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NON LINEAR ELLIPTIC EQUATIONS WITH NON-LOCAL REGIONAL OPERATORS

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

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Resumen

Esta tesis consiste de cinco partes. En la primera parte se considera el problema de Dirichlet lineal y no lineal con una difusión no local regional definido implicitamente por

$$\int_{\Omega}(-\Delta)_{\rho}^{\alpha}u(x)v(x)dx = \int_{\Omega}\int_{B(0,\rho(x))}\frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}}dzdx,$$

donde $0<\alpha<1$, $\rho\in C(\overline{\Omega})$ y $\lambda dist(x,\partial\Omega)\leq\rho(x)\leq dist(x,\partial\Omega)$ con $\lambda\in(0,1]$, $x\in\Omega$. Haciendo uso del teorema de Lax-Milgran y el Teorema del paso de la montaña se demuestra la existencia de soluciones débiles. En la segunda parte, se considera la ecuación de Schrödinger no lineal con difusión no local regional

$$\epsilon^{2\alpha}(-\Delta)_{\rho}^{\alpha}u+u=f(u)\quad\text{in}\quad\mathbb{R}^n,\quad u\in H^{\alpha}(\mathbb{R}^n), \tag{0.1}$$

donde $0<\alpha<1$, $\epsilon>0$, $n\geq 2$ y $f:\mathbb{R}\to\mathbb{R}$ es super-lineal y tiene un crecimiento sub-critico. El operador $(-\Delta)_{\rho}^{\alpha}$ es el laplaciano no local regional, con rango de alcance determinado por una función positiva $\rho\in C(\mathbb{R}^n,\mathbb{R}^+)$ y definido por

$$\int_{\mathbb{R}^n}(-\Delta)_{\rho}^{\alpha}uvdx = \int_{\mathbb{R}^n}\int_{B(0,\rho(x))}\frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}}dzdx. \tag{0.2}$$

Se prueba la existencia de solución débil para (0.1) aplicando el Teorema del paso de la montaña al funcional I_{ρ} definido en $H_{\rho}^{\alpha}(\mathbb{R}^n)$, combinado con un argumento de comparación creado por Rabinowitz. El objetivo principal de la tercera parte es estudiar el comportamiento de concentración de la solución débil de la ecuación (0.1) con $f(s)=s^p$, cuando $\epsilon\to 0$. En la cuarta parte se estudia el resultado de simetría para las soluciones ground state de (0.1). Para tal propósito, se combina los rearreglos de funciones con los métodos variacionales. Finalmente, se considera un sistema Hamiltoniano fraccionario

$${}_{t}D_{\infty}^{\alpha}({}_{-\infty}D_{t}^{\alpha}u(t))+L(t)u(t)=\ \nabla W(t,u(t)) \tag{0.3}$$

donde $\alpha\in(1/2,1)$, $t\in\mathbb{R}$, $u\in\mathbb{R}^n$, $L\in C(\mathbb{R},\mathbb{R}^{n\times n})$ es una matriz simétrica positiva definida para todo $t\in\mathbb{R}$, $W\in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ y $\nabla W(t,u)$ es el gradiente de W en u. Se demuestra que (0.3) posee al menos una solución no trivial via el Teorema del paso de la montaña.

Abstract

This thesis consists of five parts. In the first part we consider the linear and nonlinear Dirichlet problem with nonlocal regional diffusion defined by

$$\int_{\Omega}(-\Delta)^{\alpha}_{\rho}u(x)v(x)dx = \int_{\Omega}\int_{B(0,\rho(x))}\frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}}dzdx,$$

where $0<\alpha<1$, $\rho\in C(\overline{\Omega})$ and $\lambda dist(x,\partial\Omega)\leq\rho(x)\leq dist(x,\partial\Omega)$ with $\lambda\in(0,1]$, $x\in\Omega$. Using Lax-Milgran and Mountain pass Theorem we prove the existence of weak solutions. In the second part we deal with the non-linear Schrödinger equation with non-local regional diffusion

$$\epsilon^{2\alpha}(-\Delta)^{\alpha}_{\rho}u+u=f(u)\quad\text{in}\quad\mathbb{R}^n,\quad u\in H^{\alpha}(\mathbb{R}^n), \tag{0.4}$$

where $0<\alpha<1$, $\epsilon>0$, $n\geq 2$ and $f:\mathbb{R}\to\mathbb{R}$ is super-linear and has a sub-critical growth. The operator $(-\Delta)^{\alpha}_{\rho}$ is a non-local regional laplacian, with range of scope determined by the positive function $\rho\in C(\mathbb{R}^n,\mathbb{R}^+)$ and defined by

$$\int_{\mathbb{R}^n}(-\Delta)^{\alpha}_{\rho}uvdx = \int_{\mathbb{R}^n}\int_{B(0,\rho(x))}\frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}}dzdx. \tag{0.5}$$

We prove the existence of weak solution of (0.4) applying the mountain pass theorem to the functional I_ρ defined on $H^{\alpha}_{\rho}(\mathbb{R}^n)$ combined with a comparison argument devised by Rabinowitz. The main goal of the third part is to study the concentration behavior of weak solutions of the equation (0.4) with $f(s)=s^p$, when $\epsilon\to 0$. On the forth part we study the radially symmetric result for the ground state solution of (0.4). For that purpose we combine the rearrangements and variational methods. Finally, we deal with the fractional Hamiltonian system

$${}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_{t}u(t))+L(t)u(t)=\nabla W(t,u(t)) \tag{0.6}$$

where $\alpha\in(1/2,1)$, $t\in\mathbb{R}$, $u\in\mathbb{R}^n$, $L\in C(\mathbb{R},\mathbb{R}^{n\times n})$ is a symmetric and positive definite matrix for all $t\in\mathbb{R}$, $W\in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and $\nabla W(t,u)$ is the gradient of W at u. We showed that (0.6) possesses at least one nontrivial solution via Mountain pass Theorem.

En memoria de mi Padre
Julio Torres Garcia

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El Autor

El principio de la sabiduria es
el temor de Jehová.
Proverbios 1:7

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Introduction

The fractional Laplacian is an integral operator of order α, and it can be seen as the infinitesimal generator of a Lévy process, which has been found to be of interesting applications (e.g. [21], [25], [83] and the references therein). Recently, a great attention has been focused on the study of problems involving the fractional Laplacian, from a pure mathematical point of view as well as from concrete applications, since this operator naturally arises in many different contexts, such as, among the others, obstacle problems, financial market, phase transitions, anomalous diffusions, crystal dislocations, soft thin films, semipermeable membranes, flame propagations, conservation laws, ultra relativistic limits of quantum mechanics, quasi-geostrophic flows, minimal surfaces, materials science, water waves, etc... The literature is really too wide to attempt any reasonable comprehensive treatment here. We would just cite some very recent papers which analyze fractional elliptic equations [13], [14], [24], [29], [36], [37], [67], [76], [77], [78], [81].

On the other hand, research has been done in recent years regarding a fractional laplacian with restricted range of scope. We mention the work by Guan [40] and Guan and Ma [41] where they study these operators, their relation with stochastic processes and they develop integration by parts formula, and specially the work by Ishii and Nakamura [46] where the authors studied the Dirichlet problem for regional fractional laplacian modeled on the p-laplacian. These regional operators present various interesting characteristics that make them very attractive from the point of view of mathematical theory of non-local operators.

Motivated by these previous works in this document we compile the work done along three years of a supervised doctoral thesis at the Universidad de Chile. The work is composed by five chapters mainly based on three papers [38, 39, 82]. We are interested in studying some kind of differential equation when a variational version of the regional fractional laplacian is considered. More precisely we studied the following problems

0.1 Dirichlet Problem with Non Local Regional Operator

In the first part of this thesis we deal with the linear and nonlinear Dirichlet problem whit nonlocal regional operator, that is: We consider

$$\begin{aligned} (-\Delta)^{\alpha}_{\rho}u(x) &= f(x),\ x\in\Omega, \\ u(x) &= 0,\ \text{on } \partial\Omega \end{aligned} \tag{0.7}$$

and

$$\begin{aligned} (-\Delta)^{\alpha}_{\rho}u(x) &= f(x,u),\ x\in\Omega, \\ u(x) &= 0,\ \text{on } \partial\Omega \end{aligned} \tag{0.8}$$

where $0 < \alpha < 1$, $(-\Delta)_\rho^\alpha$ is the non local regional diffusion defined by

$$\int_\Omega (-\Delta)_\rho^\alpha u(x)v(x)dx = \int_\Omega \int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dzdx, \tag{0.9}$$

f is given and ρ satisfies

(ρ_0) $\rho \in C(\overline{\Omega})$

(ρ_1) $\lambda dist(x, \partial\Omega) \leq \rho(x) \leq dist(x, \partial\Omega)$ with $\lambda \in (0, 1]$, $x \in \Omega$.

The classical linear Dirichlet problem in its simplest form is described as

$$-\Delta u = f(x) \text{ en } \Omega, \quad u = 0 \text{ en } \partial\Omega, \tag{0.10}$$

where $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f : \Omega \to \mathbb{R}$ is given. Find solutions to the problem (0.10), is equivalent to find weak solutions of its variational form, that is

$$\int_\Omega \nabla u.\nabla v = \int_\Omega fv \;\; \forall v \in H_0^1(\Omega) \tag{0.11}$$

A simple method to proof the existence and uniqueness of weak solution of (0.10) is using the Lax-Milgran Theorem. Let Ω be a bounded open set and $f \in L^2(\Omega)$. Then there exists a unique weak solution $u \in H_0^1(\Omega)$ satisfying (0.11). Further u can be characterized by: $u \in H_0^1(\Omega)$ such that

$$J(u) = \min_{v \in H_0^1(\Omega)} J(v)$$

where

$$J(v) = \frac{1}{2}\int_\Omega \nabla v.\nabla v - \int_\Omega fv$$

Different thing happens when we consider the nonlinear Dirichlet problem

$$-\Delta u = f(x, u) \text{ en } \Omega, \quad u = 0 \text{ en } \partial\Omega, \tag{0.12}$$

where f is a Caratheodory function on $\Omega \times \mathbb{R}$. In this case we can not get the existence of weak solution using Lax-Milgarn Theorem but fortunately there are several tools to study problem (0.12), see [26], [52], [62], [73]. One successful method is the mountain pass theorem of Ambrosetti-Rabinowitz [7]. We know that to find weak solutions of (0.12) is equivalent to find critical points of the energy functional

$$I(u) = \frac{1}{2}\|\nabla u\|^2 - \int_\Omega F(x, u)dx \tag{0.13}$$

where

$$F(x, t) = \int_0^t f(x, s)ds$$

and the norm is that if $L^2(\Omega)$. Under the following conditions

(f_1) f is Carathéodory function i.e. $t \to f(x, t)$ is continuous a.e. $x \in \Omega$ and $x \to f(x, t)$ is measurable for all $t \in \mathbb{R}$.

(f_2) There are positive constants C_1 and C_2 such that

$$|f(x,t)| \leq C_1 + C_2|t|^s \ \ \forall x \in \overline{\Omega},\ t \in \mathbb{R},$$

where $0 \leq s < \frac{n+2\alpha}{n-2\alpha}$.

(f_3) $\frac{f(x,t)}{|t|} \to 0$, $t \to 0$, $\forall x \in \overline{\Omega}$.

(f_4) There are constants $\mu > 2$ and $r \geq 0$ such that

$$0 < \mu F(x,t) \leq t f(x,t) \ \ \forall x \in \overline{\Omega},\ \ |t| \geq r,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

Using these hypotheses the functional (0.13) be continuously differentiable on the Sobolev space $H_0^1(\Omega)$ and satisfies the mountain pass geometry conditions. This allow them to apply the mountain pass theorem to find a weak solution to (0.12).

Following the same ideas, we study the existence of weak solution for the problems (0.7) and (0.8), by using Lax-Milgran Theorem and Mountain pass Theorem.

0.2 Nonlinear Schrödinger Equation With Non-local Regional Diffusion

The aim is to study the non-linear Schrödinger equation with non-local regional diffusion

$$\epsilon^{2\alpha}(-\Delta)^\alpha_\rho u + u = f(u) \quad \text{in} \quad \mathbb{R}^n, \quad u \in H^\alpha(\mathbb{R}^n), \tag{0.14}$$

where $0 < \alpha < 1$, $\epsilon > 0$, $n \geq 2$ and $f : \mathbb{R} \to \mathbb{R}$ is super-linear and has a sub-critical growth. The operator $(-\Delta)^\alpha_\rho$ is a non-local regional laplacian, with range of scope determined by the positive function $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$ and defined by

$$\int_{\mathbb{R}^n} (-\Delta)^\alpha_\rho uv dx = \int_{\mathbb{R}^n}\int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dzdx. \tag{0.15}$$

In the context of fractional quantum mechanics, non-linear fractional Schrödinger equation has been proposed by Laskin [55], [56] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. In the last 10 years, there has been a lot of interest in the study of the fractional Schrödinger equation, see the works in [24], [32], [37], [42], [67], [76], [77]. In a recent paper Felmer, Quaas and Tan [37] considered positive solutions of nonlinear fractional Schrödinger equation

$$(-\Delta)^\alpha u + u = f(x,u) \ \text{ in } \ \mathbb{R}^n. \tag{0.16}$$

They obtained the existence of a ground state by mountain pass argument and a comparison method devised by Rabinowitz in [71] for $\alpha = 1$. They also analyzed regularity, decay and symmetry properties of these solutions, and they also studied the problem in the presence of an x dependent bounded potential. At this point it is worth mentioned the uniqueness of the ground state in one dimension and with power non-linearity obtained by Frank and Lenzmann in [36], while the multi-dimensional case was studied recently by Fall and Valdinoci in [35]. We also mention the work by Cheng [24], where the fractional Schrödinger equation with unbounded potential

$$(-\Delta)^{\alpha}u + V(x)u = |u|^{p-1}u \ \text{ in } \ \mathbb{R}^n \tag{0.17}$$

was studied. The existence of a ground state of (0.17) with unbounded potential V is obtained by Lagrange multiplier and the Nehari manifold method.

On the other hand, research has been done in recente years regarding a fractional laplacian with restricted range of scope. We mention the work by Guan [40] and Guan and Ma [41] where they study these operators, their relation with stochastic processes and they develop integration by parts formula, and specially the work by Ishii and Nakamura [46] where the authors studied the Dirichlet problem for regional fractional laplacian modeled on the p-laplacian. These regional operators present various interesting characteristics that make them very attractive from the point of view of mathematical theory of non-local operators.

In this theses we are interested in studying the non-linear Schrödinger equation when a variational version of the regional fractional laplacian is considered. We are specially interested in understanding the role of the scope function ρ on the existence of positive solution and concentration in the semi-classical limit for equation (0.14).

Now we make precise assumptions on ρ and f. For the range of scope ρ we assume $\rho : \mathbb{R}^n \to \mathbb{R}^+$ is continuous and it satisfies the following hypotheses:

(ρ_1) There are numbers $0 < \rho_0 < \rho_\infty \leq \infty$ such that

$$\rho_0 \leq \rho(x) < \rho_\infty \quad \forall x \in \mathbb{R}^n \quad \text{and} \quad \lim_{|x|\to\infty} \rho(x) = \rho_\infty.$$

(ρ_2) In case $\rho_\infty = \infty$ we further assume that there exists $a \in (0,1)$ such that

$$\limsup_{|x|\to\infty} \frac{\rho(x)}{|x|} \leq a.$$

(ρ_3) For any $x_0 \in \mathbb{R}^n$, the equation

$$|x| = \rho(x + x_0), \quad x \in \mathbb{R}^n,$$

defines an $(n-1)$-dimensional surface of class C^1 in $\mathbb{R}^n$.

Regarding the non-linearity f we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies the following hypotheses:

(f_1) $f(t) \geq 0$ if $t \geq 0$ and $f(t) = 0$ if $t \leq 0$.

(f_2) The function $t \to \frac{f(t)}{t}$ is increasing for $t > 0$ and $\lim_{t\to 0} \frac{f(t)}{t} = 0$.

(f_3) $\exists\, \theta > 2$ such that $\forall t > 0$

$$0 < \theta F(t) \leq t f(t), \qquad \text{where} \quad F(t) = \int_0^t f(\xi) d\xi.$$

(f_4) $\exists\, C > 0$ such that

$$|f(t)| \leq C(1 + |t|^p), \ \ 1 < p < \frac{n + 2\alpha}{n - 2\alpha}.$$

Before stating our results let us introduce the main ingredients involved in our approach. We let $H^\alpha(\mathbb{R}^n)$ be the usual Sobolev space equipped with the norm

$$\|u\|^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n} u(x)^2 dx. \tag{0.18}$$

Given a function ρ as above, we define

$$\|u\|_\rho^2 = \int_{\mathbb{R}^n}\int_{B(0,\rho(x))} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n} u(x)^2 dx \tag{0.19}$$

and the space

$$H_\rho^\alpha(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \,/\, \|u\|_\rho^2 < \infty\}.$$

For $u \in H_\rho^\alpha$ and f satisfying (f_1)-(f_4), we may define the functional

$$I_\rho(u) = \frac{1}{2}\|u\|_\rho^2 - \int_{\mathbb{R}^n} F(u(x))dx, \tag{0.20}$$

which is of class C^1. We say that $u \in H^\alpha(\mathbb{R}^n)$ is a weak solution of (0.14) if u is a critical point of I_ρ.

On Chapter §2 we consider the existence of ground state for (0.14) with $\epsilon = 1$. Our main existence theorem is:

Theorem 0.1. *Assume $0 < \alpha < 1$ and $n \geq 2$. If f satisfies (f_1)-(f_4) and ρ satisfies (ρ_1)-(ρ_2) then (0.14) possesses at least one non-trivial weak solution. Moreover this solution satisfies $u(x) \geq 0$ a.e. for all $x \in \mathbb{R}^n$.*

We prove the existence of weak solution of (0.14) applying the mountain pass theorem [7] to the functional I_ρ defined on $H_\rho^\alpha(\mathbb{R}^n)$. However, the direct application of the mountain pass theorem is not possible since the Palais-Smale sequences might lose compactness in the whole space $\mathbb{R}^n$. To overcome this difficulty, we use an argument devised by Rabinowitz in [71] comparing the mountain pass critical value of I_ρ with that of the functional I_{ρ_∞}, where we use the limiting value ρ_∞ given in hypothesis (ρ_1) instead of ρ in (0.20). See the work in [37], where a similar argument is used in the context of non-local operators.

On Chapter §3 we are interested in the concentration behavior of weak solutions of the equation (0.14) with $f(u) = u^p$, that is, we consider

$$\epsilon^{2\alpha}(-\Delta)_\rho^\alpha u + u = u^p, \quad \text{in} \quad \mathbb{R}^n, \quad u \in H^\alpha(\mathbb{R}^n), \tag{0.21}$$

when the positive parameter ϵ approaches zero. The function ρ, that describes the size of the ball of the influential region of the non-local operator, will certainly play a key role in deciding the concentration point of ground states of the equation. However, even though the minimum point of ρ seems to be the point of concentration, there is a non-local effect that also influence the concentration. Now we define the function that controls the concentration

$$\mathcal{H}(x) = -\frac{|S^{n-1}|}{2\alpha}\left(\frac{1}{\rho(x)^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}}\right) + \int_{\mathcal{C}^+(x)} \frac{dy}{|y|^{n+2\alpha}} - \int_{\mathcal{C}^-(x)} \frac{dy}{|y|^{n+2\alpha}},$$

where we interpret the quotient $1/\rho_\infty^{2\alpha}$ as zero, in case $\rho_\infty = \infty$. The sets $\mathcal{C}^+(x)$ and $\mathcal{C}^-(x)$ are defined as follows

$$\mathcal{C}^-(x) = \{y \in \mathbb{R}^n : \quad \rho(x+y) < |y| < \rho(x)\}$$

and

$$\mathcal{C}^+(x) = \{y \in \mathbb{R}^n : \ \ \rho(x) < |y| < \rho(x+y)\}.$$

Now we have the following theorem

Theorem 0.2. *Let $0 < \alpha < 1$, $n \geq 2$. Suppose that ρ satisfies (ρ_1)-(ρ_3) and $1 < p < \frac{n+2\alpha}{n-2\alpha}$. Then for each sequence $\epsilon_m \to 0$, there exists a subsequence (still called $\{\epsilon_m\}$) such that for every m, there is a non-negative solution $u_m = u_{\epsilon_m}$ of (0.21) that concentrates around a global minimum point x_0 of $\mathcal{H}$, as $\epsilon_m \to 0$. In more precise terms, for every $\delta > 0$ there exists $R > 0$ and $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$ we have*

$$\int_{B^c(x_0,\epsilon_m R)} u_m^2(x)dx \leq \epsilon_m^n \delta, \ \ \textit{and} \ \int_{B(x_0,\epsilon_m R)} u_m^2(x)dx \geq \epsilon_m^n C, \qquad \forall \epsilon_m \leq \epsilon_0,$$

with C a constant independent of δ and m.

The proof of this theorem again uses a comparison argument in order to obtain the concentration. Here we will compare with mountain pass critical points of the functional I defined in $H^\alpha(\mathbb{R}^n)$ as follows

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^n} F(u(x))dx. \tag{0.22}$$

On Chapter §4 we are interested on the radially symmetry result for the ground state solution given on chapter §2. During the last years, non-linear equations involving fractional Laplacian or more general integro-differential operators have been studied by many authors. In particular, great attention has been devoted to investigate symmetry results for equations involving the fractional Laplacian in $\mathbb{R}^n$, we refer to the work by Li [60] and Chen, Li and Ou [22], [23] where the method of moving planes in integral form has been developed to treat various equations and systems. See also the results obtained by Ma and Chen [61], Dipierro, Palatucci and Valdinoci [30] and Felmer, Quaas and Tan [37]. In [37], the authors used the method of moving planes in integral form to prove symmetry results for

$$(-\Delta)^\alpha u + u = f(u) \ \text{ in } \ \mathbb{R}^n, \tag{0.23}$$

taking advantage of the representation formula for u given by

$$u(x) = (\mathcal{K} * f(u))(x), \ \ x \in \mathbb{R}^n, \tag{0.24}$$

where the kernel $\mathcal{K}$, associated to the linear part of the equation, plays a key role in the moving planes argument.

In [30], for the first time using rearrangement tools and following the ideas of Berestycki and Lions [19], the authors proved existence of a nontrivial, radially symmetric, solution to

$$\begin{aligned} (-\Delta)^\alpha u + u = |u|^{p-1}u \ \ &\text{ in } \mathbb{R}^n, \\ u \in H^\alpha(\mathbb{R}^n)& \end{aligned} \tag{0.25}$$

Solutions of (0.25) can be obtained by finding critical points of the Euler-Lagrange functional I defined in the fractional Sobolev spaces $H^\alpha(\mathbb{R}^n)$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n}\left(\frac{1}{2}|u(x)|^2 - \frac{1}{p+1}|u(x)|^{p+1}\right)dx.$$

By the fractional Polya-Szegö inequality

$$\int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} u^*(x)|^2 dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} u(x)|^2 dx, \tag{0.26}$$

they noted that,

$$I(u^*) \leq I(u) \tag{0.27}$$

Therefore if there is a minimizer of I, must be a symmetric minimizer. Their proof was based on variational method, working with an appropriate constraint in order to have some compactness. This constraint can be made transparent because of the "autonomous" character of (0.25) and the fact that one can use scale changes in $\mathbb{R}^n$. In [77], Secchi looks for a radially symmetric solution of

$$(-\Delta)^\alpha u + V(x)u = f(u) \ \text{ in } \ \mathbb{R}^n, \tag{0.28}$$

where the nonlinearity f satisfies

- (i) $f : \mathbb{R} \to \mathbb{R}$ is of class $C^{1,\gamma}$ for some $\gamma > \max\{0, 1-2\alpha\}$, and odd;
- (ii) $-\infty < \liminf_{t\to 0^+} \frac{f(t)}{t} \leq \limsup_{t\to 0^+} \frac{f(t)}{t} = -m < 0$;
- (iii) $-\infty < \limsup_{t\to+\infty} \frac{f(t)}{t^{2^*_\alpha - 1}} \leq 0$;
- (iv) For some $\xi > 0$ there results $G(\xi) \int_0^\xi g(t)dt > 0$.

We note that $(i) - (iv)$ are comparable to those in [19] and f does not satisfy the usual Ambrosetti-Rabinowitz condition. Moreover V satisfies

- (V_1) $V \in C^1(\mathbb{R}^n, \mathbb{R})$, $V(x) \geq 0$ for every $x \in \mathbb{R}^n$ and this inequality is strict at some point;
- (V_2) $\|\max\{\langle \nabla V(.), 0\rangle\}\|_{L^{N/2\alpha}} < 2S$, where S is the best Sobolev constant for the critical embedding, namely
$$S = \inf_{u\in H^\alpha(\mathbb{R}^n), u\neq 0} \frac{\|(-\Delta)^{\alpha/2}u\|^2_{L^2}}{\|u\|_{L^{2^*_\alpha}}}$$
- (V_3) $\lim_{|x|\to+\infty} V(x) = 0$;
- (V_4) V is radially symmetric, i.e $V(x) = V(|x|)$.

Using the monotonicity trick method (see [47]) Secchi proves the existence of a radially symmetric solution of (0.28).

Motivated by these previous works we consider the question of symmetry for the solutions of (0.14). However, a same approach is not possible to be used for problem (0.14), since: (**1**) A similar representation formula like (0.24) is not available in general for $(-\Delta)^\alpha_\rho$. (**2**) Our problem has not the "autonomous" character because its x-dependence and we can not use scale changes in $\mathbb{R}^n$ and (**3**) To use Secchi's procedure we need more regularity in u, still regularity theory for solutions of (0.14) is a open problem.

However we can get symmetry, but our approach is different from the previous works. We use the same tools: rearrangement of functions and variational methods. The idea to prove our result, consists in replacing the path γ in the mountain pass setting by its symmetrization $\gamma^* : t \in [0,1] \to \gamma(t)^*$. Then u would be near of the set $\gamma^*([0,1])$. This idea works since rearrangements are continuous in $H^\alpha(\mathbb{R}^n)$ see [6]. We note that this idea cannot be used directly when $\alpha = 1$ and $n > 1$ since rearrangements $* : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is not continuous. See more details in the work by Van Schaftingen [84].. In fact, the method hinges on the inequality

$$I_\rho(u^*) \leq I_\rho(u) \ \text{ for all u } \in H^\alpha(\mathbb{R}^n) \tag{0.29}$$

where u^* denotes the symmetric rearrangement of $|u|$, and I_ρ is defined by

$$I_\rho(u) = \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{B(0,\rho(x))}\frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n} u^2dx\right) - \int_{\mathbb{R}^n} F(u)dx.$$

From (0.29), a natural question arise: Does the inequality

$$\int_{\mathbb{R}^n}\int_{B(0,\rho(x))}\frac{|u(^*x+z)-u^*(x)|^2}{|z|^{n+2\alpha}}dzdx \leq \int_{\mathbb{R}^n}\int_{B(0,\rho(x))}\frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}dzdx, \tag{0.30}$$

holds? The answer is true, part of the chapter §4 is dedicated to prove (0.30). Following the ideas of Algren and Lieb [6], under the following condition

(ρ_4) $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$ be a positive radially symmetry function,

first we get the regional Riesz inequality

$$\int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} u(x)v(x-y)w(y)dydx \leq \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} u^*(x)v^*(x-y)w^*(y)dydx \tag{0.31}$$

where $u, v, w \in H^\alpha(\mathbb{R}^n)$. With the help of this inequality we proved (0.30).

Finally we state our main result on Chapter §4

Theorem 0.3. *Suppose that $(\rho_{1)}, (\rho_3)$, (ρ_4) and $(f_1) - (f_4)$ hold. Then the mountain pass value is achieved by a radially symmetric function, which is a solution of (0.14). .*

0.3 Fractional Hamiltonian Systems

Fractional order models can be found to be more adequate than integer order models in some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. As a consequence, the subject of fractional differential equations is gaining more importance and attention. There has been significant development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives. For details and examples, one can see the monographs [53], [65], [70] and the papers [2], [3], [5], [8], [15], [45], [58], [64], [74], [86]. Moreover the existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic, and pseudo-almost periodic solutions have been great attention in the qualitative theory of fractional differential equations, due to its mathematical interest and applications. Some recent contributions on the existence of such solutions for abstract differential equations and fractional differential equations have been made, see [1], [3], [4], [8], [28], [33], [57], [66] for details.

Recently, also equations including both left and right fractional derivatives are discussed. Apart from their possible applications, equations with left and right derivatives is an interesting and new field in fractional differential equations theory. In this topic, many results are obtained dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by using techniques of nonlinear analysis, such as fixed point theory [11] (including Leray-Schauder nonlinear alternative), topological degree theory [48] (including co-incidence degree theory) and comparison method [88] (including upper and lower solutions and monotone iterative method) and so on.

It should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last 30 years, the critical point theory has become to a wonderful tool in studying the existence of solutions to differential equations with variational structures, we refer the reader to the books due to Mawhin and Willem [62], Rabinowitz [73] and the references listed therein.

