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FRACTIONAL REACTION-DIFFUSION PROBLEMS

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Abstract

This thesis deals with two different problems: in the first one, we study the large-time behavior of solutions of one-dimensional fractional Fisher-KPP reaction diffusion equations, when the initial condition is asymptotically front-like and it decays at infinity more slowly than a power x^b, where $b < 2\alpha$ and $\alpha \in (0,1)$ is the order of the fractional Laplacian (Chapter 2); in the second problem, we study the time asymptotic propagation of solutions to the fractional reaction diffusion cooperative systems (Chapter 3).

For the first problem, we prove that the level sets of the solutions move exponentially fast as time goes to infinity. Moreover, a quantitative estimate of motion of the level sets is obtained in terms of the decay of the initial condition.

In the second problem, we prove that the propagation speed is exponential in time, and we find a precise exponent depending on the smallest index of the fractional laplacians and of the nonlinearity, also we note that it does not depend on the space direction.

Keywords: reaction-diffusion problems, KPP, cooperative systems, fast propagation.

Résumé

Cette thèse porte sur deux problèmes différents : dans le premier, nous étudions le comportement en temps long des solutions des équations de réaction diffusion 1d-fractionnaire de type Fisher-KPP lorsque la condition initiale est asymptotiquement de type front et décroît à l'infini plus lentement que x^b, où $b < 2\alpha$ et $\alpha \in (0,1)$ est l'indice du laplacien fractionnaire (Chapitre 2). Dans le second problème, nous étudions la propagation asymptotique en temps des solutions de systèmes coopératifs de réaction-diffusion (Chapitre 3).

Dans le premier problème, nous démontrons que les ensembles de niveau des solutions se déplacent exponentiellement vite en temps quand t tend vers l'infini. De plus, une estimation quantitative du mouvement de ces ensembles est obtenue en fonction de la décroissance à l'infini de la condition initiale.

Dans le second problème, nous montrons que la vitesse de propagation est exponentielle en temps et nous trouvons un exposant précis qui dépend du plus petit ordre des laplaciens fractionnaires considérés et de la non-linéarité. Nous notons aussi que cet indice ne dépend pas de la direction spatiale de propagation.

Mots-clés: problèmes de réaction-diffusion, KPP, systèmes coopératifs, propagation rapide.

Resumen

Esta tesis trata sobre dos problemas diferentes: en el primero, se estudia el comportamiento en tiempos grandes de las soluciones de la ecuación de reacción diffusión 1d-fraccionaria del tipo Fisher-KPP cuando la condición inicial es asintóticamente como un frente y decae al infinito más lento que x^b, donde $b < 2\alpha$ y $\alpha \in (0, 1)$ es el índice del laplaciano fraccionario (Capítulo 2). En el segundo problema, se estudia la propagación asintótica en tiempo de las soluciones de sistemas cooperativos de reacción difusión (Capítulo 3).

En el primer problema, se demuestra que los conjuntos de nivel de las soluciones se desplazan exponencialmente rápido cuando el tiempo t tiende a infinito. Más aún, una estimación cuantitativa sobre el movimiento es obtenida en función del decrecimiento al infinito de la condición inicial.

En el segundo problema, se demuestra que la velocidad de propagación es exponencial en tiempo y se encuentra un exponente preciso que depende del orden más pequeño de los laplacianos fraccionarios considerados y de la nolinearidad. Se hace notar que este índice no depende de la dirección espacial de la propagación.

Palabras Clave: Problemas de reacción-difusión, KPP, sistemas cooperativos, propagación rápida.

a mis padres Yolanda y Julio

y a mi esposa Carolina.

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Introduction

Reaction-diffusion models have found widespread applicability in a surprising number of real-world models, including areas as, chemistry, biology, physics and engineering. But not only physical phenomena can be the result of a diffusive models. Stochastic processes in mathematical finance are often modeled by a Wiener process or Brownian motion, which lead to diffusive models. The simplest reaction-diffusion models are of the form

$$u_t - \Delta u = f(u) \tag{1}$$

where f is a nonlinear function representing the reaction kinetics. One of the most important examples of particular interest for us include the Fisher-KPP equation for which $f(u) = u(1-u)$. The nontrivial dynamics of these systems arises from the competition between the reaction kinetics and diffusion.

At a microscopic level, diffusion is the result of the random motion of individual particles, and the use of Laplacian operators in the model rests on the key assumption that this random motion is an stochastic Gaussian process. However, a growing number of works have shown the presence of anomalous diffusion processes, as for example Lévy processes, thus, reaction-diffusion equations with fractional Laplacian instead of standard Laplacian appear in physical models when the diffusive phenomena are better described by Lévy processes allowing long jumps, than by Brownian processes, see for example [39] for a description of some of these models. The Lévy processes occur widely in physics, chemistry and biology and recently these models that give rise to equations with the fractional Laplacians have attracted much interest.

Our particular aim on this type of anomalous diffusion problems is focussed to the study of large-time behavior of the solution of the Cauchy problem for fractional reaction-diffusion equations

$$u_t + (-\Delta)^\alpha u = f(u) \tag{2}$$

$$u(0,x) = u_0(x) \tag{3}$$

with $\alpha \in (0,1)$ in one spatial dimension, where $(-\Delta)^\alpha$ denote the fractional Laplacian. The nonlinearity f is assumed to be in the Fisher-KPP class. More precisely, the nonlinearity is assumed to have two zeros, an unstable one at $u = 0$ and a stable one at $u = 1$.

The reaction diffusion equation (1) with Fisher-KPP nonlinearity has been the subject of intense research since the seminal work by Kolmogorov, Petrovskii, and Piskunov [33]. Of particular interest are the results of Aronson and Weinberger [2] which describe the evolution of the compactly supported data. They showed that for a compactly supported initial value u_0, the movement of the fronts are linear in time. In addition, there exists a critical speed $c^* = 2\sqrt{f'(0)}$

for which the problem (2)-(3) admits planar traveling wave solutions connecting 0 and 1, that is, solutions of the form $u(t,x) = \phi(x - ct)$, which move with speeds $c \geq c^*$.

Many papers have been concerned with the large-time behavior of solutions of equation (1) or more general reaction diffusion equations with exponentially decaying initial conditions, leading to finite propagation speeds, see for example [11], [32], [35], [41] and references in [29].

In contrast with the results just mentioned, where finite speed of propagation is obtained whenever the initial value decays faster that an exponential, it is shown by Hamel and Roques [29] that when the initial condition is globally front-like, and it decays slower than any exponential then the asymptotic behavior of the front exhibits infinite speed and a very precise estimate can be obtained for the propagation of the level sets of the front in terms of the initial value, giving a precise superlinear behavior.

Moreover, Berestycki, Hamel and Roques [7] prove existence and uniqueness results for the stationary solution associated to (2) and they then analyze the behavior of the solutions of the evolution equation for large times. These results are expressed by a condition on the sign of the first eigenvalue of the associated linearized problem with periodicity condition.

Regarding (2)-(3) with $\alpha \in (0,1)$ and Fisher-KPP nonlinearity, in connection with the discussion given above for the case $\alpha = 1$, in the recent papers [14] and [15], Cabré and Roquejoffre show that for compactly supported initial value, or more generally for initial values decaying like $|x|^{-d-2\alpha}$, where d is the dimension of the spatial variable, the speed of propagation becomes exponential in time with a critical exponent $c_* = f'(0)(d+2\alpha)^{-1}$, they also show that no traveling waves exists for this equation, all results in great contrast with the case $\alpha = 1$. Moreover, if the initial data $u_0 \in [0,1]$ is nonincreasing and for $x > 0$ decay faster than $x^{-2\alpha}$, then the mass at $+\infty$ makes the front travel faster to the left, in this case with a speed $c_{**} = f'(0)/2\alpha$. All these results will be formally established in Chapter 1. Additionally we recall the earlier work in the case $\alpha \in (0,1)$ by Berestycki, Roquejoffre and Rossi [10], where it is proved that there is invasion of the unstable state by the stable one, also in [10], the authors derive a class of integro-differential reaction-diffusion equations from simple principles. They then prove an approximation result for the first eigenvalue of linear integro-differential operators of the fractional diffusion type, they also prove the convergence of solutions of fractional evolution problem to the steady state solution when the time tends to infinity. For a large class of nonlinearities, Engler [25] has proved that the invasion has unbounded speed. For another type of integro-differential equations Garnier [27] also establishes that the position of the level sets moves exponentially in time for algebraically decaying dispersal kernels. And in a recent paper Stan and Vázquez [42] study the propagation properties of nonnegative and bounded solutions of the class of reaction-diffusion equations with nonlinear fractional diffusion.

Chapter 2 is concerned with the study of the phenomena described by Hamel and Roques in [29] in the case of nonlocal diffusion, that is, when $\alpha \in (0,1)$ in equation (2)-(3). In particular, we study the asymptotic behavior of solutions with slowly decaying, globally front-like initial value. We state that, for $b < 2\alpha$, the central part of the solution moves to the right at exponential speed $f'(0)/b$, which is faster than c_{**}, the exponential speed for solutions with initial values decaying faster than $x^{-2\alpha}$. Thus we show that the exponent 2α is critical regarding the speed of propagation of the solution. Furthermore, we prove that the initial condition u_0 can be chosen so that the location of the solution u be asymptotically larger than any prescribed real-valued function.

The study of propagation fronts was also done in reaction diffusion systems, in this line, Lewis, Li and Weinberger in [37], studied spreading speeds and planar traveling waves for a particular class of cooperative reaction diffusion systems with standard diffusion by analyzing traveling waves and the convergence of initial data to wave solutions. It is shown that, for a large class of such cooperative systems, the spreading speed of the system is characterized as the slowest speed for which the system admits traveling wave solutions. Moreover, the same authors in [44] establish the existence of a explicit spreading speed σ^* for which the solution of the cooperative system spread linearly in time, when the time tends to $+\infty$.

Follow the line, it is our aim to study the spread speeds of solutions of reaction diffusion cooperative system when the standard Laplacians are replaced for instance by the fractional Laplacian with different indexes. Chapter 3 is devoted to study the time asymptotic propagation of solutions to the fractional reaction diffusion cooperative systems. We prove that the propagation speed is exponential in time, and we find a precise exponent depending on the smallest index of the fractional Laplacians and of the principal eigenvalue of the matrix $DF(0)$ where F is the nonlinearity associated to the fractional system. Also we establish the existence of a constant steady state solution for the system and we prove the convergence of the solution towards the steady solution for large times.

Chapter 1

Preliminaries and background

In this chapter, we provide some elementary properties of fractional Laplacian operators, also we introduce the main results concerning to the large time behavior of solutions of reaction-diffusion problems with Fisher-KPP nonlinearities. Our principal goal in this chapter is to note the differences between the speed propagation obtained when the problem has standard diffusion or fractional diffusion. Moreover, we present some results related with the behavior of solutions of weakly coupled reaction-diffusion equation systems.

1.1. Fractional Laplacian

Let consider the Schwartz space $\mathcal{S}$ of rapidly decaying C^1 functions in R^n, hence, for any $u \in \mathcal{S}$ and $\alpha \in (0,1)$, the fractional Laplacian denoted by $(-\triangle)^\alpha$ is defined as

$$(-\Delta)^\alpha u(x) = C(d,\alpha) P.V. \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{\mid x-y \mid^{d+2\alpha}} dy$$

where the principal value is taken as the limit of the integral over $\mathbb{R}^d \setminus B_\varepsilon(x)$ as $\varepsilon \to 0$ and $C(d,\alpha)$ is a constant that depends on α, given by

$$C(d,\alpha) = \pi^{-(2\alpha+\frac{d}{2})} \frac{\Gamma(\frac{d}{2}+\alpha)}{\Gamma(-\alpha)}$$

Let note that the principal value is not necessary in the definition if $\alpha \in (0,1/2)$. Indeed, for any $u \in \mathcal{S}$, we have

$$\int_{\mathbb{R}^d} \frac{u(x)-u(y)}{\mid x-y \mid^{d+2\alpha}} dy \leq C \int_{B_R(x)} \frac{|x-y|}{|x-y|^{d+2\alpha}} dy + \|u\|_{L^\infty(\mathbb{R}^d)} \int_{B_R(x)^c} \frac{1}{|x-y|^{d+2\alpha}} dy$$

$$= C\left(\int_{B_R(x)} \frac{1}{|x-y|^{d+2\alpha-1}}dy + \int_{B_R(x)^c} \frac{1}{|x-y|^{d+2\alpha}}dy\right)$$
$$= C\left(\int_0^R \frac{1}{r^{2\alpha}}dr + \int_R^{+\infty} \frac{1}{r^{2\alpha+1}}dr\right) < +\infty$$

where C is a positive constant.

Now, by two changes of variables, $z = y - x$ and $z' = x - y$, we can rewrite $(-\triangle)^\alpha u$ as:

$$(-\triangle)^\alpha u = \frac{1}{2}C\left(P.V.\int_{\mathbb{R}^d} \frac{u(x)-u(x+z)}{|z|^{d+2\alpha}}dz + P.V.\int_{\mathbb{R}^d} \frac{u(x)-u(x-z')}{|z'|^{d+2\alpha}}dz'\right)$$
$$= -\frac{1}{2}C \cdot P.V.\int_{\mathbb{R}^d} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2\alpha}}dy$$

Using a second order Taylor expansion, we have

$$\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2\alpha}} \leq \frac{\|D^2u\|_{L^\infty}}{|y|^{d+2\alpha-2}}$$

which is integrable near 0, thus we can write the singular integral in (1.1) as a weighted second order differential quotient

$$(-\triangle)^\alpha u = -\frac{1}{2}C\int_{\mathbb{R}^d} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2\alpha}}dy$$

To end this part, we state a result which show that the fractional Laplacian $(-\triangle)^\alpha$ can be viewed as a pseudo-differential operator of symbol $|\xi|^{2\alpha}$.

Theorem 1.1 *Let $\alpha \in (0,1)$ and let $(-\triangle)^\alpha : \mathcal{S} \to L^2(\mathbb{R}^d)$ be the fractional Laplacian operator defined by (1.1). Then, for any $u \in \mathcal{S}$,*

$$(-\Delta)^\alpha u = C\mathfrak{F}^{-1}(|\xi|^{2\alpha}(\mathfrak{F}u)), \quad \forall \xi \in \mathbb{R}^d$$

for a suitable positive constant C depending only on s and n.

1.2. Background on reaction-diffusion front propagation

Let f be a function satisfying

$$f \in C^1([0,1]), \quad f(0) = f(1) = 0 \quad \text{and} \quad 0 < f(s) \leq f'(0)s \tag{1.1}$$

We are interested in the large time behavior of solutions $u = u(t, x)$ to the Cauchy problem

$$\begin{aligned} \partial_t u - \triangle u &= f(u), & t > 0, x \in \mathbb{R}^d \quad (1.2) \\ u(0, x) &= u_0(x) & x \in \mathbb{R} \quad (1.3) \end{aligned}$$

the following result of Aronson and Weinberger [2] implies that there is a critical speed c^* such that for fairly general initial data u_0 that are non-negative and compactly supported, the invasion rate in which the stable state 1 invades the unstable state 0 is linear in time.

Theorem 1.2 *Let $c^* = 2\sqrt{f'(0)}$ and let u the solution of (1.2)-(1.3) with $u_0 \not\equiv 0$ compactly supported in $\mathbb{R}^d$ and satisfying $0 \leq u_0(x) \leq 1$. Then:*

- *if $c < c^*$, then $\lim_{t\to+\infty} \inf_{|x|\leq ct} u(t, x) = 1$,*
- *if $c > c^*$, then $\lim_{t\to+\infty} \sup_{|x|\geq ct} u(t, x) = 0$.*

Freidlin and Gärtner [28] extended the spreading properties to space periodic media and to some classes of random media using probabilistic tools, proofs using dynamical systems or PDE arguments are given in [6], [24], [46]. Hence, starting from a positive compactly supported initial data, in [28] is stated that there exists an asymptotic directional spreading speed $w^*(e) > 0$ in each direction e, in the sense that

- if $0 \leq c < w^*(e)$, then $\liminf_{t\to+\infty} u(t, x + cte) > 0$
- if $c > w^*(e)$, then $\lim_{t\to+\infty} u(t, x + cte) = 0$

locally uniformly in $x \in \mathbb{R}^d$, where $w^*(e)$ is characterized by:

$$w^*(e) = \min_{e' \in \mathbb{S}^{d-1}, e' \cdot e > 0} \frac{c^*(e')}{e' \cdot e}$$

where $c^*(e')$ is the minimal speed of pulsating traveling fronts in the direction e'.

Moreover, if we consider the case $d = 1$, it is indeed well-known that the equation $u_t = u_{xx} + f(u)$ admits a family of planar traveling waves $u(t, x) = \varphi_c(x - ct)$ for all speeds $c \geq c^* := 2\sqrt{f'(0)}$, where, for each speed $c \in [c^*, +\infty)$, the function $\phi_c : \mathbb{R} \to (0, 1)$ satisfies

$$-\phi_c'' + c\phi_c' = f(\phi_c) \quad \text{in} \quad \mathbb{R}, \qquad \phi_c(-\infty) = 1, \phi_c(+\infty) = 0,$$

Furthermore, the function ϕ_c is decreasing in $\mathbb{R}$ and unique up to shifts. Now, suppose that $u_0 : \mathbb{R} \to [0, 1]$ is uniformly continuous and satisfy

$$u_0 > 0 \quad \text{in } \mathbb{R}, \quad \lim_{x\to-\infty} u_0(x) > 0 \;\; \text{and} \;\; \lim_{x\to\infty} u_0(x) = 0 \qquad (1.4)$$

If $u_0(x)$ is equivalent as $x \to +\infty$ to a multiple of $e^{-\mu x}$ with $0 < \mu < \mu^* = \sqrt{f'(0)}$, then $u(t,x)$ converges to a finite shift of the front $\varphi_c(x-ct)$ as $t \to +\infty$, where $c = \mu + \frac{f'(0)}{\mu} > c^*$. On the other hand, if $u_0(x) = O(e^{-\mu^* x})$ as $x \to +\infty$, then $u(t,x)$ behaves as $t \to +\infty$ like $\varphi_{c^*}(x + m(t) - c^* t)$, where the shift m satisfies $m(t) = O(\ln(t))$ as $t \to +\infty$ (the limit case when $u_0 = 0$ on $[0, +\infty)$ was first treated in the seminal paper of Kolmogorov, Petrovski and Piskunov [17]). In these two situations, the location of the solution u at large time moves at a finite speed, in the sense that, for any $\lambda \in (0,1)$ and any family of real numbers $x_\lambda(t)$ such that $u(t, x_\lambda(t)) = \lambda$, then $x_\lambda(t)/t$ converges as $t \to +\infty$ to a positive constant. This constant asymptotic speed is equal to $c = \mu + \frac{f'(0)}{\mu}$ in the first case, it is equal to c^* in the second case.

Furthermore if we suppose that there exist $\delta > 0$, $s_0 \in (0,1)$ and $M \geq 0$ such that

$$f(s) \geq f'(0)s - Ms^{1+\delta} \qquad \text{for all } s \in [0, s_0] \tag{1.5}$$

and the function u_0 is assumed to decay more slowly than any exponentially decaying function as $x \to +\infty$, in the sense that

$$\forall \varepsilon > 0,\ \exists x_\varepsilon \in \mathbb{R},\ u_0(x) \geq e^{-\varepsilon x} \quad for\ x \in [x_\varepsilon, \infty) \tag{1.6}$$

Hamel and Roques [29] prove that all level sets of the solutions move infinitely fast as time goes to infinity and the locations of the level sets are expressed in terms of the decay of the initial condition. Hence, [29] contains the first systematic study of the large-time behavior of solutions of KPP equations with slowly decaying initial conditions. These results are in sharp contrast with the well-studied case of exponentially bounded initial conditions, where the solution u converge in some sense to some traveling fronts with finite speed as $t \to +\infty$.

Before to state the results, we need to introduce a few notations. For any $\lambda \in (0,1)$ and $t \geq 0$, denote by

$$E_\lambda(t) = \{x \in \mathbb{R} : u(t,x) = \lambda\}$$

the level set of u of value λ at time t. For any subset $A \subset (0,1]$, we set

$$u_0^{-1}(A) = \{x \in \mathbb{R} : u_0(x) \subset A\},$$

the inverse image of A by u_0.

Theorem 1.3 *Let u be the solution of (1.2)-(1.3), where f satisfies (1.1) and the initial condition $u_0 : \mathbb{R} \to [0,1]$ satisfies (1.4) and (1.6).*

a) Then

$$\lim_{x \to +\infty} u(t,x) = 0 \ \ \forall t \geq 0 \quad and \quad \liminf_{x \to -\infty} u(t,x) = 1 \quad as\, t \to \infty$$

b) For any given $\lambda \in (0,1)$, there is a real number $t_\lambda > 0$ such that $E_\lambda(t)$ is compact and non-empty for all $t \geq t_\lambda$, and

$$\lim_{t\to+\infty} \frac{\min E_\lambda(t)}{t} = +\infty \tag{1.7}$$

Part $b)$ of Theorem 1.3 simply says that the level sets $E_\lambda(t)$ of all level values $\lambda \in (0,1)$ move infinitely fast as $t \to +\infty$, in the average sense (1.7). As already announced above, this property is in big contrast with the finiteness of the propagation speeds of solutions which are exponentially bounded as $x \to +\infty$ at initial time.

Now, we state the main result which show a relation between the initial condition and the large-time behavior of the level sets of the solution u.

Theorem 1.4 *Let u be the solution of (1.2)-(1.3), where f satisfies (1.1), (1.5) and the initial condition $u_0 : \mathbb{R} \to [0,1]$ satisfies (1.4) and (1.6). Assume that there exists $\xi_0 \in \mathbb{R}$ such that u_0 is of class C^2 and nonincreasing on $[\xi_0, \infty)$, and $u_0''(x) = o(u_0(x))$ as $x \to \infty$.*

Then, for any $\lambda \in (0,1)$, $\delta \in (0, f'(0))$, $\gamma > 0$ and $\Gamma > 0$, there exists $\tau = \tau_{\lambda,\delta,\gamma,\Gamma} \geq t_\lambda$ such that

$$\forall t \geq \tau, \quad E_\lambda(t) \subset u_0^{-1}\{[\gamma e^{-(f'(0)+\delta)t}, \Gamma e^{-(f'(0)-\delta)t}]\} \tag{1.8}$$

Observe first that the real numbers $\gamma e^{-(f'(0)+\delta)t}$ and $\Gamma e^{-(f'(0)-\delta)t}$ belong to $(0, \sup_{\mathbb{R}} u_0)$ for large t. Now, if there is a $a \in \mathbb{R}$ such that u_0 is strictly decreasing on $[a, +\infty)$, then the inclusion of Theorem 1.4 means that

$$u_0^{-1}\left(\Gamma e^{-(f'(0)-\delta)t}\right) \leq \min E_\lambda(t) \leq \max E_\lambda(t) \leq u_0^{-1}\left(\gamma e^{-(f'(0)+\delta)t}\right) \tag{1.9}$$

for large t, where, $u_0^{-1} : (0, u_0(a)] \to [a, +\infty)$ denotes the reciprocal of the restriction of the function u_0 on the interval $[a, +\infty)$. Furthermore, if u_0 is nonincreasing over the whole real line $\mathbb{R}$, then the derivative $u_x(t,x)$ is negative for all $t > 0$ and $x \in \mathbb{R}$, from the parabolic maximum principle. Therefore, $E_\lambda(t)$ is either empty or a singleton as soon as $t > 0$. In particular, $E_\lambda(t) = \{x_\lambda(t)\}$ for all $t > t_\lambda$, where $x_\lambda(t)$ satisfies $u(t, x_\lambda(t)) = \lambda$. The inequalities (1.9) then provide lower and upper bounds of $x_\lambda(t)$ for large t. However, it is worth pointing out that Theorem 1.4 is valid for general initial conditions which decay slowly to 0 at $+\infty$ but which may not be globally nonincreasing.

Furthermore, for any $\lambda \in (0,1)$, formula (1.8) with $\gamma = \Gamma = \lambda$ can be rewritten as

$$u_0(x_\lambda(t))e^{(f'(0)-\delta)t} \leq \lambda \leq u_0(x_\lambda(t))e^{(f'(0)+\delta)t}$$

for t large enough. Roughly speaking, this means that the real numbers $x_\lambda(t)$ are then asymptotically given, in the above approximate sense, by the solution of the family of decoupled ODE's

$$\partial_t U(t,x) = f'(0)U(t,x)$$
$$U(0,x) = U_0(x)$$

parameterized by $x \in \mathbb{R}$, and then, say, by solving $U(t, x_\lambda(t)) = \lambda$. In other words, the behavior of $u(t,x)$ at large time is dominated by the reaction term, that is to say that the diffusion term plays in some sense a negligible role as compared to the growth by reaction.

Moreover, Theorem 1.4 actually yields the following corollary, which states that the level sets $E_\lambda(t)$ can be located as far to the right as wanted, provided that the initial condition is well chosen.

Corolary 1.5 *Under the assumptions and notations of Theorem 1.4, the following holds: given any function $\chi : [0,\infty) \to \mathbb{R}$ which is locally bounded, there are initial conditions u_0 such that, for any given $\lambda \in (0,1)$*

$$\min E_\lambda(t) \geq \chi(t)$$

for all t large enough.

To end this section, we present the principal results stated by Berestycki, Hamel and Roques in [7]. The authors prove existence and uniqueness results for the stationary equation and also they analyze the behavior of the solutions of the evolution equation for large times. These results are expressed by a condition on the sign of the first eigenvalue of the associated linearized problem with periodicity condition.

Thus, we are concerned with the equation

$$\partial_t u - \nabla \cdot (A(x)\nabla u) = f(x,u), \qquad t > 0, x \in \mathbb{R}^d \tag{1.10}$$

and its stationary solutions given by

$$-\nabla \cdot (A(x)\nabla u) = f(x,u), \qquad x \in \mathbb{R}^d \tag{1.11}$$

Let $L_1, ..., L_d > 0$ be d given real numbers. In the following, saying that a function $g : \mathbb{R}^d \to \mathbb{R}$ is periodic means that $g(x_1, ..., x_k + L_k, ..., x_d) = g(x_1, ..., x_d)$ for all $k = 1, ..., d$. Let us now describe the precise assumptions. Throughout the paper, the diffusion matrix field $A(x) = (a_{ij}(x))_{1\leq i,j\leq d}$ is assumed to be periodic, of class $C^{1,\alpha}$ (with $\alpha > 0$), and uniformly elliptic, in the sense that

$$\exists \alpha_0 > 0,\ \forall x \in \mathbb{R}^d,\ \forall \xi \in \mathbb{R}^d, \sum_{1\leq i,j\leq d} a_{ij}(x)\xi_i\xi_j \geq \alpha_0|\xi|^2$$

The function $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is of class $C^{0,\alpha}$ in x locally in u, locally Lipschitz continuous with respect to u, periodic with respect to x. Moreover, assume that $f(x,0) = 0$ for all $x \in \mathbb{R}^d$, that f is of class C^1 in $\mathbb{R}^d \times [0,\beta]$ (with $\beta > 0$), and set $f_u(x,0) := \lim_{s\to 0^+} f(x,s)/s$.

In several results below, the function f is furthermore assumed to satisfy

$$\forall x \in \mathbb{R}^d, s \mapsto f(x,s)/s \text{ is decreasing in } s > 0 \tag{1.12}$$

and

$$\exists M \geq 0,\ s \geq M,\ \forall x \in \mathbb{R}^d, f(x,s) \leq 0 \tag{1.13}$$

Examples of functions f satisfying (1.12 and (1.13) are functions of the type $f(x,u) = u(\mu(x) - \nu(x)u)$ or simply $f(x,u) = u(\mu(x) - u)$, where μ and ν are periodic.

Namely, we define λ_1 as the unique real number such that there exists a function $\phi > 0$ which satisfies

$$\begin{aligned} -\nabla \cdot (A(x)\nabla\phi) - f_u(x,0)\phi = \lambda_1\phi \quad &in\ \mathbb{R}^d \\ \phi > 0\ is\ periodic,\ \phi > 0,\ \|\phi^{R_0}\| = 1 & \end{aligned}$$

We are now ready to state the existence and uniqueness result on problem (1.11). Let us start with the criterion for existence.

Theorem 1.6 *1) Assume that f satisfies (1.13) and $\lambda_1 < 0$. Then, there exists a positive and periodic solution p of (1.11).*

2) Assume that f satisfies (1.12) and that $\lambda_1 \geq 0$. Then there is no positive bounded solution of (1.11) (i.e. 0 is the only nonnegative and bounded solution of (1.11)).

Next we state our uniqueness result.

Theorem 1.7 *Assume that f satisfies (1.12) and $\lambda_1 < 0$. Then, there exists at most one positive and bounded solution of (1.11). Furthermore, such a solution, if any, is periodic with respect to x. If $\lambda_1 \geq 0$ and f satisfies (1.12), then there is no nonnegative bounded solution of (1.11) other than 0.*

The core part in the proof of the above theorem consists in proving that any positive solution of (1.11) is actually bounded from below by a positive constant, which does not seem to be a straightforward property.

