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STUDY OF DIFFERENT MODELS OF THE EVOLUTION AND MOTION OF CELL POPULATIONS

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA MENCIÓN MODELACIÓN MATEMÁTICA EN COTUTELA CON LA UNIVERSITÉ PIERRE ET MARIE CURIE

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RESUMEN DE LA TESIS PARA OPTAR AL GRADO: DOCTORADO EN CIENCIAS DE LA INGENIERÍA MENCIÓN MODELACIÓN MATEMÁTICA
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En el presente trabajo hemos estudiado dos modelos de Ecuaciones diferenciales parciales diferentes aplicados a la biomatemática.

En el primero consideramos un sistema de ecuaciones parabólicas para modelar la quimiotaxis positiva de dos poblaciones unicelulares, las cuales secretan un mismo quimio-atractante. Usando el método de los momentos y un funcional de energía, logramos dar las condiciones óptimas sobre las masas iniciales para la existencia global en tiempo y blow-up de soluciones del sistema.

El segundo modelo está en el marco de la Teoría de las dinámicas adaptativas, la cual modela a diferentes escalas la evolución fenotípica de poblaciones celulares. Hemos consideramos una ecuación de Transporte, para modelar la evolución genética en el tiempo de una población celular, en la cual existe una subpoblación resistente a las condiciones ambientales. Introduciendo un parámetro pequeño y usando una ecuación auxiliar, hemos logrado demostrar que el comportamiento asintótico de las soluciones de la ecuación de Transporte corresponde a una masa de Dirac parametrizada en una función Lipschitz continua.

Hemos usado conceptos clásicos de la teoría de EDP para conseguir estos resultados, los cuales son: Funcional de Energía, Desigualdad de Hardy-Littlewood- Sobolev, Principio del Máximo, Subsolución y Supersolución.

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Chapter 1

Modeling the phenotype evolution of cell populations

1.1 Introduction

In this part, we study the evolution in time of some cell population characterized by a trait using a Transport Equation, we describe and predict the phenotype evolution in time of an organisms group differentiate by genetic trait. Mathematical equations that model phenotype structured populations are considered part of Theory of Adaptive Dynamics.

The mathematical ecology was born from the need to prove different hypothesis proposed in Biology over populations, which were confirmed using mathematical models, in particular differential equations. Differential equations, model a variety of ecological phenomena, for example population growth, species interactions, among others. To readers more interested in this type of models and their history, we invite them to see [6, 42, 45] and the reference therein.

Theory of Adaptive Dynamics appeared in the 70s and a first reference about mathematical models in this subject has been supplied by Maynard and Price in [41], but the use of Adaptive Dynamics as expression was introduced by Hofbauer-Sigmund in them book *Evolutionary Games and Population Dynamics*. The paper [41] is very important, as provides the basis to analyze mathematical models from Adaptive Dynamics point of view, because made the connection between biology and mathematical results. The formal definition of this theory is given in a large number of papers and books, for example [10, 11, 41, 45] among others, but in my opinion the easier to understand was exposed in [53] and it is defined as *"a theoretical approach for studying some of the phenotype changes that take place, over time, in evolving populations"*, this describe which is the main objective of this type of models. The emergence of this theory comes from the need to model the evolution in time of species having distinct genetic traits that allow them to survive and be maintained over time in certain environmental conditions. These ideas expand The Evolution Theory, which was proposed by Charles Darwin in 1838. Saying that expands the ideas of Darwin, in the sense that the integration of mathematical models provides a fresh look at Natural Selection Principle, since mathematics can anticipate future populations behaviors under certain environmental conditions, without being upon seeing what happens, giving formal experiments to this theory. To work in Adaptive Dynamics, we must direct our efforts in a main purpose to prove that after several reproduction cycles, total population should present the genetic or physiological structure of the stronger individuals that are survivors. The specie adaptation occurs when environmental conditions change radically and to survive, the individuals are forced to modify their conduct, as a consequence of this natural process, take place variations in their physiological and genetic structure. To model selection principle effect over population, we consider a mathematical ecological model to individuals reproduction and one introduce a new variable x to represent the phenotype diversity present in this community. To read more about Adaptive Dynamics we advise to interested readers to see [4, 5, 10, 11, 17, 19, 39, 41, 45, 53] and the reference therein.

We take into account the following basics biological hypothesis:

1. **Heritability principle**: Parents give their traits to new individuals.
2. **Selection principle**: Reproduction favors traits that are fittest to environmental conditions.
3. **Mutations o Variation principle**: New individuals can have different traits that their parents.

The combination of this three theoretical hypothesis is studied through Theory of Adaptive Dynamics, however in this work we only consider the first two.

The motivation to do this work, comes from to model a cell population for which some drug therapy was applied and exists a sub-population where the treatment did not have an effect. In medicine, these phenomena is called resistance. To obtain a model in Theory of Adaptive Dynamics, we should explain these circumstances assuming that there exists a sub-population, which presents a phenotype best adapted to environmental conditions, in our case nutrients and drug therapy. This resistant sub-population is selected when a drug targets a specific gene (Darwinian process). There are different biological theories to explain this phenomenon in a molecular level, to see more about this topic read [27, 39, 42] and the reference therein.

The resistance phenomenon is very common in different therapeutic process, for example in cancer, antiviral, antimalarial, antibacterial, poison applications (rats), pesticide, among others. In particular, in Chile, there are studies about this phenomenon, it has been found in antibacterial, HIV therapies, among others and the studies show different causes, to read more about this problem in Chile see [1] and the reference therein. Therefore, is very common meet people where a medical treatment did not produce the expected effect, for this the resistance phenomenon has attracted the attention of medical doctors and human biologists, which have gathered observations on the mechanisms explaining it.

Broadly speaking, we can explain this phenomenon considering that cell reproduction is very fast and the common drugs applied can not select which type of individuals should inhibit. Thus, based in The Evolution Theory, there is a high probability that survivors become immune to chemical after a several reproductions times, proliferating in mass, which produce a malignant and fatal cancer. When exists several types of cells differentiate by a phenotype, there are more possibilities to find a sub-populations resistant to drugs and these stronger cells reproduce uncontrollably occupying the space that others have vacated developing a tumor. In [27], have been proposed two mechanisms at which the cells can become cancerous. The first involves changes in your fitness function losing your reproduction control, this is caused by the presence of phenotype resistant to environmental conditions, as well as, the cells adapt and their proliferation is favored. The second case, is the existence of a favorable environmental conditions where the cells proliferate and accumulate a sufficient number of individuals producing mutations. In this work, we only admit the first instance in our model.

To obtain the renewal equation for this cell population, we need to define a suitable Fitness function, since being the main ingredient to describe the behavior and evolution of some individuals, to read more about this see [53].

The fitness function has different definitions in Population dynamics, these depend on type of organism that we model, but in general by its uses, we can infer that it gives a panoramic view of how the population breeds, feeds, dies and changes in several aspects. It is why, the Fitness in our equation should consider different variables that influence on cells evolution, thus we include in this function the trait variable x representing that proliferation is favoring the best adapted trait and to model environmental influence we incorporate a new term ρ. To defined ρ, we need to consider that the total mass is one of the main factors that determine the environment in which the cells live, therefore ρ should depend on total individuals mass in a period of time.

We can anticipate that in an extended period of time the population will be concentrated in the best adapted gene, thus we must obtain a Dirac mass concentrated in a singular attribute as solution representing the selection principle in this cell community.

To model the behavior of our cell population, we must consider the following biological hypothesis:

1. Asexual reproduction process.
2. Offspring have the same trait that their parents.
3. Only one-dimension strategies are considered.

In resume, we consider that organism live in the same environmental conditions for some fixed period of time where they compete for nutrients among themselves. In this first approximation, we consider that offspring has the same trait as their parents, we also define a fitness function that is trait and density dependent. In addition, we assumed there exists a unique trait best adapted at fixed environmental conditions. This first approach will give us much information about the behavior of the population and checking that the theory of evolution also applies on this scale.

We start studying in Section 1.2 a ODE to model evolution in time of a cell population characterized by a trait. Consider the follow equation

$$\left.\begin{array}{c} \frac{\partial}{\partial t} n\left(x,t\right) = \left[\frac{r(x)}{(1+\rho(t))^{\alpha}} - d - c\eta\left(x\right)\right] n\left(x,t\right) \ , \\ \\ n\left(0,x\right) = n^{0}\left(x\right) > 0 \ , \end{array}\right\} \tag{1.1}$$

where $t \geq 0$, $x \in [0,1]$, $\alpha, c, d > 0$, $n_0 \in L^1[0,1]$ and define

$$\rho\left(t\right) = \int_0^1 n\left(x,t\right) dx \ . \tag{1.2}$$

The biological meaning of the variables are:

- $n(x,t)$ cell density with trait x in t.
- $\frac{r(x)}{(1+\rho(t))^{\alpha}}$ fitness function dependent on total density.
- d average mortality rate.
- $c\eta(x)$ mortality caused by the medication to individuals with gene x.