Motivated by the above classical works, in this section we deal with the fractional Hamiltonian system

$$ {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) + L(t)u(t) = \nabla W(t, u(t)) \tag{0.32}$$

where $\alpha \in (1/2, 1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ is a symmetric matrix valued function and $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$; satisfies the following condition

(L) $L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists an $l \in C(\mathbb{R}, (0, \infty))$ such that $l(t) \to +\infty$ as $t \to \infty$ and

$$ (L(t)x, x) \geq l(t)|x|^2, \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n. \tag{0.33}$$

(W_1) $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and there is a constant $\mu > 2$ such that

$$ 0 < \mu W(t, x) \leq (x, \nabla W(t, x)), \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \setminus \{0\}.$$

(W_2) $|\nabla W(t, x)| = o(|x|)$ as $x \to 0$ uniformly with respect to $t \in \mathbb{R}$.

(W_3) There exists $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ such that

$$ |W(t, x)| + |\nabla W(t, x)| \leq |\overline{W(x)}| \text{ for every } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}.$$

In particular, if $\alpha = 1$, (0.32) reduces to the standard second order differential equation

$$ u'' - L(t)u + \nabla W(t, u) = 0, \tag{0.34}$$

where $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a given function and $\nabla W(t, u)$ is the gradient of W at u. The existence of homoclinic solution is one of the most important problems in the history of that kind of equations, and has been studied intensively by many mathematicians. Assuming that $L(t)$ and $W(t, u)$ are independent of t, or T-periodic in t, many authors have studied the existence of homoclinic solutions for (0.34) via critical point theory and variational methods.

In this case, the existence of homoclinic solution can be obtained by going to the limit of periodic solutions of approximating problems.

If $L(t)$ and $W(t,u)$ are neither autonomous nor periodic in t, this problem is quite different from the ones just described, because the lack of compacteness of the Sobolev embedding. In [72] the authors considered (0.34) without periodicity assumptions on L and W and showed that (0.34) possesses one homoclinic solution by using a variant of the mountain pass theorem without the Palais-Smale contidion. In [68], under the same assumptions of [72], the authors, by employing a new compact embedding theorem, obtained the existence of solution of (0.34).

Our goal is to show how variational methods based on Mountain pass theorem can be used to get existence results for (0.32). However, the direct application of the mountain pass theorem is not enough since the Palais-Smale sequences might lose compactness in the whole space $\mathbb{R}$. To overcome this difficulty we proof a version of compact embedding for fractional space following the ideas of [68]. We state our main existence theorem.

Theorem 0.4. *Suppose that $(L), (W_1) - (W_3)$ hold, then (0.32) possesses at least one non-trivial solution.*

Chapter 1

Dirichlet Problem with Non Local Regional Operator

In this chapter we consider the linear and nonlinear Dirichlet problem with nonlocal regional diffusion on bounded domain, namely

$$\begin{aligned} (-\Delta)^{\alpha}_{\rho} u(x) &= f(x),\ x \in \Omega, \\ u(x) &= 0,\ \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} (-\Delta)^{\alpha}_{\rho} u(x) &= f(x,u),\ x \in \Omega, \\ u(x) &= 0,\ \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

We consider the existence of weak solutions for both problems.

1.1 Preliminaries

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a Lipschitz bounded open set, $0 < \alpha < 1$ and

(ρ_0) $\rho \in C(\overline{\Omega})$

(ρ_1) $\lambda dist(x, \partial\Omega) \leq \rho(x) \leq dist(x, \partial\Omega)$ with $\lambda \in (0, 1]$, $x \in \Omega$.

The fractional Sobolev spaces of order α on Ω is defined by

$$H^{\alpha}(\Omega) = \left\{ u \in L^2(\Omega) / \ \int_{\Omega}\int_{\Omega} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2\alpha}} dz dx < +\infty \right\}$$

endowed with the norm

$$\|u\|_{H^{\alpha}} = \left(\int_{\Omega} |u(x)|^2 dx + \int_{\Omega}\int_{\Omega} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2\alpha}} dz dx \right)^{1/2} \tag{1.3}$$

Theorem 1.1. *[87] $H^{\alpha}(\Omega)$ is a Banach, separable and reflexive spaces.*

Theorem 1.2. *[29] Let $\alpha \in (0,1)$ such that $2\alpha < n$. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded open set. Then there exists a positive constant $C = C(n,\alpha)$ not depending on Ω such that, for any $u \in H^\alpha(\Omega)$, we have*

$$\|u\|^2_{L^{2^*_\alpha}(\Omega)} \leq C \int_\Omega \int_\Omega \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dzdx \tag{1.4}$$

where $2^*_\alpha = \frac{2n}{n-2\alpha}$. Consequently, the space $H^\alpha(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1, 2^*_\alpha]$ and compactly embedded for any $q \in [1, 2^*_\alpha)$.

Let $H^\alpha_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|.\|_{H^\alpha}}$, endowed with the norm

$$\|u\|_{H^\alpha_0} = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dzdx \right)^{1/2} \tag{1.5}$$

which is equivalent to (1.3). The dual of the space $H^\alpha_0(\Omega)$, is denoted by $H^{-\alpha}$.

Now we defined the spaces $H^\alpha_\rho(\Omega)$ by:

$$H^\alpha_\rho(\Omega) = \left\{ u \in L^2(\Omega) / \ \int_\Omega \int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx < +\infty \right\}$$

where ρ satisfying (ρ_0)-(ρ_1). We endowed $H^\alpha_\rho(\Omega)$ with the inner product

$$\langle u, v \rangle_{H^\alpha_\rho} = \int_\Omega u(x)v(x)dx + \int_\Omega \int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dzdx \tag{1.6}$$

inducing the following norm

$$\|u\|_{H^\alpha_\rho} = \left(\int_\Omega |u(x)|^2 dx \int_\Omega \int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx \right)^{1/2} \tag{1.7}$$

Let $H^\alpha_{0\rho}(\Omega) = \overline{C^\infty_0(\Omega)}^{\|.\|_{H^\alpha_\rho}}$ endowed with the norm

$$\|u\|^2_{H^\alpha_{0\rho}} = \int_\Omega \int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx$$

Theorem 1.3. *Let $\alpha \in (0,1)$. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a Lipschitz bounded open set. Suppose ρ satisfies (ρ_0)-(ρ_1). Then there is a constan $K = K(n, L, \alpha, \lambda)$ such that*

$$\|u\|^2_{H^\alpha_0} \leq K \|u\|^2_{H^\alpha_{0\rho}}, \ \ \forall u \in H^\alpha_{0\rho}(\Omega) \tag{1.8}$$

Proof. By definition of Lipschitz domain, for any $z \in \partial\Omega$ there is a ball $B(z, r_0)$ centered at z such that, in a suitable coordinate system $(z', z_n) = (z_1, ...z_{n-1}, z_n)$ with origin at z,

$$\Omega \cap B(z, r_0) = \{(x', x_n) / \ \ \phi(x') < x_n\} \cap B(z, r_0)$$

where ϕ is a Lipschitz function with Lipschitz constant less than equal L (independent of z) and $\phi(0) = 0$. In [31], Dyda proved the following inequality

$$\int_Q \int_Q \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dzdx \leq K \int_\Omega \int_{B(x, \lambda dist(x, \partial\Omega))} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dzdx \tag{1.9}$$

where Q is a Lipschitz box at $\partial\Omega$ defined by

$$Q = Q(z,\beta) = \{x \in \Omega / \ |x' - z'| < \beta, \ 0 < x_n - \varphi_z(x') < \beta\}$$

with $diam(Q) < \frac{r_0}{6L+9}$, where $\beta > 0$. By (ρ_1) we get

$$\int_Q \int_Q \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dzdx \le K \int_\Omega \int_{B(x,\rho(x)))} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dzdx \tag{1.10}$$

Since $\Omega = \cup Q_k$ for some increasing sequence of Lipschitz boxes Q_n, by (1.10) and Lebesgue monotone converge theorem, we get

$$\int_\Omega \int_\Omega \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dzdx \le K \int_\Omega \int_{B(0,\rho(x)))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx \tag{1.11}$$

□

Remark 1.4. By Theorem 1.3 we have the continuous embedding $H^\alpha_{0\rho}(\Omega) \hookrightarrow H^\alpha_0(\Omega)$. On the other hand we have $H^\alpha_0(\Omega) \hookrightarrow H^\alpha_\rho(\Omega)$ and the inequality

$$\|u\|^2_{H^\alpha_{0\rho}} \le \|u\|^2_{H^\alpha_0}$$

then $\|.\|_{H^\alpha_0}$ and $\|.\|_{H^\alpha_\rho}$ are a equivalent norms. By Theorem 1.2 we have

$$\begin{array}{rcll} H^\alpha_{0\rho}(\Omega) & \hookrightarrow & L^q(\Omega), & \forall\, q \in [1, 2^*_\alpha] \text{ continuously, and} \\ H^\alpha_{0\rho}(\Omega) & \hookrightarrow & L^q(\Omega), & \forall\, q \in [1, 2^*_\alpha) \text{ compactly} \end{array}$$

1.1.1 Lax-Milgran Theorem

Theorem 1.5. *Let X be a Hilbert spaces and $a(.,.)$ a continuous coercive bilinear form. Then given $f \in X$, there exists a unique $u \in X$ such that*

$$a(u,v) = (f,v) \ \ \forall\, v \in X$$

If $a(.,.)$ is also symmetric then the functional $J : X \to \mathbb{R}$ defined by

$$J(v) = \frac{1}{2}a(v,v) - (f,v)$$

attains its minimum at u.

1.1.2 Mountain Pass Theorem

Let X be a real Banach spaces and $\phi \in C^1(X,\mathbb{R})$. We say $c \in \mathbb{R}$ is critical value of ϕ if there exists $u \in X$ such that

$$\phi'(u) = 0 \ \text{ and } \ \phi(u) = c.$$

We say ϕ satisfies the Palais-Smale condition (henceforth denoted by (PS)) if any sequence $\{u_m\} \subset X$ for which

$$\phi(u_m) \ \text{ is bounded and } \ \phi'(u_m) \to 0 \ \text{ as } \ m \to \infty$$

possesses a convergent subsequence.

Theorem 1.6. (**Mountain Pass Theorem** *[73]*) *Let X be a real Banach spaces and $\phi \in C^1(X,\mathbb{R})$ satisfying (PS)-condition. Suppose $\phi(0)=0$ and*

(i) There are constants $\theta, \varrho > 0$ such that

$$\phi|_{\partial B_\theta} > \varrho$$

(ii) There is an $e \in X \backslash \overline{B_\theta}$ such that $\phi(e) \leq 0$

Then ϕ possesses a critical value $c \geq \varrho$. Moreover c can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} \phi(u)$$

where

$$\Gamma = \{\gamma \in C([0,1],X)/ \ \ \gamma(0)=0, \ \gamma(1)=e\}$$

1.2 Linear Dirichlet Problem with Non Local Regional Diffusion

In this section we consider the problem (1.1), where $f \in L^2(\Omega)$ is given and $(-\Delta)^\alpha_\rho$ is the non local regional diffusion defined by

$$\int_\Omega (-\Delta)^\alpha_\rho u(x)v(x)dx = \int_\Omega \int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dzdx. \tag{1.12}$$

where $0<\alpha<1$ and ρ satisfies (ρ_0)-(ρ_1).

Definition 1.7. $u \in H^\alpha_0(\Omega)$ is weak solution of (1.1) if

$$\int_\Omega \int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][\varphi(x+z)-\varphi(x)]}{|z|^{n+2\alpha}} dzdx = \int_\Omega f(x)\varphi(x)dx, \tag{1.13}$$

$\forall\ \varphi \in H^\alpha_0(\Omega)$.

Theorem 1.8. *Let $0<\alpha<1$, $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded open set, ρ satisfies (ρ_0) and (ρ_1) and $f \in L^2(\Omega)$. Then there is a unique weak solution $u \in H^\alpha_0(\Omega)$ satisfying (1.13).*

Proof. Let the functional

$$a(u,v) = \int_\Omega \int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dzdx.$$

We make the following statements

1. $a(.,.)$ is bilinear. Let $\beta \in \mathbb{R}$ y $u,v,w \in H^s_0(\Omega)$, the

$$\begin{aligned} a(\beta u+v,w) &= \int_\Omega \int_{B(0,\rho(x))} \frac{[(\beta u+v)(x+z)-(\beta u+v)(x)][w(x+z)-w(x)]}{|z|^{n+2\alpha}} dzdx \\ &= \beta a(u,w) + a(v,w) \end{aligned}$$

2. $a(.,.)$ is continuous. By (ρ_1), $B(x,\rho(x)) \subset \Omega$, by the Hölder inequality:

$$\begin{aligned} |a(u,v)| &= \left| \int_\Omega \int_{B(x,\rho(x))} \frac{[u(x)-u(z)][v(x)-v(z)]}{|x-z|^{n+2\alpha}} dzdx \right| \\ &\leq \int_\Omega \int_\Omega \frac{[u(x)-u(z)][v(x)-v(z)]}{|x-z|^{n+2\alpha}} dzdx \\ &\leq \left(\int_\Omega \int_\Omega \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dzdx \right)^{1/2} \left(\int_\Omega \int_\Omega \frac{|v(x)-v(z)|^2}{|x-z|^{n+2\alpha}} dzdx \right)^{1/2} \\ &\leq \|u\|_{H^\alpha(\Omega)} \|v\|_{H^\alpha(\Omega)} \end{aligned}$$

3. $a(.,.)$ is coercive. By Theorem 1.2 and 1.3 we have

$$\begin{aligned} \|u\|^2_{H^\alpha(\Omega)} &= \int_\Omega |u(x)|^2 dx + \int_\Omega \int_\Omega \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dzdx \\ &\leq (C+1) \int_\Omega \int_\Omega \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dzdx \\ &\leq K(C+1) \int_\Omega \int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx \end{aligned}$$

Then

$$\begin{aligned} a(u,u) &= \int_\Omega \int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx \\ &\geq \frac{1}{K(C+1)} \|u\|^2_{H^\alpha(\Omega)} \end{aligned}$$

Therefor $a(.,.)$ is coercive.

Then by Lax-Milgran theorem there is only solution $u \in H_0^\alpha(\Omega)$ of (**??**), and since $a(.,.)$ is symmetric we have

$$I(u) = \min_{v \in H_0^\alpha(\Omega)} I(v)$$

where

$$I(v) = \frac{1}{2} a(v,v) - \int_\Omega f v dx.$$

□

1.3 Non Linear Dirichlet Problem With Non Local Regional Operator

In this section we consider the problem (1.2), where $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a Lipschitz bounded open set, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function and $(-\Delta)^\alpha_\rho$ is defined by (1.12), where $0 < \alpha < 1$ and ρ satisfies (ρ_0)-(ρ_1)

We use the Mountain pass theorem for get a weak solution of (1.2). In order to find weak solutions of (1.2) we assume the following general hypotheses:

(f_1) f is Carathéodory function i.e.

$$\begin{array}{rcl} t & \to & f(x,t) \text{ is continuous a.e. } x \in \Omega, \\ x & \to & f(x,t) \text{ is measurable for all } t \in \mathbb{R}, \end{array}$$

(f_2) There are positive constants C_1 and C_2 such that

$$|f(x,t)| \leq C_1 + C_2|t|^s \ \ \forall x \in \overline{\Omega},\ t \in \mathbb{R},$$

where $0 \leq s < \frac{n+2\alpha}{n-2\alpha}$.

(f_3) $\frac{f(x,t)}{|t|} \to 0$, $t \to 0$, $\forall x \in \overline{\Omega}$.

(f_4) There are constants $\mu > 2$ and $r \geq 0$ such that

$$0 < \mu F(x,t) \leq t f(x,t) \ \ \forall x \in \overline{\Omega}, \ \ |t| \geq r,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

By definition of $(-\Delta)^{\alpha}_{\rho}$, we can write (1.2) in its variational form as

$$\int_{\Omega}\int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dzdx = \int_{\Omega} f(x,u(x))v(x)dx. \tag{1.14}$$

We use the mountain pass Theorem to find a critical point of the functional

$$I_{\rho}(u) = \frac{1}{2}\int_{\Omega}\int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx - \int_{\Omega} F(x,u(x))dx, \tag{1.15}$$

where $F(x,t) = \int_0^t f(x,s)ds$ is defined on $H^{\alpha}_{\rho}(\Omega)$. Moreover it can be proved that

$$I'_{\rho}(u)v = \int_{\Omega}\int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dzdx - \int_{\Omega} f(x,u)vdx,$$

for $u, v \in H^{\alpha}_{\rho}(\Omega)$. So we say that $u \in H^{\alpha}_{\rho}(\Omega)$ is critical point of I_{ρ} if

$$\int_{\Omega}\int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dzdx = \int_{\Omega} f(x,u)vdx, \ \ \forall v \in H^{\alpha}_{\rho}(\Omega).$$

It follows that u is a weak solution of (1.2) if only if u is critical point of I_{ρ}.

Lemma 1.9. *Under the conditions (ρ_0)-(ρ_1) and (f_2), I_{ρ} is well defined.*

Proof. Let $I_{\rho}(u) = I_1(u) + I_2(u)$, where

$$I_1(u) = \frac{1}{2}\int_{\Omega}\int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx, \ \ I_2(u) = \int_{\Omega} F(x,u)dx.$$

We note

$$I_1(u) \ = \ \frac{1}{2}\|u\|^2_{H^{\alpha}_{\rho}} < +\infty, \ \ \text{since} \ \ u \in H^{\alpha}_{\rho}(\Omega).$$

By another hand, is standard to prove

$$|F(x,t)| \leq C_1|t| + \frac{C_2}{s+1}|t|^{s+1}, \quad 0 \leq s < \frac{n+2\alpha}{n-2\alpha}, \ n \geq 2. \tag{1.16}$$

Then by (1.16) for $u \in H^\alpha_\rho(\Omega)$, we get

$$|I_2(u)| \leq \int_\Omega |F(x,u(x))|dx \leq C_1|\Omega| \int_\Omega |u(x)|^2 dx + \frac{C_2}{s+1}\int_\Omega |u(x)|^{s+1}dx$$

since $1 \leq s+1 < 2^*_\alpha$ by Remark 1.4 we have

$$|I_2(u)| \leq CC_1|\Omega| \|u\|^2_{H^\alpha_\rho} + \frac{CC_2}{s+1}\|u\|^{s+1}_{H^\alpha_\rho} < +\infty.$$

□

Proposition 1.10. *Under the conditions (ρ_0)-(ρ_1) and (f_2), $I_\rho \in C^1(H^\alpha_\rho(\Omega), \mathbb{R})$*

Proof. Since

$$\begin{aligned} I_1(u) &= \frac{1}{2}\int_\Omega \int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx \\ &= \frac{1}{2}\langle u, u\rangle_{H^\alpha_\rho} \end{aligned}$$

by the properties of inner product, we have

$$I_1 \in C^1(H^\alpha_\rho(\Omega), \mathbb{R})$$

and

$$I_1'(u)v = \langle u, v\rangle_{H^\alpha_\rho}$$

Let

$$I_2(u) = \int_\Omega F(x,u)dx,$$

where $F(x,t) = \int_0^t f(x,\xi)d\xi$. Let $u \in H^\alpha_\rho(\Omega)$ fixed and for each $v \in H^\alpha_\rho(\Omega)$ we consider

$$r(v) = I_2(u+v) - I_2(u) - \int_\Omega f(x,u)vdx.$$

Now we prove

$$\lim_{\|v\|_{H^\alpha_\rho} \to 0} \frac{r(v)}{\|v\|_{H^\alpha_\rho}} = 0.$$

That is, $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that

$$\|v\|_{H^\alpha_\rho} < \delta \implies |r(v)| < \epsilon \|v\|_{H^\alpha_\rho}. \tag{1.17}$$

In fact, since

$$r(v) = \int_\Omega [F(x,u+v) - F(x,u)]dx - \int_\Omega f(x,u)vdx, \tag{1.18}$$

by the fundamental Theorem of calculus we have

$$F(x,u+v)-F(x,u)=\int_0^1 \frac{d}{dt}F(x,u+tv)dt,$$

but

$$\frac{d}{dt}F(x,u+tv)=f(x,u+tv)v,$$

then

$$r(v)=\int_\Omega\left[\int_0^1 f(x,u+tv)vdt\right]dx-\int_\Omega f(x,u)vdx.$$

So

$$|r(v)|\le\int_\Omega\int_0^1|f(x,u+tv)-f(x,u)||v|dtdx. \tag{1.19}$$

Now let β such that $\frac{1}{2^*_\alpha}+\frac{1}{\beta}=1$ that is $\beta=\frac{2n}{n+2\alpha}$. Since $v\in H^\alpha_\rho(\Omega)$ and $H^\alpha_\rho(\Omega)\hookrightarrow L^{2^*_\alpha}(\Omega)$ then $v\in L^{2^*_\alpha}(\Omega)$.

Claim 1: $f(.,u(.))\in L^\beta(\Omega)$.

Proof. By (f_2) we have

$$|f(x,t)|\le C_1+C_2|t|^s,\ \ 0\le s<\frac{n+2\alpha}{n-2\alpha},$$

hence

$$\int_\Omega|f(x,u)|^\beta dx\le C_3+C_4\int_\Omega|u|^{s\beta}dx, \tag{1.20}$$

where $C_3=2^{\beta-1}C_1^\beta|\Omega|$ and $C_4=2^{\beta-1}C_2^\beta$.

Claim 2: The growth condition (f_2) is also true for

$$1\le\overline{s}<\frac{n+2\alpha}{n-2\alpha},\ \ n\ge 2$$

Proof. For $0\le s\le 1$ we have

(a) If $0\le|t|<1$

$$C_1+C_2|t|^s\le C_1+C_2 1^s\le C_1+C_2+C_5|t|^{\overline{s}}$$

where C_5 is a positive constant. Let $C_6=C_1+C_2$

$$|f(x,t)|\le C_1+C_2|t|^s\le C_6+C_5|t|^{\overline{s}},\ \ \text{where}\ \ 1\le\overline{s}<\frac{n+2\alpha}{n-2\alpha}$$

(b) If $|t|\ge 1$, $C_1+C_2|t|^s\le C_1+C_2|t|^{\overline{s}}$, hence

$$|f(x,t)|\le C_1+C_2|t|^{\overline{s}},\ \ 1\le\overline{s}<\frac{n+2\alpha}{n-2\alpha}$$

Then by (a) and (b) we have

$$|f(x,t)|\le A+B|t|^{\overline{s}},\ \ 1\le\overline{s}<\frac{n+2\alpha}{n-2\alpha}.\qquad\square$$

From here $\overline{s}$ will be denoted by s. Since $1 \leq s < \frac{n+2\alpha}{n-2\alpha}$ and by the Sobolev imbedding we have $2\alpha < n$, hence $1 < \frac{2n}{n+2\alpha} = \beta$ and

$$\beta\left(\frac{n+2\alpha}{n-2\alpha}\right) = \frac{2n}{n-2\alpha} = 2^*_\alpha.$$

Therefore by the Remark 1.4, $H^\alpha_\rho(\Omega) \hookrightarrow L^{\beta s}(\Omega)$, since $1 < \beta \leq \beta s < 2^*_\alpha$. This implies that; if $u \in H^\alpha_\rho(\Omega)$ then $u \in L^{\beta s}(\Omega)$ hence $f(.,u(.)) \in L^\beta(\Omega)$. $\square$

Let $\lambda = 2^*_\alpha$, this implies $\frac{1}{\beta} + \frac{1}{\lambda} = 1$, hence by Fubini's Theorem and Hölder inequality we get

$$\begin{aligned} |r(v)| &\leq \int_0^1 \left[\left(\int_\Omega |f(x,u+tv) - f(x,u)|^\beta dx\right)^{1/\beta} \left(\int_\Omega |v|^\lambda\right)^{1/\lambda}\right] dt \\ &\leq \int_0^1 \|f(.,u+tv) - f(.,u)\|_{L^\beta(\Omega)} \|v\|_{L^{2^*_\alpha}(\Omega)} dt. \end{aligned} \tag{1.21}$$

Moreover we note that

$$1 \leq s < 2^*_\alpha - 1 \implies \frac{1}{2^*_\alpha - 1} < \frac{1}{s} \leq 1 \implies \beta < \frac{2^*_\alpha}{s} \leq 2^*_\alpha$$

Claim 3:

$$f(.,u+tv) \to f(.,u) \text{ in } L^{2^*_\alpha/s}(\Omega) \text{ uniformly, when } v \to 0 \text{ in } H^\alpha_\rho(\Omega)$$

for $t \in [0,1]$, $\forall x \in \Omega$. Note that this claim is equivalent to

$$f(.,u+tv_k) \to f(.,u) \text{ in } L^{2^*_\alpha/s}(\Omega) \text{ uniformly, when } v_k \to 0 \in H^\alpha_\rho(\Omega),\ k \to \infty$$

for $t \in [0,1]$, $\forall x \in \Omega$
Proof. Let $\{v_k\} \in H^\alpha_\rho(\Omega)$ with $v_k \to 0$ in $H^\alpha_\rho(\Omega)$. Since

$$\frac{2^*_\alpha}{s} \leq 2^*_\alpha \implies H^\alpha_\rho(\Omega) \hookrightarrow L^{2^*_\alpha/s}(\Omega)$$

then $v_k \to 0$ in $L^{2^*_\alpha/s}(\Omega)$ so exists a subsequence $\{v_{k_j}\}$ and $g \in L^{2^*_\alpha/s}(\Omega)$ such that

$$\begin{aligned} &v_{k_j} \to 0 \ \ a.e\ x \in \Omega \\ &|v_{k_j}| \leq g(x) \ \ a.e.\ x \in \Omega \end{aligned}$$

Then

$$|u(x) + tv_{k_j}| \leq |u(x)| + |g(x)| \ \ a.e.\ x \in \ \Omega\ \forall t \in [0,1] \tag{1.22}$$

and

$$(u(x) + tv_{k_j}(x)) \to u(x) \ \ a.e.\ x \in \Omega \tag{1.23}$$

Now since f is Carathéodory and (1.23) we have

$$f(x,u(x) + tv_{k_j}(x)) \to f(x,u(x)) \ \ a.e.\ x \in \Omega$$

in other words

$$|f(x,u(x) + tv_{k_j}(x)) - f(x,u(x))|^{2^*_\alpha/s} \to 0 \ \ a.e.\ x \in \Omega$$

Now by (f_2) there exists $\phi \in L^1(\Omega)$ such that

$$|f(x,u(x)+tv_{k_j}(x)) - f(x,u(x))|^{2^*_\alpha/s} \leq \phi(x)$$

by Lebesgue dominated converge theorem

$$\lim_{k_j\to\infty}\left(\int_\Omega |f(x,u(x)+tv_{k_j}(x)) - f(x,u(x))|^{2^*_\alpha/s}dx\right) = 0.$$

□

The meaning of the claim 3 is: Given $\epsilon > 0$ $\exists\delta > 0$ such that

$$\|v\|_{H^\alpha_\rho} < \delta \implies \|f(.,u+tv) - f(.,u)\|_{L^{2^*_\alpha/s}(\Omega)} < \epsilon \text{ uniformly for } t\in[0,1].$$

Since $1 < \beta < \frac{2^*_\alpha}{s}$ and Ω is bounded, then $L^{2^*_\alpha}(\Omega) \hookrightarrow L^\beta(\Omega)$ so

$$\|v\|_{H^\alpha_\rho} < \delta \implies \|f(.,u+tv) - f(.,u)\|_{L^{\beta}(\Omega)} < K\epsilon \text{ uniformly for } t\in[0,1], \tag{1.24}$$

replacing (1.24) on (1.21) we get

$$|r(v)| \leq K\epsilon\|v\|_{L^{2^*_\alpha}(\Omega)}$$

but $H^\alpha_\rho(\Omega) \hookrightarrow L^{2^*_\alpha}(\Omega)$, hence $|r(v)| \leq CK\epsilon\|v\|_{H^\alpha_\rho}$ provided that $\|v\|_{H^\alpha_\rho} < \delta$, showing (1.17). So I_2 is Frechet differentiable and

$$I_2'(u)v = \int_\Omega f(x,u)v dx \ \ \forall v \in H^\alpha_\rho(\Omega)$$

Now we prove the continuity of I_2'. Let $\{v_k\} \in H^\alpha_\rho(\Omega)$ with $v_k \to 0$ in $H^\alpha_\rho(\Omega)$ and we use the discussion given above to prove

$$\lim_{k\to\infty}\left(\int_\Omega |f(x,u(x)+tv_k(x)) - f(x,u(x))|^{2^*_\alpha/s}dx\right) = 0$$

and by (1.24) we have

$$\|f(x,u(x)+tv_k(x)) - f(x,u(x))\|_{L^\beta(\Omega)} \to 0, \ \ v_k \to 0 \text{ in } H^\alpha_\rho(\Omega) \tag{1.25}$$

On the other hand

$$I_2'(u+v_k)\varphi - I_2'(u)\varphi = \int_\Omega f(x,u+v_k)\varphi - f(x,u)\varphi dx.$$

Let $\sigma = 2^*_\alpha$. Since $H^\alpha_\rho(\Omega) \hookrightarrow L^{2^*_\alpha}(\Omega)$, by the Hölder inequality we get

$$\begin{aligned} |\langle I_2'(u+v_k) - I_2'(u),\varphi\rangle| &\leq \left(\int_\Omega |f(x,u+tv_k) - f(x,u)|^\beta dx\right)^{1/\beta}\left(\int_\Omega |\varphi|^\sigma dx\right)^{1/\sigma} \\ &\leq C\|f(.,u+v_k) - f(.,u)\|_{L^\beta(\Omega)}\|\varphi\|_{H^\alpha_\rho}. \end{aligned}$$