Let us now consider the parabolic equation (1.10), and let $u(t,x)$ be a solution of (1.10), with initial condition $u(0,x) = u_0(x)$ in $\mathbb{R}^d$. The asymptotic behavior of $u(t,x)$ as $t \to +\infty$ is described in the following theorem:

Theorem 1.8 *Assume that f satisfies (1.11) and (1.12). Let u_0 be an arbitrary bounded and continuous function in $\mathbb{R}^d$ such that $u_0 \geq 0$, $u_0 \not\equiv 0$. Let $u(t,x)$ be the solution of (1.10) with initial datum $u(0,x) = u_0(x)$.*

1) If $\lambda_1 < 0$*, then* $u(t,x) \to p(x)$ *in* $C^2_{loc}(\mathbb{R}^d)$ *as* $t \to +\infty$*, where* p *is the unique positive solution of (1.11).*

2) If $\lambda_1 \geq 0$*, then* $u(t,x) \to 0$ *uniformly in* $\mathbb{R}^d$ *as* $t \to +\infty$*.*

1.3. Background on fractional reaction-diffusion front propagation

We are interested in the time asymptotic location of the level sets of solutions to the equation

$$\partial_t u + (-\Delta)^\alpha u = f(u), \qquad t > 0, x \in \mathbb{R}^d \tag{1.14}$$
$$u(0,x) = u_0(x) \qquad x \in \mathbb{R}^d \tag{1.15}$$

with $0 < \alpha < 1$ and f a concave function satisfying (1.1). Given $\lambda \in (0,1)$, we want to describe how the level sets $\{x \in \mathbb{R}^d : u(t,x) = \lambda\}$ spread as time goes to $+\infty$.

The results established by Cabré and Roquejoffre in [15] and announced in [14], show that there exist drastic changes in the behavior of solutions as soon as the Laplacian is replaced for instance by the fractional Laplacian $(-\triangle)^\alpha$ with $\alpha \in (0,1)$. They prove that the front position will be exponential in time, in contrast with the classical case where it is linear in time by Theorem 1.2.

For $\alpha \in (0,1)$, the fractional Laplacian is the generator for a stable Lévy process. It is reasonable to expect that the existence of jumps (or flights) in the diffusion process will accelerate the invasion of the unstable state $u = 0$ by the stable one, $u = 1$.

The first result concerns a class of initial data in $\mathbb{R}^d$, possibly discontinuous, which includes compactly supported functions. The following theorem shows that the position of all level sets moves exponentially fast in time.

Theorem 1.9 *Let* $d \geq 1$*,* $\alpha \in (0,1)$ *and* f *satisfy (1.1). Let define* $c_* = \frac{f'(0)}{d+2\alpha}$*. Let* u *be a solution of (1.14)-(1.15), where* $u_0 \not\equiv 0$*,* $0 \leq u_0 \leq 1$ *is measurable, and*

$$u_0(x) \leq C|x|^{-d-2\alpha} \quad \forall x \in \mathbb{R}^d$$

for some constant C*. Then:*

-if $c < c_*$*, then* $\lim_{t\to\infty} \inf_{|x| \leq e^{ct}} u(t,x) = 1$,

-if $c > c_*$*, then* $\lim_{t\to\infty} \sup_{|x| \geq e^{ct}} u(t,x) = 0$.

A delicate step to prove the previous theorem is the following lemma.

Lemma 1.10 *Under the assumptions of Theorem 1.9, for every $c < c_*$ there exists $\varepsilon \in (0,1)$ and $\underline{t} > 0$ such that*

$$u(t,x) \geq \varepsilon \qquad \text{for all } t \geq \underline{t} \ \text{ and } \ |x| \leq e^{ct}$$

Even if this lemma concerns initial data decaying at infinity, from it we can deduce the non-existence of planar traveling waves, under no assumption of their behavior at infinity, as in the following statement.

Proposition 1.11 *Let $d \geq 1$, $\alpha \in (0,1)$, f satisfy (1.1). Then, there exists no nonconstant planar traveling wave solution of (1.14)-(1.15). That is, all solutions of (1.14)-(1.15) taking values in $[0,1]$ and of the form $u(t,x) = \phi(x + te)$, for some vector $e \in \mathbb{R}^d$, are identically 0 or 1. Equivalently, the only solutions $\phi : \mathbb{R}^d \to [0,1]$ of*

$$(-\triangle)^\alpha \phi + e \cdot \nabla \phi = f(\phi) \qquad \text{in } \mathbb{R}^d \tag{1.16}$$

are $\phi = 0$ and $\phi = 1$.

The last statement on the elliptic equation (1.16) has an analogue for the Laplacian. As shown in [2], if $|e| < 2\sqrt{f'(0)}$ then equation (1.16) with $\alpha = 1$ admits the constants 0 and 1 as only solutions taking values in $[0,1]$.

In one space dimension, it is of interest to understand the dynamics of nonincreasing initial data. As mentioned before, for the standard Laplacian the level sets of u travel with the speed c^*, provided that u_0 decays sufficiently fast at $+\infty$. In the fractional case, the mass at $+\infty$ has an effect and what happens is not a mere copy of the result of Theorem 1.9 for compactly supported data. The mass at $+\infty$ makes the front travel faster to the left, indeed with a larger exponent than c_*.

Theorem 1.12 *Let $d = 1$, $\alpha \in (0,1)$ and f satisfy (1.1). Let define the quantity $c_{**} = \frac{f'(0)}{2\alpha}$. Let u be a solution of (1.14)-(1.15) and suppose that $0 \leq u_0 \leq 1$ is measurable and nonincreasing, $u_0 \not\equiv 0$ and*

$$u_0(x) \leq Cx^{-2\alpha} \quad \forall x > 0$$

for some constant C. Then:

-if $c < c_{**}$, then $\lim_{t\to\infty} \inf_{x \leq e^{ct}} u(t,x) = 1$,

-if $c > c_{**}$, then $\lim_{t\to\infty} \sup_{x \geq e^{ct}} u(t,x) = 0$.

Remark 1.13 Note that

$$c_{**} = \frac{f'(0)}{2\alpha} > \frac{f'(0)}{1+2\alpha} = c_*$$

where c_* is the exponent in Theorem 1.9 for $d = 1$ and compactly supported data.

Notice also the slower power decay assumed in the initial condition with respect to Theorem 1.9. One could wonder whether a model with such features is physically, or biologically relevant. In fact, this behavior is consubstantial to fast diffusion, and the model may be relevant to explain fast recolonization events in ecology.

To end this subsection, we consider the function f periodic in each x_i-variable and satisfies $f(x,0) = 0$ for all $x \in \mathbb{R}^d$, moreover

$$\forall x \in \mathbb{R}^d, s \mapsto f(x,s)/s \text{ is decreasing in } s > 0 \tag{1.17}$$

and

$$\exists M \geq 0,\ s \geq M,\ \forall x \in \mathbb{R}^d, f(x,s) \leq 0 \tag{1.18}$$

Examples of functions f satisfying (1.17 and (1.18) are functions of the type $f(x,u) = u(\mu(x) - u)$, where μ is periodic, note that if $\mu = 1$ (homogeneous media) the function f satisfies (1.1). The nonlinearity $\mu(x)u - u^2$ is often referred to as a Fisher-KPP type nonlinearity.

Let λ_1 be the principal periodic eigenvalue of the operator $(-\triangle)^\alpha - f_u(x,0)$. From [10] we know that if $\lambda_1 > 0$, every solution to (1.14)-(1.15) starting with a bounded nonnegative initial condition tends to 0 as $t \to +\infty$. Thus we assume $\lambda_1 < 0$. Then, by [10], the solution to (1.14)-(1.15) tends, as $t \to +\infty$, to the unique bounded positive steady solution to (1.14)-(1.15), denoted by u_+. By uniqueness, u_+ is periodic and the convergence holds on every compact set.

Moreover, if we consider the problem (1.14)-(1.15) with the nonlinearity $f(u) = \mu(x)u - u^2$ with μ periodic. Cabré, Coulon and Roquejoffre proved in [16] that the speed of propagation is exponential in time, with a precise exponent depending on a periodic principal eigenvalue, and that it does not depend on the space direction. This result is in contrast with the Freidlin-Gärtner formula for the standard Laplacian.

Theorem 1.14 *Assume that $\lambda_1 < 0$. Let u be the solution to (1.14)-(1.15) with u_0 piecewise continuous, nonnegative, $u_0 \not\equiv 0$, and $u_0(x) = O(|x|^{d+2\alpha})$ as $|x| \to +\infty$. Then, for every $\lambda \in (0, \min \mu)$, there exist $c_\lambda > 0$ and a time $t_\lambda > 0$ (all depending on λ and u_0) such that, for all $t > t_\lambda$,*

$$\{x \in \mathbb{R}^d |\ u(t,x) = \lambda\} \subset \{x \in \mathbb{R}^d |\ c_\lambda e^{\frac{|\lambda_1|}{d+2\alpha}} \leq |x| \leq c_\lambda^{-1} e^{\frac{|\lambda_1|}{d+2\alpha}}\}$$

Note that if $\mu = 1$ then $\lambda = -1$ and $f'(0) = 1$, therefore, the estimate obtained is much sharper than that in Theorem 1.9.

1.4. Background on reaction-diffusion systems. Spreading speeds

In the first part of this subsection, we are interested in the large time behavior of the solution $u = (u_i)_{i=1}^m$ with $m \in \mathbb{N}^*$, to the one-dimensional weakly coupled reaction-diffusion system of the form:

$$\partial_t u_i(t,x) - d_i \triangle u_i(t,x) = f_i(u(t,x)), \qquad \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R} \tag{1.19}$$

$$u_i(0,x) = u_{0i}(x), \qquad \forall x \in \mathbb{R} \tag{1.20}$$

for all $i \in \{1,...,m\}$, where each d_i is a nonnegative constant and $f = (f_i)_{i=1}^m$ is independent of x and t. If $a = (a_i)_{i=1}^m$ is a constant positive vector, we define the set of functions

$$C_a = \{u(x) : u(x) \text{ continuous and } 0 \le u(x) \le a\}$$

As general assumptions, we suppose that $f \in C([0,a])$ and the only zeros of f in C_a are 0 and a, also, the system (1.19)-(1.20) is cooperative; i.e., each $f_i(u)$ is nondecreasing in all components of u with the possible exception of the ith one.

In what follow, we present the results of Lewis, Li and Weinberger [37]. They showed the existence of a single spreading speed σ_* for which the system (1.19)-(1.20) admit a traveling wave solution $W(x - \sigma t)$ for all speeds $\sigma \ge \sigma_*$.

Before to state this result, we need some additional hypothesis.

H1. f has uniformly bounded piecewise continuous first partial derivatives for $0 \le u \le a$, and it is differentiable at 0.

H2. The Jacobian matrix $f'(0)$, whose off-diagonal entries are nonnegative, has a positive eigenvalue whose eigenvector has positive components.

Theorem 1.15 *Suppose that the function f in the system (1.19)-(1.20) satisfies the additional hypothesis (H1), (H2) and the initial condition $u_0 = (u_{0i})_{i=1}^m$ is 0 for all sufficiently large x, and that there are positive constants $0 < \rho \le \sigma < 1$ such that $0 \le u_0 \le \sigma a$ for all x and $u_0 \ge \rho a$ for all sufficiently negative x.*

Then there exist σ_*, such that for every $\sigma \ge \sigma_*$ the system (1.19)-(1.20) has a nonincreasing traveling wave solution $W(x - \sigma t)$ of speed σ with $W(-\infty) = a$ and $W(+\infty) = 0$.

Moreover, if there is a traveling wave $W(x - \sigma t)$ with $W(-\infty) = a$ such that for at least one component i

$$\liminf_{x \to +\infty} W_i(x) = 0$$

then $\sigma \ge \sigma_*$.

Note that the last statement has an analogue when we consider a single equation with standard Laplacian, in that case the limit speed is given by $2\sqrt{f'(0)}$.

The next result is stated by Lewis, Li and Weinberger in [44]. They establish the existence of a explicit spreading speed σ^* for which the solution of the system (1.19)-(1.20) spread linearly in time, when the time tends to $+\infty$.

As in the previous result, we need some additional hypothesis.

H3. $f(u)$ is piecewise continuously differentiable in u for $0 \leq u \leq a$ and differentiable at 0.

H4. The Jacobian matrix $f'(0)$ is in Frobenius form. The principal eigenvalue $\gamma_1(0)$ of its upper left diagonal block is positive and strictly larger than the principal eigenvalues $\gamma_k(0)$ of its other diagonal blocks, and there is at least one nonzero entry to the left of each diagonal block other than the first one.

Before to continue, we introduce some notations. We define the matrix

$$C_\mu = \operatorname{diag}(d_i\mu^2) + f'(0)$$

where μ is a positive number and $f'(0)$ is the Jacobian matrix with entries $f_{i,u_j}(0)$. The off-diagonal entries of C_μ are nonnegative because the system is cooperative. We also define, the matrix B_μ given by the formula

$$B_\mu = \exp(C_\mu)$$

hence, we have that the eigenvalues λ_k of B_μ are given by $\lambda_k = e^{\gamma_k(\mu)}$ where γ_k is the principal eigenvalue of the kth diagonal block of the matrix C_μ. Moreover, we define

$$\sigma^* := \inf_{\mu>0}[\gamma_1(\mu)/\mu] \tag{1.21}$$

Let $\overline{\mu} \in (0, +\infty]$ the value of μ at which this infimum in (1.21) is attained, and let $\zeta(\mu)$ be the eigenvector of B_μ which corresponds to the eigenvalue $\lambda_1(\mu)$.

Theorem 1.16 *Suppose that the function f in the system (1.19)-(1.20) satisfies the additional hypothesis (H3), (H4). Assume that $\overline{\mu}$ is finite,*

$$\gamma_1(\overline{\mu}) > \gamma_\sigma(\overline{\mu}) \qquad for\ all\ \sigma > 1$$

and

$$f(\rho\zeta(\overline{\mu})) \leq \rho f'(0)\zeta(\overline{\mu}) \qquad for\ all\ \rho > 0$$

Then there exist σ^* defined by (1.21) such that for any initial condition $u_0(x)$ in C_a which vanishes outside a bounded set, the solution of (1.19)-(1.20) has the properties that for each positive ϵ

$$\lim_{t\to+\infty}\left[\max_{|x|\geq t(\sigma^*+\epsilon)}|u(x)|\right] = 0$$

and for any strictly positive constant vector ω there is a positive R_ω with the property that if $u_0 \geq \omega$ on an interval of length $2R_\omega$, then

$$\lim_{t\to+\infty}\left[\max_{|x|\leq t(\sigma^*-\epsilon)}|a-u(x)|\right]=0$$

It is easy to see that if a function f satisfies (H1),(H3) and (H4) then σ_* in Theorem 1.14 is equal to σ^* defined by (1.21).

Following the line, Barles, Evans and Sougandis [4] present another approach to the study of the limiting behavior of the solution of certain scaled reaction-diffusion systems. Hence, they considered the system

$$\partial_t u_i^\varepsilon = \varepsilon d_i \triangle u_i^\varepsilon + \frac{1}{\varepsilon} f_i(u^\varepsilon), \qquad in\ \mathbb{R}_+^* \times \mathbb{R}^d \tag{1.22}$$

$$u_i^\varepsilon(0,x) = g_i(x), \qquad on\ x \in \mathbb{R}^d \tag{1.23}$$

for all $i \in \{1,...,m\}$. Here the constants d_k ($1 \leq k \leq m$), and the functions

$$g : \mathbb{R}^d \to \mathbb{R}^m \quad \text{and} \quad f : \mathbb{R}^m \to \mathbb{R}^m$$

are given, where we write $g = (g_i)_{i=1}^m$, $f = (f_i)_{i=1}^m$. The unknown is $u^\varepsilon = (u_i^\varepsilon)_{i=1}^m$. We will assume that

$$d_i > 0 \qquad \forall\, i \in \{1,...,m\}$$

and that the functions g, f are smooth, bounded and Lipschitz. In addition we suppose

$$g_i > 0 \qquad \forall\, i \in \{1,...,m\}$$

and

$$G_0 = \{x : g_i(x) > 0\} \qquad \forall\, i \in \{1,...,m\}$$

is a bounded, smooth subset of $\mathbb{R}^d$. Our essential assumptions all concern the reaction term f. First of all we suppose

(H1) $f(0) = 0$

and also

(H2) $f_i(u_1,...,u_{i-1},0,u_{i+1},...,u_m) > 0$ if $u_1,...,u_{i-1},u_{i+1},...,u_m \geq 0$ and $u_l > 0$ for some index $l \neq i$

Consequently, the vector field f points strictly inward along the boundary of the positivity set

$$\Pi = \{u \in \mathbb{R}^m : u_1 > 0, ..., u_m > 0\}$$

except at the point 0, which is an equilibrium point for the system (1.22)-(1.23). To ensure that our solutions u^ε do not become unbounded as $\varepsilon \to 0$, we further hypothesize

(H3) there exists a constant Λ such that $f_i(u) \leq 0$ for all $i \in \{1, ..., m\}$, if $u \in \Pi$ and $u_i \geq \Lambda$.

Next we set forth additional hypotheses which imply that the rest point 0 is unstable. Let us define the $m \times m$ matrix

$$C := Df(0)$$

Df denoting the gradient of f. Thus

$$c_{ik} = f_{i,u_k}(0), \qquad \forall 1 \leq i, k \leq m$$

We assume

(H4)

$$c_{ik} > 0, \qquad \forall 1 \leq i, k \leq m$$

and also

(H4)

$$f_i(u) \leq c_{ik} u_k, \qquad u \in \Pi, \ \forall \ 1 \leq i, k \leq m$$

where we employ the standard summation convention.

The main result stated in Theorem 1 [4], asserts that under hypotheses (F1)-(F5), $u^\varepsilon(x,t)$ converges as $\varepsilon \to 0$ to zero or not depending on whether $J(t,x) > 0$ or $J(t,x) < O$, the function J satisfying a Hamilton-Jacobi problem, whose structure we now describe. Given $p \in \mathbb{R}^d$, define the $m \times m$ matrix

$$B(p) = \mathrm{diag}(..., d_i|p|^2, ...)$$

and then set

$$A(p) = B(p) + C$$

Now the matrix $A(p)$ has positive entries, and so Perron-Frobenius theory asserts that $A(p)$ possesses a simple, real eigenvalue $\lambda_1 = \lambda_1(A)(p)$ satisfying

$$Re(\lambda) < \lambda_1$$

for all other eigenvalues λ of $A(p)$. Let us define then the Hamiltonian

$$H(p) = \lambda_1(A(p))$$

We additionally set

$$L(q) = \sup_{p\in\mathbb{R}^d}(p \cdot q - H(p))$$

L is the Lagrangian associated with H. Finally we define for each point $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ the action function

$$J(t,x) = \inf\left\{\int_0^t L(\dot{z}(s))ds \mid z(0) \in G_0, z(t) = x\right\}$$

the infimum taken over all absolutely continuous functions $z : [0,t] \to \mathbb{R}^d$, satisfying the stated initial and terminal conditions. As we will see, J turns out to be the unique solution of the Hamilton-Jacobi equation

$$\begin{array}{ll} J_t + H(DJ) = 0, & \textit{in } \mathbb{R}_+^* \times \mathbb{R}^d \\ J = 0, & \textit{on } G_0 \times \{0\} \\ J = +\infty & \textit{on } \operatorname{int}(\mathbb{R}^d - G_0) \times \{0\} \end{array}$$

in the viscosity sense.

Theorem 1.17 *Under hypotheses (F1)-(F5) we have*

- $\lim_{\varepsilon\to 0} u^\varepsilon = 0$ *uniformly on compact subset of* $\{J > 0\}$
- $\liminf_{\varepsilon\to 0} u_i^\varepsilon > 0$ *uniformly on compact subset of* $\{J < 0\}$, $\forall\, i \in \{1,...,m\}$

We interpret this theorem as describing how the Hamiltonian H, which depends upon both $C = Df(0)$ and the diffusion constants $d_l,...,d_m$, controls the instability of the equilibrium point 0.

1.5. Presentation of main results

This section is devoted to present the principal results that we will present is the following chapters. Hence, in Chapter 2, we consider the problem

$$u_t + (-\Delta)^\alpha u = f(u) \tag{1.24}$$
$$u(0,x) = u_0(x) \tag{1.25}$$

with $\alpha \in (0,1)$ in one spatial dimension. We assume that the nonlinearity in (1.24) is of Fisher-KPP type, that is, $f : [0,1] \to \mathbb{R}$ is of class C^1, concave and it satisfies

$$f(0) = f(1) = 0, \quad f'(1) < 0 < f'(0). \tag{1.26}$$

Also, we assume that the initial condition $u_0 : \mathbb{R} \to [0,1]$ is continuous and it satisfies

$$u_0 > 0 \text{ in } \mathbb{R}, \quad \lim_{x\to-\infty} u_0(x) > 0 \text{ and } \lim_{x\to\infty} u_0(x) = 0, \tag{1.27}$$

furthermore

$$\text{There exists } \xi_0 \in \mathbb{R}, \text{ such that } u_0 \text{ is non-increasing in } [\xi_0, \infty). \tag{1.28}$$

In view of the result stated in the Subsection 1.2 obtained by Hamel and Roques for slowly decaying front-like initial values in the case $\alpha = 1$, a natural question is what kind of asymptotic behavior does a solution of (1.24)-(1.25) with initial value decaying slower than a power $|x|^{-b}$ have?. The purpose of Chapter 2 is to answer this question for the case of fractional laplacian, with $\alpha \in (0,1)$ in the one dimensional case. The main result states that, for $b < 2\alpha$, the central part of the solution moves to the right at exponential speed $f'(0)/b$, which is faster than c_{**} (Theorem 1.12), the exponential speed for solutions with initial values decaying faster than $x^{-2\alpha}$. Furthermore, it is proved that the initial condition u_0 can be chosen so that the location of the solution u be asymptotically larger than any prescribed real-valued function.

The first result provides basic properties of the solutions of (1.24)-(1.25) and says that the level sets $E_\lambda(t)$ move at least exponentially fast as $t \to \infty$.

Theorem 1.18 *Let $\alpha \in (0,1)$ and $c_* = \frac{f'(0)}{2\alpha}$ and let u be the solution of (1.24)-(1.25), where f satisfies (1.26) and the initial condition $u_0 : \mathbb{R} \to [0,1]$ satisfies (1.27) and (1.28). Then u satisfies:*

a) $0 \le u(t,x) \le 1$ *for all* $(t,x) \in (0,\infty) \times \mathbb{R}$. *and*

$$\lim_{x\to+\infty} u(t,x) = 0 \;\; \forall\, t \ge 0 \quad \textit{and} \quad \lim_{t\to+\infty} \inf_{x \le e^{ct}} u(t,x) = 1 \;\; \forall c < c_*$$

b) For any given $\lambda \in (0,1)$, there is a real number $t_\lambda > 1$ such that $E_\lambda(t)$ is compact and non-empty for all $t \ge t_\lambda$.

As in Subsection 1.2, we denote by $E_\lambda(t)$ the level sets of u of value $\lambda \in (0,1)$ at time $t \ge 0$. Hence, the following result is the main theorem which states a more accurate understanding of the behavior of $E_\lambda(t)$, actually we express the motion of $E_\lambda(t)$ in terms of the behavior of the initial value u_0. For doing this we need some additional hypothesis that expresses the slow decay of the initial values:

(H1) There exists $b < 2\alpha$, such that $u_0(x) \ge x^{-b}$ for all $x \ge \xi_0$.

(H2) There exist $\rho > 1$ and $k > 0$ such that

$$\frac{u_0(\rho x)}{u_0(x)} \geq k, \qquad \text{for} \quad x \geq \xi_0.$$

Now we are in a position to state our main theorem.

Theorem 1.19 *Let $\alpha \in (0,1)$, $\lambda \in (0,1)$ and let u be the solution of (1.24)-(1.25), where f satisfies (1.26) and the initial condition $u_0 : \mathbb{R} \to [0,1]$ satisfies (1.27), (1.28) and hypothesis (H1) and (H2).*

Then, for any $\Gamma > 0$, $\gamma > 0$ and $\delta \in (0, f'(0))$, there exist $\tau = \tau(\lambda, \Gamma, \gamma, \delta, b) \geq t_\lambda$ such that

$$E_\lambda(t) \subset u_0^{-1}\{[\gamma e^{-(f'(0)+\delta)t}, \Gamma e^{-(f'(0)-\delta)t}]\}, \ \ \forall t \geq \tau,$$

where t_λ was given in Theorem 1.1.

As a corollary of this theorem, we see that by choosing the initial condition appropriately, we are able to obtain any fast fast behavior of the set $E_\lambda(t)$. In precise terms we have,

Corolary 1.20 *Under the assumptions of Theorem 1.19, given any function $\chi : [0, \infty) \to \mathbb{R}$ which is locally bounded, there are initial conditions u_0 such that, for any given $\lambda \in (0,1)$*

$$\min E_\lambda(t) \geq \chi(t)$$

for all t large enough.

The proof of Theorem 1.19 is inspired in the work by Hamel and Roques [29]. But in this case, the non-local character of the differential operator introduces a series of difficulties that were not present in the local case, also, the non-existence of traveling waves for the fractional problem, as proved in [15], implies to change various other arguments given in [29]. Theorem 1.19 and Corollary 1.20 complement the results by Cabré and Roquejoffre, where they estimate the asymptotic behavior of solutions with front-like initial values which decays faster than $x^{-2\alpha}$ as $x \to \infty$. In our case we assume the initial value decays slower than a power x^{-b}, with $b < 2\alpha$, the complementary exponents. In a sense we generalize to the case $\alpha \in (0,1)$ results proved by Hamel and Roques in [29], replacing the Laplacian by the fractional Laplacian.

In Chapter 3, we study the large time behavior of solution to the fractional cooperative system:

$$\begin{aligned} \partial_t u_i + (-\triangle)^{\alpha_i} u_i &= f_i(u), && \text{in } \mathbb{R}_+^* \times \mathbb{R}^d && (1.29) \\ u_i(0,x) &= u_{0i}(x), && \text{on } \mathbb{R}^d && (1.30) \end{aligned}$$

where $\alpha_i \in (0,1]$ for all $i \in \{1,...,m\}$ with at least one $\alpha_i \neq 1$ and $u = (u_i)_{i=1}^m$ with $m \in \mathbb{N}^*$. For what follows and without loss of generality, we suppose that $\alpha_{i+1} \leq \alpha_i$ for all $i \in \{1,...,m-1\}$ and we set $\alpha := \alpha_m < 1$.

As general assumptions, we impose the initial conditions $u_{0i} \not\equiv 0$ to be nonnegative, continuous and bounded by constants $a_i > 0$ and satisfy

$$u_{0i}(x) = O(|x|^{-(d+2\alpha_i)}) \quad as\ |x| \to \infty, \forall i \in \{1,...,m\} \tag{1.31}$$

The functions f_i satisfy

$$f_i \in C^1(\mathbb{R}^m) \text{ and } \frac{\partial f_i(u)}{\partial u_j} > 0 \ \ \forall i \neq j \tag{1.32}$$

i.e., the system (1.29)-(1.30) is cooperative. Moreover, we assume $f_i(0) = 0$ and some additional hypothesis, which are compatible with strongly coupled systems.

(H1) The principal eigenvalue λ_1 of the matrix $DF(0)$ is strictly positive, where $F = (f_i)_{i=1}^m$.

There exist positive constants δ_1 and δ_2 such that

(H2) $Df_i(0)u - f_i(u) \geq c u_i^{1+\delta_1}$.

(H3) $Df_i(0)u - f_i(u) \leq c\|u\|^{1+\delta_2}$.

(H4) F is concave, $DF(0)$ is a symmetric matrix and $\frac{\partial f_i(0)}{\partial u_i} > 0$ for all $i \in \{1,...,m\}$.

where $\delta_1, \delta_2 \geq \frac{2}{d+2\alpha}$ and $\|\cdot\|$ in (H3) is any norm on $\mathbb{R}^m$.

The purpose of this work is to understand the time asymptotic behavior of solutions to (1.29)-(1.30). We show that the speed of propagation is exponential in time, with a precise exponent depending on the smallest index α and of the principal eigenvalue λ_1 of the matrix $DF(0)$. Also we note that it does not depend on the space direction. Moreover, we prove that the solution of the system (1.29)-(1.30), tends to unique positive steady state solution as $t \to +\infty$.

Moreover, we consider the Banach space

$$C_0(\mathbb{R}^d) := \{w \text{ is continuous in } \mathbb{R}^d \text{ and } w(x) \to 0 \text{ as } |x| \to \infty\}$$

with the $L^\infty(\mathbb{R}^d)$ norm and we set $D_0(A_i)$ the domain of the operator $A_i = (-\triangle)^{\alpha_i}$ in $C_0(\mathbb{R}^d)$. In what follows we assume that $u_{0i} \in D_0(A_i)$ for all $i \in \{1,...,m\}$.

Now we are in a position to state our main theorems, which show that the functions move exponentially fast for large times.

Theorem 1.21 *Let $d \geq 1$ and assume that F satisfies (1.32), (H1), (H2) and (H3). Let u be the solution to (1.29)-(1.30) with u_0 satisfying (1.31). Then:*

a) *for every $\mu_i > 0$, there exists a constant $c > 0$ such that,*

$$u_i(t,x) < \mu_i, \quad for\ all\ t \geq \tau\ and\ |x| > ce^{\frac{\lambda_1}{d+2\alpha}t}$$

b) *there exist constants $\varepsilon_i > 0$ and $C > 0$ such that,*

$$u_i(t,x) > \varepsilon_i, \quad for\ all\ t \geq \tau\ and\ |x| < Ce^{\frac{\lambda_1}{d+2\alpha}t}$$

for all $i \in \{1, ..., m\}$ and $\tau > 0$ large enough.

To state the following result, we consider ϕ the positive constant eigenvector of $DF(0)$ associated to the first eigenvalue λ_1, where $F = (f_i)_{i=1}^m$. Thus $\lambda_1 > 0$ and $\phi > 0$ satisfy

$$(L - DF(0))\phi = -\lambda_1\phi$$
$$\phi > 0, \ \ \|\phi\| = 1$$

where $L = \mathrm{diag}((-\triangle)^{\alpha_1}, ..., (-\triangle)^{\alpha_m})$. Now, let consider the problem

$$\begin{aligned} \dot{\chi}_\varepsilon(t) &= F(\chi_\varepsilon(t)) \\ \chi_\varepsilon(0) &= \varepsilon\phi \end{aligned}$$

thus, there exists $\varepsilon' > 0$ such that, for each $\varepsilon \in (0, \varepsilon')$ we can find a constant $u_\varepsilon^+ > 0$ satisfying $\chi_\varepsilon(t) \nearrow u_\varepsilon^+$ as $t \to +\infty$, also $F(u_\varepsilon^+) = 0$. We define

$$u^+ = \inf_{\varepsilon \in (0,\varepsilon')} u_\varepsilon^+$$

since F is continuous, we deduce that $F(u^+) = 0$. Also, since the function F is positive in a small ball near to zero, we have that $u_+ > 0$.