We will prove that there is a unique positive solution of (1.1) using a fixed point method. We define the solution as the pair (ρ, n), because there is an intrinsic relation between them.
In Section 1.3, we continue adding another cell population and study the following system

$$\left.\begin{array}{c} \frac{d}{dt} n_h\left(x,t\right) = \left[\frac{r_h(x)}{(1+\rho_1(t))^{\alpha}} - d_h - c\eta_h\left(x\right)\right] n_h\left(x,t\right) \ , \\ \\ \frac{d}{dt} n_c\left(x,t\right) = \left[\frac{r_c(x)}{(1+\rho_2(t))^{\alpha}} - d_c - c\eta_c\left(x\right)\right] n_c\left(x,t\right) \ , \\ \\ n_h\left(0,x\right) = n_h^{0}\left(x\right) > 0 \ , n_c\left(0,x\right) = n_c^{0}\left(x\right) > 0 \ , \end{array}\right\} \tag{1.3}$$

where $t \geq 0$, $x \in [0,1]$, $\alpha, c, d > 0$, $n_h^0, n_c^0 \in L^1[0,1]$, n_h represent the healthy cells, in short h-cells, analogously, n_c are cancer cells, abbreviate c-cells. The biological meaning are:

- $n_h(x,t)$ h-cells density with trait x in t.
- $n_c(x,t)$ c-cells density with trait x in t.
- $\frac{r_h(x)}{(1+\rho_1(t))^{\alpha}}$ fitness function to h-cells density dependent.
- $\frac{r_c(x)}{(1+\rho_2(t))^{\alpha}}$ fitness function to c-cells density dependent.
- d_h average mortality rate to h-cells.

- d_c average mortality rate to c-cells.
- $c\eta_h$ specific mortality for h-cells with gene x.
- $c\eta_c$ specific mortality for c-cells with gene x.

We define for $i = 1, 2$,

$$\rho_i(t) = \int_0^1 [n_h(x,t)\,\Psi_{ih}(x,t) + n_c(x,t)\,\Psi_{ic}(x,t)]\,dx\,, \tag{1.4}$$

and

$$M_h(t) = \int_0^1 n_h(x,t)\,dx\,,\; M_c(t) = \int_0^1 n_c(x,t)\,dx\,. \tag{1.5}$$

Using the same arguments as in Section 1.2, we will prove there is a unique solution to (9).

In Section 1.4, we study the Transport equation

$$\left.\begin{aligned} n_t(x,t) - [g(x)n(x,t)]_x = R(x,\rho(t))n(x,t)\,, \\ \rho(t) = \textstyle\int_0^{+\infty} n(x,t)dx\,, \\ n(0,x) = n^0(x) > 0\,, \end{aligned}\right\} \tag{1.6}$$

where $t > 0$, $x \in [0, +\infty)$, $n^0 \in L^1(\mathbb{R}^+)$, $g(0) = 0$. The biological meaning of the variable are:

- x cell trait representing drug resistance.
- $n(x,t)$ cell density in time $t > 0$ with trait x.
- $\rho(t)$ total cell density in time t.
- g genetic velocity of change.

We are interesting in solutions to (1.21) such that

$$\lim_{x\to\infty} n(.,x) = 0\,, \tag{1.7}$$

integrating (1.38), we obtain the follow ODE for $\rho(t)$

$$\frac{d\rho(t)}{dt} = \int_0^{+\infty} n(x,t)R(x,\rho(t))dt\,. \tag{1.8}$$

We study the equation (1.21) to model the genetic movement when there is a velocity of variation for each trait. In Section 1.4, we focus primarily in estimate the necessary conditions to describe the asymptotic behavior for the solutions of (1.6). Based in the classical methods apply in PDE models, we will include a small ϵ parameter, which perturb our initial transport equation. This change of variables will be biologically justifies and gives important information about the cell population modeled. The classical examples about study the asymptotic behavior are showed in [22, 45] and the reference therein. Following various works for example [4, 5, 10, 11, 17, 39, 40], among others, our purpose is to prove that

$$n_\epsilon(x,t) \rightharpoonup \overline{\rho}(t)\delta_0(x-\overline{x}(t)) \,,$$

as $\epsilon \to 0$.

The method used consisted in applying the change of variables or WKB ansatz

$$n_\epsilon(x,t) = e^{\frac{u_\epsilon(x,t)}{\epsilon}} \,,$$

assuming that solutions are similar to a Gaussian function and it is simpler describes the properties of u_ϵ. We will prove that $(u_\epsilon)_\epsilon$ is a decreasing (in linear order), equilipschitz sequence, then by Ascoli' Theorem, under a subsequence its limit exits. In addition, the limit of (u_ϵ) is decreasing (in linear order), concave and nonpositive function, which allow us to elucidate the limit of n_ϵ, all technical assumptions will be presented in Section 1.4.1.

In this theory, we find several different models, for example, with a diffusive and integral terms [4, 5, 10, 11, 17, 39, 40] to model competition and populations mutations. In our cases, we start modeling a simplest case and propose in future works incorporate more terms.

Among the future work that may arise in this model, we have the following:

1. What are the necessary conditions to describe the long-time behavior of asymptotic solutions of (1.38)?
2. Consider the equation (1.21) under the mutation hypothesis, incorporating a diffusion term $\Delta n(x,t)$ or the probability kernel $K(x,y)$ in equation (1.21). The diffusion in x represents the genetic orderly movement, however the kernel models the aleatory mutation of cells with trait x to y.
3. We can consider the competition between two populations, as we make in Sections 1.2-1.3, but including a transport term. In this case we need change a fitness function to one population, follow the paper [39].
4. Another possibility, it is investigating about the Control problem. Considering the same transport equation present in (1.21) including in the fitness function two controls, that represent two different drugs. We expect to obtain a populations concentrate in two or more traits best adapted to produce the branching in the evolution of this cell population.

1.2 Cell growth model

Consider the differential equation

$$\left.\begin{array}{c}\frac{\partial}{\partial t}n\left(x,t\right)=\left[\frac{r(x)}{(1+\rho(t))^{\alpha}}-d-c\eta\left(x\right)\right]n\left(x,t\right)\ ,\\ n\left(0,x\right)=n^{0}\left(x\right)>0\ ,\end{array}\right\}\tag{1.9}$$

where $t\geq 0$, $x\in[0,1]$, $\alpha,c,d>0$, $n_0\in L^1[0,1]$ and

$$\rho\left(t\right)=\int_0^1 n\left(x,t\right)dx\ .\tag{1.10}$$

Assume that,

$$0\leq\rho\left(0\right)\leq\rho_M\ ,\tag{1.11}$$

with ρ_M holds

$$\sup_{x\in[0,1]}\left[\frac{r(x)}{(1+\rho_M)^{\alpha}}-d-c\eta(x)\right]=0\ .\tag{1.12}$$

We also suppose,

$$r\left(x\right)\geq 0\ ,\ r'\left(x\right)<0\ ,\ \eta\left(x\right)\geq 0\ ,\ \eta'\left(x\right)<0\ .\tag{1.13}$$

The solution to (2.98) is given by the representation formula

$$n\left(x,t\right)=n^0\left(x\right)\exp\int_0^t\left[\frac{r\left(x\right)}{\left(1+\rho\left(s\right)\right)^{\alpha}}-d-c\eta\left(x\right)\right]ds\ ,\tag{1.14}$$

for all $(x,t)\in\mathbf{R}^+\times[0,1]$. Therefore, for all $(x,t)\in\mathbf{R}^+\times[0,1]$

$$n\left(x,t\right)>0\ .$$

1.2.1 A priori estimates

Lemma 1 *A solution to* (2.98) *satisfies,*

$$0\leq\rho\left(t\right)\leq\rho_M\tag{1.15}$$

for all $t\geq 0$.

Proof. Integrating (2.98) we obtain

$$\frac{d}{dt}\rho\left(t\right)=\frac{\int_0^1 r\left(x\right)n\left(x,t\right)dx}{\left(1+\rho\left(t\right)\right)^{\alpha}}-d\rho\left(t\right)-c\int_0^1\eta\left(x\right)n\left(x,t\right)dx\ ,$$

then

$$\frac{d}{dt}\rho(t) \leq \sup_{x\in[0,1]} \left[\frac{r(x)}{(1+\rho(t))^{\alpha}} - d - c\eta(x)\right]\rho(t) . \tag{1.16}$$

Assume that there exists t_0 such that

$$\rho(t_0) > \rho_M \tag{1.17}$$

and by condition (1.12)

$$\sup_{x\in[0,1]} \left[\frac{r(x)}{(1+\rho(t_0))^{\alpha}} - d - c\eta(x)\right] < 0 ,$$

thus

$$\frac{d}{dt}\rho(t_0) < 0 .$$

In others words, for all $t > 0$ such that $\rho(t) > \rho_M$ then

$$\frac{d}{dt}\rho(t) < 0 ,$$

which is a contradiction. Therefore, for all $t > 0$ we have that $\rho(t) < \rho_M$.
On the other hand, by representation formula given in (1.14) we have that $\rho(t) \geq 0$. ■

1.2.2 Existence and uniqueness of solution

Consider the Banach space

$$E = C[0,T]$$

and the closed set

$$B = \left\{ u \in E : u(0) = \int_0^1 n_0(x)\,dx, 0 \leq u(t) \leq M_T \right\} , \tag{1.18}$$

for some T fixed we consider

$$M_T = \rho_M \max_{t\in[0,T]} \left\{ e^{[r(0)-d-c\eta(1)]t} \right\} .$$

Now, we define the operator $F : B \to B$

$$F(u)(t) = \int_0^1 n_0(x) \exp\left[\int_0^t \left(\frac{r(x)}{(1+u(s))^{\alpha}} - d - c\eta(x)\right) ds\right] dx . \tag{1.19}$$

Then we have the following result,

Lemma 2 *Under the assumptions* (1.10)*, F satisfies,*

1. $F(B) \subseteq B$.