Hence

$$\begin{aligned} \|I_2'(u+v_k) - I_2'(u)\|_{(H^\alpha_\rho)'} &\leq \sup_{\|\varphi\|_{H^\alpha_\rho}\leq 1} |\langle I_2'(u+v_k) - I_2'(u),\varphi\rangle| \\ &\leq C\|f(.,u+v_k) - f(.,u)\|_{L^\beta(\Omega)}, \end{aligned}$$

this implies that I_2' is continuous. Therefore $I_\rho \in C^1(H_\rho^\alpha(\Omega), \mathbb{R})$ with

$$I_\rho'(u)v = \langle u, v\rangle_{H_\rho^\alpha} - \int_\Omega f(x,u)vdx \ \ \forall v \in H_\rho^\alpha(\Omega). \tag{1.26}$$

□

Now we prove our main theorem

Theorem 1.11. *Assume that $0 < \alpha < 1$, $n \geq 2$ and suppose that ρ satisfies (ρ_0)-(ρ_1) and f satisfies (f_1)-(f_4), the equation (1.2) possesses a nontrivial weak solution $u \in H_\rho^\alpha(\Omega)$.*

Proof. We prove that I_ρ satisfies the condition of Mountain pass theorem. Let

$$I_\rho(u(x)) = \frac{1}{2}\int_\Omega \int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx - \int_\Omega F(x,u(x))dx$$

Condition 1: By (f_3) given $\epsilon > 0$ there exists a positive constant $A = A(\epsilon)$ such that

$$|F(x,t)| \leq \frac{\epsilon}{2}|t|^2 + A|t|^{s+1} \ \ \forall t \in \mathbb{R}, \ x \in \overline{\Omega}. \tag{1.27}$$

By (1.27) and Remark 1.4 we obtain

$$\begin{aligned} I_\rho(u(x)) &= \frac{1}{2}\|u\|^2_{H_\rho^\alpha} - \int_\Omega F(x,u(x))dx \\ &\geq \frac{1}{2}\|u\|^2_{H_\rho^\alpha} - \frac{\epsilon}{2}\int_\Omega |u(x)|^2 dx - A\int_\Omega |u(x)|^{s+1}dx \\ &= \|u\|^2_{H_\rho^\alpha}\left(\frac{1}{2} - \frac{C\epsilon}{2} - AC\|u\|^{s-1}_{H_\rho^\alpha}\right) \ \ \forall u \in H_\rho^\alpha(\Omega). \end{aligned}$$

Then

$$\begin{aligned} I_\rho(u) > 0 = I_\rho(0) &\iff \frac{1-C\epsilon}{2} - AC\|u\|^{s-1}_{H_\rho^\alpha} > 0 \\ &\iff 0 < \|u\|_{H_\rho^\alpha} < \left(\frac{1-C\epsilon}{2AC}\right)^{\frac{1}{s-1}}, \ \ \epsilon \text{ small enough} \end{aligned}$$

taking

$$\begin{aligned} \theta &= \|u\|_{H_\rho^\alpha} \\ \varrho &= \theta^2\left(\frac{1-KC\epsilon}{2K} - AC\theta^{s-1}\right) \end{aligned}$$

we get the first geometry condition of mountain pass

$$I_\rho(u) \geq \varrho > 0$$

Condition 2:
By (f_4) there are positive constants B_1, B_2 such that

$$F(x,t) \geq B_1|t|^\mu - B_2 \ \ \forall t \in \mathbb{R}, \ x \in \overline{\Omega}. \tag{1.28}$$

We note since $\mu > 2$, by (1.28) F is super-quadratic in t. Let $u \in H^\alpha_\rho(\Omega)\backslash\{0\}$ fixed. Then by (1.27) for $t \geq 0$ we get

$$\begin{aligned} I_\rho(tu) &= \frac{t^2}{2}\|u\|^2_{H^\alpha_\rho} - \int_\Omega F(x,tu)dx \\ &\geq \frac{t^2}{2}\|u\|^2_{H^\alpha_\rho} - \frac{\epsilon t^2}{2}\int_\Omega |u|^2dx - At^{s+1}\int_\Omega |u|^{s+1}dx \\ &\geq \frac{t^2}{2}\|u\|^2_{H^\alpha_\rho} - \frac{\epsilon C t^2}{2}\|u\|^2_{H^\alpha_\rho} - ACt^{s+1}\|u\|^{s+1}_{H^\alpha_\rho} \\ &= t^2\|u\|^2_{H^\alpha_\rho}\left(\frac{1}{2} - \frac{\epsilon C}{2} - ACt^{s-1}\|u\|^{s-1}_{H^\alpha_\rho}\right) \end{aligned}$$

Therefore

$$I_\rho(tu) > 0 \iff 0 < t < \left(\frac{1-\epsilon C}{2AC\|u\|^{s-1}_{H^\alpha_\rho}}\right)^{\frac{1}{s-1}} \tag{1.29}$$

On the other hand, by (1.28)

$$I_\rho(tu) \leq \frac{t^2}{2}\|u\|^2_{H^\alpha_\rho} - B_1t^\mu\|u\|^\mu_{L^\mu(\Omega)} + B_2|\Omega|$$

since $\mu > 2$ we obtain

$$I_\rho(tu) \to -\infty, \text{ as } t \to \infty$$

and we have $I_\rho(0) = 0$ so exists $t_0 > 0$ such that

$$\|t_0u\|_{H^\alpha_\rho} > \theta \text{ and } I_\rho(t_0u) < 0$$

and we get the second geometry condition of mountain pass.

Now we prove the Palais-Smale condition. Let $\{u_n\} \in H^\alpha_\rho(\Omega)$ such that

$$|I_\rho(u_n)| \leq M \text{ and } I'_\rho(u_n) \to 0, \quad n \to +\infty \tag{1.30}$$

we prove that $\{u_n\}$ is bounded in $H^\alpha_\rho(\Omega)$. First we note

$$I_\rho(u_n) - \frac{1}{\mu}I'_\rho(u_n)u_n = \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2_{H^\alpha_\rho} + \int_\Omega\left(\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)\right)dx.$$

Let $A_n = \{x \in \Omega / \ |u_n(x)| \geq r\}$, hence by (f_4) we get

$$\begin{aligned} I_\rho(u_n) - \frac{1}{\mu}I'_\rho(u_n)u_n &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2_{H^\alpha_\rho} + \int_{A_n}\left(\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)\right)dx \\ &\quad + \int_{A_n^c}\left(\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)\right)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2_{H^\alpha_\rho} + \int_{A_n^c}\left(\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)\right)dx. \end{aligned}$$

Now since f and F are continuous functions, the function $g(x,u_n) = \left|\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)\right|$ is continuous. Hence, since $\overline{\Omega} \times [-r,r]$ is compact, then g is bounded in $\overline{\Omega} \times [-r,r]$ namely $\exists\ N > 0$ such that

$$|g(x,u_n)| \leq N \ \ \forall\,(x,u_n) \in \overline{\Omega} \times [-r,r],$$

this implies

$$\left|\frac{1}{\mu}f(x,u_n)u_n - F(x,u_n)\right| \le N \;\; \forall\, (x,u_n) \in \overline{A_n^c}.$$

Therefore

$$I_\rho(u_n) - \frac{1}{\mu}I'_\rho(u_n)u_n \ge \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|^2_{H^\alpha_\rho} - C_7, \tag{1.31}$$

where $C_7 = N|\Omega|$. By other hand

$$I_\rho(u_n) - \frac{1}{\mu}I'_\rho(u_n)u_n \;\; \le \;\; |I_\rho(u_n)| + \frac{1}{\mu}|I'_\rho(u_n)u_n|,$$

hence by (1.30) we get

$$I_\rho(u_n) - \frac{1}{\mu}I'_\rho(u_n)u_n \le M + \frac{1}{\mu}\|u_n\|_{H^\alpha_\rho} \;\text{ for }\; n \;\text{ large enough} \tag{1.32}$$

by (1.31) and (1.32) we have

$$M + \frac{1}{\mu}\|u_n\|_{H^\alpha_\rho} \;\; \ge \;\; \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|^2_{H^\alpha_\rho} - C_7, \;\text{ for }\; n \;\text{ large enough}$$

so $\{u_n\}$ is bounded in H^α_ρ. Now we prove that $\{u_n\}$ has a subsequence convergent. Since $I'_\rho(u) \in (H^\alpha_\rho(\Omega))'$ and $H^\alpha_\rho(\Omega)$ is a Hilbert spaces, then by Riesz representation Theorem

$$I'_\rho(u)v = \langle \nabla I_\rho(u), v\rangle_{H^\alpha_\rho} \;\; \forall v \in H^\alpha_\rho(\Omega) \;\text{ with }\; \|I'_\rho(u)\|_{(H^\alpha_\rho(\Omega))'} = \|\nabla I_\rho(u)\|_{H^\alpha_\rho}. \tag{1.33}$$

Let

$$J'(u)v = \int_\Omega f(x,u)v\,dx, \tag{1.34}$$

again by Riesz representation theorem

$$J'(u)v = \langle \nabla J(u), v\rangle_{H^\alpha_\rho} \;\; \forall v \in H^\alpha_\rho(\Omega), \;\text{ and }\; \|J'(u)\|_{(H^\alpha_\rho(\Omega))'} = \|\nabla J(u)\|_{H^\alpha_\rho}. \tag{1.35}$$

Hence by (1.33) and (1.35) we get

$$\langle \nabla I_\rho(u), v\rangle_{H^\alpha_\rho} = \langle u, v\rangle_{H^\alpha_\rho} - \langle \nabla J(u), v\rangle_{H^\alpha_\rho} = \langle u - \nabla J(u), v\rangle_{H^\alpha_\rho} \;\; \forall v \in H^\alpha_\rho(\Omega)$$

so

$$\nabla I_\rho(u) = u - \nabla J(u). \tag{1.36}$$

Now we define the operator

$$\begin{aligned} T \;\; &: \;\; H^\alpha_\rho(\Omega) \to H^\alpha_\rho(\Omega) \\ &\quad\;\; u \to T(u) = \nabla J(u) \end{aligned}$$

by (1.36), $\nabla I_\rho(u) = u - T(u)$.
Claim 4: $T : H^\alpha_\rho(\Omega) \to H^\alpha_\rho(\Omega)$ is compact.
Proof. Let $\{u_n\} \in H^\alpha_\rho(\Omega)$ be a bounded sequence, we prove that the sequence $T(u_n) \in H^\alpha_\rho(\Omega)$ has a subsequence convergent. Since $H^\alpha_\rho(\Omega)$ is reflexive space, then exists a subsequence (we denote again with u_n) such that $u_n \rightharpoonup u$ in $H^\alpha_\rho(\Omega)$. By compact embedding

$$u_n \to u \;\text{ in }\; L^q(\Omega), \;\; \forall q \in [1, 2^*_\alpha). \tag{1.37}$$

Now we fixed q with $s+1 \leq q < 2^*_\alpha$ and we take r such that

$$\frac{1}{q}+\frac{1}{r}=1 \implies r=\frac{q}{q-1}.$$

We claim that

$$f(.,u_n(.)) \to f(.,u(.)) \text{ in } L^r(\Omega).$$

In fact, by (1.37) up to a subsequence

$$u_n(x) \to u(x), \quad \text{a.e. } x \in \Omega, and$$

$$|u_n(x)| \leq G(x), \quad \text{a.e } x \in \Omega, \ \forall n \in \mathbb{N} \ \text{ and } \ G \in L^q(\Omega) \tag{1.38}$$

since f is a Caratheodory function

$$f(x,u_n(x)) \to f(x,u(x)) \text{ a.e. } x \in \Omega$$

so

$$|f(x,u_n(x)) - f(x,u(x))|^{q/s} \to 0 \text{ a.e. } x \in \Omega.$$

Now by (f_2), (1.38) and the fact that Ω is bounded, there is $\phi \in L^1(\Omega)$ such that

$$|f(x,u_n(x)) - f(x,u(x))|^{q/s} \leq \phi(x)$$

the by Lebesgue dominated converge theorem

$$\lim_{n\to+\infty}\left(\int_\Omega |f(x,u_n(x)) - f(x,u(x))|^{q/s}dx\right) = 0$$

so

$$f(.,u_n(.)) \to f(.,u(.)) \text{ in } L^{q/s}(\Omega).$$

Now since $1<r<\frac{q}{s}$ and Ω is bounded, $L^{q/s}(\Omega) \hookrightarrow L^r(\Omega)$. Then

$$\|f(.,u_n(.)) - f(.,u(.))\|_{L^r} \to 0, \quad \text{when } n \to +\infty \tag{1.39}$$

We note that

$$\begin{aligned} \|T(u_n)-T(u)\|_{H^\alpha_\rho} &= \|J'(u_n)-J'(u)\|_{(H^\alpha_\rho(\Omega))'} \\ &= \sup_{\|v\|_{H^\alpha_\rho}\leq 1} |(J'(u_n)-J'(u)).v| \end{aligned}$$

moreover

$$\begin{aligned} |(J'(u_n)-J'(u)).v| &= \left|\int_\Omega [f(x,u_n)-f(x,u)]v dx\right| \\ &\leq \int_\Omega |f(x,u_n)-f(x,u)||v|dx \\ &\leq \|f(.,u_n(.))-f(.,u(.))\|_{L^r}\|v\|_{L^q} \end{aligned}$$

and since $H^\alpha_\rho(\Omega) \hookrightarrow L^q(\Omega)$ with $q \in [s+1, 2^*_\alpha]$ continuously, we get

$$|(J'(u_n)-J'(u))v| \leq C\|f(.,u_n(.))-f(.,u(.))\|_{L^r}\|v\|_{H^\alpha_\rho}$$

so

$$\|J'(u_n) - J'(u)\|_{(H^\alpha_\rho(\Omega))'} \leq C\|f(., u_n(.)) - f(., u(.))\|_{L^r}$$

that is

$$\|T(u_n) - T(u)\|_{H^\alpha_\rho} \leq C\|f(., u_n(.)) - f(., u(.))\|_{L^r} \tag{1.40}$$

then by (1.39) and (1.40)

$$\|T(u_n) - T(u)\|_{H^\alpha_\rho} \to 0 \tag{1.41}$$

so T is compact. By other hand $I'_\rho(u_n) \to 0,\ n \to +\infty \iff \|I'_\rho(u_n)\|_{(H^\alpha_\rho(\Omega))'} \to 0,\ n \to +\infty$. Hence

$$\|\nabla I_\rho(u_n)\|_{H^\alpha_\rho} \to 0,\ n \to +\infty \iff \nabla I_\rho(u_n) \to 0,\ n \to +\infty$$

being $\nabla I_\rho(u_n) = u_n - T(u_n)$, then

$$u_n = (\nabla I_\rho(u_n) + T(u_n)) \tag{1.42}$$

taking $n \to +\infty$ in (1.42) we get (up to a subsequence)

$$u_n \to T(u)$$

then we prove the PS condition. Hence by Theorem 1.6 there is a critical point of I_ρ, that is a weak solution of (1.2). $\square$

Chapter 2

Nonlinear Schrödinger Equation With Non-local Regional Diffusion

The aim of this chapter is to study the nonlinear Schrödinguer equation with non local regional diffusion. More precisely, we are concerned with solutions to the following problem

$$\begin{cases} \epsilon^{2\alpha}(-\Delta)^{\alpha}_{\rho} + u = f(u), & \text{in } \mathbb{R}^n \\ u \in H^{\alpha}_{\rho}(\mathbb{R}^n) \end{cases} \tag{2.1}$$

where $0 < \alpha < 1$, $\epsilon > 0$, $n \geq 2$ and $f : \mathbb{R} \to \mathbb{R}$ is super-linear and has a sub-critical growth. The operator $(-\Delta)^{\alpha}_{\rho}$ is a non-local regional laplacian, with range of scope determined by the positive function $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$.

In recent years, a great attention has been focused on the study of fractional and non-local operators of elliptic type, both from the mathematical and applied. These operators arise in a quite natural way in many different contexts of applications, see for example the work by Di Nezza, Patalluci and Valdinoci [29].

In the context of fractional quantum mechanics, non-linear fractional Schrödinger equation has been proposed by Laskin [55], [56] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. In the last 10 years, there has been a lot of interest in the study of the fractional Schrödinger equation, see the works in [24], [32], [37], [42] and [67]. In a recent paper Felmer, Quaas and Tan [37] considered positive solutions of nonlinear fractional Schrödinger equation

$$(-\Delta)^{\alpha}u + u = f(x, u) \text{ in } \mathbb{R}^n \tag{2.2}$$

They obtained the existence of a ground state by mountain pass argument and comparison method devised by Rabinowitz in [71] for $\alpha = 1$. They also analyzed regularity, decay and symmetry properties of these solutions, and they also studied the problem in the presence of an x dependent bounded potential. At this point it is worth mentioned the uniqueness of the ground state in one dimension and with power non-linearity was proved by Frank and Lenzmann in [36], and advances in the multi-dimensional case were obtained recently by Fall and Valdinoci in [35]. We also mention the work by Cheng [24], where the fractional Schrödinger equation with unbounded potential

$$(-\Delta)^{\alpha}u + V(x)u = |u|^{p-1}u \text{ in } \mathbb{R}^n \tag{2.3}$$

was studied. The existence of a ground state of (2.3) with unbounded potential V is obtained by Lagrange multiplier method and Nehari manifold method is used to obtain the standing wave with prescribed frequency and they proved that the standing wave is a bound state i.e, $|u| \to 0$, as $|x| \to \infty$.

On the other hand, research has been done in recente years regarding a fractional laplacian with restricted range of scope. We mention the work by Guan [40] and Guan and Ma [41] where they study these operators, their relation with stochastic processes and they develop integration by parts formula, and specially the work by Ishii and Nakamura [46] where the authors studied the Dirichlet problem for regional fractional laplacian modeled on the p-laplacian. These regional operators present various interesting characteristics that make them very attractive from the point of view of mathematical theory of non-local operators.

In this chapter we are interested in studying the non-linear Schrödinger equation when a variational version of the regional fractional laplacian is considered. We are specially interested in understanding the role of the range of scope ρ on the existence and concentration of solutions in the semi-classical limit for equation (2.1) (see chapter 2).

Now we make precise assumptions on ρ and f. For the range of scope ρ we assume $\rho : \mathbb{R}^n \to \mathbb{R}^+$ is continuous and it satisfies the following hypotheses:

(ρ_1) There are numbers $0 < \rho_0 < \rho_\infty \leq \infty$ such that

$$\rho_0 \leq \rho(x) < \rho_\infty \quad \forall x \in \mathbb{R}^n \quad \text{and} \quad \lim_{|x|\to\infty} \rho(x) = \rho_\infty.$$

(ρ_2) In case $\rho_\infty = \infty$ we further assume that there exists $a \in (0,1)$ such that

$$\limsup_{|x|\to\infty} \frac{\rho(x)}{|x|} \leq a.$$

(ρ_3) For any $x_0 \in \mathbb{R}^n$, the equation

$$|x| = \rho(x + x_0), \quad x \in \mathbb{R}^n,$$

defines an $(n-1)$-dimensional surface of class C^1 in $\mathbb{R}^n$.

Regarding the non-linearity f we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies the following hypotheses:

(f_1) $f(t) \geq 0$ if $t \geq 0$ and $f(t) = 0$ if $t \leq 0$.

(f_2) The function $t \to \frac{f(t)}{t}$ is increasing for $t > 0$ and $\lim_{t\to 0} \frac{f(t)}{t} = 0$.

(f_3) $\exists\, \theta > 2$ such that $\forall t > 0$

$$0 < \theta F(t) \leq t f(t), \qquad \text{where} \quad F(t) = \int_0^t f(\xi)d\xi.$$

(f_4) $\exists\, C > 0$ such that

$$|f(t)| \leq C(1 + |t|^p), \;\; 1 < p < \frac{n+2\alpha}{n-2\alpha}.$$

Our main existence theorem in this section is:

Theorem 2.1. *Assume $0 < \alpha < 1$ and $n \geq 2$. If f satisfies (f_1)-(f_4) and ρ satisfies (ρ_1)-(ρ_2) then (2.1) possesses at least one non-trivial weak solution. Moreover this solution satisfies $u(x) \geq 0$ a.e. for all $x \in \mathbb{R}^n$.*

2.1 Preliminary Results

2.1.1 Fractional Space

Let $0 < \alpha < 1$ and $n \geq 1$. The fractional Sobolev space of order α on $\mathbb{R}^n$ is defined by

$$H^\alpha(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) / \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} dydx < \infty \right\}$$

equipped with the norm

$$\|u\|_\alpha^2 = \int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} dydx.$$

On the other hand, using Fourier transform we may consider an alternative way of defining the Sobolev space $H^\alpha(\mathbb{R}^n)$ as

$$H^\alpha(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) / \ |\xi|^\alpha \widehat{u}(\xi) \in L^2(\mathbb{R}^n) \}$$

with inner product and norm given by

$$\langle u, v \rangle_{\widehat{H}^\alpha} = \int_{\mathbb{R}^n} (1 + |\xi|^{2\alpha}) \widehat{u}(\xi) \widehat{v}(\xi) d\xi \ \text{ and } \ \|u\|_{\widehat{H}^\alpha}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^{2\alpha}) |\widehat{u}(\xi)|^2 d\xi.$$

If we define the fractional laplacian as

$$\widehat{(-\Delta)^\alpha} u(\xi) = |\xi|^{2\alpha} \widehat{u}(\xi),$$

then it can be shown that for $0 < \alpha < 1$ there exists a constant $C(n, \alpha)$ such that for all $u, v \in H^\alpha(\mathbb{R}^n)$,

$$\begin{aligned} \int_{\mathbb{R}^n} (-\Delta)^\alpha u(x) v(x) dx &= \int_{\mathbb{R}^n} |\xi|^{2\alpha} \widehat{u}(\xi) \widehat{v}(\xi) d\xi \\ &= C(n, \alpha) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(z))(v(x) - v(z))}{|x - z|^{n+2\alpha}} dzdx. \end{aligned} \tag{2.4}$$

See [29] Proposition 3.3. Regarding this space, we recall the following embedding theorem, whose proof can be found in [29]

Theorem 2.2. *Let $\alpha \in (0, 1)$, then there exists a positive constant $C = C(n, \alpha)$ such that*

$$\|u\|_{L^{2_\alpha^*}(\mathbb{R}^n)}^2 \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} dydx \tag{2.5}$$

and then we have that $H^\alpha(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is continuous for all $q \in [2, 2_\alpha^*]$.

Moreover, $H^\alpha(\mathbb{R}^n) \hookrightarrow L^q(\Omega)$ is compact for any bounded set $\Omega \subset \mathbb{R}^n$ and for all $q \in [2, 2_\alpha^*)$, where $2_\alpha^* = \frac{2n}{n-2\alpha}$ is the critical exponent.

Now we will introduce a new regional fractional space associated to our problem. Let

$$H_\rho^\alpha(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) / \int_{\mathbb{R}^n} \int_{B(0,\rho(x))} \frac{|u(x + z) - u(x)|}{|z|^{n+2\alpha}} dzdx < +\infty \right\}$$

equipped with the norm

$$\|u\|_\rho^2 = \int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n} \int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dz dx,$$

which is induced by the following inner product

$$\langle u, v\rangle_\rho = \int_{\mathbb{R}^n} u(x)v(x)dx + \int_{\mathbb{R}^n} \int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dz dx.$$

For the space $H_\rho^\alpha(\mathbb{R}^n)$ we have the following result

Proposition 2.3. *If ρ satisfies (ρ_1) there exists a constant $C = C(n, \alpha, \rho_0) > 0$ such that*

$$\|u\|_\alpha \leq C\|u\|_\rho$$

Proof. From the definition of $\|.\|_\alpha$, given $u \in H_\rho^\alpha(\mathbb{R}^n)$ and for $\rho_0 > 0$, we have

$$\begin{aligned}\|u\|^2 &= \int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n} \int_{B(x,\rho_0)} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dz dx \\ &\quad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B(x,\rho_0)} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dz dx. \end{aligned} \tag{2.6}$$

Considering the integral over the complement of the ball $B(x, \rho_0)$, using Fubini's Theorem, we see that

$$\begin{aligned}\int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B(x,\rho_0)} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}} dz dx &= \int_{\mathbb{R}^n \setminus B(0,\rho_0)} \int_{\mathbb{R}^n} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dx dz \\ &\leq \frac{2|S^{n-1}|}{\alpha \rho_0^{2\alpha}} \|u\|_{L^2}^2. \end{aligned} \tag{2.7}$$

Using hypothesis (ρ_1) we see that $B(0, \rho_0) \subset B(0, \rho(x))$, for all $x \in \mathbb{R}^n$, then it follows from 2.6 and 2.7 that

$$\|u\|_\alpha^2 \leq C \left(\int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n} \int_{B(0,\rho(x))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dz dx \right),$$

where $C = C(n, \alpha, \rho_0) = 1 + \frac{2|S^{n-1}|}{\alpha \rho_0^{2\alpha}}$. This completes the proof. $\square$

Remark 2.4. By Proposition 2.3 we have the continuous embedding $H_\rho^\alpha(\mathbb{R}^n) \hookrightarrow H^\alpha(\mathbb{R}^n)$ and then by Theorem 2.2 we have that

- $H_\rho^\alpha(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ continuously for any $q \in [2, 2_\alpha^*]$.
- $H_\rho^\alpha(\mathbb{R}^n) \hookrightarrow L_{loc}^q(\mathbb{R}^n)$ compactly for any $q \in [2, 2_\alpha^*)$.

Remark 2.5. Since $\|u\|_\rho \leq \|u\|_\alpha$, by Proposition 2.3, under the condition (ρ_1), we have $\|\cdot\|_\rho$ and $\|\cdot\|$ are equivalent norms in $H^\alpha(\mathbb{R}^n)$.

The following lemma is a version of the concentration compactness principle proved by Coti Zelati and Rabinowitz [27].

Lemma 2.6. *Let $n \geq 2$. Assume that $\{u_k\}$ is bounded in $H^\alpha_\rho(\mathbb{R}^n)$ and it satisfies*

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^n} \int_{B(y,R)} |u_k(x)|^2 dx = 0, \tag{2.8}$$

where $R > 0$. Then $u_k \to 0$ in $L^q(\mathbb{R}^n)$ for $2 < q < 2^*_\alpha$.

Proof. Let $2 < q < 2^*_\alpha$ and consider $\theta \in (0,1)$ such that

$$\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{2^*_\alpha}.$$

Then by Hölder inequality and Theorem 2.1, for every k we have

$$\begin{aligned} \|u_k\|_{L^q(B(y,R))} &\leq \|u_k\|^{1-\theta}_{L^2(B(y,R))} \|u_k\|^\theta_{L^{2^*_\alpha}(B(y,R))} \\ &\leq C^\theta \|u_k\|^{1-\theta}_{L^2(B(y,R))} \|u_k\|^\theta_\rho \end{aligned}$$

so

$$\int_{B(y,R)} |u_k(x)|^q dx \leq C^{\theta q} \|u_k\|^{(1-\theta)q}_{L^2(B(y,R))} \|u_k\|^{q\theta}_\rho.$$

Taking $\theta = \frac{2}{q}$ and covering $\mathbb{R}^n$ with balls of radius R, in such a way that each point of $\mathbb{R}^n$ is contained in at most $n+1$ balls, we deduce that

$$\int_{\mathbb{R}^n} |u_k(x)|^q dx \leq (n+1) C^{q\theta} \sup_{y\in\mathbb{R}^n} \left(\int_{B(y,R)} |u_k(x)|^2 \right)^{\frac{q-2}{2}} \|u_k\|^2_\rho.$$

Then, by hypothesis

$$u_k \to 0, \text{ in } L^q(\mathbb{R}^n). \quad \square$$

2.1.2 The Operator $(-\Delta)^\alpha_\rho$

The spaces $H^\alpha(\mathbb{R}^n)$ and $H^\alpha_\rho(\mathbb{R}^n)$ are Hilbert spaces endowed with the inner products

$$\langle u, v\rangle = \int_{\mathbb{R}^n} u(x)v(x)dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dzdx$$

and

$$\langle u, v\rangle_\rho = \int_{\mathbb{R}^n} u(x)v(x)dx + \int_{\mathbb{R}^n}\int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dzdx,$$

respectively. Using the equivalence of the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\rho$ proved in Proposition 2.3, we can use the Lax-Milgran representation theorem (see Theorem 17.9 from [87]) to find a unique bijective linear map $\mathcal{M}_\rho : H^\alpha(\mathbb{R}^n) \to H^\alpha(\mathbb{R}^n)$ such that

$$\langle u, v\rangle_\rho = \langle \mathcal{M}_\rho u, v\rangle \text{ for all } u, v \in H^\alpha(\mathbb{R}^n).$$

Then by the definition of the inner product in $H^\alpha(\mathbb{R}^n)$ we have

$$\langle u, v\rangle_\rho = \langle \mathcal{M}_\rho u, v\rangle = \int_{\mathbb{R}^n} ((-\Delta)^\alpha \mathcal{M}_\rho u + \mathcal{M}_\rho u)(x)v(x)dx$$

and if we define the operator $(-\Delta)^\alpha_\rho : H^\alpha(\mathbb{R}^n) \to H^{-\alpha}(\mathbb{R}^n)$ by

$$(-\Delta)^\alpha_\rho = (-\Delta)^\alpha \circ \mathcal{M}_\rho + \mathcal{M}_\rho - I, \tag{2.9}$$

where I is the natural injection from $H^\alpha(\mathbb{R}^n)$ to $H^{-\alpha}(\mathbb{R}^n)$ and $H^{-\alpha}(\mathbb{R}^n)$ denotes the dual space of $H^\alpha(\mathbb{R}^n)$. With these definitions we finally have that for all $u, v \in H^\alpha(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (-\Delta)^\alpha_\rho uv dx = \int_{\mathbb{R}^n}\int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dz dx. \tag{2.10}$$

Proposition 2.7. *If ρ satisfies (ρ_1) and (ρ_2), then the operator $(-\Delta)^\alpha_\rho$ is regional, in the sense that given $u \in H^\alpha(\mathbb{R}^n)$ with compact support, there exists $R > 0$ so that for all $\varphi \in H^\alpha(\mathbb{R}^n)$ such that*

$$\mathrm{supp}(\varphi) \cap B(0,R) = \phi$$

then

$$\int_{\mathbb{R}^n} (-\Delta)^\alpha_\rho u(x)\varphi(x) dx = 0.$$

Proof. We assume that $\rho_\infty = \infty$ in (ρ_1) (the other case is similar). By (ρ_2) there exists $\bar{a} \in (0,1)$ and $R > 0$ so that $|\rho(x)| \leq \bar{a}|x|$ for all $|x| \geq \bar{R}$.