Moreover we assume that the initial condition u_0 satisfies

$$u_0 \leq u^+ \qquad \text{in } \mathbb{R}^d \tag{1.33}$$

Theorem 1.22 *Let $d \geq 1$ and assume that F satisfies (1.32), (H1), (H2), (H3) and (H4). Let u be the solution to (1.29)-(1.30) with u_0 satisfying (1.31) and (1.33). Then:*

a) *If $c < \frac{\lambda_1}{d+2\alpha}$, then*

$$\lim_{t \to +\infty} \inf_{|x| \leq e^{ct}} |u_i(t,x) - u_i^+| = 0$$

b) *If $c > \frac{\lambda_1}{d+2\alpha}$, then*

$$\lim_{t \to +\infty} \sup_{|x| \geq e^{ct}} u_i(t,x) = 0$$

for all $i \in 1, ..., m$.

Chapter 2

Fast propagation for fractional KPP equations with slowly decaying initial conditions.

Work in colaboration with Patricio Felmer.

2.1. Introduction

In this chapter we study the large-time behavior of the solution of the Cauchy problem for fractional reaction-diffusion equations

$$u_t + (-\Delta)^\alpha u = f(u) \quad (2.1)$$
$$u(0,x) = u_0(x) \quad (2.2)$$

with $\alpha \in (0,1)$ in one spatial dimension. Let us now provide a precise description of our assumptions and results. We assume that the nonlinearity in (2.1) is of Fisher-KPP type, that is, $f : [0,1] \to \mathbb{R}$ is of class C^1, concave and it satisfies

$$f(0) = f(1) = 0, \quad f'(1) < 0 < f'(0). \quad (2.3)$$

This properties mean that the growth rate $\frac{f(s)}{s}$ is maximal at $s = 0$.

We assume that the initial condition $u_0 : \mathbb{R} \to [0,1]$ is continuous and it satisfies

$$u_0 > 0 \ \text{ in } \mathbb{R}, \quad \lim_{x\to-\infty} u_0(x) > 0 \ \text{ and } \ \lim_{x\to\infty} u_0(x) = 0, \quad (2.4)$$

furthermore we assume that:

$$\text{There exists } \ \xi_0 \in \mathbb{R}, \ \text{such that } u_0 \text{ is non-increasing in } [\xi_0, \infty). \tag{2.5}$$

When u_0 satisfies the earlier conditions we say that u_0 is asymptotically front-like.

In what follows, we prove that when the initial condition is globally front-like and it decays slowly, then the asymptotic behavior of the front exhibits an exponentially fast propagation and a very precise estimate can be obtained for the propagation of the level sets of the front in terms of the initial value, giving a precise superlinear behavior.

2.2. Basic properties

We first recall the notion of mild solution that suited to our problem and we state the Comparison Principle that will be a crucial tool in our analysis. Then we present the Theorem 2.2 that follows the line of the corresponding result in [14] and [15].

In studying the existence of solution of equation (2.1)-(2.2) we first consider the linear heat equation for the fractional Laplacian

$$u_t + (-\Delta)^\alpha u = f(t,x) \tag{2.6}$$

$$u(0,x) = u_0(x), \tag{2.7}$$

whose solution may be obtained by the formula of variation of parameters or Duhamel formula

$$u(t,x) = p(t,x) * u_0(x) + \int_0^t p(t-s,x) * f(s,x) ds, \tag{2.8}$$

where the convolution is taken in the variable x. Here the kernel p is given by $p(t,x) = t^{-\frac{1}{2\alpha}} p_\alpha(t^{-\frac{1}{2\alpha}} x)$, where

$$p_\alpha(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi - |\xi|^{2\alpha}} d\xi$$

and it satisfies the following properties:

a) $p \in C((0,+\infty), \mathbb{R})$.

b) $p(t,x) \geq 0$ and $\int_{\mathbb{R}} p(t,x)dx = 1$ for all $t > 0$.

c) $p(t,\cdot) * p(s,\cdot) = p(t+s,\cdot)$ for all $t, s \in \mathbb{R}_+$

d) There exists $B > 1$ such that, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$:

$$\frac{B^{-1}}{t^{\frac{1}{2\alpha}}(1 + \mid xt^{-\frac{1}{2\alpha}} \mid^{1+2\alpha})} \leq p(t,x) \leq \frac{B}{t^{\frac{1}{2\alpha}}(1 + \mid xt^{-\frac{1}{2\alpha}} \mid^{1+2\alpha})}. \tag{2.9}$$

Now we consider the Banach space

$$\mathcal{C}_{lim} = \{w \in C(\mathbb{R}) \mid \lim_{x\to-\infty} w(x) \ \text{ and } \ \lim_{x\to+\infty} w(x) \ exist\},$$

equipped with the supremum norm. Given $u_0 \in \mathcal{C}_{lim}$, equation (2.1)-(2.2), with Fisher-KPP nonlinearities f and initial condition u_0, has a unique solution u that exists for all $x \in \mathbb{R}$ and $t \geq 0$, moreover, $u(t, \cdot) \in C([0, \infty), \mathcal{C}_{lim})$. This solution u can be obtained as the limit of the iteration scheme

$$u^{n+1}(t,x) = p(t,x) * u_0(x) + \int_0^t p(t-s,x) * f(s, u^n(s,x))ds, \tag{2.10}$$

with $u^0(t,x) = p(t,x) * u_0(x)$. The limit is uniform in x and locally in time, see [2] and [15] for details. The solution obtained in this way is called mild solution and in this chapter this will be the notion of solution we consider in all our statements.

To continue we recall the Comparison Principle, which will be frequently used in the following computations.

Theorem 2.1 (Comparison Principle) *Let $u, v \in C([0,T], \mathcal{C}_{lim})$ be mild solutions of the equations*

$$u_t + (-\triangle)^\alpha u = g(u), \quad v_t + (-\triangle)^\alpha v = h(v),$$

where $g, h : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous. If

$$g(\zeta) \leq h(\zeta), \quad \forall \zeta \in \mathbb{R}$$

and

$$u(0,x) \leq v(0,x), \quad \forall x \in \mathbb{R},$$

then

$$u(t,x) \leq v(t,x) \qquad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

For the proof of this result we refer the reader to [15] or [23].

Before to stating our results, we introduce some notation. For any $\lambda \in (0,1)$ and $t \geq 0$, we denote by

$$E_\lambda(t) = \{x \in \mathbb{R} : u(t,x) = \lambda\},$$

the level set of u of value λ at time t. For any subset $A \subset (0,1]$, we set

$$u_0^{-1}(A) = \{x \in \mathbb{R} : u_0(x) \subset A\},$$

the inverse image of A by u_0. Our first result provides basic properties of the solutions of (2.1)-(2.2) and says that the level sets $E_\lambda(t)$ move at least exponentially fast as $t \to \infty$.

Theorem 2.2 *Let $\alpha \in (0,1)$ and $c_* = \frac{f'(0)}{2\alpha}$ and let u be the solution of (2.1)-(2.2), where f satisfies (2.3) and the initial condition $u_0 : \mathbb{R} \to [0,1]$ satisfies (2.4) and (2.5). Then u satisfies:*

a) $0 \leq u(t,x) \leq 1$ for all $(t,x) \in (0,\infty) \times \mathbb{R}$ and

$$\lim_{x\to+\infty} u(t,x) = 0 \ \ \forall\, t \geq 0 \quad \textit{and} \quad \lim_{t\to+\infty} \inf_{x \leq e^{ct}} u(t,x) = 1 \ \ \forall c < c_*$$

b) For any given $\lambda \in (0,1)$, there is a real number $t_\lambda > 1$ such that $E_\lambda(t)$ is compact and non-empty for all $t \geq t_\lambda$.

Proof: Part a) We start by using the Comparison Principle, recalling that $0 \leq u_0(x) \leq 1$, to obtain that the solution u of (2.1)-(2.2) satisfies

$$0 < u(t,x) \leq 1 \quad \forall (t,x) \in [0,\infty) \times \mathbb{R}.$$

Next we analyze the limit of $u(t,x)$ as $x \to +\infty$. To do this, let us first notice that the function

$$\overline{u}(t,x) = e^{f'(0)t} \int_{\mathbb{R}} p(t,x-y) u_0(y) dy$$

is the solution of the equation

$$\begin{aligned} \overline{u}_t + (-\Delta)^\alpha \overline{u} &= f'(0)\overline{u} \\ \overline{u}(0,x) &= u_0(x). \end{aligned}$$

But, since f satisfies (2.3), it is concave and of class C^1, we have that $0 < f(s) \leq f'(0)s$ for all $s \geq 0$, therefore, we conclude that $\overline{u}$ is a supersolution of the equation (2.1)-(2.2), and then, the Comparison Principle implies that $u(t,x) \leq \overline{u}(t,x)$ for all $t \geq 0$ and $x \in \mathbb{R}$. To continue, let us define

$$C_\alpha := \int_{-\infty}^{\infty} \frac{1}{1+\mid s \mid^{1+2\alpha}} ds, \tag{2.11}$$

and notice that $C_\alpha > 1$ for all $\alpha \in (0,1)$. We may assume without loss of generality that $C_\alpha \leq B$, where B is given in (2.9).

We observe that the property is true for $t = 0$ by hypothesis on u_0. For $t > 0$, we consider $\varepsilon > 0$ and we find $M_t > 0$ such that, for each $x \geq M_t$ we have $u(\bar{t},x) < \varepsilon$. Let us start considering $\sigma > 0$ small enough such that $C_\alpha B e^{f'(0)t}\sigma < \frac{\varepsilon}{2}$ and let $\xi_1 \in [\xi_0,\infty)$ and $\xi > 0$ be such that

$$u_0(z) \leq \sigma, \quad \forall z \geq \xi_1 \quad \textit{and} \quad \int_{\xi}^{\infty} \frac{1}{1+s^{1+2\alpha}} ds < \sigma. \tag{2.12}$$

Then let us take

$$M_t := \xi_1 + \xi t^{\frac{1}{2\alpha}}$$

and consider $x \geq M_t$. Then we use the definition of $\bar{u}$ and (2.9) to find that

$$\overline{u}(t,x) \leq B\frac{e^{f'(0)t}}{t^{\frac{1}{2\alpha}}}\int_{-\infty}^{\infty}\frac{u_0(y)}{1+\mid (x-y)t^{-\frac{1}{2\alpha}}\mid^{1+2\alpha}}dy = B\frac{e^{f'(0)t}}{t^{\frac{1}{2\alpha}}}\int_{-\infty}^{\infty}\frac{u_0(x-r)}{1+\mid rt^{-\frac{1}{2\alpha}}\mid^{1+2\alpha}}dr$$

$$= Be^{f'(0)t}\int_{-\infty}^{\infty}\frac{u_0(x-st^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}}ds$$

$$= Be^{f'(0)t}\left(\int_{-\infty}^{\xi}\frac{u_0(x-st^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}}ds + \int_{\xi}^{\infty}\frac{u_0(x-st^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}}ds\right).$$

If $s \leq \xi$ we have $x - st^{\frac{1}{2\alpha}} \geq x - \xi t^{\frac{1}{2\alpha}} \geq \xi_1$. Then, using (2.12) we find

$$\begin{aligned} \bar{u}(t,x) &\leq Be^{f'(0)t}\left(\int_{-\infty}^{\xi}\frac{\sigma}{1+\mid s\mid^{1+2\alpha}}ds + \int_{\xi}^{\infty}\frac{1}{1+\mid s\mid^{1+2\alpha}}ds\right) \\ &\leq Be^{f'(0)t}\left(\sigma C_\alpha + \sigma\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $0 \leq u(t,x) < \varepsilon$. Thus we have proved that $\lim_{x\to\infty} u(t,x) = 0$.

Now we study $u(t,x)$ when $t \to \infty$. From (2.4), we may find a continuous non-increasing function $v_0 : \mathbb{R} \to [0,1]$ in $\mathcal{C}_{lim}$, such that $\mathbb{R}_+ \cap supp(v_0)$ is compact and it satisfies $0 \leq v_0(x) \leq u_0(x)$. Denote by v the solution of the Cauchy problem (2.1)-(2.2) with initial condition v_0, then by the Comparison Principle we find $v(t,x) \leq u(t,x) \leq 1$ for all $t \geq 0$ and $x \in \mathbb{R}$. Then we can use Theorem 1.5 of [15] to conclude that

$$\lim_{t\to\infty}\inf_{x\leq e^{ct}} v(t,x) = 1$$

for all $c < c_*$ and then our result follows.

Part b) From Part a) it follows that, given $d \in (0, c_*)$ and any $\lambda \in (0,1)$ there exists $t_\lambda \geq 1$ such that

$$\inf_{x\leq e^{dt}} u(t,x) > \lambda > 0 = u(t,+\infty),$$

for all $t \geq t_\lambda$. By continuity of $x \mapsto u(t,x)$ we conclude that $E_\lambda(t)$ is a non-empty compact set for all $t \geq t_\lambda$. □

Remark 2.3 As a direct consequence of Theorem 2.2, we see that

$$\lim_{t\to\infty}\frac{E_\lambda(t)}{e^{ct}} = \infty, \quad \text{for all } c < c_*. \tag{2.13}$$

2.3. Main result: Theorem 2.4

It is the purpose of our main theorem to obtain a more accurate understanding of the behavior of $E_\lambda(t)$, actually we express the motion of $E_\lambda(t)$ in terms of the behavior of the initial value u_0 and we improve the estimate for c in (2.13). For doing this we need some additional hypothesis that expresses the slow decay of the initial values:

(H1) There exists $b < 2\alpha$, such that $u_0(x) \geq x^{-b}$ for all $x \geq \xi_0$.

(H2) There exist $\rho > 1$ and $k > 0$ such that

$$\frac{u_0(\rho x)}{u_0(x)} \geq k, \qquad \text{for} \quad x \geq \xi_0.$$

Now we are in a position to state our main theorem.

Theorem 2.4 *Let $\alpha \in (0,1)$, $\lambda \in (0,1)$ and let u be the solution of (2.1)-(2.2), where f satisfies (2.3) and the initial condition $u_0 : \mathbb{R} \to [0,1]$ satisfies (2.4), (2.5) and hypothesis (H1) and (H2).*

Then, for any $\Gamma > 0$, $\gamma > 0$ and $\delta \in (0, f'(0))$, there exist $\tau = \tau(\lambda, \Gamma, \gamma, \delta, b) \geq t_\lambda$ such that

$$E_\lambda(t) \subset u_0^{-1}\{[\gamma e^{-(f'(0)+\delta)t}, \Gamma e^{-(f'(0)-\delta)t}]\}, \;\; \forall t \geq \tau,$$

where t_λ was given in Theorem 2.2.

As a corollary of this theorem, we see that by choosing the initial condition appropriately, we are able to obtain any fast fast behavior of the set $E_\lambda(t)$. In precise terms we have,

Corolary 2.5 *Under the assumptions of Theorem 2.2, given any function $\chi : [0,\infty) \to \mathbb{R}$ which is locally bounded, there are initial conditions u_0 such that, for any given $\lambda \in (0,1)$*

$$\min E_\lambda(t) \geq \chi(t)$$

for all t large enough.

The proof of Theorem 2.4 is inspired in the work by Hamel and Roques [29], by basically making two estimates to capture the set $u_0(E_\lambda(t))$, with appropriate super and sub-solutions. However, the non-local character of the differential operator introduces a series of difficulties that were not present in the local case. This is especially so in the proof of Proposition 3.1, where we have to introduce a staggered sub-solution to gain a global control in time. Moreover, the choice of ω in (2.21) is not obvious and the estimates are quite more involved. It is important to mention that the lower estimate is obtained only assuming that the initial condition u_0 satisfies only (2.4) and (2.5), see Propositions 2.6 and 2.9. Finally, we observe that the non-existence of traveling waves for the fractional problem, as proved in [15], implies to change various other arguments given in [29].

Now we would like to make some comments on hypothesis (H2). This condition complements hypotheses (H1) and it also expresses the slow decay of u_0. Actually, we observe that any power $u_0(x) = x^{-b}$ satisfies also (H2). More generally, any function $u_0 \in C^1([\xi_0, \infty))$, decreasing and convex in $[\xi_0, \infty)$, such that

$$| \frac{u_0'(x)}{u_0(x)} |= O\left(\frac{1}{x}\right) \quad as\ x \to \infty,$$

satisfies the hypothesis (H2).

Theorem 2.4 and Corollary 2.5 complement the results by Cabré and Roquejoffre, where they estimate the asymptotic behavior of solutions with front-like initial values which decays faster than $x^{-2\alpha}$ as $x \to \infty$. In our case we assume the initial value decays slower than a power x^{-b}, with $b < 2\alpha$, the complementary exponents. In a sense we generalize to the case $\alpha \in (0, 1)$ results proved by Hamel and Roques in [29], replacing the Laplacian by the fractional Laplacian.

Let us assume that the initial value is a pure power, that is, $u_0(x) = x^{-b}$, with $b < 2\alpha$, for x large. In this case we see that Theorem 2.4 implies that for all c_1 and c_2 such that

$$c_* = \frac{f'(0)}{2\alpha} < c_1 < \frac{f'(0)}{b} < c_2,$$

there is τ such that for all $x_\lambda(t) \in E_\lambda(t)$ we have

$$e^{c_1 t} \le x_\lambda(t) \le e^{c_2 t}, \qquad \text{for all } t \ge \tau.$$

These observations are in contrast with the results of Cabré and Roquejofre [15], where they showed that all solutions with front-like initial conditions decaying slower that $x^{-2\alpha}$, spread at an exponential speed c_* independent of further properties of u_0. In our case, using comparison principle and the discussion given above, we see that solutions with front-like initial conditions decaying slower than x^{-b}, with $b < 2\alpha$, spread at an exponential speed $f'(0)/b$, which is larger than c_* and depends explicitly on the exponent b.

In this sense, our results show that the exponent 2α is a critical exponent. If the initial value decays faster than $x^{-2\alpha}$ then the exponential speed is c_* and if the initial value decays slower than x^{-b}, with $b < 2\alpha$, then the exponential speed is $f'(0)/b$ or larger. Above the exponent 2α, the solution's speed of propagation starts getting influenced by the initial value, propagating faster the slower the decay is.

2.4. The lower estimate

In order to prove Theorem 2.4 we need to obtain an upper and a lower estimate for the set $E_\lambda(t)$ for t large. In this section we obtain the lower estimate. It is important to notice that in getting the lower estimate we do not require the initial condition satisfies hypothesis (H1) and (H2), but only (2.4) and (2.5).

Proposition 2.6 *Let $\alpha \in (0,1)$ and let u be the solution of (2.1)-(2.2), where f satisfies (2.3) and the initial condition u_0 satisfies (2.4) and (2.5).*

Then, for any $\Gamma > 0$, $\lambda \in (0,1)$ and $\delta \in (0, f'(0))$, there exists a time $\tau_u = \tau_u(\lambda, \Gamma, \delta) \geq t_\lambda$, such that

$$E_\lambda(t) \subset u_0^{-1}\{(0, \Gamma e^{-(f'(0)-\delta)t}]\}, \qquad \forall t \geq \tau_u. \tag{2.14}$$

For proving this proposition, we first prove a lemma where we construct an appropriate sub-solution of (2.1)-(2.2) which will enable us to prove the lower bound for small values of λ. Then we will show that such an estimate can also be done for the remaining values of $\lambda \in (0,1)$.

Let us start setting up some notation. Given $\delta \in (0, f'(0))$ we let d and δ' be such that

$$d \in (1, \frac{f'(0)}{f'(0)-\delta}) \quad \text{and} \quad \delta' = f'(0) - d(f'(0)-\delta).$$

We notice that $\delta' \in (0, f'(0))$, so we may choose ρ such that

$$f'(0) - \delta' < \rho < f'(0).$$

Next we let $s_0 \in (0,1)$ be such that $f(s_0) = \rho s_0$ and we choose $\tau > 0$ such that $\xi_* \in \mathbb{R}$ and $u_0(\xi_*) = e^{-\rho\tau} s_0$ implies $\xi_* \geq \xi_0$ and $\xi_* \geq 0$ and

$$e^{(1-\frac{1}{d})\rho\tau} C_\alpha > 2B. \tag{2.15}$$

Now we state a lemma on the existence of a small sub-solution.

Lemma 2.7 *There is $T > \tau + 1$ and a sequence of continuous functions $\underline{u}_n : [(n-1)T, nT] \to [0, s_0]$, for $n \geq 1$, such that*

$$\begin{aligned}
\underline{u}_1(0,x) &\leq u_0(x) \quad \textit{for all} \quad x \in \mathbb{R}, & (2.16)\\
(\underline{u}_n)_t + (-\Delta)^\alpha \underline{u}_n &= \rho \underline{u}_n \quad \textit{in} \quad ((n-1)T, nT) \times \mathbb{R}, & (2.17)\\
\underline{u}_{n+1}(nT,x) &\leq \underline{u}_n(nT,x) \quad \textit{for all} \quad x \in \mathbb{R}, & (2.18)\\
\lim_{x \to -\infty} \underline{u}_n(nT,x) &= s_0 & (2.19)
\end{aligned}$$

and $\underline{u}_n(t,x)$ is non-increasing in $x \in \mathbb{R}$ for all $t \in [(n-1)T, nT]$.

Proof: Let $\varepsilon > 0$ be such that $\varepsilon < \inf_{(-\infty,\xi_0)} u_0$ and let $\xi \in \mathbb{R}$ be so that $u_0(\xi) = \varepsilon$ and $u_0(x) < \varepsilon$ for all $x > \xi$. By making ε smaller if necessary, we may assume that $\xi > 0$ and we can choose $T > \tau + 1$, such that $\varepsilon = e^{-\rho T} s_0$ and let us define $\underline{u}_0(x) = \inf(u_0(x), \varepsilon)$. We let $\underline{u}_1$ be the solution of the equation

$$\begin{aligned} (\underline{u}_1)_t + (-\Delta)^\alpha \underline{u}_1 &= \rho \underline{u}_1 \\ \underline{u}_1(0,x) &= \underline{u}_0(x). \end{aligned}$$

This solution is given by

$$\underline{u}_1(t,x) = e^{\rho t} \int_{\mathbb{R}} p(t, x-y) \underline{u}_0(y) dy,$$

so that, by the election of ε and T, we have that $\underline{u}_1(t,x) \leq s_0$ for all $(t,x) \in [0,T] \times \mathbb{R}$. Moreover, we have

$$\lim_{x \to -\infty} \underline{u}_1(T,x) = \lim_{x \to -\infty} e^{\rho T} \int_{\mathbb{R}} p(t,z) \underline{u}_0(x-z) dz = e^{\rho T} \varepsilon = s_0.$$

Furthermore, since $\underline{u}_0$ is non-increasing, we see that for $x_1 \leq x_2$, we have

$$\begin{aligned} \underline{u}_1(t,x_1) &= e^{\rho t} \int_{\mathbb{R}} p(t,z) \underline{u}_0(x_1 - z) dz \\ &\geq e^{\rho t} \int_{\mathbb{R}} p(t,z) \underline{u}_0(x_2 - z) dz = \underline{u}_1(t, x_2), \end{aligned}$$

for all $t \in [0,T]$. Thus $\underline{u}_1(t,x)$ is non-increasing in x for all $t \in [0,T]$. Now we perform a recursive process to define $\underline{u}_n$ given $\underline{u}_{n-1}$, for all $n \geq 2$. We let

$$\underline{u}_{0,n-1}(x) = \inf(\underline{u}_{n-1}((n-1)T, x), \varepsilon),$$

where $\underline{u}_{n-1}((n-1)T, \cdot)$ is non-increasing and $\underline{u}_{n-1}((n-1)T, -\infty) = s_0$. Then we define $\underline{u}_n$ as the solution of

$$\begin{aligned} (\underline{u}_n)_t + (-\Delta)^\alpha \underline{u}_n &= \rho \underline{u}_n \\ \underline{u}_n((n-1)T, x) &= \underline{u}_{0,n-1}(x), \end{aligned}$$

for $(t,x) \in [(n-1)T, nT] \times \mathbb{R}$. This solution may be written as

$$\underline{u}_n(t,x) = e^{\rho(t-(n-1)T)} \int_{\mathbb{R}} p(t-(n-1)T, x-y) \underline{u}_{0,n-1}(y) dy,$$

so that, by the election of ε and T, we have that $\underline{u}_n(t,x) \leq s_0$, for all $(t,x) \in [(n-1)T, nT] \times \mathbb{R}$. Moreover, by definition

$$\underline{u}_n((n-1)T, x) = \underline{u}_{0,n-1}(x) \leq \underline{u}_{n-1}((n-1)T, x), \quad \text{for all} \quad x \in \mathbb{R}.$$

We also have

$$\lim_{x\to-\infty} \underline{u}_{0,n-1}(nT,x) = \lim_{x\to-\infty} e^{\rho T}\int_{\mathbb{R}} p(t,z)\underline{u}_{0,n-1}(x-z)dz = e^{\rho T}\varepsilon = s_0$$

and, since $\underline{u}_{n-1}((n-1)T,\cdot)$ is non-increasing, for $x_1 \leq x_2$ we obtain

$$\begin{aligned}\underline{u}_n(t,x_1) &= e^{\rho(t-(n-1)T)}\int_{\mathbb{R}} p(t-(n-1)T,z)\underline{u}_{0,n-1}(x_1-z)dz \\ &\geq e^{\rho(t-(n-1)T)}\int_{\mathbb{R}} p(t-(n-1)T,z)\underline{u}_{0,n-1}(x_2-z)dz = \underline{u}_n(t,x_2).\end{aligned}$$

Thus, $\underline{u}_n$ is non-increasing in $x \in \mathbb{R}$ for all $t \in [(n-1)T, nT]$. □

Remark 2.8 We may define the function $\underline{u} : [0,+\infty) \times \mathbb{R} \to [0, s_0]$ in such a way that, for all integer $n \geq 1$,

$$\underline{u}(t,x) = \underline{u}_n(t,x) \quad \text{for} \quad (t,x) \in [(n-1)T, nT) \times \mathbb{R}.$$

Since for all integer $n \geq 1$ the function u_n satisfies

$$(\underline{u}_n)_t + (-\Delta)^\alpha \underline{u}_n \leq f(\underline{u}_n) \quad \text{in} \quad ((n-1)T, nT) \times \mathbb{R}.$$

we may use the Comparison Principle to find that

$$\underline{u}(t,x) \leq u(t,x) \quad \text{for all} \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{2.20}$$

We finally observe that, from the monotonicity property of the functions $\underline{u}_n$, the function $\underline{u}$ is non-increasing in $x \in \mathbb{R}$ for all $t \geq 0$.

We are now in a position to prove Proposition 2.6.

Proof of Proposition 2.6: Using the notation as in the last proof, we let ω be defined as

$$\omega = \frac{e^{\frac{\rho}{d}\tau}}{B}\int_0^\infty \frac{\underline{u}_0(\xi - s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}}ds. \tag{2.21}$$

We observe that ω does not depend on λ nor Γ and that $0 < \omega < s_0$. In order to see this last fact we recall that $\tau < T$, $d > 1$ and $C_\alpha < 2B$, where C_α was defined in (2.11) and B is given in (2.9). Then

$$\omega = \frac{e^{\frac{\rho}{d}\tau}}{B}\int_0^\infty \frac{\underline{u}_0(\xi - s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}}ds \leq \frac{e^{\frac{\rho}{d}\tau}C_\alpha}{2B}\varepsilon < \frac{C_\alpha}{2B}s_0 < s_0. \tag{2.22}$$

Next, for each $t \in [\tau, \infty)$, we consider the equation for $y \in [\xi, \infty)$

$$\frac{e^{\frac{\rho}{d}t}}{B}\int_0^\infty \frac{\underline{u}_0(y - s\tau^{\frac{1}{2\alpha}})}{1+ \mid s \mid^{1+2\alpha}}ds = \omega. \tag{2.23}$$

The function $G : [\xi, \infty) \to \mathbb{R}$ given by

$$G(y) = \int_0^\infty \frac{\underline{u}_0(y - s\tau^{\frac{1}{2\alpha}})}{1+ \mid s \mid^{1+2\alpha}}ds \quad \forall y \in [\xi, \infty)$$

is clearly continuous and non-increasing, since $\underline{u}_0$ is continuous and non-increasing. Moreover, by definition of ξ we see that G is decreasing in $[\xi, \infty)$. Consequently, for every $t \in [\tau, \infty)$, equation (2.23) has a unique solution that we call $y_\omega(t)$, defining a continuous function $y_\omega : [\tau, \infty) \to [\xi, \infty)$. We see that y_ω satisfies $y_\omega(\tau) = \xi$ and it is increasing.