2. *The contraction principle.*

Proof. First, by (1.11) and (1.19) we have

$$F(u)(t) < \int_0^1 n_0(x)\, e^{[r(0)-d-c\eta(1)]t} dx \leq \rho_M \max_{t\in[0,T]} \left\{ e^{[r(0)-d-c\eta(1)]t} \right\} = M_T \, .$$

On the other hand, we obtain

$$F(u)(t) > \int_0^1 n_0(x)\, e^{-[d+c\eta(0)]t} dx \geq 0 \, .$$

Therefore, 1 makes true.
Next, we prove the contraction property. Given u_1, $u_2 \in B$ then

$$\begin{aligned} & |F(u_1)(t) - F(u_2)(t)| \\ \leq & \int_0^1 n_0(x)\, e^{-[d+c\eta(1)]t} \left| \exp \int_0^t \frac{r(x)}{(1+u_1(s))^\alpha} ds - \exp \int_0^t \frac{r(x)}{(1+u_2(s))^\alpha} ds \right| dx \\ \leq & \int_0^1 n_0(x) \left| \exp \int_0^t \frac{r(x)}{(1+u_1(s))^\alpha} ds - \exp \int_0^t \frac{r(x)}{(1+u_2(s))^\alpha} ds \right| dx \, . \end{aligned}$$

We know that exponential function is lipschitz in bounded domains, therefore

$$\left| \exp \int_0^t \frac{r(x)}{(1+u_1(s))^\alpha} ds - \exp \int_0^t \frac{r(x)}{(1+u_2(s))^\alpha} ds \right| \leq e^T \int_0^t \left| \frac{r(x)}{(1+u_1(s))^\alpha} - \frac{r(x)}{(1+u_2(s))^\alpha} \right| ds \, .$$

The function $\frac{1}{(1+u(s))^\alpha}$ is lipschitz continuous too, thus

$$\int_0^t \left| \frac{1}{(1+u_1(s))^\alpha} - \frac{1}{(1+u_2(s))^\alpha} \right| ds \leq TA \sup_{t\in[0,T]} |u_1 - u_2| \, .$$

Hence, we compute

$$\sup_{t\in[0,T]} |F(u_1) - F(u_2)| \leq r(0)\, e^T \rho_M T A \sup_{t\in[0,T]} |u_1(t) - u_2(t)| \, ,$$

then we can choose T such that

$$r(0)\, e^T \rho_M T A < 1 \, . \tag{1.20}$$

■

Theorem 3 *There exists an unique solution $n \in C\left(\mathbf{R}^{+}, L^{1}[0,1]\right)$, for which* (2.98) *and* (1.12) *makes true.*

Proof. Using Lemma 2 and Banach-Picard fixed point theorem, there is a fixed point of F on $[0,T]$. Additionally, we can iterate with the initial condition $\rho(T)$, since the choice of T is independent of the initial condition, therefore we conclude there exists a unique fixed point such that

$$\rho(t) = \int_0^1 n_0(x) \exp\left[\int_0^t \left(\frac{r(x)}{(1+\rho(t))^{\alpha}} - d - c\eta(x)\right) ds\right] dx\,,$$

which is the solution (2.98) for all $t > 0$. ■

1.3 Cell growth model for two species

In this section, we consider the following system

$$\left.\begin{array}{l} \frac{d}{dt} n_h(x,t) = \left[\frac{r_h(x)}{(1+\rho_1(t))^{\alpha}} - d_h - c\eta_h(x)\right] n_h(x,t)\,, \\ \frac{d}{dt} n_c(x,t) = \left[\frac{r_c(x)}{(1+\rho_2(t))^{\alpha}} - d_c - c\eta_c(x)\right] n_c(x,t)\,, \end{array}\right\} \quad (1.21)$$

$$n_h(0,x) = n_h^0(x) > 0\,,\, n_c(0,x) = n_c^0(x) > 0\,, \quad (1.22)$$

where n_h is h-cells density, n_c is c-cells density and $n_h^0(x)\,,\, n_c^0(x) \in L^1([0,1])$.
We also define for $i = 1, 2$,

$$\rho_i(t) = \int_0^1 \left[n_h(x,t)\,\Psi_{ih}(x,t) + n_c(x,t)\,\Psi_{ic}(x,t)\right] dx \quad (1.23)$$

and

$$M_h(t) = \int_0^1 n_h(x,t)\, dx\,,\, M_c(t) = \int_0^1 n_c(x,t)\, dx\,. \quad (1.24)$$

Furthermore, we assume that

$$0 \le M_h(0) < \frac{\rho_M^1}{\Psi_m}\,,\, 0 \le M_c(0) < \frac{\rho_M^2}{\Psi_m}\,, \quad (1.25)$$

where

$$\sup_{x\in[0,1]}\left[\frac{r_h(x)}{(1+\rho_M^1)^{\alpha}} - d_h - \eta_h(x)\right] = 0\,,\ \sup_{x\in[0,1]}\left[\frac{r_c(x)}{(1+\rho_M^2)^{\alpha}} - d_c - \eta_c(x)\right] = 0\,, \quad (1.26)$$

we define for $i = 1, 2$

$$0 < \Psi_m \le \Psi_{ih}\,,\, \Psi_{ic} \le \Psi_M\,. \quad (1.27)$$

We also suppose that

$$r_h(x) \geq 0 \, , \, r'_h(x) < 0 \, , \, r_c(x) \geq 0 \, , \, r'_c(x) < 0 \, , \, r_c(x) > r_h(x) \, , \tag{1.28}$$

$$\eta_h(x) \geq 0 \, , \, \eta'_h(x) < 0 \, , \, \eta_c(x) \geq 0 \, , \, \eta'_c(x) < 0 \, , \, \eta_c(0) > \eta_h(0) \, , \, \eta_c(1) < \eta_h(1) \, . \tag{1.29}$$

The solutions to (1.21) are given by the representation formula

$$n_h(x,t) = n_h^0(x) \exp\left[\int_0^t \left[\frac{r_h(x)}{(1+\rho_1(s))^\alpha} - d_h - c\eta_h(x)\right] ds\right] \, , \tag{1.30}$$

$$n_c(x,t) = n_c^0(x) \exp\left[\int_0^t \left[\frac{r_c(x)}{(1+\rho_2(s))^\alpha} - d_c - c\eta_c(x)\right] ds\right] \, , \tag{1.31}$$

therefore for all $(x,t) \in \mathbb{R}^+ \times [0,1]$

$$n_h(x,t) > 0 \, , \, n_c(x,t) > 0 \, .$$

1.3.1 A priori estimates

Lemma 4 *The solution to* (1.21) *satisfies,*

$$0 \leq M_h(t) < \frac{\rho_M^1}{\Psi_m} \, , \, 0 \leq M_c(t) < \frac{\rho_M^2}{\Psi_m} \, . \tag{1.32}$$

Proof. Supposing that there exists t_0 such that

$$M_h(t_0) > \frac{\rho_M^1}{\Psi_m} \, ,$$

from (1.23) we have

$$\rho_1(t_0) > \rho_M^1 \, .$$

Integrating (1.21) at the point t_0, we obtain

$$\frac{d}{dt} M_h(t_0) = \int_0^1 \left[\frac{r_h(x)}{(1+\rho_1(t_0))^\alpha} - d_h - c\eta_h(x)\right] n_h(x,t_0) \, dx \, ,$$

then

$$\frac{d}{dt} M_h(t_0) \leq \sup_{x\in[0,1]} \left[\frac{r_h(x)}{(1+\rho_1(t_0))^\alpha} - d_h - c\eta_h(x)\right] M_h(t_0) \tag{1.33}$$

and thus by (1.26) and (1.33), we can conclude

$$\frac{d}{dt} M_h(t_0) < 0 \, .$$

Consequently, if $M_h(t) > \frac{\rho_M^1}{\Psi_m}$ we conclude that

$$\frac{d}{dt}M_h(t) < 0 \,,$$

it is not possible. Hence we get

$$M_h(t) \leq \frac{\rho_M^1}{\Psi_m} \,.$$

Using an analog argument, we have

$$M_c(t) \leq \frac{\rho_M^2}{\Psi_m} \,.$$

On the other hand, by (1.30) and (1.31)

$$M_h(t) \geq 0 \,,\, M_c(t) \geq 0.$$

■

Lemma 5 *We have that ρ_i holds*

$$0 \leq \rho_i\,(t) < \frac{\Psi_M}{\Psi_m}\left[\rho_M^1 + \rho_M^2\right] \,, \tag{1.34}$$

for $i = 1, 2$.