If $\varphi \in H^\alpha(\mathbb{R}^n)$ is such that $\mathrm{supp}\varphi \cap B(0,R) = \emptyset$ then

$$\cup_{x\in \mathrm{supp}\varphi} B(x,\rho(x)) \subset \cup_{x\in\mathrm{supp}\varphi} B(x, \bar{a}|x|) \subset B(0,(1-\bar{a})R)^c.$$

On the other hand, given $u \in H^\alpha(\mathbb{R}^n)$ with compact support, there exists R_0 such that

$$\cup_{x\in\mathrm{supp} u} B(x,\rho(x)) \subset B(0,R_0).$$

Then we make R larger, if necessary, in order to get $R_0 < (1-\bar{a})R$ and thus we have that for any $(x,z) \in \mathbb{R}^n \times B(0,\rho(x))$ such that $x \in \mathrm{supp}\varphi$ or $x+z \in \mathrm{supp}\varphi$ we have $|x| \geq (1-\bar{a})R > R_0$ and then $u(x) = u(x+z) = 0$. Consequently we have that

$$\begin{aligned}&\int_{\mathbb{R}^n} (-\Delta)^\alpha_\rho u(x)v(x)dx = \\ &\int_{\mathbb{R}^n}\int_{B(0,\rho(x))} \frac{[u(x+z)-u(x)][\varphi(x+z)-\varphi(x)]}{|z|^{n+2\alpha}} dz dx = 0. \quad \square\end{aligned}$$

2.2 The Ground State

In this section, our goal is to study existence of ground states of equation 2.1, that is, non-negative solutions with lowest energy. We start making a precise definition of the notion of solutions for the equation (2.1). We have

Definition 2.8. Let ρ satisfying (ρ_1). We say that $u \in H^\alpha_\rho(\mathbb{R}^n)$ is a weak solution of (2.1) if

$$\langle u, v\rangle_\rho = \int_{\mathbb{R}^n} f(x)v(x)dx, \quad \text{for all} \quad v \in H^\alpha_\rho(\mathbb{R}^n).$$

We will prove the existence of weak solution of (2.1) applying the mountain pass theorem [7] to the functional I_ρ defined in $H^\alpha_\rho(\mathbb{R}^n)$ by

$$I_\rho(u) = \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{B(0,\rho(x))}\frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n}u(x)^2dx\right) - \int_{\mathbb{R}^n}F(u(x))dx. \quad (2.11)$$

However, the direct application of the mountain pass theorem is not enough since the Palais-Smale sequences might lose compactness in the whole space $\mathbb{R}^n$. To overcome the difficulty, we use a comparison argument devised by Rabinowitz in [71] for the Poisson equation, based on a limiting energy functional.

Using the properties of the Nemistky operators and the embeddings given in Remark 2.4, we can prove that the functional I_ρ is of class $C^1(H^\alpha_\rho(\mathbb{R}^n),\mathbb{R})$ and we have

$$I'_\rho(u(x))v(x) = \langle u,v\rangle_\rho - \int_{\mathbb{R}^n}f(u(x))v(x)dx, \ \ \forall\, v\in H^\alpha_\rho(\mathbb{R}^n).$$

We notice that a necessary condition for $u\in H^\alpha_\rho(\mathbb{R}^n)$ to be a critical point of I_ρ is $I'_\rho(u)u=0$. This condition define the Nehari manifold associated to the functional I_ρ

$$\mathcal{N}_\rho = \{u\in H^\alpha_\rho(\mathbb{R}^n)\backslash\{0\}:\ \ I'_\rho(u)u=0\}$$

and so, all non trivial solution of (2.1) belong to the Nehari manifold.

Next, from the growth conditions on f ((f_2) and (f_4)) it is standard to prove that, for any $\epsilon>0$, there exists C_ϵ such that

$$|f(t)|\le \epsilon|t| + C_\epsilon|t|^p, \ \ \forall t\in\mathbb{R}^n \quad (2.12)$$

and consequently

$$|F(t)|\le \frac{\epsilon}{2}|t|^2 + C_\epsilon|t|^{p+1}, \ \ \forall t\in\mathbb{R}^n, \quad (2.13)$$

where $1<p<\frac{n+2\alpha}{n-2\alpha}=2^*_\alpha-1$.

We start our analysis with

Lemma 2.9. *Assume the hypotheses (ρ_1), (f_1)-(f_4). For any $u\in H^\alpha_\rho(\mathbb{R}^n)\setminus\{0\}$, there is a unique $t_u=t(u)>0$ such that $t_uu\in\mathcal{N}_\rho$ and we have*

$$I_\rho(t_uu) = \max_{t\ge 0} I_\rho(tu)$$

Proof. Let $u\in H^\alpha_\rho(\mathbb{R}^n)\setminus\{0\}$ and consider the function $\psi:\mathbb{R}^+\to\mathbb{R}$ defined as

$$\psi(t) = I_\rho(tu) = \frac{t^2}{2}\|u\|^2_\rho - \int_{\mathbb{R}^n}F(tu)dx.$$

Then, by (2.13) we have

$$\int_{\mathbb{R}^n}F(u)dx \ \ \le\ \ \frac{C\epsilon}{2}\|u\|^2_\rho + CC_\epsilon\|u\|^{p+1}_\rho$$

This implies that $\psi(t)>0$, for t small. On the other hand, by (f_3) exists $A>0$ such that $F(t)\ge A|t|^\theta$, $\forall t>0$. So

$$I_\rho(tu) \ \ \le\ \ \frac{t^2}{2}\|u\|^2_{H^\alpha_\rho} - At^\theta\int_{\mathbb{R}^n}|u|^\theta \quad (2.14)$$

and since $\theta > 2$, we see that $\psi(t) < 0$ for t large. By (f_1), $\psi(0) = 0$, therefore there is $t_u = t(u) > 0$ such that

$$\psi(t_u) = \max_{t\geq 0} \psi(t) = \max_{t\geq 0} I_\rho(tu) = I_\rho(t_u u).$$

We see that $\psi'(t) = 0$ is equivalent to

$$\|u\|^2_{H^\alpha_\rho} = \int_{\mathbb{R}^n} \frac{f(tu)u}{t} dx, \tag{2.15}$$

from where, using (f_2) we prove that there is a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\rho$. $\square$

We define

$$c^*_\rho = \inf_{u\in\mathcal{N}_\rho} I_\rho(u) \tag{2.16}$$

On the other hand we consider the set of functions

$$\Gamma_\rho = \{\gamma \in C([0,1], H^\alpha_\rho(\mathbb{R}^n))/ \ \ \gamma(0) = 0, \ I_\rho(\gamma(1)) < 0\}$$

and define

$$c_\rho = \inf_{\gamma\in\Gamma_\rho} \sup_{t\in[0,1]} I_\rho(\gamma(t)) \tag{2.17}$$

Under our assumptions, certainly Γ_ρ is not empty and $c_\rho > 0$. The following lemma is crucial and it uses (f_3).

Lemma 2.10.

$$c^*_\rho = \inf_{u\in H^\alpha_\rho(\mathbb{R}^n)\setminus\{0\}} \sup_{\xi\geq 0} I_\rho(\xi u) = c_\rho. \tag{2.18}$$

Proof. We notice that I_ρ is bounded below on $\mathcal{N}_\rho$, since by (f_3), $I_\rho(u) > 0, \forall\, u \in \mathcal{N}_\rho$, so that c^*_ρ is well defined. By Lemma 2.9 for any $u \in H^\alpha_\rho(\mathbb{R}^n) \setminus \{0\}$ there is a unique $t_u = t(u) > 0$ such that $t_u u \in \mathcal{N}_\rho$, then

$$c^*_\rho \leq \inf_{u\in H^\alpha_\rho(\mathbb{R}^n)\setminus\{0\}} \max_{t\geq 0} I_\rho(tu).$$

On the other hand, for any $u \in \mathcal{N}_\rho$, we have

$$I_\rho(u) \ \ = \ \ \max_{t\geq 0} I_\rho(tu) \geq \inf_{u\in H^\alpha_\rho(\mathbb{R}^n)\setminus\{0\}} \max_{t\geq 0} I_\rho(tu)$$

so

$$c^*_\rho = \inf_{\mathcal{N}_\rho} I_\rho(u) \geq \inf_{u\in H^\alpha_\rho(\mathbb{R}^n)\setminus\{0\}} \max_{t\geq 0} I_\rho(tu),$$

therefore the first equality in (2.18) holds. Next we prove the other equality, that is $c^*_\rho = c_\rho$. We claim that for every $\gamma \in \Gamma_\rho$ there exists $t_0 \in [0,1]$ such that $\gamma(t_0) \in \mathcal{N}_\rho$.

To prove the claim, we first see that, by (2.12) and the continuous embedding $H^\alpha_\rho(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n)$ and $L^{p+1}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} f(u)u dx \leq \epsilon C\|u\|^2_{H^\alpha_\rho} + C_\epsilon C\|u\|^{p+1}_{H^\alpha_\rho}. \tag{2.19}$$

and then, for $\gamma \in \Gamma_\rho$ we have

$$\begin{aligned} I'_\rho(\gamma(t))\gamma(t) \ \ &= \ \ \|\gamma(t)\|^2_{H^\alpha_\rho} - \int_{\mathbb{R}^n} f(\gamma(t))\gamma(t) dx \\ &\geq \ \ \left(1 - \epsilon C - C_\epsilon C\|\gamma(t)\|^{p-1}_{H^\alpha_\rho}\right) \|\gamma(t)\|^2_{H^\alpha_\rho}. \end{aligned}$$

If we take $r = \left(\frac{1-\epsilon C}{C_\epsilon C}\right)^{\frac{1}{p-1}}$, then we see that

$$I'_\rho(\gamma(t))\gamma(t) > 0 \quad \forall\, t \in [0,1], \text{ such that, } \|\gamma(t)\|_{H^\alpha_\rho} < r.$$

On the other hand, using (f_3) and since $I_\rho(\gamma(1)) < 0$, we have

$$\|\gamma(1)\|^2_{H^\alpha_\rho} \;<\; \int_{\mathbb{R}^n} 2F(\gamma(t))dx < \int_{\mathbb{R}^n} \theta F(\gamma(1))dx \le \int_{\mathbb{R}^n} f(\gamma(1))\gamma(1)dx,$$

that implies $I'_\rho(\gamma(1))\gamma(1) < 0$. Thus, by the Intermediate Value Theorem, there exists $t_0 \in (t_*, 1)$ such that $I'_\rho(\gamma(t_0))\gamma(t_0) = 0$ and so $\gamma(t_0) \in \mathcal{N}_\rho$., completing the proof of the claim. From this result, $\max_{t\in[0,1]} I_\rho(\gamma(t)) \ge I_\rho(\gamma(t_0)) \ge \inf_{\mathcal{N}_\rho} I_\rho$ and then

$$c_\rho \ge c^*_\rho \tag{2.20}$$

In order to prove the other inequality we see that from (2.14), there exists t^*_u large enough such that $I_\rho(t^*_u u) < 0$. Now we define the curve $\gamma_u : [0,1] \to H^\alpha_\rho(\mathbb{R}^n)$ as $\gamma_u(t) = t(t^*_u u)$. Then $\gamma_u(0) = 0$, $I_\rho(\gamma(1)) = I_\rho(t^*_u u) < 0$ and γ_u is continuous, so that $\gamma_u \in \Gamma_\rho$. Now, by definition of γ_u,

$$\max_{t\ge 0} I_\rho(tu) \ge \max_{\xi\in[0,1]} I_\rho(\gamma_u(\xi)), \quad \forall\, H^\alpha_\rho(\mathbb{R}^n \setminus \{0\})$$

then $c^*_\rho \ge c_\rho$, completing the proof. $\square$

Remark 2.11. Since $c_\rho = \inf_{\mathcal{N}_\rho} I_\rho$ and any critical point of I_ρ lies on $\mathcal{N}_\rho$, if c_ρ is a critical value of I_ρ then it is the smallest positive critical value of I_ρ.

Lemma 2.12. *Suppose $\{u_k\} \in H^\alpha_\rho(\mathbb{R}^n)$ and there exists $b > 0$ such that*

$$I_\rho(u_k) \le b \;\text{ and }\; I'_\rho(u_k) \to 0 \tag{2.21}$$

Then either

(i) $u_k \to 0$ in $H^\alpha_\rho(\mathbb{R}^n)$, or

(ii) there is a sequence $(y_k) \in \mathbb{R}^n$, and $R, \beta > 0$ such that

$$\liminf_{k\to\infty} \int_{B(y_k,R)} |u_k(x)|^2 dx > \beta.$$

Proof. By (2.21) it is standard to check, for k large enough

$$b + \|u_k\|_{H^\alpha_\rho} \ge I_\rho(u_k) - \frac{1}{\theta} I'_\rho(u_k)u_k \ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_k\|^2_{H^\alpha_\rho} \tag{2.22}$$

and then $\{u_k\}$ is bounded in $H^\alpha_\rho(\mathbb{R}^n)$.

Suppose (ii) is not satisfied, then for any $R > 0$, (2.8) holds. Consequently by Lemma 2.6

$$\|u_k\|_{L^{p+1}} \to 0. \tag{2.23}$$

Then, noticing that

$$I'_\rho(u_k)u_k = \|u_k\|^2_{H^\alpha_\rho} - \int_{\mathbb{R}^n} f(u_k)u_k dx, \tag{2.24}$$

by (2.12) and continuous embedding we have

$$\int_{\mathbb{R}^n} f(u_k)u_k dx \leq \epsilon C\|u_k\|^2_{H^\alpha_\rho} + C_\epsilon\|u_k\|^{p+1}_{L^{p+1}}$$

where $1 < p < 2^*_\alpha - 1$. So

$$I'_\rho(u_k)u_k \geq (1-\epsilon C)\|u_k\|^2_{H^\alpha_\rho} - C_\epsilon\|u_k\|^{p+1}_{L^{p+1}}. \tag{2.25}$$

Choosing an appropriate C and using (2.21) and (2.23), we find that $u_k \to 0$ in $H^\alpha_\rho(\mathbb{R}^n)$, i.e., (i) holds. □

Our analysis require a comparison argument using a functional at infinity. When $\rho_\infty = \infty$ then the associated limiting functional is defined as

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^n} F(u(x))dx \tag{2.26}$$

and its Euler-Lagrange equations is given by

$$(-\Delta)^\alpha u + u = f(u), \quad \text{in } \mathbb{R}^n \tag{2.27}$$

This problem was studied by Felmer, Quass and Tan in [37] and they have proved the following theorem

Theorem 2.13. *[37] Under (f_1)-(f_4), I has at least one critical point with critical value c, where c is the mountain pass level defined by*

$$c = \inf_{\gamma\in\Gamma} \max_{\theta\in[0,1]} I(\gamma(\theta)) \tag{2.28}$$

and Γ is given by

$$\Gamma = \{\gamma \in C([0,1], H^\alpha(\mathbb{R}^n))/ \ \ \gamma(0) = 0, \ \ I(\gamma(1)) < 0\}.$$

When $\rho_\infty < \infty$, then we define I_{ρ_∞} as in (2.11) considering $\rho(x) = \rho_\infty$ for all $x \in \mathbb{R}^n$. Following [37] and since the functional I_{ρ_∞} is invariant under translation, we can prove that there exists at least one critical point with critical value c, given by 2.28 with I_{ρ_∞} instead of I.

Now we will prove the existence of weak solution of (2.1), using a comparison argument devised by Rabinowitz [71] and following some ideas of Felmer, Quaas and Tan [37].

Theorem 2.14. *If f satisfies (f_1)-(f_4) and ρ satisfies (ρ_1)-(ρ_2) then I_ρ has at least one critical point with critical value $c_\rho < c$.*

Proof. By definition of c_ρ in 2.18, for every sequence $\{\epsilon_k\}$, there exists a sequence of $\{u_k\}$ in $H^\alpha_\rho(\mathbb{R}^n)$ such that $\|u_k\|_\rho = 1$,

$$c_\rho \leq \max_{t\geq 0} I_\rho(tu_k) \leq c_\rho + \epsilon_k \quad \text{and} \quad \max_{t\geq 0} I_\rho(tu_k) \to c_\rho. \tag{2.29}$$

As in the proof of Lemma 2.10, associated with each u_k, there is a function $\gamma_k \in \Gamma_\rho$ such that

$$\max_{\xi\in[0,1]} I_\rho(\gamma_k(\xi)) \leq \max_{t\geq 0} I_\rho(tu_k) \leq c_\rho + \epsilon_k. \tag{2.30}$$

Now let $X = H^\alpha_\rho(\mathbb{R}^n)$, $K = [0,1]$, $K_0 = \{0,1\}$, $M = \Gamma_\rho$, $\varphi = \gamma_k$ and

$$c_1 = \max_{\gamma_k(K_0)} I_\rho = 0 < c_\rho,$$

then we can use theorem 4.3 of [62], to find a sequence $\{w_k\}$ in $H^\alpha_\rho(\mathbb{R}^n)$ and $\{\xi_k\} \subset [0,1]$ such that $I_\rho(w_k) \in (c_\rho - \epsilon_k, c_\rho + \epsilon_k)$,

$$\|w_k - \gamma_k(\xi_k)\|_{H^\alpha_\rho} \leq \epsilon_k^{1/2} \quad \text{and} \quad \|I'_\rho(w_k)\|_{(H^\alpha_\rho)'} \leq \epsilon_k^{1/2}. \tag{2.31}$$

Now, since

$$I_\rho(w_k) \to c_\rho \ \text{ in } \ \mathbb{R} \ \text{ and } \ I'_\rho(w_k) \to 0 \ \text{ in } \ (H^\alpha_\rho(\mathbb{R}^n))', \tag{2.32}$$

as in the proof of the Lemma 2.12, we show that $\{w_k\}$ is bounded in $H^\alpha_\rho(\mathbb{R}^n)$. Moreover up to a subsequence

$$w_k \rightharpoonup w \ \text{ in } \ H^\alpha_\rho(\mathbb{R}^n) \ \text{ and } \ w_k \to w \ \text{ in } \ L^{q+1}_{loc}(\mathbb{R}^n), \ \ 1 \leq q < 2^*_\alpha - 1, \tag{2.33}$$

where w is weak solution of (2.27). By Lemma 2.12, there is a sequence $\{y_k\} \subset \mathbb{R}^n$, $\beta > 0$ and $R > 0$ such that

$$\liminf_{k\to\infty} \int_{B(y_k,R)} w_k^2 dx \geq \beta. \tag{2.34}$$

If $\{y_k\}$ contains a bounded subsequence, then (2.34) guarantees that $w \neq 0$ and the results follows. If $\{y_k\}$ is an unbounded sequence, we consider the cases $\rho_\infty = \infty$ and $\rho_\infty < +\infty$ separately. First we prove that

$$c > c_\rho \tag{2.35}$$

By Theorem 2.13, I has a critical point u with critical value c. For any $y \in R^n$, we define $u_y(x) = u(x+y)$, then for any $t > 0$ we have

$$c \ = \ I(u_y) \geq I(tu_y) > I_\rho(tu_y).$$

Let $t^* > 0$ such that $t^* u_y \in \mathcal{N}_\rho$ and

$$I_\rho(t^* u_y) = \sup_{t>0} I_\rho(tu_y),$$

consequently $c > I_\rho(t^* u_y) \geq \inf_{\mathcal{N}_\rho} I_\rho(u) = c_\rho$, proving (2.35) when $\rho_\infty = \infty$. Similarly, when $\rho_\infty < \infty$ we can prove

$$c_{\rho_\infty} > c_\rho. \tag{2.36}$$

Now we may assume that, for given $R > 0$,

$$\lim_{k\to\infty} \int_{B(0,R)} |u_k|^2 dx = 0, \tag{2.37}$$

since the contrary implies that $w \neq 0$ and we finish the proof.

We analyze first the case $\rho_\infty = +\infty$. For this purpose we write

$$I_\rho(tu_k) = I(tu_k) - \frac{1}{2} \int_{\mathbb{R}^n} \int_{B^c(0,\rho(x))} \frac{|tu_k(x+z) - tu_k(x)|^2}{|z|^{n+2\alpha}} dz dx, \tag{2.38}$$

for $t \geq 0$, and we estimate the second term on the right. In first place we see that for any $\epsilon > 0$ and $\bar{t}$, there exists $R > 0$ such that

$$\int_{B^c(0,R)} \int_{B^c(0,\rho(x))} \frac{|tu_k(x+z) - tu_k(x)|^2}{|z|^{n+2\alpha}} dzdx \leq \epsilon, \tag{2.39}$$

for all $t \in [0, \bar{t}]$. In fact, by our assumption, for any $M > 0$, exists $R > 0$ such that, for $|x| > R$ we have that $\rho(x) > M$. From here, interchanging the order of integration and using the continuous embedding, we have

$$\begin{aligned}
& \int_{B^c(0,R)} \int_{B^c(0,\rho(x))} \frac{|tu_k(x+z) - tu_k(x)|^2}{|z|^{n+2\alpha}} dzdx \\
\leq & \int_{B^c(0,M)} \int_{B^c(0,R)} \frac{|tu_k(x+z) - tu_k(x)|^2}{|z|^{n+2\alpha}} dxdz \\
\leq & \int_{B^c(0,M)} \int_{\mathbb{R}^n} \frac{|tu_k(x+z) - tu_k(x)|^2}{|z|^{n+2\alpha}} dxdz \\
\leq & \frac{2\bar{t}^2 |S^{n-1}|}{\alpha M^{2\alpha}} \|u_k\|_{L^2}^2 \leq \frac{2\bar{t}^2 C |S^{n-1}|}{\alpha M^{2\alpha}} \|u_k\|_{H_\rho^\alpha}^2,
\end{aligned} \tag{2.40}$$

from were we conclude (2.39) choosing $R > 0$ large enough and recalling that $\|u_k\|_\rho = 1$. From now on we fix $R > 0$ so that (2.37) and (2.39) hold. Next we prove that

$$\lim_{k \to \infty} \int_{B(0,R)} \int_{B^c(0,\rho(x))} \frac{|tu_k(x+z) - tu_k(x)|^2}{|z|^{n+2\alpha}} dzdx = 0, \tag{2.41}$$

for all $t \in [0, \bar{t}]$. In fact, by (ρ_1) there exists $\rho_0 > 0$ such that $\rho(x) \geq \rho_0$ for all $x \in \mathbb{R}^n$, so that

$$\begin{aligned}
& \int_{B(0,R)} \int_{B^c(0,\rho(x))} \frac{|tu_k(x+z) - tu_k(x)|^2}{|z|^{n+2\alpha}} dzdx \\
\leq & \int_{B^c(0,\rho_0)} \int_{B(0,R)} \frac{|tu_k(x+z) - tu_k(x)|^2}{|z|^{n+2\alpha}} dxdz \leq \frac{2\bar{t}^2 |S^{n-1}|}{\alpha \rho_0^{2\alpha}} \|u_k\|_{L^2(B(0,R))}^2
\end{aligned} \tag{2.42}$$

and we obtain (2.41) by (2.37). Thus, by (2.38), (2.39) and (2.41) we obtain

$$I_\rho(tu_k) \geq I(tu_k) - \epsilon - \int_{B(0,R)} \int_{B^c(0,\rho(x))} \frac{|tu_k(x+z) - tu_k(x)|^2}{|z|^{n+2\alpha}} dzdx.$$

If we choose $t = t^*$ such that $I(t^* u_k) = \max_{t \geq 0} I(tu_k)$ then we see that $c_\rho \geq c - \epsilon$, from were we get a contradiction with (2.35) if we take $\epsilon > 0$ small enough.

Now we analyze the case $\rho_\infty < +\infty$. In this case we compare the functionals I_ρ and I_{ρ_∞} writting

$$I_\rho(u) = I_{\rho_\infty}(u) - \frac{1}{2} \int_{\mathbb{R}^n} \int_{B(0,\rho_\infty) \setminus B(0,\rho(x))} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dzdx. \tag{2.43}$$

By hypothesis (ρ_1), for any $\epsilon > 0$ there is $R > 0$ such that

$$0 < \rho_\infty - \rho(x) < \epsilon, \text{ whenever } |x| > R.$$

Proceeding as before, for all $t \in [0, \bar{t}]$, we obtain the estimate

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{B(0,\rho_\infty)\setminus B(0,\rho(x))} \frac{|tu_k(x+z) - tu_k(x)|^2}{|z|^{n+2\alpha}} dzdx \\ \leq \ & C(\epsilon)\|u_k\|^2_{L^2} + C\|u_k\|^2_{L^2(B(0,R))}, \end{aligned} \tag{2.44}$$

where

$$C(\epsilon) = \frac{2|S^{n-1}|\bar{t}^2}{\alpha}\left(\frac{1}{(\rho_\infty - \epsilon)^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}}\right) \quad \text{and} \tag{2.45}$$

$$C = \frac{2|S^{n-1}|\bar{t}^2}{\alpha}\left(\frac{1}{\rho_0^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}}\right). \tag{2.46}$$

Thus, we obtain

$$I_\rho(tu_k) \geq I_{\rho_\infty}(tu_k) - C(\epsilon)\|u_k\|^2_{L^2} + C\|u_k\|^2_{L^2(B(0,R))}.$$

Choosing t appropriately and ϵ small enough we conclude that

$$c_\rho > c_{\rho_\infty},$$

contradicting (2.36). To complete the proof we only need to prove that u is non-negative. Thanks to hypothesis (f_1) we see that $f(-u_-(x)) = 0$ for all $x \in \mathbb{R}^n$, so that it is enough to prove that

$$\langle u, u_- \rangle_\rho \leq -\int_{\mathbb{R}^n} u_-^2 dx. \tag{2.47}$$

Here we consider $u_- = \max\{-u, 0\}$ and $u_+ = \max\{u, 0\}$ so that $u = u_+ - u_-$. An easy computation shows that for $x, z \in \mathbb{R}^n$ we have

$$\begin{aligned} (u(x+z) - u(x))(u_-(x+z) - u_-(x)) &= -u_+(x+z)u_-(x) - u_+(x)u_-(x+z) \\ &\quad -(u_-(x+z) + u_-(x))^2 \leq 0, \end{aligned}$$

so that, by definition of the inner product $\langle \cdot, \cdot \rangle_\rho$ given in Section §2, we obtain (2.47) that proves that $u_- = 0$ a.e. in $\mathbb{R}^n$.$\square$

Chapter 3

Concentration of Positive Solutions

Let $0 < \alpha < 1$ and $n \geq 2$. In this chapter we consider the following equation

$$\begin{aligned} \epsilon^{2\alpha}(-\Delta)^{\alpha}_{\rho} u + u = u^p, \;\; \text{in} \;\; \mathbb{R}^n, \\ u \in H^{\alpha}(\mathbb{R}^n). \end{aligned} \tag{3.1}$$

where $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$, $1 < p < 2^*_{\alpha} - 1$.