Now we consider the open set Ω defined by

$$\Omega = \{(t, x) \in (\tau, \infty) \times \mathbb{R} \,|\, x < y_\omega(t)\}$$

and we claim that $\inf_{\overline{\Omega}} u > 0$. To prove the claim, we first look at $\partial\Omega$ that consists of two parts: the set of all points (t, x) for which $t \in (\tau, \infty)$ and $x = y_\omega(t)$ and the set $\{\tau\} \times (-\infty, y_\omega(\tau)]$.

i) In the first case, when $t \in (\tau, \infty)$ and $x = y_\omega(t)$, there exists $n \in \mathbb{N}$ such that $t \in [(n-1)T, nT)$. Since $\underline{u}_{0,n-1}(x)$ in Lemma 2.7 is non-increasing we have that

$$\begin{aligned}
\underline{u}(t, x) &= u_n(t, x) = e^{\rho(t-(n-1)T)}\int_{\mathbb{R}} p(t - (n-1)T, x - y)\underline{u}_{0,n-1}(y)dy \\
&\geq e^{\rho(t-(n-1)T)}\int_0^\infty p(t - (n-1)T, z)\underline{u}_{0,n-1}(x - z)dz \\
&\geq \frac{C_\alpha e^{\rho(t-(n-1)T)}}{2B}\underline{u}_{0,n-1}(x).
\end{aligned}$$

In case that $\underline{u}_{0,n-1}(x) = \varepsilon$, we conclude that

$$\underline{u}(t, x) \geq \frac{C_\alpha e^{\rho(t-(n-1)T)}}{2B}\varepsilon \geq \frac{C_\alpha \varepsilon}{2B}.$$

Otherwise, we have that $\underline{u}_{0,n-1}(x) = \underline{u}_{n-1}((n-1)T, x)$ and then, as before we obtain that

$$\begin{aligned}
\underline{u}(t, x) &\geq \frac{C_\alpha e^{\rho(t-(n-1)T)}}{2B}\underline{u}_{n-1}((n-1)T, x) \\
&\geq \frac{C_\alpha e^{\rho(t-(n-2)T)}}{2B}\int_0^\infty p(T, z)\underline{u}_{0,n-2}(x - z)dz \\
&\geq \left(\frac{C_\alpha}{2B}\right)^2 e^{\rho(t-(n-2)T)}\underline{u}_{0,n-2}(x).
\end{aligned}$$

Again, we have two cases. If $\underline{u}_{0,n-2}(x) = \varepsilon$, then we conclude that

$$\begin{aligned} \underline{u}(t,x) &\geq \left(\frac{C_\alpha}{2B}\right)^2 e^{\rho(t-(n-2)T)}\varepsilon \geq \left(\frac{C_\alpha}{2B}\right)^2 e^{\rho T}\varepsilon \\ &\geq \left(\frac{C_\alpha e^{(1-\frac{1}{d})\rho\tau}}{2B}\right)\left(\frac{C_\alpha\varepsilon}{2B}\right) \geq \frac{C_\alpha\varepsilon}{2B}, \end{aligned} \tag{2.24}$$

where we have used (2.15). Otherwise, we have that $\underline{u}_{0,n-2}(x) = \underline{u}_{n-2}((n-2)T, x)$ and then as before we have

$$\underline{u}(t,x) \geq \left(\frac{C_\alpha}{2B}\right)^2 e^{\rho(t-(n-2)T)}\underline{u}_{n-2}((n-2)T,x).$$

Repeating this procedure, we will either reach

$$\underline{u}(t,x) \geq \frac{C_\alpha\varepsilon}{2B}$$

as in (2.24), or we would have that x satisfies $\underline{u}_{0,m}(x) \neq \varepsilon$ for all $m \in \{1,2,3,...,n-1\}$. In the later case we have that $\underline{u}_{0,1}(x) = \underline{u}_1(T,x)$ and then

$$\begin{aligned} \underline{u}(t,x) &\geq \left(\frac{C_\alpha}{2B}\right)^{n-1} e^{\rho(t-T)}\underline{u}_1(T,x) \\ &\geq \left(\frac{C_\alpha}{2B}\right)^{n-1} \frac{e^{\rho t}}{B}\int_0^\infty \frac{\underline{u}_0(x - sT^{\frac{1}{2\alpha}})}{1+ \mid s \mid^{1+2\alpha}} ds \\ &\geq \left(\frac{C_\alpha}{2B}\right)^{n-1} e^{(1-\frac{1}{d})\rho(n-1)\tau}\frac{e^{\frac{\rho}{d}t}}{B}\int_0^\infty \frac{\underline{u}_0(y_\omega(t) - s\tau^{\frac{1}{2\alpha}})}{1+ \mid s \mid^{1+2\alpha}} ds \\ &= \left(\frac{C_\alpha e^{(1-\frac{1}{d})\rho\tau)}}{2B}\right)^{n-1} \omega > \omega. \end{aligned}$$

Summarizing, we have obtained that

$$\underline{u}(t, y_\omega(t)) \geq \min(\frac{C_\alpha\varepsilon}{2B}, \omega) \quad \text{for all} \quad t \geq \tau.$$

ii) In the second case, that is when $t = \tau$ and $x \in (-\infty, y_\omega(\tau)]$, we have that $x - s\tau^{\frac{1}{2\alpha}} \leq y_\omega(\tau) - s\tau^{\frac{1}{2\alpha}} = \xi - s\tau^{\frac{1}{2\alpha}}$, hence

$$\begin{aligned} \underline{u}(\tau,x) = \underline{u}_1(\tau,x) &\geq \frac{e^{\rho\tau}}{B}\int_{-\infty}^\infty \frac{\underline{u}_0(x - s\tau^{\frac{1}{2\alpha}})}{1+ \mid s \mid^{1+2\alpha}} ds \\ &\geq \frac{e^{\frac{\rho}{d}\tau}}{B}\int_0^\infty \frac{\underline{u}_0(\xi - s\tau^{\frac{1}{2\alpha}})}{1+ \mid s \mid^{1+2\alpha}} ds = \omega > 0. \end{aligned}$$

This completes the analysis on the boundary of Ω. To complete the proof we consider $(t,x) \in \Omega$, that is, $t > \tau$ and $x < y_\omega(t)$. Since $\underline{u}(t,\cdot)$ is non-increasing for each $t \geq \tau$, from i) we deduce that

$$\underline{u}(t,x) \geq \underline{u}(t, y_\omega(t)) \geq \min(\frac{C_\alpha \varepsilon}{2B}, \omega).$$

Thus, we have found $\theta > 0$ such that

$$u(t,x) \geq \underline{u}(t,x) \geq \theta, \quad \forall (t,x) \in \overline{\Omega} \tag{2.25}$$

Now we can get the upper estimate for $\lambda \in (0,\theta)$. Let $x \in E_\lambda(t)$ for $t \geq \max(\tau, t_\lambda)$, then we have

$$x > y_\omega(t) \geq \xi. \tag{2.26}$$

In fact, let us assume that $x \leq y_\omega(t)$ then $(t,x) \in \bar{\Omega}$ and, by our estimate above, we have that $u(t,x) \geq \theta$. On the other hand, by definition of $E_\lambda(t)$ we have $u(t,x) = \lambda$. Since $\lambda < \theta$, we obtain a contradiction.

Thus, from (2.26) we have that, for all $t \geq \max(\tau, t_\lambda)$ and $x \in E_\lambda(t)$

$$\begin{aligned} B\omega e^{-\frac{\rho}{d}t} &= \int_0^\infty \frac{\underline{u}_0(y_\omega(t) - s\tau^{\frac{1}{2\alpha}})}{1+ \mid s \mid^{1+2\alpha}} ds \geq \int_0^\infty \frac{\underline{u}_0(x - s\tau^{\frac{1}{2\alpha}})}{1+ \mid s \mid^{1+2\alpha}} ds \\ &\geq \int_0^\infty \frac{\underline{u}_0(x)}{1+ \mid s \mid^{1+2\alpha}} ds = \frac{C_\alpha}{2} \underline{u}_0(x) = \frac{C_\alpha}{2} u_0(x), \end{aligned} \tag{2.27}$$

where the last equality holds since $x > \xi$. From (2.27) and since $\Gamma > 0$ and $\rho > f'(0) - \delta'$, there exists $\tau_1(\lambda, \Gamma, \delta) \geq \max(\tau, t_\lambda)$ such that for all $t \geq \tau_1(\lambda, \Gamma, \delta)$ and $x \in E_\lambda(t)$,

$$u_0(x) \leq \frac{2B}{C_\alpha} \omega e^{-\frac{\rho}{d}t} \leq s_0 e^{-\frac{\rho}{d}t} \leq \Gamma e^{-\frac{f'(0)-\delta'}{d}t}.$$

Here we used (2.22). But, by definition of δ' we have $\frac{f'(0)-\delta'}{d} = f'(0) - \delta$, then we conclude that for all $t \geq \tau_1(\lambda, \Gamma, \delta)$ and $x \in E_\lambda(t)$

$$u_0(x) \leq \Gamma e^{-(f'(0)-\delta)t}.$$

In order to complete the proof of the proposition, let us now consider $\lambda \in [\theta, 1)$. Let $\underline{u}_{\theta,0}$ be the function defined by

$$\underline{u}_{\theta,0}(z) = \begin{cases} \theta & if\ z \leq 0 \\ \theta(1-z) & if\ 0 < z < 1 \\ 0 & if\ z \geq 1 \end{cases}$$

and denote by $\underline{u}_\theta$ the solution of the Cauchy problem (2.1)-(2.2) with initial condition $\underline{u}_{\theta,0}$. It follows from (2.25) that

$$u(s,x) \geq \underline{u}_{\theta,0}(x - y_\omega(s) + 1) \quad \text{for all} \quad (s,x) \in [\tau, \infty) \times \mathbb{R},$$

and then, using the Comparison Principle, we obtain

$$u(s+t,x) \geq \underline{u}_\theta(t, x - y_\omega(s) + 1), \quad \text{for all} \quad (s,x) \in [\tau,\infty) \times \mathbb{R} \text{ and } t \geq 0.$$

Now we consider $0 < c < c_* = \frac{f'(0)}{2\alpha}$ and we use Theorem 1.5 of [15] to find $T_\lambda > 0$ such that

$$\underline{u}_\theta(T_\lambda, z) > \lambda \quad \text{for all} \quad z \leq e^{cT_\lambda}.$$

We observe that T_λ may depend on θ, and thus on ε, but does not depend on s. Directly from the last two inequalities we get

$$u(s + T_\lambda, x) > \lambda, \quad \text{for all} \quad s \in [\tau, \infty) \text{ and } x \leq e^{cT_\lambda} + y_\omega(s) - 1$$

As a consequence we have that for all $t \geq \max(\tau + T_\lambda, t_\lambda)$ and $x \in E_\lambda(t)$ we obtain $x - y_\omega(t - T_\lambda) + 1 > e^{cT_\lambda}$. In fact, if $x - y_\omega(t - T_\lambda) + 1 \leq e^{cT_\lambda}$, using that $\tau \leq t - T_\lambda$, we see that $\lambda = u(t,x) = u((t - T_\lambda) + T_\lambda, x) > \lambda$, which is a contradiction. Thus, for such t and x, we have $x - y_\omega(t - T_\lambda) > e^{cT_\lambda} - 1 > 0$ and hence $x > y_\omega(t - T_\lambda)$. As a consequence,

$$\begin{aligned} B\omega e^{-\frac{\rho}{d}(t-T_\lambda)} &= \int_0^\infty \frac{\underline{u}_0(y_\omega(t-T_\lambda) - s\tau^{\frac{1}{2\alpha}})}{1+ \mid s \mid^{1+2\alpha}} ds \geq \int_0^\infty \frac{\underline{u}_0(x - s\tau^{\frac{1}{2\alpha}})}{1+ \mid s \mid^{1+2\alpha}} ds \\ &\geq \int_0^\infty \frac{\underline{u}_0(x)}{1+ \mid s \mid^{1+2\alpha}} ds = \frac{C_\alpha}{2} \underline{u}_0(x) = \frac{C_\alpha}{2} u_0(x). \end{aligned}$$

Here, the last equality is satisfied because $x > \xi$. Now we conclude as in the other case, since $\Gamma > 0$ and $\rho > f'(0) - \delta'$, there exist $\tau_2(\lambda, \Gamma, \delta) \geq \max(\tau + T_\lambda, t_\lambda)$, such that for all $t \geq \tau_2(\lambda, \Gamma, \delta)$ and $x \in E_\lambda(t)$

$$u_0(x) \leq \frac{2B}{C_\alpha} \omega e^{-\frac{\rho}{d}(t-T_\lambda)} < \Gamma e^{-\frac{f'(0)-\delta'}{d}t} = \Gamma e^{-(f'(0)-\delta)t}.$$

We complete the proof of the proposition, choosing

$$\tau_u(\lambda, \Gamma, \delta) = \max(\tau_1(\lambda, \Gamma, \delta), \tau_2(\lambda, \Gamma, \delta)). \qquad \square$$

The proof of Corollary 2.5 follows from Proposition 2.6, finding a suitable initial condition that satisfies (2.4) and (2.5). This proof follow closely the ideas by Hamel and Roques in [29].

Proof of Corollary 2.5: Let $\chi : [0, \infty) \to \mathbb{R}$ be any locally bounded function, then there exists a continuous, increasing function $g : [0, \infty) \to \mathbb{R}$ such that

$$g(z) \geq \chi(2z), \quad \forall z \geq 0.$$

Denoting by $g^{-1} : [g(0), \infty) \to [0, \infty)$ the inverse of g, we define $u_0 : \mathbb{R} \to (0, 1]$ as

$$u_0(x) = e^{-f'(0)g^{-1}(x)} \qquad \forall x \geq g(0)$$

and extended by one, to the left of $g(0)$. We easily see that $u_0 \in \mathcal{C}_{lim}$ is decreasing and that it satisfies (2.4) and (2.5).

Let u be the solution of Cauchy problem (2.1)-(2.2) with initial condition u_0 and let $\lambda \in (0, 1)$ and $\delta \in (0, \frac{f'(0)}{2})$. Moreover, let us consider $\tau_1 > 0$ large so that

$$e^{-(f'(0)-\delta)\tau_1} \leq u_0(g(0)). \tag{2.28}$$

It follows from Proposition 2.6 with $\Gamma = 1$, that there exists $\tau_2 \geq \max(\tau_1, t_\lambda)$ such that

$$y \geq u_0^{-1}(e^{-(f'(0)-\delta)t}), \qquad \forall t \geq \tau_2, \forall y \in E_\lambda(t)$$

Therefore, from (2.28) we conclude that, for each $t \geq \tau_2$

$$\min E_\lambda(t) \geq g\left(\frac{f'(0)-\delta}{f'(0)}t\right) \geq g(\frac{t}{2}) \geq \chi(t). \qquad \square$$

2.5. The upper estimate and proof of Theorem 2.4

In this section we prove the upper bound for the set $E_\lambda(t)$ and we complete the proof of Theorem 1.2. The proof of the upper bound is obtained by constructing an appropriate super-solution of (2.1)-(2.2). The construction of such super-solution strongly relies on the hypotheses (H1) and (H2). Precisely we prove

Proposition 2.9 *Let $\alpha \in (0, 1)$ and let u be the solution of (2.1)-(2.2), where f satisfies (2.3) and the initial condition u_0 satisfies (2.4), (2.5), (H1) and (H2).*

Then, for any $\gamma > 0$, $\lambda \in (0, 1)$ and $\delta \in (0, f'(0))$, there exists a time $\tau_\ell = \tau_\ell(\lambda, \gamma, \delta, b) \geq t_\lambda$ such that

$$E_\lambda(t) \subset u_0^{-1}\{[\gamma e^{-(f'(0)+\delta)t}, 1]\}, \ \ \forall t \geq \tau_\ell.$$

Proof. Let $\overline{u}$ be the solution of the problem

$$\begin{aligned} \overline{u}_t + (-\Delta)^\alpha \overline{u} &= f'(0)\overline{u} \\ \overline{u}(0, x) &= u_0(x), \end{aligned}$$

that can be expressed as

$$\overline{u}(t, x) = e^{f'(0)t} \int_{\mathbb{R}} p(t, x-y) u_0(y) dy. \tag{2.29}$$

By the assumptions on f we see that $\overline{u}$ is a super-solution for (2.1)-(2.2) and then, the Comparison Principle implies that

$$0 < u(t,x) \leq \overline{u}(t,x), \qquad \forall (t,x) \in [0,\infty) \times \mathbb{R}.$$

Since $b < 2\alpha$, there exists $a > 0$ small enough, such that $a + b \leq 2\alpha$. Let $\lambda \in (0,1)$ and $\delta \in (0, f'(0))$, then there exists $\tau_1 = \tau_1(\lambda, \delta, b) \geq t_\lambda > 1$, such that for all $t \geq \tau_1$ we have

$$\frac{\lambda}{8B^2} e^{-f'(0)t} < \inf_{x \in (-\infty, \xi_0)} \{u_0(x), \xi_0^{-b}\}, \qquad \frac{8B^2}{\lambda} < e^{\frac{\delta}{2}t} \tag{2.30}$$

and

$$\left(\frac{8B^2}{\lambda}\right)^{\frac{a}{2\alpha b}} e^{\frac{af'(0)}{2\alpha b}t} > \frac{1}{\rho - 1} \left(\frac{t}{\alpha B}\right)^{\frac{1}{2\alpha}}. \tag{2.31}$$

Here B is the constant appearing in (2.9) and ρ is given in (H2). Let $t \geq \tau_1$ and let us denote by $\varepsilon > 0$ the number such that

$$4B^2 e^{f'(0)t} \varepsilon = \frac{\lambda}{2}. \tag{2.32}$$

By (2.30) it is possible to find $\xi_1 \in [\xi_0, \infty)$, such that $u_0(\xi_1) = \varepsilon$, we assume that ξ_1 is the largest number with that property. Then, let us choose $\xi > 0$ big enough, such that

$$\int_\xi^\infty \frac{1}{s^{1+2\alpha}} ds = \frac{B\varepsilon}{2}$$

and we define $\xi_t := \xi_1 + \xi t^{\frac{1}{2\alpha}}$. For $x \geq \xi_t$ we estimate the values of $\overline{u}(t,x)$. From (2.29) we have that

$$\overline{u}(t,x) \leq B \frac{e^{f'(0)t}}{t^{\frac{1}{2\alpha}}} \int_{-\infty}^{\infty} \frac{u_0(y)}{1 + \mid (x-y) t^{-\frac{1}{2\alpha}} \mid^{1+2\alpha}} dy \leq B e^{f'(0)t} (I1 + I2)$$

where

$$I1 = \frac{1}{t^{\frac{1}{2\alpha}}} \int_{-\infty}^{\infty} \frac{u_0(x)}{1 + \mid (x-y) t^{-\frac{1}{2\alpha}} \mid^{1+2\alpha}} dy$$

and

$$I2 = \frac{1}{t^{\frac{1}{2\alpha}}} \int_{-\infty}^{\infty} \frac{\mid u_0(y) - u_0(x) \mid}{1 + \mid (x-y) t^{-\frac{1}{2\alpha}} \mid^{1+2\alpha}} dy.$$

Calculating the integrals separately, we have

$$I1 = u_0(x) \int_{-\infty}^{\infty} \frac{1}{1 + \mid r \mid^{1+2\alpha}} dr = C_\alpha u_0(x) \leq B u_0(x).$$

For $I2$ we recall the definition of ξ_1, ξ, ε and we notice that $x - st^{\frac{1}{2\alpha}} \geq \xi_t - \xi t^{\frac{1}{2\alpha}} = \xi_1$ for all $s \in (-\infty, \xi]$. With a change of variables we have then

$$\begin{aligned} I2 &= \int_{-\infty}^{\xi} \frac{\mid u_0(x - st^{\frac{1}{2\alpha}}) - u_0(x) \mid}{1 + \mid s \mid^{1+2\alpha}} ds + \int_{\xi}^{\infty} \frac{\mid u_0(x - st^{\frac{1}{2\alpha}}) - u_0(x) \mid}{1 + \mid s \mid^{1+2\alpha}} ds \\ &\leq 2\varepsilon \int_{-\infty}^{\xi} \frac{1}{1 + \mid s \mid^{1+2\alpha}} ds + 2 \int_{\xi}^{\infty} \frac{1}{1 + \mid s \mid^{1+2\alpha}} ds \\ &< 2\varepsilon B + \varepsilon B = 3\varepsilon B. \end{aligned}$$

Therefore,

$$u(t,x) < B^2 e^{f'(0)t}(u_0(x) + 3\varepsilon) \quad \forall x \geq \xi_t.$$

From here we obtain that $y < \xi_t$ for all $y \in E_\lambda(t)$. In fact, if $y \geq \xi_t$ and $y \in E_\lambda(t)$, then $y \geq \xi_t > \xi_1$ and

$$\lambda = u(t,y) < B^2 e^{f'(0)t}(u_0(y) + 3\varepsilon) \leq B^2 e^{f'(0)t}(u_0(\xi_1) + 3\varepsilon) = B^2 e^{f'(0)t} 4\varepsilon = \frac{\lambda}{2},$$

which is a contradiction. Since $u_0(\xi_1) = \varepsilon$, from (H1), (2.31) and (2.32) we see that

$$\xi_1^{\frac{a}{2\alpha}} \geq \varepsilon^{-\frac{a}{2\alpha b}} = \left(\frac{8B^2}{\lambda}\right)^{\frac{a}{2\alpha b}} e^{\frac{af'(0)}{2\alpha b}t} \geq \frac{1}{(\rho - 1)} \left(\frac{t}{\alpha B}\right)^{\frac{1}{2\alpha}}.$$

From here, since $\xi_1^{-b} \leq u_0(\xi_1) = \varepsilon$ and by the choice of τ_1 and ξ, we conclude that

$$\begin{aligned} \xi_t &= \xi_1 + \xi t^{\frac{1}{2\alpha}} = \xi_1 + \left(\frac{t}{\alpha B}\right)^{\frac{1}{2\alpha}} (u_0(\xi_1))^{-\frac{1}{2\alpha}} \\ &\leq \xi_1 + \left(\frac{1}{\alpha B}\right)^{\frac{1}{2\alpha}} t^{\frac{1}{2\alpha}} \xi_1^{\frac{b}{2\alpha}} \leq \xi_1 + (\rho - 1)\xi_1^{\frac{a}{2\alpha}} \xi_1^{\frac{b}{2\alpha}} \leq \rho \xi_1. \end{aligned}$$

Now, if ξ_2 is such that $u_0(\xi_2) = e^{-(f'(0)+\frac{\delta}{2})t}$, then since $e^{-(f'(0)+\frac{\delta}{2})t} < \varepsilon$ by (2.30) and (2.32), we have that $\xi_1 < \xi_2$. Therefore, for each $y \in E_\lambda(t)$

$$u_0(y) \geq u_0(\xi_t) \geq u_0(\rho \xi_2) \geq k u_0(\xi_2) = k e^{-(f'(0)+\frac{\delta}{2})t}.$$

Finally, making $\tau_\ell = \tau_\ell(\lambda, \gamma, \delta, b) \geq \tau_1(\lambda, \delta, b)$ larger if necessary, we may assume that if $t \geq \tau_\ell$ then $e^{\frac{\delta}{2}t} \geq \frac{\gamma}{k}$ and we find that

$$u_0(y) \in \{[\gamma e^{-(f'(0)+\delta)t}, 1]\}, \;\; \forall t \geq \tau_\ell. \qquad \square$$

Thanks to Proposition 2.6 and 2.9, we can prove Theorem 2.4 on the behavior of level sets for large times, expressed in terms of the decay of the initial condition.

Proof of Theorem 2.4: It follows directly from Proposition 2.6 and 2.9, taking

$$\tau = \tau(\lambda, \Gamma, \gamma, \delta, b) = \max(\tau_u, \tau_\ell). \qquad \square$$

Chapter 3

Exponential propagation for fractional reaction-diffusion cooperative systems

Work in colaboration with Anne-Charline Coulon.

3.1. Introduction

In this chapter, we are interested in the large time behavior of solution $u = (u_i)_{i=1}^m$ with $m \in \mathbb{N}^*$, to the fractional reaction diffusion system:

$$\partial_t u_i(t,x) + (-\triangle)^{\alpha_i} u_i(t,x) = f_i(u(t,x)), \qquad \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d \tag{3.1}$$

$$u_i(0,x) = u_{0i}(x), \qquad \forall x \in \mathbb{R}^d \tag{3.2}$$

where $\alpha_i \in (0,1]$ for all $i \in \{1,...,m\}$ with at least one $\alpha_i \neq 1$. As general assumptions, we impose the initial conditions $u_{0i} \not\equiv 0$ to be nonnegative, continuous and bounded by constants $a_i > 0$ and satisfy

$$u_{0i}(x) = O(|x|^{-(d+2\alpha_i)}) \quad as\ |x| \to \infty, \forall i \in \{1,...,m\} \tag{3.3}$$

The functions f_i satisfy

$$f_i \in C^1(\mathbb{R}^m) \text{ and } \frac{\partial f_i(u)}{\partial u_j} > 0 \;\; \forall i \neq j \tag{3.4}$$

i.e., the system (3.1)-(3.2) is cooperative. Moreover, we assume $f_i(0) = 0$.

The aim of this chapter is to understand the time asymptotic location of the level sets of solutions to (3.1)-(3.2). We show that the speed of propagation is exponential in time, with a precise exponent depending on the smallest index $\alpha := \min_i(\alpha_i)$ and of the principal eigenvalue of the matrix $DF(0)$ where $F = (f_i)_{i=1}^m$. Also we note that it does not depend on the space direction. Moreover, we prove that the solution $u(t, x)$ of (3.1)-(3.2) tends to the smallest positive constant steady state solution, as $t \to +\infty$.

For what follows and without loss of generality, we suppose that $\alpha_{i+1} \leq \alpha_i$ for all $i \in \{1, ..., m-1\}$ and we set $\alpha := \alpha_m < 1$. Before to state the main results, we need some additional hypothesis on the nonlinearities f_i, which allow us to identify precisely the propagation speed of solutions, also note that this hypothesis are compatible with strongly coupled systems.

(H1) The principal eigenvalue λ_1 of the matrix $DF(0)$ is strictly positive.

There exist positive constants δ_1 and δ_2 such that

(H2) $Df_i(0)u - f_i(u) \geq c u_i^{1+\delta_1}$.

(H3) $Df_i(0)u - f_i(u) \leq c\|u\|^{1+\delta_2}$.

(H4) F is concave, $DF(0)$ is a symmetric matrix and $\frac{\partial f_i(0)}{\partial u_i} > 0$ for all $i \in \{1, ..., m\}$.

where $\delta_1, \delta_2 \geq \frac{2}{d+2\alpha}$ and $\|\cdot\|$ in (H3) is any norm on $\mathbb{R}^m$.

Before going further on, let us state at least one example of nonlinearity F satisfying (3.4), (H1), (H2), (H3) and (H4). Let consider

$$F(u) = \begin{pmatrix} 1 - u_1^\delta & 2 \\ 2 & 1 - u_2^\delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \text{with } u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Indeed, F satisfies (3.4), $F(0) = 0$, the principal eigenvalue of $DF(0)$ is $\lambda_1 = 3$ and satisfies (H2) and (H3) with $\delta_1 = \delta_2 = \delta$, also F is concave and $DF(0)$ is symmetric.

Moreover, we consider the Banach space

$$C_0(\mathbb{R}^d) := \{w \text{ is continuous in } \mathbb{R}^d \text{ and } w(x) \to 0 \text{ as } |x| \to \infty\}$$

with the $L^\infty(\mathbb{R}^d)$ norm and we set $D_0(A_i)$ the domain of the operator $A_i = (-\triangle)^{\alpha_i}$ in $C_0(\mathbb{R}^d)$. In what follows we assume that $u_{0i} \in D_0(A_i)$ for all $i \in \{1, ..., m\}$.

Now we are in a position to state our main theorems, which show that the functions move exponentially fast for large times.

Theorem 3.1 *Let $d \geq 1$ and assume that F satisfies (3.4), (H1), (H2) and (H3). Let u be the solution to (3.1)-(3.2) with u_0 satisfying (3.3). Then:*

a) *for every $\mu_i > 0$, there exists a constant $c > 0$ such that,*

$$u_i(t,x) < \mu_i, \quad for\ all\ t \geq \tau\ and\ |x| > ce^{\frac{\lambda_1}{d+2\alpha}t}$$

b) *there exist constants $\varepsilon_i > 0$ and $C > 0$ such that,*

$$u_i(t,x) > \varepsilon_i, \quad for\ all\ t \geq \tau\ and\ |x| < Ce^{\frac{\lambda_1}{d+2\alpha}t}$$

for all $i \in \{1,...,m\}$ and $\tau > 0$ large enough.

To state the following result, we consider ϕ the positive constant eigenvector of $DF(0)$ associated to the first eigenvalue λ_1, where $F = (f_i)_{i=1}^m$. Thus $\lambda_1 > 0$ and $\phi > 0$ satisfy

$$\begin{gathered}(L - DF(0))\phi = -\lambda_1\phi \\ \phi > 0, \ \ \|\phi\| = 1\end{gathered}$$

where $L = \mathrm{diag}((-\triangle)^{\alpha_1},...,(-\triangle)^{\alpha_m})$. Now, let consider the problem

$$\begin{aligned}\dot{\chi}_\varepsilon(t) &= F(\chi_\varepsilon(t)) \\ \chi_\varepsilon(0) &= \varepsilon\phi\end{aligned}$$

thus, there exists $\varepsilon' > 0$ such that, for each $\varepsilon \in (0,\varepsilon')$ we can find a constant $u_\varepsilon^+ > 0$ satisfying $\chi_\varepsilon(t) \nearrow u_\varepsilon^+$ as $t \to +\infty$, also $F(u_\varepsilon^+) = 0$. We define

$$u^+ = \inf_{\varepsilon \in (0,\varepsilon')} u_\varepsilon^+$$

since F is continuous, we deduce that $F(u^+) = 0$. Also, since the function F is positive in a small ball near to zero, we have that $u^+ > 0$.

Moreover we assume that the initial condition u_0 satisfies

$$u_0 \leq u^+ \qquad \text{in } \mathbb{R}^d \tag{3.5}$$

Theorem 3.2 *Let $d \geq 1$ and assume that F satisfies (3.4), (H1), (H2), (H3) and (H4). Let u be the solution to (3.1)-(3.2) with u_0 satisfying (3.3) and (3.5). Then:*

a) *If* $c < \frac{\lambda_1}{d+2\alpha}$, *then*

$$\lim_{t\to+\infty} \inf_{|x|\le e^{ct}} |u_i(t,x) - u_i^+| = 0$$

b) *If* $c > \frac{\lambda_1}{d+2\alpha}$, *then*

$$\lim_{t\to+\infty} \sup_{|x|\ge e^{ct}} u_i(t,x) = 0$$

for all $i \in 1, ..., m$.