Proof. By (1.23) , (1.29)

$$\Psi_m\left[M_h\,(t) + M_c\,(t)\right] < \rho_i\,(t) < \Psi_M\left[M_h\,(t) + M_c\,(t)\right]$$

and applying Lemma 4 we obtain the result. ■

1.3.2 Existence and uniqueness of solution

We define

$$E = C\left[0, T\right] \,,$$

for some T fixed and

$$\begin{aligned} B_1 &= \left\{\rho \in E : \rho\,(0) = \int_0^1 \left[n_h^0\,(x)\,\Psi_{1h}^0\,(x) + n_c^0\,(x)\,\Psi_{1c}^0\,(x)\right] dx \,,\, 0 \leq \rho \leq M_T\right\}, \\ B_2 &= \left\{\rho \in E : \rho\,(0) = \int_0^1 \left[n_h^0\,(x)\,\Psi_{2h}^0\,(x) + n_c^0\,(x)\,\Psi_{2c}^0\,(x)\right] dx \,,\, 0 \leq \rho \leq M_T\right\}, \end{aligned}$$

where

$$M_T = K_T \Psi_M\left[\rho_M^1 + \rho_M^2\right]$$

and

$$K_T = \max\left\{\max_{t\in[0,T]}\left[\exp\left(r_h\left(0\right) - d_h - c\eta_h\left(1\right)\right)t\right], \max_{t\in[0,T]}\left[\exp\left(r_c\left(0\right) - d_c - c\eta_c\left(1\right)\right)t\right]\right\}$$

for $i = 1, 2$. Furthermore, consider the operator

$$\begin{array}{rccc} F: & B_1 \times B_2 & \rightarrow & B_1 \times B_2 \\ & (\rho_1, \rho_2) & \rightarrow & F(\rho_1, \rho_2) = (F_1(\rho_1, \rho_2), F_2(\rho_1, \rho_2)) \end{array}, \tag{1.35}$$

where

$$F_i\left(\rho_1, \rho_2\right)(t) = \int_0^1 \left[n_h\left(x,t\right)\Psi_{ih}\left(x,t\right) + n_c\left(x,t\right)\Psi_{ic}\left(x,t\right)\right]dx\ , \tag{1.36}$$

for $i = 1, 2$. Then, we have the following Lemma,

Lemma 6 *Under the assumptions* (1.26)*,* (1.27)*,* (1.28)*, F satisfies,*

1. *$F(B_1 \times B_2) \subset B_1 \times B_2$.*
2. *The contraction principle.*

Proof. First, we calculate

$$\begin{aligned} & F_i\left(\rho_1, \rho_2\right) \\ \leq & \int_0^1 \left[n_h^0\left(x\right)\Psi_{ih}\left(x,t\right)\exp\left[r_h\left(0\right) - d_h - c\eta_h\left(1\right)\right]t + n_c^0\left(x\right)\Psi_{ic}\left(x,r\right)\exp\left[r_c\left(0\right) - d_c - c\eta_c\left(1\right)\right]t\right]dx \\ \leq & M_T\ . \end{aligned}$$

On the other hand,

$$\begin{aligned} & F_i\left(\rho_1, \rho_2\right) \\ > & \Psi_m \int_0^1 \left\{n_h^0\left(x\right)\exp\left[\left(-d_h - c\eta_h\left(0\right)\right)t\right] + n_c^0\left(x\right)\exp\left[\left(-d_c - c\eta_c\left(0\right)\right)t\right]\right\}dx \\ \geq & \Psi_m\rho_m\left\{\min_{t\in[0,T]}\exp\left[\left(-d_h - c\eta_h\left(0\right)\right)t\right] + \min_{t\in[0,T]}\exp\left[\left(-d_c - c\eta_c\left(0\right)\right)t\right]\right\} \\ \geq & 0\ , \end{aligned}$$

for $i = 1, 2$. As conclusion, 1) makes true.
Next, we show that F holds the contraction property. For this purpose, consider the norm

$$\left\|\left(\rho_1, \rho_2\right) - \left(\rho_1', \rho_2'\right)\right\|_{B_1\times B_2} = \sup_{t\in[0,T]}\left|\rho_1 - \rho_1'\right| + \sup_{t\in[0,T]}\left|\rho_2 - \rho_2'\right|\ .$$

Given $(\rho_1, \rho_2), (\rho'_1, \rho'_2) \in B_1 \times B_2$ then

$$|F_i(\rho_1, \rho_2) - F_i(\rho'_1, \rho'_2)| \leq$$
$$\int_0^1 n_h^0(x) \Psi_{ih}(x,t) \left| \exp\left[\int_0^t \frac{r_h(x)}{(1+\rho_1(s))^\alpha} - d_h - c\eta_h(x)\, ds\right] - \exp\left[\int_0^t \frac{r_h(x)}{\left(1+\rho'_1(s)\right)^\alpha} - d_h - c\eta_h(x)\, ds\right]\right| dx$$
$$+ \int_0^1 n_c^0(x) \Psi_{ic}(x,t) \left| \exp\left[\int_0^t \frac{r_c(x)}{(1+\rho_2(s))^\alpha} - d_c - c\eta_c(x)\, ds\right] - \exp\left[\int_0^t \frac{r_c(x)}{\left(1+\rho'_2(s)\right)^\alpha} - d_c - c\eta_c(x)\, ds\right]\right| dx\,.$$

Cosequently, we obtain

$$|F_i(\rho_1, \rho_2) - F_i(\rho'_1, \rho'_2)|$$

$$\leq r_h(0)\, \rho_M^h \Psi_M e^T A\, |\rho_1(s) - \rho'_1(s)| + r_c(0)\, \rho_M^c \Psi_M e^T A\, |\rho_2(s) - \rho'_2(s)|\ ,$$

simultaneously it satisfies,

$$\sup_{r\in[0,R]} |F_i(\rho_1, \rho_2) - F_i(\rho\prime_1, \rho'_2)| \leq C_T \left\|(\rho_1, \rho_2) - (\rho'_1, \rho'_2)\right\|_{B_1 \times B_2}\ ,$$

where

$$C_T = \max\left[r_h(0)\, \rho_M^h \Psi_M e^T A, r_c(0)\, \rho_M^c \Psi_M e^T A\right]\ .$$

Hence, we choose $T > 0$ such that

$$C_T < 1\,, \tag{1.37}$$

obtaining the result. ∎

Theorem 7 *There exist* $n_c\,,\, n_h \in C\left(\mathbf{R}^+\,,\, L^1[0,1]\right)$*, unique solutions, for which* (1.21) *and* (1.24) *makes true.*

Proof. Using the Lemma 6 and Banach-Picard fixed point theorem there is a fixed point of F on $[0,T]$ with T defined in (1.37). Additionally, one can iterate with the initial condition $\rho(T)$, since the choice of T is not depending of the initial condition. Then we can conclude that there is a unique fixed point such that

$$F(\rho_1, \rho_2) = (F_1(\rho_1, \rho_2), F_2(\rho_1, \rho_2)) = (\rho_1, \rho_2)\ ,$$

for $t > 0$ and it is equivalent to

$$\rho_i(t) = \int_0^1 \left[n_h(x,t)\, \Psi_{ih}(x,t) + n_c(x,t)\, \Psi_{ic}(x,t)\right] dx\,,$$

for $i = 1, 2$ and it is the solution of (14). ∎

1.4 Transport Equation to cell growth

In this section we study the Transport equation

$$\left.\begin{aligned} \partial_t n(x,t) - \partial_x[g(x)n(x,t)] = R(x,\rho(t))n(x,t)\ , \\ \rho(t) = \textstyle\int_0^{+\infty} n(x,t)dx\ , \\ n(0,x) = n^0(x) > 0\ , \end{aligned}\right\} \tag{1.38}$$

where $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, $n^0 \in L^1(\mathbb{R}^+)$, we do not need to impose a boundary condition at $x = 0$, because we later assume that $g(0) = 0$.

1.4.1 Assumptions and Main results

The object in this section is to give several assumptions over coefficients and initial data, which are necessaries to obtain the asymptotic behavior of (1.38).

Fitness function

The Fitness function $R(x,\rho(t))$ model the influence of the total population on the rate of reproduction and death, which varies continuously in the variable x. Other name for R is the invasion exponent, since model the ability of individuals with trait $\overline{x}$, best adapted to environmental conditions $\overline{\rho}$ to invade all population. We assume that $R \in C^2\left(\mathbb{R}^+ \times \mathbb{R}^+\right)$ and this is a decreasing (in linear order) function (see figure 1.1). In addition, we suppose that there exists $\rho_M > 0$ such that

$$\max_{x \geq 0} R(x,\rho_M) = 0\ . \tag{1.39}$$

We also require a positive constant K_1 for which R satisfies,

$$\frac{\partial}{\partial \rho} R(x,\rho) < -K_1\ . \tag{1.40}$$

Genetic velocity of change

We define $g \in C^2(\mathbb{R}^+)$ as increasing bounded function (see figure 1.2) such that

$$g(0) = 0\ , \quad g \leq g_M\ , \quad 0 \leq g'(x) \leq 1\ , \quad |g''(x)| \leq \tfrac{K}{(1+x)^2}\ , \tag{1.41}$$

where K is a positive constant. We consider the condition $g(0) = 0$, because in the cases of closed systems one can not have entering of cells with trait $x = 0$. The shape of this functions is explained having in mind that at higher resistance (x large) implies faster velocity of change.

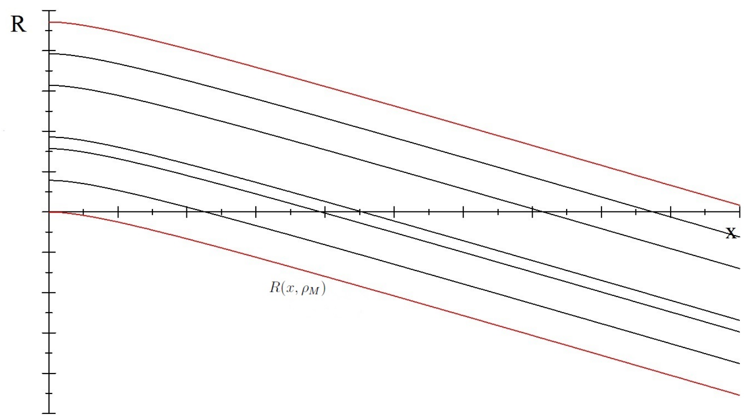

Figure 1.1: Graphic representation of fitness function R for different values of ρ .

