta}|_{\beta=0}$. Then X satisfies

$$X' = (M_2 + R(t))X \tag{2.54}$$

where M_2 is the matrix defined in (2.11) and

$$R(t) = \begin{bmatrix} v_{2,0}(t)+2 & v_{1,0}(t)-1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that there exist $C, \alpha > 0$, such that $|V_0(t) - P_2| \le Ce^{-\alpha t}$ for all $t \ge 0$, which follows for example from Lemma 2.8. Therefore $|R(t)| \le Ce^{-\alpha t}$ for all $t \ge 0$. Recall that the eigenvalues of M_2 are $\nu_1 > 0$ and ν_2, ν_3, which are complex conjugates with negative real part. Let $v^{(k)} \in \mathbb{C}^3$ be the eigenvector associated to ν_k. By Theorem 8.1 of Chapter 3 in [27], there are solutions ψ_k to

$$\psi'_k = (M_2 + R(t))\psi_k, \quad for \;\; t > 0$$

such that $\lim_{t\to\infty} \psi_k(t)e^{-\nu_k t} = v^{(k)}$. Then

$$X(t) = \sum_{k=1}^{3} c_k \psi_k$$

for some constants $c_1, c_2, c_3 \in \mathbb{C}$. Since ν_2, ν_3 have negative real parts, $\psi_k(t) \to 0$ as $t \to \infty$, for $k = 2, 3$. If $c_1 = 0$ then $X(t) \to 0$ as $t \to \infty$ and this contradicts $\frac{\partial v_{3,\beta}}{\partial \beta}|_{\beta=0}(t) = e^{2t} > 0$ for all $t \ge 0$. So $c_1 \ne 0$ and therefore

$$X(t) = c_1 v^{(1)} e^{\nu_1 t} + o(e^{\nu_1 t}) \quad as \;\; t \to \infty.$$

This shows $X(t)$ is almost parallel to $v^{(1)}$ as $t \to \infty$. Since $v^{(1)}$ is the tangent vector to $W^u(P_2)$, then $X(t)$ is not tangent to $W^s(P_2)$ for t large. On the other hand, $X = \frac{\partial V}{\partial \beta}|_{\beta=0}$ is tangent to $W^u(P_1)$. This implies $W^s(P_2)$ and $W^u(P_1)$ intersect transversally on points of $\hat{V}_0$ close to P_2. Since the flow is invertible near $\hat{V}_0$, $W^u(P_1)$ and $W^s(P_2)$ intersect transversally at every point of $\hat{V}_0$. □

We write (v_1, v_2, v_3) as points in the phase space $\mathbb{R}^3$ and let $\{e_1, e_2, e_3\}$ denote the canonical basis of $\mathbb{R}^3$.

We call $\mathcal{S} \subset \mathbb{R}^3$ a spiral around P^* if there exist independent vectors $\sigma_1, \sigma_2 \in \mathbb{R}^3$, a continuous positive function $\rho : [0, \infty) \to \mathbb{R}$ with $\rho(t) \to 0$ as $t \to \infty$, and $\omega \in \mathbb{R}$ such that

$$\mathcal{S} = \{P^* + \rho(t)\cos(\omega t)\sigma_1 + \rho(t)\sin(\omega t)\sigma_2 + o(\rho(t)) : t \ge 0\}.$$

Lemma 2.19. *$W^u(P_1) \cap \{v_3 = 2(N-2)v_1\}$ contains a spiral $\mathcal{S}$ around the point P^*.*

Proof. The linearization of (2.9) at P_2 is given by the system

$$\begin{cases} \bar{v}_1' = \bar{v}_2, \\ \bar{v}_2' = -2(N-2)\bar{v}_1 + (2-N)\bar{v}_2 + \bar{v}_3, \\ \bar{v}_3' = 2\bar{v}_3, \end{cases}$$

which is represented by the matrix M_2. Let $\bar{M}_2$ denote the matrix

$$\bar{M}_2 = \begin{bmatrix} \mathrm{Re}(\nu_2) & -\mathrm{Im}(\nu_2) & 0 \\ \mathrm{Im}(\nu_2) & \mathrm{Re}(\nu_2) & 0 \\ 0 & 0 & \nu_1 \end{bmatrix}$$

where ν_1, ν_2 are the eigenvalues (2.12). By Lemma 2.8, system (2.9) is C^1-conjugate in a neighborhood of P_2 to the flow generated by $\bar{M}_2$ around 0. More precisely, let X_t denote the flow generated by (2.9) and $Y_t = e^{\bar{M}_2 t}$. Then there are open neighborhoods $\mathcal{U}$ of P_2 and $\mathcal{V}$ of $\bar{O} = (0,0,0)$, and a C^1 diffeomorphism $\Phi : \mathcal{U} \to \mathcal{V}$ such that $Y_t(x) = \Phi \circ X_t \circ \Phi^{-1}(x)$ whenever $x \in \mathcal{V}$ and $\Phi^{-1}(x) \in \mathcal{U}$.

Let D be the 2-dimensional disk

$$D = \{V = (v_1, v_2, v_3) \ : \ v_3 = 2(N-2)v_1, \ |V - P^*| < r_0\},$$

where $r_0 > 0$ is fixed and small, so that $W^u(P_2) \cap \{v_3 = 2(N-2)v_1\}$ contains only the point P^*. This $r_0 > 0$ exists by Proposition 2.17. Also by this proposition, D is transversal to $W^u(P_2)$. Let $B^s \subset W^s(P_2) \cap \mathcal{U} \subset \{v_3 = 0\} \cap \mathcal{U}$ be an open neighborhood of P_2 relative to $W^s(P_2)$, which is diffeomorphic to a 2-dimensional disk. Define D_t as the connected component of $X_t(D) \cap \mathcal{U}$ that contains $X_t(P^*)$. We choose $\mathcal{U}$ smaller if necessary so that by the λ-Lemma of Palis [95], D_t is a C^1 manifold, which is C^1 close to B^s for t sufficiently negative. More precisely, let $\varepsilon > 0$ be small to be fixed later on. Then there exists $t_0 < 0$, $|t_0|$ large, such that for all $t \le t_0$, there is a diffeomorphism $\eta_t : D_t \to B^s$ such that $\|i' \circ \eta_t - i\|_{C^1(D_t)} \le \varepsilon$ where i, i' denote the inclusion maps. From now on we let $\mathcal{M} = D_{t_0}$.

We fix $Q \in \hat{V}_0$ such that $Q \in \mathcal{U}$ is sufficiently close to P_2. From Lemma 2.18, we can find a C^1 curve Γ contained in $W^u(P_1)$ of the form $\Gamma = \{\gamma(s) : \ |s| < \delta_0\}$ with $\gamma : (-\delta_0, \delta_0) \to \mathbb{R}^3$ a C^1 function such that $\gamma(0) = Q$ and $\gamma'(0)$ not tangent to $W^s(P_2)$ at Q. We can also assume that Γ is contained in $\mathcal{U}$ by taking δ_0 small. Choosing $\varepsilon > 0$ smaller if necessary we can assume that Γ intersects $\mathcal{M}$.

We want to prove that for $t > 0$ large, there is a point $P_t \in X_t(\Gamma) \cap \mathcal{M}$ and that the collection of points P_t describes a spiral around the point $X_{t_0}(P^*)$.

By the conjugation Φ, we will assume that P_2 is at the origin and near the origin the flow is given by $Y_t = e^{\bar{M}_2 t}$. Thus the image of $W^s(P_2) \cap \mathcal{U}$ through Φ is $\{(y_1, y_2, y_3) : y_3 = 0\}$, which is inside $\mathcal{V}$, and the image of B^s is $\{(y_1, y_2, y_3) : y_3 = 0, |y| < \delta\}$ for some $\delta > 0$.

Choosing ε small in the λ-Lemma, we can assume that the normal vector of $\widetilde{\mathcal{M}} := \Phi(\mathcal{M})$ near $\Phi(P^*)$ is almost parallel to $e_3 = (0,0,1)$. Thus by taking a subset of $\widetilde{\mathcal{M}}$, we may assume that $\widetilde{\mathcal{M}}$ is a C^1 graph with respect to the variables (y_1, y_2), that is, there exists a C^1 function $\varphi : \{\tilde{y} = (y_1, y_2) \in \mathbb{R}^2, |\tilde{y}| < \delta\} \to \mathbb{R}$ such that

$$\widetilde{\mathcal{M}} = \{(\tilde{y}, \varphi(\tilde{y})) : \tilde{y} \in \mathbb{R}^2, |\tilde{y}| < \delta\}.$$

Since $\gamma'(0)$ is not tangent to $W^s(P_2)$ at $\gamma(0)$, we have $\gamma_3'(0) \neq 0$. We may assume that $\varphi(\tilde{y}) > 0$ for $\tilde{y}$ near the origin and $\gamma_3'(0) > 0$.

We claim that for all $t > 0$ large there is a unique $s = s(t) > 0$ small so that $Y_t(\gamma(s)) \in \widetilde{\mathcal{M}}$. Indeed, this condition is equivalent to

$$e^{\nu_1 t}\gamma_3(s) = \varphi(e^{\nu_2 t}(\gamma_1(s) + i\gamma_2(s))). \tag{2.55}$$

Let $\tau = 1/t > 0$ and define, for $(\tau, s) \in (0, \delta_1) \times (-\delta_1, \delta_1)$ ($\delta_1 > 0$ a small fixed number)

$$F(\tau, s) = \gamma_3(s) - e^{-\nu_1/\tau}\varphi(e^{\nu_2/\tau}(\gamma_1(s) + i\gamma_2(s))).$$

Then, since $\nu_1 > \mathrm{Re}(\nu_2)$, F admits a C^1 extension to $\tau = 0$ and

$$F(0,s) = \gamma_3(s), \quad \frac{\partial F}{\partial \tau}(0,s) = 0, \quad \frac{\partial F}{\partial s}(0,s) = \gamma_3'(s).$$

Since $F(0,0) = 0$ and $\frac{\partial F}{\partial s}(0,0) > 0$, by the implicit function theorem, given $t > 0$ large there is a unique s small so that $F(1/t, s) = 0$. We obtain a C^1 function $s(t) > 0$ defined for all t large such that $Y_t(\gamma(s(t))) \in \widetilde{\mathcal{M}}$. Using (2.55) we see that

$$s(t) = \frac{e^{-\nu_1 t}}{\gamma_3'(0)}\varphi(0)(1 + o(1))$$

as $t \to \infty$. Writing $\nu_2 = \alpha + i\omega$, the point of intersection has the form

$$\tilde{P}_t = Y_t(\gamma(s(t))) = (0,0,\varphi(0,0)) + e^{\alpha t}\cos(\omega t)\tilde{\sigma}_1 + e^{\alpha t}\sin(\omega t)\tilde{\sigma}_2 + o(e^{\alpha t}),$$

where

$$\tilde{\sigma}_1 = \left(\gamma_1(0), \gamma_2(0), \frac{\partial \varphi}{\partial y_1}(0,0)\gamma_1(0) + \frac{\partial \varphi}{\partial y_2}(0,0)\gamma_2(0)\right),$$
$$\tilde{\sigma}_2 = \left(-\gamma_2(0), \gamma_1(0), -\frac{\partial \varphi}{\partial y_1}(0,0)\gamma_2(0) + \frac{\partial \varphi}{\partial y_2}(0,0)\gamma_1(0)\right).$$

Therefore the curve $\{\tilde{P}_t, : t > t_1\}$, where $t_1 > 0$ is large, defines a spiral contained in $\widetilde{\mathcal{M}}$. Applying the conjugation Φ^{-1} we obtain a collection of points $P_t = \Phi^{-1}(\tilde{P}_t)$ in $\mathcal{M} \cap X_t(\Gamma)$ that forms a spiral around $X_{t_0}(P^*)$. Applying the flow X_{-t_0} we see that

$$\mathcal{S} = \{X_{t-t_0}(\gamma(s(t))) : t \geq t_1\}$$

with $t_1 > 0$ large has the structure of a spiral around P^*. By construction $\mathcal{S}$ is contained in $W^u(P_1) \cap \{v_3 = 2(N-2)v_1\}$. □

Proof of Theorem 2.5. Let us define λ_* to be the height $v_3 = \lambda_*$, where $W^u(P_2)$ first intersects the boundary plane $\{v_3 = 2(N-2)v_1\}$. Define $H_\lambda = \{v_3 = \lambda\}$. If $\lambda = \lambda_*$, we know that P^* lies on the line $\{v_3 = \lambda_*, v_3 = 2(N-2)v_1\}$. From Lemma 2.19, $W^u(P_1) \cap \{v_3 = 2(N-2)v_1\}$ contains a spiral $\mathcal{S}$ around the point P^*. Since the plane H_λ is transversal to $\{v_3 = 2(N-2)v_1\}$, it is possible to show that H_{λ_*} and $\mathcal{S}$ intersect an infinite number of times, which means that problem (2.1) has infinitely many radial regular solutions; see for example Lemma 4 in [49]. If $\lambda \neq \lambda_*$, but λ is close to λ_*, we have that $H_\lambda \cap \mathcal{S}$ contains a large number of points, which means that problem (2.1) has a large number of radial regular solutions. □

2.6 Estimate the Morse index: proof of Theorem 2.6

In this section we always assume that $N \geq 10$ and prove Theorem 2.6. First we give the asymptotic behavior of a radial singular solution to problem (2.1) near the origin.

Lemma 2.20. *Assume that (λ_*, u_*) is a radial singular solution of (2.1). Then*

$$u_*(r) = -2\log r + \log\frac{2(N-2)}{\lambda_*} + r^2 + o(r^2) \quad \text{as} \quad r \to 0. \tag{2.56}$$

Proof. By Theorem 2.3, u_* is a weakly singular radial solution of (2.1). Define $v(t) = u_*(r)$ with $r = e^t$, and v_1, v_2, v_3 are given by (2.8). Therefore, from Corollary 2.15,

$$\lim_{t\to-\infty}(v_1, v_2, v_3) = (1, -2, 0).$$

By Lemma 2.8 and Lemma 2.9, we have

$$(v_1, v_2, v_3) = (1, -2, 0) + (1, 2, 4(N-1))e^{2t}\left(1 + o(e^{\delta t})\right) \quad \text{as} \quad t \to -\infty,$$

with $\delta > 0$ small. We then get

$$\begin{aligned}
u_*(r) = v(t) &= -2t + \log\frac{2(N-2)v_1(t)}{\lambda_*} \\
&= -2\log r + \log\frac{2(N-2)\left(1 + e^{2t} + o(e^{(2+\delta)t})\right)}{\lambda_*} \\
&= -2\log r + \log\frac{2(N-2)}{\lambda_*} + \log(1 + r^2 + o(r^{2+\delta})) \\
&= -2\log r + \log\frac{2(N-2)}{\lambda_*} + r^2 + o(r^2) \quad \text{as} \quad r \to 0.
\end{aligned}$$

□

For $\lambda > 0$, let us define

$$w(r) = -2\log r + \log\frac{2(N-2)}{\lambda} + \frac{\lambda}{2N}r^2, \tag{2.57}$$

Let $\rho > 0$ be a small number, which will be fixed later and let us write $c_\rho = w(\rho)$. Then w satisfies

$$\begin{cases} -\Delta w \le \lambda(e^w - 1) & \text{in } B_\rho, \\ w(\rho) = c_\rho & \text{on } \partial B_\rho, \end{cases} \tag{2.58}$$

where B_ρ is a ball with radius ρ and center at the origin.

We have the following stability property of w.

Lemma 2.21. *Suppose $N \ge 10$ and let w be defined in (2.57). There exists $\rho \in (0,1)$ small, such that w is stable in B_ρ, in the sense that*

$$\int_{B_\rho} |\nabla\varphi|^2 \ge \lambda \int_{B_\rho} e^w\varphi^2 \quad \text{for all } \varphi \in C_c^\infty(B_\rho). \tag{2.59}$$

Proof. Write $A = \frac{\lambda}{2N}$. Since $N \ge 10$,

$$\begin{aligned}
\int_{B_\rho} |\nabla\varphi|^2 - \lambda e^w\varphi^2 &= \int_{B_\rho} |\nabla\varphi|^2 - 2(N-2)\frac{\varphi^2}{r^2}e^{Ar^2} \\
&= \int_{B_\rho} \left(|\nabla\varphi|^2 - 2(N-2)\frac{\varphi^2}{r^2}\right) - 2(N-2)(A+o(1))\int_{B_\rho}\varphi^2 \\
&\ge \int_{B_\rho} \left(|\nabla\varphi|^2 - \frac{(N-2)^2}{4}\frac{\varphi^2}{r^2}\right) - 2(N-2)(A+o(1))\int_{B_\rho}\varphi^2,
\end{aligned}$$

where $o(1) \to 0$ as $\rho \to 0$. Let us recall the following improved Hardy's inequality from [15]: for $\varphi \in C_c^\infty(B_\rho)$

$$\int_{B_\rho}\left(|\nabla\varphi|^2 - \frac{(N-2)^2}{4}\frac{\varphi^2}{r^2}\right) \ge H_2\rho^{-2}\int_{B_\rho}\varphi^2,$$

where the constant H_2 is the first eigenvalue of the Laplacian in the unit ball in $N = 2$, hence it is positive and independent of N.

Choose $\rho > 0$ such that $2(N-2)(A + o(1)) \le H_2\rho^{-2}$. Then (2.59) holds. □

Lemma 2.22. *Let $\rho \in (0,1)$ be small and satisfy Lemma 2.21. Then for any radial regular solution u of (2.1) we have*

$$u(r) \le \begin{cases} w(r) & \text{in } B_\rho \\ c_\rho & \text{in } B\backslash B_\rho, \end{cases} \tag{2.60}$$

where $w(r)$ is defined in (2.57).

Proof. Arguing by contradiction, suppose there exists $r_0 \in (0, \rho)$, such that $u(r_0) = w(r_0)$. Then

$$\begin{cases} -\Delta u = \lambda(e^u - 1) & \text{in } B_{r_0}; \\ -\Delta w \leq \lambda(e^w - 1) & \text{in } B_{r_0}; \\ u = w & \text{on } \partial B_{r_0}. \end{cases} \tag{2.61}$$

Therefore,

$$\begin{cases} -\Delta(w - u) \leq \lambda\left(e^w - e^u\right) & \text{in } \ B_{r_0}, \\ w - u = 0 & \text{on } \partial B_{r_0}. \end{cases} \tag{2.62}$$

Multiplying by $(w-u)^+$ and integrating in (2.62), we obtain

$$\int_{B_{r_0}} |\nabla(w-u)^+|^2 \leq \lambda \int_{B_{r_0}} (e^w - e^u)(w-u)^+. \tag{2.63}$$

From Lemma 2.21, w is stable in B_{r_0}, by taking $\varphi = (w-u)^+$ in (2.59), we then have

$$\int_{B_{r_0}} |\nabla(w-u)^+|^2 - \lambda e^w ((w-u)^+)^2 \geq 0. \tag{2.64}$$

Combining (2.63) and (2.64), we get

$$\lambda \int_{B_{r_0}} e^w ((w-u)^+)^2 \leq \lambda \int_{B_{r_0}} (e^w - e^u)(w-u)^+.$$

We rewrite it as

$$\int_{B_{r_0}} \left[(e^w - e^u)(w-u)^+ - e^w((w-u)^+)^2\right] \geq 0.$$

By convexity, the integrand is nonpositive, therefore,

$$(e^w - e^u)(w-u)^+ - e^w((w-u)^+)^2 = 0 \qquad a.e. \quad \text{in } \ B_{r_0},$$

then

$$(w-u)^+ = 0 \qquad a.e. \quad \text{in } \ B_{r_0}.$$

It implies that $w \leq u$ in B_{r_0}, which is impossible because u is a radial regular solution. Then $u(r) \leq w(r)$ for $r \in (0, \rho)$.

Since u is a radially decreasing regular solution, $u \leq c_\rho$ in $B \backslash B_\rho$. □

Now, let (λ, u_λ) be any radial solution to (2.1) (regular or singular), and define the operator L_γ as

$$L_\gamma(\phi) = -\Delta\phi - \lambda e^{u_\lambda}\phi + \gamma\phi$$

with $\gamma > 0$ large but fixed. We have the following Lemma.

Lemma 2.23. *If $\gamma > 0$ is fixed large enough, we have:*
(a) for $N \geq 11$, $\langle L_\gamma(\phi), \phi\rangle \geq C_1\|\phi\|^2_{H_0^1(B)}$ for all $\phi \in C_c^\infty(B)$;
(b) for $N = 10$, $\langle L_\gamma(\phi), \phi\rangle \geq C_2\|\phi\|^2_{L^2(B)}$ for all $\phi \in C_c^\infty(B)$,
where C_1 and C_2 are positive constants.

Proof. For $\rho > 0$ small given in Lemma 2.21, from Lemmas 2.20 and 2.22, we have

$$\begin{aligned}
\langle L_\gamma(\phi), \phi\rangle &= \int_B L_\gamma(\phi)\phi = \int_B \left(|\nabla\phi|^2 - \lambda e^{u_\lambda}\phi^2 + \gamma\phi^2\right)\\
&= \int_B |\nabla\phi|^2 - \int_{B_\rho} \lambda e^{u_\lambda}\phi^2 - \int_{B\setminus B_\rho} \lambda e^{u_\lambda}\phi^2 + \int_B \gamma\phi^2\\
&\geq \int_B |\nabla\phi|^2 - 2(N-2)\int_{B_\rho} \frac{\phi^2}{r^2}(1 + Ar^2 + o(r^2)) - C\int_{B\setminus B_\rho} \phi^2 + \int_B \gamma\phi^2\\
&\geq \int_B \left(|\nabla\phi|^2 - 2(N-2)\frac{\phi^2}{r^2}\right) + [\gamma - \max\{2(N-2)(A+o(1)), C\}]\int_B \phi^2,
\end{aligned}$$

where $A = \frac{\lambda}{2N}$ for a radial regular solution u_λ, $A = 1$ for a radial singular solution u_λ, and $o(1) \to 0$ as $\rho \to 0$. Choose γ large such that the second term of above is nonnegative, we then get the conclusion by Hardy's inequality. □

We now define

$$\|\phi\|^2_H := \int_B \left(|\nabla\phi|^2 - \lambda e^{u_\lambda}\phi^2 + \gamma\phi^2\right)$$

which is a norm on $C_c^\infty(B)$ with associated inner product

$$(\phi, \varphi)_H = \int_B (\nabla\phi\nabla\varphi - \lambda e^{u_\lambda}\phi\varphi + \gamma\phi\varphi)$$

Completing $C_c^\infty(B)$ with respect to this norm we obtain a Hilbert space H. We denote by H^* the dual of H. We have $H_0^1(B) \subset H \subset L^2(B)$ and therefore $L^2(B) \subset H^* \subset H^{-1}(B)$. Actually by Lemma 2.23, if $N \geq 11$, the space H is just $H_0^1(B)$.

Given $h \in L^2(B) \subset H^*$ we consider the following problem

$$L_\gamma\phi = h \quad \text{in} \quad B, \qquad \text{and} \qquad \phi = 0 \quad \text{on } \partial B. \tag{2.65}$$

We say that $\phi \in H$ is a weak solution of problem (2.65) if

$$(\phi, \varphi)_H = \langle h, \varphi\rangle_{H^*,H} \qquad \text{for all } \varphi \in H.$$

By the Lax-Milgram theorem, for $h \in L^2(B)$, problem (2.65) has a unique weak solution $\phi \in H$.

Lemma 2.24. *Let $T : L^2(B) \to L^2(B)$ be the operator defined by $Th = \phi$, where ϕ is the solution of (2.65). Then T is compact and the natural embedding $H \hookrightarrow L^2(B)$ is compact.*

Proof. For $N \geq 11$, both statements hold since $T : L^2(B) \to H = H_0^1(B)$ and $H_0^1(B) \hookrightarrow L^2(B)$ is compact, by the Rellich-Kondrachov theorem. For $N = 10$, we observe that L_γ satisfies

$$\langle L_\gamma(\phi), \phi\rangle \geq c_r \|\phi\|^2_{L^r(B)} \quad \forall \phi \in C_c^\infty(B)$$

for $2 \leq r < \frac{2N}{N-2}$ where $c_r > 0$, thanks to an improved Hardy's inequality of Brezis and Vázquez [15]. Then the statements are proved in [36]. □

Proposition 2.25. *The radial singular solution (λ_*, u_*) of (2.1) has a finite Morse index.*

Proof. By Lemma 2.24, if $\gamma > 0$ is large, $(-\Delta - \lambda_* e^{u_*} + \gamma)^{-1}$ is well defined and compact from $L^2(B)$ into itself, and hence its spectrum except 0 consists of eigenvalues, and these eigenvalues form a sequence that converges to 0. Hence $-\Delta - \lambda_* e^{u_*}$ is negative definite on a finite dimensional space only. □

Next we prove a bound for the Morse index of any radial regular solution of (2.1).

Proposition 2.26. *There is an integer $K \geq 1$ independent of λ, such that for any radial regular solution u_λ of (2.1) we have*

$$1 \leq m(u_\lambda) \leq K, \tag{2.66}$$

where $m(u_\lambda)$ denotes the Morse index of u_λ.

Proof. From (2.1) we get

$$\int_B |\nabla u_\lambda|^2 = \lambda \int_B (e^{u_\lambda} - 1)u_\lambda.$$

Therefore,

$$\int_B \left(|\nabla u_\lambda|^2 - \lambda e^{u_\lambda} u_\lambda^2\right) = \lambda \int_B \left(e^{u_\lambda} - 1 - e^{u_\lambda} u_\lambda\right) u_\lambda < 0,$$

so $m(u_\lambda) \geq 1$.

We prove the proposition by contradiction. Suppose that $\{(\lambda_n, u_n)\}$ is a sequence of radial regular solutions of problem (2.1) and assume that $m(u_n) \to \infty$ as $n \to \infty$. Let us write $m(u_n) = m_n$ and

$$L_n = -\Delta - \lambda_n e^{u_n}.$$

Let

$$E_n = span\left\{\varphi \in L^2(B) \ : \ \varphi \text{ is eigenvector of } L_n \text{ with negative eigenvalue}\right\}$$

so that $dim(E_n) = m_n$. Since L_n is symmetric there exist eigenfunctions $\varphi_{1,n}, \dots \varphi_{m_n,n} \in E_n$, namely

$$\begin{cases} L_n \varphi_{i,n} = \mu_{i,n}\varphi_{i,n} & \text{in } B, \\ \varphi_{i,n} = 0 & \text{on } \partial B, \end{cases}$$

with $\mu_{i,n} < 0$, that form an orthonormal basis of E_n in $L^2(B)$ sense, that is

$$\int_B \varphi_{i,n}\varphi_{j,n} = \delta_{ij} \quad \text{for } i,j \in \{1,2,\cdots m_n\}, \tag{2.67}$$

where δ_{ij} is Kronecker's delta.

Multiplying by $\varphi_{i,n}$ and integrating on B, we find

$$\int_B \left(|\nabla\varphi_{i,n}|^2 - \lambda_n e^{u_n}\varphi_{i,n}^2\right) = \mu_{i,n}\int_B \varphi_{i,n}^2 < 0.$$

Then

$$\begin{aligned}
\int_B |\nabla\varphi_{i,n}|^2 &< \int_B \lambda_n e^{u_n}\varphi_{i,n}^2 = \int_{B_\rho} \lambda_n e^{u_n}\varphi_{i,n}^2 + \int_{B\setminus B_\rho} \lambda_n e^{u_n}\varphi_{i,n}^2 \\
&\le \int_{B_\rho} \lambda_n e^{-2\log r + \log\frac{2(N-2)}{\lambda_n} + A_n r^2}\varphi_{i,n}^2 + C\int_{B\setminus B_\rho} \varphi_{i,n}^2 \\
&= 2(N-2)\int_{B_\rho} \frac{\varphi_{i,n}^2}{r^2}(1 + A_n r^2 + o(r^2)) + C\int_{B\setminus B_\rho}\varphi_{i,n}^2 \\
&\le \frac{8}{N-2}\int_B |\nabla\varphi_{i,n}|^2 + \max\{2(N-2)(A_n + o(1)), C\}\int_B \varphi_{i,n}^2.
\end{aligned}$$

If $N \ge 11$ we deduce

$$\int_B |\nabla\varphi_{i,n}|^2 \le \frac{N-2}{N-10}\max\{2(N-2)(A_n+o(1)), C\},$$

where $A_n = \frac{\lambda_n}{2N}$. Let us assume $N \ge 11$ and leave the case $N = 10$ for later. Thus $(\varphi_{i,n})_n$ is bounded in $H_0^1(B)$. By a diagonal argument, there is a subsequence (which we write the same), such that for each $i \in \{1,2,\ldots\}$, $\varphi_{i,n} \rightharpoonup \varphi_i$ weakly in $H_0^1(B)$, $\varphi_{i,n} \to \varphi_i$ strongly in $L^2(B)$ and almost everywhere in B as $n \to +\infty$. Therefore for all $i \ge 1$,

$$\|\varphi_i\|_{H_0^1(B)} \le \liminf_{n\to+\infty}\|\varphi_{i,n}\|_{H_0^1(B)} \le C, \qquad \|\varphi_i\|_{L^2(B)} = 1.$$

Moreover, taking $n \to \infty$ in (2.67)

$$\int_B \varphi_i\varphi_j = \delta_{ij} \quad \text{for } i,j \ge 1. \tag{2.68}$$

Since $(\varphi_i)_{i\ge1}$ is bounded in $H_0^1(B)$, there is a subsequence $(\varphi_{i_j})_j$ of (φ_i) such that $\varphi_{i_j} \to \varphi$ in $L^2(B)$ as $j \to +\infty$, and $\|\varphi\|_{L^2(B)} = 1$. But from (2.68) we get

$$\int_B \varphi_{i_j}\varphi_{i_m} = 0 \quad \text{for } j \ne m.$$

Taking the limit, as $j \to +\infty$ and $m \to +\infty$, we have

$$\int_B \varphi^2 = 0,$$

which is a contradiction.

For $N = 10$, we define the Hilbert space H as the completion of $C_c^\infty(B)$ with respect to the norm

$$\|\phi\|_H^2 := \int_B \left(|\nabla \phi|^2 - \lambda_* e^{u_*} \phi^2 + \gamma \phi^2\right)$$

with $\gamma > 0$ large but fixed and u_* the radial singular solution of (2.1) with $\lambda = \lambda_*$. Then

$$\begin{aligned}\|\varphi_{i,n}\|_H^2 &= \int_B \left(|\nabla \varphi_{i,n}|^2 - \lambda_* e^{u_*} \varphi_{i,n}^2\right) + \gamma \int_B \varphi_{i,n}^2 \\ &= \mu_{i,n} \int_B \varphi_{i,n}^2 + \int_B \left(\lambda_n e^{u_n} - \lambda_* e^{u_*}\right) \varphi_{i,n}^2 + \gamma \int_B \varphi_{i,n}^2 \\ &< \int_B \left(\lambda_n e^{u_n} - \lambda_* e^{u_*}\right) \varphi_{i,n}^2 + \gamma \int_B \varphi_{i,n}^2.\end{aligned}$$

Let $\rho > 0$ be as in Lemma 2.21. Let $A_n = \frac{\lambda_n}{2N}$. From Lemma 2.20 and Lemma 2.22, we find

$$\begin{aligned}\int_B \left(\lambda_n e^{u_n} - \lambda_* e^{u_*}\right) \varphi_{i,n}^2 &= \int_{B_\rho} \left(\lambda_n e^{u_n} - \lambda_* e^{u_*}\right) \varphi_{i,n}^2 + \int_{B \setminus B_\rho} \left(\lambda_n e^{u_n} - \lambda_* e^{u_*}\right) \varphi_{i,n}^2 \\ &\le \int_{B_\rho} \left(\lambda_n e^{-2\log r + \log \frac{2(N-2)}{\lambda_n} + A_n r^2} - \lambda_* e^{-2\log r + \log \frac{2(N-2)}{\lambda_*} + r^2 + o(r^2)}\right) \varphi_{i,n}^2 \\ &\quad + C \int_{B \setminus B_\rho} \varphi_{i,n}^2 \\ &\le C \int_B \varphi_{i,n}^2.\end{aligned}$$

Thus we get

$$\|\varphi_{i,n}\|_H^2 \le (C + \gamma) \int_B \varphi_{i,n}^2 \le C.$$

That is, $(\varphi_{i,n})_n$ is bounded in H. By Lemma 2.24, the natural embedding $H \hookrightarrow L^2(B)$ is compact, so using the same argument as the case $N \ge 11$ we obtain a contradiction. This ends the proof of Proposition 2.26. □

Lemma 2.27. *Suppose that u_1, u_2 are radial regular solutions of (2.1) associated to the same parameter $\lambda > 0$. Then the graph of u_1 must intersect with the graph of u_2.*

Proof. By contradiction, assume that $u_1(r) > u_2(r)$ for any $r \in (0, 1)$, and set $v = u_1 - u_2$. By equation (2.1) we have

$$\begin{cases} -\Delta v = \lambda(e^{u_1} - e^{u_2}) > \lambda e^{u_2} v & \text{in } B; \\ v > 0 & \text{in } B; \\ v = 0 & \text{on } \partial B. \end{cases} \tag{2.69}$$

We consider the following eigenvalue problem

$$\begin{cases} -\Delta\psi = \lambda e^{u_2}\psi + \mu\psi & \text{in } B; \\ \psi > 0 & \text{in } B; \\ \psi = 0 & \text{on } \partial B. \end{cases} \tag{2.70}$$

Multiplying by ψ and v in (2.69) and (2.70) respectively, and then integrating on B, we get

$$\lambda \int_B e^{u_2}\psi v + \mu \int_B \psi v > \lambda \int_B e^{u_2}\psi v,$$

so $\mu > 0$, that is u_2 is a stable radial regular solution. Then $m(u_2) = 0$ and this contradicts Proposition 2.26. □

Proof of Theorem 2.6. The first part follows from Propositions 2.25 and 2.26.

Let K be an integer such that $m(u_\lambda) \leq K$ for any radial regular solution u_λ of (2.1) and $m(u_*) \leq K$. This integer exists by Propositions 2.25 and 2.26. Next we prove that the graph of any radial regular solution u_λ of (2.1) intersects with that of the radial singular solution u_* at most $2K+1$ times in $(0,1)$. We follow the idea of Theorem 1.2 in [71].

By contradiction, suppose that the graph of u_λ intersects with the graph of u_* at least $2K+2$ times in $(0,1)$. There are two cases: $\lambda < \lambda_*$ and $\lambda \geq \lambda_*$.

For $\lambda < \lambda_*$, we can show $m(u_\lambda) \geq K+1$, contradicting Proposition 2.26. Indeed, since the graph of (λ, u_λ) intersects with that of (λ_*, u_*) at least $2K+2$ times in $(0,1)$, there are at least $K+1$ intervals $J_i \subset (0,1)$ $(i = 1, 2, \cdots, K+1)$ such that $u_\lambda > u_*$ in J_i. Let

$$h_i = \begin{cases} u_\lambda - u_* & \text{in } J_i; \\ 0 & \text{in } (0,1)\backslash J_i. \end{cases}$$

Since u_λ and u_* satisfy equation (2.1), we have

$$\begin{aligned} -\Delta(u_\lambda - u_*) &= \lambda(e^{u_\lambda} - 1) - \lambda_*(e^{u_*} - 1) \\ &< \lambda(e^{u_\lambda} - e^{u_*}) \leq \lambda e^{u_\lambda}(u_\lambda - u_*). \end{aligned}$$

Therefore

$$Q_{u_\lambda}(h_i) = \int_B [|\nabla h_i|^2 - \lambda e^{u_\lambda} h_i^2] dx < 0.$$

Since the functions h_i, $i = 1, \ldots, K+1$ are linearly independent, we conclude that $m(u_\lambda) \geq K+1$.

For $\lambda \geq \lambda_*$, similarly we can obtain that $m(u_*) \geq K+1$. This contradicts Proposition 2.25. In fact, because the graph of u_λ intersects with that of u_* at least $2K+2$ times in $(0,1)$, there are at least $K+1$ intervals $J_k \subset (0,1)$ $(k = 1, 2, \cdots, K+1)$ such that $u_* > u_\lambda$ in J_k. Let

$$h_k = \begin{cases} u_* - u_\lambda & \text{in } J_k; \\ 0 & \text{in } (0,1)\backslash J_k. \end{cases}$$

Note that

$$-\Delta h_k < \lambda_* e^{u_*} h_k \quad in \quad J_k$$

and this implies

$$Q_{u_*}(h_k) = \int_B [|\nabla h_k|^2 - \lambda_* e^{u_*} h_k^2] dx < 0.$$

Therefore $m(u_*) \geq K + 1$.

Next we prove that the number of regular solutions to (2.1) is bounded by $(K+1)^2$ for each $\lambda \in (\lambda_0, \mu_1)$.

By contradiction, for each fixed $\lambda \in (\lambda_0, \mu_1)$, we suppose that there are at least $(K+1)^2 + 1$ radial regular solutions to (2.1), denoted by u_i $(i = 0, 1, \cdots, (K+1)^2)$. Without loss of generality, assume $u_0(0) > u_1(0) > \cdots > u_{(K+1)^2}(0)$. By Lemma 2.27, the graph of u_i, $i = 1, \cdots, (K+1)^2$, must intersect with that of u_0. Let a_i be the first point such that $u_i(a_i) = u_0(a_i)$ for $i = 1, \cdots, (K+1)^2$. Then there are the following two cases:
Case 1: There are at least $(K+1)$ different points a_i such that $u_0 - u_i > 0$ in $(0, a_i)$ and $u_i(a_i) = u_0(a_i)$.
Case 2: There exists some point $a_{i_0} \in (0, 1)$, such that there are at least $(K+1)$ regular solutions that intersect u_0 at a_{i_0}.
Case 1. We rearrange the indices so that $a_1 < \cdots < a_{K+1}$. Now $u_1(0), \ldots, u_{K+1}(0)$ are not necessarily ordered. Let $\varphi_i = (u_0 - u_i)\chi_{(0,a_i)}$. We claim that $\{\varphi_i : i = 1, 2, \cdots, (K+1)\}$ is linearly independent. Indeed, suppose that

$$\sum_{i=1}^{K+1} c_i \varphi_i = 0.$$

Since $a_{i-1} < a_i$, there exists $r_{i-1} \in (a_{i-1}, a_i)$, such that $\varphi_1(r_{i-1}) = 0,\ \varphi_2(r_{i-1}) = 0, \cdots, \varphi_{i-1}(r_{i-1}) = 0, \varphi_i(r_{i-1}) \neq 0$, then we can get $c_i = 0$, for $i = 1, 2, \cdots, (K+1)$. Then

$$\begin{aligned} Q_{u_0}(\varphi_i) &= \int_{\{|x|<a_i\}} [|\nabla \varphi_i|^2 - \lambda e^{u_0} \varphi_i^2] dx \\ &= \lambda \int_{\{|x|<a_i\}} [e^{u_0} - e^{u_i} - e^{u_0}(u_0 - u_i)](u_0 - u_i) dx < 0 \end{aligned}$$

by strict convexity and $u_0 - u_i > 0$ in $\{|x| < a_i\}$. This implies that $m(u_0) \geq K + 1$, contradicting Proposition 2.26.
Case 2. Rearranging indices, there are at least $K+1$ solutions $u_1, \cdots, u_{K+1}$ that satisfy $(u_0(r) - u_j(r)) > 0$ for $r \in (0, a_{i_0})$ and $u_j(a_{i_0}) = u_0(a_{i_0})$, $j = 1, \cdots, K+1$. Set $\varphi_j = (u_0 - u_j)\chi_{(0,a_{i_0})}$, we claim that

$$\{\varphi_j : j = 1, \cdots, K+1\} \text{ is linearly independent.} \tag{2.71}$$

Claim (2.71) together with $Q_{u_0}(\varphi_j) < 0$ yields that $m(u_0) \geq K + 1$, contradicting $1 \leq m(u_0) \leq K$.

Let us show that the claim (2.71) holds. From now on, we write $r_0 = a_{i_0}$. We assume that there exist c_j, $j = 1, \cdots, K+1$, such that

$$\sum_{j=1}^{K+1} c_j \varphi_j(r) = 0 \quad \text{for all} \quad r \in (0, r_0],$$

that is,

$$\sum_{j=1}^{K+1} c_j u_j(r) = \left(\sum_{j=1}^{K+1} c_j \right) u_0(r) \quad \text{for all} \quad r \in (0, r_0]. \tag{2.72}$$

We will deduce $c_1 = \cdots = c_{K+1} = 0$ from the following assertion:

$$\sum_{j=1}^{K+1} c_j (u_j'(r_0))^n = \left(\sum_{j=1}^{K+1} c_j \right) (u_0'(r_0))^n, \quad \text{for all integers } n \geq 0. \tag{2.73}$$

In the following we will establish (2.73). We denote $g^{(n)}$ the $n-$th derivative of g and set

$$f(u) := -\lambda(e^u - 1), \quad \forall u \in \mathbb{R}; \quad b = u_0(r_0).$$

Then $f^{(n)}(u_j(r_0)) = -\lambda e^b$ for any integer $n \geq 1$.

In order to prove (2.73), we shall show that for each $j \in \{0, 1, 2, \cdots, K+1\}$,

$$u_j^{(n)}(r_0) = P_n(u_j'(r_0)) \qquad \text{for any integer } n \geq 1, \tag{2.74}$$

where P_n is a polynomial of degree 1 for $n = 1, 2$, and of degree $n - 2$ for $n \geq 3$, whose coefficients depend only on N, n, r_0, and b.

Indeed, for $n = 1$, (2.74) is direct and for $n = 2$ this follows from equation (2.1). By induction, assume that (2.74) holds for $n = k \geq 2$. From equation (2.1), we have

$$(\Delta u_j)^{(k-1)} = (f(u_j))^{(k-1)}. \tag{2.75}$$

We see that for $n \geq 0$,

$$\begin{aligned} (\Delta u_j)^{(n)} &= u_j^{(n+2)} + \frac{N-1}{r} u_j^{(n+1)} - n \frac{N-1}{r^2} u_j^{(n)} \\ &\quad + n(n-1) \frac{N-1}{r^3} u_j^{(n-1)} - \cdots \\ &\quad + (-1)^{n-1} n! \frac{N-1}{r^n} u_j'' + (-1)^n n! \frac{N-1}{r^{n+1}} u_j', \end{aligned} \tag{2.76}$$

and by the formula for derivatives of a composition (e.g. Faà di Bruno [54]) we obtain

$$(f(u_j))^{(n)} = -\lambda e^{u_j} \sum_{\alpha_1, \ldots, \alpha_n} \frac{n!}{\alpha_1!(1!)^{\alpha_1} \alpha_2!(2!)^{\alpha_2} \cdots \alpha_n!(n!)^{\alpha_n}} \prod_{i=1}^{n} (u_j^{(i)})^{\alpha_i}, \tag{2.77}$$

where the sum ranges over integers $\alpha_1 \geq 0, \cdots, \alpha_n \geq 0$ with $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$. Using (2.75)-(2.77) with $n = k-1$ and $r = r_0$, we get

$$\begin{aligned} u_j^{(k+1)}(r_0) &= -\frac{N-1}{r_0} u_j^{(k)}(r_0) + (k-1)\frac{N-1}{r_0^2} u_j^{(k-1)}(r_0) - \cdots \\ &- (-1)^{k-2}(k-1)!\frac{N-1}{r_0^{k-1}} u_j''(r_0) - (-1)^{k-1}(k-1)!\frac{N-1}{r_0^k} u_j'(r_0) \\ &- \lambda e^b \sum_{\alpha_1,\ldots,\alpha_{k-1}} \frac{(k-1)!}{\alpha_1!(1!)^{\alpha_1}\alpha_2!(2!)^{\alpha_2}\cdots\alpha_{k-1}!((k-1)!)^{\alpha_{k-1}}} \prod_{i=1}^{k-1} (u_j^{(i)}(r_0))^{\alpha_i}, \end{aligned}$$

where the sum ranges over integers $\alpha_1 \geq 0, \cdots, \alpha_{k-1} \geq 0$ with $\alpha_1 + 2\alpha_2 + \cdots + (k-1)\alpha_{k-1} = k-1$. By the induction assumption (2.74), we have $\prod_{i=1}^{k-1} (u_j^{(i)}(r_0))^{\alpha_i}$ is a polynomial in $u_j'(r_0)$ of degree at most $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 3\alpha_5 + \cdots + (k-3)\alpha_{k-1} \leq k-1$. Thus we see the validity of (2.74).