The plan to set the Theorems 3.1 and 1.22 is organized as follows. First, we present some preliminaries in which we prove the existence and uniqueness of mild solutions for cooperative systems and also we state a comparison principle for this type of solutions, these results are based in the proofs established by Cabré and Roquejoffre in [15]. Moreover, we set algebraically upper and lower bounds for the solutions of (3.1)-(3.2), the computations are based on the results stated by Cabré, Coulon and Roquejoffre in [16]. The proofs of Theorems 3.1 and 3.2 rely on the construction of explicit classical subsolutions and supersolutions.

3.2. Mild solutions and comparison principle

In this section, we prove the existence of unique solution of the system (3.1)-(3.2). In order to prove this, we recall the notion of mild solution for the nonhomogeneous linear problem

$$\begin{aligned} \partial_t u_i + (-\triangle)^{\alpha_i} u_i &= h_i(t), \quad \text{in } (0,T) \\ u_i(0) &= u_{0i} \end{aligned} \tag{3.6}$$

where $T > 0$, $u_{0i} \in X$, and $h_i \in C([0,T], X)$ are given, where X is a Banach space. The mild solution of (3.6) is given explicitly by Duhamel principle:

$$u_i(t) = T_{t,i} u_{0i} + \int_0^t T_{t-s,i} h_i(s) ds \tag{3.7}$$

for all $t \in [0,T]$, where

$$T_{t,i} w(x) = \int_{\mathbb{R}^n} p_i(t,x) w(x-y) dy = \int_{\mathbb{R}^n} p_i(t,x-y) w(y) dy$$

and p_i is the fundamental solution of (3.6) which satisfies

a) $p_i \in C((0,+\infty), \mathbb{R})$.

b) $p_i(t,x) \ge 0$ and $\int_{\mathbb{R}} p_i(t,x) dx = 1$ for all $t > 0$.

c) $p_i(t,\cdot) * p_i(s,\cdot) = p_i(t+s,\cdot)$ for all $t, s \in \mathbb{R}_+$

d) If $\alpha_i < 1$, then there exists $B > 1$ such that, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$:

$$\frac{B^{-1}}{t^{\frac{1}{2\alpha_i}}(1+ \mid xt^{-\frac{1}{2\alpha_i}} \mid^{1+2\alpha_i})} \leq p_i(t,x) \leq \frac{B}{t^{\frac{1}{2\alpha_i}}(1+ \mid xt^{-\frac{1}{2\alpha_i}} \mid^{1+2\alpha_i})} \tag{3.8}$$

For more information about operators $T_{t,i}$ see section 2 of [15]. So, from the equation (3.7), one easily checks that $u_i \in C([0,T];X)$.

Now, we consider $G : [0,\infty) \times X^m \to X^m$, $G = (G_i(t,u))_{i=1}^m$ be a function that satisfies for all $i \in \{1,...,m\}$

$$G_i \in C^1([0,\infty) \times X^m; X) \tag{3.9}$$

$$G_i(t,\cdot) \text{ is globally Lipschitz in } X^m \text{ uniformly in } t \geq 0. \tag{3.10}$$

we note that $X^m = \prod_{i=1}^m X$ with the product norm $\|u\|_{X^m} = \sum_{i=1}^m \|u_i\|_X$ is a Banach space.

Given any $T > 0$, we are interested in the nonlinear problem

$$\begin{aligned} \partial_t u + Lu &= G(t,u), \quad \text{in } (0,T) \\ u(0) &= u_0 \end{aligned} \tag{3.11}$$

where $L = diag((-\triangle)^{\alpha_1}, ..., (-\triangle)^{\alpha_m})$ and $u = (u_i)_{i=1}^m$. It follows from (3.7) that

$$u(t) = \mathtt{T}_t u_0 + \int_0^t \mathtt{T}_{t-s} G(s,u(s))ds \tag{3.12}$$

where $\mathtt{T}_t = diag(T_{t,1}, ..., T_{t,m})$. Now, we will use a fixed point principle to prove that (3.12) has an unique solution. Define the map

$$N_{u_0}(u)(t) := \mathtt{T}_t u_0 + \int_0^t \mathtt{T}_{t-s} G(s,u(s))ds \tag{3.13}$$

It is easy to check that

$$N_{u_0} : C([0,T];X)^m \to C([0,T];X)^m$$

where $C([0,T];X)^m = \prod_{i=1}^m C([0,T];X)$ is the product space. We claim that N_{u_0} is Lipschitz in $C([0,T];X)^m$. Indeed, let $u, v \in C([0,T];X)^m$, so

$$\begin{aligned} \|N_{u_0,i}(u)(t) - N_{u_0,i}(v)(t)\|_X &\leq \int_0^t \|T_{t-s,i}(G_i(u(s)) - G_i(v(s)))\|_X ds \\ &\leq \int_0^t \|T_{t-s,i}\| \|G_i(u(s)) - G_i(v(s))\|_X ds \end{aligned}$$

$$\begin{aligned}
&\leq MLip_u(G_i)\int_0^t \|u(s)-v(s)\|_{X^m}ds \\
&= MLip_u(G_i)\int_0^t \sum_{j=1}^m \|u_j(s)-v_j(s)\|_X ds \\
&\leq tMLip_u(G_i)\|u-v\|_{C([0,T];X)^m}
\end{aligned}$$

where $M = \sup_{t\in[0,T]} \max_{i\in\{1,...,m\}} \|T_{t,i}\|$. So, from the above computations, taking the supremum in $[0,T]$ and adding in $i \in \{1,...,m\}$, we have that

$$\|N_{u_0}(u) - N_{u_0}(v)\|_{C([0,T];X)^m} \leq TM\left[\sum_{i=1}^m Lip_u(G_i)\right]\|u-v\|_{C([0,T];X)^m}$$

thus N_{u_0} is Lipschitz with constant $MT\left[\sum_{i=1}^m Lip_u(G_i)\right]$. Recall that for any strongly continuous semigroup, we have that $\|T_{t,i}\| \leq C_i e^{\omega_i t}$ for some constants C_i and ω_i. Now, it follows by induction that $(N_{u_0})^k$ is Lipschitz in $C([0,T];X)^m$ with Lipschitz constant

$$\frac{(MT)^k}{k!}\left[\sum_{i=1}^m Lip_u(G_i)\right]^k$$

where k is any positive integer. This constant is less than 1 if we take k large enough. Then, we conclude that N_{u_0} has a unique fixed point.

Moreover, if we consider the sequence of functions $(N_{u_0})^i(u^0) \in C([0,T];X)^m$, it is easy to see that there exist $u \in C([0,T];X)^m$ such that

$$u = \lim_{i\to+\infty}(N_{u_0})^i(u^0) \tag{3.14}$$

where $u^0(t) = \mathtt{T}_t u_0$, also, the limit u is the unique fixed point of N_{u_0}, and so, the unique mild solution of (3.12) for all $T > 0$.

Given $0 < T < T'$, the mild solution in $(0,T')$ must coincide in $(0,T)$ with the mild solution in this interval, by uniqueness. Thus, under assumption (3.9)-(3.10), the mild solution of (3.11) extends uniquely to all $t \in [0,\infty)$, i.e., it is global in time.

Moreover, let $u = (u_i)_{i=1}^m$ the unique solution of (3.12), if we define

$$H_i(t,w) = G_i(t,u_1,...,u_{i-1},w,u_{i+1},...,u_m)$$

we have that

$$H_i \in C^1([0,\infty)\times X;X) \tag{3.15}$$

$$H_i(t,\cdot) \text{ is globally Lipschitz in } X \text{ uniformly in } t \geq 0. \tag{3.16}$$

Consider now the problem

$$\begin{aligned} \partial_t w + (-\triangle)^{\alpha_i} w &= H_i(t,w), \qquad \text{in } (0,T) \\ w(0) &= u_{0i} \end{aligned} \tag{3.17}$$

Following the computations in section 2.3 of [15], we conclude that this problem has a unique mild solution in $C([0,T];X)$ given by $w = u_i$. Thus, if the initial datum belongs to the domain of $A_i = (-\triangle)^{\alpha_i}$ denoted by $D(A_i)$, we have further regularity in t of the mild solution $u_i = u_i(t)$. Under hypothesis (3.15)-(3.16) (here the continuous differentiability of H_i with values in X is important), the mild solution u_i of (3.17) satisfies

$$u_i \in C^1([0,T);X) \text{ and } u_i([0,T)) \subset D(A_i) \text{ if } u_{0i} \in D(A_i) \tag{3.18}$$

and it is a classical solution, i.e., a solution satisfying (3.17) pointwise for all $t \in (0,T)$. Doing the same procedure for all $i \in \{1,...,m\}$ and for all $T > 0$, we conclude that $u = (u_i)_{i=1}^m$ is a classical solution of (3.11) global in time.

Now, we set a useful fact that we need in the followings computations. If u is the solution of the system (3.11) with $u_0 \in X$ and G satisfying (3.9) and (3.10), then $\tilde{u}(t) = e^{at}u(t)$ is the mild solution of the system

$$\begin{aligned} \partial_t \tilde{u} + L\tilde{u} &= \widetilde{G}(t,\tilde{u}) \\ \tilde{u}(0) &= u_0 \end{aligned} \tag{3.19}$$

with $\widetilde{G}_i(t,\tilde{u}) = a\tilde{u}_i + e^{at}\widetilde{G}_i(t, e^{-at}\tilde{u})$ and $a \in \mathbb{R}$. This fact is proved in the same way as in [15].

We now apply all these facts to problem (3.1)-(3.2). Recall our standing assumptions for the nonlinearities $(f_i)_{i=1}^m$, from hypothesis (H2), we deduce the existence of a positive vector $M = np$ with $n > 0$ large enough and $p_i = 1$ for all $i \in \{1,...,m\}$ such that $F(M) \leq 0$ and $u_0 \leq a \leq M$. Now, we extend f_i outside of a compact set of $\mathbb{R}^m$ that contains $[0,M]$ to ensure that:

$f_i \in C^1(\mathbb{R}^m)$ is globally Lipschitz, $f_i(u)$ is nondecreasing in all components of u with the possible exception of the i^{th} one and Df_i is uniformly continuous in $\mathbb{R}^m$.

We consider the Banach space $X = C_0(\mathbb{R}^d)$ and taking $G_i(t,u)(x) := f_i(u(x))$ we can verify (3.9)-(3.10). We use that Df_i is uniformly continuous and $f_i(0) = 0$ to check that the map $u \in C_0(\mathbb{R}^d)^m \mapsto f_i(u) \in C_0(\mathbb{R}^d)$ is continuously differentiable. Thus, by the previous considerations, there is a unique mild solution u of

$$\begin{aligned} \partial_t u + Lu &= F(u), \quad \text{in } (0,\infty)\times\mathbb{R}^d \\ u(0,\cdot) &= u_0 \end{aligned} \tag{3.20}$$

for data $u_0 \in X$. Moreover, if the initial datum u_0 in (3.20) belongs to the domain $\prod_{i=1}^{m} D_0(A_i)$, where $D_0(A_i)$ is the domain of A_i in $C_0(\mathbb{R}^d)$. Then the mild solution u of (3.20) satisfies (3.18) for all $i \in \{1, ..., m\}$ and for all $T > 0$, with $D(A_i) = D_0(A_i)$ and it is a classical solution global in time.

Before to state the upper bound for the solutions, we need to establish a comparison principle for mild solutions.

Theorem 3.3 *Let $F^{1,2} = (f_i^{1,2})_{i=1}^m$ with $f_i^{1,2} \in C^1(\mathbb{R}^m)$ functions that satisfy (3.4), globally Lipschitz. Let $u^{1,2} = (u_i^{1,2})_{i=1}^m$ mild solutions of*

$$\partial_t u^1 + Lu^1 = F^1(u^1), \quad \partial_t u^2 + Lu^2 = F^2(u^2)$$

If, for all $i \in \{1, ..., m\}$, $f_i^1 \leq f_i^2$ in $\mathbb{R}^m$ and

$$u_i^1(0, \cdot) \leq u_i^2(0, \cdot), \quad belong\ to\ X$$

then

$$u_i^1(t, x) \leq u_i^2(t, x) \qquad \forall (t, x) \in [0, \infty) \times \mathbb{R}^d.$$

Proof. Taking $a = \max\{\max_i\{Lip(f_i^1)\}, \max_i\{Lip(f_i^2)\}\}$, we define

$$\tilde{f}_i^j(t, v) = av_i + e^{at} f_i^j(e^{-at} v), \qquad \forall i \in \{1, .., m\}, j = \{1, 2\}$$

The function $\tilde{f}_i^j$ is nondecreasing in its second argument, indeed

$$\frac{\partial \tilde{f}_i^j}{\partial v_k}(t, v) = a\frac{\partial v_i}{\partial v_k} + \frac{\partial f_i^j}{\partial w_k}(w), \qquad \text{with } w = e^{-at} v \text{ and } j = \{1, 2\}$$

thus,

- if $k = i$, then $\frac{\partial \tilde{f}_i^j}{\partial v_i}(t, v) = a + \frac{\partial f_i^j}{\partial w_i}(w) \geq 0$, by the choice of a.
- if $k \neq i$, then $\frac{\partial \tilde{f}_i^j}{\partial v_k}(t, v) = \frac{\partial f_i^j}{\partial w_k}(w) \geq 0$, since $\tilde{f}_i^j$ satisfy (3.4) for all $i \in \{1, .., m\}$ and $j = \{1, 2\}$.

Moreover, since $f_i^1(\cdot) \leq f_i^2(\cdot)$, we have $\tilde{f}_i^1(t, \cdot) \leq \tilde{f}_i^2(t, \cdot)$.

Now, let consider the system

$$\begin{aligned} \partial_t \tilde{u}^j + L\tilde{u}^j &= \tilde{F}^j(\tilde{u}^j) \\ \tilde{u}^j(0, \cdot) &= u_0^j \end{aligned} \tag{3.21}$$

by the previous section, we know that $\tilde{u}^j(t,x) = e^{at}u^j(t,x)$ is the solution of the system (3.21) for each $j = 1,2$, where u^j is the solution of the system (3.21) with $\tilde{F}^j$ replaced by F^j. Therefore, it is enough to prove that $\tilde{u}^1 \leq \tilde{u}^2$.

Consider the mapping N^j for $j = \{1,2\}$, defined by

$$N^j(w)(t,\cdot) := \mathtt{T}_t u_0^j(\cdot) + \int_0^t \mathtt{T}_{t-s}\widetilde{F}^j(s,w(s,\cdot))ds \tag{3.22}$$

Taking $u^{0,j}(t,\cdot) = \mathtt{T}_t u_0^j(\cdot)$, we know that $\tilde{u}^j = \lim_{n\to+\infty}(N^j)^n(u^{0,j})$, thus, using a standard induction argument, we only need to show that $(N^1)^n(u^{0,1}) \leq (N^2)^n(u^{0,2})$ on $[0,\infty)\times\mathbb{R}^d$ for all n.

Since $u_0^1 \leq u_0^2$, then $u^{0,1} \leq u^{0,2}$ on $[0,\infty)\times\mathbb{R}^d$. Now, suppose that $(N^1)^n(u^{0,1}) \leq (N^2)^n(u^{0,2})$; and by previous considerations, we have

$$\widetilde{f}_i^1(s,(N^1)^n(u^{0,1})) \leq \widetilde{f}_i^2(s,(N^1)^n(u^{0,1}))$$

and

$$\widetilde{f}_i^2(s,(N^1)^n(u^{0,1})) \leq \widetilde{f}_i^2(s,(N^2)^n(u^{0,2})).$$

Thus, for all $i \in \{1,...,m\}$

$$\begin{aligned}
(N^1)_i^{n+1}(u^{0,1}) = N_i^1[(N^1)^n(u^{0,1})] &= T_{t,i}u_{0i}^1(\cdot) + \int_0^t T_{t-s,i}\widetilde{f}_i^1(s,(N^1)^n(u^{0,1}))ds \\
&\leq T_{t,i}u_{0i}^2(\cdot) + \int_0^t T_{t-s,i}\widetilde{f}_i^2(s,(N^1)^n(u^{0,1}))ds \\
&\leq T_{t,i}u_{0i}^2(\cdot) + \int_0^t T_{t-s,i}\widetilde{f}_i^2(s,(N^2)^n(u^{0,2}))ds \\
&= N_i^2[(N^1)^n(u^{0,2})] = (N^2)_i^{n+1}(u^{0,2})
\end{aligned}$$

Hence,

$$(N^1)^{n+1}(u^{0,1}) \leq (N^2)^{n+1}(u^{0,2}), \qquad \forall (t,x) \in [0,\infty)\times\mathbb{R}^d$$

□

Remark 3.4 If we suppose $f_i^1 \leq f_i^2$ in $\mathbb{R}_+^m$ and $0 \leq u_i^1(0,\cdot) \leq u_i^2(0,\cdot)$ for all $i \in \{1,...,m\}$, we obtain the same result as in Theorem 3.3.

Since $F(0) = 0$ by the previous theorem, we conclude that the solution of the system (3.1)-(3.2), satisfies $u_i(t,x) \geq 0$ for all $(t,x) \in [0,\infty)\times\mathbb{R}^d$.

3.3. Finite time estimates

3.3.1. Upper bound

Now, we are in position to establish an algebraic upper bound for the solutions of (3.1)-(3.2). Since $u = (u_i)_{i=1}^m$ is the mild solution of system (3.1)-(3.2) with the extended function $f_i \in C^1(\mathbb{R}^m)$ which is globally Lipschitz for all $i \in \{1, ..., m\}$, we have that

$$\left|\frac{\partial f_i}{\partial u_j}(\xi)\right| \leq Lip(f_i), \qquad \forall \xi \in \mathbb{R}^m \text{ and } i, j \in \{1, ..., m\}$$

Taking $L = \max_{i \in \{1,...,m\}}\{Lip(f_i)\}$, we have

$$f_i(w) = \int_0^1 Df_i(\sigma w)d\sigma \cdot w \leq \left|\sum_{j=1}^m w_j \int_0^1 \frac{\partial f_i}{\partial w_j}(\sigma w)d\sigma\right| \leq L\sum_{j=1}^m w_j \tag{3.23}$$

for all $w \geq 0$. Let us consider $v = (v_i)_{i=1}^m$ the mild solution of the following system

$$\begin{aligned} \partial_t v + Lv &= Bv \\ v(0, \cdot) &= u_0 \end{aligned} \tag{3.24}$$

where $B = (b_{ij})_{i,j=1}^m$ is a matrix with $b_{ij} = L$ for all i, j. By (3.23) and Remark 3.4, we conclude that $u \leq v$. Moreover, since u_0 belongs to the domain $\prod_{i=1}^m D_0(A_i)$, we have that u and v are classical solutions.

Taking Fourier transforms in each term of system (3.24), we have

$$\begin{aligned} \partial_t \mathfrak{F}(v) &= (A(|\xi|) + B)\mathfrak{F}(v) \\ \mathfrak{F}(v)(0, \cdot) &= \mathfrak{F}(u_0) \end{aligned}$$

where $A(|\xi|) = \operatorname{diag}(-|\xi|^{2\alpha_1}, ..., -|\xi|^{2\alpha_m})$. Thus, we have that

$$\mathfrak{F}(v)(t, \xi) = e^{(A(|\xi|)+B)t}.\mathfrak{F}(u_0)(\xi)$$

and then

$$u(t, x) \leq v(t, x) = \mathfrak{F}^{-1}(e^{(A(|\xi|)+B)t}) * u_0(x) \tag{3.25}$$

In what follows, we prove that for each time $t > 0$, the solution of the system (3.1)-(3.2) decay as $|x|^{-d-2\alpha}$ when $|x|$ is large enough, due to the inequality and convolution in (3.25) and since u_0 satisfies (3.3), we only need to prove that each term of the matrix $\mathfrak{F}^{-1}(e^{(A(|\xi|)+B)t})$ has the desired decay. To do this, we use the technique of rotating the interval of integration by a small angle, in order to simplify the calculus at the moment to bound the integral that appear in the previous Fourier transform. Hence, before to establish the upper bound, we state the following lemma, which will help us to prove that the integral over $\mathbb{R}^d$ is equal to the integral over $e^{i\varepsilon}\mathbb{R}^d$ for some ε small enough, i.e., the bounds for the entries $h_{i,j}$ of $e^{(A(r)+B)t}$ which appear in Lemma 3.5, decay fast enough, hence we can prove that the integral over the arc obtained when we rotate the integration line, converges to zero.

Lemma 3.5 *Let $\varepsilon < \frac{\pi}{4\alpha_1}$, then, there exist a constant $c > 0$ such that*

$$|h_{i,j}(t, e^{i\varepsilon}r)| \leq e^{(c - r^{2\alpha_1}\cos(2\alpha_1\varepsilon))t}, \qquad if\ r < 1 \tag{3.26}$$

and

$$|h_{i,j}(t, e^{i\varepsilon}r)| \leq e^{(c - r^{2\alpha}\cos(2\alpha_1\varepsilon))t}, \qquad if\ r \geq 1 \tag{3.27}$$

for all $t > 0$, where $e^{(A(r)+B)t} = (h_{ij}(t,r))_{i,j=1}^{m}$.

Proof. Let consider the system

$$\begin{aligned} \partial_t w &= (A(e^{i\varepsilon}r) + B)w \\ w(0,r) &= e_j \end{aligned} \tag{3.28}$$

where e_j is the jth vector of the canonical basis of $\mathbb{R}^m$, thus, we have that

$$w(t,r) = e^{(A(e^{i\varepsilon}r)+B)t}.e_j = \left(h_{kj}(t, e^{i\varepsilon}r)\right)_{k=1}^{m}$$

Multiplying (3.28) by the conjugate transpose $\overline{w}$, we have that

$$\partial_t w.\overline{w} = -\sum_{k=1}^{m} e^{i2\alpha_k\varepsilon} r^{2\alpha_k}|w_k|^2 + Bw.\overline{w}$$

thus

$$\frac{1}{2}\partial_t|w|^2 + \sum_{k=1}^{m}\cos(2\alpha_k\varepsilon)r^{2\alpha_k}|w_k|^2 = Re(Bw.\overline{w}) \leq c|w|^2$$

for some $c > 0$. By the choice of ε and using Gronwall Lemma, we get that for all $j,k \in \{1,...,m\}$

$$|h_{k,j}(t, e^{i\varepsilon}r)| \leq |w(t,r)| \leq e^{(c - r^{2\alpha_1}\cos(2\alpha_1\varepsilon))t}, \qquad if\ r < 1$$

and

$$|h_{k,j}(t, e^{i\varepsilon}r)| \leq |w(t,r)| \leq e^{(c - r^{2\alpha}\cos(2\alpha_1\varepsilon))t}, \qquad if\ r \geq 1$$

□

Now, we divide the proof of the upper bound in two cases. First, for the sake of simplicity, we consider the one space dimension case to underline the idea of the proof. The higher space dimension case is treated after and requires the use of Bessel functions of first and third kind.

Lemma 3.6 *Let $d = 1$ and let $u = (u_i)_{i=1}^m$ the mild solution of system (3.1)-(3.2), with initial condition u_0 satisfying (3.3). Then, there exist locally bounded functions $C_i : (0, \infty) \to \mathbb{R}_+$ such that*

$$u_i(t, x) \leq \frac{C_i(t)}{|x|^{d+2\alpha}}, \qquad \forall\, t > 0, |x| >> 1$$

for all $i \in \{1, ..., m\}$.

Proof. By the convolution (3.25) and since $u_{0i}(x) = O(|x|^{-(d+2\alpha_i)})$ as $|x| \to \infty$, we only need to prove that

$$|\eta_{ij}(t, x)| \leq \frac{C_{ij}(t)}{|x|^{d+2\alpha}}, \qquad \forall\, t > 0,\ |x| >> 1$$

for all $i, j \in \{1, ...m\}$, where $\mathfrak{F}^{-1}(e^{(A(|\xi|)+B)t}) = (\eta_{ij})_{i,j=1}^m$ and C_{ij} a locally bounded function.

Now, we state several facts. If we consider $w(t) = e^{tB}e^{tA(r)}$, then w satisfies the problem

$$\begin{aligned} w'(t) &= (A(r) + B)w(t) + [e^{tB}, A(r)]e^{tA(r)} \\ w(0) &= Id \end{aligned}$$

where $[e^{tB}, A(r)] = e^{tB}A(r) - A(r)e^{tB}$, thus, by Duhamel formula we get

$$\begin{aligned} e^{t(A(r)+B)} &= e^{tB}e^{tA(r)} - \int_0^t e^{(t-s)(A(r)+B)}[e^{sB}, A(r)]e^{sA(r)}ds \\ &= C_t - D_t \end{aligned} \tag{3.29}$$

Also, we remember the Wilcox Formula [48] which states

$$\frac{\partial e^{(A(r)+B)t}}{\partial r} = \int_0^t e^{(t-s)(A(r)+B)}A'(r)e^{s(A(r)+B)}ds \tag{3.30}$$

Let $t > 0$, by the above formulas and integrating by parts, we have

$$\begin{aligned} \mathfrak{F}^{-1}(e^{(A(|\xi|)+B)t}) &= \int_{\mathbb{R}} e^{ix\xi}e^{(A(|\xi|)+B)t}d\xi = 2\int_0^\infty \cos(|x|r)e^{(A(r)+B)t}dr \\ &= -2\int_0^\infty \frac{\sin(|x|r)}{|x|}\frac{\partial e^{(A(r)+B)t}}{\partial r}dr = I_1 + I_2 + I_3 \end{aligned}$$

where

$$\begin{aligned} I_1 &= -2\int_0^\infty \frac{\sin(|x|r)}{|x|}\int_0^t C_{t-s}A'C_s(r)dsdr \\ I_2 &= -2\int_0^{|x|^{-\delta}} \frac{\sin(|x|r)}{|x|}\int_0^t \left(-C_{t-s}A'D_s(r) - D_{t-s}A'C_s(r) + D_{t-s}A'D_s(r)\right)dsdr \\ I_3 &= -2\int_{|x|^{-\delta}}^\infty \frac{\sin(|x|r)}{|x|}\int_0^t \left(-C_{t-s}A'D_s(r) - D_{t-s}A'C_s(r) + D_{t-s}A'D_s(r)\right)dsdr \end{aligned}$$

with $\delta \in (1/2, 1)$.

Let us begin with the computations of I_1. By definition, we have that

$$C_{t-s}A'C_s(r) = e^{(t-s)B}e^{(t-s)A(r)}A'(r)e^{sB}e^{sA(r)}$$

and its entries have the form:

$$\sum_{k=1}^{m}(c_{ij}^k(t,s)r^{2\alpha_k-1}e^{-(t-s)r^{2\alpha_k}})e^{-sr^{2\alpha_j}} \qquad \forall i,j \in \{1,...,m\}$$

where $c_{ij}^k(t,s)$ is some integrable function on $s \in [0,t]$ arising from products of the entries of $e^{(t-s)B}$ and e^{sB}, which have the form $(m-1+e^{mLt})$ and $(-1+e^{mLt})$. Thus, set $I_1 = ((I_1)_{ij})_{i,j=1}^m$ and take the change of variables $u = r^{2\alpha_k}|x|^{2\alpha_k}$, we obtain

$$\begin{aligned}
|(I_1)_{ij}| &= \left|\sum_{k=1}^{m}\int_0^\infty \frac{\sin(|x|r)}{|x|}\int_0^t c_{ij}^k(t,s)r^{2\alpha_k-1}e^{-(t-s)r^{2\alpha_k}}e^{-sr^{2\alpha_j}}dsdr\right| \\
&\leq \sum_{k=1}^{m}\left|\mathrm{Im}\left(\int_0^\infty \frac{e^{i|x|r}}{|x|}\int_0^t c_{ij}^k(t,s)r^{2\alpha_k-1}e^{-(t-s)r^{2\alpha_k}}e^{-sr^{2\alpha_j}}dsdr\right)\right| \\
&= \sum_{k=1}^{m}\frac{(2\alpha_k)^{-1}}{|x|^{1+2\alpha_k}}\left|\mathrm{Im}\left(\int_0^t c_{ij}^k(t,s)\int_0^\infty e^{iu^{1/2\alpha_k}}e^{-(t-s)u|x|^{-2\alpha_k}}e^{-su^{\alpha_j/\alpha_k}|x|^{-2\alpha_j}}duds\right)\right|
\end{aligned}$$

denoting

$$\sigma_k = \int_0^\infty e^{iu^{1/2\alpha_k}}e^{-(t-s)u|x|^{-2\alpha_k}}e^{-su^{\alpha_j/\alpha_k}|x|^{-2\alpha_j}}du$$

we have

$$\begin{aligned}
|(I_1)_{ij}| &= \sum_{k=1}^{m}\frac{(2\alpha_k)^{-1}}{|x|^{1+2\alpha_k}}\left|\mathrm{Im}\left(\int_0^t c_{ij}^k(t,s)\sigma_k ds\right)\right| \\
&\leq \sum_{k=1}^{m}\frac{(2\alpha_k)^{-1}}{|x|^{1+2\alpha_k}}\int_0^t c_{ij}^k(t,s)|\sigma_k|ds
\end{aligned}$$

The integral σ_k can be simplified by rotating the interval of integration by $\varepsilon < \min\left(\frac{\pi}{2}, 2\alpha\pi, \frac{\pi\alpha_j}{2\alpha_1}\right)$, thus, we get

$$\sigma_k = \int_0^\infty e^{ir^{1/2\alpha_k}e^{i\varepsilon/2\alpha_k}}e^{-(t-s)r|x|^{-2\alpha_k}e^{i\varepsilon}}e^{-sr^{\alpha_j/\alpha_k}|x|^{-2\alpha_j}e^{i\varepsilon\alpha_j/\alpha_k}}e^{i\varepsilon}dr$$

hence

$$
\begin{aligned}
|\sigma_k| &\leq \int_0^\infty e^{-r^{1/2\alpha_k}\sin(\varepsilon/2\alpha_k)} e^{-(t-s)r|x|^{-2\alpha_k}\cos(\varepsilon)} e^{-sr^{\alpha_j/\alpha_k}|x|^{-2\alpha_j}\cos(\varepsilon\alpha_j/\alpha_k)} dr \\
&\leq \int_0^\infty e^{-r^{1/2\alpha_k}\sin(\varepsilon/2\alpha_k)} dr = C(\alpha_k)
\end{aligned}
$$

Now, since $|x| > 1$

$$
|(I_1)_{ij}| = \sum_{k=1}^{m} \frac{(2\alpha_k)^{-1} C(\alpha_k)}{|x|^{1+2\alpha_k}} \int_0^t c_{ij}^k(t,s) ds \leq \frac{C(t)}{|x|^{1+2\alpha}}
$$

where $C(t)$ is a locally bounded function on $t > 0$.