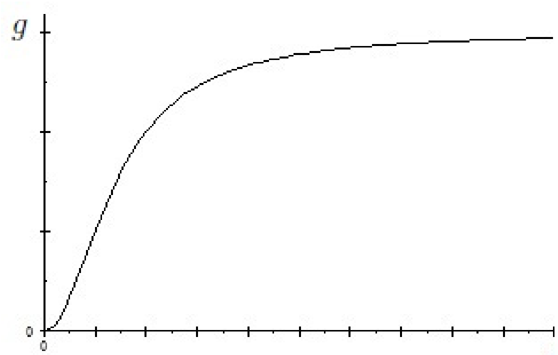

Figure 1.2: Graphic for g.

WKB Ansatz

We consider a new variable of time

$$\frac{\tau}{\epsilon} = t \; , \tag{1.42}$$

then in this work we study the asymptotic behavior for Transport PDE in its conservative form

$$\left.\begin{array}{ll} \epsilon\frac{\partial}{\partial t}n_\epsilon(x,t) - [\epsilon g(x) n_\epsilon(x,t)]_x = R(x,\rho_\epsilon(t))n_\epsilon(x,t) \; , & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \; , \\ \rho_\epsilon(t) = \int_0^{+\infty} n_\epsilon(x,t)dx \; , & t > 0 \; , \\ n_\epsilon(x,t=0) = n_\epsilon^0(x) \; , & x > 0 \; , \end{array}\right\} \tag{1.43}$$

where $0 < \epsilon < 1$, then we arrive at singular perturb transport equation, following the language using in [4.5, [22]].

We have in mind obtain a Gaussian type functions as solutions of (1.43), taking a initial data concentrated as a Gaussian function in some x_ϵ^0. We justify this change of variable having account in (1.38) that the scale on time is different at scale on x, because evolutionary time is slower than reproduction time. Using this principle, we applied the change of variable (1.42) balancing the corresponding timescales. We also should multiplied by ϵ parameter g, since represent the genetic velocity of change for individuals with trait x and it should decrease when time progresses faster.

Consider the WKB ansazt

$$n_\epsilon(x,t) = e^{\frac{u_\epsilon(x,t)}{\epsilon}} \; ,$$

which is equivalent to

$$u_\epsilon(x,t) = \epsilon \log(n_\epsilon(x,t)) \; , \tag{1.44}$$

where n_ϵ is a solution to (1.43). We replace (1.44) in (1.43), getting the Transport equation

$$\left.\begin{array}{l} \frac{\partial}{\partial t}u_\epsilon - g(x)\frac{\partial}{\partial x}u_\epsilon = R(x,\rho_\epsilon(t)) + c\epsilon g'(x) \; , \\ u_\epsilon^0(x) = \epsilon \log[n_\epsilon^0(x)] \; . \end{array}\right\} \tag{1.45}$$

To get the limit of n_ϵ, we prove that u_ϵ converges at some decreasing (in linear order), lipstchiz, concave and non positive function, the fundamental tool used to obtain the results are The Maximum Principle Theorem, Section 1.5. We do not come back on the question of existence of solutions $n_\epsilon(x,t)$, since we accept that there is a unique solution to (1.38) and for its re-scaled version (1.43).

Initial data

We also take, along this work, an initial data n^0 concentrated in x^0. This trait is the fittest in environmental conditions ρ^0, therefore holds

$$R(x^0,\rho^0) = 0 \; , \tag{1.46}$$

where R was defined before. In addition, we consider a sequence $(n_\epsilon^0, \rho_\epsilon^0)_\epsilon$ which holds

$$\rho_m \leq \int_0^\infty n_\epsilon^0(x)dx = \rho_\epsilon^0 \leq \rho_M \ , \tag{1.47}$$

for some $\rho_m > 0$. Simultaneously, this sequence satisfies,

$$\int_0^\infty n_\epsilon^0(x) R(x, \rho_\epsilon^0(x))dx = 0 \ , \tag{1.48}$$

$$n_\epsilon^0(x) \rightharpoonup \rho_0 \delta_0(x - x^0) \ , \tag{1.49}$$

where

$$\rho_\epsilon^0 \to \rho^0 \ . \tag{1.50}$$

We should assume some technical conditions over initial data, these are very important to obtain the estimates for solutions of (1.45). Here we enumerate all of them:

1. We define
$$u_\epsilon^0 = \epsilon \log[n_\epsilon^0] \ . \tag{1.51}$$
2. $$\frac{d}{dt}\rho_\epsilon(t)|_{t=0} = \int_0^{+\infty} n_\epsilon^0(x) R(x, \rho_\epsilon^0) = 0 \ . \tag{1.52}$$
3. We choice a family $(u_\epsilon^0)_\epsilon$ of initial data:
   - for which there exist A^0 , B^0 , C^0 positive constants such that
   $$-C^0(1 + x) \leq u_\epsilon^0(x) \leq B^0 - A^0 x \ . \tag{1.53}$$
   - There exists $D^0 > 0$ such that
   $$-D^0 \leq \partial_x u_\epsilon^0 \leq D^0 \ . \tag{1.54}$$
   - There exists a unique $x_\epsilon^0 > 0$ such that
   $$\max_{x \in \mathbb{R}^+} u_\epsilon^0(x) = u_\epsilon^0(x_\epsilon^0) \ . \tag{1.55}$$

Existence and Limit of ρ_ϵ

To ρ_ϵ solution of (1.8), we have the follow result:

Theorem 8 *Under assumptions 1.4.1, $\rho_\epsilon(t)$ satisfies,*

1. $\rho_m \leq \rho_\epsilon(t) \leq \rho_M$*, for all* $t > 0$.
2. $\rho_\epsilon(t)$ *is non decreasing.*
3. $(\rho_\epsilon)_\epsilon$ *is a bounded sequence in* $BV(\mathbb{R}^+)$.

As consequence there exists a sub-sequence (ρ_{ϵ_k}) such that

1. $\rho_{\epsilon_k} \to \overline{\rho}$ *in* $L^1_{loc}(\mathbb{R}^+)$.
2. $\frac{d}{dt}\rho_{\epsilon_k} \to \frac{d}{dt}\overline{\rho}$ *as measure in* $M_{loc}(\mathbb{R}^+)$.

Asymptotic Behavior

Theorem 9 *Under assumptions 1.4.1. There exists a unique $\overline{x}(t)$ lipschitz continuous function defined in $[0,T]$ such that*

$$\lim_{\epsilon\to 0} n_\epsilon(x,t) = \overline{\rho}(t)\delta_0(x-\overline{x}(t))\,, \tag{1.56}$$

in the distribution sense and

$$R(\overline{x}(t),\overline{\rho}(t)) = 0\,. \tag{1.57}$$

1.4.2 $BV_{loc}(\mathbb{R}^+)$ estimates

The first step is to prove Theorem 32 then we can pass to the limit in (1.38) as $\epsilon \to 0$. We assume the existence of ρ_ϵ in view of the methods showed in Sections 1.2-1.3.

Proof. Theorem 32 We don't re-prove that $\rho_\epsilon(t) \le \rho_M$, which follows using the same arguments than in Section 1.2-1.3. Next, we show that $\rho_\epsilon(t)$ is non decreasing. We define

$$J_\epsilon(t) = \frac{1}{\epsilon}\int_0^{+\infty} R(x,\rho_\epsilon(t))n_\epsilon(x,t)dx\,, \tag{1.58}$$

using equation (1.43), we compute

$$\begin{aligned}\frac{d}{dt}J_\epsilon(t) &= \tfrac{1}{\epsilon}J_\epsilon(t)\int_0^{+\infty}\tfrac{\partial R}{\partial\rho}n_\epsilon dx + \tfrac{1}{\epsilon}\int_0^{+\infty}R(x,\rho_\epsilon(t))\left\{\tfrac{1}{\epsilon}R(x,\rho_\epsilon(t))n_\epsilon + c[g(x)n_\epsilon]_x\right\}dx\\ &= \tfrac{1}{\epsilon}J_\epsilon(t)\int_0^{+\infty}\tfrac{\partial R}{\partial\rho}n_\epsilon dx + \int_0^{+\infty}\left\{\tfrac{1}{\epsilon^2}R(x,\rho_\epsilon(t))^2 n_\epsilon - c\tfrac{1}{\epsilon}R_x(x,\rho_\epsilon(t))g(x)n_\epsilon\right\}dx\,,\end{aligned}$$

hence

$$\frac{d}{dt}J_\epsilon(t) \ge \frac{1}{\epsilon}J_\epsilon\int_0^{+\infty}\frac{\partial R}{\partial\rho}n_\epsilon\,. \tag{1.59}$$

Applying conditions (1.48)-(1.52) in (1.59), we have that $J_\epsilon(t)\ge 0$. As a consequence $\rho_\epsilon(t)\ge\rho_m$ by (1.47). ■

1.4.3 Regularity properties for u_ϵ

In this subsection, we study the regularity for u_ϵ solution of (1.45), we prove that u_ϵ is bounded by decreasing functions and estimate the bounds for $\partial_x u_\epsilon$. Then (u_ϵ) is an equilipschitz decreasing sequence of functions in $C^0_{loc}(\mathbb{R}^+\times\mathbb{R}^+)$, thus by Ascoli' Theorem, under a sub-sequence, its limit exists.
By definition of R, we can assume there exist A^1 , B^1 , C^1 , D^1 positive constants such that for all $\rho\in[\rho_m,\rho_M]$

$$-C_1(1+x) \le R(x,\rho)+\epsilon c g'(x) \le B_1 - A_1 x \tag{1.60}$$

and

$$-D_1 \le \partial_x[R(x,\rho)+\epsilon c g'(x)] \le D_1\,. \tag{1.61}$$

Theorem 10 *Considering assumptions 1.4.1. Then u_ϵ solution of* (1.45) *satisfies,*