Next we prove that (2.73) holds, again by induction. From (2.72), we have

$$\sum_{j=1}^{K+1} c_j u_j^{(n)}(r_0) = \left(\sum_{j=1}^{K+1} c_j\right) u_0^{(n)}(r_0) \text{ for any integer } n \geq 0, \tag{2.78}$$

and so (2.73) holds for $n = 0, 1$. Suppose (2.73) holds for $n = k$. By equation (2.1), we get

$$(\Delta u_j)^{(n)} = (f(u_j))^{(n)}. \tag{2.79}$$

Since $u_j(r_0) = u_0(r_0)$ for $j = 1, 2, \cdots, K+1$, from (2.76)-(2.79), we obtain for any integer $n \geq 0$,

$$\sum_{j=1}^{K+1} c_j \left((u_j'(r_0))^n + A_{j,n}\right) = \left(\sum_{j=1}^{K+1} c_j\right) \left((u_0'(r_0))^n + A_{0,n}\right) \tag{2.80}$$

where

$$A_{j,n} = \sum_{\alpha_1,\ldots,\alpha_n} \frac{n!}{\alpha_1!(1!)^{\alpha_1}\alpha_2!(2!)^{\alpha_2}\cdots\alpha_n!(n!)^{\alpha_n}} \prod_{i=1}^{n} (u_j^{(i)}(r_0))^{\alpha_i}$$

and the sum ranges over integers $0 \leq \alpha_1 < n, \alpha_2 \geq 0, \cdots, \alpha_n \geq 0$ with $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$. In writing (2.80) we have used again the formula for the n-th order derivative of a composition, where we have isolated one term. Consider (2.80) for $n = k+1$. By (2.74) we know that $\prod_{i=1}^{k+1} (u_j^{(i)}(r_0))^{\alpha_i}$ is a polynomial in $u_j'(r_0)$ of degree at most

$$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 3\alpha_5 + \cdots + (k-1)\alpha_{k+1}.$$

Since $0 \leq \alpha_1 < k+1$, we see that

$$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 3\alpha_5 + \cdots + (k-1)\alpha_{k+1} < \alpha_1 + 2\alpha_2 + \cdots + (k+1)\alpha_{k+1} = k+1$$

and therefore $A_{j,n}$ can be expressed as a polynomial in $u_j'(r_0)$ of degree at most k. Thus by the induction assumption, we have

$$\sum_{j=1}^{K+1} c_j A_{j,n} = \left(\sum_{j=1}^{K+1} c_j\right) A_{0,n}$$

and so (2.73) holds for any integer $n \geq 0$.

Finally we turn to the proof of (2.71), namely the linear independence of φ_j, $j = 1, \cdots, K+1$. We denote $u_0'(r_0) = d_0$, $u_j'(r_0) = d_j$ for $j = 1, 2, \cdots, K+1$. For $n = 1, 2, \cdots, K+1$, we can rewrite (2.73) as

$$\begin{pmatrix} d_1 - d_0 & d_2 - d_0 & \cdots & d_{K+1} - d_0 \\ d_1^2 - d_0^2 & d_2^2 - d_0^2 & \cdots & d_{K+1}^2 - d_0^2 \\ d_1^3 - d_0^3 & d_2^3 - d_0^3 & \cdots & d_{K+1}^3 - d_0^3 \\ \vdots & \vdots & \ddots & \vdots \\ d_1^{K+1} - d_0^{K+1} & d_2^{K+1} - d_0^{K+1} & \cdots & d_{K+1}^{K+1} - d_0^{K+1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{K+1} \end{pmatrix} = 0. \tag{2.81}$$

A calculation shows that the determinant of the coefficient matrix of (2.81) is equal to a $(K+2) \times (K+2)$ Vandermonde determinant and the value is

$$\prod_{0 \leq j < i \leq K+1} (d_i - d_j) \neq 0.$$

Thus $c_1 = c_2 = \cdots = c_{K+1} = 0$ and this ends the proof of Theorem 2.6. □

2.7 Appendix

Proof of Proposition 2.1. Suppose u is a classical solution of (2.1). Let $\phi_1 > 0$ be the first eigenfunction of $-\Delta$ corresponding to the first eigenvalue μ_1. Multiplying problem (2.1) by ϕ_1 and integrating over B, we find

$$\mu_1 \int_B u\phi_1 = \lambda \int_B (e^u - 1)\phi_1 > \lambda \int_B u\phi_1.$$

Thus $\lambda < \mu_1$.

Multiplying problem (2.1) by $x \cdot \nabla u$, and integrating over B, we have

$$-\int_B \Delta u (x \cdot \nabla u) = \lambda \int_B (e^u - 1)(x \cdot \nabla u). \tag{2.82}$$

But

$$
\begin{aligned}
-\int_B \Delta u(x \cdot \nabla u) &= -\frac{1}{2}\int_{\partial B} |\nabla u|^2 x \cdot \nu + \left(1 - \frac{N}{2}\right)\int_B |\nabla u|^2 \\
&\leq \left(1 - \frac{N}{2}\right)\int_B |\nabla u|^2, \qquad (2.83)
\end{aligned}
$$

since $x \cdot \nu \geq 0$ on ∂B. Moreover,

$$
\lambda \int_B (e^u - 1)(x \cdot \nabla u) = -\lambda N \int_B (e^u - 1 - u). \qquad (2.84)
$$

From (2.82)-(2.84), we get

$$
\left(\frac{N}{2} - 1\right)\int_B |\nabla u|^2 \leq \lambda N \int_B (e^u - 1 - u).
$$

We rewrite the above inequality as

$$
\frac{N-2}{4}\int_B |\nabla u|^2 \leq \lambda N \int_B (e^u - 1 - u) - \frac{N-2}{4}\int_B |\nabla u|^2.
$$

Multiplying equation (2.1) by u and substituting we get

$$
\frac{N-2}{4}\int_B |\nabla u|^2 \leq \lambda \int_B \left[N(e^u - 1 - u) - \frac{N-2}{4}(e^u - 1)u\right].
$$

The integrand on the right hand is negative for $u \geq C_0$, with C_0, a positive constant, so the integral can be restricted to the region $\{x : u(x) \leq C_0\}$ and in this region

$$
N(e^u - 1 - u) - \frac{N-2}{4}(e^u - 1)u \leq C_1 u^2.
$$

Thus

$$
\frac{N-2}{4}\int_B |\nabla u|^2 \leq \lambda C_1 \int_B u^2 \leq \lambda C_2 \int_B |\nabla u|^2,
$$

where $C_1 > 0$, $C_2 > 0$. This implies that $u = 0$ if $0 < \lambda < \frac{N-2}{4C_2}$. □

Chapter 3

A new critical curve for the Lane-Emden system

3.1 Introduction

We consider the Lane-Emden system

$$\begin{cases} -\Delta u = v^p, \ u > 0 & \text{in } \mathbb{R}^N, \\ -\Delta v = u^q, \ v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{3.1}$$

where $N \geq 1$ and $p \geq q \geq 1$. Introduced independently by Mitidieri [85] and Van der Vorst [114], the Sobolev critical hyperbola plays a crucial role in the analysis of (3.1). In particular, Mitidieri [86] (see also Serrin and Zou [109]) proved that (3.1) has a nontrivial radially symmetric solution if and only if (p, q) lies on or above the hyperbola i.e. when

$$\frac{1}{p+1} + \frac{1}{q+1} \leq 1 - \frac{2}{N}. \tag{3.2}$$

The Lane-Emden conjecture states that such a result should continue to hold for any positive solution (not necessarily radially symmetric). See Souplet [110] and the references therein for the progress on this conjecture.

In this chapter we characterize the stability of radially symmetric solutions of the Lane-Emden system (3.1). Let us now recall the definition of stable solution of system (3.1), see also Definition 1.15.

Definition 3.1. *A solution (u, v) to (3.1) is stable if there exists a positive supersolution of the linearized system i.e. if there exists $(\phi, \psi) \in C^2(\mathbb{R}^N)^2$ such that*

$$\begin{cases} -\Delta \phi \geq p v^{p-1} \psi & \text{in } \mathbb{R}^N, \\ -\Delta \psi \geq q u^{q-1} \phi & \text{in } \mathbb{R}^N, \\ \quad \phi, \psi > 0 & \text{in } \mathbb{R}^N. \end{cases}$$

Let us also recall that if (3.2) holds, then

$$(u_s, v_s) = (a|x|^{-\alpha}, b|x|^{-\beta}), \quad x \in \mathbb{R}^N \setminus \{0\} \tag{3.3}$$

is a weak solution of (3.1) provided

$$\alpha = \frac{2(p+1)}{pq-1}\,, \quad \beta = \frac{2(q+1)}{pq-1} \tag{3.4}$$

and $a = (ST^p)^{\frac{1}{pq-1}}$, $b = (S^qT)^{\frac{1}{pq-1}}$, $S = \alpha(N-2-\alpha)$, $T = \beta(N-2-\beta)$.

Our main result states that the stability of a radial solution of the Lane-Emden system is determined by the position of the exponents (p, q) with respect to a new critical curve, which we call "Joseph and Lundgren", since the exponent introduced by these authors in [73] is the intersection of the curve with the diagonal $p = q$.

Theorem 3.2. *Assume $p \geq q \geq 1$.*

(i) If $N \geq 11$ and (p, q) lies on or above the Joseph-Lundgren critical curve i.e.

$$\Big[\frac{(N-2)^2 - (\alpha-\beta)^2}{4}\Big]^2 \geq pq\alpha\beta(N-2-\alpha)(N-2-\beta), \tag{3.5}$$

then any radially symmetric solution (u, v) of (3.1) is stable and satisfies

$$u < u_s \quad \text{and} \quad v < v_s \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

where (u_s, v_s) is the singular solution given by (3.3) and α, β are the scaling exponents given by (3.4).

(ii) If $N \leq 10$ or if $N \geq 11$ and (3.5) fails, then there is no stable radially symmetric solution of (3.1).

Remark 3.3. *Equation (3.5) is derived by studying the stability of the singular solution (u_s, v_s) given by (3.3).*

Remark 3.4. • *The above theorem was first proved by Cowan for $1 \leq N \leq 10$, $p \geq q \geq 2$ and (u, v) not necessarily radial. See [30].*

• *In the case $p = q$, using Remarks 1.1(a) and 2.1(a) of Souplet [110] and Farina's seminal work for the case of a single equation [55], part (ii) of the theorem readily follows. The result continues to hold for possibly nonradial solutions, assumed to be stable only outside a compact set.*

• *In the biharmonic case $q = 1$, the theorem was first proved by Karageorgis [74] using the asymptotics found by Gazzola and Grunau in [63].*

• *In all the other cases, only partial results were known. To the authors knowledge, the state of the art for nonradial solutions is contained in the following references: Wei and D.*

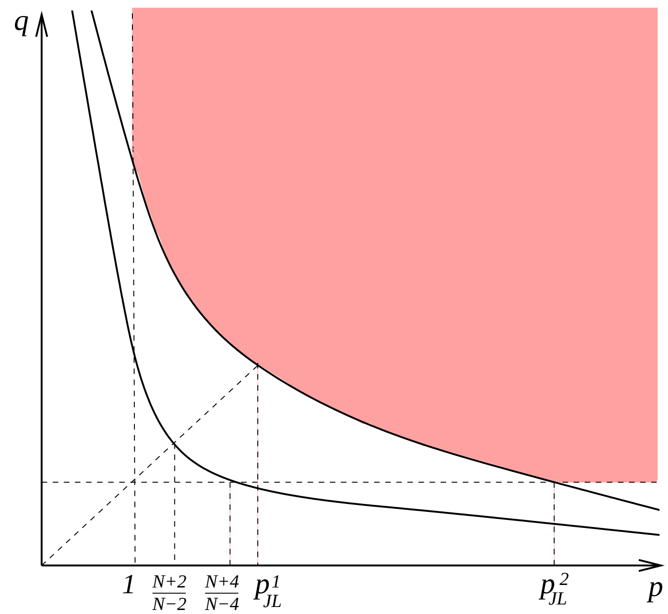

Figure 4: The stable region (shaded) for radially symmetric solutions of the Lane-Emden system (3.1).

Ye [117], Wei, Xu and Yang [115], Hajlaoui, A. Harrabi and D. Ye [72] for the biharmonic case, and Cowan [30] for the general case. We believe that the methods of the paper [52] by Goubet, Warnault and two of the authors should slightly improve the known results (and coincide with [72] in the biharmonic case).

- *Our result does not cover the case where one of the exponents is less than 1.*
- *The left hand-side in (3.5) is related to the following Hardy-Rellich inequality:*

$$\int_{\mathbb{R}^N} |x|^{2-\gamma}|\Delta\varphi|^2 dx \geq C_\gamma \int_{\mathbb{R}^N} |x|^{-2-\gamma}\varphi^2 dx. \tag{3.6}$$

The optimal constant C_γ in the class of radially symmetric functions $\varphi = \varphi(|x|)$ is given by

$$C_\gamma = \inf_{\substack{\varphi\in C_c^\infty(\mathbb{R}^N\setminus\{0\})\\ 0\neq\varphi=\varphi(|x|)}} \frac{\displaystyle\int_\Omega |x|^{2-\gamma}|\Delta\varphi|^2 dx}{\displaystyle\int_\Omega |x|^{-2-\gamma}\varphi^2 dx} = \Big[\frac{(N-2)^2-\gamma^2}{4}\Big]^2, \tag{3.7}$$

and the above infimum is never achieved. See Caldiroli and Musina [20]. We remark that the optimal constant C_γ in (3.7) corresponds to the left hand-side in (3.5) with $\gamma = \alpha - \beta \in [0, 2)$.

As an immediate corollary of Theorem 3.2 and standard blow-up analysis, we obtain the following regularity result.

Corollary 3.5. *Let B denote the unit ball of $\mathbb{R}^N$, $N \geq 1$, $\lambda, \mu > 0$. Let $f, g \in C^1(\mathbb{R})$ be two nondecreasing functions such that $f(0) \geq 0$, $g(0) > 0$, $f'(0)g'(0) > 0$ and*

$$\lim_{t\to+\infty} \frac{f'(t)}{t^{p-1}} = a, \qquad \lim_{t\to+\infty} \frac{g'(t)}{t^{q-1}} = b$$

for some $a, b > 0$, $p \geq q \geq 1$, $pq > 1$. Then, any extremal solution to the system

$$\begin{cases} -\Delta u = \lambda f(v), \ u > 0 & \text{in } B, \\ -\Delta v = \mu g(u), \ v > 0 & \text{in } B, \\ \ \ u = v = 0 & \text{on } \partial B \end{cases} \tag{3.8}$$

is bounded if either $N \leq 10$ or if $N \geq 11$ and (p, q) lies below the Joseph-Lundgren critical curve i.e. (3.5) *fails.*

For the notion of extremal solution for systems, we refer to Montenegro [88]. See also Cowan [29] for partial results on general domains. The proof is a straightforward adaptation of Theorem 1.9 in [37], using the version of the blow-up technique introduced by Polacik, Quittner and Souplet [100], so we skip it.

3.2 Preliminary results

The following three results will serve for the purpose of comparing solutions. In the lemma below, we say that a solution is strictly stable in a bounded region $\Omega \subset \mathbb{R}^N$ if the principal eigenvalue of the linearized equation with Dirichlet boundary conditions in Ω is strictly positive.

Lemma 3.6. *Let $(u, v) \in C^2(\mathbb{R}^N)^2$ be a stable solution of (3.1). Then, given any bounded domain $\Omega \subset \mathbb{R}^N$, (u, v) is strictly stable in Ω. In particular, the linearized operator satisfies the maximum principle, that is, any pair $(\phi, \psi) \in C^2(\overline{\Omega})^2$ such that*

$$\begin{cases} -\Delta\phi \geq pv^{p-1}\psi & \text{in } \Omega, \\ -\Delta\psi \geq qu^{q-1}\phi & \text{in } \Omega, \\ \ \ \phi, \psi \geq 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies $\phi, \psi \geq 0$ in Ω.

Proof. Since (u, v) is stable in $\mathbb{R}^N$, the linearized equation has a strict supersolution in Ω. As observed by Sweers [111] and Busca-Sirakov [18], this implies in turn that the principal eigenvalue of the linearized operator with Dirichlet boundary conditions in Ω is strictly positive and equivalently that the maximum principle holds.

In the next lemma, we say that a solution is minimal if it lies below any (local) supersolution of the same equation. See e.g. [50] for the notion of minimal solution.

Lemma 3.7. *Assume $p \geq q \geq 1$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $a, b \in C(\partial\Omega)$, $a, b \geq 0$. If $(u, v) \in C^2(\overline{\Omega})^2$ is a strictly stable solution of*

$$\begin{cases} -\Delta u = v^p & \text{in } \Omega, \\ -\Delta v = u^q & \text{in } \Omega, \\ \quad u = a(x)\,, v = b(x) & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

then (u, v) is minimal.

Proof. Assume that (u, v) is a strictly stable solution of (3.9). By the maximum principle,

$$u \geq \min_{\partial\Omega} a\,, \quad v \geq \min_{\partial\Omega} b \quad \text{ in } \Omega.$$

In particular, there exists the minimal solution (u_m, v_m) of (3.9) and

$$u \geq u_m \geq \min_{\partial\Omega} a\,, \quad v \geq v_m \geq \min_{\partial\Omega} b \quad \text{ in } \Omega.$$

Set $\phi = u - u_m$, $\psi = v - v_m$. Then, $\phi, \psi \geq 0$ in Ω and, since $p \geq q \geq 1$,

$$\begin{cases} -\Delta\phi = v^p - v_m^p \leq pv^{p-1}\psi & \text{in } \Omega, \\ -\Delta\psi = u^q - u_m^q \leq qu^{q-1}\phi & \text{in } \Omega, \\ \quad \phi = \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Since (u, v) is strictly stable, the maximum principle holds and implies that $\phi, \psi \leq 0$ in Ω. It follows that $\phi \equiv \psi \equiv 0$, that is, $u = u_m$ and $v = v_m$. □

As an immediate consequence of the two previous lemmas, we obtain

Corollary 3.8. *Let $(u, v) \in C^2(\mathbb{R}^N)^2$ be a stable solution of (3.1) and let (u_s, v_s) be the singular solution defined by (3.3). If there exists $R > 0$ such that $u(R) \leq u_s(R)$ and $v(R) \leq v_s(R)$, then*

$$u < u_s \quad \textit{and} \quad v < v_s \quad \textit{in } B_R \setminus \{0\}.$$

Proof. Since $u_s(0) = v_s(0) = \infty$, there exists $r \in (0, R)$ such that

$$u < u_s \quad \text{and} \quad v < v_s \quad \text{in } \overline{B_r} \setminus \{0\}. \tag{3.10}$$

We next apply Lemma 3.7 for $\Omega = B_R \setminus \overline{B_r}$, $a(x) = u$, $b(x) = v$. Thus (u, v) is the minimal solution of (3.9) and $u < u_s$, $v < v_s$ in $B_R \setminus \overline{B_r}$. This last inequality together with (3.10) yield the conclusion. □

3.2.1 Stability of the singular solution.

In this part we investigate the stability of the singular solution (u_s, v_s) given by (3.3).

Proposition 3.9. *The following are equivalent:*

(i) The singular solution (u_s, v_s) is stable in $\mathbb{R}^N \setminus \{0\}$;

(ii) The singular solution (u_s, v_s) is stable outside of some compact set;

(iii) (p, q) satisfies (3.5).

Proof. Since the implication $(i) \Rightarrow (ii)$ is trivial, we only need to prove the implications

$$(ii) \Rightarrow (iii) \Rightarrow (i)$$

Assume first that (ii) holds, that is, the singular solution (u_s, v_s) is stable outside of a compact set. Thus, (u_s, v_s) is stable in $\mathbb{R}^N \setminus \overline{B_r}$ for some $r > 0$. By scale invariance, (u_s, v_s) is stable in $\mathbb{R}^N \setminus \overline{B_\rho}$ for all $\rho > 0$.

Set $\gamma = \alpha - \beta$, where α, β are the scaling exponents given by (3.4) and let K_1, K_2 be the constants such that

$$pv_s^{p-1} = K_1|x|^{-2+\gamma} \qquad \text{and} \qquad qu_s^{q-1} = K_2|x|^{-2-\gamma}.$$

Then, (p, q) satisfies (3.5) if and only if

$$C_\gamma \geq K_1K_2,$$

where C_γ is given by (3.7). Assume by contradiction that (p, q) does not satisfy (3.5). Then, we may find an open annular region $\Omega = B_{R_1} \setminus \overline{B_{R_2}}$ such that

$$\lambda := \min_{\varphi \in H\setminus\{0\}} \frac{\displaystyle\int_\Omega |x|^{2-\gamma}|\Delta\varphi|^2 dx}{\displaystyle\int_\Omega |x|^{-2-\gamma}\varphi^2 dx} < K_1K_2, \tag{3.11}$$

where H is the space of radial functions φ such that $\int_\Omega |x|^{2-\gamma}|\Delta\varphi|^2 dx < +\infty$ and $\varphi = 0$ on $\partial\Omega$. Let $\varphi > 0$ be a minimizer of (3.11), so that letting $\psi = |x|^{2-\gamma}(-\Delta\varphi)$, we have

$$\begin{cases} -\Delta\varphi = |x|^{-2+\gamma}\psi, \ \varphi > 0 & \text{in } \Omega, \\ -\Delta\psi = \lambda|x|^{-2-\gamma}\varphi, \ \psi > 0 & \text{in } \Omega, \\ \ \varphi = \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Since (u_s, v_s) is strictly stable in Ω, thanks to [111, Theorem 1.1], there also exists $(\tilde\varphi, \tilde\psi) \in C^2(\overline{\Omega})^2$ such that

$$\begin{cases} -\Delta\tilde\varphi = K_1|x|^{-2+\gamma}\tilde\psi, \ \tilde\varphi > 0 & \text{in } \Omega, \\ -\Delta\tilde\psi = K_2|x|^{-2-\gamma}\tilde\varphi + 1, \ \tilde\psi > 0 & \text{in } \Omega, \\ \ \tilde\varphi = \tilde\psi = 0 & \text{on } \partial\Omega. \end{cases}$$

A straightforward integration by part shows that φ and $\tilde{\varphi}$ satisfy

$$\langle \varphi, \tilde{\varphi} \rangle := \int_{\Omega} |x|^{2-\gamma} \Delta\varphi \Delta\tilde{\varphi} dx \leq 0$$

which is impossible, since both ψ and $\tilde{\psi}$ are positive. Hence (p, q) satisfies (3.5) and we have proved that (ii) implies (iii).

Assume now (iii). It is easy to see that

$$\phi(x) = \frac{4K_1}{(N-2-\gamma)(N-2+\gamma)} |x|^{-\frac{N-2-\gamma}{2}} , \quad \psi(x) = |x|^{-\frac{N-2+\gamma}{2}} \tag{3.12}$$

satisfy

$$\begin{aligned} -\Delta\phi &= p v_s^{p-1} \psi \\ -\Delta\psi &\geq q u_s^{q-1} \phi \end{aligned} \tag{3.13}$$

in $\mathbb{R}^N \setminus \{0\}$, which means that (u_s, v_s) is stable in $\mathbb{R}^N \setminus \{0\}$. $\square$

3.3 Proof of Theorem 3.2

We start this section with the following simple remark.

Remark 3.10. *Let (u, v) be a radially symmetric solution of (3.1). Then*

$$\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = 0.$$

To see this, we first note that (u, v) satisfies

$$\begin{cases} -(r^{N-1}u')' = r^{N-1}v^p & \text{for all } r \geq 0, \\ -(r^{N-1}v')' = r^{N-1}u^q & \text{for all } r \geq 0. \end{cases} \tag{3.14}$$

This implies that $r \longmapsto r^{N-1}u'(r)$ and $r \longmapsto r^{N-1}v'(r)$ are decreasing on $[0, \infty)$ and so $u', v' \leq 0$ in $[0, \infty)$. Thus, u and v are decreasing in $[0, \infty)$. Hence, there exist

$$\ell_1 := \lim_{r\to\infty} u(r) \in [0, \infty) , \quad \ell_2 := \lim_{r\to\infty} v(r) \in [0, \infty),$$

and $u \geq \ell_1$, $v \geq \ell_2$ in $[0, \infty)$.

If $\ell_2 > 0$, then the first equation in (3.14) implies

$$-(r^{N-1}u')' \geq Cr^{N-1} \quad \text{for all } r \geq 0,$$

where $C = \ell_2^p > 0$. Integrating twice over $[0, r]$ in the above inequality we deduce

$$-u(r) + u(0) \geq \frac{C}{2N} r^2 \to \infty \quad \text{as } r \to \infty,$$

contradiction. Thus, $\ell_2 = 0$ and similarly $\ell_1 = 0$ which proves our claim.

Assume (p, q) satisfies (3.5). Then by Proposition 3.9, the singular solution (u_s, v_s) is stable in $\mathbb{R}^N \setminus \{0\}$.

Theorem 3.2(i) follows from the proposition below.

Proposition 3.11. *Assume (p, q) satisfies (3.5). Then for any radially symmetric solution (u, v) of (3.1), we have*

$$u < u_s \quad \text{and} \quad v < v_s \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{3.15}$$

Proof. Assume by contradiction that there exists a radially symmetric solution (u, v) of (3.1) for which (3.15) fails to hold and set

$$U = u_s - u\ , \quad V = v_s - v.$$

Since (3.15) is not fulfilled, U' and V' must change sign in $(0, \infty)$. Indeed, otherwise $U' < 0$ or $V' < 0$ in $(0, \infty)$ which implies (since $U(\infty) = V(\infty) = 0$) that $u_s \geq u$ or $v_s \geq v$ in $(0, \infty)$. Now, the maximum principle yields $u_s \geq u$ and $v_s \geq v$ in $(0, \infty)$ and this contradicts our assumption.

Let $r_1 > 0$ (resp. $r_2 > 0$) be the first zero of U' (resp. V'). Thus

$$U' < 0 \text{ in } (0, r_1),\ U'(r_1) = 0, \quad V' < 0 \text{ in } (0, r_2),\ V'(r_2) = 0.$$

Without losing the generality, we may assume $r_2 \geq r_1$. Set next

$$r_3 := \inf\{r > 0 : V(r) < 0\} \in (0, \infty]$$

and we claim that $r_3 < r_1$. If $r_3 \geq r_1$ then $V > 0$ in $(0, r_1)$ which means

$$v < v_s \quad \text{in } (0, r_1). \tag{3.16}$$

Integrating in (3.1) and using (3.16) we find

$$(r^{N-1}u')' = -r^{N-1}v^p > -r^{N-1}v_s^p = (r^{N-1}u_s')' \quad \text{in } (0, r_1).$$

Integrating the above inequality over $[0, r_1]$ we find $u'(r_1) > u_s'(r_1)$ which contradicts $U'(r_1) = 0$. Hence $r_3 \in (0, r_1)$. Similarly we define

$$r_4 := \inf\{r > 0 : U(r) < 0\} \in (0, \infty]$$

and as before we deduce $r_4 \in (0, r_2)$. In fact, we show that $r_4 \leq r_1$. Assuming the contrary, that is, $r_4 > r_1$, we find $r_1 < r_4 < r_2$. Further, since $V' < 0$ in $(0, r_2)$ we deduce $V(r) < V(r_3) = 0$ for all $r \in (r_3, r_2)$ so $v_s < v$ in (r_3, r_2). Therefore,

$$(r^{N-1}u')' = -r^{N-1}v^p < -r^{N-1}v_s^p = (r^{N-1}u_s')' \quad \text{in } (r_3, r_2).$$

Integrating over $[r_1, r]$, $r_1 < r < r_2$, and using $U'(r_1) = 0$ we obtain $u'(r) < u_s'(r)$ for all $r \in (r_1, r_2)$. This means that U is increasing in (r_1, r_2). In particular, $U(r_1) < U(r_4) = 0$. On the other hand, from the definition of r_4 we have $U(r_1) > 0$, contradiction. We have thus obtained $r_3 < r_1$, $r_4 \le r_1 \le r_2$ which yield

$$U(r_1) \le 0,\ U'(r_1) = 0,\ V(r_1) < 0,\ V'(r_1) \le 0. \tag{3.17}$$

Next, let (ϕ, ψ) be defined by (3.12) and recall that (ϕ, ψ) solves the linearized equation (3.13) in $\mathbb{R}^N \setminus \{0\}$. Also, since $p \ge q \ge 1$, (U, V) satisfies

$$\begin{cases} -\Delta U \le p v_s^{p-1} V \\ -\Delta V \le q u_s^{q-1} U \end{cases} \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{3.18}$$

We multiply the equations in (3.13) by V and U, and the two equations in (3.18) by ψ and ϕ respectively. Integrating over B_r, $r > 0$, we find

$$\int_{B_r} (-\Delta U)\psi \le \int_{B_r} (-\Delta \phi) V \quad \text{and} \quad \int_{B_r} (-\Delta V)\phi \le \int_{B_r} (-\Delta \psi) U.$$

Adding the above inequalities we deduce

$$\int_{B_r} \Big(V\Delta\phi - \phi\Delta V \Big) + \int_{B_r} \Big(U\Delta\psi - \psi\Delta U \Big) \le 0 \quad \text{for all } r > 0,$$

that is,

$$\int_{\partial B_r} \Big(V\frac{\partial \phi}{\partial \nu} - \phi\frac{\partial V}{\partial \nu} \Big) + \int_{\partial B_r} \Big(U\frac{\partial \psi}{\partial \nu} - \psi\frac{\partial U}{\partial \nu} \Big) \le 0 \quad \text{for all } r > 0.$$

Since U, V, ϕ, ψ are radially symmetric, this yields

$$V\phi' - \phi V' + U\psi' - \psi U' \le 0 \quad \text{in } (0, \infty). \tag{3.19}$$

Now, let us remark that $\phi, \psi > 0$ and $\phi', \psi' < 0$ in $(0, \infty)$. Combining this fact with (3.17) we deduce that (3.19) does not hold ar $r = r_1$, a contradiction. Hence $u < u_s$ and $v < v_s$ in $\mathbb{R}^N \setminus \{0\}$. □

Assume next that (3.5) fails to hold. We establish first the following result.

Proposition 3.12. *Assume (p, q) does not satisfy (3.5). Then, for any stable solution (u, v) of (3.1) we have*

$$u < u_s \quad \text{and} \quad v < v_s \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Proof. Assume by contradiction that $u - u_s$ changes sign in $\mathbb{R}^N \setminus \{0\}$. Then $v - v_s$ also changes sign in $\mathbb{R}^N \setminus \{0\}$ for otherwise $v - v_s \le 0$ in $\mathbb{R}^N \setminus \{0\}$ implies

$$-\Delta(u - u_s) = v^p - v_s^p \le 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Also $u-u_s<0$ in a neighborhood of the origin and by Remark 3.10 we have $u(x)-u_s(x)\to 0$ as $|x|\to\infty$. By the maximum principle, we deduce $u-u_s\le 0$ in $\mathbb{R}^N\setminus\{0\}$ which contradicts our assumption.

Hence $u-u_s$ and $v-v_s$ change sign on $(0,\infty)$. Denote by r_1 (resp. r_2) the first sign-changing zero of $u-u_s$ (resp. $v-v_s$). From Corollary 3.8, $u-u_s$ (resp. $v-v_s$) cannot be zero in a whole neighborhood of r_1 (resp. r_2). Without losing generality, we may assume that $r_1\le r_2$.

We claim that $u-u_s$ has a second sign-changing point $r_3>r_1$. Indeed, otherwise $u-u_s\ge 0$ in $\mathbb{R}^N\setminus B_{r_1}$ which by the maximum principle implies that $v-v_s\ge 0$ in $\mathbb{R}^N\setminus B_{r_2}$. Therefore, $u\ge u_s$, $v\ge v_s$ in $\mathbb{R}^N\setminus B_{r_2}$ which implies that (u_s,v_s) is a stable solution of (3.1) in $\mathbb{R}^N\setminus B_{r_2}$ and thus, contradicts Proposition 3.9. Hence, there exists $r_3>r_1$ a second sign-changing point of $u-u_s$. Further, we must have $r_3\ge r_2$ for otherwise $r_1<r_3<r_2$. Then $u(r_3)=u_s(r_3)$ and $v(r_3)<v_s(r_3)$ which by Corollary 3.8 yields $u<u_s$, $v<v_s$ in $B_{r_3}\setminus\{0\}$. But this is impossible since $u(r_1)=u_s(r_1)$. Thus, $r_3\ge r_2$.

We next claim that $v-v_s$ has a second sign-changing point $r_4>r_2$. As before, if this is not true, then $v-v_s\ge 0$ in $\mathbb{R}^N\setminus B_{r_2}$ and by the maximum principle we find $u-u_s\ge 0$ in $\mathbb{R}^N\setminus B_{r_3}$. Then $u\ge u_s$, $v\ge v_s$ in $\mathbb{R}^N\setminus B_{r_3}$, so (u_s,v_s) is stable in $\mathbb{R}^N\setminus B_{r_3}$ which contradicts Proposition 3.9.

We show next that $r_4\ge r_3$. Assuming the contrary we have $r_2<r_4<r_3$. At this stage, two cases may occur:

Case 1: $v\le v_s$ in (r_4,r_3). Remark that $u(r_3)=u_s(r_3)$ and $v(r_3)\le v_s(r_3)$. By Corollary 3.8 we deduce $u<u_s$ in B_{r_3} which is impossible since $u(r_1)=u_s(r_1)$.

Case 2: $v-v_s$ has a third sign-changing point $\rho\in(r_4,r_3)$. Then $v-v_s>0$ on (r_2,r_4) and $v-v_s<0$ on (r_4,ρ). On the other hand,

$$-\Delta(v-v_s)=u^q-u_s^q\ge 0 \quad \text{in } B_\rho\setminus\overline{B}_{r_4}$$

and $v-v_s=0$ on $\partial(B_\rho\setminus B_{r_4})$. The maximum principle yields $v-v_s>0$ on (r_4,ρ), a contradiction. We have proved that $r_4\ge r_3$.

We claim that $u-u_s$ has a third sign-changing point $r_5>r_3$. Indeed, if this is not true, then $u-u_s\le 0$ in $\mathbb{R}^N\setminus B_{r_3}$ and by the maximum principle we have $v-v_s\le 0$ in $\mathbb{R}^N\setminus B_{r_4}$. Hence $u\le u_s$, $v\le v_s$ in $\mathbb{R}^N\setminus B_{r_4}$ which combined with Corollary 3.8 produces $u<u_s$, $v<v_s$ in B_{r_4}. This is clearly impossible since $u(r_1)=u_s(r_1)$. Hence, $u-u_s$ has a third sign-changing point $r_5>r_3$.

If $r_5\le r_4$ then

$$-\Delta(u-u_s)=v^p-v_s^p\ge 0 \quad \text{in } B_{r_5}\setminus\overline{B}_{r_3}$$

and $u-u_s=0$ on $\partial(B_{r_5}\setminus B_{r_3})$. By the maximum principle we infer that $u-u_s\ge 0$ in $B_{r_5}\setminus B_{r_3}$ which implies $u-u_s\ge 0$ in $B_{r_5}\setminus B_{r_1}$. This contradicts the fact that $r_3\in(r_1,r_5)$ is a sign-changing point of $u-u_s$.

If $r_5 > r_4$ then $u(r_4) \le u_s(r_4)$ and $v(r_4) = v_s(r_4)$. By Corollary 3.8 we deduce $u < u_s$, $v < v_s$ in B_{r_4} which is again a contradiction. □

We are now ready to complete the proof of Theorem 3.2(ii). We adapt an idea introduced in [39]. Assume there exists a positive stable radially symmetric solution (u, v) of (3.1) and set

$$M_1 = \sup_{r\in(0,\infty)} \frac{u(r)}{u_s(r)}\,, \quad M_2 = \sup_{r\in(0,\infty)} \frac{v(r)}{v_s(r)}.$$

By Proposition 3.12 we have $M_1, M_2 \le 1$. Since $\lim_{r\to\infty} u(r) = 0$, u coincides with the Newtonian potential of v^p. Hence

$$\begin{aligned} u(x) &= c_N \int_{\mathbb{R}^N} |x-y|^{2-N} v^p(y)dy \\ &\le M_2^p \left\{ c_N \int_{\mathbb{R}^N} |x-y|^{2-N} v_s^p(y)dy \right\} = M_2^p u_s(x). \end{aligned}$$

Thus, $M_1 \le M_2^p$ and similarly $M_2 \le M_1^q$. It follows that $M_1 \le M_1^{pq}$. So, since $pq > 1$ we have either $M_1 = 0$ or $M_1 = 1$. If $M_1 = 0$ then $u \equiv 0$ and this yields $v \equiv 0$ which is impossible. Therefore $M_1 = 1$ and similarly $M_2 = 1$, i.e.

$$\sup_{r\in(0,\infty)} \frac{u(r)}{u_s(r)} = \sup_{r\in(0,\infty)} \frac{v(r)}{v_s(r)} = 1.$$

By the strong maximum principle, (u, v) cannot touch (u_s, v_s), so there exists a sequence $\{R_k\}$ converging to $+\infty$ such that

$$\lim_{k\to\infty} \frac{u(R_k)}{u_s(R_k)} = 1. \tag{3.20}$$

Define

$$u_k(r) = R_k^\alpha u(R_k r)\,, \quad v_k(r) = R_k^\beta v(R_k r) \quad r \ge 0.$$

By scale invariance we have

$$0 < u_k < u_s\,, \quad 0 < v_k < v_s \quad \text{in } \mathbb{R}^N \setminus \{0\} \tag{3.21}$$

and (u_k, v_k) solves the Lane-Emden system (3.1) in $\mathbb{R}^N \setminus \{0\}$. By elliptic regularity, $\{(u_k, v_k)\}$ converges uniformly in $C^2_{loc}(\mathbb{R}^N \setminus \{0\})$ to a solution $(\widetilde{u}, \widetilde{v})$ of (3.1) which, in view of (3.21), also satisfies

$$0 \le \widetilde{u} \le u_s\,, \quad 0 \le \widetilde{v} \le v_s \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Let us remark that by (3.20) we have

$$\widetilde{u}(1) = \lim_{k\to\infty} u_k(1) = \lim_{k\to\infty} R_k^a u(R_k) = \lim_{k\to\infty} R_k^a u_s(R_k) = u_s(1).$$

On the other hand,

$$\begin{cases} -\Delta(\widetilde{u} - u_s) = \widetilde{v}^p - v_s^p \le 0 & \text{in } \mathbb{R}^N \setminus \{0\}, \\ \lim_{|x|\to 0} (\widetilde{u} - u_s) \le 0\,, \quad \lim_{|x|\to\infty} (\widetilde{u} - u_s) \le 0. \end{cases}$$

By the strong maximum principle we deduce that $\widetilde{u} \equiv u_s$ in $\mathbb{R}^N \setminus \{0\}$. This is impossible, since $\widetilde{u}$ is a stable solution by construction while u_s is unstable when (3.5) fails.

Chapter 4

Multiplicity of solutions to nearly critical elliptic equation in the bounded domain of $\mathbb{R}^3$

4.1 Introduction

We are interested in the following semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = u^p + \lambda u^q, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, λ is a positive parameter and $p > q > 1$.

Existence and multiplicity of solutions to (4.1) have been studied intensively by many authors for the exponents p and q in different ranges. Ambrosetti, Brézis and Cerami [2], using the method of sub and super solutions, established that for $0 < q < 1$ and $p > 1$ arbitrary, there exists $\Lambda > 0$ such that problem (4.1) has a minimal solution u_λ for $\lambda \in (0, \Lambda)$, and u_λ is increasing with respect to λ; for $\lambda = \Lambda$, problem (4.1) has at least one weak soltuion; for all $\lambda > \Lambda$, problem (4.1) has no solution. Moreover, using variational tools, the authors [2] also showed that if $0 < q < 1 < p \leq 5$, for all $\lambda \in (0, \Lambda)$, problem (4.1) has a second solution.

Let us also mention the question of existence and multiplicity of solutions to (4.1) for $q = 1$.

(a) If $1 < p < 5$, for $0 < \lambda < \mu_1$, where μ_1 is the first eigenvalue of $-\Delta$ under Dirichlet boundary condition, a solution can be found by the standard constrained minimization procedure thanks to compactness of Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$.

(b) If $p \geq 5$, this case is more delicate, since for $p = 5$ the Sobolev embedding loses compactness while for $p > 5$ Sobolev embedding fails. Pohozaev [99] proved that if Ω

is strictly star-shaped, then there is no solution of (4.1) if $\lambda \leq 0$ and $p \geq 5$. For the supercritical case, del Pino, Dolbeault and Musso [44], established existence and multiplicity of solutions to problem (4.1) when p is supercritical but sufficiently close to 5. For $p = 5$, the great contribution to this case was the pioneering work of Brézis and Nirenberg [14]. They obtained that if $q = 1$, (4.1) has a solution if and only if $\lambda \in (\frac{1}{4}\mu_1, \mu_1)$ when Ω is a ball. Brézis and Nirenberg [14] obtained the following results for the case $q > 1$: if $1 < q \leq 3$, there exists a solution if and only if $\lambda > 0$ is large enough. If $3 < q < 5$, (4.1) has a solution for every $\lambda > 0$. In addition, when Ω is a ball, they gave the following conjecture, which based on numerical computations.

If $q = 3$, there is some $\tilde{\lambda}$ such that

- for $\lambda > \tilde{\lambda}$, there is a unique solution of (4.1);
- for $\lambda \leq \tilde{\lambda}$, there is no solution of (4.1).

If $1 < q < 3$, there is some $\tilde{\lambda}$ such that

- for $\lambda > \tilde{\lambda}$, there are two solutions of (4.1);
- for $\lambda = \tilde{\lambda}$, there is a unique solution of (4.1);
- for $\lambda < \tilde{\lambda}$, there is no solution of (4.1).

Afterwards, Atkinson and Peletier [6] proved the nonuniqueness of solutions to (4.1) conjectured by Brézis and Nirenberg for $N = 3$, $p = 5$ and $1 < q < 3$. Not restricting to integer values of N, they established for $2 < N < 4$, $p = \frac{N+2}{N-2}$ and $1 < q < \frac{6-N}{N-2}$, then there exists some $\tilde{\lambda} > 0$ such that (4.1) has at least two solutions for any $\lambda > \tilde{\lambda}$, and it has no solution for $\lambda < \tilde{\lambda}$. Rey [103] provided another partial answer to above conjecture. He obtained that for $p = 5$ and $2 < q < 3$, $\lambda > 0$ large enough, problem (4.1) has at least $Cat(\Omega) + 1$ solutions, where Ω is any smooth and bounded domain in $\mathbb{R}^3$ and $Cat(\Omega)$ denotes Ljusternik-Schnirelman category of Ω.