We want to see the concentration behavior of weak solution of (3.1) in the sense that: For all $\delta > 0$, there exists $R > 0$ and $\epsilon_0 > 0$ such that

$$\int_{B^c(x_0, \epsilon_m R)} u_m^2(x) dx \leq \epsilon_m^n \delta \;\; \text{and} \;\; \int_{B(x_0, \epsilon_m R)} u_m^2(x) dx \geq \epsilon_m^n C, \;\; \forall \epsilon_m \leq \epsilon_0$$

We start rescaling equation (3.1), for this purpose we define $\rho_\epsilon(x) = \frac{1}{\epsilon}\rho(\epsilon x)$ and $(-\Delta)^{\alpha}_{\rho_\epsilon}$ the operator defined by (2.10) changing ρ by ρ_ϵ. We then consider the rescaled equation

$$(-\Delta)^{\alpha}_{\rho_\epsilon} v(x) + v(x) = v^p(x), \;\; \text{in} \;\;\; \mathbb{R}^n. \tag{3.2}$$

We see that given a weak solution u of (3.1) then $v_\epsilon(x) = u(\epsilon x)$ is a solution of (3.2) and reciprocally. In fact, by definition (2.10) and changing variables, we have that for every test function φ

$$\begin{aligned} & \int_{\mathbb{R}^n} ((-\Delta)^{\alpha}_{\rho_\epsilon} v_\epsilon(x)) \varphi_\epsilon(x) dx \\ = \;& \epsilon^{2\alpha} \int_{\mathbb{R}^n} \int_{B(0,\rho(\epsilon x))} \frac{[u(\epsilon x + w) - u(\epsilon x)][\varphi(\epsilon x + w) - \varphi(\epsilon x)]}{|w|^{n+2\alpha}} dw dx \\ = \;& \epsilon^{2\alpha} \int_{\mathbb{R}^n} (-\Delta)^{\alpha}_{\rho} u(\epsilon x) \varphi_\epsilon(x) dx, \end{aligned}$$

where $\varphi_\epsilon(x) = \varphi(\epsilon x)$. From here we conclude that

$$(-\Delta)^{\alpha}_{\rho_\epsilon} v_\epsilon(x) = (-\Delta)^{\alpha}_{\rho} u(\epsilon x) = (u(\epsilon x))^p - u(\epsilon x).$$

In order to study equations (3.1) and (3.2) it is convenient to use the ϵ-dependent Hilbert space $H^{\alpha}_{\rho_\epsilon}(\mathbb{R}^n)$ defined by

$$H^{\alpha}_{\rho,\epsilon}(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \;\; \int_{\mathbb{R}^n} \int_{B(0, \frac{1}{\epsilon}\rho(\epsilon x))} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dz dx < +\infty \right\},$$

with inner product $\langle\cdot,\cdot\rangle_{\rho_\epsilon}$ defined by

$$\langle u,v\rangle_{H^\alpha_{\rho,\epsilon}} = \int_{\mathbb{R}^n}\int_{B(0,\frac{1}{\epsilon}\rho(\epsilon x))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n} uvdx.$$

As a consequence of Proposition 2.3 we have

$$\|u\| \le C\|u\|_{\rho_\epsilon}, \tag{3.3}$$

where $C = C(\epsilon, n, \alpha, \rho_0) > 0$, which implies the continuous embedding $H^\alpha_{\rho_\epsilon}(\mathbb{R}^n) \hookrightarrow H^\alpha(\mathbb{R}^n)$. As a consequence of Remark 2.4 we have also that

$$\begin{aligned} H^\alpha_{\rho_\epsilon}(\mathbb{R}^n) &\hookrightarrow L^q(\mathbb{R}^n) \text{ continuously for any } q\in[2,2^*_\alpha] \quad \text{and} \\ H^\alpha_{\rho_\epsilon}(\mathbb{R}^n) &\hookrightarrow L^q_{loc}(\mathbb{R}^n) \text{ compactly for any } q\in[2,2^*_\alpha). \end{aligned}$$

We define the energy functional associated with (3.2),

$$I_{\rho_\epsilon}(u) = \frac{1}{2}\|u\|_{\rho_\epsilon} - \frac{1}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx, \tag{3.4}$$

which is well defined in $H^\alpha_{\rho_\epsilon}(\mathbb{R}^n)$, $I_{\rho_\epsilon} \in C^1(H^\alpha_{\rho_\epsilon}(\mathbb{R}^n),\mathbb{R})$, and the critical points of I_{ρ_ϵ} are the weak solutions of (3.2). We further introduce

$$\mathcal{N}_{\rho_\epsilon} = \{v\in H^\alpha_{\rho_\epsilon}(\mathbb{R}^n)\setminus\{0\}: \quad I'_{\rho_\epsilon}(v)v = 0\},$$

$$\Gamma_{\rho_\epsilon} = \{\gamma\in C([0,1],H^\alpha_{\rho_\epsilon}(\mathbb{R}^n)): \quad \gamma(0)=0, \quad I_{\rho_\epsilon}(\gamma(1))<0\}$$

and the mountain pass minimax value

$$c_{\rho_\epsilon} = \inf_{\gamma\in\Gamma_{\rho_\epsilon}}\max_{t\in[0,1]} I_{\rho_\epsilon}(\gamma(t)).$$

From Lemma 2.10 we also have

$$0 < c_{\rho_\epsilon} = \inf_{v\in\mathcal{N}_{\rho_\epsilon}} I_{\rho_\epsilon}(v) = \inf_{v\in H^\alpha_{\rho_\epsilon}(\mathbb{R}^n)\setminus\{0\}}\max_{t\ge 0} I_{\rho_\epsilon}(tv). \tag{3.5}$$

We note that, the function ρ, that describes the size of the ball of the influential region of the non-local operator, will certainly play a key role in deciding the concentration point of ground states of the equation. However, even though the minimum point of ρ seems to be the point of concentration, there is a non-local effect that also influence the concentration. Now we define the function that controls the concentration

$$\mathcal{H}(x) = -\frac{|S^{n-1}|}{2\alpha}\left(\frac{1}{\rho(x)^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}}\right) + \int_{\mathcal{C}^+(x)}\frac{dy}{|y|^{n+2\alpha}} - \int_{\mathcal{C}^-(x)}\frac{dy}{|y|^{n+2\alpha}}, \tag{3.6}$$

where we interpret the quotient $1/\rho_\infty^{2\alpha}$ as zero, in case $\rho_\infty = \infty$. The sets $\mathcal{C}^+(x)$ and $\mathcal{C}^-(x)$ are defined as follows

$$\mathcal{C}^-(x) = \{y\in\mathbb{R}^n: \quad \rho(x+y) < |y| < \rho(x)\}$$

and

$$\mathcal{C}^+(x) = \{y\in\mathbb{R}^n: \quad \rho(x) < |y| < \rho(x+y)\}.$$

Now we are in a position to state our main theorem in this section

Theorem 3.1. *Let $0 < \alpha < 1$, $n \geq 2$. Suppose that ρ satisfies (ρ_1)-(ρ_3) and $1 < p < \frac{n+2\alpha}{n-2\alpha}$. Then for each sequence $\epsilon_m \to 0$, there exists a subsequence (still called $\{\epsilon_m\}$) such that every m, there is a non-negative solution $u_m = u_{\epsilon_m}$ of (3.1) that concentrates around a global minimum point x_0 of $\mathcal{H}$, as $\epsilon_m \to 0$. In more precise terms, for every $\delta > 0$ there exists $R > 0$ and $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$ we have*

$$\int_{B^c(x_0,\epsilon_m R)} u_m^2(x)dx \leq \epsilon_m^n \delta, \quad \text{and} \quad \int_{B(x_0,\epsilon_m R)} u_m^2(x)dx \geq \epsilon_m^n C, \qquad \forall \epsilon_m \leq \epsilon_0,$$

with C a constant independent of δ and m.

The proof of this theorem again uses a comparison argument in order to obtain the concentration. For this purpose we consider the functional

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} u^{p+1}dx,$$

whose critical point are the solutions of

$$(-\triangle)^\alpha u + u = u^p, \text{in } \mathbb{R}^n. \tag{3.7}$$

We also consider the critical value

$$c^* = \inf_{\mathcal{N}} I$$

where $\mathcal{N} = \{u \in H^\alpha(\mathbb{R}^n) \setminus \{0\} : \quad I'(u)u = 0\}$ is the Nehari manifold. Moreover c^* can be characterized as follow

$$c^* = c = \inf_{u \in H^\alpha(\mathbb{R}^n)\setminus\{0\}} \max_{\xi \geq 0} I(\xi u)$$

where

$$c = \inf_{\gamma \in \Gamma} \sup_{\xi \in [0,1]} I(\gamma(\xi)),$$

and

$$\Gamma = \{\gamma \in C([0,1], H^\alpha(\mathbb{R}^n)) : \quad \gamma(0) = 0, \quad I(\gamma(1)) < 0\}$$

3.1 Asymptotic Values Of The Functional When $\epsilon \to 0$

In this section we make a preliminary analysis of the asymptotic behavior of the functional associated to equation (3.1) when $\epsilon \to 0$. In this and next section we consider the power function $f(s) = s^p$ to prove Theorem 3.1. For simplicity, we prefer to treat only the power function, but all the arguments can be adapted to deal with a general f satisfying the hypotheses (f_1)-(f_4). We start with some basic properties of the function $\mathcal{H}$.

Lemma 3.2. *Assuming ρ satisfies $(\rho_1) - (\rho_3)$, the function $\mathcal{H}$ is continuous and*

$$\lim_{|x| \to \infty} \mathcal{H}(x) = 0. \tag{3.8}$$

Moreover, there exists $x_0 \in \mathbb{R}^n$ such that

$$\inf_{x \in \mathbb{R}^n} \mathcal{H}(x) = \mathcal{H}(x_0) < 0. \tag{3.9}$$

Proof. Hypothesis (ρ_3) implies that the function $\mathcal{H}$ is continuous in $\mathbb{R}^n$. By definition of the sets $\mathcal{C}^+(x)$ and $\mathcal{C}^-(x)$, when $\rho_\infty < \infty$, we see that

$$\lim_{|x|\to\infty} \operatorname{meas}(\mathcal{C}^+(x)) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \operatorname{meas}(\mathcal{C}^-(x)) = 0.$$

In case $\rho_\infty = \infty$, we easily see that, for every $M > 0$ we have that $\mathcal{C}^+(x) \subset B^c(0, M)$ if $|x|$ is large enough. For a similar statement with $\mathcal{C}^-(x)$, we use hypothesis (ρ_2) to get that, for $|x|$ large enough,

$$|x + y| \geq \frac{1-a}{2}|x|, \quad \text{for all} \quad y \in \mathcal{C}^-(x).$$

This implies that, for every $M > 0$ we have that $\mathcal{C}^-(x) \subset B^c(0, M)$ if $|x|$ is large enough. Thus we conclude that (3.8) holds.

Next we see that

$$\begin{aligned}
\mathcal{H}(x) &= -\frac{|S^{n-1}|}{2\alpha}\left(\frac{1}{\rho(x)^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}}\right) + \frac{1}{2}\int_{\mathcal{C}^+(x)} \frac{dy}{|y|^{n+2\alpha}} - \frac{1}{2}\int_{\mathcal{C}^-(x)} \frac{dy}{|y|^{n+2\alpha}} \\
&\leq -\frac{|S^{n-1}|}{2\alpha}\left(\frac{1}{\rho(x)^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}}\right) + \int_{\mathcal{C}^+(x)} \frac{dy}{|y|^{n+2\alpha}} < 0,
\end{aligned}$$

where the last inequality follows from the fact that $\mathcal{C}^+(x) \subset B^c(0, \rho(x))$ and $\mathcal{C}^+(x)$ is a bounded set. From here and (3.8) the existence of a global minimum is a consequence of the continuity of $\mathcal{H}$. $\square$

Along this section we will consider a sequence of functions $\{w_m\} \subset H^\alpha(\mathbb{R}^n)$ such that $\|w_m - w\|_{L^2(\mathbb{R}^2)} \to 0$, where $w \in H^\alpha(\mathbb{R}^n)$. We will also consider sequences $\{z_m\} \subset \mathbb{R}^n$ and $\{\epsilon_m\} \subset \mathbb{R}$ and assume that $\epsilon_m \to 0$ as $m \to \infty$. We define $\overline{\rho}_m$ as

$$\overline{\rho}_m(x) = \frac{1}{\epsilon_m}\rho(\epsilon_m x + \epsilon_m z_m), \tag{3.10}$$

and we consider the functional I_ρ defined in (2.11), with $F(t) = |t|^{p+1}/(p+1)$ and for different scope functions ρ, in particular for ρ_∞/ϵ_m constant and $\bar{\rho}_m$ defined in (3.10). In case $\epsilon_m z_m \to \bar{x}$ we will also consider the functional with $\rho(\bar{x})/\epsilon_m$. Thus, in this section we will be considering the functionals

$$I_{\frac{\rho_\infty}{\epsilon_m}}, I_{\frac{\rho(\bar{x})}{\epsilon_m}} \quad \text{and} \quad I_{\bar{\rho}_m}.$$

We will also consider the functional I in $\mathbb{R}^n$ (with $\rho \equiv \infty$) defined in (2.26). The following theorem is a key to understand the concentration phenomenon for equation (3.1).

Theorem 3.3. *Under hypotheses $(\rho_1) - (\rho_3)$, we assume as above that $w_m, w \in H^\alpha(\mathbb{R}^n)$ are such that $\|w_m - w\|_{L^2(\mathbb{R}^2)} \to 0$ and $\epsilon_m \to 0$, as $m \to \infty$. Then we have:*

i) If $\epsilon_m z_m \to \bar{x}$ then

$$\lim_{m\to\infty} \frac{I_{\overline{\rho}_m}(w_m) - I_{\frac{\rho_\infty}{\epsilon_m}}(w_m)}{\epsilon_m^{2\alpha}} = \|w\|_{L^2}^2 \mathcal{H}(\overline{x}) \qquad \textit{and} \tag{3.11}$$

ii) If $|\epsilon_m| z_m \to \infty$ then

$$\lim_{m\to\infty} \frac{I_{\overline{\rho}_m}(w_m) - I_{\frac{\rho_\infty}{\epsilon_m}}(w_m)}{\epsilon_m^{2\alpha}} = 0. \tag{3.12}$$

In various stages of the proof of this theorem it will be convenient to replace the function w_m by the truncated limit $w_R(x) = w(x)\chi_{B(0,R)}(x)$, where $R > 0$. We clearly have

Lemma 3.4. *For all $\delta > 0$ there exist $m(\delta) > 0$ and $R(\delta) > 0$ such that*

$$\int_{\mathbb{R}^n} |w_m(x) - w_R(x)|^2 dx < \delta, \quad \textit{whenever} \;\; m > m(\delta), \;\; R > R(\delta).$$

Proof. Simply consider

$$|w_m(x) - w_R(x)|^2 \leq 2|w_m(x) - w(x)|^2 + 2|w(x) - w_R(x)|^2. \quad \square$$

In order to prove Theorem 3.3 we first prove several lemmas under the hypotheses (i), that is

$$\epsilon_m z_m \to \overline{x} \;\text{ as }\; m \to \infty.$$

We analyze the cases $\rho_\infty = +\infty$ and $\rho_\infty < +\infty$ separately. It will be convenient to decompose the problem considering

$$I_{\overline{\rho}_m}(w_m) - I_{\frac{\rho_\infty}{\epsilon_m}}(w_m) = I_{\overline{\rho}_m}(w_m) - I_{\frac{\rho(\overline{x})}{\epsilon_m}}(w_m) - \left(I_{\frac{\rho_\infty}{\epsilon_m}}(w_m) - I_{\frac{\rho(\overline{x})}{\epsilon_m}}(w_m) \right). \tag{3.13}$$

For the second term of the right hand side we have

Lemma 3.5. *Under assumption of Theorem 3.3 and assuming (i)*

$$\lim_{m\to\infty} \frac{I_{\frac{\rho_\infty}{\epsilon_m}}(w_m) - I_{\frac{\rho(\overline{x})}{\epsilon_m}}(w_m)}{\epsilon_m^{2\alpha}} = \frac{|S^{n-1}|}{2\alpha}\|w\|_{L^2}^2 \left(\frac{1}{\rho(\overline{x})^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}} \right). \tag{3.14}$$

In case $\rho_\infty = +\infty$, here we write $I_{\frac{\rho_\infty}{\epsilon_m}} = I$ and $1/\rho_\infty^{2\alpha} = 0$.

Proof. We first consider the case $\rho_\infty = +\infty$. We have

$$\begin{aligned}
& I(w_m) - I_{\frac{\rho(\overline{x})}{\epsilon_m}}(w_m) \\
= \;& \frac{1}{2}\int_{\mathbb{R}^n}\int_{B^c(0,\frac{\rho(\overline{x})}{\epsilon_m})} \frac{|w_m(x+z) - w_m(x)|^2}{|z|^{n+2\alpha}} dz dx \\
= \;& \frac{1}{2}\int_{B^c(0,\frac{\rho(\overline{x})}{\epsilon_m})}\int_{\mathbb{R}^n} \frac{w_m^2(x+z) - 2w_m(x+z)w_m(z) + w_m^2(x)}{|z|^{n+2\alpha}} dx dz \\
= \;& \frac{|S^{n-1}|}{2\alpha}\|w_m\|_{L^2}^2 \frac{\epsilon_m^{2\alpha}}{\rho(\overline{x})^{2\alpha}} - \int_{B^c(0,\frac{\rho(\overline{x})}{\epsilon_m})}\int_{\mathbb{R}^n} w_m(x+z)w_m(x)dx \frac{dz}{|z|^{n+2\alpha}}.
\end{aligned}$$

If we denote by E_m the second term above and we consider $R > 0$, we have

$$\begin{aligned}
& |E_m - \int_{B^c(0,\frac{\rho(\overline{x})}{\epsilon_m})}\int_{\mathbb{R}^n} w_R(x+z)w_R(x)dx \frac{dz}{|z|^{n+2\alpha}}| \\
\leq \;& (\|w_m\|_{L^2} + \|w_R\|_{L^2})\|w_m - w_R\|_{L^2} \int_{B^c(0,\frac{\rho(\overline{x})}{\epsilon_m})} \frac{dz}{|z|^{n+2\alpha}} \\
= \;& \frac{\epsilon_m^{2\alpha}}{\rho(\bar{x})^{2\alpha}}(\|w_m\|_{L^2} + \|w_R\|_{L^2})\|w_m - w_R\|_{L^2}.
\end{aligned}$$

From here, using Lemma 3.4 and the fact that $w_R(\cdot)$ y $w_R(\cdot+z)$ have disjoint supports if $|z|>2R$, we obtain that $\lim_{m\to\infty} E_m = 0$, which implies the result.

In case $\rho_\infty < +\infty$ we proceed similarly, noticing that we have to replace the integral over $B^c(0,\frac{\rho(\overline{x})}{\epsilon_m})$ by an integral over $B(0,\frac{\rho_\infty}{\epsilon_m})\setminus B(0,\frac{\rho(\overline{x})}{\epsilon_m})$ and compute accordingly. $\square$

Now we consider the first term in (3.13), for which it is convenient to write

$$\begin{aligned} \mathcal{I} &= \frac{I_{\bar\rho_m}(w_m) - I_{\rho(\overline{x})/\epsilon_m}(w_m)}{\epsilon_m^{2\alpha}} \qquad (3.15)\\ &= \frac{1}{2\epsilon_m^{2\alpha}}\int_{\mathbb{R}^n}\int_{A(\frac{\rho(\overline{x})}{\epsilon_m},\bar\rho_m(x))}\frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}}dzdx \\ &\quad -\frac{1}{2\epsilon_m^{2\alpha}}\int_{\mathbb{R}^n}\int_{A(\bar\rho_m(x),\frac{\rho(\overline{x})}{\epsilon_m})}\frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}}dzdx \\ &= \mathcal{I}_1+\mathcal{I}_2. \end{aligned}$$

Here, and in what follows, we denote by $A(a,b)$ the annulus $B(0,b)\setminus B(0,a)$ and we notice that $A(a,b)=\emptyset$ when $a\ge b$.

We start our analysis with $\mathcal{I}_1$ and for this purpose, we first consider the second and third term in the expansion of the quadratic expression $|w_m(x+z)-w_m(x)|^2$.

Lemma 3.6. *Under assumption of Theorem 3.3 and assuming (i) we have*

$$\lim_{m\to\infty}\frac{1}{\epsilon_m^{2\alpha}}\int_{\mathbb{R}^n}\int_{A(\frac{\rho(\overline{x})}{\epsilon_m},\bar\rho_m(x))}\frac{w_m(x+z)w_m(x)}{|z|^{n+2\alpha}}dzdx = 0$$

and

$$\lim_{m\to\infty}\frac{1}{\epsilon_m^{2\alpha}}\int_{\mathbb{R}^n}\int_{A(\frac{\rho(\overline{x})}{\epsilon_m},\bar\rho_m(x))}\frac{w_m^2(x)}{|z|^{n+2\alpha}}dzdx = 0.$$

Proof. The first limit is obtained using the arguments given in the proof of Lemma 3.5. To study the second limit we see that

$$\begin{aligned} &\frac{1}{\epsilon_m^{2\alpha}}\int_{\mathbb{R}^n}\int_{A(\frac{\rho(\overline{x})}{\epsilon_m},\bar\rho_m(x))}\frac{w_m^2(x)}{|z|^{n+2\alpha}}dzdx \\ =\ & \frac{|S^{n-1}|}{2\alpha}\int_{\mathbb{R}^n} w_m^2(x)\left(\frac{1}{\rho(\overline{x})^{2\alpha}}-\frac{1}{\rho(\epsilon_m x+\epsilon_m z_m)^{2\alpha}}\right)_+dx \\ \le\ & \frac{|S^{n-1}|}{2\alpha}\int_{\mathbb{R}^n} w_R^2(x)\left(\frac{1}{\rho(\overline{x})^{2\alpha}}-\frac{1}{\rho(\epsilon_m x+\epsilon_m z_m)^{2\alpha}}\right)_+dx \\ &+\frac{|S^{n-1}|}{2\alpha\rho_0^{2\alpha}}(\|w_m\|_{L^2}+\|w_R\|_{L^2})\|w_m-w_R\|_{L^2}, \end{aligned}$$

where $R>0$. By the continuity of ρ and the fact that $\epsilon_m z_m\to\bar x$ as $m\to\infty$, we see that

$$\lim_{m\to\infty}\left(\frac{1}{\rho(\overline{x})^{2\alpha}}-\frac{1}{\rho(\epsilon_m x+\epsilon_m z_m)^{2\alpha}}\right)_+ = 0,$$

uniformly in $B(0,R)$. From here, Lemma 3.4 and the inequality above, the result follows. $\square$

Next we consider the first term in the expansion of $|w_m(x+z)-w_m(x)|^2$.

Lemma 3.7. *Under assumption of Theorem 3.3 and assuming (i)*

$$\lim_{m\to\infty} \frac{1}{\epsilon_m^{2\alpha}} \int_{\mathbb{R}^n} \int_{A(\frac{\rho(\overline{x})}{\epsilon_m},\bar{\rho}_m(x))} \frac{w_m(x+z)^2}{|z|^{n+2\alpha}} dzdx = \|w\|_{L^2} \int_{\mathcal{C}^+(\bar{x})} \frac{dz}{|z|^{n+2\alpha}}.$$

Proof. To start we consider

$$\frac{1}{\epsilon_m^{2\alpha}} \int_{\mathbb{R}^n} \int_{A(\frac{\rho(\overline{x})}{\epsilon_m},\bar{\rho}_m(x))} \frac{w_m(x+z)^2}{|z|^{n+2\alpha}} dzdx = E_m^1 + E_m^2,$$

where

$$E_m^1 = \frac{1}{\epsilon_m^{2\alpha}} \int_{\mathbb{R}^n} \int_{A(\frac{\rho(\overline{x})}{\epsilon_m},\bar{\rho}_m(x))} \frac{w_R(x+z)^2}{|z|^{n+2\alpha}} dzdx, \tag{3.16}$$

and E_m^2 is the error term. For E_m^2 we have

$$\begin{aligned} |E_m^2| &\leq \frac{1}{\epsilon_m^{2\alpha}} \int_{\mathbb{R}^n} \int_{B^c(0,\frac{\rho(\overline{x})}{\epsilon_m}))} \frac{|w_m(x+z)^2 - w_R(x+z)^2|}{|z|^{n+2\alpha}} dzdx \\ &= \frac{1}{\epsilon_m^{2\alpha}} \int_{B^c(0,\frac{\rho(\overline{x})}{\epsilon_m}))} \int_{\mathbb{R}^n} |w_m(x+z)^2 - w_R(x+z)^2| dx \frac{dz}{|z|^{n+2\alpha}} \\ &\leq \frac{|S^{n-1}|}{2\alpha\rho(\bar{x})^{2\alpha}} (\|w_m\|_{L^2} + \|w_R\|_{L^2}) \|w_m - w_R\|_{L^2}. \end{aligned} \tag{3.17}$$

Next we consider E_m^1 and we observe that

$$\begin{aligned} E_m^1 &= \frac{1}{\epsilon_m^{2\alpha}} \int_{\mathbb{R}^n} \int_{B(-x,R)\cap A(\frac{\rho(\overline{x})}{\epsilon_m},\bar{\rho}_m(x))} \frac{w(x+z)^2}{|z|^{n+2\alpha}} dzdx \\ &= \frac{1}{\epsilon_m^{2\alpha}} \int_{\Omega_m^+} \int_{B(-x,R)\cap A(\frac{\rho(\overline{x})}{\epsilon_m},\bar{\rho}_m(x))} \frac{w(x+z)^2}{|z|^{n+2\alpha}} dzdx, \end{aligned} \tag{3.18}$$

where

$$\Omega_m^+ = \{x \in \mathbb{R}^n : \ \frac{\rho(\overline{x})}{\epsilon_m} - R < |x| < \frac{\rho(\epsilon_m x + \epsilon_m z_m)}{\epsilon_m} + R\}. \tag{3.19}$$

On the other hand we see that for any (x, z) such that $|x + z| < R$ we have

$$\frac{1}{(|x|+R)^{n+2\alpha}} \leq \frac{1}{|z|^{n+2\alpha}} \leq \frac{1}{(|x|-R)^{n+2\alpha}}. \tag{3.20}$$

Therefore

$$\begin{aligned} E_m^1 &\leq \frac{1}{\epsilon_m^{2\alpha}} \int_{\Omega_m^+} \frac{1}{(|x|-R)^{n+2\alpha}} \int_{B(-x,R)\cap A(\frac{\rho(\overline{x})}{\epsilon_m},\bar{\rho}_m(x))} w(x+z)^2 dzdx \\ &\leq \frac{\|w_R\|_{L^2}}{\epsilon_m^{2\alpha}} \left(\int_{M_m} \frac{dx}{(|x|-R)^{n+2\alpha}} + \int_{N_m} \frac{dx}{(|x|-R)^{n+2\alpha}}, \right) \end{aligned} \tag{3.21}$$

where the sets M_m and N_m are defined as follows

$$\begin{aligned} M_m &= \{x \in \Omega_m^+ : \ B(-x,R) \subset A(\frac{\rho(\overline{x})}{\epsilon_m}, \bar{\rho}_m(x))\} \\ &= \{x \in \mathbb{R}^n : \ \frac{\rho(\overline{x})}{\epsilon_m} + R < |x| < \bar{\rho}_m(x) - R\} \end{aligned}$$

and

$$\begin{aligned} N_m &= \{x \in \Omega_m^+ \setminus M_m : \ B(-x,R) \cap A(\frac{\rho(\overline{x})}{\epsilon_m}, \bar{\rho}_m(x)) \neq \emptyset\} \\ &= \{x \in \mathbb{R}^n : \ \frac{\rho(\overline{x})}{\epsilon_m} - R < |x| < \frac{\rho(\overline{x})}{\epsilon_m} + R\} \cup \\ & \quad \{x \in \mathbb{R}^n : \ \bar{\rho}_m(x) - R < |x| < \bar{\rho}_m(x) + R\} = N_m^1 \cup N_m^2. \end{aligned}$$

Similarly, from (3.18), (3.19) and (3.20) we find that

$$\begin{aligned} E_m^1 &\geq \frac{1}{\epsilon_m^{2\alpha}} \int_{\Omega_m^+} \frac{1}{(|x|+R)^{n+2\alpha}} \int_{B(-x,R)\cap A(\frac{\rho(\overline{x})}{\epsilon_m}, \bar{\rho}_m(x))} w(x+z)^2 dz dx \\ &\geq \frac{\|w_R\|_{L^2}}{\epsilon_m^{2\alpha}} \int_{M_m} \frac{dx}{(|x|+R)^{n+2\alpha}}. \end{aligned} \tag{3.22}$$

In order to complete the analysis of E_m^1 we just need to look at the limit of the integrals. We recall that, by hypothesis (ρ_3), the set defined by the equation

$$\rho(y+\bar{x}) = |y|,$$

is an $(n-1)$-dimensional surface and that we are assuming that $\lim_{m\to\infty} \epsilon_m z_m = \bar{x}$. So we have

$$\begin{aligned} & \lim_{m\to\infty} \frac{1}{\epsilon_m^{2\alpha}} \int_{N_m^1} \frac{dx}{(|x|-R)^{n+2\alpha}} \\ = & \lim_{m\to\infty} \int_{\{|\rho(y+\epsilon_m z_m)-|y||<\epsilon_m R\}} \frac{dy}{(|y|-\epsilon_m R)^{n+2\alpha}} = 0. \end{aligned}$$

Using similar arguments we obtain

$$\lim_{m\to\infty} \frac{1}{\epsilon_m^{2\alpha}} \int_{N_m^2} \frac{dx}{(|x|-R)^{n+2\alpha}} = 0$$

and

$$\lim_{m\to\infty} \frac{1}{\epsilon_m^{2\alpha}} \int_{M_m} \frac{dx}{(|x|\pm R)^{n+2\alpha}} = \int_{\mathcal{C}^+} \frac{dy}{(|y|)^{n+2\alpha}},$$

completing the proof of the lemma.□

Using Lemmas 3.6 and 3.7 we conclude that

$$\begin{aligned} \lim_{m\to 0} \mathcal{I}_1 &= \lim_{m\to 0} \frac{1}{2\epsilon_m^{2\alpha}} \int_{\mathbb{R}^n} \int_{A(\frac{\rho(\overline{x})}{\epsilon_m}, \bar{\rho}_m(x))} \frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}} dz dx \\ &= \frac{\|w\|_{L^2}}{2} \int_{\mathcal{C}^+(\bar{x})} \frac{dz}{|z|^{n+2\alpha}}. \end{aligned} \tag{3.23}$$