1. *u_ϵ is uniformly lipschitz, uniformly locally bound and it has linear decay at infinity.*
2. *Under a sub sequence, $u_\epsilon \to u$ in $C^0_{loc}(\mathbb{R}^+ \times \mathbb{R}^+)$.*
3. *u solve,*
$$\left.\begin{array}{c} \frac{\partial u}{\partial t} - g(x)\frac{\partial u}{\partial x} = R(x, \overline{\rho}(t))\,, \\ \\ u(x, t=0) = u^0(x)\,. \end{array}\right\} \tag{1.62}$$
4. *$u(x,.)$ solution of* (1.62) *belongs $C^1(\mathbb{R}^+)$.*

Some estimates using the Characteristic Method

Consider the change of variable

$$\left.\begin{array}{c} \dot{X}(t;y) = -g(X(t;y))\,, \\ X(t=0;y) = y\,, \end{array}\right\} \tag{1.63}$$

replacing (1.63) in (1.45), we obtain

$$\frac{d}{dt}[u_\epsilon(t, X(t;y))] = R(X(t;y), \rho(t)) + \epsilon g'(X(t;y))\,,$$

therefore obtain the representation formula for u_ϵ

$$u_\epsilon(X(t;y), t) = u^0_\epsilon(y) + \int_0^t R(X(s;y), \rho(s)) + \epsilon g'(X(s;y))ds\,. \tag{1.64}$$

This representation formula is useful to observe that the comparison principle holds, for more details see Section 1.5. In a similar way, one can obtain the representation formula for u, concluding that $u \in C^1$, proving the statement 4.

L^∞ and Lipstchitz estimates

Lemma 11 *Under assumptions 1.4.1 , there are positive constants M_0 , L_0 , K_0 , α_0 , β_0 such that*

$$-M_0(1+x)e^{\beta_0 t} \leq u_\epsilon \leq (L_0 - K_0 x)(1 + \alpha_0 t)\,, \tag{1.65}$$

where u_ϵ is solution of (1.45) .

Proof.

First step: Subsolution. Defining

$$\underline{h}(x,t) = -M_0(1+x)e^{\beta_0 t} \tag{1.66}$$

and replacing in (1.45), we have

$$\partial_t \underline{h} - g(x)\,\partial_x \underline{h} - [R(x,\rho_\epsilon(t)) + \epsilon g'(x)]$$

$$= -\beta_0 e^{\beta_0 t} M_0 (1+x) + g(x) M_0 e^{\beta_0 t} - R(x,\rho_\epsilon(t)) - \epsilon g'(x)$$

$$\leq -\beta_0 e^{\beta_0 t} M_0 (1+x) + g(x) M_0 e^{\beta_0 t} + C_1 (1+x)\,,$$

we should choose M_0 , β_0 such that

$$\left.\begin{aligned} -\beta_0 M_0 + C_1 \leq 0\,, \\ -\beta_0 e^{\beta_0 t} M_0 + c\|g\|_\infty M_0 e^{\beta_0 T} + C_1 \leq 0\,, \end{aligned}\right\} \tag{1.67}$$

Consequently, by (1.60) and (1.67) we have that

$$-\beta_0 e^{\beta_0 t} M_0 (1+x) + g(x) M_0 e^{\beta_0 t} - R(x,\rho_\epsilon(t)) - \epsilon g'(x) \leq 0\,,$$

it follows that $\underline{h}$ is a subsolution of (1.45) and holds

$$\partial_t [\underline{h} - u_\epsilon] - g(x)\,\partial_x [\underline{h} - u_\epsilon] \leq 0\,.$$

Using (1.53), we can apply Lemma 19 obtaining

$$\underline{h}(x,t) \leq u_\epsilon(x,t)\ .$$

Second step: Supersolution. In the same way, we define

$$\overline{h}(x,t) = (L_0 - K_0 x)(1 + \alpha_0 t)\,. \tag{1.68}$$

Replacing in (1.45)

$$\partial_t \overline{h} - g(x)\,\partial_x \overline{h} - [R(x,\rho_\epsilon(t)) + \epsilon g'(x)]$$

$$= \alpha_0 (L_0 - K_0 x) + g(x) K_0 (1 + \alpha_0 t) - [R(x,\rho_\epsilon) + \epsilon c g'(x)]$$

$$\geq \alpha_0 (L_0 - K_0 x) + g(x) K_0 (1 + \alpha_0 t) - [B_1 - A_1 x]\,,$$

we should choose α_0 , L_0 and K_0 such that

$$\alpha_0 L_0 \geq B_1\,,\ \alpha_0 K_0 \leq A_1\,. \tag{1.69}$$

Conditions (1.60)-(1.69) imply

$$\alpha_0 (L_0 - K_0 x) + g(x) K_0 (1 + \alpha_0 t) - [B_1 - A_1 x] \geq 0\,, \tag{1.70}$$

resulting that $\overline{h}$ is a supersolution of (1.45) and holds

$$\partial_t \left[u_\epsilon - \overline{h}\right] - g(x)\,\partial_x \left[u_\epsilon - \overline{h}\right] \leq 0\,.$$

As consequence of (1.53), we can apply Lemma 19 concluding

$$u_\epsilon(x,t) \leq \overline{h}(x,t)\,.$$

■

Lemma 12 *Under initial assumptions 1.4.1 , there are M_1 , α_1 such that*

$$-M_1 e^{\alpha_1 t} \leq \partial_x u_\epsilon(x,t) \leq M_1 e^{\alpha_1 t}\,, \tag{1.71}$$

where u_ϵ is solution of (1.45).

Proof. Differentiating once in x (1.45) and calling $v_\epsilon = \partial_x u_\epsilon$, we have

$$\partial_t v_\epsilon - g(x)\partial_x v_\epsilon - g'(x)v_\epsilon - \partial_x[R(x,\rho_\epsilon(t)) + \epsilon c g'(x)] = 0\,. \tag{1.72}$$

First step: Supersolution. We define

$$\overline{h}(x,t) = M_1 e^{\alpha_1 t}\,, \tag{1.73}$$

replacing (1.73) in (1.72), we obtain the equality

$$\begin{aligned} &\partial_t \overline{h} - g(x)\partial_x \overline{h} - g'(x)\overline{h} - \partial_x[R(x,\rho_\epsilon(t)) + \epsilon c g'(x)] \\ &= \alpha_1 M_1 e^{\alpha_1 t} - g'(x) M_1 e^{\alpha_1 t} - D_1\,. \end{aligned} \tag{1.74}$$

Then, we must choose $\alpha_1 > 1$, M_1 such that

$$\alpha_1 M_1 e^{\alpha_1 t} - g'(x) M_1 e^{\alpha_1 t} - D_1 \quad \geq M_1\left(\alpha_1 - g'(x)\right)e^{\alpha_1 t} - D_1 \quad \geq 0\,. \tag{1.75}$$

It suffices to conclude that $\overline{h}$ is a supersolution, by (1.54), we can apply Lemma 20, resulting that

$$\partial_x u_\epsilon(x,t) \leq M_1 e^{\alpha_1 t}\,.$$

Second step: Subsolution. Defining

$$\underline{h}(x,t) = -M_1 e^{\alpha_1 t}\,, \tag{1.76}$$

replacing (1.76) in (1.72), we have

$$\begin{aligned} &\partial_t \underline{h} - g(x)\partial_x \underline{h} - g'(x)\underline{h} - \partial_x[R(x,\rho_\epsilon(t)) + \epsilon c g'(x)] \\ &= -\alpha_1 M_1 e^{\alpha_1 t} + g'(x) M_1 e^{\alpha_1 t} + D_1\,, \end{aligned} \tag{1.77}$$

thus result the same condition obtaining before

$$-\alpha_1 M_1 e^{\alpha_1 t} + g'(x) M_1 e^{\alpha_1 t} + D_1 \leq 0\,.$$

Therefore $\underline{h}$ is a subsolution, then the result follows.

■

Concavity estimates for u

We can assume that there exists K_2 , H positive constants such that

$$\begin{gathered} R_{xx}(x,\rho) + cg''(x)\partial_x u \le \tfrac{-K_2}{(1+x)^2} \ , \\ u^0_{xx} \le \tfrac{-H}{(1+x)^2} \ . \end{gathered} \tag{1.78}$$

Lemma 13 *Under assumptions 1.4.1 and* (1.78)*, there is H positive constant such that*

$$\partial_{xx} u(x,t) \le \frac{-H}{(1+x)^2} \ . \tag{1.79}$$

Proof. Differentiating once in (1.62) we have

$$\partial_t[\partial_x u] - g(x)\partial_{xx} u - g'(x)\partial_x u - \partial_x R(x,\rho(t)) = 0,$$

differentiating once again, we get

$$\partial_t[\partial_{xx} u] - g''(x)\partial_x u - g'(x)\partial_{xx} u - g'(x)\partial_{xx} u - g(x)\partial_x[\partial_{xx} u] - R_{xx}(x,\rho(t)) = 0$$

and naming $\partial_{xx} u = w$, we have

$$\partial_t w - 2g'(x)w - g(x)\partial_x w - g''(x)\partial_x u - R_{xx}(x,\rho(t)) = 0 \ . \tag{1.80}$$