The purpose of this chapter is to establish multiplicity of solutions to problem (4.1) when p approaches to the critical exponent from below. Namely, we consider

$$\begin{cases} -\Delta u = u^{5-\varepsilon} + \lambda u^q, \quad u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $1 < q < 3$, $\lambda > 0$ and $\varepsilon > 0$. In the following, we write $p = 5 - \varepsilon$. It is known that the solutions to problem (4.2) correspond to the critical points of the following functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} - \frac{\lambda}{q+1}\int_\Omega |u|^{q+1}, \quad u \in H_0^1(\Omega). \tag{4.3}$$

In order to state our results, we introduce some notations. Let us consider Green's function $G(x, y)$, solution for any given $y \in \Omega$ of

$$\begin{cases} -\Delta_x G(x, y) = \delta_y(x) & \text{in } \Omega; \\ G(x, y) = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

and its regular part $H(x,y) = \frac{1}{4\pi|x-y|} - G(x,y)$. Then $H(x,y)$ satisfies

$$\begin{cases} -\Delta_x H(x,y) = 0 & \text{in } \Omega; \\ H(x,y) = \frac{1}{4\pi|x-y|} & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

The Robin's function of Ω is defined as $R(x) = H(x,x)$, where $H(x,y)$, $x,y \in \Omega$ is given by (4.5), so $R(x)$ is smooth, $R(x) \to +\infty$ as $x \to \partial\Omega$, and it is positive by the maximum principle. Thus $R(x)$ has a minimum in Ω, and hence it has at least one critical point $\xi_0 \in \Omega$.

Regarding $\varepsilon > 0$ as a small parameter, we construct a large solution. Our results can be stated as follows.

Theorem 4.1. *Let $1 < q < 3$, there exists $\lambda_0 > 0$, depending on Ω, q, and $\varepsilon_0 > 0$, such that for any given $\lambda \geq \lambda_0$, $\varepsilon \in (0, \varepsilon_0)$, problem (4.2) has at least two solutions. One of them is the mountain pass solution u_1, the other one is the large solution u_2, which has the form of*

$$u_2(x) = 3^{\frac{1}{4}} \frac{(\Lambda_* \varepsilon)^{\frac{1}{2}}}{((\Lambda_*\varepsilon)^2 + |x - \xi_*|^2)^{\frac{1}{2}}}(1 + o(1)), \tag{4.6}$$

satisfying

$$J(u_2) = \frac{\sqrt{3}}{4}\pi^2 - a_2 \varepsilon \log \varepsilon + O(\varepsilon), \tag{4.7}$$

where $a_2 > 0$ and $\Lambda_* > 0$ and $\xi_* \to \xi_0$, $o(1) \to 0$ uniformly in $\bar{\Omega}$ as $\varepsilon \to 0$.

Next, we use λ as parameter to construct a third solution for $2 < q < 3$.

Theorem 4.2. *Assume that $2 < q < 3$. There exist $\hat{\lambda} \geq \lambda_0$ and $\delta_0 > 0$, such that for any $\lambda \geq \hat{\lambda}$ satisfying*

$$0 < \varepsilon \lambda^{\frac{2}{3-q}} \log \lambda < \delta_0, \tag{4.8}$$

then for all sufficiently small $\varepsilon > 0$, problem (4.2) has at least three solutions.

In the case $1 < q \leq 2$, it is also possible to find a third solution but the proof is more delicate and will be addressed in future work.

We now mention some contributions to multiplicity of solutions to equations with two powers in the whole space $\mathbb{R}^N$ with $N \geq 3$. Recently, Dávila, del Pino and Guerra [35] studied nonuniqueness of positive solution of the following problem

$$-\Delta u + u = u^p + \lambda u^q, \quad u > 0 \quad \text{in } \mathbb{R}^3; \qquad u(z) \to 0 \ \text{ as } \ |z| \to \infty. \tag{4.9}$$

More precisely, the authors obtained at least three solutions to problem (4.9) if $1 < q < 3$, $\lambda > 0$ is sufficiently large and fixed, and $p < 5$ is close enough to 5.

This chapter is organized as follows, in Section 4.2, we compute the energy asymptotic expansion. We build the large solution in Section 4.3 and prove Theorem 4.1. We prove Theorem 4.2 in Section 4.4.

4.2 The asymptotic expansion

We recall that, according to [19], the functions

$$w_{\mu,\xi}(x) = 3^{\frac{1}{4}} \frac{\mu^{\frac{1}{2}}}{(\mu^2 + |x-\xi|^2)^{\frac{1}{2}}} \qquad \mu > 0, \quad \xi \in \mathbb{R}^3,$$

are the only solutions (except translations) of the problem

$$-\Delta w = w^5, \quad w > 0 \quad \text{in} \quad \mathbb{R}^3. \tag{4.10}$$

As $\xi \in \Omega$ and μ goes to zero, these functions provide us with approximate solutions to the problem that we are interested in. However, in view of the Dirichlet boundary condition, the approximate solution needs to be improved.

From now on we assume that $\xi \in \Omega$ and is far from the boundary of Ω, that is, there exists $\delta > 0$ such that

$$d(\xi, \partial\Omega) \geq \delta. \tag{4.11}$$

Let $U_{\mu,\xi}(x)$ be the unique solution of

$$\begin{cases} -\Delta U_{\mu,\xi} = w_{\mu,\xi}^5 & \text{in } \Omega; \\ U_{\mu,\xi} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

We have the following estimates.

Lemma 4.3. *Let $d(\xi, \partial\Omega) \geq \delta$ for some $\delta > 0$, for $\mu > 0$ small enough, one has*
(a) $0 < U_{\mu,\xi}(x) \leq w_{\mu,\xi}(x)$,
(b) $U_{\mu,\xi}(x) = w_{\mu,\xi}(x) - 4\pi 3^{\frac{1}{4}} \mu^{\frac{1}{2}} H(x,\xi) + O(\mu^{\frac{5}{2}})$.

Proof. By the maximum principle, we obtain (a). Now we define

$$D(x) = U_{\mu,\xi}(x) - w_{\mu,\xi}(x) + 4\pi 3^{\frac{1}{4}} \mu^{\frac{1}{2}} H(x,\xi).$$

Observe that for $x \in \partial\Omega$, as $\mu \to 0$,

$$\begin{aligned} D(x) &= U_{\mu,\xi}(x) - w_{\mu,\xi}(x) + 4\pi 3^{\frac{1}{4}} \mu^{\frac{1}{2}} H(x,\xi) \\ &= 3^{\frac{1}{4}} \mu^{\frac{1}{2}} \left[\frac{1}{|x-\xi|} - \frac{1}{(\mu^2 + |x-\xi|^2)^{\frac{1}{2}}} \right] \sim \mu^{\frac{5}{2}} |x-\xi|^{-3}. \end{aligned}$$

Then $D(x)$ satisfies

$$\begin{cases} -\Delta D = 0 & \text{in } \Omega; \\ D = O(\mu^{\frac{5}{2}}) \quad \textit{as} \quad \mu \to 0 & \text{on } \partial\Omega. \end{cases} \tag{4.13}$$

Therefore (b) follows from the maximum principle. □

In the following we write $U = U_{\mu,\xi}$, we now compute the energy expansion $J(U)$, where $J(u)$ is defined by (4.3).

Lemma 4.4. *Let $d(\xi, \partial\Omega) \geq \delta$, assume that $\mu > 0$ is small enough, then we have if $2 < q < 3$,*

$$J(U) = a_0 + a_1\mu H(\xi,\xi) - a_2\varepsilon \log\mu + a_3\varepsilon - \lambda a_4 \mu^{\frac{5-q}{2}} + O(\lambda\mu^{\frac{q+1}{2}}) + O(\mu^2) + o(\varepsilon). \quad (4.14)$$

If $q = 2$,

$$J(U) = a_0 + a_1\mu H(\xi,\xi) - a_2\varepsilon \log\mu + a_3\varepsilon - \lambda a_5 \mu^{\frac{3}{2}} \log\mu + O(\lambda\mu^{\frac{3}{2}}) + O(\mu^2) + o(\varepsilon). \quad (4.15)$$

If $1 < q < 2$,

$$J(U) = a_0 + a_1\mu H(\xi,\xi) - a_2\varepsilon \log\mu + a_3\varepsilon - \lambda a_6 \mu^{\frac{q+1}{2}} + O(\lambda\mu^{\frac{5-q}{2}}) + O(\mu^2) + o(\varepsilon), \quad (4.16)$$

where $o(\varepsilon)$ is uniform in the C^1-sense on the point ξ satisfying (4.11) as $\varepsilon \to 0$, and a_i, $i = 0, 1, \ldots, 6$, are some constants.

Proof. We write $J(U) = J_5(U) + (J_p(U) - J_5(U)) + J_\lambda(U)$, where

$$J_p(U) = \frac{1}{2}\int_\Omega |\nabla U|^2 - \frac{1}{p+1}\int_\Omega U^{p+1} \quad \text{and} \quad J_\lambda(U) = -\frac{\lambda}{q+1}\int_\Omega U^{q+1}.$$

Since U satisfies $-\Delta U = w_{\mu,\xi}^5$ in Ω and $U = 0$ on $\partial\Omega$, we write $U = \pi_{\mu,\xi} + w_{\mu,\xi}$, then

$$\begin{aligned} J_5(U) &= \frac{1}{2}\int_\Omega |\nabla U|^2 - \frac{1}{6}\int_\Omega U^6 = \frac{1}{2}\int_\Omega w_{\mu,\xi}^5 U - \frac{1}{6}\int_\Omega U^6 \\ &= \frac{1}{2}\int_\Omega w_{\mu,\xi}^5 (\pi_{\mu,\xi} + w_{\mu,\xi}) - \frac{1}{6}\int_\Omega (\pi_{\mu,\xi} + w_{\mu,\xi})^6 \\ &= \frac{1}{3}\int_\Omega w_{\mu,\xi}^6 - \frac{1}{2}\int_\Omega w_{\mu,\xi}^5 \pi_{\mu,\xi} - \frac{1}{6}\int_\Omega \left[(\pi_{\mu,\xi} + w_{\mu,\xi})^6 - w_{\mu,\xi}^6 - 6w_{\mu,\xi}^5 \pi_{\mu,\xi}\right] \\ &:= I - II + \mathcal{R}_1. \end{aligned} \quad (4.17)$$

By the mean theorem, we find

$$\begin{aligned} \mathcal{R}_1 &= -\frac{1}{6}\int_\Omega \left[(\pi_{\mu,\xi} + w_{\mu,\xi})^6 - w_{\mu,\xi}^6 - 6w_{\mu,\xi}^5 \pi_{\mu,\xi}\right] \, dx \\ &= -5\int_\Omega \int_0^1 (w_{\mu,\xi} + t\pi_{\mu,\xi})^4 \pi_{\mu,\xi}^2 (1-t) \, dt dx = O(\mu^2). \end{aligned}$$

Now we expand the other two terms in the right hand side of (4.17).

$$\begin{aligned} I &= \frac{1}{3}\int_\Omega w_{\mu,\xi}^6 \, dx = \frac{1}{3}\left(\int_{\mathbb{R}^3} 3^{\frac{3}{2}} \frac{1}{(1+|z|^2)^3} \, dz - \int_{\mathbb{R}^3 \setminus \frac{\Omega - \xi}{\mu}} 3^{\frac{3}{2}} \frac{1}{(1+|z|^2)^3} \, dz\right) \\ &= a_0 + O(\mu^3), \end{aligned}$$

where $a_0 = \frac{\sqrt{3}\pi^2}{4}$. Moreover from Lemma 4.3, we have

$$
\begin{aligned}
II &= \frac{1}{2}\int_\Omega w_{\mu,\xi}^5 \pi_{\mu,\xi}\, dx = \frac{1}{2}\mu^{\frac{1}{2}} \int_{\frac{\Omega-\xi}{\mu}} 3^{\frac{5}{4}} \frac{1}{(1+|z|^2)^{\frac{5}{2}}} \pi_{\mu,\xi}(\mu z+\xi)\, dz \\
&= \frac{1}{2}\int_{\frac{\Omega-\xi}{\mu}} w_{1,0}^5(z) \left[-4\pi 3^{\frac{1}{4}}\mu\left[H(\xi,\xi)+O(\mu|z|)+o(\mu)\right]+O(\mu^3)\right] dz \\
&= -\mu H(\xi,\xi)a_1 + \mathcal{R}_2,
\end{aligned}
$$

where $a_1 = 2\pi 3^{\frac{1}{4}} \int_{\mathbb{R}^3} w_{1,0}^5(z)\, dz = 8\sqrt{3}\pi^2$ and

$$
\begin{aligned}
\mathcal{R}_2 &= 2\pi 3^{\frac{1}{4}} \Bigg(\mu H(\xi,\xi)\int_{\mathbb{R}^3\setminus\frac{\Omega-\xi}{\mu}} w_{1,0}^5(z)\, dz - O(\mu^2)\int_{\frac{\Omega-\xi}{\mu}} w_{1,0}^5(z)|z| \\
&\quad - \int_{\frac{\Omega-\xi}{\mu}} w_{1,0}^5(z)[o(\mu^2)+O(\mu^3)]\Bigg)\, dz = O(\mu^2).
\end{aligned}
$$

Thus we get the following expansion

$$
J_5(U) = a_0 + a_1\mu H(\xi,\xi) + O(\mu^2). \tag{4.18}
$$

By Taylor expansion in p, we get

$$
\begin{aligned}
J_p(U) - J_5(U) &= \frac{1}{6}\int_\Omega U^6 - \frac{1}{6-\varepsilon}\int_\Omega U^6 U^{-\varepsilon} \\
&= \frac{1}{6}\int_\Omega U^6 - \left[\frac{1}{6}+\frac{1}{36}\varepsilon + o(\varepsilon)\right]\int_\Omega U^6\left(1-\varepsilon\log U + o(\varepsilon)\right) \\
&= \varepsilon\left[\frac{1}{6}\int_\Omega U^6\log U - \frac{1}{36}\int_\Omega U^6\right] + o(\varepsilon) \\
&= \varepsilon\left[\frac{1}{6}\int_\Omega w_{\mu,\xi}^6 \log w_{\mu,\xi} - \frac{1}{36}\int_\Omega w_{\mu,\xi}^6 + O(\mu\log\mu)\right] + o(\varepsilon) \\
&= (-a_2\log\mu + a_3)\varepsilon + o(\varepsilon),
\end{aligned} \tag{4.19}
$$

where $a_2 = \frac{1}{12}\int_{\mathbb{R}^3} w_{1,0}^6(z)\, dz = \frac{\sqrt{3}\pi^2}{16}$ and $a_3 = \frac{1}{36}\int_{\mathbb{R}^3} w_{1,0}^6(z)[6\log(w_{1,0}(z)) - 1]\, dz$.

Finally we compute $J_\lambda(U)$. If $2 < q < 3$,

$$
\begin{aligned}
J_\lambda(U) &= -\frac{\lambda}{q+1}\int_\Omega U^{q+1}\, dx = -\frac{\lambda}{q+1}\int_\Omega w_{\mu,\xi}^{q+1}\, dx + O(\lambda\mu^{\frac{q+1}{2}}) \\
&= -\lambda\mu^{\frac{5-q}{2}}\Bigg[\frac{1}{q+1}\int_{\mathbb{R}^3} 3^{\frac{q+1}{4}}\frac{1}{(1+|z|^2)^{\frac{q+1}{2}}} \\
&\quad - \frac{1}{q+1}\int_{\mathbb{R}^3\setminus\frac{\Omega-\xi}{\mu}} 3^{\frac{q+1}{4}}\frac{1}{(1+|z|^2)^{\frac{q+1}{2}}}\Bigg] + O(\lambda\mu^{\frac{q+1}{2}})
\end{aligned}
$$

$$= -\lambda a_4 \mu^{\frac{5-q}{2}} + O(\lambda \mu^{\frac{q+1}{2}}), \tag{4.20}$$

where $a_4 = \frac{1}{q+1}\int_{\mathbb{R}^3} w_{1,0}^{q+1}(z)\,dz = \frac{3^{\frac{q+1}{4}}\pi^{\frac{3}{2}}\Gamma(\frac{q-2}{2})}{(q+1)\Gamma(\frac{q+1}{2})}$. If $q = 2$,

$$\begin{aligned} J_\lambda(U) &= -\frac{\lambda}{3}\mu^{\frac{3}{2}}\int_{\frac{\Omega-\xi}{\mu}} 3^{\frac{3}{4}}\frac{1}{(1+|z|^2)^{\frac{3}{2}}}\,dz + O(\lambda\mu^{\frac{3}{2}}) \\ &= -\lambda a_5 \mu^{\frac{3}{2}}\log\mu + O(\lambda\mu^{\frac{3}{2}}), \end{aligned} \tag{4.21}$$

where $a_5 = -2\pi 3^{-\frac{1}{4}}$, here we use the fact $\int_0^a \frac{r^2}{(1+r^2)^{3/2}}dr = \log(a+\sqrt{1+a^2}) - \frac{a}{\sqrt{1+a^2}}$. If $1 < q < 2$,

$$\begin{aligned} J_\lambda(U) &= -\frac{\lambda}{q+1}\int_\Omega \left[w_{\mu,\xi}(x) - 4\pi 3^{\frac{1}{4}}\mu^{\frac{1}{2}}H(x,\xi) + O(\mu^{\frac{5}{2}})\right]^{q+1} \\ &= -\mu^{\frac{q+1}{2}}\frac{\lambda}{q+1}\int_\Omega \Bigg\{3^{\frac{1}{4}}\left[\frac{1}{(\mu^2+|x-\xi|^2)^{\frac{1}{2}}} - \frac{1}{|x-\xi|}\right] \\ &\qquad +4\pi 3^{\frac{1}{4}}G(x,\xi) + O(\mu^2)\Bigg\}^{q+1} \\ &= -\lambda\mu^{\frac{q+1}{2}}a_6 + O(\lambda\mu^{\frac{5-q}{2}}), \end{aligned} \tag{4.22}$$

where $a_6 = \frac{1}{q+1}(4\pi 3^{\frac{1}{4}})^{q+1}\int_\Omega G^{q+1}(x,\xi)\,dx$. From (4.18)- (4.22), we obtain C^0-estimate of the energy expansion. By the same way we can get the C^1-estimate also holds. □

4.3 Construct the large solution

In this section, by Lyapunov-Schmidt reduction procedure, we build a large solution for $\lambda \geq 0$ given and $\varepsilon > 0$ small enough. Then we prove Theorem 4.1.

4.3.1 The first approximate solution and the linearized problem

If u is a solution of (4.1), via the change of variables

$$v(y) = \varepsilon^\kappa u(\varepsilon y), \quad \kappa = \frac{2}{p-1}, \quad y \in \Omega_\varepsilon,$$

where $\Omega_\varepsilon = \frac{\Omega}{\varepsilon}$. Then $v(y)$ satisfies

$$\begin{cases} -\Delta v = f_\varepsilon(v), \quad v > 0 & \text{in } \Omega_\varepsilon; \\ v = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{4.23}$$

where and in the following we denote $f_\varepsilon(v) = v^p + \lambda\varepsilon^\alpha v^q$ with $\alpha = \frac{2(p-q)}{p-1}$.

Define the function

$$V(y) \equiv V_{\Lambda,\xi'}(y) = \varepsilon^{\frac{1}{2}} U_{\mu,\xi}(\varepsilon y), \quad \Lambda = \frac{\mu}{\varepsilon}, \quad \xi' = \frac{\xi}{\varepsilon}, \quad y \in \Omega_\varepsilon, \tag{4.24}$$

where $U_{\mu,\xi}$ is the solution of (4.12). Then $V(y)$ satisfies

$$\begin{cases} -\Delta V(y) = w^5_{\Lambda,\xi'}(y) & \text{in } \Omega_\varepsilon; \\ V(y) = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

We note that assumption (4.11) is equivalent to

$$d(\xi', \partial\Omega_\varepsilon) \geq \frac{\delta}{\varepsilon}. \tag{4.25}$$

We assume that

$$\hat{\delta} < \Lambda < \frac{1}{\hat{\delta}}, \tag{4.26}$$

with $\hat{\delta} > 0$ small but fixed.

From Lemma 4.3, for ξ' and Λ satisfying (4.25) and (4.26), we have

$$0 < V(y) \leq w_{\Lambda,\xi'}(y) \quad \text{in } \Omega_\varepsilon. \tag{4.27}$$

$$V(y) = w_{\Lambda,\xi'}(y) - 4\pi 3^{\frac{1}{4}} \Lambda^{\frac{1}{2}} \varepsilon H(\varepsilon y, \varepsilon\xi') + O(\varepsilon^3) \quad \text{in } \Omega_\varepsilon, \quad \text{as } \varepsilon \to 0. \tag{4.28}$$

We next look for a solution of (4.23) of the form

$$v(y) = V(y) + \phi(y),$$

where V is given by (4.24) and ϕ is a small term. We can rewrite (4.23) as

$$\begin{cases} L_\varepsilon(\phi) = N(\phi) + R & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{4.29}$$

where

$$L_\varepsilon(\phi) = -\Delta\phi - f'_\varepsilon(V)\phi, \quad N(\phi) = f_\varepsilon(V+\phi) - f_\varepsilon(V) - f'_\varepsilon(V)\phi, \quad R = \Delta V + f_\varepsilon(V).$$

We first consider the linearized problem at V and we invert it in an orthogonal space. More precisely, we consider the following problem: $h \in L^\infty(\Omega_\varepsilon)$ being given, find a solution ϕ which satisfies

$$\begin{cases} -\Delta\phi - (5-\varepsilon)V^{4-\varepsilon}\phi - \lambda q \varepsilon^\alpha V^{q-1}\phi = h + \sum_{i=0}^{3} c_i w^4_{\Lambda,\xi'} Z_i & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \phi w^4_{\Lambda,\xi'} Z_i = 0 \quad i = 0,1,2,3, \end{cases} \tag{4.30}$$

for some numbers c_i $(i = 0, 1, 2, 3)$, where Z_i are defined by

$$Z_0 = \frac{\partial V}{\partial \Lambda}, \qquad Z_i = \frac{\partial V}{\partial \xi'_i}, \quad i = 1, 2, 3.$$

Then $Z_i (i = 0, 1, 2, 3)$ satisfy

$$\begin{cases} -\Delta Z_i = 5 w_{\Lambda,\xi'}^4 \tilde{Z}_i & \text{in } \Omega_\varepsilon; \\ Z_i = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

with $\tilde{Z}_0 = \frac{\partial w_{\Lambda,\xi'}}{\partial \Lambda}$, and $\tilde{Z}_i = \frac{\partial w_{\Lambda,\xi'}}{\partial \xi'_i}$ for $i = 1, 2, 3$.

Our next aim is to prove that problem (4.30) has a unique solution with uniform bounds in some appropriate norms. For f a function in Ω_ε, we define the following weighted $L^\infty-$norms

$$\|f\|_* = \sup_{y \in \Omega_\varepsilon} (1 + |y - \xi'|^2)^{\frac{\theta - 2}{2}} |f(y)|, \tag{4.31}$$

and

$$\|f\|_{**} = \sup_{y \in \Omega_\varepsilon} (1 + |y - \xi'|^2)^{\frac{\theta}{2}} |f(y)|, \tag{4.32}$$

where θ satisfies

$$2 < \theta < 3. \tag{4.33}$$

Observe that the first norm $\| \cdot \|_*$ is equivalent to $\|w_{\Lambda,\xi'}^{-(\theta-2)} f\|_\infty$ and the second norm $\| \cdot \|_{**}$ is equivalent to $\|w_{\Lambda,\xi'}^{-\theta} f\|_\infty$ uniformly with respect to Λ and ξ'.

Proposition 4.5. *Let $\lambda > 0$ be fixed and ξ', Λ satisfy (4.25), (4.26), then there exists $\varepsilon_0 > 0$ and a constant $C > 0$, such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in L^\infty(\Omega_\varepsilon)$ with $\|h\|_{**} < +\infty$, problem (4.30) has a unique solution $\phi := T_\varepsilon(h)$ with $\|\phi\|_* < +\infty$. Moreover,*

$$\|\phi\|_* \leq C\|h\|_{**}, \qquad |c_i| \leq C\|h\|_{**}. \tag{4.34}$$

The argument of its proof follows from the ideas of M. del Pino et al. in [45] and Rey et al. in [105].

We first prove a priori estimate for solutions of the following problem

$$\begin{cases} -\Delta\phi - (5 - \varepsilon) V^{4-\varepsilon} \phi = h + \sum_{i=0}^{3} c_i w_{\Lambda,\xi'}^4 Z_i & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \phi w_{\Lambda,\xi'}^4 Z_i = 0 \quad i = 0, 1, 2, 3. \end{cases} \tag{4.35}$$

Lemma 4.6. *Under the conditions of Proposition 4.5, then there exists $C > 0$ such that if $\varepsilon > 0$ is sufficiently small, for any h, ϕ satisfying (4.35), we have*

$$\|\phi\|_* \leq C\|h\|_{**}, \qquad |c_i| \leq C\|h\|_{**}.$$

Proof. The proof follows from the following lemma. □

Lemma 4.7. *Assume ϕ_ε solves (4.35) for $h = h_\varepsilon$. If $\|h_\varepsilon\|_{**} \to 0$ as $\varepsilon \to 0$, then $\|\phi_\varepsilon\|_* \to 0$.*

Proof. For $0 < \rho < \theta - 2$, we define

$$\|f\|_\rho = \sup_{y\in\Omega_\varepsilon}(1 + |y - \xi'|^2)^{\frac{\theta-2-\rho}{2}}|f(y)|.$$

Claim: $\|\phi_\varepsilon\|_\rho \to 0$ as $\varepsilon \to 0$.

Indeed, by contradiction, we may assume that $\|\phi_\varepsilon\|_\rho = 1$. Multiplying the first equation in (4.35) by Z_j and integrating on Ω_ε, we get

$$\int_{\Omega_\varepsilon}\left(-\Delta Z_j - (5-\varepsilon)V^{4-\varepsilon}Z_j\right)\phi_\varepsilon - \int_{\Omega_\varepsilon} h_\varepsilon Z_j = \sum_{i=0}^{3} c_i \int_{\Omega_\varepsilon} w^4_{\Lambda,\xi'}Z_iZ_j.$$

Since

$$\begin{aligned}
&\int_{\Omega_\varepsilon}\left(-\Delta Z_j - (5-\varepsilon)V^{4-\varepsilon}Z_j\right)\phi_\varepsilon = \int_{\Omega_\varepsilon}\left(5w^4_{\Lambda,\xi'}\tilde{Z}_j - (5-\varepsilon)V^{4-\varepsilon}Z_j\right)\phi_\varepsilon \\
= &\int_{\Omega_\varepsilon}\left[5w^4_{\Lambda,\xi'}\tilde{Z}_j - (5-\varepsilon)\left(w^{4-\varepsilon}_{\Lambda,\xi'} + O(\varepsilon)\right)\left(\tilde{Z}_j + O(\varepsilon)\right)\right]\phi_\varepsilon \\
= &\; O(\varepsilon)\|\phi_\varepsilon\|_\rho\int_{\Omega_\varepsilon}\frac{1}{(1+|y-\xi'|^2)^{\frac{5-\varepsilon}{2}}}\frac{1}{(1+|y-\xi'|^2)^{\frac{\theta-2-\rho}{2}}} \\
= &\; o(\|\phi_\varepsilon\|_\rho),
\end{aligned}$$

$$\int_{\Omega_\varepsilon} h_\varepsilon Z_j \;\leq\; \|h_\varepsilon\|_{**}\int_{\Omega_\varepsilon} w^\theta_{\Lambda,\xi'}(\tilde{Z}_j + O(\varepsilon)) = O(\|h_\varepsilon\|_{**}),$$

and

$$\int_{\Omega_\varepsilon} w^4_{\Lambda,\xi'}Z_iZ_j \;=\; \delta_{ij}\int_{\Omega_\varepsilon} w^4_{\Lambda,\xi'}(\tilde{Z}_i + O(\varepsilon))^2 = \delta_{ij}(\gamma_i + o(1)),$$

where δ_{ij} is Kronecker's delta and γ_i $(i = 0, 1, 2, 3)$ are strictly positive constants. Consequently, inverting the quasi-diagonal linear system solved by the c_i's, we find

$$c_i = O(\|h_\varepsilon\|_{**}) + o(\|\phi_\varepsilon\|_\rho). \tag{4.36}$$

In particular, $c_i = o(1)$ as $\varepsilon \to 0$. Moreover, the first equation in (4.35) can be written as

$$\phi_\varepsilon(x) = \int_{\Omega_\varepsilon} G_\varepsilon(x,y)\left[(5-\varepsilon)V^{4-\varepsilon}(y)\phi_\varepsilon(y) + h_\varepsilon(y) + \sum_{i=0}^{3} c_i w^3_{\Lambda,\xi'}(y)Z_i(y)\right]\, dy, \tag{4.37}$$

where $G_\varepsilon(x,y)$ is the Green's function of $-\Delta$ in Ω_ε with Dirichlet boundary condition, which satisfies

$$G_\varepsilon(x,y) = \varepsilon G(\varepsilon x, \varepsilon y) \le \frac{C}{|x-y|}.$$

In the following, we use the following basic estimate, which was proved in the Appendix B [116]: for any $0<\sigma<1$, there is a constant $C>0$ such that

$$\int_{\mathbb{R}^3} \frac{1}{|z-y|}\frac{1}{(1+|y|)^{2+\sigma}}\, dy \le \frac{C}{(1+|z|)^\sigma}.$$

Hence we have

$$\begin{aligned}
&\left|\int_{\Omega_\varepsilon} G_\varepsilon(x,y)V^{4-\varepsilon}(y)\phi_\varepsilon(y)\, dy\right| \le C\int_{\Omega_\varepsilon}\frac{1}{|x-y|}\left|w^{4-\varepsilon}_{\Lambda,\xi'}(y)\phi_\varepsilon(y)\right|\, dy\\
\le\ & C\|\phi_\varepsilon\|_\rho \int_{\Omega_\varepsilon}\frac{1}{|x-y|}\frac{1}{(1+|y-\xi'|^2)^{\frac12(4-\varepsilon)}}\frac{1}{(1+|y-\xi'|^2)^{\frac{\theta-2-\rho}{2}}}\, dy\\
\le\ & C\|\phi_\varepsilon\|_\rho \int_{\Omega_\varepsilon}\frac{1}{|(x-\xi')-(y-\xi')|}\frac{1}{(1+|y-\xi'|)^{2+\theta-2}}\frac{1}{(1+|y-\xi'|)^{2-\rho-\varepsilon}}\, dy\\
\le\ & C\|\phi_\varepsilon\|_\rho \int_{\mathbb{R}^3}\frac{1}{|(x-\xi')-(y-\xi')|}\frac{1}{(1+|y-\xi'|)^{2+\theta-2}}\, dy\\
\le\ & C\|\phi_\varepsilon\|_\rho\left(1+|x-\xi'|^2\right)^{-\frac{\theta-2}{2}},
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
&\left|\int_{\Omega_\varepsilon} G_\varepsilon(x,y)h_\varepsilon(y)\, dy\right| \le C\|h_\varepsilon\|_{**}\int_{\Omega_\varepsilon}\frac{1}{|x-y|}\frac{1}{(1+|y-\xi'|^2)^{\frac{\theta}{2}}}\, dy\\
\le\ & C\|h_\varepsilon\|_{**}\int_{\mathbb{R}^3}\frac{1}{|(x-\xi')-(y-\xi')|}\frac{1}{(1+|y-\xi'|)^{2+\theta-2}}\, dy\\
\le\ & C\|h_\varepsilon\|_{**}\left(1+|x-\xi'|^2\right)^{-\frac{\theta-2}{2}},
\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}
&\left|\int_{\Omega_\varepsilon} G_\varepsilon(x,y)w^4_{\Lambda,\xi'}(y)Z_i(y)\, dy\right| \le C\int_{\Omega_\varepsilon}\frac{1}{|x-y|}\frac{1}{(1+|y-\xi'|^2)^{\frac52}}\, dy\\
\le\ & C\int_{\Omega_\varepsilon}\frac{1}{|x-y|}\frac{1}{(1+|y-\xi'|)^{2+\theta-2}}\frac{1}{(1+|y-\xi'|)^{5-\theta}}\, dy
\end{aligned}$$

$$\leq \quad C\left(1+|x-\xi'|^2\right)^{-\frac{\theta-2}{2}}. \tag{4.40}$$

Then from (4.37)-(4.40), we get

$$|\phi_\varepsilon(x)| \leq C\left(\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**} + |c_i|\right)\left(1+|x-\xi'|^2\right)^{-\frac{\theta-2}{2}}, \tag{4.41}$$

which yields that

$$\left(1+|x-\xi'|^2\right)^{\frac{\theta-2-\rho}{2}} |\phi_\varepsilon(x)| \leq C\left(1+|x-\xi'|^2\right)^{-\frac{\rho}{2}}. \tag{4.42}$$

Moreover, $\|\phi_\varepsilon\|_\rho = 1$ and (4.42) imply that there exist $R > 0$, $\gamma > 0$ independent of ε such that

$$\|\phi_\varepsilon\|_{L^\infty(B_R(\xi'))} > \gamma. \tag{4.43}$$

Set $\bar{\phi}_\varepsilon(y) = \phi_\varepsilon(y-\xi')$, by local elliptic estimate, passing to a subsequence of $(\bar{\phi}_\varepsilon)_\varepsilon$, still denote $(\bar{\phi}_\varepsilon)_\varepsilon$, such that $(\bar{\phi}_\varepsilon)_\varepsilon$ converges uniformly on any compact set of $\mathbb{R}^3$ to a nontrivial solution of

$$-\Delta\bar{\phi} = 5w_{\Lambda,0}^4\bar{\phi} \qquad \text{for some} \ \ \Lambda > 0.$$

It is well known that [104],

$$\bar{\phi} = \alpha_0\frac{\partial w_{\Lambda,0}}{\partial\Lambda} + \sum_{i=1}^{3}\alpha_i\frac{\partial w_{\Lambda,0}}{\partial y_i}.$$

Recall that

$$\int_{\Omega_\varepsilon}\phi_\varepsilon w_{\Lambda,\xi'}^4 Z_i = 0 \qquad \text{for } i = 0,1,2,3.$$

By dominated convergence, we find that

$$\alpha_0\int_{\mathbb{R}^3}\left(\frac{\partial w_{\Lambda,0}}{\partial\Lambda}\right)^2 w_{\Lambda,0}^4 = 0 \qquad \text{and} \qquad \alpha_i\int_{\mathbb{R}^3}\left(\frac{\partial w_{\Lambda,0}}{\partial y_i}\right)^2 w_{\Lambda,0}^4 = 0, \qquad \text{for } i = 1,2,3.$$

So $\alpha_i = 0$ for $i = 0,1,2,3$ and $\bar{\phi} = 0$, this contradicts (4.43). Therefore we get $\|\phi_\varepsilon\|_\rho \to 0$ as $\varepsilon \to 0$. Finally, from (4.36) and (4.41), we have

$$\|\phi_\varepsilon\|_* \leq C(\|h_\varepsilon\|_{**} + \|\phi_\varepsilon\|_\rho).$$

Hence $\|\phi_\varepsilon\|_* \to 0$ as $\varepsilon \to 0$. □

Lemma 4.8. *Let $\lambda > 0$ be fixed and ξ', Λ satisfy (4.25), (4.26), there exists $C > 0$ such that if $\varepsilon > 0$ is sufficiently small, for any h, ϕ satisfying (4.30), we have*

$$\|\phi\|_* \leq C\|h\|_{**}, \qquad |c_i| \leq C\|h\|_{**}.$$

Proof. We claim that $\|V^{q-1}\phi\|_{**} \le C\varepsilon^{q-3}\|\phi\|_*$. Since $V \le w_{\Lambda,\xi'}$, we only need to show that

$$\|w_{\Lambda,\xi'}^{q-1}\phi\|_{**} \le C\varepsilon^{q-3}\|\phi\|_*.$$

In fact,

$$\begin{aligned}
\|w_{\Lambda,\xi'}^{q-1}\phi\|_{**} &= \sup_{y\in\Omega_\varepsilon}(1+|y-\xi'|^2)^{\frac{\theta}{2}}|w_{\Lambda,\xi'}(y)|^{q-1}|\phi(y)| \\
&\le \|\phi\|_* \sup_{y\in\Omega_\varepsilon}(1+|y-\xi'|^2)|w_{\Lambda,\xi'}(y)|^{q-1} \\
&\le \|\phi\|_* \sup_{y\in\Omega_\varepsilon}(1+|y-\xi'|^2)^{1-\frac{q-1}{2}} \\
&\le \|\phi\|_* \sup_{y\in\Omega_\varepsilon}|y-\xi'|^{3-q} \le C\varepsilon^{q-3}\|\phi\|_*
\end{aligned}$$

By the first estimate in Lemma 4.6, we get

$$\|\phi\|_* \le C\|h\|_{**} + C\varepsilon^\alpha\|V^{q-1}\phi\|_{**} \le C\|h\|_{**} + C\varepsilon^{\alpha+q-3}\|\phi\|_*.$$

Recall that $\alpha = \frac{5-q}{2} + O(\varepsilon)$, we have that $\alpha + q - 3 > 0$. Thus we get $\|\phi\|_* \le C\|h\|_{**}$.

Similarly, we obtain $|c_i| \le C\|h\|_{**}$. □

Proof of Proposition 4.5. By Lemma 4.8, we get the estimates in (4.34). Now we prove existence and uniqueness of solution to (4.30). We consider the Hilbert space

$$H = \left\{\phi \in H_0^1(\Omega_\varepsilon) \;:\; \int_{\Omega_\varepsilon}\phi w_{\Lambda,\xi'}^4 Z_i = 0,\ i = 0,1,2,3\right\}$$

with inner product

$$\langle\phi,\psi\rangle = \int_{\Omega_\varepsilon}\nabla\phi\nabla\psi.$$

Then problem (4.30) is equivalent to find $\phi \in H$ such that

$$\langle\phi,\psi\rangle = \int_{\Omega_\varepsilon}\left[(5-\varepsilon)V^{4-\varepsilon}\phi + \lambda q\varepsilon^\alpha V^{q-1}\phi + h\right]\psi, \quad \text{for } \forall\psi \in H. \tag{4.44}$$

By the Riesz representation theorem, (4.44) is equivalent to solve

$$\phi = K(\phi) + \tilde{h} \tag{4.45}$$

with $\tilde{h} \in H$ depending linearly on h, and $K : H \to H$ being a compact operator. Fredholm's alternative guarantees that there is a unique solution to problem (4.45) for any h provided that

$$\phi = K(\phi) \tag{4.46}$$

has only the zero solution in H. (4.46) is equivalent to problem (4.30) with $h = 0$. If $h = 0$, the first estimate in (4.34) implies that $\phi = 0$. This completes the proof.

For later purpose, it is important to understand the differentiability of the operator T_ε with respect to Λ, ξ'. Consider the L_*^∞(resp. L_{**}^∞) functions defined on Ω_ε with $\|\cdot\|_*$ norm (resp. $\|\cdot\|_{**}$ norm). We have the following result.

Proposition 4.9. *Under the conditions of Proposition 4.5, the map* $(\Lambda, \xi') \mapsto T_\varepsilon(h)$ *is* C^1 *with respect to* Λ, ξ' *in the considered region and the* L^∞_* *norm. Moreover,*

$$\|\partial_\Lambda T_\varepsilon(h)\|_* \le C\|h\|_{**}, \qquad \|\partial_{\xi'} T_\varepsilon(h)\|_* \le C\|h\|_{**}. \tag{4.47}$$

Proof. T_ε is C^1 with respect to Λ and ξ' follows from the smoothness of K and $\tilde{h}$, which occur in the implicit definition (4.45) of $\phi = T_\varepsilon(h)$, with respect to these variables. Differentiating (4.30) with respect to $\xi'_k (k = 1, 2, 3)$, set $\phi = T_\varepsilon(h)$, $Y = \partial_{\xi'_k}\phi$ and $d_i = \partial_{\xi'_k} c_i$, $k = 1, 2, 3$, then Y satisfies

$$\begin{cases} -\Delta Y - (5-\varepsilon)V^{4-\varepsilon}Y - \lambda q \varepsilon^\alpha V^{q-1} Y = \bar{h} + \sum\limits_{i=0}^{3} d_i w^4_{\Lambda,\xi'} Z_i & \text{in } \Omega_\varepsilon; \\ Y = 0 \ \text{ on } \partial\Omega_\varepsilon; \quad \int_{\Omega_\varepsilon} \left[\phi \, \partial_{\xi'_k}\left(w^4_{\Lambda,\xi'}Z_i\right) + Y w^4_{\Lambda,\xi'} Z_i\right] = 0 & i = 0, \cdots, 3, \end{cases} \tag{4.48}$$

where

$$\bar{h} = (5-\varepsilon)(4-\varepsilon)V^{3-\varepsilon}Z_i\phi + \lambda q(q-1)\varepsilon^\alpha V^{q-2} Z_i \phi + \sum_{i=0}^{3} c_i \partial_{\xi'_k}\left(w^4_{\Lambda,\xi'}Z_i\right).$$

Set $\eta = Y - \sum\limits_{j=0}^{3} b_j Z_j$, where $b_j \in \mathbb{R}$ is chosen such that

$$\int_{\Omega_\varepsilon} \eta w^4_{\Lambda,\xi'} Z_i = 0,$$

that is, b_j satisfies

$$\sum_{j=0}^{3} b_j \int_{\Omega_\varepsilon} w^4_{\Lambda,\xi'} Z_i Z_j = \int_{\Omega_\varepsilon} Y w^4_{\Lambda,\xi'} Z_i. \tag{4.49}$$

Since this system is almost diagonal, it has a unique solution and we have

$$|b_j| \le C\|\phi\|_*. \tag{4.50}$$

Moreover, η satisfies

$$\begin{cases} -\Delta \eta - (5-\varepsilon)V^{4-\varepsilon}\eta - \lambda q \varepsilon^\alpha V^{q-1}\eta = g + \sum\limits_{i=0}^{3} d_i w^4_{\Lambda,\xi'} Z_i & \text{in } \Omega_\varepsilon; \\ \eta = 0 & \text{on } \partial\Omega_\varepsilon; \\ \int_{\Omega_\varepsilon} \eta w^4_{\Lambda,\xi'} Z_i = 0 \quad i = 0, 1, 2, 3, \end{cases} \tag{4.51}$$

with

$$g = \sum_{j=0}^{3} b_j \left[-\Delta Z_j - (5-\varepsilon)V^{4-\varepsilon}Z_j - \lambda q \varepsilon^\alpha V^{q-1} Z_j\right] + \bar{h}.$$

By Proposition 4.5, we have that $\eta = T_\varepsilon(g)$ and

$$\|\eta\|_* \leq C\|g\|_{**}. \tag{4.52}$$

On the other hand, we have

$$\begin{aligned} \|g\|_{**} \leq & \sum_{j=0}^{3} |b_j| \left\| -\Delta Z_j - (5-\varepsilon)V^{4-\varepsilon} Z_j - \lambda q \varepsilon^\alpha V^{q-1} Z_j \right\|_{**} \\ & + C\|V^{3-\varepsilon} Z_i \phi\|_{**} + C\varepsilon^\alpha \|V^{q-2} Z_i \phi\|_{**} \\ & + \sum_{i=0}^{3} |c_i| \left\| \partial_{\xi_k'} \left(w^4_{\Lambda,\xi'} Z_i \right) \right\|_{**}. \end{aligned}$$

Now we estimate all terms in the right hand side in above inequality. We have

$$\begin{aligned} & \left\| -\Delta Z_j - (5-\varepsilon)V^{4-\varepsilon} Z_j - \lambda q \varepsilon^\alpha V^{q-1} Z_j \right\|_{**} \\ & \leq C \left\| w^{-\theta}_{\Lambda,\xi'} \left[-\Delta Z_j - (5-\varepsilon)V^{4-\varepsilon} Z_j - \lambda q \varepsilon^\alpha V^{q-1} Z_j \right] \right\|_\infty \leq C, \end{aligned}$$

$$\left\| V^{3-\varepsilon} Z_i \phi \right\|_{**} \leq C \left\| w^{-\theta}_{\Lambda,\xi'} V^{3-\varepsilon} Z_i \phi \right\|_\infty \leq C\|\phi\|_* \|w^{1-\varepsilon}_{\Lambda,\xi'} Z_i\|_\infty \leq C\|\phi\|_*,$$

and

$$\varepsilon^\alpha \|V^{q-2} Z_i \phi\|_{**} \leq C\varepsilon^\alpha \left\| w^{-\theta}_{\Lambda,\xi'} V^{q-2} Z_i \phi \right\|_\infty \leq C\varepsilon^{\alpha+q-3} \|\phi\|_* = o(\|\phi\|_*).$$

From (4.34), we find

$$\sum_{i=0}^{3} |c_i| \left\| \partial_{\xi_k'} \left(w^4_{\Lambda,\xi'} Z_i \right) \right\|_{**} \leq C\|h\|_{**} \|w^{-\theta}_{\Lambda,\xi'} \partial_{\xi_k'} \left(w^4_{\Lambda,\xi'} Z_i \right) \|_\infty \leq C\|h\|_{**}.$$

Thus we get

$$\|\eta\|_* \leq C\|h\|_{**}. \tag{4.53}$$

By (4.50), (4.53) and $\|Z_j\|_* \leq C$, we obtain that

$$\|\partial_{\xi_k'} \phi\|_* \leq \sum_{j=0}^{3} |b_j| \|Z_j\|_* + \|\eta\|_* \leq C(\|\phi\|_* + \|h\|_{**}) \leq C\|h\|_{**}.$$

Similarly, we can get the estimate for $\|\partial_\Lambda \phi\|_*$ in (4.47). □

4.3.2 The nonlinear problem

In this subsection, our purpose is to study the nonlinear problem. First, we estimate $\|R\|_{**}$, $\|\partial_\Lambda R\|_{**}$ and $\|\partial_{\xi'} R\|_{**}$.