Let continue with the computations of $I_2 = ((I_2)_{ij})_{i,j=1}^m$, in what follows we consider the matrix norm

$$
\|A\| = \max_{\|y\|=1} \|Ay\| \quad with \quad \|y\| = \left[\sum_{i=1}^{m} |y_i|^2\right]^{\frac{1}{2}}
$$

thus, for all $i, j \in \{1, ..., m\}$

$$
\begin{aligned}
|(I_2)_{ij}| &\leq \|I_2\| \\
&\leq \frac{2}{|x|} \int_0^{|x|^{-\delta}} \int_0^t \|C_{t-s}A'D_s\| + \|D_{t-s}A'C_s\| + \|D_{t-s}A'D_s\| ds dr
\end{aligned}
\tag{3.31}
$$

computing each norm and since $r \in [0, |x|^{-\delta}]$, we have that $\|A(r)\| \leq r^{2\alpha}$ and $\|A'(r)\| \leq Cr^{2\alpha-1}$, thus

$$
\begin{aligned}
\|C_{t-s}A'D_s\| &= \|e^{(t-s)B} e^{(t-s)A(r)} A'(r) \int_0^s e^{(s-w)(A(r)+B)} [e^{wB}, A(r)] e^{wA(r)} dw\| \\
&\leq 2s\|A'(r)\|\|A(r)\| e^{(\|A(r)\|+\|B\|)t} \\
&\leq 2Cse^{(1+\|B\|)t} r^{4\alpha-1}
\end{aligned}
$$

in the same way we prove that $\|D_{t-s}A'C_s\| \leq 2C(t-s)e^{(1+\|B\|)t} r^{4\alpha-1}$ and $\|D_{t-s}A'D_s\| \leq 4C(t-s)se^{(1+\|B\|)t} r^{6\alpha-1}$.

By (3.31) and since $\delta \in (1/2, 1)$, we conclude

$$
|(I_2)_{ij}| \leq \frac{2C}{|x|^{1+4\delta\alpha}} (t^2 + \frac{t^3}{3}) e^{(1+\|B\|)t} \leq \frac{C(t)}{|x|^{1+2\alpha}}
$$

Now, we compute the bound for the first term of I_3. Rotating the interval of integration by $\varepsilon < \min(\pi, \frac{\pi}{4\alpha_1})$, we get

$$\begin{aligned} I_3^1 &= 2\int_{|x|^{-\delta}}^{\infty} \frac{\sin(|x|r)}{|x|} \int_0^t C_{t-s}A'D_s ds dr = 2\mathrm{Im}\left(\int_{|x|^{-\delta}}^{\infty} \frac{e^{i|x|r}}{|x|} \int_0^t C_{t-s}A'D_s ds dr\right) \\ &= 2\mathrm{Im}\left(\int_0^{\varepsilon} \frac{e^{i|x|^{1-\delta}e^{i\theta}}}{|x|} \int_0^t C_{t-s}A'D_s(|x|^{-\delta}e^{i\theta})|x|^{-\delta} i e^{i\theta} ds d\theta\right) \\ &\qquad +2\mathrm{Im}\left(\int_{|x|^{-\delta}}^{\infty} \frac{e^{i|x|re^{i\varepsilon}}}{|x|} \int_0^t C_{t-s}A'D_s(re^{i\varepsilon})e^{i\varepsilon} ds dr\right) \\ &:= 2\mathrm{Im}(\sigma_1) + 2\mathrm{Im}(\sigma_2) \end{aligned}$$

Hence, to get a bound for $\sigma_1 = ((\sigma_1)_{ij})_{i,j=1}^m$, we need to state a bound for each term that appear in $(\sigma_1)_{ij}$. By abuse of notation, we call σ to each one of these terms. Hence, the general form for σ is

$$\begin{aligned} \sigma = \int_0^{\varepsilon} \int_0^t \int_0^s & \frac{e^{i|x|^{1-\delta}e^{i\theta}}}{|x|} C(t-s)C(w)e^{-(t-s)|x|^{-2\delta\alpha_j}e^{i2\alpha_j\theta}} |x|^{-\delta(2\alpha_j-1)} e^{i(2\alpha_j-1)\theta} \\ & h_{p,q}(s-w, |x|^{-\delta}e^{i\varepsilon})|x|^{-2\delta\alpha_l} e^{i2\alpha_l\theta} e^{-w|x|^{-2\delta\alpha_k}e^{i2\alpha_k\theta}} |x|^{-\delta} i e^{i\theta} dw ds d\theta \end{aligned}$$

where $C(t-s)$ and $C(w)$ are positive and integrable functions. Now, since $|x| > 1$ and by (3.26) of Lemma 3.5

$$\begin{aligned} |\mathrm{Im}(\sigma)| &\leq \int_0^{\varepsilon} \int_0^t \int_0^s \frac{e^{-|x|^{1-\delta}\sin(\theta)}}{|x|} C(t-s)C(w)e^{-(t-s)|x|^{-2\delta\alpha_j}\cos(2\alpha_j\theta)} |x|^{-\delta(2\alpha_j-1)} \\ &\qquad e^{(c-r^{2\alpha_1}\cos(2\alpha_1\varepsilon))(s-w)} |x|^{-2\delta\alpha_l} e^{-w|x|^{-2\delta\alpha_k}\cos(2\alpha_k\theta)} |x|^{-\delta} dw ds d\theta \\ &\leq \frac{1}{|x|^{1+2\delta(\alpha_j+\alpha_l)}} \int_0^{\varepsilon} \int_0^t \int_0^s C(t-s)C(w)e^{c(s-w)} dw ds d\theta \\ &\leq \frac{\varepsilon C(t)}{|x|^{1+2\alpha}} \end{aligned}$$

then, we conclude

$$|\mathrm{Im}((\sigma_1)_{ij})| \leq \frac{C(t)}{|x|^{1+2\alpha}}$$

As in the proof of σ_1, we continue calling by σ each term that appears in $(\sigma_2)_{ij}$, thus, the general form for σ is

$$\begin{aligned} \sigma = \int_{|x|^{-\delta}}^{\infty} \int_0^t \int_0^s & \frac{e^{i|x|re^{i\varepsilon}}}{|x|} C(t-s)C(w)e^{-(t-s)r^{2\alpha_j}e^{i2\alpha_j\varepsilon}} r^{2\alpha_j-1} e^{i(2\alpha_j-1)\varepsilon} \\ & h_{p,q}(s-w, re^{i\varepsilon})r^{2\alpha_l} e^{i2\alpha_l\varepsilon} e^{-wr^{2\alpha_k}e^{i2\alpha_k\varepsilon}} e^{i\varepsilon} dw ds dr \end{aligned}$$

then

$$|\text{Im}(\sigma)| \leq \frac{e^{-|x|^{1-\delta}\sin(\varepsilon)}}{|x|} \int_0^\infty \int_0^t \int_0^s C(t-s)C(w)e^{-(t-s)r^{2\alpha_j}\cos(2\alpha_j\varepsilon)} r^{2\alpha_j - 1} |h_{p,q}(s-w, re^{i\varepsilon})| r^{2\alpha_l} e^{-wr^{2\alpha_k}\cos(2\alpha_k\varepsilon)} dwdsdr$$

now, breaking the first integral in two terms and bounding, we have by (3.26) in Lemma 3.5 that

$$\begin{aligned}
&\int_0^1 \int_0^t \int_0^s C(t-s)C(w)e^{-(t-s)r^{2\alpha_j}\cos(2\alpha_j\varepsilon)} r^{2\alpha_j-1} |h_{p,q}(s-w, re^{i\varepsilon})| r^{2\alpha_l} e^{-wr^{2\alpha_k}\cos(2\alpha_k\varepsilon)} dwdsdr \\
&\leq \int_0^1 \int_0^t \int_0^s C(t-s)C(w) r^{2(\alpha_j+\alpha_l)-1} e^{-(t-s)r^{2\alpha_j}\cos(2\alpha_j\varepsilon)} e^{(c-r^{2\alpha_1}\cos(2\alpha_1\varepsilon))(s-w)} e^{-wr^{2\alpha_k}\cos(2\alpha_k\varepsilon)} dwdsdr \\
&\leq \int_0^t \int_0^s C(t-s)C(w)e^{c(s-w)} dwds \int_0^1 r^{2(\alpha_j+\alpha_l)-1} dr \\
&= C(t)
\end{aligned}$$

and applying (3.27) of Lemma 3.5, we have

$$\begin{aligned}
&\int_1^\infty \int_0^t \int_0^s C(t-s)C(w)e^{-(t-s)r^{2\alpha_j}\cos(2\alpha_j\varepsilon)} r^{2\alpha_j-1} |h_{p,q}(s-w, re^{i\varepsilon})| r^{2\alpha_l} e^{-wr^{2\alpha_k}\cos(2\alpha_k\varepsilon)} dwdsdr \\
&\leq \int_1^\infty \int_0^t \int_0^s C(t-s)C(w) r^{2(\alpha_j+\alpha_l)-1} e^{-(t-s)r^{2\alpha_j}\cos(2\alpha_j\varepsilon)} e^{(c-r^{2\alpha}\cos(2\alpha_1\varepsilon))(s-w)} e^{-wr^{2\alpha_k}\cos(2\alpha_k\varepsilon)} dwdsdr \\
&\leq \int_0^t \int_0^s C(t-s)C(w)e^{c(s-w)} dwds \int_1^\infty r^{2(\alpha_j+\alpha_l)-1} e^{-tr^{2\alpha}\cos(2\alpha_1\varepsilon)} dr \\
&\leq C(t)
\end{aligned}$$

Hence,

$$|\text{Im}(\sigma)| \leq \frac{C(t)e^{-|x|^{1-\delta}\sin(\varepsilon)}}{|x|}$$

then, since $\delta \in (1/2, 1)$ and $|x| >> 1$, we get that

$$|\text{Im}((\sigma_2)_{ij})| \leq \frac{C(t)}{|x|^{1+2\alpha}}$$

To conclude, we can do a similar proof for the others two terms of I_3. □

Now, we state the proof of Lemma 3.6 in the higher space dimension case, i.e. when $d > 1$ and $\alpha := \alpha_m \leq ... \leq \alpha_1 \leq 1$. Let note that this case requires the use of Bessel functions of first and third kind.

Proof. As in the previous proof, we only need to prove that

$$|x|^{d+2\alpha}|\eta_{ij}(t,x)| \leq C_{ij}(t), \qquad \forall t > 0,\ |x| > 1$$

for all $i, j \in \{1, ...m\}$, where $\mathfrak{F}^{-1}(e^{(A(|\xi|)+B)t}) = (\eta_{ij})_{i,j=1}^m$ and C_{ij} a locally bounded function.

Let $t > 0$ and $|x| > 1$, using the spherical coordinates system in dimension $d > 1$ and the definition of Bessel Function of first kind [1], we have

$$\begin{aligned}
\mathfrak{F}^{-1}(e^{(A(|\xi|)+B)t}) &= \int_{\mathbb{R}^d} e^{ix\xi} e^{(A(|\xi|)+B)t} d\xi \\
&= C_d \int_0^\infty \int_{-1}^1 e^{(A(r)+B)t} \cos(|x|rs) r^{d-1}(1-s^2)^{\frac{d-3}{2}} ds dr \\
&= \frac{C_d}{|x|^{\frac{d}{2}-1}} \int_0^\infty e^{(A(r)+B)t} J_{\frac{d}{2}-1}(|x|r) r^{\frac{d}{2}} dr
\end{aligned}$$

setting $\rho = |x|$, doing a change of variables and integrating by parts

$$\begin{aligned}
|x|^{d+2\alpha}\mathfrak{F}^{-1}(e^{(A(|\xi|)+B)t}) &= C_d \rho^{\frac{d}{2}+2\alpha+1} \int_0^\infty e^{(A(r)+B)t} J_{\frac{d}{2}-1}(\rho r) r^{\frac{d}{2}} dr \\
&= C_d \rho^{\frac{d}{2}} \int_0^\infty e^{(A(\frac{r}{\rho})+B)t} J_{\frac{d}{2}-1}(r) r^{\frac{d}{2}} dr \\
&= -C_d \rho^{\frac{d}{2}-1} \int_0^\infty r^{\frac{d}{2}} J_{\frac{d}{2}}(r) \left.\frac{\partial e^{(A(s)+B)t}}{\partial s}\right|_{s=\frac{r}{\rho}} dr \\
&= Re\left[-C_d \rho^{\frac{d}{2}-1} \int_0^\infty r^{\frac{d}{2}} H^{(1)}_{\frac{d}{2}}(r) \left.\frac{\partial e^{(A(s)+B)t}}{\partial s}\right|_{s=\frac{r}{\rho}} dr\right]
\end{aligned}$$

where $H^{(1)}_\nu = J_\nu + iY_\nu$ is the Bessel Function of third kind [1]. Now, using (3.29) and the notation given in (3.30), we have

$$|x|^{d+2\alpha}|\eta_{ij}(t,x)| \leq C_d\left[|(I_1)_{ij}| + |(I_2)_{ij}| + |(I_3)_{ij}| + |(I_4)_{ij}|\right]$$

for all $i, j \in \{1, ..., m\}$, where

$$I_n = ((I_n)_{ij})_{i,j=1}^m = \rho^{\frac{d}{2}-1} \int_0^\infty r^{\frac{d}{2}} H_{\frac{d}{2}}^{(1)}(r) \int_0^t J_n(r\rho^{-1}) ds dr$$

for all $n \in \{1, 2, 3, 4\}$, with $J_1 = -C_{t-s}A'C_s$, $J_2 = C_{t-s}A'D_s$, $J_3 = D_{t-s}A'C_s$ and $J_4 = -D_{t-s}A'D_s$.

Let us begin with the computations of I_1. The general term has the form

$$\begin{aligned}(I_1)_{ij} &= \sum_{k=1}^m \int_0^\infty \rho^{2\alpha-1} r^{\frac{d}{2}} H_{\frac{d}{2}}^{(1)}(r) \int_0^t c_{ij}^k(t,s)(r/\rho)^{2\alpha_k-1} e^{-(t-s)(r/\rho)^{2\alpha_k}} e^{-s(r/\rho)^{2\alpha_j}} ds dr \\ &:= \sum_{k=1}^m \int_0^t c_{ij}^k(t,s)\sigma_k ds\end{aligned}$$

where $c_{ij}^k(t,s)$ is some integrable function on $s \in [0,t]$ formed by the product of the terms of $e^{(t-s)B}$ and e^{sB}. The integral term σ_k can be estimated by rotating the interval of integration by $\varepsilon < \min(\frac{\pi}{2}, \frac{\pi}{2\alpha_1})$, thus, we get

$$\sigma_k = \rho^{2(\alpha-\alpha_k)} \int_0^\infty (re^{i\varepsilon})^{\frac{d}{2}+2\alpha_k-1} H_{\frac{d}{2}}^{(1)}(re^{i\varepsilon}) e^{-(t-s)(r/\rho)^{2\alpha_k} e^{i2\alpha_k\varepsilon}} e^{-s(r/\rho)^{2\alpha_j} e^{i2\alpha_j\varepsilon}} e^{i\varepsilon} dr$$

since $\rho = |x| > 1$ and $\alpha \leq \alpha_k$

$$\begin{aligned}|\sigma_k| &\leq \int_0^\infty r^{\frac{d}{2}+2\alpha_k-1} \left| H_{\frac{d}{2}}^{(1)}(re^{i\varepsilon}) \right| e^{-(t-s)(r/\rho)^{2\alpha_k}\cos(2\alpha_k\varepsilon)} e^{-s(r/\rho)^{2\alpha_j}\cos(2\alpha_j\varepsilon)} dr \\ &\leq \int_0^\infty r^{\frac{d}{2}+2\alpha_k-1} \left| H_{\frac{d}{2}}^{(1)}(re^{i\varepsilon}) \right| dr = C(\alpha_k)\end{aligned}$$

the last integral is finite since $H_{\frac{d}{2}}^{(1)}(z) \sim -\frac{i}{\pi}\Gamma(d/2)(z/2)^{-d/2}$ on the interval $[0, R]$ and also $|H_{\frac{d}{2}}^{(1)}(re^{i\varepsilon})| \leq \frac{c}{\sqrt{r}} e^{-r\sin(\varepsilon)}$ in $(R, +\infty)$ for some $R > 0$ large enough [30]. Then,

$$|(I_1)_{ij}| \leq \sum_{k=1}^m C(\alpha_k) \int_0^t c_{ij}^k(t,s) ds = C_{ij}(t)$$

where $C_{ij}(t)$ is a positive positive locally bounded function on $t > 0$.

Now, let continue with the bound for I_2. Hence, by rotating the interval of integration by $\varepsilon < \min(\frac{\pi}{2}, \frac{\pi}{4\alpha_1})$, we get

$$I_2 = \rho^{2\alpha-1}\int_0^\infty (re^{i\varepsilon})^{\frac{d}{2}} H^{(1)}_{\frac{d}{2}}(re^{i\varepsilon})\int_0^t C_{t-s}A'D_s(r\rho^{-1}e^{i\varepsilon})e^{i\varepsilon}dsdr$$

Hence, to get a bound for $I_2 = ((I_2)_{ij})_{i,j=1}^m$, we need to state a bound for each term that appear in $(I_2)_{ij}$, by abuse of notation, we call σ to each one of these terms. Hence, the general form of σ is

$$\sigma = \rho^{2\alpha-1}\int_0^\infty\int_0^t\int_0^s (re^{i\varepsilon})^{\frac{d}{2}} H^{(1)}_{\frac{d}{2}}(re^{i\varepsilon})C(t-s)C(w)e^{-(t-s)(r/\rho)^{2\alpha_j}e^{i2\alpha_j\varepsilon}}(r/\rho)^{2\alpha_j-1}$$
$$e^{i(2\alpha_j-1)\varepsilon}h_{p,q}(s-w, r\rho^{-1}e^{i\varepsilon})(r/\rho)^{2\alpha_l}e^{i2\alpha_l\varepsilon}e^{-w(r/\rho)^{2\alpha_k}e^{i2\alpha_k\varepsilon}}e^{i\varepsilon}dwdsdr$$

then

$$|\sigma| \leq \rho^{2(\alpha-\alpha_j-\alpha_l)}\int_0^\infty\int_0^t\int_0^s r^{\frac{d}{2}}\left|H^{(1)}_{\frac{d}{2}}(re^{i\varepsilon})\right|C(t-s)C(w)e^{-(t-s)(r/\rho)^{2\alpha_j}\cos(2\alpha_j\varepsilon)}$$
$$r^{2\alpha_j-1}|h_{p,q}(s-w, r\rho^{-1}e^{i\varepsilon})|r^{2\alpha_l}e^{-w(r/\rho)^{2\alpha_k}\cos(2\alpha_k\varepsilon)}dwdsdr$$

now, doing a similar procedure as in the last part of the previous proof, by (3.26), (3.27) of Lemma 3.5 and since $\rho = |x| > 1$, we have

$$\begin{aligned}|\sigma| &\leq 2\int_0^t\int_0^s C(t-s)C(w)e^{c(s-w)}dwds\int_0^\infty r^{\frac{d}{2}+2(\alpha_j+\alpha_l)-1}\left|H^{(1)}_{\frac{d}{2}}(re^{i\varepsilon})\right|dr\\ &= \widetilde{C}_{ij}(t)\end{aligned}$$

Hence, we get

$$|(I_2)_{ij}| \leq C_{ij}(t)$$

To conclude, we can do a similar proof for I_3 and I_4. □

Now, we present an alternative simple proof of Lemma 3.6, for the particular case in which $\alpha := \alpha_i < 1$ for all $i \in \{1, ..., m\}$, this proof is done for $d \geq 1$. In this case, since we are working with a unique index α, we can bound directly in the iteration process (3.32), to prove that the solution of the system (3.1)-(3.2) decay as $|x|^{-d-2\alpha}$ when $|x|$ is large enough for all $t > 0$.

Proof. As in the previous cases, we have that

$$|f_i(u)| = \left|\int_0^1 Df_i(\sigma u)d\sigma\cdot u\right| \leq \left|\sum_{j=1}^m u_j\int_0^1\frac{\partial f_i}{\partial u_j}(\sigma u)d\sigma\right| \leq L\sum_{j=1}^m |u_j|$$

where $L = \max_{i\in\{1,...,m\}}\{Lip(f_i)\}$. Following (3.14), we have that $u = \lim_{i\to+\infty} u^i$ where

$$u^i = N^i_{u_0}(u^0) = N_{u_0}(u^{i-1}), \quad \text{with } \ u^0(t) = (T_t u_{0i})_{i=1}^m \tag{3.32}$$

Using the iterative process (3.32) and the semigroup properties of the operator T_t, we have that for all $i \in \{1, ..., m\}$ and $n \in \mathbb{N}$

$$|u_i^n(t,x)| \leq \left(1 + (mLt) + \frac{(mLt)^2}{2!} + ... + \frac{(mLt)^n}{n!}\right) \sum_{j=1}^m T_t u_{0j}(x) \tag{3.33}$$

where $u^n = (u_i^n)_{i=1}^m$. Also, we know that

$$\|u^n - u\|_{C([0,\infty),X)^m} \to 0, \quad \textit{when } \ n \to +\infty$$

where $X = C_0(\mathbb{R}^d)$. Then, we deduce that

$$|u_i^n(t,x)| \to |u_i(t,x)| = u_i(t,x) \quad \textit{when } \ n \to +\infty$$

for all $(t,x) \in [0,\infty) \times \mathbb{R}^d$ and $i \in \{1, ..., m\}$. Taking the limit when $n \to +\infty$ in (3.33), we conclude that

$$u_i(t,x) \leq e^{mLt} \sum_{j=1}^m T_t u_{0j}(x), \qquad (t,x) \in [0,\infty) \times \mathbb{R}^d \tag{3.34}$$

Now, by hypothesis (3.3), we have that there exists $r_i > 0$ large enough, such that

$$u_{0i}(x) \leq C_i |x|^{-d-2\alpha}, \qquad if \ \ |x| > r_i$$

also we know that $0 \leq u_{0i} \leq a_i$. Thus, if $|x| > 2r_i$ and $t > 0$

$$\begin{aligned} T_t u_{0i}(x) &\leq \int_{\mathbb{R}^d} \frac{t^{-\frac{d}{2\alpha}} B u_{0i}(y)}{1 + (t^{-\frac{1}{2\alpha}}|x-y|)^{d+2\alpha}} dy \\ &\leq \int_{\{|y|\leq |x|/2\}} \frac{t^{-\frac{d}{2\alpha}} B u_{0i}(y)}{1 + (t^{-\frac{1}{2\alpha}}|x-y|)^{d+2\alpha}} dy \\ &\qquad + \int_{\{|y|>|x|/2\}} \frac{t^{-\frac{d}{2\alpha}} B u_{0i}(y)}{1 + (t^{-\frac{1}{2\alpha}}|x-y|)^{d+2\alpha}} dy \\ &:= I_1 + I_2 \end{aligned}$$

If $|y| \leq |x|/2$, we have $|x-y| \geq |x| - |y| \geq \frac{|x|}{2}$, then

$$I_1 \leq \int_{\{|y|\leq|x|/2\}} \frac{t^{-\frac{d}{2\alpha}} B u_{0i}(y)}{1 + (t^{-\frac{1}{2\alpha}} \frac{|x|}{2})^{d+2\alpha}} dy \leq \frac{2^{d+2\alpha} t B}{|x|^{d+2\alpha}} \int_{\{|y|\leq|x|/2\}} u_{0i}(y) dy$$

moreover, $\int_{\{|y|<r_i\}} u_{0i}(y)dy \leq a_i\omega(0,r_i)$ and

$$\int_{\{r_i\leq|y|\leq|x|/2\}} u_{0i}(y)dy \leq \int_{\{r_i\leq|y|\leq|x|/2\}} \frac{C_i}{|y|^{d+2\alpha}}dy \leq \frac{C_i}{2\alpha r_i^{2\alpha}}$$

then,

$$I_1 \leq \frac{2^{d+2\alpha}tB}{|x|^{d+2\alpha}}\left(a_i\omega(0,r_i)+\frac{C_i}{2\alpha r_i^{2\alpha}}\right)$$

Now, if $|y|>|x|/2$, we have that $u_{0i}(y) \leq \frac{C_i}{|y|^{d+2\alpha}} \leq \frac{2^{d+2\alpha}C_i}{|x|^{d+2\alpha}}$, hence

$$I_2 \leq \frac{2^{d+2\alpha}C_iB}{|x|^{d+2\alpha}}\int_{\mathbb{R}^d}\frac{ds}{1+|s|^{d+2\alpha}} := \frac{2^{d+2\alpha}C_iB}{|x|^{d+2\alpha}}C_\alpha$$

Therefore, we conclude that, if $|x|\geq 2r_i$

$$T_tu_{0i} \leq \frac{2^{d+2\alpha}tB}{|x|^{d+2\alpha}}\left(a_i\omega(0,r_i)+\frac{C_i}{2\alpha r_i^{2\alpha}}\right)+\frac{2^{d+2\alpha}C_iB}{|x|^{d+2\alpha}}C_\alpha$$

Otherwise if $|x|<2r_i$

$$\begin{aligned} T_tu_{0i}(x) &\leq \int_{\mathbb{R}^d}\frac{t^{-\frac{d}{2\alpha}}Bu_{0i}(y)}{1+(t^{-\frac{1}{2\alpha}}|x-y|)^{d+2\alpha}}dy \\ &\leq a_iC_\alpha B \\ &\leq \frac{a_iC_\alpha B(2r_i)^{d+2\alpha}}{|x|^{d+2\alpha}} \end{aligned}$$

Thus, we conclude that there exist a constant $\overline{C}_i$ such that

$$T_tu_{0i}(x) \leq \overline{C}_i(1+t)|x|^{-d-2\alpha}$$

Using (3.34), we have that

$$u_i(t,x) \leq \frac{C(t)}{|x|^{d+2\alpha}}, \qquad \forall (t,x)\in(0,\infty)\times\mathbb{R}^d$$

where $C(t)=(1+t)e^{mLt}\sum_{i=1}^m C_i$. □

3.3.2. Comparison principle for classical solutions

Now, we state the following comparison principle for classical solutions. This result will be useful to deal with sub and super solutions. Indeed, we have not devised a mild representation for them, so we can not apply Theorem 3.3 directly.