Let consider

$$\overline{w}(x,t) = \frac{-H}{(1+x)^2} \tag{1.81}$$

and replacing in (1.80) we have

$$\begin{aligned} & -2g'(x)\overline{w} - g(x)\partial_x \overline{w} - g''(x)\partial_x u - R_{xx}(x,\rho(t)) \\ &= \tfrac{2g'(x)H}{(1+x)^2} - \tfrac{2g(x)H}{(1+x)^3} - g''(x)\partial_x u - R_{xx}(x,\rho(t)) \\ &\ge \tfrac{2g'(x)H}{(1+x)^2} - \tfrac{2g(x)H}{(1+x)^3} + \tfrac{K_2}{(1+x)^2} \\ &\ge -\tfrac{2g(x)H}{(1+x)^3} + \tfrac{K_2}{(1+x)^2} \\ &\ge -\tfrac{2g(x)H}{(1+x)^2} + \tfrac{K_2}{(1+x)^2} \ . \end{aligned}$$

Then, if we choose $H > 0$ such that

$$g(x) \le \frac{K_2}{2H} \ ,$$

$\overline{w}$ is a supersolution of (1.80) and we can apply Lemma 20 . ■

1.4.4 Asymptotic behavior

Existence of an optimal trait

Lemma 14 *Under assumptions* (1.4.1). *Then* $u_\epsilon(x,t)$ *solution of* (1.45) *satisfies,*

$$\lim_{\epsilon\to 0}\max_{x\in\mathbb{R}^+} u_\epsilon(x,t) = 0\,,$$

for all $t\in[0,+\infty)$*, thus* $\max_{x\in\mathbb{R}^+} u(x,t)=0$ *and there is a unique* $\overline{x}(t)$ *such that*

$$u\left(\overline{x}(t),t\right)=0\,. \tag{1.82}$$

Proof. Let us fix $t>0$ and suppose

$$\lim_{\epsilon\to 0}\max_{x\in\mathbb{R}^+} u_\epsilon(x,t) = u(\overline{x}(t),t) > 0\,,$$

thus there exists $h>0$ such that

$$u(x,t)>0\,, \tag{1.83}$$

for all $x\in A=[\overline{x}(t)-h,\overline{x}(t)+h]$. We know that $(u_\epsilon)_\epsilon$ is equilipstichiz, then, under a subsequence, converges uniformly upon A. Hence by uniform convergence definition, there exists ϵ_0 such that for all $\epsilon<\epsilon_0$,

$$u_{\epsilon_0}(x,t)>0\,.$$

for all $x\in A$. Resulting that,

$$\int_A e^{\frac{u_\epsilon}{\epsilon}}dx \le |A|e^{\frac{\max_{x\in A}u_\epsilon(x,t)}{\epsilon}} \to +\infty\,. \tag{1.84}$$

It is a contradiction, since

$$\int_A e^{\frac{u_\epsilon}{\epsilon}}dx \le \rho_M\,.$$

On the other hand, assume that

$$\lim_{\epsilon\to 0}\max_{x\in\mathbb{R}^+} u_\epsilon(x,t) < 0\,,$$

then there exists ϵ_0 such that

$$\max_{x\in\mathbb{R}^+} u_\epsilon(x,t) < 0\,,$$

for all $\epsilon<\epsilon_0$. Taking L large enough and by (1.65)

$$\int_L^{+\infty} e^{\frac{u_\epsilon(x,t)}{\epsilon}}dx \le \int_L^{+\infty} e^{\frac{(L_0-K_0x)(1+\alpha_0)t}{\epsilon}}dx = \frac{\epsilon}{(1+\alpha_0 t)K_0}e^{\frac{(L_0-K_0L)(1+\alpha_0)t}{\epsilon}}\,. \tag{1.85}$$

Further, we have that

$$\int_0^L e^{\frac{u_\epsilon(x,t)}{\epsilon}}dx \le Le^{\frac{\max_{x>0}u_\epsilon(x,t)}{\epsilon}}\,. \tag{1.86}$$

Now, by (1.85) - (1.86) it follows that

$$\rho_m \leq \lim_{\epsilon\to 0}\int_0^{+\infty} e^{\frac{u_\epsilon(x,t)}{\epsilon}}dx \leq \lim_{\epsilon\to 0}\left[\frac{\epsilon}{(1+\alpha_0 t)K_0}e^{\frac{(L_0-K_0L)(1+\alpha_0)t}{\epsilon}} + Le^{\frac{\max_{x\in\mathbb{R}^+} u_\epsilon(x,t)}{\epsilon}}\right] = 0\,.$$

It is a contradiction, since by definition $\rho_m > 0$. ■

With Lemma 14, we conclude that the statement (1.57) in Theorem 9, which follows from standard analysis of convergence to Dirac masses. Indeed $n_\epsilon(x,t)$ vanishes exponentially fast away from $\overline{x}(t)$. As consequence, we have the following result,

Lemma 15 *Under assumptions 1.4.1 . The function $u(x,t)$ solution of* (1.62) *satisfies,*

$$\partial_x u\left(\overline{x}(t),t\right) = 0\,,\ \partial_t u\left(\overline{x}(t),t\right) = 0 \tag{1.87}$$

a.e. in time. Then

$$R(\overline{x}(t),\overline{\rho}(t)) = 0\,. \tag{1.88}$$

a.e. in time.

Proof. We replace (1.87) in (1.62) obtaining (1.88). ■

In this part, we obtain a representation formula for $\overline{x}(t)$, calling Canonical equation in Adaptive Dynamics, with which we can determinate that $\overline{x}(t)$ is a Lipschitz bounded function, obtaining the result presented below.

Lemma 16 (Canonical equation) *Under assumptions 1.4.1 , $\overline{x}(t)$ is a bounded Lipschitz function.*

Proof. By property, we have that

$$\partial_x u(\overline{x}(t),t) = 0\,, \tag{1.89}$$

differentiating in time (1.89) we have

$$\frac{d}{dt}[\partial_x u(\overline{x}(t),t)] = \partial_t[\partial_x u(\overline{x}(t),t)] + \partial_{xx}u(\overline{x}(t),t)\dot{\overline{x}}(t) = 0\,.$$

Next, differentiate in x (1.62) obtaining

$$\partial_t[\partial_x u(x,t)] - g'(x)\partial_x u(x,t) - g(x)\partial_{xx}u(x,t) = \partial_x R(x,\rho(t))\,.$$

Evaluating in $x = \overline{x}(t)$ we have

$$\partial_t[\partial_x u(\overline{x}(t),t)] - g(\overline{x}(t))\partial_{xx}u(\overline{x}(t),t) = \partial_x R(\overline{x}(t),\overline{\rho}(t))\,.$$

Concluding that the Canonical equation for $\overline{x}(t)$ is

$$\dot{\overline{x}}(t) = -g(\overline{x}(t)) - \left[\partial_x R(\overline{x}(t),\overline{\rho}(t))\right]\left[\partial_{xx}u(\overline{x}(t),t)\right]^{-1}\,.$$

■

By section 1.4.3 $(u_\epsilon)_{\epsilon>0}$ under a subsequence has a limit. Its limit $u(x,t)$ is non positive concave function decreasing, at least in linear order, which has a unique maximum point $\overline{x}(t)$ for all $t \geq 0$. Then, for ϵ small enough u_ϵ is a concave function and we can applied the classical arguments to conclude that

$$\lim_{\epsilon\to 0} \frac{\int_0^{+\infty} e^{\frac{u_\epsilon(x,t)}{\epsilon}} \phi(x)dx}{\int_0^{+\infty} e^{\frac{u_\epsilon(x,t)}{\epsilon}} dx} = \phi(\overline{x}(t)) \ ,$$

in other words, as we anticipated

$$\lim_{\epsilon\to 0} n_\epsilon(x,t) = \overline{\rho}(t)\delta_0(x-\overline{x}(t)) \ .$$

Stationary Solution

In this part, the objective is to show the necessaries conditions, for which there is a unique solution for the stationary problem of (1.38). We consider the stationary equation of (1.38)

$$-\partial_x \left[n\left(x\right) g\left(x\right)\right] = R(x,\rho) n\left(x\right) \ , \} \tag{1.90}$$

where ρ holds

$$\int_0^{+\infty} n(x)dx = \rho \ .$$

First, we define the weak solution of (1.90). Let $\phi \in C^1\left(\mathbb{R}^+\right)$ test function. Multiplying by ϕ and integrating upon $[0,+\infty)$, result that

$$-\int_0^{+\infty} \partial_x[n(x)g(x)]\phi(x)dx = \int_0^{+\infty} n(x)R(x,\rho)\phi(x)dx \ . \tag{1.91}$$

By initial assumptions (Section 1.4.1), we know that $g(0) = 0$. In addition, we search solutions such that $\lim_{x\to+\infty} n(x) = 0$, thus using by parts integration formula in (1.91), we obtain

$$\int_0^{+\infty} n(x)g(x)\phi'(x)dx = \int_0^{+\infty} n(x)R(x,\rho)\phi(x)dx \ . \tag{1.92}$$

The equality (1.92) gives the definition to the weak solution of (1.90).

Lemma 17 *Under hypothesis 1.4.1 and assuming that*

$$\max_{x\in\mathbb{R}^+} R(x,\rho_M) = R(0,\rho_M) = 0 \, . \tag{1.93}$$

Then, the function

$$n(x) = \rho_M \delta_0(x) \, , \tag{1.94}$$

is the unique solution of (1.90).