Lemma 4.10. *Assume* $1 < q < 3$, *let* $\lambda > 0$ *be fixed and* ξ', Λ *satisfy (4.25), (4.26), then choosing* $2 < \theta < 3$ *appropriately in the norms (4.31), (4.32), there exists a constant* $C > 0$ *independent of* ξ', Λ, *such that*

$$\|R\|_{**} \leq C\varepsilon, \qquad \|\partial_\Lambda R\|_{**} \leq C\varepsilon, \qquad \|\partial_{\xi'} R\|_{**} \leq C\varepsilon, \tag{4.54}$$

for $\varepsilon > 0$ *small enough.*

Proof. Recall that $R = V^{5-\varepsilon} - w^5_{\Lambda,\xi'} + \lambda\varepsilon^\alpha V^q$. By (4.28), $V = w_{\Lambda,\xi'} + O(\varepsilon)$. Consequently,

$$\begin{aligned} |V^{5-\varepsilon} - w^5_{\Lambda,\xi'}| &\leq |V^{5-\varepsilon} - w^{5-\varepsilon}_{\Lambda,\xi'}| + |w^{5-\varepsilon}_{\Lambda,\xi'} - w^5_{\Lambda,\xi'}| \\ &\leq C\varepsilon \left(w^{4-\varepsilon}_{\Lambda,\xi'} + w^5_{\Lambda,\xi'} |\log w_{\Lambda,\xi'}| \right). \end{aligned}$$

Thus for $2 < \theta < 3$,

$$\begin{aligned} \|V^{5-\varepsilon} - w^5_{\Lambda,\xi'}\|_{**} &\leq C\|w^{-\theta}_{\Lambda,\xi'}(V^{5-\varepsilon} - w^5_{\Lambda,\xi'})\|_\infty \\ &\leq C\varepsilon \sup_{\Omega_\varepsilon} w^{-\theta}_{\Lambda,\xi'}(w^{4-\varepsilon}_{\Lambda,\xi'} + w^5_{\Lambda,\xi'}|\log w_{\Lambda,\xi'}|) \leq C\varepsilon. \end{aligned}$$

Moreover,

$$\|\lambda\varepsilon^\alpha V^q\|_{**} \leq C\lambda\varepsilon^\alpha \|w^{-\theta}_{\Lambda,\xi'} V^q\|_\infty \leq C\lambda\varepsilon^\alpha \sup_{\Omega_\varepsilon} |w^{q-\theta}_{\Lambda,\xi'}| \leq \begin{cases} C\lambda\varepsilon^\alpha & \text{if } q > \theta; \\ C\lambda\varepsilon^{\alpha+q-\theta} & \text{if } q \leq \theta. \end{cases}$$

Note that $\alpha = \frac{5-q}{2} + O(\varepsilon)$, we choose $2 < \theta < \frac{3+q}{2}$, so $\alpha + q - \theta > 1$. Therefore we get the first estimate in (4.54). Furthermore

$$\partial_\Lambda R = (5-\varepsilon)V^{4-\varepsilon} Z_0 - 5w^4_{\Lambda,\xi'}\tilde{Z}_0 + \lambda q\varepsilon^\alpha V^{q-1} Z_0,$$

and

$$\partial_{\xi'_i} R = (5-\varepsilon)V^{4-\varepsilon} Z_i - 5w^4_{\Lambda,\xi'}\tilde{Z}_i + \lambda q\varepsilon^\alpha V^{q-1} Z_i, \quad i = 1,2,3.$$

By similar computations, we can get the rest estimates in (4.54). $\square$

Now we consider the following problem

$$\begin{cases} -\Delta\phi - (5-\varepsilon)V^{4-\varepsilon}\phi - \lambda q\varepsilon^\alpha V^{q-1}\phi = N(\phi) + R + \sum_{i=0}^{3} c_i w^4_{\Lambda,\xi'} Z_i & \text{in } \Omega_\varepsilon; \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \phi w^4_{\Lambda,\xi'} Z_i = 0 \quad i = 0,1,2,3. \end{cases} \tag{4.55}$$

Proposition 4.11. *There exists $C>0$ independent of ξ', Λ satisfying (4.25), (4.26), such that for $\varepsilon>0$ small enough, there exists a unique solution $\phi=\phi(\Lambda,\xi')$ of problem (4.55), satisfying*

$$\|\phi\|_* \le C\varepsilon. \tag{4.56}$$

Proof. By Proposition 4.5, problem (4.55) can be written as the fixed point problem

$$\phi = T_\varepsilon(N(\phi)+R) := A_\varepsilon(\phi).$$

Define

$$\mathcal{F}_M = \{\phi \in H_0^1(\Omega_\varepsilon)\cap L^\infty(\Omega_\varepsilon) \;:\; \|\phi\|_* \le M\varepsilon\}$$

with $M>0$ large but fixed which will be chosen later. Then A_ε sends $\mathcal{F}_M$ into itself.

Indeed, we have

$$\|A_\varepsilon(\phi)\|_* = \|T_\varepsilon(N(\phi)+R)\|_* \le C(\|N(\phi)\|_{**}+\|R\|_{**}). \tag{4.57}$$

Moreover,

$$\begin{aligned}
&\|N(\phi)\|_{**} = \left\|\int_0^1 [f_\varepsilon'(V+t\phi)-f_\varepsilon'(V)]\,\phi\;dt\right\|_{**}\\
&\le C\left\|w_{\Lambda,\xi'}^{-2}\int_0^1 |f_\varepsilon'(V+t\phi)-f_\varepsilon'(V)|\;dt\right\|_\infty \|\phi\|_*\\
&\le C\left(\left\|w_{\Lambda,\xi'}^{-2}[(V+|\phi|)^{4-\varepsilon}-V^{4-\varepsilon}]\right\|_\infty + \lambda\varepsilon^\alpha\left\|w_{\Lambda,\xi'}^{-2}[(V+|\phi|)^{q-1}-V^{q-1}]\right\|_\infty\right)\|\phi\|_*.
\end{aligned} \tag{4.58}$$

Since

$$\begin{aligned}
&\left\|w_{\Lambda,\xi'}^{-2}[(V+|\phi|)^{4-\varepsilon}-V^{4-\varepsilon}]\right\|_\infty \le C\left\|w_{\Lambda,\xi'}^{-2}\left(w_{\Lambda,\xi'}^{3-\varepsilon}|\phi|+|\phi|^{4-\varepsilon}\right)\right\|_\infty\\
\le\;& C\left\|w_{\Lambda,\xi'}^{\theta-1-\varepsilon}\right\|_\infty\|\phi\|_* + C\left\|w_{\Lambda,\xi'}^{(\theta-2)(4-\varepsilon)-2}\right\|_\infty\|\phi\|_*^{4-\varepsilon}\\
\le\;& C\varepsilon^{\theta-1-\varepsilon}\|\phi\|_* + C\varepsilon^{\min\{(\theta-2)(4-\varepsilon)-2,0\}}\|\phi\|_*^{4-\varepsilon}.
\end{aligned} \tag{4.59}$$

On the other hand, by Lemma 2.2 in [77], we have

$$||V+\phi|^{q-1}-|V|^{q-1}| \le C\begin{cases}|V|^{q-2}|\phi|+|\phi|^{q-1} & \text{if } 2\le q<3;\\ \min\{|V|^{q-2}|\phi|,|\phi|^{q-1}\} & \text{if } 1<q<2.\end{cases}$$

Thus for $1<q<2$,

$$\begin{aligned}
\left\|w_{\Lambda,\xi'}^{-2}[(V+|\phi|)^{q-1}-V^{q-1}]\right\|_\infty &\le C\min\left\{\|w_{\Lambda,\xi'}\|_\infty^{q-4+\theta-2}\|\phi\|_*, \|w_{\Lambda,\xi'}\|_\infty^{(\theta-2)(q-1)-2}\|\phi\|_*^{q-1}\right\}\\
&\le C\min\left\{\varepsilon^{q+\theta-6}\|\phi\|_*, \varepsilon^{(\theta-2)(q-1)-2}\|\phi\|_*^{q-1}\right\}.
\end{aligned} \tag{4.60}$$

For $2 \leq q < 3$,

$$\begin{aligned} \left\| w_{\Lambda,\xi'}^{-2}[(V+|\phi|)^{q-1} - V^{q-1}] \right\|_\infty &\leq C \left\| w_{\Lambda,\xi'}^{-2}[w_{\Lambda,\xi'}^{q-2}|\phi| + |\phi|^{q-1}] \right\|_\infty \\ &\leq C\varepsilon^{q+\theta-6}\|\phi\|_* + C\varepsilon^{(\theta-2)(q-1)-2}\|\phi\|_*^{q-1}. \end{aligned} \tag{4.61}$$

From (4.58)-(4.61), if $1 < q < 3$, for $\phi \in \mathcal{F}_M$, then we have

$$\|N(\phi)\|_{**} \leq C\varepsilon^\tau \|\phi\|_*, \quad \text{with some } \tau > 0. \tag{4.62}$$

Thus by (4.54), (4.57) and (4.62), we find for $\phi \in \mathcal{F}_M$,

$$\|A_\varepsilon(\phi)\|_* \leq C(\varepsilon^\tau\|\phi\|_* + \varepsilon) \leq C(M\varepsilon^\tau + 1)\varepsilon.$$

Choosing M large such that $C(M\varepsilon^\tau + 1) \leq M$. It implies that $A_\varepsilon(\mathcal{F}_M) \subset \mathcal{F}_M$.

Next we show that A_ε is a contraction map. For $\phi_1, \phi_2 \in \mathcal{F}_M$,

$$\begin{aligned} \|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* &\leq C\|N(\phi_1) - N(\phi_2)\|_{**} \\ &= C\|[f'_\varepsilon(V + t\phi_1 + (1-t)\phi_2) - f'_\varepsilon(V)](\phi_1 - \phi_2)\|_{**} \\ &= \|[f'_\varepsilon(V+\tilde{\phi}) - f'_\varepsilon(V)](\phi_1 - \phi_2)\|_{**} \\ &\leq C \left\| w_{\Lambda,\xi'}^{-\theta}[f'_\varepsilon(V+\tilde{\phi}) - f'_\varepsilon(V)](\phi_1 - \phi_2) \right\|_\infty \\ &\leq C \left\| w_{\Lambda,\xi'}^{-2}[f'_\varepsilon(V+\tilde{\phi}) - f'_\varepsilon(V)] \right\|_\infty \|\phi_1 - \phi_2\|_*, \end{aligned}$$

where $\tilde{\phi} = t\phi_1 + (1-t)\phi_2 \in \mathcal{F}_M$ for $t \in (0,1)$. It can be easily checked that

$$\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* \leq C\varepsilon^\tau\|\phi_1 - \phi_2\|_*, \quad \text{with some } \tau > 0.$$

It yields that A_ε has a unique fixed point in $\mathcal{F}_M$. Hence problem (4.55) has a unique solution ϕ such that $\|\phi\|_* \leq C\varepsilon$, for some $C > 0$. □

Proposition 4.12. *The solution $\phi(\Lambda, \xi')$ constructed in Proposition 4.11 is C^1 with respect to Λ and ξ' in the considered region. Moreover,*

$$\|\partial_\Lambda \phi\|_* \leq C\varepsilon, \qquad \|\partial_{\xi'}\phi\|_* \leq C\varepsilon. \tag{4.63}$$

Proof. We write

$$B(\Lambda, \xi', \phi) = \phi - T_\varepsilon(N(\phi) + R), \tag{4.64}$$

we have

$$B(\Lambda, \xi', \phi) = 0, \tag{4.65}$$

and

$$\partial_\phi B(\Lambda, \xi', \phi)[\psi] = \psi - \partial_\phi[T_\varepsilon(N(\phi) + R)]\psi = \psi - T_\varepsilon[\partial_\phi(N(\phi))\psi]. \tag{4.66}$$

By a direct calculation, we get

$$\|T_\varepsilon[\partial_\phi(N(\phi))\psi]\|_* \le C\|\partial_\phi(N(\phi))\psi\|_{**} \le C\|w_{\Lambda,\xi'}^{-2}\partial_\phi(N(\phi))\|_\infty\|\psi\|_* \le C\varepsilon^\tau\|\psi\|_*.$$

with $\tau > 0$. Therefore

$$\|\partial_\phi B(\Lambda,\xi',\phi)[\psi]\|_* \le (1+C\varepsilon^\tau)\|\psi\|_*.$$

It follows that for $\varepsilon > 0$ small enough, $\partial_\phi B(\Lambda,\xi',\phi)$ is invertible in $\|\cdot\|_*$ with uniformly bounded inverse. It also depends continuously on its parameters. Let us differentiate (4.64) with respect to ξ' and by (4.66), we have

$$\partial_{\xi'}B(\Lambda,\xi',\phi) = -(\partial_{\xi'}T_\varepsilon)(N(\Lambda,\xi',\phi)+R) - T_\varepsilon((\partial_{\xi'}N)(\Lambda,\xi',\phi)+\partial_{\xi'}R), \tag{4.67}$$

where all the previous expressions depend continuously on their parameters. Hence the implicit function theorem implies that $\phi = \phi(\Lambda,\xi')$ is C^1 with respect to Λ,ξ' in the considered region.

Moreover, differentiating (4.65) with respect to ξ', we get

$$\partial_{\xi'}\phi = -(\partial_\phi B(\Lambda,\xi',\phi))^{-1}\partial_{\xi'}B(\Lambda,\xi',\phi).$$

By (4.67), (4.47) and (4.34), we get

$$\|\partial_{\xi'}\phi\|_* \le C(\|N(\phi)\|_{**} + \|R\|_{**} + \|(\partial_{\xi'}N)(\Lambda,\xi',\phi)\|_{**} + \|\partial_{\xi'}R\|_{**}) \le C\varepsilon.$$

Similarly, we can get $\|\partial_\Lambda\phi\|_* \le C\varepsilon$. □

4.3.3 The reduced functional

We have solved the nonlinear problem (4.55). In order to find a solution to problem (4.23), we need to find Λ and ξ' such that

$$c_i(\Lambda,\xi') = 0 \quad \text{for } i = 0,1,2,3. \tag{4.68}$$

The energy functional to problem (4.23) is given by

$$I(v) = \frac{1}{2}\int_{\Omega_\varepsilon}|\nabla v|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon}|v|^{p+1} - \lambda\frac{\varepsilon^\alpha}{q+1}\int_{\Omega_\varepsilon}|v|^{q+1}$$

Set

$$\mathcal{I}(\Lambda,\xi') = I\left(V_{\Lambda,\xi'}(y) + \phi_{\Lambda,\xi'}(y)\right), \tag{4.69}$$

where $V_{\Lambda,\xi'}$ is defined in (4.24) and $\phi_{\Lambda,\xi'}$ is solved by Proposition 4.11. We have the following fact.

Lemma 4.13. *Let ξ' and Λ satisfy (4.25) and (4.26). Then the functional $\mathcal{I}(\Lambda,\xi')$ is of class C^1. Moreover, for all $\varepsilon > 0$ sufficiently small, the function $v(y) = V_{\Lambda,\xi'}(y) + \phi_{\Lambda,\xi'}(y)$ is a solution to problem (4.23) if and only if (Λ,ξ') is a critical point of $\mathcal{I}(\Lambda,\xi')$.*

Proof. As a consequence of Proposition 4.12, we can get the map $(\Lambda, \xi') \mapsto \mathcal{I}(\Lambda, \xi')$ is of class C^1. For $k \in \{1, 2, 3\}$, we have

$$\begin{aligned}
\partial_{\xi'_k} \mathcal{I}(\Lambda, \xi') &= DI(V_{\Lambda,\xi'} + \phi_{\Lambda,\xi'}) \left[\frac{\partial V_{\Lambda,\xi'}}{\partial \xi'_k} + \frac{\partial \phi_{\Lambda,\xi'}}{\partial \xi'_k} \right] \\
&= \sum_{i=0}^{3} c_i \int_{\Omega_\varepsilon} w^4_{\Lambda,\xi'} Z_i \left[\frac{\partial V_{\Lambda,\xi'}}{\partial \xi'_k} + \frac{\partial \phi_{\Lambda,\xi'}}{\partial \xi'_k} \right] \\
&= \sum_{i=0}^{3} c_i \int_{\Omega_\varepsilon} w^4_{\Lambda,\xi'} Z_i Z_k \left(1 + o(1)\right),
\end{aligned}$$

here we use the fact that $\|\partial_{\xi'_k} \phi_{\Lambda,\xi'}\|_* = O(\varepsilon)$. Similarly, we find

$$\partial_\Lambda \mathcal{I}(\Lambda, \xi') = \sum_{i=0}^{3} c_i \int_{\Omega_\varepsilon} w^4_{\Lambda,\xi'} Z_i Z_0 \left(1 + o(1)\right),$$

where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly for the norm $\|\cdot\|_*$. It defines an almost diagonal linear equation system for c_i. Thus (Λ, ξ') is a critical point of $\mathcal{I}(\Lambda, \xi')$ if and only if $c_i = 0$ for $i = 0, 1, 2, 3$. This ends the proof of Lemma. □

Lemma 4.14. *As $\varepsilon \to 0$, we have the following expansion*

$$\mathcal{I}(\Lambda, \xi') - I(V_{\Lambda,\xi'}) = o(\varepsilon),$$

where $o(\varepsilon)$ is in the C^1- sense uniformly on ξ', Λ satisfying (4.25),(4.26).

Proof. For notation simplicity, we write $V_{\Lambda,\xi'}$ by V, and $\phi_{\Lambda,\xi'}$ by ϕ. By the Taylor expansion and the fact that $DI(V_{\Lambda,\xi'} + \phi_{\Lambda,\xi'})[\phi] = 0$, we have

$$\begin{aligned}
& \mathcal{I}(\Lambda, \xi') - I(V_{\Lambda,\xi'}) \\
= \ & I\left(V + \phi\right) - I(V) = \int_0^1 D^2 I\left(V + t\phi\right)[\phi, \phi] t \, dt \\
= \ & \int_0^1 \left[\int_{\Omega_\varepsilon} \left(|\nabla \phi|^2 - p(V + t\phi)^{p-1} \phi^2 - \lambda \varepsilon^\alpha q (V + t\phi)^{q-1} \phi^2 \right) dy \right] t \, dt \\
= \ & \int_0^1 \left\{ \int_{\Omega_\varepsilon} \left(p \left[V^{p-1} - (V + t\phi)^{p-1} \right] \phi^2 + \left[R + N(\phi) \right] \phi \right.\right. \\
& \qquad \left.\left. + \lambda \varepsilon^\alpha q \left[V^{q-1} - (V + t\phi)^{q-1} \right] \phi^2 \right) dy \right\} t \, dt \\
\le \ & C \int_{\Omega_\varepsilon} |V^{p-1} - (V + \phi)^{p-1}| \phi^2 \, dy + C \lambda \varepsilon^\alpha \int_{\Omega_\varepsilon} |V^{q-1} - (V + \phi)^{q-1}| \phi^2 \, dy \\
& + \int_{\Omega_\varepsilon} |R| \, |\phi| \, dy + \int_{\Omega_\varepsilon} |N(\phi)| \, |\phi| \, dy \\
:= \ & I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{4.70}$$

where

$$
\begin{aligned}
I_1 &\leq C\int_{\Omega_\varepsilon}(V^{3-\varepsilon}|\phi|+|\phi|^{4-\varepsilon})\phi^2\,dy \leq C\int_{\Omega_\varepsilon}(w_{\Lambda,\xi'}^{3-\varepsilon}|\phi|^3+|\phi|^{6-\varepsilon})\,dy\\
&\leq C\|\phi\|_*^3\int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^{3-\varepsilon+3(\theta-2)}\,dy + C\|\phi\|_*^{6-\varepsilon}\int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^{(6-\varepsilon)(\theta-2)}\,dy\\
&\leq C\|\phi\|_*^3 \leq C\varepsilon^3 = o(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= C\lambda\varepsilon^\alpha\int_{\Omega_\varepsilon}|V^{q-1}-(V+\phi)^{q-1}||\phi|^2\,dy\\
&\leq C\lambda\varepsilon^\alpha\begin{cases}\|\phi\|_*^3\int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^{q-2+3(\theta-2)}dy+\|\phi\|_*^{q+1}\int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^{(q+1)(\theta-2)}dy & \text{if } 2\leq q<3;\\ \\ \min\left\{\|\phi\|_*^3\int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^{q-2+3(\theta-2)}dy, \|\phi\|_*^{q+1}\int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^{(q+1)(\theta-2)}dy\right\} & \text{if } 1<q<2.\end{cases}\\
&\leq C\lambda\varepsilon^{\min\{\alpha+q-2+3(\theta-2),\ \alpha+q-2+(q+1)(\theta-2)\}} = o(\varepsilon),
\end{aligned}
$$

since $\|R\|_{**}\leq C\varepsilon$, $\|N(\phi)\|_{**}\leq C\varepsilon^\tau\|\phi\|_*$ and $\|\phi\|_*\leq C\varepsilon$, we get

$$
\begin{aligned}
I_3 &= \int_{\Omega_\varepsilon}|R|\ |\phi|\ dy = \int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^{-\theta}|R|w_{\Lambda,\xi'}^{-(\theta-2)}|\phi|w_{\Lambda,\xi'}^{2\theta-2}\,dy\\
&\leq C\|R\|_{**}\|\phi\|_*\int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^{2\theta-2}\,dy \leq C\varepsilon^{2\theta-5}\|R\|_{**}\|\phi\|_* \leq C\varepsilon^{2\theta-3} = o(\varepsilon),
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \int_{\Omega_\varepsilon}|N(\phi)|\ |\phi|\ dy = \int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^{-\theta}|N(\phi)|w_{\Lambda,\xi'}^{-(\theta-2)}|\phi|w_{\Lambda,\xi'}^{2\theta-2}\,dy\\
&\leq C\|N(\phi)\|_{**}\|\phi\|_*\int_{\Omega_\varepsilon} w_{\Lambda,\xi'}^{2\theta-2}\,dy\\
&\leq C\varepsilon^{2\theta-5}\|N(\phi)\|_{**}\|\phi\|_* \leq C\varepsilon^{2\theta+\tau-3} = o(\varepsilon).
\end{aligned}
$$

Therefore,

$$\mathcal{I}(\Lambda,\xi') - I(V_{\Lambda,\xi'}) = o(\varepsilon).$$

where $o(\varepsilon)$ is uniform in the C^1-sense for ξ', Λ satisfying (4.25),(4.26). By a similar way, we can obtain

$$D_{(\Lambda,\xi')}\left(\mathcal{I}(\Lambda,\xi')-I(V)\right) = o(\varepsilon).$$

This ends the proof of Lemma. □

Lemma 4.15. *Under the change of variable (4.24), as $\varepsilon\to 0$, we have*

$$I(V_{\Lambda,\xi'}) = J(U_{\mu,\xi}) + c_0\varepsilon\log\varepsilon + o(\varepsilon), \tag{4.71}$$

where $o(\varepsilon)$ is in the C^1- sense uniformly on ξ', Λ satisfying (4.25),(4.26), c_0 is a positive constant.

Proof. In fact,

$$
\begin{aligned}
I(V_{\Lambda,\xi'}) &= \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla V_{\Lambda,\xi'}(y)|^2 dy - \frac{1}{p+1}\int_{\Omega_\varepsilon} |V_{\Lambda,\xi'}(y)|^{p+1} dy - \lambda \frac{\varepsilon^\alpha}{q+1}\int_{\Omega_\varepsilon} |V_{\Lambda,\xi'}(y)|^{q+1} dy \\
&= \frac{1}{2}\varepsilon^3 \int_{\Omega_\varepsilon} |\nabla U_{\mu,\xi}(\varepsilon y)|^2 dy - \frac{\varepsilon^{\frac{p+1}{2}}}{p+1}\int_{\Omega_\varepsilon} |U_{\mu,\xi}(\varepsilon y)|^{p+1} dy - \lambda \frac{\varepsilon^{\alpha+\frac{q+1}{2}}}{q+1}\int_{\Omega_\varepsilon} |U_{\mu,\xi}(\varepsilon y)|^{q+1} dy \\
&= \frac{1}{2}\int_{\Omega} |\nabla U_{\mu,\xi}(x)|^2 dx - \frac{1}{p+1}\varepsilon^{\frac{p+1}{2}-3}\int_{\Omega} |U_{\mu,\xi}(x)|^{p+1} dx - \lambda \frac{\varepsilon^{\alpha+\frac{q+1}{2}-3}}{q+1}\int_{\Omega} |U_{\mu,\xi}(x)|^{q+1} dx \\
&= \frac{1}{2}\int_{\Omega} |\nabla U_{\mu,\xi}(x)|^2 dx - \frac{\varepsilon^{-\frac{\varepsilon}{2}}}{p+1}\int_{\Omega} |U_{\mu,\xi}(x)|^{p+1} dx - \lambda \frac{\varepsilon^{\frac{1-q}{2}\frac{\varepsilon}{4-\varepsilon}}}{q+1}\int_{\Omega} |U_{\mu,\xi}(x)|^{q+1} dx \\
&= \frac{1}{2}\int_{\Omega} |\nabla U_{\mu,\xi}(x)|^2 \, dx - \frac{1}{p+1}\int_{\Omega} U_{\mu,\xi}(x)^{p+1} \, dx - \frac{\lambda}{q+1}\int_{\Omega} U_{\mu,\xi}(x)^{q+1} \, dx \\
&\quad + \frac{1}{p+1}\left[1-\varepsilon^{-\frac{\varepsilon}{2}}\right]\int_{\Omega} U_{\mu,\xi}(x)^{p+1} \, dx + \frac{\lambda}{q+1}\left[1-\varepsilon^{\frac{1-q}{2}\frac{\varepsilon}{4-\varepsilon}}\right]\int_{\Omega} U_{\mu,\xi}(x)^{q+1} \, dx \\
&= J(U_{\mu,\xi}) + \frac{1}{p+1}\left[1-\varepsilon^{-\frac{\varepsilon}{2}}\right]\int_{\Omega} U_{\mu,\xi}(x)^{p+1} \, dx + \frac{\lambda}{q+1}\left[1-\varepsilon^{\frac{1-q}{2}\frac{\varepsilon}{4-\varepsilon}}\right]\int_{\Omega} U_{\mu,\xi}(x)^{q+1} \, dx.
\end{aligned}
$$

While,

$$
\begin{aligned}
\frac{1}{p+1}\left[1-\varepsilon^{-\frac{\varepsilon}{2}}\right]\int_{\Omega} U_{\mu,\xi}(x)^{p+1} \, dx &= \left(\frac{1}{6}+\frac{1}{36}\varepsilon + o(\varepsilon)\right)\left[\frac{1}{2}\varepsilon\log\varepsilon + o(\varepsilon\log\varepsilon)\right] \\
&\quad \times \int_{\Omega} U_{\mu,\xi}(x)^6 \left[1-\varepsilon\log U_{\mu,\xi}(x) + o(\varepsilon)\right] \\
&= \frac{1}{12}\varepsilon\log\varepsilon \int_{\Omega} U_{\mu,\xi}(x)^6 dx + o(\varepsilon) \\
&= c_0\varepsilon\log\varepsilon + o(\varepsilon),
\end{aligned}
$$

with $c_0 = \frac{1}{12}\int_{\mathbb{R}^3} w_{\mu,\xi}(x)^6 dx$. Moreover,

$$
\begin{aligned}
&\frac{\lambda}{q+1}\left[1-\varepsilon^{\frac{1-q}{2}\frac{\varepsilon}{4-\varepsilon}}\right]\int_{\Omega} U_{\mu,\xi}(x)^{q+1} \, dx \\
= \ &\frac{\lambda(q-1)}{8(q+1)}\varepsilon\log\varepsilon \int_{\Omega} w_{\mu,\xi}(x)^{q+1} dx + o(\varepsilon) = o(\varepsilon),
\end{aligned}
$$

where $o(\varepsilon)$ is in the C^1- sense uniformly on ξ', Λ satisfying (4.25),(4.26). Thus (4.71) holds. $\square$

Proof of Theorem 4.1. Since $p = 5-\varepsilon$ with $\varepsilon > 0$ is the subcritical exponent and Ω is a smooth bounded domain, for $\lambda > 0$ fixed, by the mountain pass theorem [102, Theorem 2.2], problem (4.2) has a mountain pass solution, denoted u_1. The mountain pass critical value is given by

$$0 < c_m = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)),$$

where
$$\Gamma = \Big\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\Big\}$$
with $e \in H_0^1(\Omega)$ such that $J(e) < 0$. Moreover we have the following assertion:

there exists $\lambda_0 > 0$, depending on Ω, q, such that for any $\lambda \geq \lambda_0$ and $\varepsilon \geq 0$, we have $J(u_1) < \frac{\sqrt{3}}{4}\pi^2$.

Indeed fixed $u_0 \in H_0^1(\Omega) \setminus \{0\}$ with $u_0 \geq 0$ in Ω, we have
$$J(tu_0) = \frac{t^2}{2}\int_\Omega |\nabla u_0|^2 - \frac{t^{p+1}}{p+1}\int_\Omega |u_0|^{p+1} - \frac{\lambda t^{q+1}}{q+1}\int_\Omega |u_0|^{q+1}.$$
First note that $\lim_{t\to+\infty} J(tu_0) = -\infty$, thus there exists $t_\lambda \geq 0$ such that $J(t_\lambda u_0) = \max\limits_{t\geq 0} J(tu_0)$. Moreover t_λ satisfies
$$\begin{aligned} t_\lambda^2 \int_\Omega |\nabla u_0|^2 &= t_\lambda^{p+1}\int_\Omega |u_0|^{p+1} + \lambda t_\lambda^{q+1}\int_\Omega |u_0|^{q+1} \\ &\geq t_\lambda^{p+1}\int_\Omega |u_0|^{p+1}, \end{aligned} \tag{4.72}$$
which implies that $t_\lambda \leq \left\{\frac{\int_\Omega |\nabla u_0|^2}{\int_\Omega |u_0|^{p+1}}\right\}^{\frac{1}{p-1}}$. It follows that
$$\lim_{\lambda\to+\infty} t_\lambda = 0 \tag{4.73}$$
If (4.73) fails, then there exists some sequence $t_{\lambda_n} \to t_0 > 0$ as $\lambda_n \to +\infty$. By the first equality in (4.72), we get
$$\lim_{n\to+\infty} t_{\lambda_n}^2 \int_\Omega |\nabla u_0|^2 = \lim_{n\to+\infty}\left(t_{\lambda_n}^{p+1}\int_\Omega |u_0|^{p+1} + \lambda_n t_{\lambda_n}^{q+1}\int_\Omega |u_0|^{q+1}\right) = +\infty,$$
which leads to a contradiction, since $\{t_{\lambda_n}\}$ is bounded.

Therefore, there exists $\lambda_0 > 0$, which depends on Ω, q, by Lemma 4.18 and (4.73), for $\lambda \geq \lambda_0$, we have
$$\begin{aligned} 0 < c_m &\leq \max_{t\geq 0} J(tu_0) = J(t_\lambda u_0) \\ &= \frac{t_\lambda^2}{2}\int_\Omega |\nabla u_0|^2 - \frac{t_\lambda^{p+1}}{p+1}\int_\Omega |u_0|^{p+1} - \frac{\lambda t_\lambda^{q+1}}{q+1}\int_\Omega |u_0|^{q+1} \\ &\leq \frac{t_\lambda^2}{2}\int_\Omega |\nabla u_0|^2 - \frac{t_\lambda^{p+1}}{p+1}\int_\Omega |u_0|^{p+1} \to 0. \end{aligned}$$
In particular, $J(u_1) < \frac{\sqrt{3}}{4}\pi^2$.

Next we prove existence of the large solution of (4.2). By Lemma 4.13, we know that $u(\varepsilon y) = \varepsilon^{-\kappa}\left(V_{\Lambda,\xi'}(y) + \phi_{\Lambda,\xi'}(y)\right)$ is a solution to problem (4.2) if and only if (Λ, ξ') is a critical point of $\mathcal{I}(\Lambda, \xi')$. So we have to prove existence of the critical point of $\mathcal{I}(\Lambda, \xi')$.

From Lemma 4.14 and (4.71), we have

$$\mathcal{I}(\Lambda, \xi') = J(U_{\mu,\xi}) + c_0 \varepsilon \log \varepsilon + o(\varepsilon).$$

This together with Lemma 4.4 yields that for $2 < q < 3$,

$$\mathcal{I}(\Lambda, \xi') = a_0 + \varepsilon\varphi(\Lambda, \xi) + a_3\varepsilon + o(\varepsilon), \tag{4.74}$$

where

$$\varphi(\Lambda, \xi) = a_1 \Lambda H(\xi, \xi) - a_2 \log \Lambda,$$

with constants $a_1, a_2 > 0$ being given in Lemma 4.4, and $o(\varepsilon)$ is uniform in the C^1 sense for ξ', Λ in the considered region.

Define

$$\widetilde{\mathcal{I}}(\Lambda, \xi') = \frac{1}{\varepsilon}\mathcal{I}(\Lambda, \xi') - \frac{a_0}{\varepsilon} - a_3.$$

Then we have

$$\widetilde{\mathcal{I}}(\Lambda, \xi') = \varphi(\Lambda, \xi) + o(1), \tag{4.75}$$

where $\xi' = \frac{\xi}{\varepsilon}$ and $o(1)$ is in the C^1- sense uniformly on ξ', Λ satisfying (4.25),(4.26). Since the function $H(\xi, \xi)$ has at least one critical point, denoted by ξ_0, with $H(\xi_0, \xi_0) > 0$, then (Λ_0, ξ_0), with $\Lambda_0 = \frac{a_2}{a_1 H(\xi_0,\xi_0)}$, is a nondegenerate critical point of $\varphi(\Lambda, \xi)$. It follows that the local degree $\deg(\nabla\varphi(\Lambda, \xi), \mathcal{O}, 0)$ is well defined and is nonzero, where $\mathcal{O}$ is arbitrary small neighborhood of (Λ_0, ξ_0). So $\deg(\nabla\widetilde{\mathcal{I}}(\Lambda, \xi'), \mathcal{O}, 0) \neq 0$ for $\varepsilon > 0$ small enough. Hence we find a critical point (Λ_*, ξ'_*) of $\widetilde{\mathcal{I}}(\Lambda, \xi')$, such that $(\Lambda_*, \xi'_*) \to (\Lambda_0, \xi'_0)$ with $\xi'_0 = \frac{\xi_0}{\varepsilon}$ as $\varepsilon \to 0$. Then (Λ_*, ξ'_*) is also a critical point of $\mathcal{I}(\Lambda, \xi')$. Thus we get that

$$u_2(x) = \varepsilon^{-\kappa} \left(V_{\Lambda_*, \xi'_*} + \phi_{\Lambda_*, \xi'_*}\right) \left(\frac{x}{\varepsilon}\right)$$

is the solution of problem (4.2). Recalling that $\kappa = \frac{2}{p-1} = \frac{1}{2} + \frac{1}{8}\varepsilon + o(\varepsilon)$, then by above construction and Lemma 4.4, we get (4.6) and (4.7).

Similarly, we can get existence of the large solution to problem (4.2) for $q = 2$ and $1 < q < 2$.

4.4 Proof of Theorem 4.2

In this section, we assume $2 < q < 3$, the aim is to construct the third solution by regarding $\lambda > 0$ as a large parameter. Set

$$\varrho = \lambda^{-\frac{2}{3-q}}.$$

We observe that $\varrho \to 0$ as $\lambda \to \infty$. Taking the following change of variable

$$\tilde{w}(y) = \varrho^{\frac{2}{4-\varepsilon}} u(\varrho y).$$

If u is a solution of problem (4.2), then $\tilde{w}$ satisfies

$$\begin{cases} -\Delta \tilde{w} = \tilde{w}^{5-\varepsilon} + \lambda \varrho^m \tilde{w}^q, \quad w > 0 & \text{in } \Omega_\varrho; \\ \tilde{w} = 0 & \text{on } \partial\Omega_\varrho, \end{cases} \tag{4.76}$$

where $\Omega_\varrho = \frac{\Omega}{\varrho}$ and $m = \frac{2(1-q)}{4-\varepsilon} + 2 = \frac{5-q}{2} + \frac{1-q}{8}(1+o(1))\varepsilon$. We observe that

$$\lambda \varrho^m = \lambda \varrho^{\frac{5-q}{2} + \frac{1-q}{8}(1+o(1))\varepsilon} \leq C\varrho \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{4.77}$$

For $\varepsilon > 0$ small and $\lambda > 0$ large enough, (4.10) is the limit equation of problem (4.76).

Let $U_{\mu,\xi}(x)$ be the unique solution of (4.12), we define in Ω_ϱ the function

$$\widetilde{W}_{\Lambda,\xi'}(y) = \varrho^{\frac{1}{2}} U_{\mu,\xi}(\varrho y), \quad \Lambda = \frac{\mu}{\varrho}, \quad \xi' = \frac{\xi}{\varrho}. \tag{4.78}$$

Then $\widetilde{W}_{\Lambda,\xi'}(y)$ satisfies

$$-\Delta \widetilde{W}_{\Lambda,\xi'}(y) = w^5_{\Lambda,\xi'}(y), \quad \text{in } \Omega_\varrho; \quad \widetilde{W}_{\Lambda,\xi'}(y) = 0 \quad \text{on } \partial\Omega_\varrho.$$

We assume that, for $\delta > 0$ small but fixed,

$$d(\xi', \partial\Omega_\varrho) \geq \frac{\delta}{\varrho} \qquad \text{and} \quad \delta < \Lambda < \frac{1}{\delta}. \tag{4.79}$$

From Lemma 4.3, we have

$$\widetilde{W}_{\Lambda,\xi'}(y) = w_{\Lambda,\xi'}(y) - 4\pi 3^{\frac{1}{4}} \Lambda^{\frac{1}{2}} \varrho H(\varrho y, \varrho \xi') + O(\varrho^3) \quad \text{in} \quad \Omega_\varrho, \quad \text{as} \quad \varrho \to 0.$$

We will look for a solution of (4.76) of the form

$$\tilde{w}(y) = \widetilde{W}_{\Lambda,\xi'}(y) + \tilde{\phi}(y),$$

where $\widetilde{W}_{\Lambda,\xi'}(y)$ is defined by (4.78) and $\tilde{\phi}$ is a small term. Then problem (4.76) becomes

$$\begin{cases} L_1(\tilde{\phi}) = N_1(\tilde{\phi}) + R_1 & \text{in } \Omega_\varrho; \\ \tilde{\phi} = 0 & \text{on } \partial\Omega_\varrho, \end{cases} \tag{4.80}$$

where

$$L_1(\tilde{\phi}) = -\Delta\tilde{\phi} - g'(\widetilde{W}_{\Lambda,\xi'})\tilde{\phi}, \qquad \text{with } \ g(w) = w^{5-\varepsilon} + \lambda\varrho^m w^q. \tag{4.81}$$

$$N_1(\tilde{\phi}) = g(\widetilde{W}_{\Lambda,\xi'} + \tilde{\phi}) - g(\widetilde{W}_{\Lambda,\xi'}) - g'(\widetilde{W}_{\Lambda,\xi'})\tilde{\phi}, \quad R_1 = \Delta\widetilde{W}_{\Lambda,\xi'} + g(\widetilde{W}_{\Lambda,\xi'}) \tag{4.82}$$

Next, we search for $\tilde{\phi}$ by the fixed point argument. For f a function in Ω_ϱ, we define the same weighted $L^\infty-$norms as (4.31) and (4.32). Namely,

$$\|f\|_{*,\varrho} = \sup_{y\in\Omega_\varrho} (1 + |y - \xi'|^2)^{\frac{\theta-2}{2}} |f(y)|, \tag{4.83}$$

and

$$\|f\|_{**,\varrho} = \sup_{y\in\Omega_\varrho}(1+|y-\xi'|^2)^{\frac{\theta}{2}}|f(y)|, \tag{4.84}$$

where θ satisfies

$$2 < \theta < q. \tag{4.85}$$

Then we can get

$$\|R_1\|_{**,\varrho} \le C\varrho, \quad \|D_{(\Lambda,\xi')}R_1\|_{**,\varrho} \le C\varrho. \tag{4.86}$$

In fact, we note that $\widetilde{W}_{\Lambda,\xi'}(y) = w_{\Lambda,\xi'}(y) + O(\varrho)$,

$$\begin{aligned}
\|R_1\|_{**,\varrho} &= \left\|\Delta\widetilde{W}_{\Lambda,\xi'} + \widetilde{W}^{5-\varepsilon}_{\Lambda,\xi'} + \lambda\varrho^m\widetilde{W}^q_{\Lambda,\xi'}\right\|_{**,\varrho} = \left\|\widetilde{W}^{5-\varepsilon}_{\Lambda,\xi'} - w^5_{\Lambda,\xi'}(y) + \lambda\varrho^m\widetilde{W}^q_{\Lambda,\xi'}\right\|_{**,\varrho} \\
&\le C\varrho\|w_{\Lambda,\xi'}(y)^4\|_{**,\varrho} + C\varepsilon\|w_{\Lambda,\xi'}(y)^5\log(w_{\Lambda,\xi'}(y))\|_{**,\varrho} + \lambda\varrho^m\left\|w^q_{\Lambda,\xi'}\right\|_{**,\varrho} \\
&\le C\varrho\|w_{\Lambda,\xi'}(y)^{4-\theta}\|_\infty + C\varepsilon\|w_{\Lambda,\xi'}(y)^{5-\theta}\log(w_{\Lambda,\xi'}(y))\|_\infty + C\lambda\varrho^m\left\|w^{q-\theta}_{\Lambda,\xi'}\right\|_\infty \\
&\le C\varrho\varrho^{4-\theta} + C\varepsilon\varrho^{5-\theta}|\log\varrho| + C\lambda\varrho^m\varrho^{q-\theta} \le C\varrho,
\end{aligned}$$

since $2 < \theta < q$. We get the first estimate in (4.86). By similar computations, we can get $\|D_{(\Lambda,\xi')}R_1\|_{**,\varrho} \le C\varrho$.

Now we consider the following problem

$$\begin{cases}
-\Delta\tilde\phi - g'(\widetilde{W}_{\Lambda,\xi'})\tilde\phi = N_1(\tilde\phi) + R_1 + \sum_{i=0}^{3} d_i w^4_{\Lambda,\xi'}\bar Z_i & \text{in } \Omega_\varrho; \\
\tilde\phi = 0 & \text{on } \partial\Omega_\varrho; \\
\int_{\Omega_\varrho}\tilde\phi w^4_{\Lambda,\xi'}\bar Z_i = 0 \quad i = 0,1,2,3,
\end{cases} \tag{4.87}$$

for some numbers d_i $(i = 0,1,2,3)$, where $\bar Z_i$ are defined by

$$\bar Z_0 = \frac{\partial\widetilde{W}_{\Lambda,\xi'}}{\partial\Lambda}, \qquad \bar Z_i = \frac{\partial\widetilde{W}_{\Lambda,\xi'}}{\partial\xi'_i}, \quad i = 1,2,3.$$

By similar processes in Section 4.3, we have the following result.