In a complete analogous way we can prove that

$$\begin{aligned} \lim_{m\to 0} \mathcal{I}_2 &= \lim_{m\to 0} \frac{1}{2\epsilon_m^{2\alpha}} \int_{\mathbb{R}^n} \int_{A(\bar{\rho}_m(x), \frac{\rho(\overline{x})}{\epsilon_m})} \frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}} dz dx \\ &= -\frac{\|w\|_{L^2}}{2} \int_{\mathcal{C}^-(\bar{x})} \frac{dz}{|z|^{n+2\alpha}}. \end{aligned} \tag{3.24}$$

Proof of Theorem 3.3. The proof of (i) is a consequence of (3.13), (3.15), (3.23), (3.24) and Lemma 3.5. Now we consider (ii) in case $\rho_\infty = \infty$. We have

$$\begin{aligned}\frac{|I(w_m) - I_{\bar\rho_m}(w_m)|}{\epsilon_m^{2\alpha}} &= \frac{1}{2\epsilon_m^{2\alpha}}\int_{B(-z_m,\frac{R}{\epsilon_m})}\int_{B^c(0,\bar\rho_m(x))}\frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}}dzdx \\ &\quad + \frac{1}{2\epsilon_m^{2\alpha}}\int_{B^c(-z_m,\frac{R}{\epsilon_m})}\int_{B^c(0,\bar\rho_m(x))}\frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}}dzdx \\ &= E_m^1 + E_m^2,\end{aligned}$$

where $R>0$. By hypothesis (ρ_1) and since $\rho_\infty=\infty$, we have that for any $M>0$ there is $R>0$ and m_0 large enough such that

$$\rho(\epsilon_m x + \epsilon_m z_m) \geq \rho_0 \quad \text{if} \quad |x+z_m| \leq \frac{R}{\epsilon_m}$$

and

$$\rho(\epsilon_m x + \epsilon_m z_m) \geq M \quad \text{if} \quad |x+z_m| > \frac{R}{\epsilon_m}, \quad \text{for all} \quad m \geq m_0.$$

Consequently

$$\begin{aligned} E_m^1 &\leq \frac{1}{\epsilon_m^{2\alpha}}\int_{B(-z_m,\frac{R}{\epsilon_m})}\int_{B^c(0,\frac{\rho_0}{\epsilon_m})}\frac{w_m^2(x+z)+w_m^2(x)}{|z|^{n+2\alpha}}dzdx \\ &\leq \frac{|S^{n-1}|}{\alpha\rho_0^{2\alpha}}(\|w_m\|_{L^2}+\|w_R\|_{L^2})\|w_m-w_R\|_{L^2} \\ &\quad + \frac{1}{\epsilon_m^{2\alpha}}\int_{B(-z_m,\frac{R}{\epsilon_m})}\int_{B^c(0,\frac{\rho_0}{\epsilon_m})}\frac{w_R^2(x+z)}{|z|^{n+2\alpha}}dzdx \\ &\quad + \frac{|S^{n-1}|}{\alpha\rho_0^{2\alpha}}\|w_R(.-z_m)\|^2_{L^2(B(0,\frac{R}{\epsilon_m}))}. \end{aligned} \tag{3.25}$$

We observe that if $|x+z|<R$ and $|x+z_m|<R/\epsilon_m$ then $|z|>|z_m|/2$, where we may need to make m larger. Then we further look at the integral above

$$\begin{aligned}\int_{B(-z_m,\frac{R}{\epsilon_m})}\int_{B^c(0,\frac{\rho_0}{\epsilon_m})}\frac{w_R^2(x+z)}{|z|^{n+2\alpha}}dzdx &= \int_{B(-z_m,\frac{R}{\epsilon_m})}\int_{B(-x,R)}\frac{w^2(x+z)}{|z|^{n+2\alpha}}dzdx \\ &\leq \epsilon_m^{2\alpha}2^{n+2\alpha}R^n\|w\|^2\frac{|S^{n-1}|}{n}\frac{1}{|\epsilon_m z_m|^{n+2\alpha}}. \end{aligned} \tag{3.26}$$

On the other hand

$$\begin{aligned} E_m^2 &\leq \frac{1}{2\epsilon_m^{2\alpha}}\int_{B^c(-z_m,\frac{R}{\epsilon_m})}\int_{B^c(0,\frac{M}{\epsilon_m})}\frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}}dzdx \\ &\leq \frac{|S^{n-1}|}{2\alpha M^{2\alpha}}\|w_m\|^2_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{3.27}$$

From Lemma 3.4, (3.25)-(3.27) we can argue that (3.11) holds.

In the case $\rho_\infty<\infty$, by hypothesis (ρ_1) we have that, for any $\delta>0$ there is $R>0$ and m large enough such that

$$\rho(\epsilon_m x + \epsilon_m z_m) \geq \rho_\infty - \delta \quad \text{if} \quad |x+z_m| > \frac{R}{\epsilon_m}.$$

Consequently

$$\frac{|I_{\frac{\rho_\infty}{\epsilon_m}}(w_m)-I_{\bar\rho_m}(w_m)|}{\epsilon_m^{2\alpha}} \le \frac{1}{2\epsilon_m^{2\alpha}}\int_{B(-z_m,\frac{R}{\epsilon_m})}\int_{B^c(0,\frac{\rho_0}{\epsilon_m})}\frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}}dzdx + E_m^3, \tag{3.28}$$

where

$$\begin{aligned} E_m^3 &= \frac{1}{2\epsilon_m^{2\alpha}}\int_{B^c(-z_m,\frac{R}{\epsilon_m})}\int_{B(0,\frac{\rho_\infty}{\epsilon_m})\setminus B(0,\frac{\rho_\infty-\delta}{\epsilon_m})}\frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}}dzdx\\ &\le \frac{|S^{n-1}|}{2\alpha}\|w_m\|^2_{L^2(\mathbb{R}^n)}\left(\frac{1}{(\rho_\infty-\delta)^{2\alpha}}-\frac{1}{\rho_\infty^{2\alpha}}\right). \end{aligned} \tag{3.29}$$

The integral in (3.28) is estimated exactly as E_m^1. From here and (3.28) we can argue that (3.11) holds, completing the proof.□

Before leaving this section we prove a lemma that will be useful later. We recall that, by Lemma 2.9, for any $w_m \in H^\alpha_{\rho_m}(\mathbb{R}^n)$, there is a unique $t_m = t(w_m) > 0$ such that $t_m w_m \in \mathcal{N}_{\rho_m}$ and

$$I_{\rho_m}(t_m w_m) = \max_{t\ge 0} I_{\rho_m}(t w_m). \tag{3.30}$$

Let $\mathcal{N}$ the Nehari manifold associated to the limit problem, that is

$$\mathcal{N} = \{u \in H^\alpha(\mathbb{R}^n)\setminus\{0\} / \ I'(u)u = 0\}.$$

Lemma 3.8. *Assume $w_m \to w$ in $H^\alpha(\mathbb{R}^n)$ and $w \in \mathcal{N}$, then*

$$\lim_{m\to\infty} t_m = 1.$$

Proof. In fact, by definition of t_m we have

$$t_m^{1-p}\|u_m\|^2 = \int_{\mathbb{R}^n}|u_m|^{p+1}dx. \tag{3.31}$$

Since $w_m \to w \in \mathcal{N}$ we have that $\|w_m\|^2 \to \|w\|^2$, $\int_{\mathbb{R}^n} u_m^{p+1}dx \to \int_{\mathbb{R}^n} u^{p+1}dx$ and w is non-zero. Thus t_m converges to $\bar t$ and $\bar t = 1$. □

3.2 Concentration Behaviour

In this section we complete our study on the concentration behavior for ground states of equation (3.1) and we prove Theorem 3.1. Now we start the proof of Theorem 3.1 with some preliminary lemmas.

Lemma 3.9. *Suppose (ρ_1) holds. Then*

$$\lim_{\epsilon\to 0^+} c_{\rho_\epsilon} = c. \tag{3.32}$$

Proof. Since we obviously have

$$\int_{\mathbb{R}^n}\int_{B(0,\rho_\epsilon(x))}\frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}dzdx \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}dzdx,$$

for all $u\in H^\alpha_{\rho_\epsilon}(\mathbb{R}^n)$, then we have $I_{\rho_\epsilon}(u)\le I(u)$ and therefore

$$\limsup_{\epsilon\to 0^+} c_{\rho_\epsilon}\le c. \tag{3.33}$$

Now consider $v_\epsilon\in H^\alpha_{\rho_\epsilon}$ a solution of equation (3.2) with critical value c_{ρ_ϵ}, then $c_{\rho_\epsilon}=I_{\rho_\epsilon}(u_\epsilon)=\max_{t\ge 0}I_{\rho_\epsilon}(tu_\epsilon)$ and $I'_{\rho_\epsilon}(v_\epsilon)v_\epsilon=0$, so that

$$\int_{\mathbb{R}^n}\int_{B(0,\rho_\epsilon(x))}\frac{|v_\epsilon(x+z)-v_\epsilon(x)|^2}{|z|^{n+2\alpha}}dzdx+\int_{\mathbb{R}^n}v_\epsilon^2dx=\int_{\mathbb{R}^n}v_\epsilon^{p+1}dx \tag{3.34}$$

and then

$$c_{\rho_\epsilon}=\frac{p-1}{2(p+1)}\left(\int_{\mathbb{R}^n}\int_{B(0,\rho_\epsilon(x))}\frac{|v_\epsilon(x+z)-v_\epsilon(x)|^2}{|z|^{n+2\alpha}}dzdx+\int_{\mathbb{R}^n}v_\epsilon^2dx\right). \tag{3.35}$$

From here and using that c_{ρ_ϵ} by (3.33), we see that $\|v_\epsilon\|_{\rho_\epsilon}$ is bounded when $\epsilon\to 0^+$. Next we see that for any $t>0$ we have

$$c_{\rho_\epsilon} \;\ge\; I(tv_\epsilon)-\frac{1}{2}\int_{\mathbb{R}^n}\int_{B^c(0,\rho_\epsilon(x))}\frac{t^2|v_\epsilon(x+z)-v_\epsilon(x)|^2}{|z|^{n+2\alpha}}dzdx.$$

Using hypothesis (ρ_1) and estimating the second term in the right side as in (2.40) and (2.42), we find

$$c_{\rho_\epsilon}\ge I(tv_\epsilon)-\frac{Ct^2|S^{n-1}|\|v_\epsilon\|^2_{H^\alpha_{\rho_\epsilon}}}{\alpha\rho_0^{2\alpha}}\epsilon^{2\alpha}.$$

Now choosing $t=t^*_\epsilon>0$, such that $I(t^*_\epsilon v_\epsilon)=\max_{t\ge 0}I(tv_\epsilon)$ we find

$$c_{\rho_\epsilon}\ge c-\frac{C(t^*_\epsilon)^2|S^{n-1}|\|v_\epsilon\|^2_{H^\alpha_{\rho_\epsilon}}}{\alpha\rho_0^{2\alpha}}\epsilon^{2\alpha},$$

from where $\liminf_{\epsilon\to 0^+}c_{\rho_\epsilon}\ge c$. Combining with (3.33) we get (3.32). □

Lemma 3.10. *If v_ϵ is s family solutions of (3.2) with critical value c_{ρ_ϵ}, then there exists a family $\{y_\epsilon\}$ and positive constants R and β such that*

$$\liminf_{\epsilon\to 0^+}\int_{B(y_\epsilon,R)}v_\epsilon^2(x)dx\ge\beta>0. \tag{3.36}$$

Proof. If not, there exists a sequence $v_k=v_{\epsilon_k}$ such that

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^n}\int_{B(y,R)}v_k^2(x)dx=0,$$

the by Lemma 2.6, we have $v_k\to 0$ in $L^q(\mathbb{R}^n)$ for any $2<q<2^*_\alpha$. However, this is impossible because by (3.34), (3.35) and Lemma 3.9

$$\frac{p-1}{2(p+1)}\int_{\mathbb{R}^n}v_\epsilon^{p+1}(x)dx=c_{\rho_\epsilon}\to c,\quad\text{as }\epsilon\to 0.\quad\square$$

Now let
$$w_\epsilon(x) = v_\epsilon(x + y_\epsilon) = u_\epsilon(\epsilon x + \epsilon y_\epsilon), \tag{3.37}$$
then by (3.39),
$$\liminf_{\epsilon\to 0^+} \int_{B(0,R)} w_\epsilon^2(x)dx \geq \beta > 0. \tag{3.38}$$
To continue, we consider the rescaled scope function $\overline{\rho}_\epsilon$, as defined in (3.10),
$$\bar{\rho}_\epsilon(x) = \frac{1}{\epsilon}\rho(\epsilon x + \epsilon y_\epsilon)$$
and then w_ϵ satisfies the equation
$$(-\Delta)^\alpha_{\bar{\rho}_\epsilon} w_\epsilon(x) + w_\epsilon(x) = w_\epsilon^p(x), \ \text{ in } \ \mathbb{R}^n. \tag{3.39}$$

Now we prove the convergence of w_ϵ as $\epsilon \to 0$.

Lemma 3.11. *For every sequence $\{\epsilon_m\}$ there is a subsequence, we keep calling the same, so that $w_{\epsilon_m} = w_m \to w$ in $H^\alpha(\mathbb{R}^n)$, when $m \to \infty$, where w is a solution of (2.27).*

Proof. From (3.35) we see that $\{w_\epsilon\}$ is bounded, and then, for every sequence $\{\epsilon_m\}$ there is a subsequence, we keep calling the same, so that $w_{\epsilon_m} = w_m \rightharpoonup w$, which satisfies equation (2.27). To prove the convergence of this sequence we use its weak convergence together with (3.35) and Lemma 3.9 to get
$$\begin{aligned}
\|w\| &\leq \liminf_{m\to\infty}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n} w_m^2(x)dx\right)\\
&\leq \limsup_{m\to\infty}\left(\int_{\mathbb{R}^n}\int_{B(0,\overline{\rho}_{\epsilon_m}(x))}\frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n} w_m^2(x)dx\right)\\
&\quad + \limsup_{m\to\infty}\int_{\mathbb{R}^n}\int_{B^c(0,\overline{\rho}_{\epsilon_m}(x))}\frac{|w_m(x+z)-w_m(x)|^2}{|z|^{n+2\alpha}}dzdx\\
&\leq \limsup_{m\to\infty}\frac{2(p+1)}{p-1}c_{\rho_{\epsilon_m}} + \limsup_{m\to\infty}\frac{2K|S^{n-1}|}{\alpha\rho_0^{2\alpha}}\epsilon_m^{2\alpha}\\
&= \frac{2(p+1)}{p-1}c = \|w\|.
\end{aligned}$$
Here K is an estimate for $\|w_m\|$. Thus $\|w_m\|^2 \to \|w\|^2$ and hence $w_m \to w$ in $H^\alpha(\mathbb{R}^n)$. $\square$

We are now in a position to complete the proof of our second main theorem.

Proof of Theorem 1.2. We first obtain an upper bound for the critical values $c_{\rho_{\epsilon_m}} = c_m$, for the sequence $\{\epsilon_m\}$ given in Lemma 3.11. Next we consider the scope function
$$\tilde{\rho}_m(x) = \frac{1}{\epsilon_m}\rho(\epsilon_m x + x_0),$$
where x_0 is a global minimum point of $\mathcal{H}$, see Lemma 3.2. To continue, we consider the function $w_m = w_{\epsilon_m}$ as given in (3.37) and let $t_m > 0$ such that $t_m w_m \in \mathcal{N}_{\tilde{\rho}_m}$. According to Lemma 3.11, $\{w_m\}$ converges to $w \in \mathcal{N}$, then $t_m \to 1$ and $t_m w_m \to w$.

Now we apply Theorem 3.3 to obtain that

$$c_m \le I_{\tilde{\rho}_m}(t_m w_m) = I_{\frac{\rho_\infty}{\epsilon_m}}(t_m w_m) + \epsilon_m^{2\alpha}\left(\|w\|_{L^2}^2 \mathcal{H}(x_0) + o(1)\right). \tag{3.40}$$

We have used part (i) of Theorem 3.3 with $z_m = x_0/\epsilon_m$.

On the other hand, since $w_m \in H^\alpha(\mathbb{R}^n)$ is a critical point of $I_{\bar{\rho}_m}$, we have that

$$c_m = I_{\bar{\rho}_m}(w_m) \ge I_{\bar{\rho}_m}(t_m w_m). \tag{3.41}$$

We write $y_m = y_{\epsilon_m}$. If $\epsilon_m |y_m| \to \infty$, then we may apply part (ii) of Theorem 3.3 with $z_m = y_m$ in (3.41) and obtain that

$$c_m \ge I_{\frac{\rho_\infty}{\epsilon_m}}(t_m w_m) + \epsilon_m^{2\alpha} o(1),$$

which contradicts (3.40). We conclude then, that $\{\epsilon_m y_m\}$ is bounded and that, for a subsequence, $\epsilon_m y_m \to \bar{x}$, for some $\bar{x} \in \mathbb{R}^n$. Now we apply Theorem 3.3 again, but now part (i) with $z_m = y_m$ in (3.41), and we obtain that

$$c_m \ge I_{\frac{\rho_\infty}{\epsilon_m}}(t_m w_m) + \epsilon_m^{2\alpha}\left(\|w\|_{L^2}^2 \mathcal{H}(\bar{x}) + o(1)\right). \tag{3.42}$$

From (3.40) and (3.42) we finally get that

$$\|w\|_{L^2}^2 \mathcal{H}(\overline{x}) + o(1) \le \|w\|_{L^2}^2 \mathcal{H}(x_0) + o(1)$$

and taking the limit as $m \to \infty$, we get

$$\mathcal{H}(\overline{x}) \le \mathcal{H}(x_0) \tag{3.43}$$

completing the proof of the theorem. □

Chapter 4

Radial symmetry

In this chapter we consider the nonlinear Schrödinger equation with fractional regional diffusion

$$(-\Delta)^{\alpha}_{\rho} u + u = f(u), \ \ x \in \mathbb{R}^n. \tag{4.1}$$

and the nonlinear Schrödinger equation with fractional laplacian and external potential

$$(-\Delta)^{\alpha} u + V(x)u = f(u), \ \ x \in \mathbb{R}^n, \tag{4.2}$$

We prove that the solution of (4.1) and (4.2) are symmetric. A naive idea to get our result, consists in replacing the path γ by is its symmetrization $\gamma^* : t \in [0,1] \to \gamma(t)^*$. Then the mountain pass solution u of (4.1) and (4.2) would be near of the set $\gamma^*([0,1])$. This idea works since $*$ is continuous in $H^{\alpha}(\mathbb{R}^n)$. This is not true in the entire case since, when $n > 1$ the symmetry rearrangement $* : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is not continuous, for more details see [84].

4.1 Symmetry Rearrangement

In this section we remember some fact of rearrangement of sets and functions. We just enunciated some properties, the proofs of these properties can be seen on [50], [52], [59], [79]. We present a new type of Riesz and Polya-Szegö inequality with range of scope determined by a radial symmetry positive function.

Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set and denote the measure of A by $|A|$. Define the symmetrization A^* of A to be the closed ball centered at the origin such with the same measure as A. Thus in one dimension

$$A^* := [-\frac{|A|}{2}, \frac{|A|}{2}]$$

and in n dimensions, if we define $\omega(n)$ to be the volume of the unit ball in $\mathbb{R}^n$, then for $A \subset \mathbb{R}^n$

$$A^* := B(0, (|A|/\omega(n))^{1/n}).$$

Let $u : \mathbb{R}^n \to \mathbb{R}$ a Borel measurable function, then u is said to vanish at infinity if

$$|\{x : |u(x)| > t\}| < \infty \ \text{ for all } \ t > 0$$

The symmetric decreasing rearrangement of a characteristic function χ_A is defined as

$$\chi_A^* := \chi_{A^*}$$

We now use that any non negative function can be expressed as an integral of the characteristic functions of the sets $\{u \geq t\}$ (which is a standard abbreviation for $\{x : u(x) \geq t\}$) as follows

$$u(x) = \int_0^{u(x)} 1dt = \int_0^\infty \chi_{\{u \geq t\}}(x)dt. \tag{4.3}$$

Note that this, along with Fubini's theorem, implies

$$\begin{aligned} \int_{\mathbb{R}^n} u(x)dx &= \int_{\mathbb{R}^n} \int_0^\infty \chi_{\{u \geq t\}}(x)dtdx \\ &= \int_0^\infty |\{x : u(x) \geq t\}|dt. \end{aligned}$$

Now if $u : \mathbb{R}^n \to \mathbb{R}$ is a Borel measurable function vanishing at infinity we define

$$u^*(x) = \int_0^\infty \chi^*_{\{|u| \geq t\}}(x)dt \tag{4.4}$$

The rearrangement u^* has a number of properties, see [59]:

(i) u^* is nonnegative.

(ii) u^* is radially symmetric and nonincreasing, i.e:

$$|x| \leq |y| \text{ implies } u^*(y) \leq u^*(x)$$

(iii) u^* is a lower semicontinuos function.

(iv) The level sets of u^* are simply the rearrangement of the level set of u, i.e

$$\{x : u^*(x) > t\} = \{x : |u(x)| > t\}^*.$$

an important consequence of this is the equimeasurability of the function u and u^*, i.e

$$|\{u^* > t\}| = |\{|u| > t\}| \text{ for all } t > 0.$$

(v) For any positive monotone function ϕ, we have

$$\int_{\mathbb{R}^n} \phi(|u(x)|)dx = \int_{\mathbb{R}^n} \phi(u^*(x))dx.$$

In particular, $u^* \in L^p(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and

$$\|u\|_{L^p} = \|u^*\|_{L^p}$$

(vi) Let $V(|x|) \geq 0$ be a spherically symmetric increasing function on $\mathbb{R}^n$. If u is a nonnegative function on $\mathbb{R}^n$, vanishing at infinity the

$$\int_{\mathbb{R}^n} V(|x|)|u^*(x)|^2 dx \leq \int_{\mathbb{R}^n} V(|x|)|u(x)|^2 dx$$

(vii) **Riesz' rearrangement inequality.** Let u, v, w be nonnegative measurable functions on $\mathbb{R}^n$ that vanish at infinity. Then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x)v(x-y)w(y)dydx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u^*(x)v^*(x-y)w^*(y)dydx$$

4.2 Rearrangement Inequalities

Rearrangements have long been a basic tool in the calculus of variations and in the theory of partial differential equations arising as Euler-Lagrange equations of variational problems. The basic Polya-Szegö inequality claims that the symmetric decreasing rearrangement diminishes the L^2-norm of the gradient of a function u:

$$\int_{\mathbb{R}^n} |\nabla u^*(x)|^2 dx \le \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \tag{4.5}$$

where u^* represent the symmetric decreasing rearrangement of u, see [50].

The inequality (4.5), together with its several variants [50], is a powerful key to a number of variational problems of geometric and functional nature, concerning extremal properties of domains and functions. Besides optimal Sobolev embeddings, classical isoperimetric inequalities in mathematical physics and sharp eigenvalue inequalities fall within these results; a priori estimates for solutions to elliptic problems in sharp form are also a closely related topic [52].

The fractional version of (4.5), namely

$$\int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} u^*(x)|^2 dx \le \int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} u(x)|^2 dx, \tag{4.6}$$

was proved by Almgren and Lieb [6] using a rearrangement inequality for convex integrands. Recently Park [69] proved this inequality using Fourier analysis.

We notice that inequality (4.6) is a principal key to get a radially symmetric minimizer. So, an interesting problem is to find if the following inequality is true

$$\int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \frac{|u^*(x+z)-u^*(x)|^2}{|z|^{n+2\alpha}} dz dx \le \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dz dx, \tag{4.7}$$

This is a second main goal in this paper. We want to get a regional version of Riesz and Polya-Szegö inequality with a radial symmetric positive scope function $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$. In next section, following the ideas of Almgren and Lieb [6] we prove the following inequalities: let u, v, w be nonnegative measurable functions on $\mathbb{R}^n$ vanishing at infinity, then

$$\int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} u(x)v(x-z)w(z) dy dx \le \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} u^*(x)v^*(x-z)w^*(z) dz dx, \tag{4.8}$$

and (4.7).

4.2.1 Regional Rearrangement inequalities

In this section we will present and prove a new type of Riesz and Polya-Szegö rearrangement inequalities. This new inequalities are essential to prove symmetry result for fractional elliptic problems. To prove that inequalities we follow the ideas of Algren and Lieb [6] and Lieb and Loss [59]. First we prove the regional Riesz inequality and for this we need the following lemma

Lemma 4.1. *Let u be nonnegative measurable function on $\mathbb{R}^n$ that vanish at infinity and $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$ be a positive radially symmetric function. Then for each $x \in \mathbb{R}^n$*

$$\big(u(y)\chi_{B(0,\rho(|x|))}(y)\big)^* \le u^*(y)\chi_{B(0,\rho(|x|))}(y) \tag{4.9}$$

Proof. First we note that for measurable sets A and B we have

$$(A \cap B)^* \subset A^* \cap B^*. \tag{4.10}$$

In fact, since $A \cap B \subset A$ and $A \cap B \subset B$, then $|A \cap B| \leq |A|$ and $|A \cap B| \leq |A|$. Therefore

$$\begin{aligned} &B(0, (|A \cap B|/w(n))^{1/n}) \subset B(0, (|A|/w(n))^{1/n}), \quad \text{and} \\ &B(0, (|A \cap B|/w(n))^{1/n}) \subset B(0, (|B|/w(n))^{1/n}). \end{aligned}$$

This implies

$$B(0, (|A \cap B|/w(n))^{1/n}) \subset B(0, (|A|/w(n))^{1/n}) \cap B(0, (|B|/w(n))^{1/n}),$$

hence $(A \cap B)^* \subset A^* \cap B^*$.

Now, we note that (4.9) follow from:

$$\{(u(y)\chi_{B(0,\rho(|x|))}(y))^* > t\} \subseteq \{[u^*(y)\chi_{B(0,\rho(|x|))}(y)] > t\} \ \text{ for all } \ t.$$

In fact, by (iv) and (4.10) we have

$$\begin{aligned} \{(u\chi_{B(0,\rho(|x|))})^* > t\} &= \{u\chi_{B(0,\rho(|x|))} > t\}^* \\ &= [\{u > t\} \cap B(0, \rho(|x|))]^* \\ &\subseteq \{u > t\}^* \cap B(0, \rho(|x|)) \\ &= \{u^* > t\} \cap B(0, \rho(|x|)) \\ &= \{u^*\chi_{B(0,\rho(|x|))} > t\}. \end{aligned}$$

Hence

$$\begin{aligned} \left(u\chi_{B(0,\rho(|x|))}\right)^*(y) &= \int_0^\infty \chi_{\{(u\chi_{B(0,\rho(|x|))})^* > t\}}(y)dt \\ &\leq \int_0^\infty \chi_{\{u^*\chi_{B(0,\rho(|x|))}\} > t}(y)dt \\ &= (u^*\chi_{B(0,\rho(|x|))})(y) \end{aligned}$$

This proves our inequality. □

With this lemma we are ready to prove our regional Riesz inequality

Theorem 4.2. (Regional Riesz Rearrangement Inequality) *Let u, v, w be nonnegative measurable functions on $\mathbb{R}^n$ that vanish at infinity and $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$ be a positive radially symmetric function. Then*

$$\int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} u(x)v(x-y)w(y)dydx \leq \int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} u^*(x)v^*(x-y)w^*(y)dydx \tag{4.11}$$

Proof. By Riesz rearrangement inequality and Lemma 4.1, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} u(x)v(x-y)w(y)dydx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x)v(x-y)w(y)\chi_{B(0,\rho(|x|))}(y)dydx \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u^*(x)v^*(x-y)(w\chi_{B(0,\rho(|x|))})^*(y)dydx \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u^*(x)v^*(x-y)w^*(y)\chi_{B(0,\rho(|x|))}(y)dydx \\ &= \int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} u^*(x)v^*(x-y)w^*(y)dydx. \end{aligned}$$

□

Now we are going to prove the regional version of Polya-Szegö inequality. We consider the functional $E_\rho : H^\alpha_\rho(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$E_\rho[u] = \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx.$$

We will get a representation of E_ρ that will be useful for our purpose. We start with the representation formula

$$\frac{1}{|z|^{n+2\alpha}} = \frac{1}{\Gamma\left(\frac{n+2\alpha}{2}\right)}\int_0^\infty e^{-t|z|^2} t^{\frac{n+2\alpha}{2}-1} dt, \tag{4.12}$$

whose proof can be found in [6]. Using, (4.12) and Fubini's theorem we have

$$\begin{aligned} E[u] &= \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} |u(x+z)-u(x)|^2 \frac{1}{|z|^{n+2\alpha}} dzdx \\ &= \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} |u(x+z)-u(x)|^2 \frac{1}{\Gamma\left(\frac{n+2\alpha}{2}\right)}\int_0^\infty e^{-t|z|^2} t^{\frac{n+2\alpha}{2}-1} dtdzdx \\ &= \frac{1}{\Gamma\left(\frac{n+2\alpha}{2}\right)}\int_0^\infty \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} |u(x+z)-u(x)|^2 e^{-t|z|^2} dzdx t^{\frac{n+2\alpha}{2}-1} dt \end{aligned}$$

From here we define

$$I_t[u] = \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} |u(x+z)-u(x)|^2 e^{-t|z|^2} dzdx, \quad t>0. \tag{4.13}$$

Theorem 4.3. (Regional Polya-Szegö inequality) *Let $0<\alpha<1$ and $u \in H^\alpha_\rho(\mathbb{R}^n)$ and $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$ be a positive radially symmetric function. Then*

$$E_\rho[u^*] \le E_\rho(u). \tag{4.14}$$

Proof. First notice that $|u(x+z)-u(x)| \ge ||u(x+z)|-|u(x)||$, then without loss of generality we may assume that u is non-negative. Furthermore, in view of (4.13) we only need to prove that

$$I_t[u^*] \le I_t[u], \quad \forall t>0. \tag{4.15}$$

Let $\phi(t) = |t|^2$. We can write $\phi = \phi_+ + \phi_-$ where

$$\phi_\pm(t) = \begin{cases} \phi(t), & \text{if } \pm t \ge 0, \\ 0, & \text{if } \pm t \le 0. \end{cases}$$

We decompose $I_t = I_t^+ + I_t^-$ accordingly. Bellow we prove the assertion of the theorem with I_t replaced by I_t^+. The assertion for I_t^- is similar and hence the result for the original I_t follows.