Theorem 3.7 *Let* $u = (u_i)_{i=1}^m$ *and* $v = (v_i)_{i=1}^m$ *functions in* $C^1([0,T]; C_0(\mathbb{R}^d))^m$*,* f_i *satisfies (3.4) and*

$$\partial_t u_i + (-\triangle)^{\alpha_i} u_i \leq f_i(u), \quad \partial_t v_i + (-\triangle)^{\alpha_i} v_i \geq f_i(v)$$

If, for all $i \in \{1, ..., m\}$

$$u_i(0,x) \leq v_i(0,x), \quad \forall x \in \mathbb{R}^d$$

also

$$u_i(t,x) = O(|x|^{-(d+2\alpha)}) \ \ and \ \ v_i(t,x) = O(|x|^{-(d+2\alpha)}) \quad as\ |x| \to \infty$$

for all $t \in [0,T]$*. Then*

$$u(t,x) \leq v(t,x) \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^d.$$

Proof. Let us take $w_i = u_i - v_i$, then w_i satisfy $w_i(0,x) \leq 0$ and

$$\begin{aligned} \partial_t w_i + (-\triangle)^{\alpha_i} w_i &\leq f_i(u) - f_i(v) \\ &= \int_0^1 \nabla f_i(\sigma u + (1-\sigma)v) d\sigma.(u-v) \\ &= \int_0^1 \nabla f_i(\zeta_\sigma) d\sigma.w \end{aligned} \tag{3.35}$$

where $\zeta_\sigma = \sigma u + (1-\sigma)v$. By hypothesis, we have that

$$w_i \in C^1([0,T]; C_0(\mathbb{R}^d)) \tag{3.36}$$

then there exist positive constants $C_1(T)$ and $C_2(T)$ such that

$$|w_i(t,x)| \leq C_1(T) \quad \text{and} \quad |\partial_t w_i(t,x)| \leq C_2(T) \tag{3.37}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$. Moreover, by the decay assumptions on the functions u and v, we have that

$$w_i(t,x) = O(|x|^{-(d+2\alpha)}) \quad \forall\, t \in [0,T]\ as\ |x| \to \infty \tag{3.38}$$

Thus, it is easy to see that

$$\int_{\mathbb{R}^d} |w_i(t,x)||w_j(t,x)| dx \leq C_{ij}(T) \qquad \forall\, t \in [0,T] \tag{3.39}$$

where $C_{ij}(T)$ are constants that depend of T. Now, let w_i^+ be the positive part of w_i, we want to prove that

$$\partial_t \left[\int_{\mathbb{R}^d} (w_i^+)^2 dx \right] = \int_{\mathbb{R}^d} \partial_t \left[(w_i^+)^2 \right] dx \tag{3.40}$$

which is quite simple, because, from (3.36), we deduce that $(w_i^+)^2$ and $\partial_t\left[(w_i^+)^2\right]$ are continuous in $(0,T)\times\mathbb{R}^d$ and

$$\left|\partial_t\left[(w_i^+)^2\right]\right| = 2\left|w_i^+\partial_t w_i\right| \leq 2C_2(T)\left|w_i\right| \leq Cg(x) \tag{3.41}$$

the last inequality and the existence of the integrable function g follows from (3.37) and (3.38), thus we conclude (3.40). Now, multiplying each term of the equation (3.35) by the positive part of w_i and integrating over $\mathbb{R}^d$, we have that

$$0 \leq \int_{\mathbb{R}^d} w_i^+(-\triangle)^{\alpha_i} w_i dx \leq \int_{\mathbb{R}^d} w_i^+ \int_0^1 \nabla f_i(\zeta_\sigma)d\sigma.w dx - \int_{\mathbb{R}^d} w_i^+\partial_t w_i dx \tag{3.42}$$

by (3.39) and (3.41), we get

$$\int_{\mathbb{R}^d} w_i^+(-\triangle)^{\alpha_i} w_i dx < \infty$$

Now, since all the above integral exist and having in mind that $f_i \in C^1(\mathbb{R}^m)$ for all $i \in \{1,...,m\}$, from (3.42), we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^d} w_i^+\partial_t w_i dx + \int_{\mathbb{R}^d} w_i^+(-\triangle)^{\alpha_i} w_i dx \\
&\qquad \leq \int_{\mathbb{R}^d} w_i^+ \int_0^1 \nabla f_i(\zeta_\sigma)d\sigma.w dx \\
&\qquad = \int_{\mathbb{R}^d} w_i^+ \int_0^1 \partial_i f_i(\zeta_\sigma)d\sigma w_i dx + \sum_{j=1,i\neq j}^m \int_{\mathbb{R}^d} w_i^+ \int_0^1 \partial_j f_i(\zeta_\sigma)d\sigma w_j dx \\
&\qquad = \int_{\mathbb{R}^d}\int_0^1 \partial_i f_i(\zeta_\sigma)d\sigma (w_i^+)^2 dx + \sum_{j=1,i\neq j}^m \int_{\mathbb{R}^d}\int_0^1 \partial_j f_i(\zeta_\sigma)d\sigma w_i^+ w_j^+ dx \\
&\qquad\qquad - \sum_{j=1,i\neq j}^m \int_{\mathbb{R}^d}\int_0^1 \partial_j f_i(\zeta_\sigma)d\sigma w_i^+ w_j^- dx
\end{aligned}$$

Since $w_i^+\partial_t w_i = \frac{1}{2}\partial_t(w_i^+)^2$ and $\partial_j f_i(\zeta_\sigma) > 0$, by (3.40) and (3.42), we get

$$\begin{aligned}
\frac{1}{2}\partial_t\left[\int_{\mathbb{R}^d}(w_i^+)^2 dx\right] &\leq \int_{\mathbb{R}^d}\int_0^1 \partial_i f_i(\zeta_\sigma)d\sigma (w_i^+)^2 dx + \sum_{j=1,i\neq j}^m \int_{\mathbb{R}^d}\int_0^1 \partial_j f_i(\zeta_\sigma)d\sigma w_i^+ w_j^+ dx \\
&\leq \int_{\mathbb{R}^d}\int_0^1 \partial_i f_i(\zeta_\sigma)d\sigma (w_i^+)^2 dx + \frac{1}{2}\sum_{j=1,i\neq j}^m \int_{\mathbb{R}^d}\int_0^1 \partial_j f_i(\zeta_\sigma)d\sigma (w_i^+)^2 dx \\
&\qquad + \frac{1}{2}\sum_{j=1,i\neq j}^m \int_{\mathbb{R}^d}\int_0^1 \partial_j f_i(\zeta_\sigma)d\sigma (w_j^+)^2 dx \\
&\leq C\sum_{j=1}^m \int_{\mathbb{R}^d}(w_j^+)^2 dx
\end{aligned}$$

Doing this procedure for each $i \in \{1, ..., m\}$ and adding

$$\partial_t \left[\sum_{j=1}^{m} \int_{\mathbb{R}^d} (w_j^+)^2 dx \right] \leq C \sum_{j=1}^{m} \int_{\mathbb{R}^d} (w_j^+)^2 dx$$

By the Gronwall inequality

$$\sum_{j=1}^{m} \int_{\mathbb{R}^d} (w_j^+)^2 dx \leq e^{\int_0^t C ds} \sum_{j=1}^{m} \int_{\mathbb{R}^d} (w_j^+(0,x))^2 dx = 0$$

so,

$$\int_{\mathbb{R}^d} (w_j^+)^2 dx = 0, \qquad \forall j \in \{1, ..., m\}$$

then, we conclude

$$w_j(t,x) \leq 0, \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^d, \forall j$$

□

Remark 3.8 Note that as a consequence of Lemma 3.6 (in the case $d = 1$ and $d > 1$), we have the enough regularity to apply Theorem 3.7 to the solution of the problem (3.1)-(3.2).

3.3.3. Lower bound

The following is an important result needed to prove the Theorem 3.1, which sets an algebraically lower bound for the solutions of the cooperative system (3.1)-(3.2). This result is valid for any dimension d and for any index α_i.

Lemma 3.9 *Let $u = (u_i)_{i=1}^m$ the solution of the system (3.1)-(3.2), with initial condition u_0 satisfying (3.3) and f_i satisfying (3.4). Then, there exist constants $\sigma_i > 0$ and $\tau_1 > 0$ such that*

$$u_i(t + t_0, x) \geq \frac{C_i t e^{-\sigma_i t}}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}, \qquad \forall i \in \{1, ..., m\}$$

for all $t_0 > 0$, $x \in \mathbb{R}^d$ and $t \geq \tau_1$, with C_i positive constants that depend of t_0.

Proof. Step 1. From hypothesis (H2), we deduce the existence of a positive vector $M = mp$ with $m > 0$ large enough and $p_i = 1$ for all $i \in \{1, ..., m\}$ such that $F(M) \leq 0$, thus taking m large if necessary we have $u_0 \leq a \leq M$, furthermore, M is a supersolution of (3.1)-(3.2), then by Theorem 3.7, we conclude that $0 \leq u(t,x) \leq M$.

Now, since $f_i(0) = 0$, we have that

$$f_i(u) = \int_0^1 Df_i(\sigma u)d\sigma \cdot u = \sum_{j=1}^{m} u_j \int_0^1 \frac{\partial f_i}{\partial u_j}(\zeta_\sigma)d\sigma$$

where $\zeta_\sigma = \sigma u \in [0, u(t,x)] \subset [0, M]$ and $\frac{\partial f_i}{\partial u_j} : [0, M] \to \mathbb{R}$ is continuous for all $i, j \in \{1, ..., m\}$, using the fact that the system is cooperative, there exist constants $\gamma_{ij} > 0$ such that

$$\left|\frac{\partial f_i}{\partial u_i}(\zeta_\sigma)\right| \leq \gamma_{ii} \quad \text{and} \quad \gamma_{ij} \leq \frac{\partial f_i}{\partial u_j}(\zeta_\sigma) \ \text{ for all } \ i \neq j$$

So, taking $t_0 > 0$ fixed, we have that

$$\partial_t u_m + (-\triangle)^{\alpha_m} u_m = f_m(u) \geq \int_0^1 \frac{\partial f_m}{\partial u_m}(\zeta_\sigma)d\sigma u_m \geq -\gamma_{mm} u_m$$

for all $x \in \mathbb{R}^d$ and $t \geq t_0$, by the maximum principle of reaction diffusion equations, we have that

$$u_m(t + t_0, x) \geq e^{-\gamma_{mm} t}(p_m(t, \cdot) * u_m(t_0, \cdot))(x), \qquad \text{for all } t \geq 0$$

Since $u_m(t_0, \cdot) \not\equiv 0$ is continuous and nonnegative, we can find $\xi \in \mathbb{R}^d$ fixed such that $u_m(t_0, y) \geq C$ for all $y \in B_R(\xi)$ for some $R > 0$ and $C > 0$.

Taking $|x| > R$, $t \geq 1$ and using that $\alpha := \alpha_m < 1$

$$\begin{aligned}
(p_m(t, \cdot) * u_m(t_0, \cdot))(x) &= \int_{\mathbb{R}^d} p_m(t, x - y)u_m(t_0, y)dy \\
&\geq \int_{|y-\xi|\leq R} Cp_m(t, x - y)dy \\
&\geq C \int_{|y-\xi|\leq R} \frac{B^{-1}t}{t^{\frac{d}{2\alpha}+1} + |x - y|^{d+2\alpha}} dy \\
&= C \int_{|z|\leq R} \frac{B^{-1}t}{t^{\frac{d}{2\alpha}+1} + |x - \xi - z|^{d+2\alpha}} dz
\end{aligned}$$

also, $|x - \xi - z| \leq |x| + |\xi| + |z| \leq |x| + cR + R \leq (2 + c)|x|$, with the constant $c = |\xi|/R$, so

$$t^{\frac{d}{2\alpha}+1} + |x - \xi - z|^{d+2\alpha} \leq (2 + c)^{d+2\alpha} t^{\frac{d}{2\alpha}+1} + (2 + c)^{d+2\alpha}|x|^{d+2\alpha}$$

then

$$\begin{aligned}
(p_m(t, \cdot) * u_m(t_0, \cdot))(x) &\geq \frac{CB^{-1}}{(2 + c)^{d+2\alpha}} \int_{|z|\leq R} \frac{t}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}} dz \\
&= \frac{\widetilde{C}t}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}
\end{aligned}$$

Now, if $|x| \leq R$ and $t \geq 1$, taking $\rho > 0$ such that $\text{supp}(u_m(t_0, \cdot)) \cap B_\rho(0) \neq \emptyset$

$$\begin{aligned}(p_m(t, \cdot) * u_m(t_0, \cdot))(x) &\geq \int_{B_\rho(0)} \frac{B^{-1} t u_m(t_0, y)}{t^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}} dy \\ &\geq \frac{B^{-1}}{t^{\frac{d}{2\alpha}+1} + (R+\rho)^{d+2\alpha}} \int_{B_\rho(0)} u_m(t_0, y) dy \\ &\geq \overline{C} e^{-t}\end{aligned}$$

for some small constant $\overline{C} > 0$. Moreover, since $t \geq 1$

$$(p_m(t, \cdot) * u_m(t_0, \cdot))(x) \geq \overline{C} e^{-t} \geq \frac{\overline{C} t e^{-t}}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}$$

Then, there exist $C_m = C_m(t_0) > 0$ such that

$$u_m(t + t_0, x) \geq \frac{C_m t e^{-\sigma_m t}}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}, \qquad \forall x \in \mathbb{R}^d, t \geq 1$$

with $\sigma_m = \gamma_{mm} + 1$.

Step 2. To finish the proof is necessary to find a lower bound of the convolution between $p_i(1, \cdot)$ and $\frac{t}{t^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}}$ for all $t \geq 1$, Thus, by the same computations as above, it is possible to find a constant $C > 0$ such that

$$\int_{\mathbb{R}^d} \frac{t e^{-\frac{|y|^2}{4}}}{t^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}} dy \geq \frac{C t e^{-t}}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}, \qquad \forall x \in \mathbb{R}^d, t \geq 1$$

and

$$\int_{\mathbb{R}^d} \frac{1}{1 + |y|^{d+2\alpha_i}} \left[\frac{t}{t^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}} \right] dy \geq \frac{C t e^{-t}}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}, \qquad \forall x \in \mathbb{R}^d, t \geq 1$$

Step 3. Now, we set $i \in \{1, ..., m-1\}$, we have that

$$\begin{aligned}\partial_t u_i + (-\triangle)^{\alpha_i} u_i &= f_i(u) \geq \int_0^1 \frac{\partial f_i}{\partial u_m}(\zeta_\sigma) d\sigma u_m + \int_0^1 \frac{\partial f_i}{\partial u_i}(\zeta_\sigma) d\sigma u_i \\ &\geq \gamma_{im} u_m - \delta_i u_i\end{aligned}$$

for all $x \in \mathbb{R}^d$ and $t \geq t_0$, where $\delta_i \geq \max(\gamma_{ii}, \sigma_m + 2)$. Then, by the maximum principle of reaction diffusion equation and Duhamel formula, we have that

$$u_i(t+t_0,x) \geq e^{-\delta_i t}(H_i(t,\cdot) * u_i(t_0,\cdot))(x) + \gamma_{im} e^{-\delta_i t} \int_0^t \int_{\mathbb{R}^d} H_i(t-s,y) u_m(s+t_0, x-y) e^{\delta_i s} dyds$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, where

$$H_i(t,x) = \begin{cases} \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} & if\ \alpha_i = 1 \\ p_i(t,x) & if\ \alpha_i \in (0,1). \end{cases}$$

So, taking $t \geq \tau_1$ for any $\tau_1 \geq 3$

$$\begin{aligned} u_i(t+t_0,x) &\geq \gamma_{im} e^{-\delta_i t} \int_0^t \int_{\mathbb{R}^d} H_i(t-s,y) u_m(s+t_0,x-y) e^{\delta_i s} dyds \\ &\geq C_m \gamma_{im} e^{-\delta_i t} \int_1^{t-1} \int_{\mathbb{R}^d} H_i(t-s,y) \frac{s e^{(\delta_i - \sigma_m)s}}{s^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}} dyds \end{aligned}$$

Now, we have two cases. The first one, if $\alpha_i = 1$, then

$$u_i(t+t_0,x) \geq \frac{C_m \gamma_{im}}{(4\pi)^{\frac{d}{2}}} \frac{e^{-\delta_i t}}{t^{\frac{d}{2}}} \int_1^{t-1} e^{(\delta_i - \sigma_m)s} \int_{\mathbb{R}^d} \frac{s e^{-\frac{|y|^2}{4}}}{s^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}} dyds$$

and if $\alpha_i \in (0,1)$, then

$$u_i(t+t_0,x) \geq \frac{C_m \gamma_{im}}{B} \frac{e^{-\delta_i t}}{t^{\frac{d}{2\alpha_i}}} \int_1^{t-1} e^{(\delta_i - \sigma_m)s} \int_{\mathbb{R}^d} \frac{1}{1+|y|^{d+2\alpha_i}} \left[\frac{s}{s^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}} \right] dyds$$

in any cases, thanks to the Step 2, we can bound as follows

$$\begin{aligned} u_i(t+t_0,x) &\geq C_i \frac{e^{-\delta_i t}}{t^{\frac{d}{2\alpha}}} \int_1^{t-1} \frac{s e^{(\delta_i - \sigma_m} - 1)s}{s^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}} ds \\ &\geq C_i \frac{e^{-\delta_i t}(e^{t-1} - e)}{t^{\frac{d}{2\alpha}}(t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha})} \\ &\geq \frac{C_i t e^{-\sigma_i t}}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}} \qquad \forall x \in \mathbb{R}^d, t \geq \tau_1 \end{aligned}$$

with τ_1 larger if necessary and taking $\sigma_i := \delta_i$.

Then, we proved that there exist constants $C_i > 0$ and $\sigma_i > 0$ such that

$$u_i(t+t_0,x) \geq \frac{C_i t e^{-\sigma_i t}}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}, \qquad \forall i \in \{1,...,m\}$$

for all $x \in \mathbb{R}^d$ and $t \geq \tau_1$. □

3.4. Proof of Theorem 3.1

In order to prove Theorem 3.1, we need to construct explicit sub and super solution, which will have the form of the following vector field

$$v(t,x) = a\left(1 + b(t)|x|^{\delta(d+2\alpha)}\right)^{-\frac{1}{\delta}}\phi \tag{3.43}$$

where $b(t)$ is a time continuous function and $\phi \in \mathbb{R}^m$ is the normalized principal eigenvector of $DF(0)$ associated to the principal eigenvalue λ_1 and $\delta \in \mathbb{R}_+$ is taken as in the hypothesis (H2) and (H3). Note that, since the system is cooperative, $\frac{\partial f_j}{\partial u_i}(0) > 0$ for all $i \neq j \in \{1, ..., m\}$, by Perron-Frobenius Theorem, we can take $\phi > 0$.

The following result allow us to understand the behavior of the fractional laplacian $(-\triangle)^{\alpha_i}$ on the function v defined by (3.43). The proof of this result is based in a result proved by Bonforte and vázquez in [13], also a similar result was announced in [16].

Lemma 3.10 *Let v be defined as in (3.43). Then, there exist a constant $D > 0$ such that*

$$|\,(-\triangle)^{\alpha_i}v_i\,| \leq Db(t)^{\frac{2\alpha_i}{\delta(d+2\alpha)}}v_i, \qquad in\ \mathbb{R}^d$$

with $\alpha_i \in (0,1]$, for all $i \in \{1, ..., m\}$.

Proof. In the case $\alpha_i \in (0,1)$ with $\delta \geq \frac{2}{d+2\alpha}$, we only need to prove

$$|\,(-\triangle)^{\alpha_i}w(x)\,| \leq Dw(x) \tag{3.44}$$

where $w(x) = (1+ \mid x \mid^{\delta(d+2\alpha)})^{-\frac{1}{\delta}}$. Indeed, note that $v(t,x) = aw(b(t)^{\frac{1}{\delta(d+2\alpha)}}x)\phi$, also, since $w \in C^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ we have that $(-\triangle)^{\alpha_i}w \in L^\infty(\mathbb{R}^d)$ and it is $2\alpha_i$-homogeneous, thus

$$(-\triangle)^{\alpha_i}v_i(t,x) = a\phi_i(-\triangle)^{\alpha_i}w(b(t)^{\frac{1}{\delta(d+2\alpha)}}x) = a\phi_i b(t)^{\frac{2\alpha_i}{\delta(d+2\alpha)}}(-\triangle)^{\alpha_i}w(y)$$

where $y = b(t)^{\frac{1}{\delta(d+2\alpha)}}x$. Moreover $w(x) \leq |x|^{-(d+2\alpha)}$ and $|D^2w(x)| \leq c|x|^{-(d+2\alpha+2)}$ for $|x|$ large enough.

Since $(-\triangle)^{\alpha_i}w \in L^\infty(\mathbb{R}^d)$, taking D large enough, it is sufficient to prove the result for large values of $|x|$. We have to estimate

$$|\,(-\triangle)^{\alpha_i}w(x)\,| = C(d,\alpha_i)\left|P.V.\int_{\mathbb{R}^d}\frac{w(x)-w(y)}{|x-y|^{d+2\alpha_i}}dy\right|$$

hence

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{w(x)-w(y)}{|x-y|^{d+2\alpha_i}}dy &= \int_{|y|>3|x|/2}\frac{w(x)-w(y)}{|x-y|^{d+2\alpha_i}}dy + \int_{\{|x|\leq 2|y|\leq 3|x|\}\setminus B_{|x|/2}(x)}\frac{w(x)-w(y)}{|x-y|^{d+2\alpha_i}}dy \\ &\quad + \int_{B_{|x|/2}(x)}\frac{w(x)-w(y)}{|x-y|^{d+2\alpha_i}}dy + \int_{|y|\leq|x|/2}\frac{w(x)-w(y)}{|x-y|^{d+2\alpha_i}}dy \\ &:= I_1 + I_2 + I_3 + I_4\end{aligned}$$

In the first integral, since $|y| > 3|x|/2$ we have $w(y) \leq w(x)$ and then $|w(x) - w(y)| \leq w(x)$, hence

$$|I_1| = \left| \int_{|y|>3|x|/2} \frac{w(x) - w(y)}{|x-y|^{d+2\alpha_i}} dy \right| \leq C w(x) \int_{3|x|/2}^{\infty} \frac{1}{r^{1+2\alpha_i}} dr \leq \frac{C_1}{|x|^{d+2\alpha+2\alpha_i}}$$

since $w(x) \leq |x|^{-(d+2\alpha)}$ for $|x|$ large enough.

Following a similar idea as in the above computation, since $|y| > 3|x|/2$ we have $w(y) \leq w(x/2)$ and then $|w(x) - w(y)| \leq w(x/2)$, also note that $w(x/2) \leq |x/2|^{-(d+2\alpha)}$ for $|x|$ large enough, hence

$$|I_2| = \left| \int_{\{|x| \leq 2|y| \leq 3|x|\} \setminus B_{|x|/2}(x)} \frac{w(x) - w(y)}{|x-y|^{d+2\alpha_i}} dy \right| \leq \frac{C w(x/2)}{(|x|/2)^{d+2\alpha_i}} \int_{|x|/2}^{3|x|/2} r^{d-1} dr \leq \frac{C_2}{|x|^{d+2\alpha+2\alpha_i}}$$

since $|x-y| \geq |x|/2$.

Before to continue, note that

$$P.V. \int_{B_{|x|/2}(x)} \frac{\nabla w(x) \cdot (x-y)}{|x-y|^{d+2\alpha}} dy = 0$$

moreover, if $|x-y| < |x|/2$ then $|x|/2 < |y| < 3|x|/2$, therefore

$$|\partial^2_{ij} w(y)| \leq \frac{c}{|y|^{d+2\alpha+2}} \leq \frac{c 2^{d+2\alpha+2}}{|x|^{d+2\alpha+2}}$$

for all $y \in B_{|x|/2}(x)$. Now we can estimate I_3 as follows:

$$\begin{aligned}
|I_3| &= \left| \int_{B_{|x|/2}(x)} \frac{w(x) - w(y)}{|x-y|^{d+2\alpha_i}} dy \right| \\
&= \left| \int_{B_{|x|/2}(x)} \frac{\nabla w(x) \cdot (x-y)}{|x-y|^{d+2\alpha_i}} dy + \int_{B_{|x|/2}(x)} \frac{(x-y)^t D^2 w(\overline{x})(x-y)}{|x-y|^{d+2\alpha_i}} dy \right| \\
&\leq \sup_{1 \leq i,j \leq d} \|\partial_{ij} w\|_{L^\infty(B_{|x|/2}(x))} \left| \int_{B_{|x|/2}(x)} \frac{1}{|x-y|^{d+2\alpha_i-2}} dy \right| \\
&\leq \frac{C}{|x|^{d+2\alpha+2}} \int_0^{|x|/2} \frac{1}{r^{2\alpha_i-1}} dr \\
&\leq \frac{C}{|x|^{d+2\alpha+2}} \left(\frac{|x|}{2} \right)^{2-2\alpha_i} \\
&= \frac{C_3}{|x|^{d+2\alpha_i+2\alpha}}
\end{aligned}$$

It only remains to estimate the fourth integral. Note that $|y| \leq |x|/2$ implies $w(x) \leq w(y)$ which gives $|w(x) - w(y)| \leq w(y)$, thus

$$|I_4| \leq \int_{|y|\leq|x|/2} \frac{|w(x) - w(y)|}{|x-y|^{d+2\alpha_i}} dy \leq \frac{2^{d+2\alpha_i}}{|x|^{d+2\alpha_i}} \int_{\mathbb{R}^d} w(y)dy \leq \frac{C_4}{|x|^{d+2\alpha_i}}$$

since $|x-y| \geq |x|/2$ and $|x|$ is large enough. Therefore, we conclude that

$$|\,(-\triangle)^{\alpha_i} w(x)\,| \leq \frac{C_5}{|x|^{d+2\alpha}} \leq Dw(x)$$

since $|x|$ is large enough and for some constant D.

Now, for the case in which $\alpha_i = 1$ for some i and $\delta \geq \frac{2}{d+2\alpha}$, by a similar analysis as in the previous case, we only need to prove

$$|\,-\triangle w(x)\,| \leq Dw(x) \tag{3.45}$$

where $w(x) = (1+ |\,x\,|^{\delta(d+2\alpha)})^{-\frac{1}{\delta}}$. In what follows, we prove (3.45)

$$\begin{aligned} |-\triangle v| &\leq \frac{1}{(1+|x|^{\delta(d+2\alpha)})^{\frac{1}{\delta}}} \left[\frac{C_1 |x|^{2(\delta(d+2\alpha)-1)}}{(1+|x|^{\delta(d+2\alpha)})^2} + \frac{C_2 |x|^{\delta(d+2\alpha)-2}}{1+|x|^{\delta(d+2\alpha)}} \right] \\ &\leq \frac{1}{(1+|x|^{\delta(d+2\alpha)})^{\frac{1}{\delta}}} \left[C_1|x|^{-2} + C_2|x|^{-2} \right] \\ &\leq Dv(x) \end{aligned}$$

since $|x|$ is large and for some constant D. □

In what follows, we will use the results of previous sections to obtain appropriate classical sub and super solutions of the system (3.1)-(3.2) with the form of the vector field (3.43), then we use the comparison principle to obtain the desired results.

We divide the proof of Theorem 3.1 in two lemmas.

Lemma 3.11 *Let $d \geq 1$ and assume that F satisfies (3.4), (H1) and (H2). Let u be the solution to (3.1)-(3.2) with u_0 satisfying (3.3). Then, for every $\mu = (\mu_i)_{i=1}^m > 0$, there exist $c > 0$ such that, for all $t > \tau$*

$$\left\{ x \in \mathbb{R}^d \mid\ |x| > c e^{\frac{\lambda_1}{d+2\alpha} t} \right\} \subset \left\{ x \in \mathbb{R}^d \mid\ u(t,x) < \mu \right\}$$

with $\tau > 0$ large enough.

Proof: We consider the function

$$\overline{u}(t,x) = \overline{a}\left(1+\overline{b}(t) \mid x \mid^{\delta_1(d+2\alpha)}\right)^{-\frac{1}{\delta_1}} \phi$$

where $\phi \in \mathbb{R}^m$ is the normalized principal eigenvector of $DF(0)$ associated to the principal eigenvalue λ_1. The idea is adjust $\overline{a} > 0$ and $\overline{b}(t)$ asymptotically proportional to $e^{-\delta_1\lambda_1 t}$ so that the function $\overline{u}$ serves as supersolution of the problem (3.1)-(3.2).

Before to continue, we choose a constant $\overline{B} < (1 + D\lambda_1^{-1})^{-\frac{\delta_1(d+2\alpha)}{2\alpha}}$ where $D > 0$ is given in Lemma 3.10. Now, we consider the problem

$$-\overline{b}'(t) - \delta_1 D\overline{b}(t)^{\frac{2\alpha}{\delta_1(d+2\alpha)}+1} - \delta_1\lambda_1\overline{b}(t) = 0, \;\; \overline{b}(0) = (-D\lambda_1^{-1} + \overline{B}^{-\frac{2\alpha}{\delta_1(d+2\alpha)}})^{-\frac{\delta_1(d+2\alpha)}{2\alpha}}$$

which has a solution given by $\overline{b}(t) = (-D\lambda_1^{-1} + \overline{B}^{-\frac{2\alpha}{\delta_1(d+2\alpha)}} e^{\frac{2\alpha\lambda_1}{d+2\alpha}t})^{-\frac{\delta_1(d+2\alpha)}{2\alpha}}$, note that, $0 \leq \overline{b}(t) \leq \overline{b}(0) \leq 1$.

Using Lemma 3.10, note that for all $i \in \{1, ..., m\}$

$$\begin{aligned}
\partial_t\overline{u}_i + (-\triangle)^{\alpha_i}\overline{u}_i - f_i(\overline{u}) &= \partial_t\overline{u}_i + (-\triangle)^{\alpha_i}\overline{u}_i - Df_i(0)\overline{u} + [Df_i(0)\overline{u} - f_i(\overline{u})] \\
&\geq -\frac{\overline{a}\phi_i\overline{b}'(t)|x|^{\delta_1(d+2\alpha)}}{\delta_1(1+\overline{b}(t)|x|^{\delta_1(d+2\alpha)})^{\frac{1}{\delta_1}+1}} - D\overline{b}(t)^{\frac{2\alpha_i}{\delta_1(d+2\alpha)}}\overline{u}_i - \lambda_1\overline{u}_i + \frac{c\phi_i^{1+\delta_1}\overline{a}^{1+\delta_1}}{(1+\overline{b}(t)|x|^{\delta_1(d+2\alpha)})^{\frac{1}{\delta_1}+1}} \\
&\geq \frac{\overline{a}\phi_i}{\delta_1(1+\overline{b}(t)|x|^{\delta_1(d+2\alpha)})^{\frac{1}{\delta_1}+1}}\left\{-\overline{b}'(t) - \delta_1 D\overline{b}(t)^{\frac{2\alpha}{\delta_1(d+2\alpha)}+1} - \delta_1\lambda_1\overline{b}(t)\right\}|x|^{\delta_1(d+2\alpha)} \\
&\quad + \frac{\overline{a}\phi_i}{(1+\overline{b}(t)|x|^{\delta_1(d+2\alpha)})^{\frac{1}{\delta_1}+1}}\left\{-D\overline{b}(t)^{\frac{2\alpha}{\delta_1(d+2\alpha)}} - \lambda_1 + c\phi_i^{\delta_1}\overline{a}^{\delta}\right\}
\end{aligned} \tag{3.46}$$

in the last inequality we use that $\alpha \leq \alpha_i$ and $\overline{b}(t) \leq 1$. Thus, taking

$$\overline{a} \geq \frac{(D+\lambda_1)^{\frac{1}{\delta_1}}}{c^{\frac{1}{\delta_1}}\min_i(\phi_i)}$$

the right hand side of inequality (3.46) is bigger than or equal to 0 for all $t > 0$.