Proposition 4.16. *Assume Λ and ξ' satisfy (4.79), for $\lambda > 0$ large enough, there exists a unique solution $\tilde\phi = \tilde\phi(\Lambda,\xi')$ of problem (4.87), which is C^1 with respect to Λ and ξ'. Moreover,*

$$\|\tilde\phi\|_{*,\varrho} \le C\varrho, \quad \|D_{(\Lambda,\xi')}\tilde\phi\|_{*,\varrho} \le C\varrho. \tag{4.88}$$

In order to find solutions to problem (4.76), we only need to find Λ and ξ' such that

$$d_i(\Lambda, \xi') = 0 \quad \text{for } i = 0, 1, 2, 3. \tag{4.89}$$

The energy functional of problem (4.76) is given by

$$E(\tilde{w}) = \frac{1}{2}\int_{\Omega_\varrho} |\nabla \tilde{w}|^2 - \frac{1}{6-\varepsilon}\int_{\Omega_\varrho} |\tilde{w}|^{6-\varepsilon} - \lambda \varrho^m \frac{1}{q+1}\int_{\Omega_\varrho} |\tilde{w}|^{q+1}$$

Set

$$\mathcal{E}(\Lambda, \xi') = E\left(\widetilde{W}_{\Lambda,\xi'}(y) + \tilde{\phi}_{\Lambda,\xi'}(y)\right), \tag{4.90}$$

where $\widetilde{W}_{\Lambda,\xi'}(y)$ is defined by (4.78) and $\tilde{\phi}_{\Lambda,\xi'}$ is the solution to problem (4.87), which is solved by Proposition 4.16.

Lemma 4.17. *Under the assumptions of Proposition 4.16, $\mathcal{E}(\Lambda, \xi')$ is of class C^1. Moreover, for all $\lambda > 0$ sufficiently large, the function $\tilde{w}(y) = \widetilde{W}_{\Lambda,\xi'}(y) + \tilde{\phi}_{\Lambda,\xi'}(y)$ is a solution to problem (4.76) if and only if (Λ, ξ') is a critical point of $\mathcal{E}(\Lambda, \xi')$.*

The proof of this lemma is similar to Lemma 4.13. Using the same arguments as Lemma 4.14 and (4.71), we can obtain as $\lambda \to \infty$,

$$\mathcal{E}(\Lambda, \xi') = J(U_{\mu,\xi}) + o(\varrho), \tag{4.91}$$

where $o(\varrho)$ is in the C^1- sense uniformly on ξ', Λ satisfying (4.79).

Proof of Theorem 4.2. Suppose (4.79) holds, recall that $\mu = \Lambda\varrho$ and $\xi = \varrho\xi' \in \Omega$. Then for ε and λ satisfying (4.8), from Lemma 4.4 and (4.91), we obain

$$\mathcal{E}(\Lambda, \xi') = a_0 + \psi_q(\Lambda, \xi)\varrho + o(\varrho), \tag{4.92}$$

where

$$\psi_q(\Lambda, \xi) := a_1 H(\xi, \xi)\Lambda - a_4 \Lambda^{\frac{5-q}{2}} \quad \text{for } 2 < q < 3$$

with $a_1 > 0, a_4 > 0$ being given in Lemma 4.4.

Define

$$\widetilde{\mathcal{E}}(\Lambda, \xi') = \frac{1}{\varrho}\left[\mathcal{E}(\Lambda, \xi') - a_0\right].$$

Then we have

$$\widetilde{\mathcal{E}}(\Lambda, \xi') = \psi_q(\Lambda, \xi) + o(1),$$

with $o(1) \to 0$ as $\lambda \to \infty$, uniformly in the C^1- sense for ξ', Λ satisfying (4.79).

Next we find the critical point of $\mathcal{E}(\Lambda, \xi')$. Since for $2 < q < 3$,

$$\Lambda_{0,q} := \left(\frac{2a_1 H(\xi, \xi)}{a_4(5-q)}\right)^{\frac{2}{3-q}}$$

satisfies $\partial_\Lambda \psi_q(\Lambda, \xi)|_{\Lambda=\Lambda_{0,q}} = 0$. Moreover, $H(\xi, \xi)$ has a critical point ξ_0, with $H(\xi_0, \xi_0) > 0$, then $(\Lambda_{0,q}, \xi_0)$ is a nondegenerate critical point of $\psi_q(\Lambda, \xi)$. Thus there is a critical point (Λ_1, ξ_1') of $\mathcal{E}(\Lambda, \xi')$, such that $(\Lambda_1, \xi_1') \to (\Lambda_{0,q}, \xi_0')$ with $\xi_0' = \frac{\xi_0}{\varrho}$. Therefore, by Lemma 4.17,

$$u_3(x) = \varrho^{-\frac{2}{4-\varepsilon}} \left(\widetilde{W}_{\Lambda_1, \xi_1'} \left(\frac{y}{\varrho} \right) + \tilde{\phi}_{\Lambda_1, \xi_1'} \left(\frac{y}{\varrho} \right) \right)$$

is a solution of (4.2). By above construction, we have

$$u_3(x) = 3^{\frac{1}{4}} \frac{(\Lambda_1 \lambda^{-\frac{2}{3-q}})^{\frac{1}{2}}}{((\Lambda_1 \lambda^{-\frac{2}{3-q}})^2 + |x - \xi_1|^2)^{\frac{1}{2}}} (1 + o(1)), \tag{4.93}$$

where $o(1) \to 0$ uniformly in $\bar{\Omega}$ when λ is large enough and satisfies (4.8), and $(\Lambda_1, \xi_1) \to (\Lambda_{0,q}, \xi_0)$. Moreover $J(u_3) > \frac{\sqrt{3}}{4}\pi^2$. In fact, we can easily check that

$$J(u_3(x)) = a_0 + \psi_q(\Lambda_{0,q}, \xi_0)\varrho + o(\varrho),$$

where

$$\begin{aligned}
\psi_q(\Lambda_{0,q}, \xi_0) &= a_1 H(\xi_0, \xi_0)\Lambda_{0,q} - a_4 \Lambda_{0,q}^{\frac{5-q}{2}} \\
&= a_1 H(\xi_0, \xi_0) \left(\frac{2a_1 H(\xi_0, \xi_0)}{a_4(5-q)} \right)^{\frac{2}{3-q}} - a_4 \left(\left(\frac{2a_1 H(\xi_0, \xi_0)}{a_4(5-q)} \right)^{\frac{2}{3-q}} \right)^{\frac{5-q}{2}} \\
&= a_1 H(\xi_0, \xi_0) \left(\frac{2a_1 H(\xi_0, \xi_0)}{a_4(5-q)} \right)^{\frac{2}{3-q}} \frac{3-q}{5-q} > 0.
\end{aligned}$$

So we get

$$J(u_3) > a_0 = \frac{\sqrt{3}}{4}\pi^2.$$

Basing on Theorem 4.1 which provides two solutions, by comparing the energy of these solutions, we conclude the result.

4.5 Appendix

Lemma 4.18. *For all $\varepsilon > 0$, we have*

$$c_m = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) = \inf_{u \in \mathcal{N}(\Omega)} J(u) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \max_{t \geq 0} J(tu),$$

where $\mathcal{N}(\Omega) = \left\{ u \in H_0^1(\Omega) : \int_\Omega |\nabla u|^2 = \int_\Omega |u|^{p+1} + \lambda \int_\Omega |u|^{q+1} \right\}$.

Proof. The argument follows from [118]. For the reader's convenience, we prove it here. Let $\varepsilon > 0$ be fixed, we claim

$$\inf_{u\in\mathcal{N}(\Omega)} J(u) = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \max_{t\geq 0} J(tu). \tag{4.94}$$

Let $u \in H_0^1(\Omega) \setminus \{0\}$ be fixed, define $\Phi(t) = J(tu)$ for $t \geq 0$. Then we have that $\Phi(0) = 0$, $\Phi(t) > 0$ for small $t > 0$ and $\Phi(t) < 0$ for $t > 0$ large enough. Thus $\max\limits_{[0,+\infty)} \Phi(t)$ is achieved.

We observe that $\Phi'(t) = 0$ implies

$$\|u\|^2_{H_0^1(\Omega)} = t^{p-1}\int_\Omega |u|^{p+1} + \lambda t^{q-1}\int_\Omega |u|^{q+1}.$$

Set $\psi(t) = t^{p-1}\int_\Omega |u|^{p+1} + \lambda t^{q-1}\int_\Omega |u|^{q+1}$, obviously, $\psi(t)$ is an increasing function of t. Therefore there is a unique point t=t(u) such that $\Phi'(t(u)) = 0$ and $t(u)u \in \mathcal{N}(\Omega)$. Now we prove that $\mathcal{N}(\Omega)$ is radially homeomorphic to $H_0^1(\Omega) \setminus \{0\}$. It is enough to prove that $t : H_0^1(\Omega) \setminus \{0\} \to \mathbb{R}^+$ is continuous. Indeed, assume that $u_n \to u$ in $H_0^1(\Omega) \setminus \{0\}$, then $u_n \to u$ in $H_0^1(\Omega)$ and $u_n \to u$ in $L^s(\Omega)$ for $s \leq 6$. Moreover,

$$\int_\Omega |\nabla u_n|^2 = t^{p-1}(u_n)\int_\Omega |u_n|^{p+1} + \lambda t^{q-1}(u_n)\int_\Omega |u_n|^{q+1}, \tag{4.95}$$

thus $\{t(u_n)\}_n$ is bounded in $\mathbb{R}^+$, then there exists a subsequence of $\{t(u_n)\}_n$, still denoted by $\{t(u_n)\}_n$, such that $t(u_n) \to t_0$ as $n \to +\infty$. By taking the limit in (4.95), we get

$$\int_\Omega |\nabla u|^2 = t_0^{p-1}\int_\Omega |u|^{p+1} + \lambda t_0^{q-1}\int_\Omega |u|^{q+1}.$$

Hence $t(u) = t_0$, where $t_0 u \in \mathcal{N}(\Omega)$.

Since $J(tu) < 0$ for $u \in H_0^1(\Omega) \setminus \{0\}$ and t is large, we obtain

$$\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \leq \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \max_{t\geq 0} J(tu).$$

Finally, we show that

$$\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \geq \inf_{u\in\mathcal{N}(\Omega)} J(u).$$

It is sufficient to prove that $\gamma([0,1]) \cap \mathcal{N}(\Omega) \neq \emptyset$ for all $\gamma \in \Gamma$. In fact,

$$\begin{aligned} \Psi(u) : &= \int_\Omega |\nabla u|^2 - \int_\Omega |u|^{p+1} - \lambda\int_\Omega |u|^{q+1} \\ &= 2J(u) + \frac{1-p}{p+1}\int_\Omega |u|^{p+1} + \lambda\frac{1-q}{q+1}\int_\Omega |u|^{q+1}. \end{aligned}$$

It is easy to check that there exists $\rho_0 > 0$ such that

$$\Psi(u) > 0 \quad \text{for all} \quad 0 < \|u\|_{H_0^1(\Omega)} \leq \rho_0.$$

For any $\gamma \in \Gamma$, we have $\Psi(\gamma(0)) = 0$ and $\Psi(\gamma(1)) < 2J(\gamma(1)) < 0$. Therefore there exists $t_1 \in [0,1]$, such that $\|\gamma(t_1)\|_{H_0^1(\Omega)} > \rho_0$ and $\Psi(\gamma(t_1)) = 0$. So $\gamma(t_1) \in \gamma([0,1]) \cap \mathcal{N}(\Omega)$. We complete the proof.

□

Chapter 5

Bubble tower solutions for supercritical elliptic problem in $\mathbb{R}^N$

5.1 Introduction

We are interested in the elliptic equation

$$\begin{cases} -\Delta u + u = u^p + \lambda u^q, \qquad u > 0 \qquad \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{5.1}$$

where $N \geq 3$, $\lambda > 0$ and $1 < q < p$. This problem arises in the study of standing waves of a nonlinear Schrödinger equation with two power type nonlinearities, see for example Tao, Visan and Zhang [113].

If $p = q$, equation (5.1) reduces to

$$\begin{cases} -\Delta u + u = u^p, \qquad u > 0 \qquad \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{5.2}$$

after a suitable scaling.

Thanks to the classical result of Gidas, Ni and Nirenberg [68], solutions of (5.1) and (5.2) are radially symmetric about some point, which we will assume is always the origin.

It is well known that problem (5.2) has a solution if and only if $1 < p < \frac{N+2}{N-2}$. Existence was proved by Berestycki and Lions [10], while non-existence from the Pohozaev identity [99]. Uniqueness also holds and was fully settled by Kwong [76], after a series of contributions [22, 80, 96, 97, 94, 93]. See also Felmer, Quaas, Tang and Yu [57] for further properties.

Concerning (5.1), the work of Berestycki and Lions [10] is still applicable if $1 < q < p < \frac{N+2}{N-2}$, and one obtains existence of a solution. If $p, q \geq \frac{N+2}{N-2}$ there is no solution, again from the Pohozaev identity.

Recently, Dávila, del Pino and Guerra [35] proved that uniqueness does not hold in general for (5.1), if $1 < q < p < \frac{N+2}{N-2}$. More precisely if $N = 3$, the authors obtained at least three solutions to problem (5.1) if $1 < q < 3$, $\lambda > 0$ is sufficiently large and fixed, and $p < 5$ is close enough to 5.

Let us mention some contributions to the question of existence for (5.1) when one exponent is subcritical and other is critical or supercritical. If $1 < q < p = \frac{N+2}{N-2}$ in (5.1), Alves, de Morais Filho and Souto [1] proved:

- when $N \geq 4$, there exists a nontrivial classical solution for all $\lambda > 0$ and $1 < q < \frac{N+2}{N-2}$;
- when $N = 3$, there exists a nontrivial classical solution for all $\lambda > 0$ and $3 < q < 5$;
- when $N = 3$, there exists a nontrivial classical solution for $\lambda > 0$ large enough and $1 < q \leq 3$.

Moreover, Ferrero and Gazzola [56] proved that for $q < \frac{N+2}{N-2} \leq p$, there exists $\bar{\lambda} > 0$, such that if $\lambda > \bar{\lambda}$, then (5.1) has at least one solution, while for $q < \frac{N+2}{N-2} < p$, there exists $0 < \underline{\lambda} < \bar{\lambda}$ such that if $\lambda < \underline{\lambda}$, then there is no solution.

In this chapter, we are interested in multiplicity of solutions of (5.1), and for this we take an asymptotic approach, that is, we consider

$$\begin{cases} -\Delta u + u = u^p + \lambda u^q, \quad u > 0 & \text{in } \mathbb{R}^N; \\ u(z) \to 0 \quad \text{as} \quad |z| \to \infty, \end{cases} \tag{5.3}$$

where $p = p^* + \varepsilon$, with $p^* = \frac{N+2}{N-2}$, $\lambda > 0$ and $\varepsilon > 0$ are parameters, and q satisfies

$$1 < q < \frac{N+2}{N-2} \quad \text{if } N \geq 4; \qquad 3 < q < 5 \quad \text{if } N = 3. \tag{5.4}$$

Our result can be stated as follows.

Theorem 5.1. *Let $\lambda > 0$ and let q satisfy (5.4). Given an integer $k \geq 1$, then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there is a solution $u_\varepsilon(z)$ of problem (5.3) of the form*

$$u_\varepsilon(z) = (N(N-2))^{\frac{N-2}{4}} \sum_{j=1}^{k} \frac{\varepsilon^{-[(j-1)+\frac{2}{p^*-q}]}(\Lambda_j^*)^{-\frac{N-2}{2}}}{\left(1 + \varepsilon^{-\frac{4}{N-2}[(j-1)+\frac{1}{p^*-q}]}(\Lambda_j^*)^{-2}|z|^2\right)^{\frac{N-2}{2}}}(1 + o(1)), \tag{5.5}$$

where the constants $\Lambda_j^* > 0$, $j = 1, 2, \ldots, k$, can be computed explicitly and depend on k, N, q.

Remark 5.2. *The expansion (5.5) is valid if*

$$\frac{1}{C}\varepsilon^{\frac{2}{N-2}[(i-1)+\frac{1}{p^*-q}]} \leq |z| \leq C\varepsilon^{\frac{2}{N-2}[(i-1)+\frac{1}{p^*-q}]},$$

with some $i \in \{1, 2, \cdots, k\}$, and $o(1) \to 0$ uniformly as $\varepsilon \to 0$ in this region.

The solutions described in this result behave like a superposition of "bubbles" of different blow-up orders centered at the origin, and hence have been called bubble-tower solutions. By bubbles we mean the functions

$$w_\mu(z) = \alpha_N \frac{\mu^{\frac{N-2}{2}}}{(\mu^2 + |z|^2)^{\frac{N-2}{2}}}, \quad \text{with} \quad \alpha_N = (N(N-2))^{\frac{N-2}{4}},$$

where $\mu > 0$, which are the unique positive solutions of

$$-\Delta w = w^{p^*} \quad \text{in } \mathbb{R}^N,$$

(except translations). Based on numerical simulations, in Figures 5 and 6, we describe qualitatively the bifurcation diagrams of solutions for problem (5.3) where q satisfies (5.4). The solutions from Theorem 5.1 (for $k = 1, 2$), are also marked in the diagrams.

Bubble-tower solutions were found by del Pino, Dolbeault and Musso [43] for a slightly supercritical Brezis-Nirenberg problem in a ball, and after that have been studied intensively [21, 44, 46, 48, 64, 65, 83, 89, 91, 98]. In particular we mention the work of Campos [21] who considered the existence of bubble-tower solutions to a problem related to ours:

$$\begin{cases} -\Delta u = u^{p^* \pm \varepsilon} + u^q, \quad u > 0 \quad \text{in } \mathbb{R}^N; \\ u(z) \to 0 \ \text{ as } \ |z| \to \infty, \end{cases}$$

with $\frac{N}{N-2} < q < p^* = \frac{N+2}{N-2}$, $N \geq 3$.

The proof of our result starts with a variation of the so-called Emden-Fowler transformation, which reduces the problem of finding k-bubble solution to the problem of finding a k-bump solution of a second-order ordinary differential equation in $\mathbb{R}$. After a Lyapunov-Schmidt reduction procedure, see for example [58, 83, 21], the problem becomes to find a critical point of some functional depending on k real parameters.

In Section 5.2, we give Emden-Fowler transformation for problem (5.3) and build the first approximate solution to the ODE. We study the linearized problem at an approximate solution and nonlinear problem in Sections 5.3 and 5.4. In Section 5.5, we study the finite-dimensional variational reduction problem and prove Theorem 5.1. We leave some of the estimates in the Appendix.

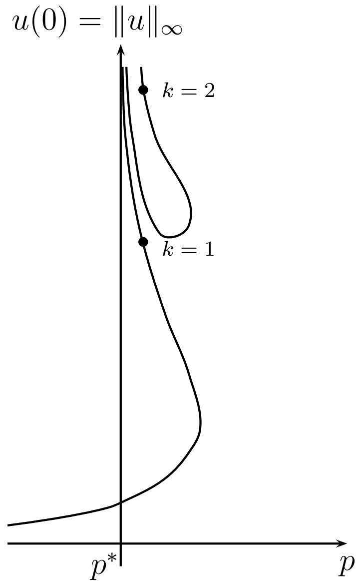

Figure 5: Bifurcation diagram $u(0)$ vs. p for solutions of (5.3) for λ sufficiently large and fixed, and q satisfying (5.4).

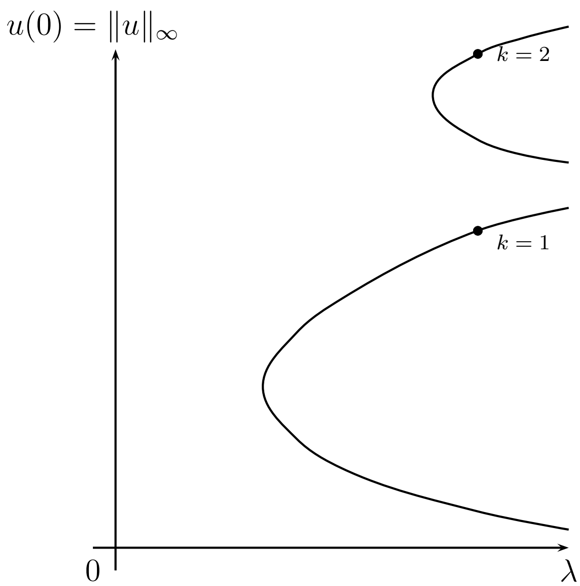

Figure 6: Bifurcation diagram $u(0)$ vs. λ for solutions of (5.3) with $p = p^* + \varepsilon$, $\varepsilon > 0$ small and fixed, and q satisfying (5.4).

5.2 The first approximate solution

In this section, we build the first approximate solution to (5.3). In order to do this, we introduce the solutions of problem

$$-\Delta w = w^{p^*} \quad \text{in } \mathbb{R}^N,$$

which are given by

$$w_\mu(z) = \alpha_N \frac{\mu^{\frac{N-2}{2}}}{(\mu^2 + |z|^2)^{\frac{N-2}{2}}}$$

with $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$ and any parameter $\mu > 0$.

Let us define U_μ as the unique solution of the following problem

$$\begin{cases} -\Delta U_\mu + U_\mu = w_\mu^{p^*} \quad \text{in } \mathbb{R}^N; \\ U_\mu(z) \to 0 \ \text{ as } \ |z| \to \infty. \end{cases} \tag{5.6}$$

We write

$$U_\mu(z) = w_\mu(z) + R_\mu(z).$$

Then $R_\mu(z)$ satisfies

$$\begin{cases} -\Delta R_\mu(z) + R_\mu(z) = -w_\mu(z) \quad \text{in } \mathbb{R}^N; \\ R_\mu(z) \to 0 \ \text{ as } \ |z| \to \infty. \end{cases} \tag{5.7}$$

We have the following result, whose proof is postponed to the Appendix.

Lemma 5.3. *Assume $0 < \mu \le 1$, we have*

(a) $0 < U_\mu(z) \le w_\mu(z)$, for $z \in \mathbb{R}^N$.

(b) One has

$$U_\mu(z) \le C\mu^{\frac{N-2}{2}} |z|^{-(N+2)}, \qquad \text{for } |z| \ge R,$$

where R is a large positive number but fixed.

(c)Given any $\mu > 0$ small, we have

(i) If $|z| \ge 1$, then

$$|R_\mu(z)| \le C \frac{\mu^{\frac{N-2}{2}}}{|z|^{N-2}} \quad \text{for} \quad N \ge 3. \tag{5.8}$$

(ii) If $|z| \le \frac{\mu}{2}$, then

$$|R_\mu(z)| \ \le \ C \begin{cases} \mu^{-\frac{N-6}{2}} & \text{for } N \ge 5; \\ \mu \log \frac{1}{\mu} & \text{for } N = 4; \\ \mu^{\frac{1}{2}} & \text{for } N = 3. \end{cases} \tag{5.9}$$

If $\frac{\mu}{2} \leq |z| \leq 1$, *then*

$$|R_\mu(z)| \leq C \begin{cases} \mu^{-\frac{N-6}{2}} \frac{1}{(1+|\frac{z}{\mu}|^2)^{\frac{N-4}{2}}} & \text{for } N \geq 5; \\ \mu \log \frac{1}{|z|} & \text{for } N = 4; \\ \mu^{\frac{1}{2}} & \text{for } N = 3. \end{cases} \tag{5.10}$$

We define the following Emden-Fowler transformation

$$v(x) = \mathcal{T}(u(r)) = \left(\frac{p^*-1}{2}\right)^{\frac{2}{p^*-1}} r^{\frac{2}{p^*-1}} u(r) \tag{5.11}$$

with

$$r = |z| = e^{-\frac{p^*-1}{2}x}, \quad x \in (-\infty, +\infty). \tag{5.12}$$

Using this transformation, finding a radial solution $u(r)$ to problem (5.3) corresponds to that of solving the problem

$$\begin{cases} \mathcal{L}_0(v) = \alpha_\varepsilon e^{\varepsilon x} v^{p^*+\varepsilon} + \lambda \beta_N e^{-(p^*-q)x} v^q & \text{in } (-\infty, +\infty); \\ v(x) > 0 \quad \text{for} \quad x \in (-\infty, +\infty); \\ v(x) \to 0 \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{5.13}$$

where

$$\mathcal{L}_0(v) = -v'' + v + \left(\frac{2}{N-2}\right)^2 e^{-\frac{4}{N-2}x} v, \tag{5.14}$$

$$\alpha_\varepsilon = \left(\frac{p^*-1}{2}\right)^{-\frac{2\varepsilon}{p^*-1}}, \qquad \beta_N = \left(\frac{p^*-1}{2}\right)^{\frac{2(p^*-q)}{p^*-1}}.$$

We observe that $\mathcal{L}_0$ is the transformed operator associated to $-\Delta + Id$. Moreover,

$$W(x-\xi) = \mathcal{T}(w_\mu)(r) = \left(\frac{4N}{N-2}\right)^{\frac{N-2}{4}} e^{-(x-\xi)} \left(1 + e^{-\frac{4}{N-2}(x-\xi)}\right)^{-\frac{N-2}{2}}$$

with $\mu = e^{-\frac{2}{N-2}\xi}$, is the unique solution of the problem

$$\begin{cases} W'' - W + W^{p^*} = 0 & \text{in } (-\infty, +\infty); \\ W'(0) = 0; \\ W(x) > 0, \quad W(x) \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{5.15}$$

Note that $W(x) = O(e^{-|x|})$.

Define the function

$$V_\xi(x) = \mathcal{T}(U_\mu)(r), \quad \text{with } r = e^{-\frac{p^*-1}{2}x}, \quad \mu = e^{-\frac{2}{N-2}\xi}.$$

Then $V_\xi(x)$ is the solution of the problem

$$\begin{cases} \mathcal{L}_0 V_\xi(x) = W(x-\xi)^{p^*} & \text{in } (-\infty, +\infty); \\ V_\xi(x) \to 0 \ \text{ as } \ |x| \to \infty. \end{cases} \tag{5.16}$$

We write

$$V_\xi(x) = W(x-\xi) + R_\xi(x), \tag{5.17}$$

where $R_\xi(x) = \mathcal{T}(R_\mu)(r)$. By the Emden-Fowler transformation and as a consequence of Lemma 5.3, we have the following estimates.

Lemma 5.4. *For $\xi > 0$, we have*

(a) $0 < V_\xi(x) \leq W(x-\xi) = O(e^{-|x-\xi|}), \qquad$ *for* $x \in \mathbb{R}$.

(b)

$$V_\xi(x) \leq C e^{\frac{N+6}{N-2}x} e^{-\xi}, \qquad \text{for } -\infty < x \leq -\frac{N-2}{2}\log R, \tag{5.18}$$

for $R > 0$ is a fixed large number as in Lemma 5.3.

(c) For $N \geq 3$, there is a positive constant C, such that

$$|R_\xi(x)| \leq C \begin{cases} e^{-|x-\xi|} & \text{if } \ x \leq 0; \\ e^{-|x-\xi|} e^{-\frac{2}{N-2}\min\{x,\xi\}} & \text{if } \ x \geq 0. \end{cases} \tag{5.19}$$

Define

$$Z_\xi(x) := \partial_\xi V_\xi(x) = \partial_\xi W(x-\xi) + \partial_\xi R_\xi(x).$$

Note that $\partial_\xi W(x-\xi) = O(e^{-|x-\xi|})$ and

$$\partial_\xi W(x-\xi) = -\frac{2}{N-2}\mu \mathcal{T}\left(\partial_\mu w_\mu(r)\right), \tag{5.20}$$

$$Z_\xi(x) = -\frac{2}{N-2}\mu \mathcal{T}\left(\widetilde{Z}_\mu(r)\right) \qquad \text{with} \quad \widetilde{Z}_\mu(z) = \partial_\mu U_\mu(z), \tag{5.21}$$

$$\partial_\xi R_\xi(x) = -\frac{2}{N-2}\mu \mathcal{T}\left(\partial_\mu R_\mu(r)\right). \tag{5.22}$$

Then from (5.102), (5.22) and Lemma 5.4 (c), we have for $N \geq 3$,

$$|\partial_\xi R_\xi(x)| \leq C \begin{cases} e^{-|x-\xi|} & \text{if } \ x \leq 0; \\ e^{-|x-\xi|} e^{-\frac{2}{N-2}\min\{x,\xi\}} & \text{if } \ x \geq 0. \end{cases} \tag{5.23}$$

Therefore

$$Z_\xi(x) = O(e^{-|x-\xi|}) \quad \text{for } \forall\ x \in \mathbb{R}. \tag{5.24}$$

Moreover, from (5.103) and (5.21), we find

$$|Z_\xi(x)| \le Ce^{\frac{N+6}{N-2}x} e^{-\xi}, \qquad \text{for} \quad -\infty < x \le -\frac{N-2}{2}\log R, \tag{5.25}$$

for $R > 0$ is a fixed large number.

Let $\eta > 0$ be a small but fixed number. Given an integer number k, let Λ_j, for $j = 1, \cdots, k$, be positive numbers and satisfy

$$\eta < \Lambda_j < \frac{1}{\eta}. \tag{5.26}$$

Set

$$\mu_1 = \varepsilon^{\frac{2}{(N+2)-(N-2)q}} \Lambda_1 \quad \text{and} \quad \mu_j = \varepsilon^{\frac{2}{N-2}(j-1)+\frac{2}{(N+2)-(N-2)q}} \Lambda_j \tag{5.27}$$

for $j = 2, \cdots, k$. We observe that

$$\frac{\mu_{j+1}}{\mu_j} = \varepsilon^{\frac{2}{N-2}} \frac{\Lambda_{j+1}}{\Lambda_j}, \qquad j = 1, \cdots, k-1. \tag{5.28}$$

Define k points in $\mathbb{R}$ as

$$\mu_j = e^{-\frac{2}{N-2}\xi_j}, \qquad j = 1, \ldots, k.$$

Then we have that

$$0 < \xi_1 < \xi_2 < \cdots < \xi_k.$$

and

$$\begin{cases} \xi_1 = -\frac{1}{p^*-q}\log\varepsilon - \frac{N-2}{2}\log\Lambda_1, \\ \xi_j - \xi_{j-1} = -\log\varepsilon - \frac{N-2}{2}\log\frac{\Lambda_j}{\Lambda_{j-1}}, \quad j = 2, \ldots, k, \end{cases} \tag{5.29}$$

Set

$$W_j = W(x - \xi_j), \quad R_j = R_{\xi_j}(x), \quad V_j = W_j + R_j, \quad V = \sum_{j=1}^{k} V_j. \tag{5.30}$$

We look for a solution of (5.3) of the form $u = \sum\limits_{j=1}^{k} U_{\mu_j} + \psi$ corresponding to find a solution of (5.13) of the form

$$v = V + \phi,$$

where V is given by (5.30) and $\phi = \mathcal{T}(\psi)$ is a small term. Thus problem (5.13) becomes

$$\begin{cases} \mathcal{L}_\varepsilon(\phi) = N(\phi) + E \qquad \text{in } (-\infty, +\infty); \\ \phi(x) > 0 \quad \text{for} \quad x \in (-\infty, +\infty); \\ \phi(x) \to 0 \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{5.31}$$

where

$$\mathcal{L}_\varepsilon(\phi) = \mathcal{L}_0(\phi) - \alpha_\varepsilon (p^* + \varepsilon) e^{\varepsilon x} V^{p^*+\varepsilon-1}\phi - \lambda q \beta_N e^{-(p^*-q)x} V^{q-1}\phi, \tag{5.32}$$

$$\begin{aligned} N(\phi) &= \alpha_\varepsilon e^{\varepsilon x} \left[(V+\phi)^{p^*+\varepsilon} - V^{p^*+\varepsilon} - (p^*+\varepsilon) V^{p^*+\varepsilon-1}\phi \right] \\ &\quad + \lambda \beta_N e^{-(p^*-q)x} \left[(V+\phi)^q - V^q - q V^{q-1}\phi \right], \end{aligned} \tag{5.33}$$

and

$$\begin{aligned} E &= \alpha_\varepsilon e^{\varepsilon x} V^{p^*+\varepsilon} - \mathcal{L}_0(V) + \lambda \beta_N e^{-(p^*-q)x} V^q \\ &= \alpha_\varepsilon e^{\varepsilon x} V^{p^*+\varepsilon} - \sum_{j=1}^{k} W_j^{p^*} + \lambda \beta_N e^{-(p^*-q)x} V^q, \end{aligned} \tag{5.34}$$

where $\mathcal{L}_0$ is defined by (5.14).

5.3 The linear problem

In order to solve problem (5.31), we consider first the following problem: given points $\xi = (\xi_1, \dots, \xi_k)$, finding a function ϕ such that for certain constants $c_1, c_2, \dots, c_k$,

$$\begin{cases} \mathcal{L}_\varepsilon(\phi) = N(\phi) + E + \sum\limits_{j=1}^{k} c_j Z_j \quad \text{in } (-\infty, +\infty); \\ \lim\limits_{|x|\to\infty} \phi(x) = 0; \\ \int_{\mathbb{R}} Z_j \phi = 0, \quad \forall\, j = 1, \dots, k, \end{cases} \tag{5.35}$$

where $Z_j(x) = Z_{\xi_j}(x) = \partial_{\xi_j} V_{\xi_j}(x)$ for $j = 1, 2, \cdots, k$.

To solve (5.35), it is important to understand its linear part, thus we consider the following problem: given a function h, finding ϕ such that

$$\begin{cases} \mathcal{L}_\varepsilon(\phi) = h + \sum\limits_{j=1}^{k} c_j Z_j \quad \text{in } (-\infty, +\infty); \\ \lim\limits_{|x|\to\infty} \phi(x) = 0; \\ \int_{\mathbb{R}} Z_j \phi = 0, \quad \forall\, j = 1, \dots, k, \end{cases} \tag{5.36}$$

for certain constants c_j.

Now we analyze invertibility properties of the operator $\mathcal{L}_\varepsilon$ under the orthogonality conditions. Let σ satisfy

$$0 < \sigma < \min\left\{q-1, 1, \frac{(N+2)(2q-1)}{N+6}, \frac{3q-p^*}{2}\right\}. \tag{5.37}$$

We define the real number M as follows

$$M = \begin{cases} 0 & \text{if } 1 \geq \frac{4}{N-2}+\sigma; \\ \max\{0,\gamma\} & \text{if } 1 \leq \frac{4}{N-2}+\sigma, \end{cases} \tag{5.38}$$

where γ satisfies

$$\left(1-\left(\frac{4}{N-2}+\sigma\right)^2\right)e^{-\frac{4}{N-2}\gamma} = -\frac{1}{2}\left(\frac{2}{N-2}\right)^2.$$

We define the following norms for a function φ defined on $\mathbb{R}$,

$$\|\varphi\|_* = \sup_{x\leq -M} e^{-(\frac{4}{N-2}+\sigma)x}e^{\sigma\xi_1}|\varphi(x)| + \sup_{x\in\mathbb{R}}\left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|}\right)^{-1}|\varphi(x)|, \tag{5.39}$$

and

$$\|\varphi\|_{**} \;=\; \sup_{x\in\mathbb{R}}\left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|}\right)^{-1}|\varphi(x)|. \tag{5.40}$$

The following result holds.

Proposition 5.5. *There exist positive numbers ε_0, and $C>0$ such that if the points $0<\xi_1<\xi_2<\ldots<\xi_k$ satisfy (5.29), then for all $0<\varepsilon<\varepsilon_0$ and all functions $h\in C(\mathbb{R};\mathbb{R})$ with $\|h\|_{**}<+\infty$, problem (5.36) has a unique solution $\phi =: T_\varepsilon(h)$ with $\|\phi\|_*<+\infty$. Moreover,*

$$\|\phi\|_* \leq C\|h\|_{**} \quad \text{and} \quad |c_j| \leq C\|h\|_{**}. \tag{5.41}$$

We first consider a simpler problem

$$\begin{cases} \mathcal{L}_0(\phi) - \alpha_\varepsilon(p^*+\varepsilon)e^{\varepsilon x}V^{p^*+\varepsilon-1}\phi = h + \sum_{j=1}^k c_j Z_j \ \text{ in } (-\infty,+\infty); \\ \lim_{|x|\to\infty} \phi(x) = 0; \\ \int_{\mathbb{R}} Z_j\phi = 0, \quad \forall\ j=1,\ldots,k, \end{cases} \tag{5.42}$$

for certain constants c_j, here $\mathcal{L}_0$ is defined by (5.14).

Lemma 5.6. *Under the assumptions of Proposition 5.5, then for all $0<\varepsilon<\varepsilon_0$ and any h, ϕ solution of (5.42), we have*

$$\|\phi\|_* \leq C\|h\|_{**}, \tag{5.43}$$

and

$$|c_j| \leq C\|h\|_{**}. \tag{5.44}$$

Proof. To prove (5.43), by contradiction, we suppose that there exist sequences ϕ_n, h_n, ε_n and c_j^n that satisfy (5.42), with

$$\|\phi_n\|_* = 1, \quad \|h_n\|_{**} \to 0, \quad \varepsilon_n \to 0.$$

We get a contradiction by the following steps.

Step 1: $c_j^n \to 0$ as $n \to +\infty$.

Multiplying (5.42) by Z_i^n and integrating by parts twice, we get that

$$\begin{aligned} & \sum_{j=1}^{k} c_j^n \int_{\mathbb{R}} Z_j^n Z_i^n \\ = & -\int_{\mathbb{R}} h_n Z_i^n + \int_{\mathbb{R}} \left[\mathcal{L}_0(Z_i^n) - \alpha_{\varepsilon_n}(p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^*+\varepsilon_n-1} Z_i^n\right] \phi_n. \end{aligned} \tag{5.45}$$

Note that

$$\int_{\mathbb{R}} Z_j^n Z_i^n = C\delta_{ij} + o(1), \tag{5.46}$$

where δ_{ij} is Kronecker's delta. Then (5.45) defines a linear system in the c_j's which is almost diagonal as $n \to \infty$.

Since $Z_i^n(x) = \partial_{\xi_i^n} V_{\xi_i^n}(x) = O(e^{-|x-\xi_i^n|})$, we then have

$$\begin{aligned} \left|\int_{\mathbb{R}} h_n Z_i^n\right| & \leq C\|h_n\|_{**} \int_{\mathbb{R}} \left(\sum_{j=1}^{k} e^{-\sigma|x-\xi_j^n|}\right) e^{-|x-\xi_i^n|} dx \\ & \leq Ck\|h_n\|_{**} \int_{\mathbb{R}} e^{-|y|} dy \leq C\|h_n\|_{**}. \end{aligned} \tag{5.47}$$

Moreover, Z_i^n satisfy

$$\mathcal{L}_0(Z_i^n) = p^* W^{p^*-1}(x - \xi_i^n) \partial_{\xi_i^n} W(x - \xi_i^n),$$

so we get

$$\begin{aligned} & \left|\int_{\mathbb{R}} \left[\mathcal{L}_0(Z_i^n) - \alpha_{\varepsilon_n}(p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^*+\varepsilon_n-1} Z_i^n\right] \phi_n\right| \\ = & \left|\int_{\mathbb{R}} \left[p^* W(x - \xi_i^n)^{p^*-1} \partial_{\xi_i^n} W(x - \xi_i^n) - \alpha_{\varepsilon_n}(p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^*+\varepsilon_n-1} \partial_{\xi_i^n} W(x - \xi_i^n)\right] \phi_n \right. \\ & \left. + \int_{\mathbb{R}} [\alpha_{\varepsilon_n}(p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^*+\varepsilon_n-1} \partial_{\xi_i^n} R_{\xi_i^n}(x)] \phi_n \right| \\ = & \ o(1)\|\phi_n\|_*. \end{aligned} \tag{5.48}$$

From (5.45)-(5.48), we obtain

$$|c_j^n| \leq C\|h_n\|_{**} + o(1)\|\phi_n\|_*. \tag{5.49}$$

Thus $\lim_{n\to\infty} c_j^n = 0$.

Step 2: For any $L > 0$, any $l \in \{1, 2, \cdots, k\}$, we have

$$\sup_{x\in[\xi_l^n - L, \xi_l^n + L]} |\phi_n(x)| \to 0, \quad \text{as } n \to \infty. \tag{5.50}$$

Indeed, suppose not, we assume that there exist $L > 0$ and some $l \in \{1, 2, \cdots, k\}$ such that

$$|\phi_n(x_{n,l})| \geq c > 0, \quad \text{for some } x_{n,l} \in [\xi_l^n - L, \xi_l^n + L].$$

By elliptic estimates, there is a subsequence of ϕ_n converging uniformly on compact sets to a nontrivial bounded solution $\tilde{\phi}$ of

$$\mathcal{L}_0(\tilde{\phi}) = p^* W^{p^*-1}(x - \xi_l)\tilde{\phi},$$

where $\xi_l = \lim_{n\to\infty} \xi_l^n$. By nondegeneracy [104], it is well known that $\tilde{\phi} = cZ_l$ for some constant $c \neq 0$. But taking the limit in the orthogonality condition $\int_{\mathbb{R}} Z_l^n \phi_n = 0$, we obtain $\tilde{\phi} = 0$, which is a contradiction. Thus (5.50) holds.

Step 3: We prove that $\|\phi_n\|_* \to 0$ as $n \to \infty$.

Claim: For any $L > 0$ and $j \in \{1, 2, \cdots, k\}$, we have

$$\sup_{\mathbb{R}\setminus\cup_{j=1}^k[\xi_j^n - L, \xi_j^n + L]} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j^n|}\right)^{-1} |\phi_n(x)| \to 0, \tag{5.51}$$

and

$$\sup_{x\leq -M} e^{-(\frac{4}{N-2}+\sigma)x} e^{\sigma\xi_1^n} |\phi_n(x)| \to 0, \tag{5.52}$$

as $n \to +\infty$.

By the definition of $\|\cdot\|_*$ in (5.39), using (5.50), (5.51) and (5.52), we get that $\|\phi_n\|_* \to 0$ as $n \to \infty$.

Now we prove the above claim. We note that

$$h_n + \sum_{j=1}^k c_j^n Z_j^n \leq (C_0\|h_n\|_{**} + o(\|\phi_n\|_*)) \sum_{j=1}^k e^{-\sigma|x-\xi_j^n|},$$

where C_0 is a positive constant.