Since $\phi_+(0) = 0$, we have that

$$\begin{aligned} \phi_+(u(x+z)-u(x)) &= \int_0^{u(x+z)-u(x)} \phi_+'(t)dt \\ &= \int_{u(x)}^{u(x+z)} \phi_+'(u(x+z)-t)dt \\ &= \int_0^\infty \phi_+'(u(x+z)-t)\chi_{\{u\le t\}}(x)dt. \end{aligned}$$

Then, by Fubini's theorem

$$
\begin{aligned}
I_t^+[u] &= \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \phi_+'(u(x+z)-u(x))e^{-t|z|^2}dzdx \\
&= \int_0^\infty\int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \phi_+'(u(x+z)-t)e^{-t|z|^2}\chi_{\{u\le t\}}(x)dzdxdt. \qquad (4.16)
\end{aligned}
$$

Now

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \phi_+'(u(x+z)-t)e^{-t|z|^2}\chi_{\{u\le t\}}(x)dzdx \\
=& \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \phi_+'(u(x+z)-t)e^{-t|z|^2}(1-\chi_{\{u>t\}}(x))dzdx \\
=& \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \phi_+'(u(x+z)-t)e^{-t|z|^2}dzdx \\
&-\int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \phi_+'(u(x+z)-t)e^{-t|z|^2}\chi_{\{u>t\}}(x)dzdx. \qquad (4.17)
\end{aligned}
$$

We notice that the function $g(s)=\phi_+'(s-t)$ is increasing and $\left(e^{-t|z|^2}\right)^*=e^{-t|z|^2}$, for all $t>0$, then by property (v) we have

$$\left(\phi_+'(u(x)-t)\right)^*=\phi_+'(u^*-t).$$

From here, (4.16), (4.17) and Theorem 4.2 we find that

$$I_t^+[u^*]\le I_t^+[u], \quad \forall t>0.$$

Since we also have $I_t^-[u^*]\le I_t^-[u]$, we conclude. □

Remark 4.4. Theorem 4.3 implies the non-expansivity of symmetric decreasing rearrangement of the regional fractional Sobolev norm in $H_\rho^\alpha(\mathbb{R}^n)$, that is, for $u\in H_\rho^\alpha(\mathbb{R}^n)$

$$\|u^*\|_{H_\rho^\alpha}\le\|u\|_{H_\rho^\alpha}. \qquad (4.18)$$

Finally we recall a result proved by Almgren and Lieb in [6], which is acrucial ingredient to prove our main theorem in the next section.

Theorem 4.5. *For each $0<\alpha<1$ and each $n\ge1$, the map $\mathfrak{R}: H^\alpha(\mathbb{R}^n)\to H^\alpha(\mathbb{R}^n)$, defined as $\mathfrak{R}u=u^*$, is continuous and, as a consequence, $\mathfrak{R}: H_\rho^\alpha(\mathbb{R}^n)\to H_\rho^\alpha(\mathbb{R}^n)$ is also continuous.*

4.3 Symmetry Result

In this section, we will deal with the following problem

$$
\begin{gathered}
(-\Delta)_\rho^\alpha u+u=f(u) \;\text{ in } \mathbb{R}^n, \\
u\in H_\rho^\alpha(\mathbb{R}^n).
\end{gathered} \qquad (4.19)
$$

where ρ satisfies

(ρ_1) There are numbers $0 < \rho_0 < \rho_\infty \leq \infty$ such that

$$\rho_0 \leq \rho(x) < \rho_\infty \quad \forall x \in \mathbb{R}^n \quad \text{and} \quad \lim_{|x|\to\infty} \rho(x) = \rho_\infty.$$

(ρ_2) In case $\rho_\infty = \infty$ we further assume that there exists $a \in (0,1)$ such that

$$\limsup_{|x|\to\infty} \frac{\rho(x)}{|x|} \leq a.$$

(ρ_3) $\rho(x) = \rho(|x|)$

Regarding the nonlinearity f we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies the following hypotheses:

(f_1) $f(t) \geq 0$ if $t \geq 0$ and $f(t) = 0$ if $t \leq 0$.

(f_2) The function $t \to \frac{f(t)}{t}$ is increasing for $t > 0$ and $\lim_{t\to 0} \frac{f(t)}{t} = 0$.

(f_3) $\exists \theta > 2$ such that $\forall t > 0$

$$0 < \theta F(t) \leq t f(t), \quad \textit{where} \quad F(t) = \int_0^t f(s)ds.$$

(f_4) $\exists C > 0$ such that

$$|f(t)| \leq C(1 + |t|^p), \;\; 1 < p < \frac{n+2\alpha}{n-2\alpha}.$$

Associated to (4.35) we have the following functional

$$I_\rho(u) = \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx + \int_{\mathbb{R}^n} |u(x)|^2 dx\right) - \int_{\mathbb{R}^n} F(u(x))dx \quad (4.20)$$

We note, by the regional Polya-Szegö inequality, if u^* is the symmetry rearrangement of u we have

$$I_\rho(u^*) \leq I_\rho(u). \quad (4.21)$$

We want to prove that the mountain pass solution u given in Chapter 2 is radial. It should be possible to have more information on the symmetry of u since we have (4.21). A naive idea consists in replacing the path γ by is its symmetrization $\gamma^* : t \in [0,1] \to \gamma(t)^*$. Then u given by Theorem 2.14 would be near of the set $\gamma^*([0,1])$. This idea works since $*$ is continuous in $H^\alpha_\rho(\mathbb{R}^n)$. This is not true in the entire case since, when $n > 1$ the symmetry rearrangement $* : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is not continuous, for more details see [84].

Now we state and prove our main theorem:

Theorem 4.6. *Suppose that ($\rho_{1)}$) $-$ (ρ_3) and (f_1) $-$ (f_4) hold. Then the mountain pass value is achieved by a radially symmetric function, which is a solution of (4.35).*

Proof. Under (f_1)-(f_4), (ρ_1)-(ρ_2), in [38] we have proved that I_ρ satisfies the mountain pass geometry condition with mountain pass level

$$c_\rho = \inf_{\gamma\in\Gamma_\rho} \sup_{t\in[0,1]} I_\rho(\gamma(t)),$$

where $\Gamma_\rho = \{\gamma \in C([0,1], H^\alpha_\rho(\mathbb{R}^n))/ \ \ \gamma(0) = 0,\ I_\rho(\gamma(1)) < 0\}$. By definition of c_ρ, for any $n \in \mathbb{N}$, there is $\gamma_n \in \Gamma_\rho$ such that

$$\sup_{t\in[0,1]} I_\rho(\gamma_n(t)) \leq c_\rho + \frac{1}{n^2}. \tag{4.22}$$

Now, defining $\gamma_n^*(t) = [\gamma_n(t)]^*$, we see that Theorem 4.5 and the fact that $I_\rho(\gamma_n^*(1)) \leq I_\rho(\gamma_n(1)) < 0$ imply that $\gamma_n^* \in \Gamma_\rho$. Moreover, by the regional Polya-Zsegö inequality proved in Theorem 4.3, we have

$$I_\rho(\gamma_n^*(t)) \leq I_\rho(\gamma_n(t)),\ \ \forall t \in [0,1].$$

So

$$\sup_{t\in[0,1]} I_\rho(\gamma_n^*(t)) \leq c_\rho + \frac{1}{n^2}. \tag{4.23}$$

Then by Theorem 4.3 of [62], there is a sequence $u_n \in H^\alpha_\rho(\mathbb{R}^n)$ and $\xi_n \in [0,1]$ such that

$$\|u_n - \gamma_n^*(\xi_n)\|_{H^\alpha_\rho} \leq \frac{1}{n}, \tag{4.24}$$

$$I_\rho(u_n) \in (c_\rho - \frac{1}{n^2}, c_\rho + \frac{1}{n^2}) \quad \text{and} \tag{4.25}$$

$$\|I'_\rho(u_n)\|_{(H^\alpha_\rho)'} \leq \frac{1}{n}. \tag{4.26}$$

Following the ideas of the proof of Theorem 1.1 of [38] we can show that: $u_n \to u$ in $H^\alpha_\rho(\mathbb{R}^n)$, $I_\rho(u) = c_\rho$, $I'_\rho(u) = 0$ and

$$\lim_{n\to\infty} \|u - \gamma_n^*(\xi_n)\|_{H^\alpha_\rho} = 0. \tag{4.27}$$

The last equality shows that $u = u^*$. $\square$

We can use the same ideas with equation

$$(-\Delta)^\alpha u + V(x)u = f(u),\ \ x \in \mathbb{R}^n. \tag{4.28}$$

In [76], Secchi studied (4.28) with an $x-$dependence nonlinearity $f(x,t)$. Using the approach of Rabinowitz in [73], namely a comparison argument, Secchi proved the existence of a ground state solution of (4.28), when $f(x,t)$ is super-linear and has a sub-critical growth. On the other hand, in [77] Secchi considered the existence of radially symmetric solution of (4.28) under some weaker conditions on f, using the monotonicity trick of Struwe and Jeanjean [47].

Now, our purpose is to prove the symmetry result for (4.28) using the approach discussed above. For that purpose we consider that the nonlinearity f satisfies (f_1)-(f_4) and regarding the potential V we assume

(V_1) $V \in C(\mathbb{R}^n)$ and $\inf_{\mathbb{R}^n} V(x) = V_0 > 0$

(V_2) $\lim_{|x|\to\infty} V(x) = V_\infty$.

(V_3) V is radially symmetric and increasing.

Weak solutions to (4.28) are critical points of the functional $I_V : H_V^\alpha(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$I_V(u) = \frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}dydx + \int_{\mathbb{R}^n} V(|x|)|u(x)|^2 dx\right) - \int_{\mathbb{R}^n} F(u(x))dx \quad (4.29)$$

We state our main Theorem

Theorem 4.7. *Suppose that $(f_1)-(f_4)$ and $(V_1)-(V_3)$ hold. Then the mountain pass value is achieved by a radially symmetric function, which is a solution of (4.28).*

Proof. Under (f_1)-(f_4), (V_1)-(V_2) we find that I_V satisfies the mountain pass geometry conditions, using the proof of Secchi in [76] with minor modifications. The mountain pass level for I_V is given by

$$c_V = \inf_{\gamma\in\Gamma_V} \sup_{t\in[0,1]} I_V(\gamma(t)),$$

where Γ_V is defined as usual. By definition of c_V, for any $n \in \mathbb{N}$, there is $\gamma_n \in \Gamma_V$ such that

$$\sup_{t\in[0,1]} I_V(\gamma_n(t)) \le c_V + \frac{1}{n^2}. \quad (4.30)$$

Now, let $\gamma_n^*(t) = [\gamma_n(t)]^*$. By the continuity of rearrangements in $H_V^\alpha(\mathbb{R}^n)$ we have that $\gamma_n^* \in \Gamma_V$. Moreover, by the fractional Polya-Szegö inequality and taking into account that V satisfies (V_3), we have

$$I_V(\gamma_n^*(t)) \le I_V(\gamma_n(t)), \quad \forall t \in [0,1].$$

So

$$\sup_{t\in[0,1]} I_V(\gamma_n^*(t)) \le c_V + \frac{1}{n^2}. \quad (4.31)$$

By Theorem 4.3 in [62], there is a sequence $u_n \in H_\rho^\alpha(\mathbb{R}^n)$ and $\xi_n \in [0,1]$ such that

$$\|u_n - \gamma_n^*(\xi_n)\|_{H_V^\alpha} \le \frac{1}{n}, \quad (4.32)$$

$$I_V(u_n) \in (c_V - \frac{1}{n^2}, c_V + \frac{1}{n^2}), \quad (4.33)$$

$$\|I_V'(u_n)\|_{(H_V^\alpha)'} \le \frac{1}{n}. \quad (4.34)$$

Following the ideas of the proof of Theorem 5.2 of [76], we can show that $u_n \to u$, $I_V(u) = c_V$, $I_V'(u)u = 0$ and finally that

$$\lim_{n\to\infty} \|u - \gamma_n^*(\xi_n)\|_{H_V^\alpha} = 0, \quad (4.35)$$

concluding the proof. $\square$

Chapter 5

Fractional Hamiltonian Systems

Fractional order models can be found to be more adequate than integer order models in some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. As a consequence, the subject of fractional differential equations is gaining more importance and attention. There has been significant development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives. For details and examples, one can see the monographs [53], [65], [70] and the papers [2], [3], [5], [8], [15], [45], [58], [64], [74], [86]. Moreover the existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic, and pseudo-almost periodic solutions have been great attention in the qualitative theory of fractional differential equations, due to its mathematical interest and applications. Some recent contributions on the existence of such solutions for abstract differential equations and fractional differential equations have been made, see [1], [3], [4], [8], [28], [33], [57], [66] for details.

Recently, also equations including both left and right fractional derivatives are discussed. Apart from their possible applications, equations with left and right derivatives is an interesting and new field in fractional differential equations theory. In this topic, many results are obtained dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by using techniques of nonlinear analysis, such as fixed point theory [11] (including Leray-Schauder nonlinear alternative), topological degree theory [48] (including co-incidence degree theory) and comparison method [88] (including upper and lower solutions and monotone iterative method) and so on.

It should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last 30 years, the critical point theory has become to a wonderful tool in studying the existence of solutions to differential equations with variational structures, we refer the reader to the books due to Mawhin and Willem [62], Rabinowitz [73] and the references listed therein.

Motivated by the above classical works, in recent paper [49], for the first time, Jiao and Zhou showed that the critical point theory is an effective approach to tackle the existence of solutions for the following fractional boundary value problem

$$\begin{aligned} {}_tD_T^\alpha({}_0D_t^\alpha u(t)) = \nabla F(t,u(t)),\ \text{a.e. } t\in[0,T], \\ u(0)=u(T)=0. \end{aligned} \tag{5.1}$$

and obtained the existence of at least one nontrivial solution. We note that it is not easy to use the critical point theory to study (5.1), since it is often very difficult to establish a suitable space and variational functional for the fractional boundary value problem.

In this section we deal with the fractional Hamiltonian system

$${}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) + L(t)u(t) = \nabla W(t,u(t)) \tag{5.2}$$

where $\alpha\in(1/2,1)$, $t\in\mathbb{R}$, $u\in\mathbb{R}^n$, $L\in C(\mathbb{R},\mathbb{R}^{n\times n})$ is a symmetric matrix valued function and $W:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$; satisfies the following condition

(L) $L(t)$ is positive definite symmetric matrix for all $t\in\mathbb{R}$ and there exists an $l\in C(\mathbb{R},(0,\infty))$ such that $l(t)\to+\infty$ as $t\to\infty$ and

$$(L(t)x,x)\geq l(t)|x|^2,\ \text{ for all } t\in\mathbb{R} \text{ and } x\in\mathbb{R}^n. \tag{5.3}$$

(W_1) $W\in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and there is a constant $\mu>2$ such that

$$0<\mu W(t,x)\leq(x,\nabla W(t,x)),\ \text{ for all } t\in\mathbb{R} \text{ and } x\in\mathbb{R}^n\setminus\{0\}.$$

(W_2) $|\nabla W(t,x)|=o(|x|)$ as $x\to0$ uniformly with respect to $t\in\mathbb{R}$.

(W_3) There exists $\overline{W}\in C(\mathbb{R}^n,\mathbb{R})$ such that

$$|W(t,x)|+|\nabla W(t,x)|\leq|\overline{W(x)}|\ \text{ for every } x\in\mathbb{R}^n \text{ and } t\in\mathbb{R}.$$

In particular, if $\alpha=1$, (5.2) reduces to the standard second order differential equation

$$u''-L(t)u+\nabla W(t,u)=0, \tag{5.4}$$

where $W:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is a given function and $\nabla W(t,u)$ is the gradient of W at u. The existence of homoclinic solution is one of the most important problems in the history of that kind of equations, and has been studied intensively by many mathematicians. Assuming that $L(t)$ and $W(t,u)$ are independent of t, or T-periodic in t, many authors have studied the existence of homoclinic solutions for (5.4) via critical point theory and variational methods. In this case, the existence of homoclinic solution can be obtained by going to the limit of periodic solutions of approximating problems.

If $L(t)$ and $W(t,u)$ are neither autonomous nor periodic in t, this problem is quite different from the ones just described, because the lack of compacteness of the Sobolev embedding. In [72] Rabinowitz and Tanaka considered (5.4) without periodicity assumptions on L and W and showed that (5.4) possesses one homoclinic solution by using a variant of the mountain pass theorem without the Palais-Smale contidion. In [68], under the same assumptions of [72], Omana and Willem, by employing a new compact embedding theorem, obtained the existence of solution of (5.4) by mountain pass Theorem.

Our goal in this section is to show how variational methods based on Mountain pass theorem can be used to get existence results for (5.2). However, the direct application of the mountain pass theorem is not enough since the Palais-Smale sequences might lose compactness in the whole space $\mathbb{R}$. To overcome this difficulty we proof a version of compact embedding for fractional space following the ideas of [68]. We state our main existence theorem.

Theorem 5.1. *Suppose that $(L), (W_1)-(W_3)$ hold, then (5.2) possesses at least one nontrivial solution.*

5.1 Preliminary Results

5.1.1 Liouville-Weyl Fractional Calculus

The Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ are defined as

$$ {}_{-\infty}I_x^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}u(\xi)d\xi \tag{5.5}$$

$$ {}_{x}I_{\infty}^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(\xi-x)^{\alpha-1}u(\xi)d\xi \tag{5.6}$$

The Liouville-Weyl fractional derivative of order $0 < \alpha < 1$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals

$$ {}_{-\infty}D_x^{\alpha}u(x) = \frac{d}{dx}{}_{-\infty}I_x^{1-\alpha}u(x) \tag{5.7}$$

$$ {}_{x}D_{\infty}^{\alpha}u(x) = -\frac{d}{dx}{}_{x}I_{\infty}^{1-\alpha}u(x) \tag{5.8}$$

The definitions (5.7) and (5.8) may be written in an alternative form:

$$ {}_{-\infty}D_x^{\alpha}u(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{u(x)-u(x-\xi)}{\xi^{\alpha+1}}d\xi \tag{5.9}$$

$$ {}_{x}D_{\infty}^{\alpha}u(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{u(x)-u(x+\xi)}{\xi^{\alpha+1}}d\xi \tag{5.10}$$

We establish the Fourier transform properties of the fractional integral and fractional differential operators. Recall that the Fourier transform $\widehat{u}(w)$ of $u(x)$ is defined by

$$\widehat{u}(w) = \int_{-\infty}^{\infty} e^{-ix.w}u(x)dx.$$

Let $u(x)$ be defined on $(-\infty, \infty)$. Then the Fourier transform of the Liouville-Weyl integral and differential operator satisfies

$$ \widehat{{}_{-\infty}I_x^{\alpha}u(x)}(w) = (iw)^{-\alpha}\widehat{u}(w), \quad \widehat{{}_{x}I_{\infty}^{\alpha}u(x)}(w) = (-iw)^{-\alpha}\widehat{u}(w) \tag{5.11}$$

$$ \widehat{{}_{-\infty}D_x^{\alpha}u(x)}(w) = (iw)^{\alpha}\widehat{u}(w), \quad \widehat{{}_{x}D_{\infty}^{\alpha}u(x)}(w) = (-iw)^{\alpha}\widehat{u}(w) \tag{5.12}$$

5.1.2 Fractional Derivative Spaces

In this section we introduce some fractional spaces for more detail see [34]. Let $\alpha > 0$, define the semi-norm

$$|u|_{I^\alpha_{-\infty}} = \|{}_{-\infty}D^\alpha_x u\|_{L^2}$$

and norm

$$\|u\|_{I^\alpha_{-\infty}} = \left(\|u\|^2_{L^2} + |u|^2_{I^\alpha_{-\infty}}\right)^{1/2}, \tag{5.13}$$

and let

$$I^\alpha_{-\infty} = \overline{C^\infty_0(\mathbb{R},\mathbb{R}^n)}^{\|\cdot\|_{I^\alpha_{-\infty}}}.$$

Now we define the fractional Sobolev space $H^\alpha(\mathbb{R},\mathbb{R}^n)$ in terms of the Fourier transform. Let $0 < \alpha < 1$, let the semi-norm

$$|u|_\alpha = \||w|^\alpha \widehat{u}\|_{L^2} \tag{5.14}$$

and norm

$$\|u\|_\alpha = \left(\|u\|^2_{L^2} + |u|^2_\alpha\right)^{1/2},$$

and let

$$H^\alpha = \overline{C^\infty_0(\mathbb{R},\mathbb{R}^n)}^{\|\cdot\|_\alpha}.$$

We note a function $u \in L^2(\mathbb{R},\mathbb{R}^n)$ belong to $I^\alpha_{-\infty}$ if and only if

$$|w|^\alpha \widehat{u} \in L^2(\mathbb{R},\mathbb{R}^n). \tag{5.15}$$

Especially

$$|u|_{I^\alpha_{-\infty}} = \||w|^\alpha \widehat{u}\|_{L^2}. \tag{5.16}$$

Therefore $I^\alpha_{-\infty}$ and H^α are equivalent with equivalent semi-norm and norm. Analogous to $I^\alpha_{-\infty}$ we introduce I^α_∞. Let the semi-norm

$$|u|_{I^\alpha_\infty} = \|{}_xD^\alpha_\infty u\|_{L^2}$$

and norm

$$\|u\|_{I^\alpha_\infty} = \left(\|u\|^2_{L^2} + |u|^2_{I^\alpha_\infty}\right)^{1/2}, \tag{5.17}$$

and let

$$I^\alpha_\infty = \overline{C^\infty_0(\mathbb{R},\mathbb{R}^n)}^{\|\cdot\|_{I^\alpha_\infty}}.$$

Moreover $I^\alpha_{-\infty}$ and I^α_∞ are equivalent , with equivalent semi-norm and norm, see [34] for more details.

Now we give the prove of the Sobolev lemma.

Theorem 5.2. *If $\alpha > \frac{1}{2}$, then $H^\alpha \subset C(\mathbb{R},\mathbb{R}^n)$ and there is a constant $C = C_\alpha$ such that*

$$\sup_{x\in\mathbb{R}} |u(x)| \le C\|u\|_\alpha \tag{5.18}$$

Proof. By the Fourier inversion theorem, if $\widehat{u} \in L^1(\mathbb{R})$, then u is continuous and

$$\sup_{x\in\mathbb{R}} |u(x)| \le \|\widehat{u}\|_{L^1}.$$

Hence, to prove the theorem it is enough to prove that $\|\widehat{u}\|_{L^1} \leq \|u\|_\alpha$. So by Schwarz inequality, we have

$$\begin{aligned}\int_{\mathbb{R}} |\widehat{u}(w)|dw &= \int_{\mathbb{R}} (1+|w|^2)^{\alpha/2}|\widehat{u}(w)|\frac{1}{(1+|w|^2)^{\alpha/2}}dw \\ &\leq \left(\int_{\mathbb{R}} (1+|w|^{2\alpha})|\widehat{u}(w)|^2 dw\right)^{1/2}\left(\int_{\mathbb{R}} (1+|w|^2)^{-\alpha}dw\right)^{1/2}.\end{aligned}$$

The first integral on the right is $\|u\|_\alpha^2$, so the theorem boils down to the fact

$$\int_{\mathbb{R}} (1+|w|^2)^{-\alpha}dw < \infty$$

precisely when $\alpha > \frac{1}{2}$. $\Box$

Remark 5.3. If $u \in H^\alpha$, then $u \in L^q(\mathbb{R},\mathbb{R}^n)$ for all $q \in [2,\infty)$, since

$$\int_{\mathbb{R}} |u(x)|^q dx \leq \|u\|_\infty^{q-2}\|u\|_{L^2}^2$$

In what follows, we introduce the fractional space in which we will construct the variational framework of (5.2). Let

$$X^\alpha = \left\{u \in H^\alpha | \quad \int_{\mathbb{R}} [|_{-\infty}D_t^\alpha u(t)|^2 + L(t)u(t).u(t)]dt < \infty\right\},$$

then X^α is a reflexive and separable Hilbert space with the inner product

$$\langle u,v\rangle_{X^\alpha} = \int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha u(t),\ {}_{-\infty}D_t^\alpha v(t)) + L(t)u(t).v(t)dt$$

and the corresponding norm

$$\|u\|_{X^\alpha}^2 = \langle u,u\rangle_{X^\alpha}.$$

Lemma 5.4. *Suppose L satisfies (L). Then X^α is continuously embedded in H^α.*

Proof. Since $l \in C(\mathbb{R},(0,\infty))$ and l is coercive, then $l_{min} = \min_{t\in\mathbb{R}} l(t)$ exists, so we have

$$(L(t)u(t),u(t)) \geq l(t)|u(t)|^2 \geq l_{min}|u(t)|^2,\ \forall t \in \mathbb{R}.$$

Then

$$\begin{aligned}l_{min}\|u\|_\alpha^2 &= l_{min}\left(\int_{\mathbb{R}} |_{-\infty}D_t^\alpha u(t)|^2 + |u(t)|^2 dt\right) \\ &\leq l_{min}\int_{\mathbb{R}} |_{-\infty}D_t^\alpha u(t)|^2 dt + \int_{\mathbb{R}} (L(t)u(t),u(t))dt\end{aligned}$$

So

$$\|u\|_\alpha^2 \leq K\|u\|_{X^\alpha}^2 \tag{5.19}$$

where $K = \frac{\max\{l_{min},1\}}{l_{min}}$. $\Box$

The main difficulty in dealing with the existence of solution for (5.2) is the lack of compactness of the Sobolev embedding. To overcome this difficulty under the assumptions of Theorem 5.1, we employ the following compact embedding Lemma.

Lemma 5.5. *Suppose L satisfies (L). Then the imbedding of X^α in $L^2(\mathbb{R},\mathbb{R}^n)$ is compact.*

Proof. We note first that by Lemma 5.4 and Remark 5.3 we have

$$X^\alpha \hookrightarrow L^2(\mathbb{R}) \text{ is continuous.}$$

Now, let $(u_k) \in X^\alpha$ be a sequence such that $u_k \rightharpoonup u$ in X^α. We will show that $u_k \to u$ in $L^2(\mathbb{R})$. Suppose, without loss of generality, that $u_k \to 0$ in X^α. The Banach-Steinhaus Theorem implies that

$$A = \sup_k \|u_k\|_{X^\alpha} < +\infty$$

Let $\epsilon > 0$; there is $T_0 < 0$ such that $\frac{1}{l(t)} \leq \epsilon$ for all t such that $t \leq T_0$. Similarly, there is $T_1 > 0$, such that $\frac{1}{l(t)} \leq \epsilon$ for all $t \geq T_1$. Sobolev's Theorem (see e.g. [80]) implies that $u_k \to 0$ uniformly on $\overline{\Omega} = [T_0, T_1]$, so there is a k_0 such that

$$\int_\Omega |u_k(t)|^2 dt \leq \epsilon, \text{ for all } k \geq k_0. \tag{5.20}$$

Since $\frac{1}{l(t)} \leq \epsilon$ on $(-\infty, T_0]$ we have

$$\int_{-\infty}^{T_0} |u_k(t)|^2 dt \leq \epsilon \int_{-\infty}^{T_0} l(t)|u_k(t)|^2 dt \leq \epsilon A^2. \tag{5.21}$$

Similarly, since $\frac{1}{l(t)} \leq \epsilon$ on $[T_1, +\infty)$, we have

$$\int_{T_1}^{+\infty} |u_k(t)|^2 dt \leq \epsilon A^2. \tag{5.22}$$

Combining (5.20), (5.21) and (5.22) we get $u_k \to 0$ in $L^2(\mathbb{R},\mathbb{R}^n)$. $\square$

Lemma 5.6. *There are constants $c_1 > 0$ and $c_2 > 0$ such that*

$$W(t,u) \geq c_1|u|^\mu, \quad |u| \geq 1 \tag{5.23}$$

and

$$W(t,u) \leq c_2|u|^\mu, \quad |u| \leq 1 \tag{5.24}$$

Proof. By (W_1) we note that

$$\mu W(t,\sigma u) \leq (\sigma u, \nabla W(t,\sigma u)).$$

Let $f(\sigma) = W(t,\sigma u)$, then

$$\frac{d}{d\sigma}\left(f(\sigma)\sigma^{-\mu}\right) \geq 0 \tag{5.25}$$

Now we consider two cases

Case 1. $|u| \leq 1$. In this case we integrate (5.25), from 1 until $\frac{1}{|u|}$ and we get

$$W(t,u) \leq W(t,\frac{u}{|u|})|u|^\mu. \tag{5.26}$$

Case 2. $|u| \geq 1$. In this case we integrate (5.25), from $\frac{1}{|u|}$ until 1 and we get

$$W(t,u) \geq |u|^{\mu} W(t, \frac{u}{|u|}). \tag{5.27}$$

Now, since $u \in \mathbb{R}^n$, $\frac{u}{|u|} \in B(0,1)$. So, since W is continuous and $B(0,1)$ is compact, there are $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \leq W(t,u) \leq c_2, \quad \text{for every } u \in B(0,1).$$

Therefore we get the affirmation of the Lemma. $\square$

Remark 5.7. By Lemma 5.6, we have

$$W(t,u) = o(|u|^2) \text{ as } u \to 0 \text{ uniformly in } t \in \mathbb{R} \tag{5.28}$$

In addition, by (W_2), we have, for any $u \in \mathbb{R}^n$ such that $|u| \leq M_1$, there exists some constant $d > 0$ (dependent on M_1) such that

$$|\nabla W(t, u(t))| \leq d|u(t)| \tag{5.29}$$

Similar to Lemma 2 of [68], we can get the following result.