To end the proof, for any $t_0 > 0$ fixed, we can take $\overline{a}$ satisfying the above condition and $t_1 > t_0$ such that

$$\overline{u}_i(t_1, x) \geq u_i(t_0, x), \qquad \forall x \in \mathbb{R}^d, \forall i \in \{1, ..., m\}$$

note that, this is possible due to Lemma 3.6 (in the case $d = 1$ and $d > 1$). Therefore, we conclude that $\overline{u}$ is a supersolution to (3.1)-(3.2) for all $t \geq t_1$. Thus, using Theorem 3.7, we get for all $t \geq t_0$

$$\overline{u}_i(t + t_1 - t_0, x) \geq u_i(t, x), \qquad \forall x \in \mathbb{R}^d, \forall i \in \{1, ..., m\} \tag{3.47}$$

Now, given any $(\mu_i)_{i=1}^m > 0$, we define

$$c_i^{d+2\alpha} := \frac{\overline{a}\phi_i e^{\lambda_1(t_1-t_0)}}{\mu_i \overline{B}^{\frac{1}{\delta_1}}}$$

and since $\overline{B}e^{-\delta_1\lambda_1(t+t_1-t_0)} \le \overline{b}(t+t_1-t_0)$, we have that, if

$$|x| > c_i e^{\frac{\lambda_1}{d+2\alpha}t} \qquad\qquad \text{for all } i \in \{1,..,m\}$$

then

$$|x|^{\delta_1(d+2\alpha)} > \frac{\overline{a}^{\delta_1}\phi_i^{\delta_1}}{\mu_i^{\delta_1}\overline{B}} e^{\delta_1\lambda_1(t+t_1-t_0)}$$

Therefore

$$\begin{aligned}\frac{\overline{a}^{\delta_1}\phi_i^{\delta_1}}{\mu_i^{\delta_1}} &< \overline{B}e^{-\delta_1\lambda_1(t+t_1-t_0)}|x|^{\delta_1(d+2\alpha)} \\ &\le \overline{b}(t+t_1-t_0)|x|^{\delta_1(d+2\alpha)} \\ &\le 1+\overline{b}(t+t_1-t_0)|x|^{\delta_1(d+2\alpha)}\end{aligned}$$

Thus, using (3.47)

$$u_i(t,x) \le \overline{u}_i(t+t_1-t_0,x) = \frac{\overline{a}\phi_i}{(1+\overline{b}(t+t_1-t_0)|x|^{\delta_1(d+2\alpha)})^{\frac{1}{\delta_1}}} < \mu_i$$

By the above computations, and taking $c = \max_i\{c_i\}$, we conclude that

$$\left\{x \in \mathbb{R}^d \mid \ |x| > ce^{\frac{\lambda_1}{d+2\alpha}t}\right\} \subset \left\{x \in \mathbb{R}^d \mid \ u_i(t,x) < \mu_i\right\}$$

for all $t > \tau$, with $\tau := t_0$.

□

Lemma 3.12 *Let $d \ge 1$ and assume that F satisfies 3.4, (H1) and (H3). Let u be the solution to (3.1)-(3.2) with u_0 satisfying (3.3). Then, for all $i \in \{1,...,m\}$, there exist constants $\varepsilon_i > 0$ and $C > 0$ such that,*

$$u_i(t,x) > \varepsilon_i, \quad \textit{for all } t \ge \tau \textit{ and } |x| < Ce^{\frac{\lambda_1}{d+2\alpha}t}$$

with $\tau > 0$ large enough.

Proof: As in the previous proof, we consider the function

$$\underline{u}(t,x) = \underline{a}\left(1+\underline{b}(t)|x|^{\delta_2(d+2\alpha)}\right)^{-\frac{1}{\delta_2}}\phi$$

where $\phi \in \mathbb{R}^m$ is the normalized principal eigenvector of $DF(0)$ associated to the principal eigenvalue λ_1, $\underline{a} > 0$ and $\underline{b}(t)$ asymptotically proportional to $e^{-\delta_2\lambda_1 t}$. In the following we choose $\underline{a}$ and $\underline{b}$ such that the function $\underline{u}$ serves as subsolution of the problem (3.1)-(3.2).

Taking for the moment $\underline{B} \leq (L\lambda_1^{-1})^{-\frac{\delta_2(d+2\alpha)}{2\alpha}}$ with any constant $L \geq \max\{D, \lambda_1\}$. We consider the function $\underline{b}(t) = (L\lambda_1^{-1} + \underline{B}^{-\frac{2\alpha}{\delta_2(d+2\alpha)}} e^{\frac{2\alpha\lambda_1}{d+2\alpha}t})^{-\frac{\delta_2(d+2\alpha)}{2\alpha}}$ which a solution of the problem

$$-\underline{b}'(t) + \delta_2 L\underline{b}(t)^{\frac{2\alpha}{\delta_2(d+2\alpha)}+1} - \delta_2\lambda_1\underline{b}(t) = 0, \;\; \underline{b}(0) = (L\lambda_1^{-1} + \underline{B}^{-\frac{2\alpha}{\delta_2(d+2\alpha)}})^{-\frac{\delta_2(d+2\alpha)}{2\alpha}}$$

also, note that $\underline{b}(t) \leq \underline{b}(0) \leq 1$, since $L > \lambda_1$.

Using Lemma 3.10, we have that for all $i \in \{1,...,m\}$

$$\begin{aligned}
\partial_t\underline{u}_i + (-\triangle)^{\alpha_i}\underline{u}_i - f_i(\underline{u}) &\leq \partial_t\underline{u}_i + (-\triangle)^{\alpha_i}\underline{u}_i - Df_i(0)\underline{u} + \mid Df_i(0)\underline{u} - f_i(\underline{u}) \mid \\
&\leq -\frac{\underline{a}\phi_i\underline{b}'(t)|x|^{\delta_2(d+2\alpha)}}{\delta_2(1+\underline{b}(t)|x|^{\delta_2(d+2\alpha)})^{\frac{1}{\delta_2}+1}} + D\underline{b}(t)^{\frac{2\alpha_i}{\delta_2(d+2\alpha)}}\underline{u}_i - \lambda_1\underline{u}_i + \frac{c\underline{a}^{1+\delta_2}}{(1+\underline{b}(t)|x|^{\delta_2(d+2\alpha)})^{\frac{1}{\delta_2}+1}} \\
&\leq \frac{\underline{a}\phi_i}{\delta_2(1+\underline{b}(t)|x|^{\delta_2(d+2\alpha)})^{\frac{1}{\delta_2}+1}}\left\{-\underline{b}'(t) + \delta_2 L\underline{b}(t)^{\frac{2\alpha}{\delta_2(d+2\alpha)}+1} - \delta_2\lambda_1\underline{b}(t)\right\}|x|^{\delta_2(d+2\alpha)} \\
&\qquad + \frac{\underline{a}\phi_i}{(1+\underline{b}(t)|x|^{\delta_2(d+2\alpha)})^{\frac{1}{\delta_2}+1}}\left\{L\underline{b}(t)^{\frac{2\alpha}{\delta_2(d+2\alpha)}} - \lambda_1 + \frac{c\underline{a}^{\delta_2}}{\phi_i}\right\}
\end{aligned} \tag{3.48}$$

in the last inequality we use that $\alpha \leq \alpha_i$ and $\underline{b}(t) \leq 1$. Thus, taking

$$\underline{a} \leq \left(\frac{\min_i\{\phi_i\}\lambda_1}{2c}\right)^{\frac{1}{\delta_2}}$$

the right hand side of inequality (3.48) is less than or equal to 0 for all $t > 0$.

Since, $\underline{u}_i(0,\cdot) \leq u_{0i}$ may not hold for all $i \in \{1,...,m\}$, we look for a time $t_1 \geq \max\{\tau_1, 2L\lambda_1^{-1}\}$, where τ_1 was defined in Lemma 3.9. Moreover, for any $t_0 > 0$ fixed, we know that

$$u_i(t_1+t_0,x) \geq \frac{c_i t_1 e^{-\sigma_i t_1}}{t_1^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}, \qquad \forall i \in \{1,...,m\}, \forall x \in \mathbb{R}^d$$

Consequently, we choose

$$\underline{a} = \frac{\min_i\{c_i\}e^{-\max_i\{\sigma_i\}t_1}}{2\max_i\{\phi_i\}t_1^{\frac{d}{2\alpha}}}, \qquad \underline{B} = \left(\frac{2}{t_1}\right)^{\frac{\delta_2(d+2\alpha)}{2\alpha}}$$

taking t_1 large if necessary, such that, the requirements

$$\underline{a} \leq \left(\frac{\min_i\{\phi_i\}\lambda_1}{2c}\right)^{\frac{1}{\delta_2}} \quad \text{and} \quad \underline{B} \leq (L\lambda_1^{-1})^{-\frac{\delta_2(d+2\alpha)}{2\alpha}}$$

are satisfied.

By the election of $\underline{a}$ and $\underline{B}$, we deduce that

- since
$$\underline{a} \leq \frac{c_i e^{-\sigma_i t_1}}{2\phi_i t_1^{\frac{d}{2\alpha}}}, \qquad \text{then} \quad \underline{a}\phi_i t_1^{\frac{d}{2\alpha}+1} \leq \frac{c_i}{2} t_1 e^{-\sigma_i t_1}$$

- By the election of t_1, we deduce that $t_1 \geq 2L\lambda_1^{-1}$, thus
$$\underline{B}^{\frac{2\alpha}{\delta_2(d+2\alpha)}} = \frac{2}{t_1} \geq \frac{1}{t_1 - L\lambda_1^{-1}}$$
therefore,
$$b(0)^{-\frac{1}{\delta_2}} = \left[\left(L\lambda_1^{-1} + \underline{B}^{-\frac{2\alpha}{\delta_2(d+2\alpha)}}\right)^{\frac{\delta_2(d+2\alpha)}{2\alpha}}\right]^{\frac{1}{\delta_2}} \leq t_1^{\frac{d+2\alpha}{2\alpha}}$$
and
$$\frac{2\underline{a}\phi_i}{c_i e^{-\sigma_i t_1} t_1} \leq \frac{1}{t_1^{\frac{d+2\alpha}{2\alpha}}} \leq b(0)^{\frac{1}{\delta_2}}$$

Then, we deduce

$$\frac{c_i}{2} t_1 e^{-\sigma_i t_1} - \underline{a}\phi_i t_1^{\frac{d}{2\alpha}+1} \geq 0 \geq \left[\underline{a}\phi_i - \frac{c_i}{2} b(0)^{\frac{1}{\delta_2}} e^{-\sigma_i t_1} t_1\right] |x|^{d+2\alpha}$$

so,

$$\frac{c_i}{2} t_1 e^{-\sigma_i t_1} (1 + b(0)^{\frac{1}{\delta_2}} |x|^{d+2\alpha}) \geq \underline{a}\phi_i (t_1^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha})$$

but

$$(1 + b(0)|x|^{\delta_2(d+2\alpha)})^{\frac{1}{\delta_2}} \geq \frac{1 + b(0)^{\frac{1}{\delta_2}} |x|^{d+2\alpha}}{2}$$

Therefore,

$$\frac{c_i t_1 e^{-\sigma_i t_1}}{t_1^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}} \geq \frac{\underline{a}\phi_i}{(1+b(0)|x|^{\delta_2(d+2\alpha)})^{\frac{1}{\delta_2}}}$$

and then we get

$$u_i(t_1+t_0, x) \geq \underline{u}_i(0, x), \qquad \forall x \in \mathbb{R}^d,\ i \in \{1, ..., m\}$$

By Theorem 3.7, we have for all $i \in \{1, ..., m\}$

$$u_i(t, x) \geq \underline{u}_i(t - t_1 - t_0, x), \qquad \forall x \in \mathbb{R}^d, t \geq t_1 + t_0 \tag{3.49}$$

Let us define

$$\varepsilon_i = \frac{\underline{a}\phi_i}{2^{\frac{1}{\delta_2}}} \quad \text{and} \quad C^{d+2\alpha} = e^{-\lambda_1(t_1+t_0)}\underline{B}^{-\frac{1}{\delta_2}}$$

Then, if $t \geq t_1 + t_0$ and $|x| \leq Ce^{\frac{\lambda_1}{d+2\alpha}t}$, we get that $u_i(t, x) \geq \varepsilon_i$ for all $i \in \{1, ..., m\}$.

Indeed, taking $t \geq t_1 + t_0$ and $|x| \leq Ce^{\frac{\lambda_1}{d+2\alpha}t}$, we have that

$$|x|^{\delta_2(d+2\alpha)} \leq C^{\delta_2(d+2\alpha)} e^{\delta_2\lambda_1 t} = \underline{B}^{-1} e^{\delta_2\lambda_1(t-t_1-t_0)}$$

Since $\underline{b}(t) \leq Be^{-\delta_2\lambda_1 t}$, then

$$\underline{b}(t - t_1 - t_0)|x|^{\delta_2(d+2\alpha)} \leq 1$$

Using (3.49), we conclude

$$u_i(t, x) \geq \underline{u}_i(t - t_1 - t_0, x) = \frac{\underline{a}\phi_i}{(1+\underline{b}(t-t_1-t_0)|x|^{\delta_2(d+2\alpha)})^{\frac{1}{\delta_2}}} \geq \frac{\underline{a}\phi_i}{2^{\frac{1}{\delta_2}}} = \varepsilon_i$$

for all $t > \tau$, with $\tau := t_1 + t_0$. □

To conclude this section, the proof of Theorem 3.1 follows directly from Lemmas 3.11 and 3.12.

3.5. Proof of Theorem 3.2

In this subsection, we prove that under some appropriate assumptions on the nonlinearity and the initial datum, it is possible to state that the solution $u(t, x)$ to (3.1)-(3.2) tends, as $t \to +\infty$, to the smallest constant positive steady solution of (3.1)-(3.2). In order to prove this result, let define by ϕ the positive constant eigenvector of $DF(0)$ associated to the first eigenvalue λ_1, where $F = (f_i)_{i=1}^m$. Thus λ_1 and $\phi > 0$ satisfy

$$\begin{array}{c}(L - DF(0))\phi = -\lambda_1 \phi \\ \phi > 0, \ \ \|\phi\| = 1\end{array}$$

where $L = \mathrm{diag}((-\triangle)^{\alpha_1}, ..., (-\triangle)^{\alpha_m})$. Now, let consider the problem

$$\begin{array}{rcl}\dot{\chi}_\varepsilon(t) &=& F(\chi_\varepsilon(t)) \\ \chi_\varepsilon(0) &=& \varepsilon\phi\end{array} \tag{3.50}$$

thus, there exists $\varepsilon' > 0$ such that, for each $\varepsilon \in (0, \varepsilon')$ we can find a constant $u_\varepsilon^+ > 0$ satisfying $\chi_\varepsilon(t) \nearrow u_\varepsilon^+$ as $t \to +\infty$, also $F(u_\varepsilon^+) = 0$. We define

$$u^+ = \inf_{\varepsilon \in (0,\varepsilon')} u_\varepsilon^+$$

since F is continuous, we deduce that $F(u^+) = 0$. Also, since the function F is positive in a small ball near to zero, we deduce that $u_+ > 0$.

Moreover we assume that the initial condition u_0 satisfies

$$u_0 \leq u^+ \qquad \text{in } \mathbb{R}^d$$

We prove Theorem 3.2 through a succession of lemmas. Let $B_R(0)$ be the open ball of $\mathbb{R}^d$, with center 0 and radius R. Let us call u_R the unique solution of the elliptic system

$$\begin{array}{rcll}(-\triangle)^{\alpha_i} u_i^R &=& f_i(u^R), & \forall x \in B_R(0) \\ u^R &=& 0 & \textit{on } \mathbb{R}^d \setminus B_R(0) \\ u^R &>& 0 & \textit{on } B_R(0)\end{array} \tag{3.51}$$

Lemma 3.13 *Let $d \geq 1$, $\varepsilon > 0$ and assume that F satisfies (3.4), (H1), (H2), (H3) and (H4). There exists $R > 0$ such that the solution v^R of the system*

$$\begin{array}{rcll}\partial_t v_i^R + (-\triangle)^{\alpha_i} v_i^R &=& f_i(v^R), & \forall t > 0, x \in B_R(0) \\ v^R(t,x) &=& 0 & \textit{on } t \geq 0, x \in \mathbb{R}^d \setminus B_R(0) \\ 0 < v^R(0,x) &\leq& \min(\varepsilon, u^R), & \textit{on } x \in B_R(0)\end{array} \tag{3.52}$$

satisfies

$$\lim_{t \to +\infty} v^R(t,x) = u^R(x) \qquad \forall\, x \in B_1(0)$$

Proof: Let ϕ^R be the positive eigenvalue associated to λ_R in the ball $B_R(0)$, thus ϕ^R and λ_R satisfy

$$(L - DF(0))\phi^R = \lambda_R \phi^R \qquad in\ B_R(0)$$
$$\phi^R > 0\ \ in\ \ B_R(0),\ \phi^R = 0\ \ in\ \ \mathbb{R}^d \setminus B_R(0),\ \ \|\phi^R\| = 1$$

Now, following the same computations as in [10], we can deduce that λ_R given by the minimum of

$$\frac{\frac{1}{2}\sum_{i=1}^m \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \frac{(\phi_i(x)-\phi_i(y))^2}{|x-y|^{d+2\alpha}} dy\right) dx - \int_{B_R(0)} [DF(0)\phi(x)] \cdot \phi(x)\, dx}{\sum_{i=1}^m \int_{B_R(0)} \phi_i(x)^2\, dx}$$

taken over all functions $\phi \in C^1(B_R(0)) \cap C(\overline{B}_R(0))$, $\phi \not\equiv 0$, vanishing on $\partial B_R(0)$ and extended by 0 outside $B_R(0)$, converges to $-\lambda_1$ when R goes to infinite, moreover, by hypothesis (H1) we have that $\lambda_1 > 0$. Thus, we can find $R > 0$ large enough such that $\lambda_R < 0$.

Since u^R and v^R satisfy (3.51) and (3.52) in the ball $B_R(0)$, and $u^R = v^R = 0$ in $\mathbb{R}^d \setminus B_R(0)$, then both functions satisfy the system (3.1)-(3.2) with initial conditions $v^R(0,\cdot)$ and $u^R(\cdot)$ respectively, moreover $v^R(0,\cdot) \leq u^R(\cdot)$ in $\mathbb{R}^d$. Thus, by Theorem 3.7 we have that $v^R(t,x) \leq u^R(x)$ for all $R > 0$, $t > 0$ and $x \in \mathbb{R}^d$.

Let w^R be the solution of

$$\begin{aligned} \partial_t w_i^R + (-\triangle)^{\alpha_i} w_i^R &= f_i(w^R), && \forall t > 0, x \in B_R(0) \\ w^R(t,x) &= 0 && on\ t \geq 0, x \in \mathbb{R}^d \setminus B_R(0) \\ w^R(0,x) &= k\phi^R(x) && on\ x \in B_R(0) \end{aligned} \tag{3.53}$$

Taking $k > 0$, we deduce

$$\begin{aligned} f_i(k\phi^R) &\geq kDf_i(0)\phi^R - ck^{1+\delta_2}\|\phi^R\|^{1+\delta_2} \\ &= kDf_i(0)\phi^R - ck^{1+\delta_2} \end{aligned}$$

Therefore, it follows from the above inequality and by the definition of ϕ^R that

$$k(-\triangle)^{\alpha_i}\phi_i^R - \underline{f}_i(k\phi^R) = k\left(\lambda_R \phi_i^R + ck^{\delta_2}\right) \leq 0, \qquad in\ B_R(0)$$

for all $i \in \{1, ..., m\}$, taking k small enough and since $\lambda_R < 0$. Then $k\phi^R$ is a subsolution of (3.51) in the ball $B_R(0)$. Thus w^R is nondecreasing in time t. Moreover, taking $k > 0$ small if necessary, $w^R(0,x) \leq v^R(0,x)$ in $\mathbb{R}^d$, thus by Theorem 3.7

$$w^R(t,x) \leq v^R(t,x), \qquad \forall\, t > 0, x \in B_R(0)$$

Finally, one has

$$w^R(t,x) \le v^R(t,x) \le u^R(x), \qquad \forall\, t>0, x \in B_R(0)$$

Since w^R is nondecreasing in time t, standard elliptic estimates imply that w^R converges locally to a stationary solution $w^\infty (\le u^R)$ of (3.53). But since u^R is the unique solution of (3.51), we conclude that $w^\infty = u^R$ in $B_1(0)$. □

Remark 3.14 Let note that for each $y \in \mathbb{R}^d$, if $x \in B_1(y)$ then $x - y \in B_1(0)$. Thus taking $\sigma = (\sigma)_{i=1}^m > 0$, as a consequence of Lemma 3.13, there exist $R > 0$ and $T_\sigma > 0$ that not depend of y, such that, for all $t \ge T_\sigma$

$$|v_i^R(t, x-y) - u_i^R(x-y)| \le \sigma_i \qquad \forall\, x \in B_1(y)$$

for each $i \in \{1, ..., m\}$.

The proof of Theorem 3.2 essentially relies on the following property in which we prove that any steady state solution of (3.1)-(3.2) is bounded from below away from zero.

Lemma 3.15 *Let $d \ge 1$ and assume that F satisfies (3.4), (H1), (H2), (H3) and (H4). Let v be a positive bounded solution of*

$$(-\triangle)^{\alpha_i} v_i \;=\; f_i(v), \qquad \forall\, i \in \{1, ..., m\} \tag{3.54}$$

Then, there exists $\varepsilon > 0$ small enough such that $v \ge \varepsilon\phi$ in $\mathbb{R}^d$.

Proof: In what follows we prove that there exists a constant vector $k > 0$ such that $v \ge k$ in $\mathbb{R}^d$. Let $y \in \mathbb{R}^d$ be any arbitrary fixed vector, we note that $v(\cdot + y)$ continue satisfying (3.54), moreover, for each $R > 0$, there exists a constant $k_{y,R} > 0$ such that $v(x+y) \ge k_{y,R}$ for all $x \in B_R(0)$.

Now, let consider the system

$$\begin{aligned}
\partial_t w_i^R + (-\triangle)^{\alpha_i} w_i^R &= f_i(w^R), && \forall t>0, x \in B_R(0) \\
w^R(t,x) &= 0 && \text{on } t \ge 0, x \in \mathbb{R}^d \setminus B_R(0) \\
0 < w^R(0,x) &\le \min(k_{y,R}, u^R), && \text{on } x \in B_R(0)
\end{aligned} \tag{3.55}$$

Since $v(\cdot + y) \ge w^R(0,x)$ in $\mathbb{R}^d$, by Theorem 3.7, we have that

$$v(x+y) \ge w^R(t,x) \qquad \forall (t,x) \in [0,\infty) \times \mathbb{R}^d \tag{3.56}$$

Now, by Lemma 3.13, there exists $R > 0$ large enough such that $w^R(t,x)$ converges to $u^R(x)$, as $t \to +\infty$ for all $x \in B_1(0)$. Hence, taking the limit when t tends to $+\infty$ in (3.56), we have that

$$v(x+y) \ge u^R(x) \qquad \forall\, x\; \in B_1(0)$$

Furthermore, taking $x = 0$ and since $y \in \mathbb{R}^d$ is arbitrary, we conclude

$$v(y) \geq u^R(0) := k \qquad \forall\, y \ \in \mathbb{R}^d$$

Finally, we take $\varepsilon > 0$ small enough such that $k \geq \varepsilon\phi$. □

Now, we establish the following result in which we state that u^+ is the smaller stationary solution of (3.1)-(3.2).

Lemma 3.16 *Let $d \geq 1$ and assume that F satisfies (3.4), (H1), (H2), (H3) and (H4). If v is a positive continuous solution of (3.54) such that $v \leq u^+$ in $\mathbb{R}^d$, then $v \equiv u^+$.*

Proof: First, since $F(u^+) = 0$, then u^+ satisfies the system (3.54). Now, by Lemma 3.15, there exists $\varepsilon > 0$ such that $v \geq \varepsilon\phi$ in $\mathbb{R}^d$. By Theorem 3.3, we deduce that

$$v(x) \geq \chi_\varepsilon(t) \qquad \forall\, t \geq 0,\ x \in \mathbb{R}^d \tag{3.57}$$

where the function χ_ε satisfies

$$\begin{aligned} \dot{\chi}_\varepsilon(t) &= F(\chi_\varepsilon(t)) \\ \chi_\varepsilon(0) &= \varepsilon\phi \end{aligned}$$

Thus, taking $t \to +\infty$, from (3.57) and the definition of u^+, we deduce that

$$v(x) \geq u^+_\varepsilon \geq u^+$$

Since $v \leq u^+$, we conclude that $v \equiv u^+$. □

In the following result we state a relation between the stationary solution in the ball $B_R(0)$ and the stationary solution in the whole space.

Lemma 3.17 *Let $d \geq 1$ and assume that F satisfies (3.4), (H1), (H2), (H3) and (H4). Let u^R be the unique solution of the system*

$$\begin{aligned} (-\triangle)^{\alpha_i} u_i^R &= f_i(u^R), & \forall x \in B_R(0) \\ u^R &= 0 & \text{on } \mathbb{R}^d \setminus B_R(0) \\ u^R &> 0 & \text{on } B_R(0) \end{aligned} \tag{3.58}$$

then, u^R converges to u^+ as $R \to +\infty$, locally on compact sets.

Proof: Let $R < R'$ and $x \in B_{R'} \setminus B_R$, thus $u^R(x) = 0$, $f_i(u^R) = 0$ and $(-\triangle)^{\alpha_i} u_i^R(x) \leq 0$, then we have that u^R is a subsolution of (3.58) on $B_{R'}$, hence, we conclude $u^R \leq u^{R'}$ and therefore the sequence $\{u^R\}$ is nondecreasing in R. Moreover, since u^+ is a supersolution of (3.58) for all $R > 0$, we have that $u^R \leq u^+$ for all radius R. Hence, the sequence $\{u^R\}$ is nondecreasing, bounded and by elliptic estimates converges in compact sets to a positive solution $\tilde{v} \leq u^+$ of (3.54). Thus, by Lemma 3.16 we conclude that $\tilde{v} = u^+$. □

Remark 3.18 As a consequence of Lemma 3.17, for each $\sigma = (\sigma)_{i=1}^m > 0$ and $y \in \mathbb{R}^d$, there exists $R_\sigma > 0$ that not depend of y, such that, for all $R \geq R_\sigma$

$$|u_i^R(x-y) - u_i^+| \leq \sigma_i \qquad \forall\, x \in B_1(y)$$

for each $i \in \{1, ..., m\}$.

Now, we can prove our main result.

Theorem 3.19 *Let $d \geq 1$ and assume that F satisfies (3.4), (H1), (H2), (H3) and (H4). Let u be the solution to (3.1)-(3.2) with u_0 satisfying (3.3) and (3.5). Then:*

a) *If $c < \frac{\lambda_1}{d+2\alpha}$, then*

$$\lim_{t\to+\infty} \inf_{|x|\leq e^{ct}} |u_i(t,x) - u_i^+| = 0$$

b) *If $c > \frac{\lambda_1}{d+2\alpha}$, then*

$$\lim_{t\to+\infty} \sup_{|x|\geq e^{ct}} u_i(t,x) = 0$$

for all $i \in 1, ..., m$.

Proof: First, since $u_0(x) \leq u^+$ and u^+ satisfies the equation (3.1), by Theorem 3.3, we deduce that $u(t,x) \leq u^+$. Now, let $c < \frac{\lambda_1}{d+2\alpha}$, we take $c < c_1 < c_2 < \frac{\lambda_1}{d+2\alpha}$ fixed, thus by Theorem 3.1, there exists $\tau > 0$ and $\varepsilon = (\varepsilon_i)_{i=1}^m$, such that

$$u_i(s,x) > \varepsilon_i, \quad \text{for all } s \geq \tau \text{ and } |x| \leq e^{c_2 s} \tag{3.59}$$

where $u = (u_i)_{i=1}^m$ is the solution of (3.1)-(3.2).

Let $\sigma > 0$, by the Remarks 3.14 and 3.18, we can find $R_\sigma > 0$ and $T_\sigma > 0$ large enough such that for $R \geq R_\sigma$ and $s \geq T_\sigma$, we have

$$|u_i^R(x-y) - u_i^+| \leq \frac{\sigma_i}{2} \quad \text{and} \quad |v_i^R(s, x-y) - u_i^R(x-y)| \leq \frac{\sigma_i}{2} \tag{3.60}$$

for all $y \in \mathbb{R}^d$, $x \in B_1(y)$ and $i \in \{1, ..., m\}$.

In what follows, taking $R \geq R_\sigma$ and τ large if necessary such that

$$R < e^{c_2\tau} - e^{c_1\tau}, \qquad e^{c_1 T_\sigma} < e^{(c_1-c)\tau}$$

we consider $y \in \{z : |z| + R \leq e^{c_2 s}\}$ with $s \geq \tau$. Then by (3.59), $v^R(0, \cdot - y)$ defined on $B_R(y)$ as in the Lemma 3.13 is a subsolution of the system (3.1)-(3.2) for times larger than s and for all $x \in \mathbb{R}^d$. Thus, by Theorem 3.7 and (3.60), we have that

$$u_i(\omega + s, x) \geq u_i^R(x-y) - \frac{\sigma_i}{2}, \quad \text{for all } \omega \geq T_\sigma \text{ and } x \in B_1(y)$$

Moreover, since $R \geq R_\sigma$ and taking $\omega = T_\sigma$

$$u_i(s + T_\sigma, x) \geq u_i^+ - \sigma_i, \quad \text{for all } x \in B_1(y) \tag{3.61}$$

Furthermore, since $\{z : |z| \leq e^{c_1 s}\}$ is a compact set, we can find a finite number of vectors $y_1, ..., y_k$, such that $\bigcup_{i=1}^k B_1(y_i)$ cover $\{z : |z| \leq e^{c_1 s}\}$. Thus, we have

$$u_i(s + T_\sigma, x) \geq u_i^+ - \sigma_i \quad \text{for all } |x| \leq e^{c_1 s}$$

Then, taking $t = s + T_\sigma \geq \tau + T_\sigma$

$$u_i(t, x) \geq u_i^+ - \sigma_i \quad \text{for all } |x| \leq e^{-c_1 T_\sigma} e^{c_1 t}$$

thus, we conclude the proof of part a), taking $\tau_\sigma := \tau + T_\sigma$ and by election of τ, we have that

$$u_i(t, x) \geq u_i^+ - \sigma_i \quad \text{for all } |x| \leq e^{ct}$$

To prove part b), we use directly Lemma 3.11. □

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