For $x \in \mathbb{R}\setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L]$, let us define

$$\tilde{\psi}_n(x) \;=\; \left(C_0\|h_n\|_{**} + e^{\sigma L} \sup_{\cup_{j=1}^k[\xi_j^n - L, \xi_j^n + L]} |\phi_n(x)| + o(\|\phi_n\|_*)\right) \sum_{j=1}^k e^{-\sigma|x-\xi_j^n|}$$

$$+\varrho\sum_{j=1}^{k} e^{-\bar{\sigma}|x-\xi_j^n|}$$

with $\varrho > 0$ small but fixed and $0 < \bar{\sigma} < \sigma$. Then by choosing suitable large $L > 0$, we get

$$\begin{aligned} & \mathcal{L}_0(\tilde{\psi}_n(x)) - \alpha_{\varepsilon_n}(p^* + \varepsilon_n)e^{\varepsilon_n x}V^{p^*+\varepsilon_n-1}\tilde{\psi}_n(x) \\ \geq\ & \mathcal{L}_0(\phi_n(x)) - \alpha_{\varepsilon_n}(p^* + \varepsilon_n)e^{\varepsilon_n x}V^{p^*+\varepsilon_n-1}\phi_n(x). \end{aligned}$$

On the other hand, we have that for any $L > 0$ and $j \in \{1, 2, \cdots, k\}$,

$$\tilde{\psi}_n(\xi_j^n - L) \geq \phi_n(\xi_j^n - L) \quad \text{and} \quad \tilde{\psi}_n(\xi_j^n + L) \geq \phi_n(\xi_j^n + L).$$

Moreover, there exists $R > 0$ large enough, such that

$$\tilde{\psi}_n(R) \geq \phi_n(R),$$

and

$$\tilde{\psi}_n(-R) \geq \phi_n(-R).$$

By the maximum principle, we get

$$\phi_n(x) \leq \tilde{\psi}_n(x) \quad \text{for} \ \ x \in [-R, R] \setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L].$$

Similarly, we obtain $\phi_n(x) \geq -\tilde{\psi}_n(x)$ for $x \in [-R, R] \setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L]$. Thus

$$|\phi_n(x)| \leq \tilde{\psi}_n(x) \quad \text{for} \ \ x \in [-R, R] \setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L].$$

Letting $R \to +\infty$, we get

$$|\phi_n(x)| \leq \tilde{\psi}_n(x) \quad \text{for} \ \ x \in \mathbb{R} \setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L].$$

Letting $\varrho \to 0$, for $x \in \mathbb{R} \setminus \cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L]$, we have that

$$|\phi_n(x)| \leq \left(C_0\|h_n\|_{**} + e^{\sigma L} \sup_{\cup_{j=1}^k [\xi_j^n - L, \xi_j^n + L]} |\phi_n(x)| + o(\|\phi_n\|_*) \right) \sum_{j=1}^{k} e^{-\sigma|x-\xi_j^n|}.$$

So (5.51) holds.

For $x \leq -M$, let $\rho > 0$ small and $C_1 > 0$ be chosen later, we define

$$\psi_n(x) = C_1 \left(C_0\|h_n\|_{**} + o(\|\phi_n\|_*) \right) e^{(\frac{4}{N-2}+\sigma)x} e^{-\sigma\xi_1^n} + \rho e^{\frac{4}{N-2}x}.$$

According to the definition of M in (5.38), we then have

$$\begin{aligned} & \mathcal{L}_0(\psi_n(x)) - \alpha_{\varepsilon_n}(p^* + \varepsilon_n)e^{\varepsilon_n x}V^{p^*+\varepsilon_n-1}\psi_n(x) \\ \geq\ & \frac{1}{2}\left(\frac{2}{N-2}\right)^2 \frac{1}{k} C_1(C_0\|h_n\|_{**} + o(\|\phi_n\|_*)) \sum_{j=1}^{k} e^{-\sigma|x-\xi_j^n|}. \end{aligned}$$

Choosing C_1 such that $\frac{1}{2}\left(\frac{2}{N-2}\right)^2\frac{1}{k}C_1 \geq 1$. then

$$\begin{aligned} & \mathcal{L}_0(\psi_n(x)) - \alpha_{\varepsilon_n}(p^*+\varepsilon_n)e^{\varepsilon_n x}V^{p^*+\varepsilon_n-1}\psi_n(x) \\ \geq\ & (C_0\|h_n\|_{**} + o(\|\phi_n\|_*))\sum_{j=1}^{k} e^{-\sigma|x-\xi_j^n|} \geq h_n + \sum_{j=1}^{k} c_j^n Z_j^n \\ =\ & \mathcal{L}_0(\phi_n(x)) - \alpha_{\varepsilon_n}(p^*+\varepsilon_n)e^{\varepsilon_n x}V^{p^*+\varepsilon_n-1}\phi_n(x). \end{aligned}$$

Moreover, by (5.51), we can find

$$\psi_n(-M) \geq \phi_n(-M),$$

and there exists $R>0$ large enough, such that

$$\psi_n(-R) \geq \phi_n(-R).$$

By the maximum principle, we get

$$\phi_n(x) \leq \psi_n(x) \quad \text{for} \quad x \in [-R,-M].$$

By a similar argument, we obtain $\phi_n(x) \geq -\psi_(x)$ for $x \in [-R,-M]$. Thus

$$|\phi_n(x)| \leq \psi_n(x) \quad \text{for} \quad x \in [-R,-M].$$

Let $R \to +\infty$, we get

$$|\phi_n(x)| \leq \psi_n(x) \quad \text{for} \quad x \in [-\infty,-M].$$

Let $\rho \to 0$, we have

$$|\phi_n(x)| \leq C_1\left(C_0\|h_n\|_{**} + o(\|\phi_n\|_*)\right)e^{(\frac{4}{N-2}+\sigma)x}e^{-\sigma\xi_1^n} \quad \text{for} \quad x \in [-\infty,-M].$$

So we obtain that (5.52) holds.

Moreover, estimate (5.44) follows from (5.49) and (5.43). □

Proof of Proposition 5.5. From Lemma 5.6, for ϕ and h satisfying (5.36), we then have

$$\|\phi\|_* \leq C\left(\|h\|_{**} + \|e^{-(p^*-q)x}V^{q-1}\phi\|_{**}\right). \tag{5.53}$$

and

$$|c_j| \leq C\left(\|h\|_{**} + \|e^{-(p^*-q)x}V^{q-1}\phi\|_{**}\right). \tag{5.54}$$

In order to establish (5.41), it is sufficient to show that

$$\|e^{-(p^*-q)x}V^{q-1}\phi\|_{**} \leq o(1)\|\phi\|_*. \tag{5.55}$$

Indeed,

$$
\begin{aligned}
\|e^{-(p^*-q)x}V^{q-1}\phi\|_{**} &\le \sup_{x\le -M}\left(\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|}\right)^{-1}\left|e^{-(p^*-q)x}V^{q-1}\phi\right| \\
&\quad + \sup_{x\ge -M}\left(\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|}\right)^{-1}\left|e^{-(p^*-q)x}V^{q-1}\phi\right| \\
: &= Q_1+Q_2.
\end{aligned} \tag{5.56}
$$

Now we estimate Q_1 and Q_2 respectively, we first have

$$
\begin{aligned}
Q_1 &\le C\sup_{x\le -M} e^{\sigma|x-\xi_1|}|\phi(x)|e^{-(p^*-q)x}V^{q-1} \\
&\le C\sup_{x\le -M} e^{-(\frac{4}{N-2}+\sigma)x}e^{\sigma\xi_1}|\phi(x)|e^{\frac{4}{N-2}x}e^{-(p^*-q)x}\sum_{j=1}^{k} e^{-(q-1)|x-\xi_j|} \\
&\le C\sup_{x\le -M} e^{-(\frac{4}{N-2}+\sigma)x}e^{\sigma\xi_1}|\phi(x)|e^{2(q-1)x}e^{-(q-1)\xi_1} \\
&\le Ce^{-(q-1)\xi_1}\sup_{x\le -M} e^{-(\frac{4}{N-2}+\sigma)x}e^{\sigma\xi_1}|\phi(x)|.
\end{aligned} \tag{5.57}
$$

For Q_2, if $-M\le x\le \xi_1$, then we have

$$
\begin{aligned}
e^{-(p^*-q)x}V^{q-1} &\le \sum_{j=1}^{k} e^{-(p^*-q)x}e^{-(q-1)|x-\xi_j|}\le Ce^{(2q-p^*-1)x}e^{-(q-1)\xi_1} \\
&\le C\max\left\{e^{-(p^*-q)\xi_1}, e^{-(q-1)\xi_1}\right\}.
\end{aligned}
$$

If $x\ge \xi_1$, then we have

$$
e^{-(p^*-q)x}V^{q-1}\le \sum_{j=1}^{k} e^{-(p^*-q)x}e^{-(q-1)|x-\xi_j|}\le Ce^{-(p^*-q)x}\le Ce^{-(p^*-q)\xi_1}.
$$

Thus we find

$$
Q_2\le C\max\left\{e^{-(p^*-q)\xi_1}, e^{-(q-1)\xi_1}\right\}\sup_{x\ge -M}\left(\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|}\right)^{-1}|\phi(x)|. \tag{5.58}
$$

From (5.56)-(5.58), we get

$$
\|e^{-(p^*-q)x}V^{q-1}\phi\|_{**}\le C\max\left\{e^{-(p^*-q)\xi_1}, e^{-(q-1)\xi_1}\right\}\|\phi\|_* = o(1)\|\phi\|_*.
$$

So estimate (5.55) holds.

We now prove existence and uniqueness of solution to (5.36). Consider the Hilbert space

$$
H=\left\{\phi\in H^1(\mathbb{R})\ :\ \int_{\mathbb{R}} Z_j\phi = 0,\quad \forall\ j=1,2,\dots,k\right\}
$$

with inner product

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}} (\phi'\psi' + \phi\psi) dx.$$

Then problem (5.42) is equivalent to find $\phi \in H$ such that

$$\begin{aligned} \langle \phi, \psi \rangle \ &= \ \int_{\mathbb{R}} \Big[\alpha_\varepsilon (p^* + \varepsilon) V^{p^*+\varepsilon-1} \phi + \lambda q \beta_N e^{-(p^*-q)x} V^{q-1} \phi \\ &\quad + \left(\frac{2}{N-2} \right)^2 e^{-\frac{4}{N-2}x} \phi + h \Big] \psi dx \end{aligned} \tag{5.59}$$

for all $\psi \in H$. By the Riesz representation theorem, (5.59) is equivalent to solve

$$\phi = K(\phi) + \tilde{h} \tag{5.60}$$

with $\tilde{h} \in H$ depending linearly on h, and $K : H \to H$ being a compact operator. Fredholm's alternative yields there is a unique solution to problem (5.60) for any h provided that

$$\phi = K(\phi) \tag{5.61}$$

has only the zero solution in H. (5.61) is equivalent to problem (5.36) with $h = 0$. If $h = 0$, estimate (5.41) implies that $\phi = 0$. This ends the proof.

Now we study the differentiability of the operator T_ε with respect to $\xi = (\xi_1, \dots, \xi_k)$. Consider the Banach space

$$\mathcal{C}_* = \{ f \in C(\mathbb{R}) \ : \ \|f\|_{**} < \infty \}$$

endowed with the $\|\cdot\|_{**}$ norm. The following result holds.

Proposition 5.7. *Under the assumption of Proposition 5.5, the map $\xi \mapsto T_\varepsilon$ is of class C^1. Moreover,*

$$\|D_\xi T_\varepsilon(h)\|_* \leq C\|h\|_{**} \tag{5.62}$$

uniformly on the vectors ξ which satisfy (5.29).

Proof. Fix $h \in \mathcal{C}_*$ and let $\phi = T_\varepsilon(h)$ for $\varepsilon < \varepsilon_0$. Let us recall that ϕ satisfies

$$\begin{cases} \mathcal{L}_\varepsilon(\phi) = h + \sum_{j=1}^k c_j Z_j \quad \text{in } (-\infty, +\infty); \\ \lim_{|x|\to\infty} \phi(x) = 0; \\ \int_{\mathbb{R}} Z_j \phi = 0, \quad \forall\, j = 1, \dots, k, \end{cases}$$

for certain constants c_j. Differentiating above equation with respect to ξ_l, $l \in \{1, \dots, k\}$. Set $Y = \partial_{\xi_l} \phi$ and $d_j = \partial_{\xi_l} c_j$, we have

$$\begin{cases} \mathcal{L}_\varepsilon(Y) = \overline{h} + \sum_{j=1}^k d_j Z_j \quad \text{in } (-\infty, +\infty); \\ \lim_{|x|\to\infty} Y(x) = 0; \\ \int_{\mathbb{R}} Y Z_j + \phi \partial_{\xi_l} Z_j = 0, \quad \forall\, j = 1, \dots, k, \end{cases}$$

where

$$\begin{aligned}\overline{h} &= \alpha_\varepsilon(p^*+\varepsilon)(p^*+\varepsilon-1)e^{\varepsilon x}V^{p^*+\varepsilon-2}Z_l\phi \\ &\quad +\lambda q(q-1)\beta_N e^{-(p^*-q)x}V^{q-2}Z_l\phi + c_l\partial_{\xi_l}Z_l.\end{aligned}$$

Let $\eta = Y - \sum\limits_{i=1}^{k} b_i Z_i$, where $b_i \in \mathbb{R}$ is chosen such that

$$\int_{\mathbb{R}} \eta Z_j = 0,$$

that is,

$$\sum_{i=1}^{k} b_i \int_{\mathbb{R}} Z_i Z_j = \int_{\mathbb{R}} Y Z_j = \int_{\mathbb{R}} \partial_{\xi_l}\phi Z_j = -\int_{\mathbb{R}} \phi \partial_{\xi_l} Z_j. \tag{5.63}$$

This is an almost diagonal system, it has a unique solution and we have

$$|b_i| \le C\|\phi\|_*. \tag{5.64}$$

Moreover, η satisfies

$$\begin{cases} \mathcal{L}_\varepsilon(\eta) = g + \sum\limits_{j=1}^{k} d_j Z_j \quad \text{ in } (-\infty,+\infty); \\ \lim\limits_{|x|\to\infty} \eta(x) = 0; \\ \int_{\mathbb{R}} \eta Z_j = 0, \quad \forall\ j = 1,\ldots,k, \end{cases} \tag{5.65}$$

with

$$g = \overline{h} - \sum_{i=1}^{k} b_i \mathcal{L}_\varepsilon(Z_i).$$

From Proposition 5.5, there is a unique solution $\eta = T_\varepsilon(g)$ to (5.65) and

$$\|\eta\|_* \le C\|g\|_{**}. \tag{5.66}$$

On the other hand, we have

$$\begin{aligned}\|g\|_{**} &\le C\|e^{\varepsilon x}V^{p^*+\varepsilon-2}Z_l\phi\|_{**} + C\|e^{-(p^*-q)x}V^{q-2}Z_l\phi\|_{**} \\ &\quad +\|c_l\partial_{\xi_l}Z_l\|_{**} + \sum_{i=1}^{k}|b_i|\|\mathcal{L}_\varepsilon(Z_i)\|_{**} \\ &\quad \le C(\|\phi\|_* + |c_l| + |b_i|) \le C\|h\|_{**},\end{aligned} \tag{5.67}$$

because $|b_i| \le C\|\phi\|_*$, $\|\phi\|_* \le C\|h\|_{**}$, $|c_l| \le C\|h\|_{**}$ and

$$\|\mathcal{L}_\varepsilon(Z_i)\|_{**} = \big\|p^* W(x-\xi_i)^{p^*-1}\partial_{\xi_i}W(x-\xi_i)$$

$$\begin{aligned}
&-\alpha_\varepsilon(p^*+\varepsilon)e^{\varepsilon x}V^{p^*+\varepsilon-1}Z_i - \lambda q\beta_N e^{-(p^*-q)x}V^{q-1}Z_i\Big\|_{**}\\
\leq\ & C\|W(x-\xi_i)^{p-1}\partial_{\xi_i}W(x-\xi_i)\|_{**}\\
&+C\|e^{\varepsilon x}V^{p^*+\varepsilon-1}Z_i\|_{**} + C\|e^{-(p^*-q)x}V^{q-1}Z_i\|_{**}\\
\leq\ & C.
\end{aligned}$$

By (5.64), (5.66), (5.67) and $\|Z_i\|_* \leq C$, we obtain that

$$\|\partial_{\xi_l}\phi\|_* \leq \|\eta\|_* + \sum_{i=1}^{k}|b_i|\|Z_i\|_* \leq C\|h\|_{**}.$$

Besides $\partial_{\xi_l}\phi$ depends continuously on ξ in the considered region for this norm. □

5.4 The nonlinear problem

In this section, our purpose is to study the nonlinear problem. We first have the validity of the following result.

Lemma 5.8. *We have*

$$\|N(\phi)\|_{**} \leq C\left(\|\phi\|_*^{\min\{p^*,2\}} + \|\phi\|_*^{\min\{q,2\}}\right); \tag{5.68}$$

and

$$\|\partial_\phi N(\phi)\|_{**} \leq C\left(\|\phi\|_*^{\min\{p^*-1,1\}} + \|\phi\|_*^{\min\{q-1,1\}}\right). \tag{5.69}$$

Proof. We have

$$\begin{aligned}
N(\phi) &= \alpha_\varepsilon e^{\varepsilon x}\left[(V+\phi)^{p^*+\varepsilon} - V^{p^*+\varepsilon} - (p^*+\varepsilon)V^{p^*+\varepsilon-1}\phi\right]\\
&\quad +\lambda\beta_N e^{-(p^*-q)x}\left[(V+\phi)^q - V^q - qV^{q-1}\phi\right]\\
&= \alpha_\varepsilon e^{\varepsilon x}(p^*+\varepsilon)\int_0^1\left[(V+t\phi)^{p^*+\varepsilon-1} - V^{p^*+\varepsilon-1}\right]\phi\,dt\\
&\quad +\lambda q\beta_N e^{-(p^*-q)x}\int_0^1\left[(V+t\phi)^{q-1} - V^{q-1}\right]\phi\,dt.
\end{aligned}$$

Then

$$\begin{aligned}
&\|N(\phi)\|_{**}\\
=\ & \alpha_\varepsilon(p^*+\varepsilon)\sup_{x\in\mathbb{R}}\left(\sum_{j=1}^{k}e^{-\sigma|x-\xi_j|}\right)^{-1} e^{\varepsilon x}\left|\int_0^1\left[(V+t\phi)^{p^*+\varepsilon-1} - V^{p^*+\varepsilon-1}\right]\phi\,dt\right|\\
&+\lambda q\beta_N\sup_{x\in\mathbb{R}}\left(\sum_{j=1}^{k}e^{-\sigma|x-\xi_j|}\right)^{-1} e^{-(p^*-q)x}\left|\int_0^1\left[(V+t\phi)^{q-1} - V^{q-1}\right]\phi\,dt\right|
\end{aligned}$$

$$:= \quad N_1 + N_2.$$

We assume that $\|\phi\|_* \le 1$, by Lemma 5.15 in the Appendix, if $p^* \ge 2$, we have

$$\begin{aligned}
N_1 &\le C \sup_{x\in\mathbb{R}} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|}\right)^{-1} e^{\varepsilon x} V^{p^*+\varepsilon-2}|\phi|^2 + C \sup_{x\in\mathbb{R}} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|}\right)^{-1} e^{\varepsilon x}|\phi|^{p^*+\varepsilon} \\
&\le C \sup_{x\le -M} e^{\sigma|x-\xi_1|} e^{\varepsilon x} V^{p^*+\varepsilon-2} e^{(\frac{8}{N-2}+2\sigma)x} e^{-2\sigma\xi_1} \left[e^{-(\frac{4}{N-2}+\sigma)x} e^{\sigma\xi_1}|\phi|\right]^2 \\
&\quad + C \sup_{x\ge -M} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|}\right) e^{\varepsilon x} V^{p^*+\varepsilon-2} \left[\left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|}\right)^{-1} |\phi|\right]^2 \\
&\quad + C \sup_{x\le -M} e^{\sigma|x-\xi_1|} e^{\varepsilon x} e^{(\frac{4}{N-2}+\sigma)(p^*+\varepsilon)x} e^{-(p^*+\varepsilon)\sigma\xi_1} \left[e^{-(\frac{4}{N-2}+\sigma)x} e^{\sigma\xi_1}|\phi|\right]^{p^*+\varepsilon} \\
&\quad + C \sup_{x\ge -M} \left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|}\right)^{p^*+\varepsilon-1} e^{\varepsilon x} \left[\left(\sum_{j=1}^k e^{-\sigma|x-\xi_j|}\right)^{-1} |\phi|\right]^{p^*+\varepsilon} \\
&\le C\|\phi\|_*^2 + C\|\phi\|_*^{p^*+\varepsilon} \le C\|\phi\|_*^2.
\end{aligned}$$

Similarly, if $1 < p^* < 2$, we find that

$$N_1 \le C\|\phi\|_*^{p^*}.$$

Thus we get

$$N_1 \le C\|\phi\|_*^{\min\{p^*,2\}}.$$

Moreover, we can conclude that

$$N_2 \le C\|\phi\|_*^{\min\{q,2\}}.$$

Thus we get (5.68).

We differentiate $N(\phi)$ with respect to ϕ, we have

$$\begin{aligned}
\partial_\phi N(\phi) &= \alpha_\varepsilon (p^*+\varepsilon) e^{\varepsilon x} \left[(V+\phi)^{p^*+\varepsilon-1} - V^{p^*+\varepsilon-1}\right] \\
&\quad + \lambda\beta_N q e^{-(p^*-q)x} \left[(V+\phi)^{q-1} - V^{q-1}\right].
\end{aligned}$$

By a similar argument as $\|N(\phi)\|_{**}$, (5.69) holds. □

Lemma 5.9. *Let σ be a positive number which satisfies (5.37) and $0 < \xi_1 < \xi_2 < \ldots < \xi_k$ satisfying (5.29). If q satisfies (5.4), then there exist $\tau \in (\frac{1}{2}, 1)$ and a constant $C > 0$, such that*

$$\|E\|_{**} \le C\varepsilon^\tau, \qquad \|\partial_\xi E\|_{**} \le C\varepsilon^\tau. \tag{5.70}$$

Proof. We have

$$\begin{aligned}
E &= \alpha_\varepsilon e^{\varepsilon x} V^{p^*+\varepsilon} - \sum_{j=1}^k W_j^{p^*} + \lambda\beta_N e^{-(p^*-q)x} V^q \\
&= \alpha_\varepsilon e^{\varepsilon x}\left(V^{p^*+\varepsilon} - V^{p^*}\right) + (\alpha_\varepsilon e^{\varepsilon x}-1)V^{p^*} + \left(V^{p^*} - \left(\sum_{j=1}^k W_j\right)^{p^*}\right) \\
&\quad + \left(\left(\sum_{j=1}^k W_j\right)^{p^*} - \sum_{j=1}^k W_j^{p^*}\right) + \lambda\beta_N e^{-(p^*-q)x} V^q \\
&:= E_1 + E_2 + E_3 + E_4 + E_5. \qquad (5.71)
\end{aligned}$$

Estimate of E_1:

$$\begin{aligned}
|E_1| &= \left|\alpha_\varepsilon e^{\varepsilon x}\left(V^{p^*+\varepsilon} - V^{p^*}\right)\right| = \left|\varepsilon\alpha_\varepsilon e^{\varepsilon x}\int_0^1 V^{p^*+t\varepsilon}\log V\,dt\right| \\
&\le C\varepsilon e^{\varepsilon x} V^\varepsilon V^{p^*}|\log V| \le C\varepsilon V^{p^*}|\log V| \le C\varepsilon\sum_{j=1}^k e^{-\sigma|x-\xi_j|}. \qquad (5.72)
\end{aligned}$$

Estimate of E_2: by the Taylor expansion, we have

$$\begin{aligned}
|E_2| &= \left|(\alpha_\varepsilon e^{\varepsilon x}-1)V^{p^*}\right| = \left|\left(\left(\frac{p^*-1}{2}\right)^{-\frac{2\varepsilon}{p^*-1}} e^{\varepsilon x} - 1\right)V^{p^*}\right| \\
&= \left|\left[\left(1-\varepsilon\frac{2}{p^*-1}\log\frac{p^*-1}{2} + o(\varepsilon)\right)e^{\varepsilon x} - 1\right]V^{p^*}\right| \\
&= \left(\varepsilon x\int_0^1 e^{t\varepsilon x}\,dt + O(\varepsilon)e^{\varepsilon x}\right)V^{p^*-\sigma}V^\sigma \\
&\le C\varepsilon|\log\varepsilon|\sum_{j=1}^k e^{-\sigma|x-\xi_j|}. \qquad (5.73)
\end{aligned}$$

Estimate of E_3: since

$$|E_3| = \left|V^{p^*} - \left(\sum_{j=1}^k W_j\right)^{p^*}\right| \le CV^{p^*-1}\sum_{j=1}^k |R_{\xi_j}(x)|.$$

Thanks to Lemma 5.4, for $x \le 0$, we have

$$|E_3| \le CV^{p^*-1}\sum_{j=1}^k e^{-|x-\xi_j|} \le CV^{p^*-1}e^{-\xi_1} \le C\varepsilon^{\frac{1}{p^*-q}}\sum_{j=1}^k e^{-\sigma|x-\xi_j|}.$$

For $0 \le x \le \xi_1$,

$$\begin{aligned} |E_3| &\le CV^{p^*-1}\sum_{j=1}^{k} e^{-|x-\xi_j|}e^{-\frac{2}{N-2}\min\{x,\xi_j\}} \\ &\le C\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|}\begin{cases} \varepsilon^{\frac{2}{N+2-(N-2)q}} & \text{if } N\ge 4; \\ \varepsilon^{\frac{1}{5-q}} & \text{if } N=3. \end{cases} \end{aligned}$$

If $x \ge \xi_1$, for $0 < \sigma < p^* - 1$, we have

$$\begin{aligned} |E_3| &\le CV^{p^*-1}\sum_{j=1}^{k} e^{-|x-\xi_j|}e^{-\frac{2}{N-2}\min\{x,\xi_j\}} \\ &\le CV^{p^*-1}e^{-\frac{2}{N-2}\xi_1} \le C\varepsilon^{\frac{2}{N+2-(N-2)q}}\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|}. \end{aligned}$$

Therefore, for $x \in \mathbb{R}$, we get

$$|E_3| \le C\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|}\begin{cases} \varepsilon^{\frac{2}{N+2-(N-2)q}} & \text{if } N\ge 4; \\ \varepsilon^{\frac{1}{5-q}} & \text{if } N=3. \end{cases} \tag{5.74}$$

Estimate of E_4: if $-\infty < x \le \frac{\xi_1+\xi_2}{2}$, we have

$$\begin{aligned} |E_4| &= \left| \left(\sum_{j=1}^{k} W_j\right)^{p^*} - W_1^{p^*} - \sum_{j=2}^{k} W_j^{p^*} \right| \\ &\le \left| \left(\sum_{j=1}^{k} W(x-\xi_j)\right)^{p^*} - W(x-\xi_1)^{p^*} \right| + \left| \sum_{j=2}^{k} W(x-\xi_j)^{p^*} \right| \\ &\le p^*\left(\sum_{j=1}^{k} W(x-\xi_j)\right)^{p^*-1} \sum_{j=2}^{k} W(x-\xi_j) + \sum_{j=2}^{k} W(x-\xi_j)^{p^*} \\ &= p^*\left(\sum_{j=1}^{k} W(x-\xi_j)\right)^{p^*-1-\theta} \left(\sum_{j=1}^{k} W(x-\xi_j)\right)^{\theta} \sum_{j=2}^{k} W(x-\xi_j) + \sum_{j=2}^{k} W(x-\xi_j)^{p^*} \end{aligned}$$

with θ satisfying $0 < \theta < p^* - 1 - \sigma$. Since

$$\left(\sum_{j=1}^{k} W(x-\xi_j)\right)^{\theta} \sum_{j=2}^{k} W(x-\xi_j) \le \sum_{j=1}^{k} W(x-\xi_j)^{\theta} \sum_{j=2}^{k} W(x-\xi_j)$$

$$\le C\sum_{j=1}^{k} e^{-\theta|x-\xi_j|}\sum_{j=2}^{k} e^{-|x-\xi_j|} \le Ce^{-\theta|x-\xi_1|}\sum_{j=2}^{k} e^{-|x-\xi_j|}$$

$$
\begin{aligned}
&= C\sum_{j=2}^{k} e^{-\theta|x-\xi_1|}e^{-\theta|x-\xi_j|}e^{-(1-\theta)|x-\xi_j|} \\
&= C\sum_{j=2}^{k} e^{-\theta(|x-\xi_1|+|x-\xi_j|)}e^{-(1-\theta)|x-\xi_j|} \le C\sum_{j=2}^{k} e^{-\theta|\xi_1-\xi_j|}e^{-(1-\theta)\frac{\xi_2-\xi_1}{2}} \\
&\le Ce^{-\theta(\xi_2-\xi_1)}e^{-(1-\theta)\frac{\xi_2-\xi_1}{2}} = Ce^{-\frac{1+\theta}{2}(\xi_2-\xi_1)} \le C\varepsilon^{\frac{1+\theta}{2}}.
\end{aligned}
$$

Here we use $|x-\xi_1| \le |x-\xi_j|$, $|x-\xi_j| \ge \frac{\xi_2-\xi_1}{2}$ and $|\xi_1-\xi_j| \ge \xi_2-\xi_1$ for $j = 2, \cdots, k$. Moreover,

$$
\begin{aligned}
\sum_{j=2}^{k} W(x-\xi_j)^{p^*} &\le C\sum_{j=2}^{k} e^{-p^*|x-\xi_j|} = C\sum_{j=2}^{k} e^{-\sigma|x-\xi_j|}e^{-(p^*-\sigma)|x-\xi_j|} \\
&\le \sum_{j=2}^{k} e^{-\sigma|x-\xi_j|}e^{-(p^*-\sigma)\frac{\xi_2-\xi_1}{2}} \\
&\le C\varepsilon^{\frac{p^*-\sigma}{2}}\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|}.
\end{aligned}
$$

Thus

$$
|E_4| \le C\varepsilon^{\frac{1+\theta}{2}}\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|}, \quad \text{for} \quad -\infty < x \le \frac{\xi_1+\xi_2}{2},
$$

Similarly, for $\frac{\xi_{l-1}+\xi_l}{2} \le x \le \frac{\xi_l+\xi_{l+1}}{2}$ with $l = 2, \cdots, k-1$, and $x \ge \frac{\xi_{k-1}+\xi_k}{2}$, we get

$$
|E_4| \le C\varepsilon^{\frac{1+\theta}{2}}\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|}.
$$

Therefore, for $x \in \mathbb{R}$, we have

$$
|E_4| \le C\varepsilon^{\frac{1+\theta}{2}}\sum_{j=1}^{k} e^{-\sigma|x-\xi_j|}, \tag{5.75}
$$

with $0 < \theta < p^* - 1 - \sigma$.

Estimate of E_5:

$$
|E_5| = \left|\lambda q\beta_N e^{-(p^*-q)x}V^q\right| \le CV^\sigma e^{-(p^*-q)x}V^{q-\sigma}.
$$

If $-\infty < x \le -\frac{N-2}{2}\log R$ with $R > 0$ large but fixed as in Lemma 5.3, for $0 < \sigma < \frac{(N+2)(2q-1)}{N+6}$, from (5.18), we have

$$
|E_5| \le CV^\sigma e^{-(p^*-q)x}\left(\sum_{j=1}^{k} e^{\frac{N+6}{N-2}x}e^{-\xi_j}\right)^{q-\sigma}
$$

$$\leq \quad CV^\sigma e^{-(q-\sigma)\xi_1} \leq C\varepsilon^{\frac{q-\sigma}{p^*-q}} \sum_{j=1}^k e^{-\sigma|x-\xi_j|}.$$

If $-\frac{N-2}{2}\log R \leq x \leq \xi_1$, we have

$$\begin{aligned} |E_5| &\leq CV^\sigma e^{-(p^*-q)x} e^{-(q-\sigma)|x-\xi_1|} \\ &\leq CV^\sigma \begin{cases} e^{(p^*-2q+\sigma)\frac{N-2}{2}\log R} e^{-(q-\sigma)\xi_1} & \text{if } p^*-2q+\sigma \geq 0; \\ e^{-(p^*-2q+\sigma)\xi_1} e^{-(q-\sigma)\xi_1} & \text{if } p^*-2q+\sigma < 0. \end{cases} \\ &\leq C\max\{\varepsilon, \varepsilon^{\frac{q-\sigma}{p^*-q}}\} \sum_{j=1}^k e^{-\sigma|x-\xi_j|}. \end{aligned}$$

If $x \geq \xi_1$, we find

$$|E_5| \leq CV^\sigma e^{-(p^*-q)x} V^{q-\sigma} \leq CV^\sigma e^{-(p^*-q)\xi_1} \leq C\varepsilon \sum_{j=1}^k e^{-\sigma|x-\xi_j|}.$$

Thus, for $x \in \mathbb{R}$, we get that

$$|E_5| \leq C\max\{\varepsilon, \varepsilon^{\frac{q-\sigma}{p^*-q}}\} \sum_{j=1}^k e^{-\sigma|x-\xi_j|}. \tag{5.76}$$

From (5.71)-(5.76), for $0 < \theta < p^* - 1 - \sigma$ and σ satisfying (5.37), we have

$$\|E\|_{**} \leq C \begin{cases} \max\left\{\varepsilon|\log\varepsilon|,\ \varepsilon^{\frac{2}{N+2-(N-2)q}},\ \varepsilon^{\frac{1+\theta}{2}}, \varepsilon^{\frac{q-\sigma}{p^*-q}}\right\} & \text{if } N \geq 4; \\ \max\left\{\varepsilon|\log\varepsilon|, \varepsilon^{\frac{1}{5-q}}, \varepsilon^{\frac{1+\theta}{2}}, \varepsilon^{\frac{q-\sigma}{p^*-q}}\right\} & \text{if } N = 3. \end{cases}$$

Therefore, if q satisfies (5.4), we find that there exists $\tau \in (\frac{1}{2}, 1)$ such that

$$\|E\|_{**} \leq C\varepsilon^\tau.$$

Differentiating E with respect to ξ_i $(i = 1, 2\cdots, k)$, we have

$$\begin{aligned} \partial_{\xi_i} E &= \alpha_\varepsilon(p^*+\varepsilon)e^{\varepsilon x} V^{p^*+\varepsilon-1}\partial_{\xi_i} V - p^* \sum_{j=1}^k W(x-\xi_j)^{p^*-1}\partial_{\xi_i} W(x-\xi_j) \\ &\quad + \lambda\beta_N q e^{-(p^*-q)x} V^{q-1}\partial_{\xi_i} V \end{aligned}$$

The proof of estimate for $\|\partial_\xi E\|_{**}$ is similar to $\|E\|_{**}$. □

Proposition 5.10. *Assume that $0 < \xi_1 < \xi_2 < \ldots < \xi_k$ satisfy (5.29), then there exists $C > 0$ such that for $\varepsilon > 0$ small enough, there exists a unique solution $\phi = \phi(\xi)$ to problem (5.35) with*

$$\|\phi\|_* \leq C\varepsilon^\tau, \tag{5.77}$$

for some $\tau \in (\frac{1}{2}, 1)$*, satisfying Lemma 5.9. Moreover, the map* $\xi \mapsto \phi(\xi)$ *is of class* C^1 *for the* $\|\cdot\|_*$ *norm and*

$$\|\partial_\xi \phi\|_* \leq C\varepsilon^\tau. \tag{5.78}$$

Proof. Problem (5.35) is equivalent to solve a fixed point problem

$$\phi = T_\varepsilon(N(\phi) + E) := A_\varepsilon(\phi).$$

We will show that the operator A_ε is a contraction map in a proper region. Set

$$\mathcal{F}_\gamma = \{\phi \in C(\mathbb{R}) : \|\phi\|_* \leq \gamma\varepsilon^\tau\}$$

where $\gamma > 0$ will be chosen later.

For $\phi \in \mathcal{F}_\gamma$, by Lemmas 5.68 and 5.9, we get

$$\begin{aligned} \|A_\varepsilon(\phi)\|_* &= \|T_\varepsilon(N(\phi) + E)\|_* \leq C\|N(\phi)\|_{**} + \|E\|_{**} \\ &\leq C\left((\gamma\varepsilon^\tau)^{\min\{p^*,2\}} + (\gamma\varepsilon^\tau)^{\min\{q,2\}} + \varepsilon^\tau\right) \\ &= C\left(\gamma^{\min\{p^*,2\}}\varepsilon^{\min\{p^*-1,1\}\tau} + \gamma^{\min\{q,2\}}\varepsilon^{\min\{q-1,1\}\tau} + 1\right)\varepsilon^\tau. \end{aligned}$$

Then we have $A_\varepsilon(\phi) \in \mathcal{F}_\gamma$ for $\phi \in \mathcal{F}_\gamma$, by choosing γ large enough but fixed.

Moreover, for $\phi_1, \phi_2 \in \mathcal{F}_\gamma$, by writing

$$N(\phi_1) - N(\phi_2) = \int_0^1 N'(\phi_2 + t(\phi_1 - \phi_2))dt(\phi_1 - \phi_2).$$

By Proposition 5.5, using (5.69) we find

$$\begin{aligned} &\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* \leq C\|N(\phi_1) - N(\phi_2)\|_{**} \\ \leq\ & C\left(\left(\max_{i=1,2}\|\phi_i\|_*\right)^{\min\{p^*-1,1\}} + \left(\max_{i=1,2}\|\phi_i\|_*\right)^{\min\{q-1,1\}}\right)\|\phi_1 - \phi_2\|_* \\ \leq\ & C\varepsilon^\kappa\|\phi_1 - \phi_2\|_* \end{aligned}$$

with $\kappa > 0$, this yields that A_ε is a contraction map from $\mathcal{F}_\gamma$ to $\mathcal{F}_\gamma$. Thus A_ε has a unique fixed point in $\mathcal{F}_\gamma$.

Now we consider the differentiability of $\xi \mapsto \phi(\xi)$. We write

$$B(\xi, \phi) := \phi - T_\varepsilon(N(\phi) + E).$$

First we observe that $B(\xi, \phi) = 0$. Moreover,

$$\partial_\phi B(\xi, \phi)[\theta] = \theta - T_\varepsilon(\theta(\partial_\phi(N(\phi)))) \equiv \theta + M(\theta),$$

where

$$M(\theta) = -T_\varepsilon(\theta(\partial_\phi(N(\phi)))).$$

By a direct computation, we get

$$\|M(\theta)\|_* \le C\|\theta(\partial_\phi(N(\phi)))\|_{**} \le C\varepsilon^\kappa\|\theta\|_*.$$

So for ε small enough, the operator $\partial_\phi B(\xi,\phi)$ is invertible with uniformly bounded inverse in $\|\cdot\|_*$. It also depends continuously on its parameters. Let us differentiate with respect to ξ, we have

$$\partial_\xi B(\xi,\phi) = -(\partial_\xi T_\varepsilon)(N(\phi)+E) - T_\varepsilon((\partial_\xi N)(\xi,\phi)+\partial_\xi E),$$

where all these expressions depend continuously on their parameters. The implicit function theorem yields that $\phi(\xi)$ is of class C^1 and

$$\partial_\xi\phi = -(\partial_\phi B(\xi,\phi))^{-1}[\partial_\xi B(\xi,\phi)]$$

so that

$$\|\partial_\xi\phi\|_* \le C\left(\|N(\phi)\|_{**} + \|E\|_{**} + \|(\partial_\xi N)(\xi,\phi)\|_{**} + \|\partial_\xi E\|_{**}\right) \le C\varepsilon^\tau,$$

since

$$\begin{aligned}\partial_\xi N(\xi,\phi) \;&=\; \alpha_\varepsilon(p^*+\varepsilon)e^{\varepsilon x}\left[(V+\phi)^{p^*+\varepsilon-1} - V^{p^*+\varepsilon-1} - (p^*+\varepsilon-1)V^{p^*+\varepsilon-2}\phi\right]\partial_\xi V\\ &\quad+\lambda\beta_N q e^{-(p^*-q)x}\left[(V+\phi)^{q-1} - V^{q-1} - (q-1)V^{q-2}\phi\right]\partial_\xi V,\end{aligned}$$

then it is easily checked that

$$\|\partial_\xi N(\xi,\phi)\|_{**} \le C\|\phi\|_* \le C\varepsilon^\tau.$$

□

5.5 The finite dimensional variational reduction

According to the results of the previous section, our problem has been reduced to find points $\xi = (\xi_1, \xi_2, \cdots, \xi_k)$ such that

$$c_j(\xi) = 0 \qquad \text{for all } j = 1,\ldots,k. \tag{5.79}$$

If (5.79) holds, then $v = V + \phi$ is a solution to (5.13), and $u = \sum_{j=1}^{k} U_{\mu_j} + \psi$ is the solution to problem (5.3), with $\psi = \mathcal{T}^{-1}(\phi)$.

Define the function $\mathcal{I}_\varepsilon : (\mathbb{R}^+)^k \to \mathbb{R}$ as

$$\mathcal{I}_\varepsilon(\xi) := I_\varepsilon(V+\phi).$$

where V is defined by (5.30) and I_ε is the energy functional of (5.13) defined as

$$\begin{aligned}I_\varepsilon(v) \;&=\; \frac{1}{2}\int_{-\infty}^{+\infty}(|v'(x)|^2+|v|^2)dx + \frac{1}{2}\left(\frac{2}{N-2}\right)^2\int_{-\infty}^{+\infty} e^{-\frac{4}{N-2}x}v^2dx\\ &\quad-\frac{1}{p^*+\varepsilon+1}\alpha_\varepsilon\int_{-\infty}^{+\infty} e^{\varepsilon x}|v|^{p^*+\varepsilon+1}dx - \frac{1}{q+1}\lambda\beta_N\int_{-\infty}^{+\infty} e^{-(p^*-q)x}|v|^{q+1}dx.\end{aligned}$$

We have the following fact.

Lemma 5.11. *The function $V+\phi$ is a solution to (5.13) if and only if $\xi=(\xi_1,\dots,\xi_k)$ is a critical point of $\mathcal{I}_\varepsilon(\xi)$, where $\phi=\phi(\xi)$ is given by Proposition 5.10.*

Proof. For $s\in\{1,2,\dots,k\}$, we have

$$\begin{aligned}\partial_{\xi_s}\mathcal{I}_\varepsilon(\xi) &= \partial_{\xi_s}(I_\varepsilon(V+\phi)) = DI_\varepsilon(V+\phi)[\partial_{\xi_s}V+\partial_{\xi_s}\phi]\\ &= \sum_{j=1}^k c_j\int_{\mathbb{R}} Z_j[\partial_{\xi_s}V+\partial_{\xi_s}\phi]\\ &= \sum_{j=1}^k c_j\left(\int_{\mathbb{R}} Z_jZ_s dx + o(1)\right)\end{aligned}$$

where $o(1)\to 0$ as $\varepsilon\to 0$ uniformly for the norm $\|\cdot\|_*$. This implies that the above relations define an almost diagonal homogeneous linear equation system for the c_j. Thus ξ is the critical point of $\mathcal{I}_\varepsilon$ if and only if $c_j=0$ for all $j=1,2,\dots,k$. □

Lemma 5.12. *The following expansion holds*

$$\mathcal{I}_\varepsilon(\xi) = I_\varepsilon(V) + o(\varepsilon),$$

as $\varepsilon\to 0$, $o(\varepsilon)$ is uniform in the C^1-sense on the vectors ξ satisfying (5.29).

Proof. By the fact that $DI_\varepsilon(V+\phi)[\phi]=0$ and using the Taylor expansion, we have

$$\begin{aligned}\mathcal{I}_\varepsilon(\xi)-I_\varepsilon(V) &= I_\varepsilon(V+\phi)-I_\varepsilon(V) = \int_0^1 D^2I_\varepsilon(V+t\phi)[\phi^2]tdt\\ &= \int_0^1 tdt\int_{-\infty}^{+\infty}(N(\phi)+E)\phi dx\\ &\quad +(p^*+\varepsilon)\alpha_\varepsilon\int_0^1 tdt\int_{-\infty}^{+\infty} e^{\varepsilon x}\left[V^{p^*+\varepsilon-1}-(V+t\phi)^{p^*+\varepsilon-1}\right]\phi^2dx\\ &\quad +\lambda\beta_N q\int_0^1 tdt\int_{-\infty}^{+\infty} e^{-(p^*-q)x}\left[V^{q-1}-(V+t\phi)^{q-1}\right]\phi^2dx\end{aligned}$$

and since $\|\phi\|_*\le C\varepsilon^\tau$ and $\|E\|_{**}\le C\varepsilon^\tau$ with $\tau>\frac12$, we get

$$\mathcal{I}_\varepsilon(\xi)-I_\varepsilon(V) = O(\varepsilon^{2\tau}) = o(\varepsilon)$$

uniformly on the points ξ satisfying (5.29).