Lemma 5.8. *Suppose that (L), (W_1)-(W_2) are satisfied. If $u_k \rightharpoonup u$ in X^{α}, then $\nabla W(t,u_k) \to \nabla W(t,u)$ in $L^2(\mathbb{R},\mathbb{R}^n)$.*

Proof. Assume that $u_k \rightharpoonup u$ in X^{α}. Then there exists a constant $d_1 > 0$ such that, by Banach-Steinhaus Theorem and (5.18),

$$\sup_{k\in\mathbb{N}} \|u_k\|_{\infty} \leq d_1, \quad \|u\|_{\infty} \leq d_1.$$

By (W_2), for any $\epsilon > 0$ there is $\delta > 0$ such that

$$|u_k| < \delta \ \text{ implies } \ |\nabla W(t,u_k)| \leq \epsilon |u_k|$$

and by (W_3) there is $M > 0$ such that

$$|\nabla W(t,u_k)| \leq M, \quad \text{forall } \delta < |u_k| \leq d_1.$$

Therefore, there exists a constant $d_2 > 0$ such that

$$|\nabla W(t,u_k(t))| \leq d_2|u_k(t)|, \quad |\nabla W(t,u(t))| \leq d_2|u(t)|$$

for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Hence,

$$|\nabla W(t,u_k(t)) - \nabla W(t,u(t))| \leq d_2(|u_k(t)| + |u(t)|) \leq d_2(|u_k(t) - u(t)| + 2|u(t)|),$$

Since, by Lemma 5.5, $u_k \to u$ in $L^2(\mathbb{R},\mathbb{R}^n)$, passing to a subsequence if necessary, it can be assumed that

$$\sum_{k=1}^{\infty} \|u_k - u\|_{L^2} < \infty$$

But this implies $u_k(t) \to u(t)$ almost everywhere $t \in \mathbb{R}$ and

$$\sum_{k=1}^{\infty} |u_k(t) - u(t)| = v(t) \in L^2(\mathbb{R}, \mathbb{R}^n).$$

Therefore

$$|\nabla W(t, u_k(t)) - \nabla W(t, u(t))| \leq d_2(v(t) + 2|u(t)|).$$

Then, using the Lebesgue's convergence theorem, the Lemma is proved. □

Now we are going to establish the corresponding variational framework to obtain the existence of solutions for (5.2). Define the functional $I : X^\alpha \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} \left[\frac{1}{2}|{}_{-\infty}D_t^\alpha u(t)|^2 + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t))\right] dt \\ &= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, u(t))dt \end{aligned} \tag{5.30}$$

Lemma 5.9. *Under the conditions of Theorem 5.1, we have*

$$I'(u)v = \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty} D_t^\alpha v(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t))]\, dt \tag{5.31}$$

for all $u, v \in X^\alpha$, which yields that

$$I'(u)u = \|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} (\nabla W(t, u(t)), u(t))dt. \tag{5.32}$$

Moreover, I is a continuously Fréchet-differentiable functional defined on X^α, i.e., $I \in C^1(X^\alpha, \mathbb{R})$.

Proof. We firstly show that $I : X^\alpha \to \mathbb{R}$. By (5.28), there is a $\delta > 0$ such that $|u| \leq \delta$ implies that

$$W(t, u) \leq \epsilon|u|^2 \text{ for all } t \in \mathbb{R} \tag{5.33}$$

Let $u \in X^\alpha$, then $u \in C(\mathbb{R}, \mathbb{R}^n)$, the space of continuous function $u \in \mathbb{R}$ such that $u(t) \to 0$ as $|t| \to +\infty$. Therefore there is a constant $R > 0$ such that $|t| \geq R$ implies $|u(t)| \leq \delta$. Hence, by (5.33), we have

$$\int_{\mathbb{R}} W(t, u(t)) \leq \int_{-R}^{R} W(t, u(t))dt + \epsilon \int_{|t|\geq R} |u(t)|^2 dt < +\infty. \tag{5.34}$$

Combining (5.30) and (5.34), we show that $I : X^\alpha \to \mathbb{R}$.
Now we prove that $I \in C^1(X^\alpha, \mathbb{R})$. Rewrite I as follows

$$I = I_1 - I_2,$$

where

$$I_1 = \frac{1}{2}\int_{\mathbb{R}} [|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t))]dt, \quad I_2 = \int_{\mathbb{R}} W(t, u(t))dt$$

It is easy to check that $I_1 \in C^1(X^\alpha, \mathbb{R})$ and

$$I_1'(u)v = \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty} D_t^\alpha v(t)) + (L(t)u(t), v(t))]\, dt. \tag{5.35}$$

Thus it is sufficient to show this is the case for I_2. In the process we will see that

$$I_2'(u)v = \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt, \tag{5.36}$$

which is defined for all $u, v \in X^\alpha$. For any given $u \in X^\alpha$, let us define $J(u) : X^\alpha \to \mathbb{R}$ as follows

$$J(u)v = \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt, \ \ \forall v \in X^\alpha.$$

It is obvious that $J(u)$ is linear. Now we show that $J(u)$ is bounded. Indeed, for any given $u \in X^\alpha$, by (5.29), there is a constant $d_3 > 0$ such that

$$|\nabla W(t, u(t))| \leq d_3|u(t)|,$$

which yields that, by the Hölder inequality and Lemma 5.4

$$\begin{aligned} |J(u)v| &= \left| \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt \right| \leq d_3 \int_{\mathbb{R}} |u(t)||v(t)|dt \\ &\leq \frac{d_3}{l_{min}} \|u\|_{X^\alpha} \|v\|_{X^\alpha}. \end{aligned} \tag{5.37}$$

Moreover, for u and $v \in X^\alpha$, by Mean Value Theorem, we have

$$\int_{\mathbb{R}} W(t, u(t) + v(t))dt - \int_{\mathbb{R}} W(t, u(t))dt = \int_{\mathbb{R}} (\nabla W(t, u(t) + h(t)v(t)))dt,$$

where $h(t) \in (0, 1)$. Therefore, by Lemma 5.5 and the Hölder inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}} (\nabla W(t, u(t) + h(t)v(t)), v(t))dt - \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt \\ &\quad = \int_{\mathbb{R}} (\nabla W(t, u(t)) + h(t)v(t) - \nabla W(t, u(t)), v(t))dt \to 0 \end{aligned} \tag{5.38}$$

as $v \to 0$ in X^α. Combining (5.37) and (5.38), we see that (5.36) holds. It remains to prove that I_2' is continuous. Suppose that $u \to u_0$ in X^α and note that

$$\begin{aligned} \sup_{\|v\|_{X^\alpha}=1} |I_2'(u)v - I_2'(u_0)v| &= \sup_{\|v\|_{X^\alpha}=1} \left| \int_{\mathbb{R}} (\nabla W(t, u(t)) - \nabla W(t, u_0(t)), v(t))dt \right| \\ &\leq \sup_{\|v\|_{X^\alpha}=1} \|\nabla W(., u(.)) - \nabla W(., u_0(.))\|_{L^2} \|v\|_{L^2} \\ &\leq \frac{1}{\sqrt{l_{\min}}} \|\nabla W(., u(.)) - \nabla W(., u_0(.))\|_{L^2} \end{aligned}$$

By lemma 5.5, we obtain that $I_2'(u)v - I_2'(u_0)v \to 0$ as $\|u\|_{X^\alpha} \to \|u_0\|_{X^\alpha}$ uniformly with respect to v, which implies the continuity of I_2' and $I \in C^1(X^\alpha, \mathbb{R})$. □

Lemma 5.10. *Under the conditions of* $(L) - (W_2)$, *I satisfies the (PS) condition.*

Proof. Assume that $(u_k)_{k\in\mathbb{N}} \in X^\alpha$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $C_1 > 0$ such that

$$|I(u_k)| \leq C_1, \ \ \|I'(u_k)\|_{(X^\alpha)^*} \leq C_1 \tag{5.39}$$

for every $k \in \mathbb{N}$.
We firstly prove that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in X^α. By (5.30), (5.32) and (W_1), we have

$$\begin{aligned} C_1 + \|u_k\|_{X^\alpha} &\geq I(u_k) - \frac{1}{\mu} I'(u_k)u_k \\ &= \left(\frac{\mu}{2} - 1\right)\|u_k\|^2_{X^\alpha} - \int_{\mathbb{R}} [W(t, u_k(t)) - \frac{1}{\mu}(\nabla W(t, u_k(t)), u_k(t))]dt \\ &\geq \left(\frac{\mu}{2} - 1\right)\|u_k\|^2_{X^\alpha}. \end{aligned} \tag{5.40}$$

Since $\mu > 2$, the inequality (5.40) shows that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in X^α. So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u$ in X^α and hence, by Lemma 5.5, $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^n)$. It follows from the definition of I that

$$\begin{aligned} &(I'(u_k) - I'(u))(u_k - u) \\ &= \|u_k - u\|^2_{X^\alpha} - \int_{\mathbb{R}} [\nabla W(t, u_k) - \nabla W(t, u)](u_k - u)dt. \end{aligned} \tag{5.41}$$

Since $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^n)$, we have (see lemma 5.8) $\nabla W(t, u_k(t)) \to \nabla W(t, u(t))$ in $L^2(\mathbb{R}, \mathbb{R}^n)$. Hence

$$\int_{\mathbb{R}} (\nabla W(t, u_k(t)) - \nabla W(t, u(t)), u_k(t) - u(t))dt \to 0$$

as $k \to +\infty$. So (5.41) implies

$$\|u_k - u\|_{X^\alpha} \to 0 \ \text{ as } \ k \to +\infty.$$

□
Now we are in the position to give the proof of Theorem 5.1. We divide the proof into several steps.
Proof of theorem 5.1.
Step 1. It is clear that $I(0) = 0$ and $I \in C^1(X^\alpha, \mathbb{R})$ satisfies the (PS) condition by Lemma 5.9 and 5.10.
Step 2. Now We show that there exist constant $\rho > 0$ and $\beta > 0$ such that I satisfies the condition (i) of Theorem 1.6. By Lemma 5.5, there is a $C_0 > 0$ such that

$$\|u\|_{L^2} \leq C_0\|u\|_{X^\alpha}.$$

On the other hand by Theorem 5.2, there is $C_\alpha > 0$ such that

$$\|u\|_\infty \leq C_\alpha\|u\|_{X^\alpha}.$$

By (5.28), for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$W(t, u(t)) \leq \epsilon|u(t)|^2 \ \text{ wherever } \ |u(t)| < \delta.$$

Let $\rho = \frac{\delta}{C_\alpha}$ and $\|u\|_{X^\alpha} \leq \rho$; we have $\|u\|_\infty \leq \frac{\delta}{C_\alpha}.C_\alpha = \delta$. Hence

$$|W(t, u(t))| \leq \epsilon|u(t)|^2 \ \text{ for all } t \in \mathbb{R}.$$

Integrating on $\mathbb{R}$, we get

$$\int_{\mathbb{R}} W(t,u(t))dt \le \epsilon\|u\|_{L^2}^2 \le \epsilon C_0^2\|u\|_{X^\alpha}^2$$

So, if $\|u\|_{X^\alpha} = \rho$, then

$$I(u) = \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t,u(t))dt \ge (\frac{1}{2} - \epsilon C_0^2)\|u\|_{X^\alpha}^2 = (\frac{1}{2} - \epsilon C_0^2)\rho^2.$$

And it suffices to choose $\epsilon = \frac{1}{4C_0^2}$ to get

$$I(u) \ge \frac{\rho^2}{4C_0^2} = \beta > 0 \tag{5.42}$$

Step 3. It remains to prove that there exists an $e \in X^\alpha$ such that $\|e\|_{X^\alpha} > \rho$ and $I(e) \le 0$, where ρ is defined in Step 2. Consider

$$I(\sigma u) = \frac{\sigma^2}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t,\sigma u(t))dt$$

for all $\sigma \in \mathbb{R}$. By (5.23), there is $c_1 > 0$ such that

$$W(t,u(t)) \ge c_1|u(t)|^\mu \text{ for all } |u(t)| \ge 1. \tag{5.43}$$

Take some $u \in X^\alpha$ such that $\|u\|_{X^\alpha} = 1$. Then there exists a subset Ω of positive measure of $\mathbb{R}$ such that $u(t) \ne 0$ for $t \in \Omega$. Take $\sigma > 0$ such that $\sigma|u(t)| \ge 1$ for $t \in \Omega$. Then by (5.43), we obtain

$$I(\sigma u) \le \frac{\sigma^2}{2} - c_1\sigma^\mu \int_\Omega |u(t)|^\mu dt. \tag{5.44}$$

Since $c_1 > 0$ and $\mu > 2$, (5.44) implies that $I(\sigma u) < 0$ for some $\sigma > 0$ with $\sigma|u(t)| \ge 1$ for $t \in \Omega$ and $\|\sigma u\|_{X^\alpha} > \rho$, where ρ is defined in Step 2. By Theorem 1.6, I possesses a critical value $c \ge \beta > 0$ given by

$$c = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} I(\gamma(s))$$

where

$$\Gamma = \{\gamma \in C([0,1],X^\alpha) : \ \gamma(0) = 0,\ \gamma(1) = e\}.$$

Hence there is $u \in X^\alpha$ such that

$$I(u) = c, \quad I'(u) = 0$$

□

Bibliography

[1] R. Agarwal, B. de Andrade and C. Cuevas, *"On type of periodicity and ergodicity to a class of fractional order differential equations"*, Adv. Difference Equ. ID 179750, 25 pages, 2010.

[2] R. Agarwal, M. Benchohra, and S. Hamani, *"Boundary value problems for fractional differential equations"*, Georg. Math. J. Vol. 16, No 3, 401-411, 2009.

[3] R. Agarwal, M. Belmekki and M. Benchohra, *"A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative"*, Adv. Difference Equ. Vol. 9, 47 pages, 2009.

[4] R. Agarwal, J. dos Santos and C. Cuevas, *"Analytic resolvent operator and existence results for fractional integro-differential equations"*, J. Abs. Diff. Equ. Vol. 2, No 2, 26-47, 2012.

[5] O. Agrawal, J. Tenreiro Machado and J. Sabatier, *"Fractional derivatives and their application: Nonlinear dynamics"*, Springer-Verlag, Berlin, 2004.

[6] F. Algrem and E. Lieb, *"Symmetric decreasing rearrangement is sometimes continuous"*, J. Amer. Math. Soc. **2**, 683-773(1989).

[7] A. Ambrosetti and P. Rabinowitz, *"Dual variational methods in critical points theory and applications"*, J. Func. Anal. **14**, 349-381(1973).

[8] V. Anh and R. Mcvinish, *"Fractional differential equations driven by Lévy noise"*, J. Appl. Math. and Stoch. Anal. Vol. 16, No 2, 97-119, 2003.

[9] T. Atanackovic and B. Stankovic, *"On a class of differential equations with left and right fractional derivatives"*, ZAMM., **87**, 537-539(2007).

[10] R. Blumenthal and R. Getoor, *"Some theorems on stable processes"*, Trans. Am. Math. Soc. **95**, 263-273(1960).

[11] Z. Bai and H. Lü, *"Positive solutions for boundary value problem of nonlinear fractional differential equation"*, J. Math. Anal. Appl.., Vol. 311, 495-505, 2005.

[12] D. Baleanu and J. Trujillo, *"On exact solutions of a class of fractional Euler-Lagrange equations"*, Nonlinear Dyn., **52**, 331-335(2008).

[13] B. Barrios, E. Colorado, A. de Pablo and U. Sánchez, *"On some critical problems for the fractional laplacian operator"* J. Differential Equations, **252**, No 11, 6133-6162 (2012).

[14] C. Brändle, E. Colorado, A. de Pablo and U. Sánchez, *"A concave-convex elliptic problem involving the fractional Laplacian"*, Proc. Roy. Soc. Edinburgh Sect. A, **143** No 1, 39-71 (2013).

[15] M. Benchohra, J. Henderson, S. Ntouyas, and A. Ouahab, *"Existence results for fractional order functional differential equations with infinite delay"*, J. of Math. Anal. Appl. Vol. 338, No 2, 1340-1350, 2008.

[16] D. Benson, S. Wheatcraft and M. Meerschaert, *"Application of a fractional advection-dispersion equation"*, Water Resour. Res. **36**, 1403-1412 (2000).

[17] D. Benson, S. Wheatcraft and M. Meerschaert, *"The fractional-order governing equation of Lévy motion"*, Water Resour. Res. **36**, 1413-1423 (2000).

[18] D. Benson, R. Schumer and M. Meerschaert, *"Fractional dispersion, Lévy motion, and the MADE tracer test"*, Transp. Porous Med. **42**, 211-240 (2001).

[19] H. Berestycki and P.-L. Lions, *"Nonlinear scalar field equations. I. Existence of a ground state"*, Arch. Rational Mech. Anal. **82**, No 4, 313-345(1983).

[20] L. Caffarelli and L. Silvestre, *"Regularity theory for fully nonlinear integro-differential equations"*, Comm. Pure App. Math. **62**, 597-638(2009).

[21] L. Caffarelli, J. Roquejoffre, and Y. Sire, *"Variational problems with free boundaries for the fractional Laplacian"*, Journal of the European Mathematical Society, **12** (5), 1151-1179 (2010).

[22] W. Chen, C. Li and B. Ou, *"Qualitative properties of solutions for an integral equation"*, Discrete and Continuous Dynamical Systems, **12** No 2, 347-354 (2005).

[23] W. Chen, C. Li and B. Ou, *"Classification of solutions for an integral equation"*, Comm. Pure Appl. Math., **59**, 330-343 (2006).

[24] M. Cheng, *"Bound state for the fractional Schrödinger equation with undounded potential"*, J. Math. Phys. **53**, 043507 (2012).

[25] R. Cont and P. Tankov, *"Financial modeling with jump processes"*, Chapman & Hall/CRC, 2004.

[26] D. Costa, *"An invitation to variational methods in differential equations"*, Birkhäuser, Boston, 2007.

[27] V. Coti Zelati and P. Rabinowitz, *"Homoclinic type solutions for a semilinear elliptic pde on $\mathbb{R}^n$"*, Comm. Pure Appl. Math., **45**, 1217-1269(1992).

[28] C. Cuevas, G. N'Guérékata and A. Sepulveda, *"Pseudo almost automorphic solutions to fractional differential and integro-differential equations"*, CAA. Vol.16, No 1, 131-152, 2012.

[29] E. Di Nezza, G. Patalluci and E. Valdinoci, *"Hitchhiker's guide to the fractional Sobolev spaces"*, Bull. Sci. Math., **136**, 5, 521-573(2012).

[30] S. Dipierro, G. Palatucci and E. Valdinoci, *"Existence and symmetry results for a Schroödinger type problem involving the fractional laplacian"*, Le Matematiche, **LXVIII**, 201-216 (2013).

[31] B. Dyda, *"On comparabiblity of integral forms"*, J. Math. Anal. Appli. **318**, 564-577(2006).

[32] J. Dong and M.Xu, *"Some solutions to the space fractional Schrödinger equation using momentum representation method"*, J. Math. Phys. **48**, 072105 (2007).

[33] A. El-Sayed, *"Fractional order evolution equations"*, J. Frac. Cal. Vol. 7, 89-100, 1995.

[34] V. Ervin and J. Roop, *"Variational formulation for the stationary fractional advection dispersion equation"*, Numer. Meth. Part. Diff. Eqs, **22**, 58-76(2006).

[35] M. Fall and E. Valdinoci, *"Uniqueness and nondegeneracy of positive solutions of $(-\Delta)^\alpha u + u = u^p$ in $\mathbb{R}^n$ when α is close to 1"*, arXiv:1301.4868v2 [math.AP] 14 Jul 2013.

[36] R. Frank and E. Lenzmann, *"Uniqueness of non-linear ground states for fractional Laplacians in $\mathbb{R}$"*, Acta Math., **210** No 2, 261-318 (2013).

[37] P. Felmer, A. Quaas and J. Tan *"Positive solutions of nonlinear Schrödinguer equation with the fractional laplacian"*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, **142**, No 6, 1237-1262(2012).

[38] P. Felmer and C. Torres, *"Non-linear Schrödinger equation with non-local regional diffusion"*, preprint.

[39] P. Felmer and C. Torres, *"Radial symmetry of ground state for a fractional Nonlinear Schroödinger Equation"*, preprint.

[40] Q-Y. Guan *"Integration by Parts Formula for Regional Fractional Laplacian."* Commun. Math. Phys. 266, 289-329 (2006).

[41] Q-Y. Guan, Z.M. Ma *"The reflected α-symmetric stable processes and regional fractional Laplacian."* Probab. Theory Relat. Fields 134(4), 649-694 (2006)

[42] X. Guo and M. Xu, *"Some physical applications of fractional Schrödinger equation"*, J. Math. Phys. **47**, 082104 (2006).

[43] H. Hajaiej and C. Stuart, *"Existence and non-existence of Schwarz symmetric ground states for elliptic eigenvalue problems"*, Ann. Mat. Pura Appl. **184**, 297-314 (2005).

[44] D. Haroske and H. Triebel *"Distributions, sobolev spaces, elliptic equations"*, European Mathematical Society, Switzerland, 2008.

[45] R. Hilfer, *"Applications of fractional calculus in physics"*, World Scientific, Singapore, 2000.

[46] H. Ishii and G. Nakamura, *"A class of integral equations and approximation of p-Laplace equations"*, Calc. Var. **37**, 485-522(2010).

[47] L. Jeanjean, *“On the existence of bounded Palais-Smale sequence and application to a Ladesman.Lazer-type problem set on $\mathbb{R}^n$”*, Proc. Roy. Soc. Edinburgh Sect. A, **129**, 787-809(1999).

[48] W. Jang, *“The existence of solutions for boundary value problems of fractional differential equations at resonance”*, Nonlinear Anal. Vol. 74, 1987-1994, 2011.

[49] F. Jiao and Y. Zhou, *“Existence results for fractional boundary value problem via critical point theory”*, Intern. Journal of Bif. and Chaos, **22**, No 4, 1-17(2012).

[50] B. Kawohl, *“Rearrangements and convexity of level sets in PDE”* Lect. Notes Math. 1150, Springer-Verlag, Berlin (1985)

[51] S. Kesavan, *“Symmetrization and applications”* World Scientific, Hackensack, NJ (2006).

[52] S. Kesavan, *“Topics in functional analysis and applications”*, Wiley Eastern Limited, 1989.

[53] A. Kilbas, H. Srivastava and J. Trujillo, *“Theory and applications of fractional differential equations”*, North-Holland Mathematics Studies, vol 204, Amsterdam, 2006.

[54] M. Klimek, *“Existence and uniqueness result for a certain equation of motion in fractional mechanics”*, Bull. Polish Acad. Sci. Tech. Sci., **58**, No 4, 573-581(2010).

[55] N. Laskin, *“Fractional quantum mechanics and Lévy path integrals”*, Phys. Lett. A **268**, 298 - 305 (2000).

[56] N. Laskin, *“Fractional Schrödinger equation”*. Phys. Rev. E **66**, 056108 (2002).

[57] V. Lakshmikantham, *“Theory of fractional functional differential equations”*, Non. Anal. Vol. 69, 3337-3343, 2008.

[58] V. Lakshmikantham and A. Vatsala, *“Basic theory of fractional differential equations”*, Non. Anal. Vol. 69, No 8, 2677-2682, 2008.

[59] E. Lieb and M. Loss, *“Analysis”* Grad. Stud. Math., vol. **14**, Amer. Math. Soc., Providence, RI (2001).

[60] Y. Li, *“Remark on some conformally invariant integral equations: the method fo moving spheres”*, J. Eur. Math. Soc., **6**, 153-180 (2004).

[61] L. Ma and D. Chen, *“Radial symmetry and monotonicity for an integral equation”*, J. Math. Anal. Appl., **342**, 943-949 (2008).

[62] J. Mawhin and M. Willen, *“Critical point theory and Hamiltonian systems”*, Applied Mathematical Sciences 74, Springer, Berlin, 1989.

[63] W. McLean, *“Strongly elliptic systems and boundary integral equations ”*, Cambridge University Press, 2000.

[64] R. Metsler and J. Klafter, *“The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics”*, J. Phys. A, **37**, 161-208(2004).

[65] K. Miller and B. Ross, *"An introduction to the fractional calculus and fractional differential equations"*, Wiley and Sons, New York, 1993.

[66] G. N'Guérékata, *"A Cauchy problem for some fractional abstract differential equation with nonlocal conditions"*, Non. Anal. Vol. 70, 1873-1876, 2009.

[67] E. de Oliveira, F. Costa, and J. Vaz, *"The fractional Schrödinger equation for delta potentials"*, J. Math. Phys. **51**, 123517 (2010).

[68] W. Omana and M. Willem *"Homoclinic orbits for a class of Hamiltonian systems"*, Differ. Integr. Equ., **5**, No 5, 1115-1120(1992).

[69] Y. Park, *"Fractional Polya-Zsegö inequality"*, J. Chungcheong Math. Soc. **24**, No 2, 267-271(2011).

[70] I. Podlubny, *"Fractional differential equations"*, Academic Press, New York, 1999.

[71] P. Rabinowitz, *"On a class of nonlinear Schrödinguer equations"*, ZAMP, **43**, 270-291(1992).

[72] P. Rabinowitz and K. Tanaka *"Some result on connecting orbits for a class of Hamiltonian systems"*, Math. Z., **206**, 473-499(1991).

[73] P. Rabinowitz, *"Minimax method in critical point theory with applications to differential equations"*, CBMS Amer. Math. Soc., No **65**, 1986.

[74] J. Sabatier, O. Agrawal and J. Tenreiro Machado, *"Advances in fractional calculus. Theoretical developments and applications in physics and engineering"*, Springer-Verlag, Berlin, 2007.

[75] S. Samko, A. Kilbas and O. Marichev *"Fractional integrals and derivatives: Theory and applications"*, Gordon and Breach, New York, 1993.

[76] S. Secchi, *"Ground state solutions for nonlinear fractional Schrödinger equation in $\mathbb{R}^n$"*, J. Math. Phys., **54**, 031501(2013).

[77] S. Secchi, *"On fractional Schrödinger equation in $\mathbb{R}^n$ without the Ambrosetti-Rabinowitz condition"*, preprint. arXiv:1210.0755v1 [math.AP] 2 Oct 2012.

[78] R. Servadei and E. Valdinoci, *"Mountain Pass solutions for non-local elliptic operators"* J. Math. Anal. Appl. **389**, 887-898 (2012).

[79] B. Simon, *"Convexity: An analytic viewpoint"*. Cambridge Tracts in Math. 187, Cambridge University Press, (2011).

[80] C. Stuart, *"Bifurcation into spectral gaps"*, Société Mathématique de Belgique, 1995.

[81] J. Tan, *"The Brezis-Nirenberg type problem involving the square root of the Laplacian"*, Calc. Var. Partial Differential Equations **42**, No 1, 21-41(2012).

[82] C. Torres, *"Existence of solutions for fractional Hamiltonian systems"*, Electronic J. Diff. Eq. **2013**, No 259, 1-12(2013).

[83] E. Valdinoci, *"From the long jump random walk to the fractional Laplacian"*, Boletín de la Sociedad Española de Matemática Aplicada, **49**, 33-44 (2009).

[84] J. Van Schaftingen, *"Symmetrization and minimax principle"*, Comm. Contemporary Math., **7**, No 4, 463-481(2005).

[85] X. Wang, *"On concentration of positive bound states of nonlinear Schrödinger equations"*, Commun. Math. Phys. **153**, 229-244(1993).

[86] B. West, M. Bologna and P. Grigolini, *"Physics of fractal operators"*, Springer-Verlag, Berlin, 2003.

[87] J. Wloka, *"Partial differential equations"*, Cambridge University Press, 1987.

[88] S. Zhang, *"Existence of a solution for the fractional differential equation with nonlinear boundary conditions"*, Comput. Math. Appl. Vol. 61, 1202-1208, 2011.