Moreover, differentiating with respect to ξ_s, we have

$$\partial_{\xi_s}\left(\mathcal{I}_\varepsilon(\xi)-I_\varepsilon(V)\right) = \int_0^1\int_{-\infty}^{+\infty}\partial_{\xi_s}[(N(\phi)+E)\phi]tdxdt$$

$$+\alpha_\varepsilon(p^*+\varepsilon)\int_0^1 tdt\int_{-\infty}^{+\infty} e^{\varepsilon x}\partial_{\xi_s}\left(\left[V^{p^*+\varepsilon-1}-(V+t\phi)^{p^*+\varepsilon-1}\right]\phi^2\right)dx$$

$$+\lambda\beta_N q\int_0^1 tdt\int_{-\infty}^{+\infty} e^{-(p^*-q)x}\partial_{\xi_s}\left(\left[V^{q-1}-(V+t\phi)^{q-1}\right]\phi^2\right)dx.$$

By the fact that $\|\partial_\xi\phi\|_*\le C\varepsilon^\tau$ and $\|\partial_\xi E\|_{**}\le C\varepsilon^\tau$ with $\tau>\frac{1}{2}$, we deduce that

$$\partial_{\xi_s}\left(\mathcal{I}_\varepsilon(\xi)-I_\varepsilon(V)\right)=O(\varepsilon^{2\tau})=o(\varepsilon).$$

□

Now we consider the energy functional of problem (5.3), which is defined by

$$J(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+u^2)-\frac{1}{p^*+1+\varepsilon}\int_{\mathbb{R}^N}|u|^{p^*+1+\varepsilon}-\frac{\lambda}{q+1}\int_{\mathbb{R}^N}|u|^{q+1}.$$

By a direct calculation, we have that

$$I_\varepsilon(V)=\left(\frac{2}{N-1}\right)^{N-1}\frac{1}{\omega_{N-1}}J(U), \tag{5.80}$$

where V is defined by (5.30), ω_{N-1} is the volume of the unit sphere in $\mathbb{R}^N$, and

$$U(z)=\sum_{j=1}^k U_{\mu_j}(z),$$

with U_{μ_j} satisfying (5.6).

We give the following expansion of $J(U)$, whose proof is in the Appendix.

Lemma 5.13. *Assume that (5.26) and (5.29) hold, then we have the following expansion:*

$$J(U)=a_1+a_2\varepsilon-\varphi(\Lambda_1,\cdots,\Lambda_k)\varepsilon+a_3\varepsilon\log\varepsilon+o(\varepsilon), \tag{5.81}$$

where

$$\varphi(\Lambda_1,\cdots,\Lambda_k)=a_4\Lambda_1^{\frac{N+2-(N-2)q}{2}}-a_5\sum_{i=1}^k\log\Lambda_i+a_6\sum_{l=1}^{k-1}\left(\frac{\Lambda_{l+1}}{\Lambda_l}\right)^{\frac{N-2}{2}}, \tag{5.82}$$

and as $\varepsilon\to 0$, $o(\varepsilon)$ is uniform in the C^1-sense on the Λ_i's satisfying (5.26), and

$$\begin{aligned} a_1&=\frac{k}{N}\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}dz,\\ a_2&=\frac{k}{(p^*+1)^2}\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}dz\\ &\quad-\frac{k}{p^*+1}\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}\log\frac{\alpha_N}{(1+|z|^2)^{\frac{N-2}{2}}}dz,\\ a_3&=\frac{(N-2)^2}{4N}\left(\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}dz\right) \end{aligned}$$

$$\times \sum_{i=1}^{k} \left(\frac{2(i-1)}{N-2} + \frac{2}{N+2-(N-2)q} \right),$$
$$a_4 = \frac{\lambda}{q+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{(N-2)(q+1)}{2}}} dz,$$
$$a_5 = \frac{(N-2)^2}{4N} \left(\alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^N} dz \right),$$
$$a_6 = \alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} \frac{1}{|z|^{N-2}} dz.$$

Proof of Theorem 5.1. Thanks to Lemma 5.11, we know that

$$u = \sum_{j=1}^{k} U_{\mu_j} + \psi \quad \text{with } \psi = \mathcal{T}^{-1}(\phi)$$

is a solution to problem (5.3) if and only if ξ is a critical point of $\mathcal{I}_\varepsilon(\xi)$, where the existence of ϕ is guaranteed by Proposition 5.10.

Finding a critical point of $\mathcal{I}_\varepsilon(\xi)$ is equivalent to find that of $\widetilde{\mathcal{I}}_\varepsilon(\xi)$, which is defined as

$$\widetilde{\mathcal{I}}_\varepsilon(\xi) = -\left(\frac{N-1}{2}\right)^{N-1} \frac{\omega_{N-1}}{\varepsilon} \mathcal{I}_\varepsilon(\xi) + \frac{a_1}{\varepsilon} + a_2 + a_3 \log \varepsilon.$$

On the other hand, from Lemmas 5.12 and 5.13, using (5.80), we have

$$\begin{aligned} \mathcal{I}_\varepsilon(\xi) &= I_\varepsilon(V) + o(\varepsilon) = \left(\frac{2}{N-1}\right)^{N-1} \frac{1}{\omega_{N-1}} J(U) + o(\varepsilon) \\ &= \left(\frac{2}{N-1}\right)^{N-1} \frac{1}{\omega_{N-1}} \left[a_1 + a_2\varepsilon - \varphi(\Lambda_1, \cdots, \Lambda_k)\varepsilon + a_3 \varepsilon \log \varepsilon\right] + o(\varepsilon), \end{aligned}$$

as $\varepsilon \to 0$, where $\varphi(\Lambda)$ is defined by (5.82) and $o(\varepsilon)$ is uniform in the C^1-sense. Then we have

$$\widetilde{\mathcal{I}}_\varepsilon(\xi) = \varphi(\Lambda) + o(1), \tag{5.83}$$

where $o(1)$ is uniform in the C^1-sense as $\varepsilon \to 0$.

We set $s_1 = \Lambda_1$, $s_j = \frac{\Lambda_j}{\Lambda_{j-1}}$, then we can write $\varphi(\Lambda_1, \cdots, \Lambda_k)$ as

$$\begin{aligned} \varphi(s_1, \cdots, s_k) &= a_4 s_1^{\frac{N+2-(N-2)q}{2}} - a_5 \log s_1 - a_5 \sum_{j=2}^{k} \log \Lambda_j + a_6 \sum_{j=2}^{k} s_j^{\frac{N-2}{2}} \\ &= a_4 s_1^{\frac{N+2-(N-2)q}{2}} - a_5 k \log s_1 \\ &\quad - \sum_{j=2}^{k} \left[a_5(k-j+1)\log s_j - a_6 s_j^{\frac{N-2}{2}}\right] \end{aligned}$$

$$:= \quad \tilde{\varphi}_1 - \sum_{j=2}^{k} \tilde{\varphi}_j,$$

with

$$\tilde{\varphi}_1 = a_4 s_1^{\frac{N+2-(N-2)q}{2}} - a_5 k \log s_1$$

and

$$\tilde{\varphi}_j = a_5(k-j+1)\log s_j - a_6 s_j^{\frac{N-2}{2}}, \quad j = 2, \dots, k.$$

We note that

$$\bar{s}_1 = \left(\frac{2a_5 k}{a_4(N+2-(N-2)q)}\right)^{\frac{2}{N+2-(N-2)q}} \tag{5.84}$$

is the critical point of $\tilde{\varphi}_1$, and

$$\bar{s}_j = \left(\frac{2a_5(k-j+1)}{(N-2)a_6}\right)^{\frac{2}{N-2}}, \qquad j = 2, \cdots, k, \tag{5.85}$$

is the critical point of $\tilde{\varphi}_j$. Moreover

$$\tilde{\varphi}_1''(\bar{s}_1) < 0, \quad \tilde{\varphi}_j''(\bar{s}_j) < 0, \quad j = 2, \cdots, k.$$

So $(\bar{s}_1, \bar{s}_2, \dots, \bar{s}_k)$ is a nondegenerate critical point of $\varphi(s_1, \cdots, s_k)$. Thus

$$\Lambda^* := (\bar{s}_1, \bar{s}_2\bar{s}_1, \bar{s}_3\bar{s}_2\bar{s}_1, \cdots, \bar{s}_k \times \cdots \times \bar{s}_2\bar{s}_1)$$

is a nondegenerate critical point of $\varphi(\Lambda)$. It follows that the local degree $deg(\nabla\varphi(\Lambda), \mathcal{O}, 0)$ is well defined and is nonzero, here $\mathcal{O}$ is an arbitrarily small neighborhood of Λ^*. Hence from (5.83), for ε small enough, we have that

$$deg(\nabla_\xi \widetilde{\mathcal{I}}_\varepsilon(\xi), \bar{\mathcal{O}}, 0) \neq 0,$$

with $\bar{\mathcal{O}}$ is a small neighborhood of $\xi^* = (\xi_1^*, \dots, \xi_k^*)$, where

$$\xi_j^* = \left[(j-1) + \frac{1}{p^*-q}\right] \log\frac{1}{\varepsilon} - \frac{N-2}{2} \log\left(\bar{s}_j \bar{s}_{j-1} \cdots \bar{s}_1\right), \text{ for } \forall\ j = 1, \dots, k.$$

So ξ^* is a critical point of $\widetilde{\mathcal{I}}_\varepsilon(\xi)$, which implies there is a critical point of $\mathcal{I}_\varepsilon$.

Furthermore, if for some i, $|x-\xi_i| \le C_0$ with some $C_0 > 0$, then we have $|\phi| = o(W(x-\xi_i))$. Thus $\psi(|z|) = \mathcal{T}^{-1}(\phi(x)) = o(w_{\mu_i})$ for $\frac{1}{C}\mu_i \le |z| \le C\mu_i$. Moreover, from (c) of Lemma 5.3, we get that $R_{\mu_i} = o(w_{\mu_i})$ for $\frac{1}{C}\mu_i \le |z| \le C\mu_i$. Therefore we obtain (5.5) holds with

$$\Lambda_j^* = \bar{s}_j \bar{s}_{j-1} \cdots \bar{s}_1, \quad j = 1, \dots, k,$$

where $\bar{s}_j$ are given by (5.84) and (5.85). This finishes the proof. □

5.6 Appendix

5.6.1 Some useful tools

In this subsection, we first give some useful Lemmas here, we use them for the later purpose.

Lemma 5.14. *[116] For any $0 < \sigma < N-2$, there is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|y|)^{2+\sigma}}\, dy \leq \frac{C}{(1+|z|)^{\sigma}}.$$

Lemma 5.15. *For any $a \in \mathbb{R}$ and $b \in \mathbb{R}$, we have*

$$||a+b|^q - |a|^q| \leq C \begin{cases} |a|^{q-1}|b| + |b|^q & \text{if } q \geq 1; \\ \min\{|a|^{q-1}|b|, |b|^q\} & \text{if } 0 < q < 1. \end{cases}$$

5.6.2 Proof of Lemma 5.3

In order to prove Lemma 5.3, we introduce the Green function. For a fixed $z \in \mathbb{R}^N$, let $G(z, y)$ be the Green function of $-\Delta + Id$, which satisfies

$$\begin{aligned} -\Delta G(z,y) + G(z,y) &= \delta_z(y) \quad \text{in } \mathbb{R}^N \\ G(z,y) \to 0 & \qquad\quad |y| \to \infty. \end{aligned}$$

We have the following result.

Lemma 5.16. *We have*

$$|G(z,y)| \leq \frac{C}{|y-z|^{N-2}} \qquad \text{for } \ 0 < |y-z| \leq 1, \tag{5.86}$$

and

$$|G(z,y)| \leq C|y-z|^{\frac{1-N}{2}} e^{-|y-z|} \qquad \text{for } \ |y-z| \geq 1. \tag{5.87}$$

Proof. By radial symmetry, we can write $G(z,y) = G(r)$ with $r = |y-z|$. Since $G(r)$ is singular at zero and tends to zero at infinity, we can verify that G is given by

$$G(r) = \frac{N-2}{(2\pi)^{\frac{N}{2}} \Gamma(\frac{N}{2})^2} r^{\frac{2-N}{2}} K_{\frac{N-2}{2}}(r),$$

where $K_{\frac{N-2}{2}}(r)$ is a Modified Bessel Function of the Second Kind, see [68]. For $N = 3$, the function G has the explicit form $G(r) = \frac{e^{-r}}{4\pi r}$. In general, we have that $K_{\frac{N-2}{2}}(r) \sim \frac{\Gamma(\frac{N-2}{2})}{2}(\frac{2}{r})^{\frac{N-2}{2}}$ for r close to 0, and $K_{\frac{N-2}{2}}(r) \sim \sqrt{\frac{\pi}{2r}} e^{-r}$ for r large. Using these estimates, we obtain the result. □

Proof of Lemma 5.3. (a) It is a direct consequence of the maximum principle.

(b) Define the barrier function $Q(z)=\mu^{\frac{N-2}{2}}|z|^{-(N+2)}$. It satisfies $-\Delta Q(z)+Q(z)\geq c\mu^{\frac{N-2}{2}}|z|^{-(N+2)}$ for all $|z|\geq R$ with $R>0$ a large constant, here c is positive constant. Since $Q(z)=\mu^{\frac{N-2}{2}}R^{-(N+2)}$ for $|z|=R$ and $U_\mu(z)\leq w_\mu(z)\leq\alpha_N\mu^{\frac{N-2}{2}}|z|^{-(N-2)}$ for all $|z|\geq 0$. Set $\varphi(z)=AQ(z)-U_\mu(z)$ for some constant $A>0$, we then have $-\Delta\varphi(z)+\varphi(z)\geq 0$ for $|z|\geq R$, and $\varphi(z)\geq 0$ for $|z|=R$ by choosing suitable constant A. By the maximum principle we get $U_\mu(z)\leq AQ(z)=A\mu^{\frac{N-2}{2}}|z|^{-(N+2)}$ for $|z|\geq R$.

(c) Set $B_1(z)=\{y:\ |y-z|\leq 1\}$, by Lemma 5.16, we have

$$\begin{aligned}|R_\mu(z)| &\leq \int_{\mathbb{R}^N}|G(y-z)|w_\mu(y)dy\\ &\leq C\int_{B_1(z)}\frac{1}{|y-z|^{N-2}}\frac{\mu^{\frac{N-2}{2}}}{(\mu^2+|y|^2)^{\frac{N-2}{2}}}dy\\ &\quad +C\int_{\mathbb{R}^N\setminus B_1(z)}|y-z|^{\frac{1-N}{2}}e^{-|y-z|}\frac{\mu^{\frac{N-2}{2}}}{(\mu^2+|y|^2)^{\frac{N-2}{2}}}dy\\ &:= I_1(z)+I_2(z).\end{aligned}\tag{5.88}$$

(i) We may assume that $|z|\geq 2$, we first estimate $I_1(z)$. For $y\in B_1(z)$, we have $|y|\geq|z|-1\geq\frac{|z|}{2}$. Therefore

$$\begin{aligned}I_1(z) &\leq C\frac{\mu^{\frac{N-2}{2}}}{(\mu^2+|\frac{z}{2}|^2)^{\frac{N-2}{2}}}\int_{B_1(z)}\frac{1}{|y-z|^{N-2}}dy\\ &= C\frac{\mu^{\frac{N-2}{2}}}{(\mu^2+|\frac{z}{2}|^2)^{\frac{N-2}{2}}}\int_{B_1(0)}\frac{1}{|z|^{N-2}}dz\\ &\leq C\frac{\mu^{\frac{N-2}{2}}}{(\mu^2+|\frac{z}{2}|^2)^{\frac{N-2}{2}}}\leq C\frac{\mu^{\frac{N-2}{2}}}{|z|^{N-2}}.\end{aligned}\tag{5.89}$$

Now let us estimate I_2. Set $\tilde{y}=\frac{y}{\mu}$, $\tilde{z}=\frac{z}{\mu}$ and $d=\frac{1}{2}|\tilde{z}|$, we have

$$\begin{aligned}I_2(z) &= C\mu^{\frac{N+2}{2}}\int_{\mathbb{R}^N\setminus B_{\frac{1}{\mu}}(\tilde{z})}|\mu(\tilde{y}-\tilde{z})|^{\frac{1-N}{2}}e^{-\mu|\tilde{y}-\tilde{z}|}\frac{1}{(1+|\tilde{y}|^2)^{\frac{N-2}{2}}}d\tilde{y}\\ &\leq C\mu^{\frac{N+2}{2}}\int_{B_d(0)}|\mu(\tilde{y}-\tilde{z})|^{\frac{1-N}{2}}e^{-\mu|\tilde{y}-\tilde{z}|}\frac{1}{(1+|\tilde{y}|)^{N-2}}d\tilde{y}\\ &\quad +C\mu^{\frac{N+2}{2}}\int_{B_d(\tilde{z})\setminus B_{\frac{1}{\mu}}(\tilde{z})}|\mu(\tilde{y}-\tilde{z})|^{\frac{1-N}{2}}e^{-\mu|\tilde{y}-\tilde{z}|}\frac{1}{(1+|\tilde{y}|)^{N-2}}d\tilde{y}\\ &\quad +C\mu^{\frac{N+2}{2}}\int_{\mathbb{R}^N\setminus(B_d(\tilde{z})\cup B_d(0))}|\mu(\tilde{y}-\tilde{z})|^{\frac{1-N}{2}}e^{-\mu|\tilde{y}-\tilde{z}|}\frac{1}{(1+|\tilde{y}|)^{N-2}}d\tilde{y}\\ &:= I_{2,1}+I_{2,2}+I_{2,3}.\end{aligned}\tag{5.90}$$

Note that for $|y-z|\geq 1$, we have $|y-z|^{\frac{1-N}{2}}e^{-|y-z|}\leq \frac{1}{|y-z|^s}$ for any $s>0$. If $\tilde{y}\in B_d(0)$, we have $|\tilde{y}-\tilde{z}|\geq|\tilde{z}|-|\tilde{y}|\geq d$, then

$$\begin{aligned}
I_{2,1} &\leq C\mu^{\frac{N+2}{2}}\int_{B_d(0)}\frac{1}{\mu^N|\tilde{y}-\tilde{z}|^N}\frac{1}{(1+|\tilde{y}|)^{N-2}}d\tilde{y}\\
&\leq C\mu^{\frac{N+2}{2}}\frac{1}{\mu^N d^N}\int_{B_d(0)}\frac{1}{(1+|\tilde{y}|)^{N-2}}d\tilde{y}\\
&\leq C\frac{\mu^{\frac{N-2}{2}}}{|z|^{N-2}}. \qquad (5.91)
\end{aligned}$$

If $\tilde{y}\in B_d(\tilde{z})\backslash B_{\frac{1}{\mu}}(\tilde{z})$, we have $1+|\tilde{y}|>|\tilde{y}|=|\tilde{z}+\tilde{y}-\tilde{z}|\geq|\tilde{z}|-|\tilde{y}-\tilde{z}|\geq d$, thus

$$I_{2,2}\leq C\mu^{\frac{N+2}{2}}\int_{B_d(\tilde{z})\backslash B_{\frac{1}{\mu}}(\tilde{z})}\frac{1}{\mu^{N+1}|\tilde{y}-\tilde{z}|^{N+1}}\frac{1}{(1+|\tilde{y}|)^{N-2}}d\tilde{y}\leq C\frac{\mu^{\frac{N-2}{2}}}{|z|^{N-2}}. \qquad (5.92)$$

If $\tilde{y}\in\mathbb{R}^N\backslash(B_d(\tilde{z})\cup B_d(0))$, we have $|\tilde{y}-\tilde{z}|\geq d=\frac{1}{2}|\tilde{z}|$, $|\tilde{y}|\geq d=\frac{1}{2}|\tilde{z}|$. We find that if $|\tilde{y}|\geq 2|\tilde{z}|$, then $|\tilde{y}-\tilde{z}|\geq|\tilde{y}|-|\tilde{z}|\geq\frac{1}{2}|\tilde{y}|$. If $\frac{1}{2}|\tilde{z}|\leq|\tilde{y}|\leq 2|\tilde{z}|$, then $|\tilde{y}-\tilde{z}|\geq d=\frac{1}{2}|\tilde{z}|\geq\frac{1}{4}|\tilde{y}|$. Thus,

$$I_{2,3}\leq C\mu^{\frac{N+2}{2}}\int_{\mathbb{R}^N\backslash(B_d(\tilde{z})\cup B_d(0))}\frac{1}{\mu^N|\tilde{y}-\tilde{z}|^N}\frac{1}{(1+|\tilde{y}|)^{N-2}}d\tilde{y}\leq C\frac{\mu^{\frac{N-2}{2}}}{|z|^{N-2}}. \qquad (5.93)$$

From (5.90)-(5.93), we obtain that

$$I_2\leq C\frac{\mu^{\frac{N-2}{2}}}{|z|^{N-2}}.$$

Combing this with (5.89), we get that (5.8).

(ii) First we suppose that $|z|\leq\frac{\mu}{2}$,

$$\begin{aligned}
I_1(z) &= C\int_{B_1(z)}\frac{1}{|y-z|^{N-2}}\frac{\mu^{\frac{N-2}{2}}}{(\mu^2+|y|^2)^{\frac{N-2}{2}}}dy\\
&= C\mu^{-\frac{N-6}{2}}\int_{B_{\frac{1}{\mu}}(0)}\frac{1}{|\tilde{y}|^{N-2}}\frac{1}{(1+|\tilde{y}+\tilde{z}|)^{N-2}}d\tilde{y}\\
&\leq C\mu^{-\frac{N-6}{2}}\int_{B_{\frac{1}{\mu}}(0)}\frac{1}{|\tilde{y}|^{N-2}}\frac{1}{(1+|\tilde{y}|)^{N-2}}d\tilde{y}\\
&\leq C\begin{cases}\mu^{-\frac{N-6}{2}} & \text{if } N\geq 5;\\ \mu\log\frac{1}{\mu} & \text{if } N=4;\\ \mu^{\frac{1}{2}} & \text{if } N=3.\end{cases} \qquad (5.94)
\end{aligned}$$

We now assume $\frac{\mu}{2}\leq|z|\leq 1$, we have

$$I_1=C\mu^{-\frac{N-6}{2}}\int_{B_{\frac{1}{\mu}}(\tilde{z})}\frac{1}{|\tilde{y}-\tilde{z}|^{N-2}}\frac{1}{(1+|\tilde{y}|)^{N-2}}d\tilde{y} \qquad (5.95)$$

with $\tilde{z} = \frac{z}{\mu}$. Let $d = \frac{1}{2}|\tilde{z}|$, then

$$B_{\frac{1}{\mu}}(\tilde{z}) = B_d(\tilde{z}) \cup \left(B_{\frac{1}{\mu}}(\tilde{z}) \cap B_d(0)\right) \cup \left(B_{\frac{1}{\mu}}(\tilde{z}) \backslash (B_d(\tilde{z}) \cup B_d(0))\right).$$

For $\tilde{y} \in B_d(\tilde{z})$, we have $|\tilde{y} - \tilde{z}| \le d, \quad |\tilde{y}| \ge |\tilde{z}| - |\tilde{z} - \tilde{y}| \ge d$, so

$$\begin{aligned} &\mu^{-\frac{N-6}{2}} \int_{B_d(\tilde{z})} \frac{1}{|\tilde{y} - \tilde{z}|^{N-2}} \frac{1}{(1 + |\tilde{y}|)^{N-2}} d\tilde{y} \\ \le\ & C\mu^{-\frac{N-6}{2}} \frac{1}{d^{N-2}} \int_{B_d(\tilde{z})} \frac{1}{|\tilde{y} - \tilde{z}|^{N-2}} d\tilde{y} \le C\mu^{-\frac{N-6}{2}} \frac{1}{d^{N-4}}. \end{aligned} \tag{5.96}$$

Moreover, if $\tilde{y} \in B_d(0)$, then $|\tilde{y} - \tilde{z}| \ge |\tilde{z}| - |\tilde{y}| \ge d$. Thus

$$\begin{aligned} &\mu^{-\frac{N-6}{2}} \int_{B_d(0)} \frac{1}{|\tilde{y} - \tilde{z}|^{N-2}} \frac{1}{(1 + |\tilde{y}|)^{N-2}} d\tilde{y} \\ \le\ & C\mu^{-\frac{N-6}{2}} \frac{1}{d^{N-2}} \int_{B_d(0)} \frac{1}{(1 + |\tilde{y}|)^{N-2}} d\tilde{y} \le C\mu^{-\frac{N-6}{2}} \frac{1}{d^{N-4}}. \end{aligned} \tag{5.97}$$

Finally, if $\tilde{y} \in B_{\frac{1}{\mu}}(\tilde{z}) \backslash (B_d(\tilde{z}) \cup B_d(0))$, then we have $|\tilde{y} - \tilde{z}| \ge C|\tilde{y}|$. As a result,

$$\begin{aligned} &C\mu^{-\frac{N-6}{2}} \int_{B_{\frac{1}{\mu}}(\tilde{z}) \backslash (B_d(\tilde{z}) \cup B_d(0))} \frac{1}{|\tilde{y} - \tilde{z}|^{N-2}} \frac{1}{(1 + |\tilde{y}|)^{N-2}} d\tilde{y} \\ \le\ & C\mu^{-\frac{N-6}{2}} \int_{B_{\frac{1}{\mu}}(\tilde{z}) \backslash B_d(0)} \frac{1}{|\tilde{y}|^{N-2}} \frac{1}{(1 + |\tilde{y}|)^{N-2}} d\tilde{y} \\ \le\ & C \begin{cases} \mu^{-\frac{N-6}{2}} \frac{1}{(1+|\frac{z}{\mu}|^2)^{\frac{N-4}{2}}} & \text{if } N \ge 5; \\ \mu \log \frac{1}{|z|} & \text{if } N = 4; \\ \mu^{\frac{1}{2}}(1 - |z|) & \text{if } N = 3. \end{cases} \end{aligned} \tag{5.98}$$

Now we estimate $I_2(z)$ for $|z| \le 1$. We assume that $|y - z| \ge 2$, then $|y| \ge 1$. Therefore

$$\int_{\mathbb{R}^N \backslash B_2(z)} |y - z|^{\frac{1-N}{2}} e^{-|y-z|} \frac{\mu^{\frac{N-2}{2}}}{(\mu^2 + |y|^2)^{\frac{N-2}{2}}} dy \le C\mu^{\frac{N-2}{2}} \tag{5.99}$$

From (5.94) and (5.99), we get (5.9). (5.10) follows from (5.95)-(5.99). □

Set

$$\widetilde{Z}_\mu(z) = \partial_\mu U_\mu(z), \qquad \overline{Z}_\mu(z) = \partial_\mu w_\mu(z),$$

then $\tilde{Z}_\mu(z)$ satisfies

$$\begin{cases} -\Delta \widetilde{Z}_\mu + \widetilde{Z}_\mu = \frac{N+2}{N-2} w_\mu^{\frac{4}{N-2}} \overline{Z}_\mu & \text{in } \mathbb{R}^N; \\ \widetilde{Z}_\mu(z) \to 0 \ \text{ as } \ |z| \to \infty. \end{cases} \tag{5.100}$$

We can write

$$\widetilde{Z}_\mu(z) = \overline{Z}_\mu(z) + \partial_\mu R_\mu(z),$$

then $\partial_\mu R_\mu(z)$ satisfies

$$\begin{cases} -\Delta(\partial_\mu R_\mu(z)) + \partial_\mu R_\mu(z) = -\partial_\mu w_\mu(z) & \text{in } \mathbb{R}^N; \\ \partial_\mu R_\mu(z) \to 0 \ \text{ as } \ |z| \to \infty. \end{cases} \tag{5.101}$$

We observe that $|-\partial_\mu w_\mu(z)| \leq C\mu^{-1} w_\mu$, then we have

Corollary 5.17. *One has*

$$|\partial_\mu R_\mu(z)| \leq C\mu^{-1}|R_\mu(z)| \quad \textit{for } \forall\ z \in \mathbb{R}^N. \tag{5.102}$$

Moreover, by the maximum principle, we have that

$$|\widetilde{Z}_\mu(z)| \leq C\mu^{\frac{N-4}{2}}|z|^{-(N+2)}, \qquad \textit{for } \ |z| \geq R, \tag{5.103}$$

where R is a large positive number but fixed in Lemma 5.3.

5.6.3 Expansion of energy

Finally, we compute the expansion of energy functional $J(U)$.

Proof of Lemma 5.13.

$$\begin{aligned} J(U) &= \left[\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla U|^2 + U^2) - \frac{1}{p^*+1}\int_{\mathbb{R}^N} U^{p^*+1}\right] \\ &\quad + \left[\frac{1}{p^*+1}\int_{\mathbb{R}^N} U^{p^*+1} - \frac{1}{p^*+1+\varepsilon}\int_{\mathbb{R}^N} U^{p^*+1+\varepsilon}\right] - \frac{\lambda}{q+1}\int_{\mathbb{R}^N} U^{q+1} \\ &:= J_1 + J_2 + J_3, \end{aligned} \tag{5.104}$$

where $U = \sum_{j=1}^{k} U_{\mu_j}$ with $U_{\mu_j} = w_{\mu_j} + R_{\mu_j}$.

Step 1. *We expand J_1.*

$$\begin{aligned} J_1 &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla U|^2 + U^2) - \frac{1}{p^*+1}\int_{\mathbb{R}^N} U^{p^*+1} \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left(\left|\nabla\left(\sum_{j=1}^{k} U_{\mu_j}\right)\right|^2 + \left(\sum_{j=1}^{k} U_{\mu_j}\right)^2\right) - \frac{1}{p^*+1}\int_{\mathbb{R}^N}\left(\sum_{j=1}^{k} U_{\mu_j}\right)^{p^*+1} \\ &= \frac{1}{2}\sum_{j=1}^{k}\int_{\mathbb{R}^N}(|\nabla U_{\mu_j}|^2 + U_{\mu_j}^2) + \sum_{i,j=1,\ i>j}^{k}\int_{\mathbb{R}^N}(\nabla U_{\mu_j}\nabla U_{\mu_i} + U_{\mu_j}U_{\mu_i}) \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{p^*+1}\int_{\mathbb{R}^N}\left(\sum_{j=1}^k U_{\mu_j}\right)^{p^*+1} \\
= & \ \frac{1}{2}\sum_{j=1}^k\int_{\mathbb{R}^N} w_{\mu_j}^{p^*}U_{\mu_j}dz+\sum_{i,j=1,\ i>j}^k\int_{\mathbb{R}^N} w_{\mu_i}^{p^*}U_{\mu_j}dz \\
& -\frac{1}{p^*+1}\int_{\mathbb{R}^N}\left[\left(\sum_{j=1}^k U_{\mu_j}\right)^{p^*+1}-\sum_{j=1}^k U_{\mu_j}^{p^*+1}-(p^*+1)\sum_{i,j=1,\ i>j}^k U_{\mu_i}^{p^*}U_{\mu_j}\right]dz \\
& -\frac{1}{p^*+1}\sum_{j=1}^k\int_{\mathbb{R}^N} U_{\mu_j}^{p^*+1}dz-\sum_{i,j=1,\ i>j}^k\int_{\mathbb{R}^N} U_{\mu_i}^{p^*}U_{\mu_j}dz \\
= & \ \sum_{j=1}^k\left[\frac{1}{2}\int_{\mathbb{R}^N} w_{\mu_j}^{p^*}U_{\mu_j}dz-\frac{1}{p^*+1}\int_{\mathbb{R}^N} U_{\mu_j}^{p^*+1}dz\right]-\sum_{i,j=1,\ i>j}^k\int_{\mathbb{R}^N}\left(U_{\mu_i}^{p^*}-w_{\mu_i}^{p^*}\right)U_{\mu_j}dz \\
& -\frac{1}{p^*+1}\int_{\mathbb{R}^N}\left[\left(\sum_{j=1}^k U_{\mu_j}\right)^{p^*+1}-\sum_{j=1}^k U_{\mu_j}^{p^*+1}-(p^*+1)\sum_{i,j=1,\ i>j}^k U_{\mu_i}^{p^*}U_{\mu_j}\right]dz \\
:= & \ J_{1,1}+J_{1,2}+J_{1,3}.
\end{aligned} \tag{5.105}
$$

Now we estimate each term $J_{1,i}$, $i=1,2,3$.

$$
\begin{aligned}
J_{1,1} = & \ \sum_{j=1}^k\left[\frac{1}{2}\int_{\mathbb{R}^N} w_{\mu_j}^{p^*+1}dz+\frac{1}{2}\int_{\mathbb{R}^N} w_{\mu_j}^{p^*}R_{\mu_j}dz-\frac{1}{p^*+1}\int_{\mathbb{R}^N} w_{\mu_j}^{p^*+1}dz\right. \\
& \left.-\frac{1}{p^*+1}\int_{\mathbb{R}^N}(U_{\mu_j}^{p^*+1}-w_{\mu_j}^{p^*+1})dz\right] \\
= & \ \sum_{j=1}^k\left[\frac{1}{N}\int_{\mathbb{R}^N} w_{\mu_j}^{p^*+1}dz+\frac{1}{2}\int_{\mathbb{R}^N} w_{\mu_j}^{p^*}R_{\mu_j}dz-\frac{1}{p^*+1}\int_{\mathbb{R}^N}(U_{\mu_j}^{p^*+1}-w_{\mu_j}^{p^*+1})dz\right],
\end{aligned} \tag{5.106}
$$

where

$$
\int_{\mathbb{R}^N} w_{\mu_j}^{p^*+1}=\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}dz, \tag{5.107}
$$

and from Lemma 5.3 and (5.27), if $N\geq 5$, for $j\in\{1,\cdots,k\}$, then we have

$$
\begin{aligned}
& \int_{\mathbb{R}^N} w_{\mu_j}^{p^*}(z)R_{\mu_j}(z)dz\leq\int_{\mathbb{R}^N} w_{\mu_j}^{p^*}(x)|R_{\mu_j}(z)|dz \\
\leq & \ C\mu_j^{-\frac{N-6}{2}}\int_{|z|\leq\frac{\mu_j}{2}}\frac{\mu_j^{\frac{N+2}{2}}}{(\mu_j^2+|z|^2)^{\frac{N+2}{2}}}dz+C\int_{\frac{\mu_j}{2}\leq|z|\leq 1}\frac{\mu_j^{\frac{N+2}{2}}}{(\mu_j^2+|z|^2)^{\frac{N+2}{2}}}\frac{\mu_j^{-\frac{N-6}{2}}}{(1+|\frac{z}{\mu_j}|^2)^{\frac{N-4}{2}}}dz
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_{|z|\ge 1}\frac{\mu_j^{\frac{N+2}{2}}}{(\mu_j^2+|z|^2)^{\frac{N+2}{2}}}\frac{\mu_j^{\frac{N-2}{2}}}{|z|^{N-2}}dz\\
\le\ & C\mu_j^2=o(\varepsilon).
\end{aligned}
$$

If $N=4$, for $1<q<3$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} w_{\mu_j}^{p^*}(z)R_{\mu_j}(z)dz\\
\le\ & C\mu_j^2\log\frac{1}{\mu_j}\int_{|z|\le\frac12}\frac{1}{(1+|z|^2)^3}dz+C\mu_j^2\int_{\frac12\le|z|\le\frac{1}{\mu_j}}\frac{1}{(1+|z|^2)^3}\log\frac{1}{\mu_j|z|}dz\\
&+C\int_{|z|\ge\frac{1}{\mu_j}}\frac{1}{(1+|z|^2)^3}\frac{1}{|z|^2}dz\\
\le\ & C\mu_j^2\log\frac{1}{\mu_j}=o(\varepsilon).
\end{aligned}
$$

If $N=3$, for $3<q<5$, we get

$$
\int_{\mathbb{R}^N} w_{\mu_j}^{p^*}(z)R_{\mu_j}(z)dz\le C\mu_j^{\frac12}\int_{|z|\le1}\frac{\mu_j^{\frac52}}{(\mu_j^2+|z|^2)^{\frac52}}dz+C\int_{|z|\ge1}\frac{\mu_j^{\frac52}}{(\mu_j^2+|z|^2)^{\frac52}}\frac{\mu_j^{\frac12}}{|z|}\le C\mu_j=o(\varepsilon).
$$

As a result, if q satisfies (5.4), then we have

$$
\sum_{j=1}^k\int_{\mathbb{R}^N} w_{\mu_j}^{p^*}R_{\mu_j}dz=o(\varepsilon). \tag{5.108}
$$

Moreover, by Lemma 5.15 and Lemma 5.3, a simple calculation yields that

$$
\int_{\mathbb{R}^N}(U_{\mu_j}^{p^*+1}-w_{\mu_j}^{p^*+1})dz\le C\int_{\mathbb{R}^N}\left[w_{\mu_j}^{p^*}|R_{\mu_j}|+|R_{\mu_j}|^{p^*+1}\right]dz=o(\varepsilon). \tag{5.109}
$$

Thus from (5.106)-(5.109), we find

$$
J_{1,1}=\frac{k}{N}\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}dz+o(\varepsilon). \tag{5.110}
$$

Estimate of $J_{1,2}$. From Lemma 5.3 and (5.27), for $i>j$, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N}(U_{\mu_i}^{p^*}-w_{\mu_i}^{p^*})U_{\mu_j}\ &\le\ \int_{\mathbb{R}^N}\left||w_{\mu_i}+R_{\mu_i}|^{p^*}-w_{\mu_i}^{p^*}\right|(w_{\mu_j}+|R_{\mu_j}|)dz\\
&\le\ C\int_{\mathbb{R}^N}\left(|w_{\mu_i}|^{p^*-1}|R_{\mu_i}|+|R_{\mu_i}|^{p^*}\right)(w_{\mu_j}+|R_{\mu_j}|)dz\\
&\le\ C\int_{\mathbb{R}^N}|w_{\mu_i}|^{p^*-1}w_{\mu_j}|R_{\mu_i}|dz+C\int_{\mathbb{R}^N}|w_{\mu_i}|^{p^*-1}|R_{\mu_i}||R_{\mu_j}|dz
\end{aligned}
$$

$$+C\int_{\mathbb{R}^N} w_{\mu_j}|R_{\mu_i}|^{p^*}dz + C\int_{\mathbb{R}^N}|R_{\mu_i}|^{p^*}|R_{\mu_j}|dz = o(\varepsilon).$$

So

$$J_{1,2} = o(\varepsilon). \tag{5.111}$$

Next we estimate $J_{1,3}$.

Given $\delta > 0$ small but fixed. Let $\mu_1, \cdots, \mu_k$ be given by (5.26), and set $\mu_0 = \frac{\delta^2}{\mu_1}$ and $\mu_{k+1} = 0$. Define the following annulus

$$A_i := B(0, \sqrt{\mu_i\mu_{i-1}})\backslash B(0, \sqrt{\mu_i\mu_{i+1}}), \quad \text{for} \quad i = 1, \cdots, k.$$

We observe that $B(0,\delta) = \bigcup_{i=1}^k A_i$. On each A_i, the leading term in $\sum\limits_{j=1}^k U_{\mu_j}$ is U_{μ_i}.

$$\begin{aligned}
-(p^*+1)J_{1,3} &= \sum_{l=1}^k \int_{A_l}\left[\left(\sum_{j=1}^k U_{\mu_j}\right)^{p^*+1} - \sum_{j=1}^k U_{\mu_j}^{p^*+1} - (p^*+1)\sum_{i,j=1,\ i>j}^k U_{\mu_i}^{p^*}U_{\mu_j}\right]dz \\
&\quad + \int\limits_{\mathbb{R}^N\backslash B(0,\delta)}\left[\left(\sum_{j=1}^k U_{\mu_j}\right)^{p^*+1} - \sum_{j=1}^k U_{\mu_j}^{p^*+1} - (p^*+1)\sum_{i,j=1,\ i>j}^k U_{\mu_i}^{p^*}U_{\mu_j}\right]dz \\
&:= L_1 + L_2,
\end{aligned} \tag{5.112}$$

where

$$\begin{aligned}
L_2 &= \int\limits_{\mathbb{R}^N\backslash B(0,\delta)}\left[\left(\sum_{j=1}^k U_{\mu_j}\right)^{p^*+1} - \sum_{j=1}^k U_{\mu_j}^{p^*+1} - (p^*+1)\sum_{i,j=1,\ i\neq j}^k U_{\mu_i}^{p^*}U_{\mu_j}\right]dz \\
&\quad +(p^*+1)\sum_{i,j=1,\ i<j}^k \int\limits_{\mathbb{R}^N\backslash B(0,\delta)} U_{\mu_i}^{p^*}U_{\mu_j}dz.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{i,j=1,\ i<j}^k \int\limits_{\mathbb{R}^N\backslash B(0,\delta)} U_{\mu_i}^{p^*}U_{\mu_j}dz &\leq \sum_{i,j=1,\ i<j}^k \int\limits_{\mathbb{R}^N\backslash B(0,\delta)} w_{\mu_i}^{p^*}w_{\mu_j}dz \\
&\leq C\sum_{i,j=1,\ i<j}^k\left(\frac{\mu_j}{\mu_i}\right)^{\frac{N-2}{2}}\int\limits_{\mathbb{R}^N\backslash B(0,\frac{\delta}{\mu_i})}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}\frac{1}{((\frac{\mu_j}{\mu_i})^2+|z|^2)^{\frac{N-2}{2}}}dz \\
&= o(\varepsilon),
\end{aligned}$$

and

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N\setminus B(0,\delta)} \left[\left(\sum_{j=1}^{k} U_{\mu_j}\right)^{p^*+1} - \sum_{j=1}^{k} U_{\mu_j}^{p^*+1} - (p^*+1) \sum_{i,j=1,\ i\neq j}^{k} U_{\mu_i}^{p^*} U_{\mu_j}\right] dz \\
\leq\ & C\sum_{j=1}^{k} \int\limits_{\mathbb{R}^N\setminus B(0,\delta)} U_{\mu_j}^{p^*+1} dz + C\sum_{i,j=1}^{k} \int\limits_{\mathbb{R}^N\setminus B(0,\delta)} U_{\mu_i}^{p^*} U_{\mu_j} dz \leq C\mu_1^N = o(\varepsilon).
\end{aligned}$$

Thus

$$L_2 = o(\varepsilon). \tag{5.113}$$

On the other hand, let us estimate each integral on A_l, we have

$$\begin{aligned}
&\int_{A_l} \left[\left(U_{\mu_l} + \sum_{j=1,j\neq l}^{k} U_{\mu_j}\right)^{p^*+1} - U_{\mu_l}^{p^*+1} - \sum_{j=1,j\neq l}^{k} U_{\mu_j}^{p^*+1} - (p^*+1) \sum_{i,j=1,\ i>j}^{k} U_{\mu_i}^{p^*} U_{\mu_j}\right] dz \\
=\ & \int_{A_l} \left[\left(U_{\mu_l} + \sum_{j=1,j\neq l}^{k} U_{\mu_j}\right)^{p^*+1} - U_{\mu_l}^{p^*+1} - (p^*+1) U_{\mu_l}^{p^*} \sum_{j=1,j\neq l}^{k} U_{\mu_j}\right] dz \\
& - \sum_{j=1,j\neq l}^{k} \int_{A_l} U_{\mu_j}^{p^*+1} dz - (p^*+1) \int_{A_l} \left[\sum_{i,j=1,\ i>j}^{k} U_{\mu_i}^{p^*} U_{\mu_j} - U_{\mu_l}^{p^*} \sum_{j=1,j\neq l}^{k} U_{\mu_j}\right] dz \\
:=\ & L_{1,1} + L_{1,2} + L_{1,3}.
\end{aligned} \tag{5.114}$$

We estimate $L_{1,i}$ for $i = 1, 2, 3$ in (5.114). We first estimate $L_{1,2}$.

$$\begin{aligned}
|L_{1,2}| &= \sum_{j=1,j\neq l}^{k} \int_{A_l} U_{\mu_j}^{p^*+1} dz \leq \sum_{j=1,j\neq l}^{k} \int_{A_l} w_{\mu_j}^{p^*+1} dz \\
&= \begin{cases} O\left(\left(\frac{\mu_l}{\mu_j}\right)^{\frac{N}{2}}\right) & \text{if } j \leq l-1 < l; \\ O\left(\left(\frac{\mu_j}{\mu_l}\right)^{\frac{N}{2}}\right) & \text{if } j \geq l+1 > l. \end{cases} \\
&= o(\varepsilon).
\end{aligned} \tag{5.115}$$

Moreover,

$$\begin{aligned}
-\frac{1}{p^*+1} L_{1,3} &= \int_{A_l} \left[\sum_{i,j=1,\ i>j}^{k} U_{\mu_i}^{p^*} U_{\mu_j} - U_{\mu_l}^{p^*} \sum_{j=1,j\neq l}^{k} U_{\mu_j}\right] dz \\
&= -\sum_{j=1,\ j>l}^{k} \int\limits_{A_l} U_{\mu_l}^{p^*} U_{\mu_j} dz + \sum_{i,j=1,\ i\neq l, i>j}^{k} \int\limits_{A_l} U_{\mu_i}^{p^*} U_{\mu_j} dz \\
&= -\sum_{j=1,\ j>l}^{k} \int\limits_{A_l} (U_{\mu_l}^{p^*} - w_{\mu_l}^{p^*}) U_{\mu_j} dz - \sum_{j=1,\ j>l}^{k} \int\limits_{A_l} w_{\mu_l}^{p^*} U_{\mu_j} dz
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{i,j=1,\ i\neq l,i>j}^{k} \int_{A_l} U_{\mu_i}^{p^*} U_{\mu_j} dz \\
= & - \sum_{j=1,\ j>l}^{k} \int_{A_l} (U_{\mu_l}^{p^*} - w_{\mu_l}^{p^*}) w_{\mu_j} dz - \sum_{j=1,\ j>l}^{k} \int_{A_l} w_{\mu_l}^{p^*} w_{\mu_j} dz \\
& - \sum_{j=1,\ j>l}^{k} \int_{A_l} U_{\mu_l}^{p^*} R_{\mu_j} dz + \sum_{i,j=1,\ i\neq l,i>j}^{k} \int_{A_l} U_{\mu_i}^{p^*} U_{\mu_j} dz \\
:= & \ M_1 + M_2 + M_3 + M_4. \qquad (5.116)
\end{aligned}
$$

First, we have

$$
\begin{aligned}
-M_1 &= \sum_{j=1,\ j>l}^{k} \int_{A_l} (U_{\mu_l}^{p^*} - w_{\mu_l}^{p^*}) w_{\mu_j} dz \\
&\leq \sum_{j=1,j>l}^{k} \int_{A_l} \left| |w_{\mu_l} + R_{\mu_l}|^{p^*} - w_{\mu_l}^{p^*} \right| w_{\mu_j} dz \\
&\leq \sum_{j=1,j>l}^{k} \int_{A_l} \left(w_{\mu_l}^{p^*-1} w_{\mu_j} |R_{\mu_l}| + w_{\mu_j} |R_{\mu_l}|^{p^*} \right) dz = o(\varepsilon). \qquad (5.117)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
-M_2 &= \sum_{j=1,\ j>l}^{k} \int_{A_l} w_{\mu_l}^{p^*} w_{\mu_j} dz \\
&= \sum_{j=1,\ j>l}^{k} \alpha_N^{p^*+1} \left(\frac{\mu_j}{\mu_l}\right)^{\frac{N-2}{2}} \int_{\sqrt{\frac{\mu_{l+1}}{\mu_l}} \leq |z| \leq \sqrt{\frac{\mu_{l-1}}{\mu_l}}} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} \frac{1}{((\frac{\mu_j}{\mu_l})^2 + |z|^2)^{\frac{N-2}{2}}} dz \\
&= \sum_{j=1,\ j>l}^{k} \left(\frac{\mu_j}{\mu_l}\right)^{\frac{N-2}{2}} \left[\alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} \frac{1}{|z|^{N-2}} dz + o(1) \right]. \qquad (5.118)
\end{aligned}
$$

Next, it holds

$$
- M_3 = \sum_{j=1,\ j>l}^{k} \int_{A_l} U_{\mu_l}^{p^*} R_{\mu_j} dz \leq \sum_{j=1,\ j>l}^{k} \int_{A_l} w_{\mu_l}^{p^*} |R_{\mu_j}| dz = o(\varepsilon). \qquad (5.119)
$$

Finally, we have

$$
M_4 = \sum_{i,j=1,\ i\neq l,i>j}^{k} \int_{A_l} U_{\mu_i}^{p^*} U_{\mu_j} dz \leq \sum_{i,j=1,\ i\neq l,i>j}^{k} \int_{A_l} w_{\mu_i}^{p^*} w_{\mu_j} dz = o(\varepsilon). \qquad (5.120)
$$

In fact, if $i > j$,

$$\begin{aligned}
\int\limits_{A_l} w_{\mu_i}^{p^*} w_{\mu_j} &= \alpha_N^{p^*+1} \int\limits_{\sqrt{\mu_l\mu_{l+1}}\le|z|\le\sqrt{\mu_l\mu_{l-1}}} \frac{\mu_i^{\frac{N+2}{2}}}{(\mu_i^2+|z|^2)^{\frac{N+2}{2}}} \frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2+|z|^2)^{\frac{N-2}{2}}} dz \\
&= \alpha_N^{p^*+1} \left(\frac{\mu_i}{\mu_j}\right)^{\frac{N-2}{2}} \int\limits_{\frac{\sqrt{\mu_l\mu_{l+1}}}{\mu_i}\le|z|\le\frac{\sqrt{\mu_l\mu_{l-1}}}{\mu_i}} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} \frac{1}{(1+(\frac{\mu_i}{\mu_j})^2|z|^2)^{\frac{N-2}{2}}} dz. \\
&\le C\left(\frac{\mu_i}{\mu_j}\right)^{\frac{N-2}{2}} \begin{cases} \left(\frac{\mu_l\mu_{l-1}}{\mu_i^2}\right)^{\frac{N}{2}} & \text{if } \ i\le l-1<l; \\ \left(\frac{\mu_i^2}{\mu_l\mu_{l-1}} - \frac{\mu_i^2}{\mu_l\mu_{l+1}}\right) & \text{if } \ i\ge l+1>l, \end{cases} \\
&= o(\varepsilon).
\end{aligned} \tag{5.121}$$

Thus, by (5.116)-(5.120) and (5.27), we obtain

$$L_{1,3} = \begin{cases} (p^*+1)\varepsilon\left(\frac{\Lambda_{l+1}}{\Lambda_l}\right)^{\frac{N-2}{2}} \alpha_N^{p^*+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} \frac{1}{|z|^{N-2}} dz + o(\varepsilon) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } l=1,\cdots,k-1; \\ o(\varepsilon) \qquad\qquad\qquad\qquad\qquad\quad \text{if } l=k. \end{cases} \tag{5.122}$$

Now we estimate $L_{1,1}$ in (5.114). By the mean value theorem, for some $t\in[0,1]$, we have

$$\begin{aligned}
L_{1,1} &= \int_{A_l}\left[\left(U_{\mu_l} + \sum_{j=1,j\ne l}^{k} U_{\mu_j}\right)^{p^*+1} - U_{\mu_l}^{p^*+1} - (p^*+1)U_{\mu_l}^{p^*}\sum_{j=1,j\ne l}^{k} U_{\mu_j}\right] \\
&= \frac{p^*(p^*+1)}{2}\int_{A_l}\left(U_{\mu_l} + t\sum_{j=1,j\ne l}^{k} U_{\mu_j}\right)^{p^*-1}\left(\sum_{j=1,j\ne l}^{k} U_{\mu_j}\right)^2 \\
&\le C\sum_{j=1,j\ne l}^{k}\int_{A_l} w_{\mu_l}^{p^*-1}w_{\mu_j}^2 + C\sum_{i,j=1,\ i,j\ne l}^{k}\int_{A_l} w_{\mu_i}^{p^*-1}w_{\mu_j}^2 \\
&\le C\sum_{j=1,j\ne l}^{k}\left(\int_{A_l} w_{\mu_l}^{p^*}w_{\mu_j}\right)^{\frac{p^*-1}{p^*}}\left(\int_{A_l} w_{\mu_j}^{p^*+1}\right)^{\frac{1}{p^*}} \\
&+ C\sum_{i,j=1,\ i,j\ne l}^{k}\left(\int_{A_l} w_{\mu_i}^{p^*+1}\right)^{\frac{p^*-1}{p^*+1}}\left(\int_{A_l} w_{\mu_j}^{p^*+1}\right)^{\frac{2}{p^*+1}} \\
&= o(\varepsilon).
\end{aligned} \tag{5.123}$$

Therefore, by (5.112)-(5.115), (5.122) and (5.123), we have

$$J_{1,3} = -\varepsilon\sum_{l=1}^{k-1}\left(\frac{\Lambda_{l+1}}{\Lambda_l}\right)^{\frac{N-2}{2}} \alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}\frac{1}{|z|^{N-2}}dz + o(\varepsilon). \tag{5.124}$$

From (5.105), (5.110), (5.111) and (5.124), we get

$$\begin{aligned} J_1 &= \frac{k}{N}\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}dz \\ &-\varepsilon\sum_{l=1}^{k-1}\left(\frac{\Lambda_{l+1}}{\Lambda_l}\right)^{\frac{N-2}{2}}\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}\frac{1}{|z|^{N-2}}dz+o(\varepsilon). \end{aligned} \tag{5.125}$$

Step 2. *We estimate J_2.*

The Taylor expansion gives that

$$\begin{aligned} J_2 &= \frac{1}{p^*+1}\int_{\mathbb{R}^N}U^{p^*+1}-\frac{1}{p^*+1+\varepsilon}\int_{\mathbb{R}^N}U^{p^*+1+\varepsilon} \\ &= \frac{1}{p^*+1}\int_{\mathbb{R}^N}U^{p^*+1}-\left(\frac{1}{p^*+1}-\frac{1}{(p^*+1)^2}\varepsilon+o(\varepsilon)\right)\int_{\mathbb{R}^N}U^{p^*+1}(1+\varepsilon\log U+o(\varepsilon)) \\ &= \varepsilon\left[\frac{1}{(p^*+1)^2}\int_{\mathbb{R}^N}U^{p^*+1}-\frac{1}{p^*+1}\int_{\mathbb{R}^N}U^{p^*+1}\log U\right]+o(\varepsilon), \end{aligned} \tag{5.126}$$

where

$$\int_{\mathbb{R}^N}U^{p^*+1} = k\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}+o(\varepsilon), \tag{5.127}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N}U^{p^*+1}\log U &= \sum_{l=1}^{k}\int_{A_l}\left(\sum_{j=1}^{k}w_{\mu_j}\right)^{p^*+1}\log\left(\sum_{j=1}^{k}w_{\mu_j}\right) \\ &+\sum_{l=1}^{k}\int_{A_l}\left[\left(\sum_{j=1}^{k}U_{\mu_j}\right)^{p^*+1}\log\left(\sum_{j=1}^{k}U_{\mu_j}\right)\right. \\ &\qquad\left.-\left(\sum_{j=1}^{k}w_{\mu_j}\right)^{p^*+1}\log\left(\sum_{j=1}^{k}w_{\mu_j}\right)\right] \\ &+\int_{\mathbb{R}^N\setminus B(0,\delta)}U^{p^*+1}\log U := D_1+D_2+D_3. \end{aligned} \tag{5.128}$$

Since

$$\begin{aligned} D_1 &= \sum_{l=1}^{k}\int_{A_l}\left(w_{\mu_l}+\sum_{j=1,j\neq l}^{k}w_{\mu_j}\right)^{p^*+1}\log\left(w_{\mu_l}+\sum_{j=1,j\neq l}^{k}w_{\mu_j}\right) \\ &= \alpha_N^{p^*+1}\sum_{l=1}^{k}\mu_l^{-N}\int_{A_l}\left(\frac{1}{(1+|\frac{z}{\mu_l}|^2)^{\frac{N-2}{2}}}+\mu_l^{\frac{N-2}{2}}\sum_{j=1,j\neq l}^{k}\frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2+|z|^2)^{\frac{N-2}{2}}}\right)^{p^*+1} \end{aligned}$$

$$\times \log\left[\alpha_N \mu_l^{-\frac{N-2}{2}}\left(\frac{1}{(1+|\frac{z}{\mu_l}|^2)^{\frac{N-2}{2}}}+\mu_l^{\frac{N-2}{2}}\sum_{j=1,j\neq l}^{k}\frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2+|z|^2)^{\frac{N-2}{2}}}\right)\right]dz$$

$$= \alpha_N^{p^*+1}\sum_{l=1}^{k}\int_{\sqrt{\frac{\mu_{l+1}}{\mu_l}}\le|z|\le\sqrt{\frac{\mu_{l-1}}{\mu_l}}}\left(\frac{1}{(1+|z|^2)^{\frac{N-2}{2}}}+\mu_l^{\frac{N-2}{2}}\sum_{j=1,j\neq l}^{k}\frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2+\mu_l^2|z|^2)^{\frac{N-2}{2}}}\right)^{p^*+1}$$

$$\times \log\left[\alpha_N \mu_l^{-\frac{N-2}{2}}\left(\frac{1}{(1+|z|^2)^{\frac{N-2}{2}}}+\mu_l^{\frac{N-2}{2}}\sum_{j=1,j\neq l}^{k}\frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2+\mu_l^2|z|^2)^{\frac{N-2}{2}}}\right)\right]dz$$

$$= -\frac{N-2}{2}\left(\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}dz\right)\sum_{l=1}^{k}\log\mu_l$$

$$+k\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}\log\frac{\alpha_N}{(1+|z|^2)^{\frac{N-2}{2}}}dz+O(\varepsilon|\log\varepsilon|). \tag{5.129}$$

By the mean value theorem, we have

$$D_2 \le \sum_{l=1}^{k}\int_{A_l}\left(\sum_{j=1}^{k}w_{\mu_j}\right)^{p^*}\left[(p^*+1)\log\left(\sum_{j=1}^{k}w_{\mu_j}\right)+1\right]\sum_{j=1}^{k}R_{\mu_j}dz = O(\varepsilon|\log\varepsilon|). \tag{5.130}$$

Moreover,

$$D_3 \le C\sum_{j=1}^{k}\int_{\mathbb{R}^N\setminus B(0,\delta)}w_{\mu_j}^{p^*+1}\log(w_{\mu_j}+\sum_{i=1,i\neq j}^{k}w_{\mu_i})dz = O(\varepsilon|\log\varepsilon|). \tag{5.131}$$

Thus from (5.126)-(5.131), we get

$$J_2 = \varepsilon\frac{k}{(p^*+1)^2}\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}dz$$

$$-\varepsilon\frac{k}{p^*+1}\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}\log\frac{\alpha_N}{(1+|z|^2)^{\frac{N-2}{2}}}dz$$

$$+\varepsilon\frac{(N-2)^2}{4N}\left(\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}dz\right)\sum_{i=1}^{k}\log\Lambda_i$$

$$+\frac{(N-2)^2}{4N}\left(\alpha_N^{p^*+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}dz\right)$$

$$\times\sum_{i=1}^{k}\left(\frac{2(i-1)}{N-2}+\frac{2}{N+2-(N-2)q}\right)\varepsilon\log\varepsilon+o(\varepsilon). \tag{5.132}$$

Step 3. *Let us estimate J_3.*

$$-(q+1)J_3 = \lambda\sum_{l=1}^{k}\int_{A_l}\left[\left(U_{\mu_l}+\sum_{j=1,j\neq l}^{k}U_{\mu_j}\right)^{q+1}-U_{\mu_l}^{q+1}-(q+1)U_{\mu_l}^{q}\sum_{j=1,j\neq l}^{k}U_{\mu_j}\right]$$

$$\begin{aligned}
&+\lambda\sum_{l=1}^{k}\int_{A_l}U_{\mu_l}^{q+1}+\lambda(q+1)\sum_{l=1}^{k}\int_{A_l}\sum_{j=1,j\neq l}^{k}U_{\mu_l}^{q}U_{\mu_j}+\lambda\int_{\mathbb{R}^N\setminus B(0,\delta)}\left(\sum_{j=1}^{k}U_{\mu_j}\right)^{q+1}\\
:=\ & J_{3,1}+J_{3,2}+J_{3,3}+J_{3,4}.
\end{aligned}$$

By the mean value theorem, for some $t\in[0,1]$, we have

$$\begin{aligned}
J_{3,1} &= \lambda\frac{q(q+1)}{2}\int_{A_l}\left(U_{\mu_l}+t\sum_{j=1,j\neq l}^{k}U_{\mu_j}\right)^{q-1}\left(\sum_{j=1,j\neq l}^{k}U_{\mu_j}\right)^{2}\\
&\leq C\lambda\sum_{j=1,j\neq l}^{k}\int_{A_l}w_{\mu_l}^{q-1}w_{\mu_j}^{2}+C\lambda\sum_{i,j=1,\ i,j\neq l}^{k}\int_{A_l}w_{\mu_i}^{q-1}w_{\mu_j}^{2}.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{j=1,j\neq l}^{k}\int_{A_l}w_{\mu_l}^{q-1}w_{\mu_j}^{2} &= \sum_{j=1,j\neq l}^{k}\int_{A_l}(w_{\mu_l}^{q-1}w_{\mu_j}^{\frac{q-1}{q}})w_{\mu_j}^{\frac{q+1}{q}}\\
&\leq \sum_{j=1,j\neq l}^{k}(\int_{A_l}(w_{\mu_l}^{q}w_{\mu_j})^{\frac{q-1}{q}}(\int_{A_l}w_{\mu_j}^{q+1})^{\frac{1}{q}}, \qquad (5.133)
\end{aligned}$$

and

$$\sum_{i,j=1,\ i,j\neq l}^{k}\int_{A_l}w_{\mu_i}^{q-1}w_{\mu_j}^{2} \leq \sum_{i,j=1,\ i,j\neq l}^{k}\left(\int_{A_l}w_{\mu_i}^{q+1}\right)^{\frac{q-1}{q+1}}\left(\int_{A_l}w_{\mu_j}^{q+1}\right)^{\frac{2}{q+1}}. \qquad (5.134)$$

If $j>l$, then

$$\begin{aligned}
&\int_{A_l}w_{\mu_l}^{q}w_{\mu_j}dz=\alpha_N^{q+1}\int_{\sqrt{\mu_l\mu_{l+1}}\leq|z|\leq\sqrt{\mu_l\mu_{l-1}}}\frac{\mu_l^{\frac{N-2}{2}q}}{(\mu_l^2+|z|^2)^{\frac{N-2}{2}q}}\frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2+|z|^2)^{\frac{N-2}{2}}}dz\\
=\ &\left(\frac{\mu_j}{\mu_l}\right)^{\frac{N-2}{2}}\mu_l^{-\frac{N-2}{2}q+\frac{N+2}{2}}\left[\alpha_N^{q+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N-2}{2}q}}\frac{1}{|z|^{N-2}}dz+o(1)\right]. \qquad (5.135)
\end{aligned}$$

If $l<j$, then

$$\begin{aligned}
&\int_{A_l}w_{\mu_l}^{q}w_{\mu_j}dx=\alpha_N^{q+1}\int_{\sqrt{\mu_l\mu_{l+1}}\leq|z|\leq\sqrt{\mu_l\mu_{l-1}}}\frac{\mu_l^{\frac{N-2}{2}q}}{(\mu_l^2+|z|^2)^{\frac{N-2}{2}q}}\frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2+|z|^2)^{\frac{N-2}{2}}}dz\\
=\ &\left(\frac{\mu_l}{\mu_j}\right)^{\frac{N-2}{2}}\mu_l^{-\frac{N-2}{2}q+\frac{N+2}{2}}\alpha_N^{q+1}\int_{\sqrt{\frac{\mu_{l+1}}{\mu_l}}\leq|z|\leq\sqrt{\frac{\mu_{l-1}}{\mu_l}}}\frac{1}{(1+|z|^2)^{\frac{N-2}{2}q}}\frac{1}{(1+(\frac{\mu_l}{\mu_j})^2|z|^2)^{\frac{N-2}{2}}}dz
\end{aligned}$$

$$\leq \left(\frac{\mu_l}{\mu_j}\right)^{\frac{N-2}{2}} \mu_l^{-\frac{N-2}{2}q+\frac{N+2}{2}} \alpha_N^{q+1} \int\limits_{\sqrt{\frac{\mu_{l+1}}{\mu_l}} \leq |z| \leq \sqrt{\frac{\mu_{l-1}}{\mu_l}}} \frac{1}{(1+|z|^2)^{\frac{N-2}{2}q}} dz. \tag{5.136}$$

For $i \neq l$, we have

$$\int\limits_{A_l} w_{\mu_i}^{q+1} \leq C\mu_i^{-\frac{N-2}{2}q+\frac{N+2}{2}} \begin{cases} (\frac{\mu_l}{\mu_i})^{\frac{N}{2}} & \text{if } \ i \leq l-1 < l; \\ (\frac{\mu_i^2}{\mu_l \mu_{l-1}})^{\frac{N-2}{2}q-1} & \text{if } \ i \geq l+1 > l. \end{cases} \tag{5.137}$$

From (5.133)-(5.137), (5.4) and (5.27), we get $J_{3,1} = o(\varepsilon)$.

Moreover,

$$J_{3,2} = \lambda \sum_{l=1}^{k} \int_{A_l} w_{\mu_l}^{q+1} + \lambda \sum_{l=1}^{k} \int_{A_l} (U_{\mu_l}^{q+1} - w_{\mu_l}^{q+1}).$$

Since by (5.27), we have

$$\begin{aligned} \sum_{l=1}^{k} \int_{A_l} w_{\mu_l}^{q+1} &= \sum_{l=1}^{k} \mu_l^{N-\frac{(N-2)(q+1)}{2}} \int\limits_{\sqrt{\frac{\mu_{l+1}}{\mu_l}} \leq |z| \leq \sqrt{\frac{\mu_{l-1}}{\mu_l}}} \frac{1}{(1+|z|^2)^{\frac{(N-2)(q+1)}{2}}} dz \\ &= \sum_{l=1}^{k} \mu_l^{\frac{N+2-(N-2)q}{2}} \left(\int\limits_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{(N-2)(q+1)}{2}}} dz + o(1) \right) \\ &= \mu_1^{\frac{N+2-(N-2)q}{2}} \left(\int\limits_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{(N-2)(q+1)}{2}}} dz + o(1) \right) \\ &= \varepsilon \Lambda_1^{\frac{N+2-(N-2)q}{2}} \int\limits_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{(N-2)(q+1)}{2}}} dz + o(\varepsilon), \end{aligned}$$

and From Lemma 5.15 and Lemma 5.3, we can easily check that

$$\int_{A_l} |U_{\mu_l}^{q+1} - w_{\mu_l}^{q+1}| \leq \int_{A_l} \left(w_{\mu_l}^q |R_{\mu_l}| + |R_{\mu_l}|^{q+1} \right) dz = o(\varepsilon).$$

So we find

$$J_{3,2} = \varepsilon \Lambda_1^{\frac{N+2-(N-2)q}{2}} \lambda \int\limits_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{(N-2)(q+1)}{2}}} dz + o(\varepsilon). \tag{5.138}$$

From (5.135) and (5.136), we have

$$J_{3,3} \leq C\lambda \sum_{l=1}^{k} \int_{A_l} \sum_{j=1, j\neq l}^{k} U_{\mu_l}^q U_{\mu_j} \leq C\lambda \sum_{l=1}^{k} \int_{A_l} \sum_{j=1, j\neq l}^{k} w_{\mu_l}^q w_{\mu_j} = o(\varepsilon).$$

Finally,

$$\begin{aligned} J_{3,4} &= \lambda \int_{\mathbb{R}^N \setminus B(0,\delta)} \left(\sum_{j=1}^{k} U_{\mu_j} \right)^{q+1} \leq C \sum_{j=1}^{k} \int_{\mathbb{R}^N \setminus B(0,\delta)} w_{\mu_j}^{q+1} dz \\ &\leq C \sum_{j=1}^{k} \mu_j^{\frac{N+2-(N-2)q}{2}} \int_{\frac{\delta}{\mu_j}}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{\frac{(N-2)(q+1)}{2}}} dr = o(\varepsilon). \end{aligned}$$

Thus we get

$$J_3 = -\varepsilon \Lambda_1^{\frac{N+2-(N-2)q}{2}} \frac{\lambda}{q+1} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{(N-2)(q+1)}{2}}} dz + o(\varepsilon). \tag{5.139}$$

From (5.104), (5.125), (5.132) and (5.139), we obtain (5.81) holds. □

References

[1] C. O. Alves, D. C. de Morais Filho & M. A. S. Souto, *Radially Symmetric Solutions for a Class of Critical Exponent Elliptic Problems in* $\mathbb{R}^N$, Electron. J. Differential Equations 1996, no. 07, pp. 1-12.

[2] A. Ambrosetti, H. Brézis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. 122 (1994), 519-543.

[3] A. Ambrosetti, A. Malchiodi, Nonlinear analysis and semilinear elliptic problems, Cambridge studies in advanced mathematics 104.

[4] G. Arioli, F. Gazzola, H-C Grunau, *Entire solutions for a semilinear fourth order elliptic problem with exponential nonlinearity*, J. Differential Equations 230(2006), no. 2, 743-770.

[5] G. Arioli, F. Gazzola, H-C Grunau, E. Mitidieri, *A semilinear fourth order elliptic problem with exponential nonlinearity*, SIAM J. Math. Anal. 36(2005), no. 4, 1226-1258.

[6] F. V. Atkinson, L. A. Peletier, *Emden-Fowler equations involving critical exponents*, Nonlinear Anal. 10 (1986), no. 8, 755-776.

[7] A. Bahri, J. M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. 41 (1988), no.3, 253-294.

[8] R. Bamón, I. Flores, M. del Pino, *Ground states of semilinear elliptic equations: A geometric approach*, Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (5) (2000), 551-581.

[9] G. R. Belickiĭ, N. Forms, Invariants and Local Mappings, Naukova Dumka, Kiev, 1979, P.176.

[10] H. Beresticki, P. L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. 82 (1983), no. 4, 313-345.

[11] E. Berchio, F. Gazzola, D. Pierotti, Gelfand type elliptic problems under Steklov boundary conditions, Ann. Inst. Henri Poincaré, Analyse non Linéaire, vol. 27 (2010), 315-335.

[12] G. Bratu, *Sur les équations intégrales non linéaires*, Bull. Soc. Math. France, 42(1914), 113-142.

[13] H. Brézis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, *Blow-up for $u_t - \triangle u = g(u)$ revisited*, Advances in Diff. Eq. 1 (1996), pp. 73-90.

[14] H. Brézis, L. Nirenberg, *Positive solutions of non-linear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. 36 (4) (1983), 437-477.

[15] H. Brézis, J. Luis Vázquez, *Blow-up solutions of some nonlinear elliptic problems*, Revista Matemática de la Universidad Complutense de Madrid, 10(1997), no. 2, 444-469.

[16] C. Budd, J. Norbury, *Semilinear elliptic equations and supercritical growth*, J. Differential Equations 68 (1987) no. 2, 169-197.

[17] J. Busca, R. Manásevich, *A Liouville type theorem for Lane-Emden systems*, Indiana Univ. Math. J. 51(2002), 37-51.

[18] J. Busca, B. Sirakov, *Harnack type estimates for nonlinear elliptic systems and applications*, Ann. Inst. H. Poincaré Anal. Non Linaire 21 (2004), no. 5, 543-590.

[19] L. Caffarelli, B. Gidas, and J. Spruck, *Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. 42 (1989), 271-297.

[20] P. Caldiroli, R. Musina, *Rellich inequalities with weights*, Calc. Var. Partial Differential Equations 45 (2012), no. 1-2, 147-164.

[21] J. Campos, *Bubble-tower phenomena in a semilinear elliptic equation with mixed Sobolev growth*, Nonlinear Anal. 68 (2008), no. 5, 1382-1397.

[22] C.V. Coffman, *On the positive solutions of boundary-value problems for a class of nonlinear differential equations.* J. Differential Equations 3 (1967), 92-111.

[23] S. Chandrasekhar, An Introduction to the Study of Stellar Structure, Dover Publications Inc., New York, NY, 1957.

[24] K. C. Chang, Methods in Nonlinear Analysis. Springer, 2005.

[25] W. Chen and C. Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. 63 (1991), 615-622.

[26] W. Chen and C. Li, *An integral system and the Lane-Emden conjecture*, Discrete Contin. Dyn. Syst. 24(4), 1167-1184, 2009.

[27] E. A. Coddington, N. Levinson, Theory of ordinary differential equations, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[28] D. S. Cohen and H. B. Keller, *Some positone problems suggested by nonlinear heat generation*, J. Math. Mech. 16 (1967), 1361-1376.

[29] C. Cowan, *Regularity of stable solutions of a Lane-Emden type system*, http://arxiv.org/abs/1206.4273 (19 june 2012).

[30] C. Cowan, *Liouville theorems for stable Lane-Emden systems and biharmonic problems*, Nonlinearity, 26(8)(2013), 2357-2371.

[31] M. G. Crandall, P. H. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rational Mech. Anal. 58 (1975), no. 3, 207-218.

[32] M. G. Crandall, P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, Journal of Functional Analysis 8(1971), 321-340.

[33] E. N. Dancer, A. Farina, *On the classification of solutions of $-\Delta u = e^u$ on $\mathbb{R}^N$: stability outside a compact set and applications*, Proc. Amer. Math. Soc. 137 (2009), no. 4, 1333-1338.

[34] J. Dávila, Singular solutions of semi-linear elliptic problems, Handbook of differential equations: stationary partial differential equations. Vol. VI, 83176, Handb. Differ. Equ. Elsevier/North-Holland, Amsterdam, 2008.

[35] J. Dávila, M. del Pino, I. Guerra, *Non-uniqueness of positive ground states of nonlinear schrödinger equations*, Proc. London Math. Soc. (3) 106 (2013), 318-344.

[36] J. Dávila, L. Dupaigne, *Comparison results for PDEs with a singular potential*, Proc. Roy. Soc. Edinburgh, Sect. A 133 (2003), no. 1, 61-83.

[37] J. Dávila, L. Dupaigne, A. Farina, *Partial regularity of finite Morse index solutions to the Lane-Emden equation*, J. Funct. Anal. 261 (2011), no. 1, 218-232.

[38] J. Dávila, L. Dupaigne, I, Guerra, and M. Montenegro, *Stable solutions for the bilaplacian with exponential nonlinearity*, SIAM J. Math. Anal. 39 (2007), no. 2, 565-592.

[39] J. Dávila, L. Dupaigne and M. Montenegro, *The extremal solution of a boundary reaction problem*, Commun. Pure Appl. Anal. 7 (2008), no. 4, 795-817.

[40] J. Dávila, I. Flores, I. Guerra, *Multiplicity of solutions for a fourth order problem with exponential nonlinearity*, J. Differential Equations 247 (2009), no. 11, 3136–3162.

[41] J. Dávila, I. Flores, I. Guerra, *Multiplicity of solutions for a fourth order equation with power-type nonlinearity*, Math. Ann. 348 (2010), no. 1, 143-193.

[42] D. G. de Figueiredo, P.L. Felmer, *A Liouville type theorem for elliptic systems*, Ann. Scuola Norm. Sup. Pisa, 21 (1994), 387-397.

[43] M. del Pino, J. Dolbeault, M. Musso, *"Bubble-tower" radial solutions in the slightly supercritical Brezis-Nirenberg problem*, J. Differential Equations 193 (2003), no. 2, 280-306.

[44] M. del Pino, J. Dolbeault, M. Musso, *The Brezis-Nirenberg problem near criticality in dimension 3*, J. Math. Pures Appl. (9) 83 (2004), no. 12, 1405-1456.

[45] M. del Pino, P. Felmer, M. Musso, *Two-bubble solutions in the super-critical Bahri-Coron's problem*, Calc. Var. Partial Differential Equations 20 (2004), no. 2, 231-233.

[46] M. del Pino, I. Guerra, *Ground states of a prescribed mean curvature equation*, J. Differential Equations 241 (2007), no. 1, 112-129.

[47] M. del Pino, M. Musso, Bubbling in Nonlinear Elliptic Problems Near Criticality Handbook of Differential Equations, Stationary Partial Differential Equations, Vol. 3 (2006), 215-316.

[48] M. del Pino, M. Musso, A. Pistoia, *Super-critical boundary bubbling in a semilinear Neumann problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005), no. 1, 45-82.

[49] J. Dolbeault, I. Flores, *Geometry of phase space and solutions of semilinear elliptic equations in a ball*, Trans. Amer. Math. Soc. 359 (9) (2007), 4073-4087.

[50] S. Dumont, L. Dupaigne, O. Goubet, and V. Rădulescu, *Back to the Keller-Osserman condition for boundary blow-up solutions*, Adv. Nonlinear Stud. 7 (2007), no. 2, 271-298.

[51] L. Dupaigne, Stable Solutions of Elliptic Partial Differential Equations. CHAPMAN HALL/CRC Monographs and Surveys in Pure and Applied Mathematics, 2011.

[52] L. Dupaigne, M. Ghergu, O. Goubet, and G. Warnault, *The Gelfand problem for the biharmonic operator*, Archive for Rational Mechanics and Analysis June 2013, Volume 208, Issue 3, pp 725-752.

[53] L. Dupaigne and G. Nedev, *Semilinear elliptic PDEs with a singular potential*, Advances in Differential Equations 7(2002), no. 8, 973-1002.

[54] F. Faà di Bruno, Note Sur une nouvelle formule de calcul differentiel, The Quarterly Journal of Pure and Applied Mathematics 1(1857), 359-360.

[55] A. Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of* $\mathbb{R}^N$, J. Math. Pures Appl. (9) 87 (2007), no. 5, 537-561.

[56] A. Ferrero, F. Gazzola, *On subcriticality assumptions for the existence of ground states of quasilinear elliptic equations*, Adv. Differential Equations 8 (2003), 1081-1106.

[57] P. L. Felmer, A. Quaas, M. Tang, J. Yu, *Monotonicity properties for ground states of the scalar field equation.* Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008), no. 1, 105-119.

[58] A. Floer, A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with bounded potential*, J. Funct. Anal. 69(3)(1986), 397-408.

[59] I. Flores, *A resonance phenomenon for ground states of an elliptic equation of Emden-Fowler type*, J. Differential Equations 198 (2004), 1-15.

[60] I. Flores, *Singular solutions of the Brezis-Nirenberg problem in a ball*, Comm. Pure Appl. Anal. 8 (2) (2009), 673-682.

[61] R. H. Fower, *Further studies of Emden's and similar differential equations*, Quart. J. Math, Oxford Series, 2(1931), 259-288.

[62] D. A. Frank-Kamenetshii, Diffusion and Heat Exchange in Chemical Kinetics, Princeton Univ. Press, Princeton, NJ, 1955.

[63] F. Gazzola and H-C. Grunau, *Radial entire solutions for supercritical biharmonic equations*, Math. Ann. 334 (2006), no. 4, 905-936.

[64] Y. Ge, R. Jing, F. Pacard, *Bubble towers for supercritical semilinear elliptic equations*, J. Funct. Anal. 221 (2005), no. 2, 251-302.

[65] Y. Ge, M. Musso, A. Pistoia, *Sign changing tower of bubbles for an elliptic problem at the critical exponent in pierced non-symmetric domains*, Comm. Partial Differential Equations 35 (2010), no. 8, 1419-1457.

[66] I. M. Gelfand, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Transl. 29 (2) (1963), 295-381.

[67] B. Gidas, W. M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 68(1979), 209-243.

[68] B. Gidas, W. M. Ni, and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^N$, Mathematical analysis and applications, Part A, Academic Press, 1981, 7, 369-402.

[69] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. 34(4)(1981), 525-598.

[70] D. Gilbarg, N. S. Trudinger, Elliptic Partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[71] Z. Guo, J. Wei, *Global solution branch and Morse Index estimates of a semilinear elliptic equation with super-critical Exponential*, Transactions of the American Mathematical Society 363 (2011), no. 9, 4777-4799.

[72] H. Hajlaoui, A. Harrabi, and D. Ye, *On stable solutions of biharmonic problem with polynomial growth*, arXiv:1211.2223.

[73] D. D. Joseph, T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Ration. Mech. Anal. 49 (1972), 241-269.

[74] P. Karageorgis, *Stability and intersection properties of solutions to the nonlinear biharmonic equation*, Nonlinearity 22 (2009), no. 7, 1653-1661.

[75] H. B. Keller and J. Keener, *Positive solutions of convex nonlinear eigenvalue problems*, J. Differ. Eq. 16 (1974), 103-125.

[76] M.K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbf{R^n}$. Arch. Rational Mech. Anal. 105 (1989), no. 3, 243-266.

[77] Y. Y. Li, *On a singularly perturbed equation with Neumann boundary condition*. Comm. Partial Differential Equations 23(1998), 487-545.

[78] C. S. Lin, W. M. Ni, *A counterexample to the nodal line conjecture and a related semilinear equation*, Proc. Amer. Math. Soc. 102(2)(1988), 271-277.

[79] J. Liouville, *Sur l'equation aux différences partielles* $\frac{d^2 \log \lambda}{dudv} \pm \frac{\lambda}{2a^2} = 0$, J. Math. Pure Appl 36(1853), 71-72.

[80] K. McLeod, J. Serrin, *Uniqueness of positive radial solutions of* $\Delta u + f(u) = 0$ *in* $\mathbf{R^n}$. Arch. Rational Mech. Anal. 99 (1987), no. 2, 115-145.

[81] F. Merle, L. A. Peletier, *Positive solutions of elliptic equations involving supercritical growth*, Proc. Roy. Soc. Edinburgh Sect. A 118 (1991), no. 1-2, 49-62.

[82] F. Merle, L. A. Peletier, J. Serrin, *A bifurcation problem at a singular limit*, Indiana Univ. Math. J. 43 (1994), no. 2, 585-609.

[83] A. M. Micheletti, M. Musso, A. Pistoia, *Super-position of spikes for a slightly supercritical elliptic equation in* $\mathbb{R}^N$, Discrete Contin. Dyn. Syst. 12 (2005), no. 4, 747-760.

[84] F. Mignot, J. P. Puel, *Sur une classe de problemes non lineaires aveac nonlinearite positive, croissante, convexe*, Comm. P. D. E. 5(1980), 791-836.

[85] E. Mitidieri, *A Rellich type identity and applications*, Comm. Partial Differential Equations 18 (1993), no. 1-2, 125-151.

[86] E. Mitidieri, *Nonexistence of positive solutions of semilinear elliptic systems in* $\mathbb{R}^N$, Differential Integral Equations 9 (1996), no. 3, 465-479.

[87] E. Mitidieri, S. I. Pokhozhaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. Tr. Mat. Inst. Steklova 234 (2001), 1-384; translation in Proc. Steklov Inst. Math., 234(3) (2001), 1-362.

[88] M. Montenegro, *Minimal solutions for a class of elliptic systems*, Bull. London Math. Soc. 37 (2005), no. 3, 405-416.

[89] M. Musso, A. Pistoia, *Sign changing solutions to a nonlinear elliptic problem involving the critical Sobolev exponent in pierced domains*, J. Math. Pures Appl. (9) 86 (2006), no. 6, 510-528.

[90] M. Musso, A. Pistoia, *Multispike solutions for a nonlinear elliptic problem involving the critical Sobolev exponent*, Indiana University Math. Journal, 51 (2002), no. 3, 541-579.

[91] M. Musso, A. Pistoia, *Tower of bubbles for almost critical problems in general domains*, J. Math. Pures Appl. (9) 93 (2010), no. 1, 1-40.

[92] K. Nagasaki, T. Suzuki, Spectral and related properties about the Emden-Fowler equation $-\Delta u = \lambda e^u$ on circular domains. Math. Ann. 299(1) (1994), 1-15

[93] W. M. Ni, *Uniqueness of solutions of nonlinear Dirichlet problems.* J. Differential Equations 50 (1983), no. 2, 289-304.

[94] W. M. Ni, R. D. Nussbaum, *Uniqueness and nonuniqueness for positive radial solutions of* $\Delta u + f(u, r) = 0$. Comm. Pure Appl. Math. 38 (1985), no. 1, 67-108.

[95] J. Palis, W. de Melo, Geometric theory of dynamical systems: an introduction, Springer-Verlag, New York, Heidelberg, Berlin (1982).

[96] L.A. Peletier, J. Serrin, *Uniqueness of positive solutions of semilinear equations in* $\mathbf{R^n}$. Arch. Rational Mech. Anal. 81 (1983), no. 2, 181-197.

[97] L.A. Peletier, J. Serrin, *Uniqueness of nonnegative solutions of semilinear equations in* $\mathbf{R^n}$. J. Differential Equations 61 (1986), no. 3, 380-397.

[98] A. Pistoia, T. Weth, *Sign changing bubble tower solutions in a slightly subcritical semilinear Dirichlet problem*, Ann. Inst. H. Poincaré Anal. Non Linaire 24 (2007), no. 2, 325-340.

[99] S. I. Pohožaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR 165 (1965), 36-39.

[100] P. Poláčik, P. Quittner, and P. Souplet, *Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and system- s*, Duke Math. J. 139 (2007), no. 3, 555-579.

[101] P. Quittner, P. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states. Birkhuser Advanced Texts: Basler Lehrbcher. [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser Verlag, Basel, 2007. xii+584 pp. ISBN: 978-3-7643-8441-8.

[102] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, CBMS Regional Conference Series in Mathematics, vol. 65, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.

[103] O. Rey, *Bifurcation from infinity in a nonlinear elliptic equation involving the limiting Sobolev exponent*, Duke Math. J. 60 (1990), no. 3, 815-861.

[104] O. Rey, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal. 89 (1990), 1-52.

[105] O. Rey, J. Wei, *Blowing up solutions for an elliptic Neumann problem with sub- or supercritical nonlinearity. I.* $N = 3$, J. Funct. Anal. 212 (2004), no. 2, 472-499.

[106] J. Serrin, H. Zou, *Non-existence of positive solutions of Lane-Emden system*, Differential Integral Equations 9(1996), 635-653.

[107] J. Serrin, H. Zou, *Existence of positive solutions of Lane-Emden system*, Atti Semin. Mat. Fis. Univ. Modena. Suppl. 46(1998), 369-380.

[108] J. Serrin, H. Zou, *The existence of positive entire solutions of elliptic Hamiltonian system*, Comm. Part. Diff. Eq. 23(1998), 577-599.

[109] J. Serrin and H. Zou, *Non-existence of positive solutions of semilinear ellip- tic systems*, A tribute to Ilya Bakelman (College Station, TX, 1993), Discourses Math. Appl. vol. 3, Texas A & M Univ., College Station, TX, 1994, pp. 55-68.

[110] P. Souplet, *The proof of the Lane-Emden conjecture in four space dimensions*, Adv. Math. 221 (2009), no. 5, 1409-1427.

[111] G. Sweers, *Strong positivity in* $C(\bar{\Omega})$ *for elliptic systems*, Math. Z. 209 (1992), no. 2, 251-271.

[112] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) 110 (1976), 353-372.

[113] T. Tao, M. Visan, X. Zhang, *The nonlinear Schrödinger equation with combined power-type nonlinearities.* Comm. Partial Differential Equations 32 (2007), no. 7-9, 1281-1343.

[114] R. C. A. M. Van der Vorst, *Variational identities and applications to differential systems*, Arch. Rational Mech. Anal. 116 (1992), no. 4, 375-398.

[115] J. Wei, X. Xu, and W. Yang, *On the classification of stable solutions to biharmonic problems in large dimensions*, Pacific Journal of Mathematics 263(2013), no. 2, 495-512.

[116] J. Wei, S. Yan, *Infinitely many solutions for the prescribed scalar curvature problem on* S^N, J. Funct. Anal. 258 (2010), no. 9, 3048-3081.

[117] J. Wei and D. Ye, *Liouville Theorems for finite Morse index solutions of Biharmonic problem*, Mathematische Annalen 356(2013), no. 4, 1599-1612.

[118] M. Willem, Minimax Theorems, Birkhauser, Boston, Basel, Berlin, 1996.

[119] H. Zou, *Symmetry of ground states of semilinear elliptic equations with mixed Sobolev growth*, Indiana Univ. Math. J. 45(1996), 221-240.