Proof. Taking a test function $\phi(x) \in C^1(\mathbb{R}^+)$ we have

$$\int_0^{+\infty} R(x,\rho_M)\phi(x)n(x)dx = \rho_M R(0,\rho_M)\phi(0) = 0\,. \tag{1.95}$$

On the other hand, by hypothesis over g (Section 1.4.1) we obtain

$$-c\int_0^{+\infty} [g(x)n(x)]_x\phi(x)dx = c\rho_M g(0)\phi'(0) = 0\,. \tag{1.96}$$

Therefore, the pair $(\rho_M, \rho_M\delta_0(x))$ is a solution to (1.90) since holds (1.92).
Now, suppose there exists a solution for some $\rho^* < \rho_M$. Consider $n(x) = \rho^*\delta_0(x - x_{\rho^*})$, where x_{ρ^*} satisfies

$$R(x_{\rho^*}, \rho^*) = 0\,. \tag{1.97}$$

By definition (1.92), x_{ρ^*} must verify that

$$g(x_{\rho^*}) = 0\,.$$

But, by hypothesis over g (Section 1.4.1), $g(x) = 0$ if and only if $x = 0$. Therefore $n(x) = \rho^*\delta_0(x - x_{\rho^*})$ does not hold (1.92).
■

Remark 18 *To obtain the long-time behavior of asymptotic solutions of* (1.38)*, we need to give stronger estimates upon $\overline{\rho}(t)$ and $\overline{x}(t)$, because we should control in some sense the following limits*

$$\begin{gathered} \lim_{t\to+\infty}\overline{\rho}(t)\,, \\ \lim_{t\to+\infty}\overline{x}(t)\,, \\ \lim_{t\to+\infty}R(\overline{x}(t),\overline{\rho}(t))\,, \end{gathered} \tag{1.98}$$

since we only have proved that: $\overline{x}(t)$ is a lipschitz, bounded function and $\rho(t)$ is increasing, bounded in $BV_{loc}(\mathbb{R}^+)$.

1.5 Maximum Principle

We need to apply the maximum principle in 1.4.3 to obtain some important estimations over u_ϵ such as: regularity and convexity. Therefore in this section proceed to show this principle, which are simple yet highly significant.

Theorem 19 *Consider $g \in C^1(\mathbb{R}^+)$ increasing bounded positive function. Let v that satisfies*

$$\left.\begin{array}{c} \frac{\partial v}{\partial t} - g(x)\frac{\partial v}{\partial x} = f(x,t) \le 0\,, \\ v^0(x) \le 0\,. \end{array}\right\} \tag{1.99}$$

Then, for all $t \in [0,T]$ we have that

$$v(x,t) \le 0\,.$$

Proof. Follow the method of characteristic, we considered the change of variable

$$\frac{dX}{dt} = -g(X(t)) \tag{1.100}$$

for some initial condition given. By Picard's theorem there exits a unique solution in $C^1(0, T)$ and by Inverse function Theorem, this change of variable has a inverse, then we obtain

$$\left.\begin{array}{c} \frac{d}{dt}v(X(t), t) = f(X(t), t) < 0 \,, \\ v^0(X(t)) \leq 0 \,. \end{array}\right\}$$

Therefore,

$$v(X(t), t) \leq v^0(X(t)) < 0$$

and the results follows.

■

Theorem 20 *Consider* $g \in C^1(\mathbb{R}^+)$ *increasing bounded positive function. Let* v *which satisfies*

$$\left.\begin{array}{c} \frac{\partial v}{\partial t} - g(x)\frac{\partial v}{\partial x} - g'(x)v = f(x, t) < 0 \,, \\ v^0(x) \leq 0 \,. \end{array}\right\} \tag{1.101}$$

Then, for all $t \in [0, T]$ *we have that*

$$v(x, t) \leq 0 \,.$$

Proof. Using the same change of variable defined by (1.100)

$$\left.\begin{array}{c} \frac{d}{dt}v(X(t), t) - g'(X(t))v(X(t), t) = f(X(t), t) < 0 \,, \\ v^0(X(t)) \leq 0 \,. \end{array}\right\}$$

Therefore, applying Gronwall's inequality

$$v(X(t), t) \leq v^0(x(0))e^{c\int_0^t g'(X(s))ds} < 0 \,.$$

■

Chapter 2

Modeling two chemotactic species interactions

2.1 Introduction

In this chapter, mainly studied the conditions for global existence and blow-up for solutions to Keller- Segel type parabolic pde system in $\mathbb{R}^2$.

This chapter had been published in the following papers (see appendix):

- E. ESPEJO, K. VILCHES, C. CONCA (2012), Sharp conditon for blow-up and global existence in a two species chemotactic Keller-Segel system in $\mathbb{R}^2$ *European J. Appl. Math*
- C. CONCA, E. ESPEJO, K. VILCHES (2011), Remarks on the blow-up and global existence for a two species chemotactic Keller-Segel system in $\mathbb{R}^2$. *European J. Appl. Math*, Available on CJO 2011 doi:10.1017/S0956792511000258.

Motivation to study such models was born in 1960 where Evelyn Fox Keller joint with Lee Segel, physicist and applied mathematician respectively, studied the behavior of very particular amoeba, called Dictyostelium discoideum or Slime mold, which was discovered by K. B. Raper in 1935. The Keller-Segel model was proposed to describe the behavior of this amoeba. The slime mold is a unicellular organism that detects an extracellular signal and transforms it into an intracellular. This signal activates oriented cell movement toward its gradient, which is known as aggregation process. The signal is a chemical secreted by themselves called cyclic Adenosine Mono phosphate (cAMP). Chemotaxis is the word used to describe this phenomenon and it makes reference to the aggregation of organisms sensitive at gradient of a chemical substance.

We also found this kind of behavior: in mammals, fish, birds, bacteria, humans, etc., all species tend to aggregate or repel depending on the environment where they are and their needs. For this reason, with the help of the biologists, we could find new applications for the Keller-Segel model.

Since its discovery, scientists have had an special interest in understand this type of organization, because through comprehension of this amoebas could better perceive the emergence of life millions of years ago. Adequately comprehension of these amoebae can help us to understand how from simplest organisms higher intelligence and greater physiological complexity were formed. The scientist believe that life as we know it arises from the existence of bacteria and by changes in their environment, were forced to cluster forming more complex cells.

Biologists were believing that the organization of this amoeba existed leaders or pacemakers, which have to decide when the movement would occur. They said that this particulars individuals should define the direction followed by others. Although, experiments and observations showed that the chemotaxis process was produced by changes in the environment, but the main cause apparently was a very large number of individuals and this allowed to build a pseudoplasmoid, where each individual is preserved and remain together to survive. E. Keller proposed a mathematical model to prove that the movement was not directed by a particular amoeba, however, this is a great example of an organization coordinate to survive. The fundamental condition that produces Chemotaxis in this organization was discovered by mathematicians and there exists an important relation between the quantity of chemical substances (cAMP) and initial total mass, which determines when aggregation occurs or not, in mathematical language this mean Blow-up or Existence global in time of solutions.

The behavior of this organization has been described by the classical mathematical model in chemotaxis introduced by E.F. Keller and L.A. Segel in [34] and it is the following parabolic system:

$$\left.\begin{array}{r} u_t = \nabla\cdot(\mu\nabla u - \chi u\nabla v)\,,\, x\in\Omega\,,\, t>0\,, \\ v_t = \gamma\Delta v - \beta v + \alpha u\,,\, x\in\Omega\,,\, t>0\,, \end{array}\right\} \tag{2.1}$$

where the biological meaning of variables are:

- $u(x,t)$ cell density
- $v(x,t)$ concentration of the chemical at point x in time t
- χ chemotactic sensitivity
- γ diffusion coefficient of the chemo-attractant
- μ diffusion coefficient of the cell density
- β rate of consumption
- α rate of production

subject to homogeneous Neumann boundary conditions over $\Omega\subset\mathbb{R}^N$, which has smooth boundary $\partial\Omega$, positive initial data $u(x,0)=u_0$ and $v(x,0)=v_0$ is considered, furthermore all parameters are positive.

It was conjectured by S. Childress & J.K. Percus in [13], that in a two-dimensional domain there exists a critical number C such that if $\int u_0(x)dx < C$ then the solution exists globally in time and if $\int u_0(x)dx > C$ blow-up happens. To different versions of the Keller-Segel model the conjecture has been essentially proved, finding the critical value $C = 8\pi/\chi$. To interest readers that want to make a complete review of this topic we refer the papers [31, 32] and the references therein.

In the case of several chemotactic species a new question arises:

Is there a critical curve in the plane of initial masses $\theta_1\theta_2$ delimiting on one side global existence and blow-up on the other side?

This question was already formulated by G.Wolansky in [54] and from Theorem 5 we have the following result:

Theorem 21 *Consider the system*

$$\left.\begin{array}{rcl} \partial_t u_1 &=& \Delta u_1 - \chi_1 \nabla\cdot(u_1\nabla v)\,, \\ \partial_t u_2 &=& \mu_2\Delta u_2 - \chi_2 \nabla\cdot(u_2\nabla v)\,, \\ 0 &=& \Delta v + u_1 + u_2 - v\,, \end{array}\right\}$$

along with Dirichlet boundary conditions for v and initial radial data: $u_1(0,\cdot) = \varphi$, $u_2(0,\cdot) = \psi$, $v(0,\cdot) = \phi$, with $\varphi, \psi, \phi \geq 0$ on the two-dimensional disc of radius 1. Further, let θ_1, θ_2 be the total preserved masses of the chemotactic species. Also assume that

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1+\theta_2)^2 > 0\,,\ \theta_1 < \frac{8\pi}{\chi_1}\,,\ \theta_2 < \frac{8\pi}{\chi_2}\ . \tag{2.2}$$

Then for $(u_1(0,.)\,,\,u_2(0,.)) \in Y_N$ with

$$Y_N = \left\{u_1, u_2 : B(0) \to \mathbb{R}^+ : \int u_i = \theta_i\,,\ \int_{B_1(0)} u_i \log u_i < \infty\right\}\,,$$

there exists a global in time classical solution.

A lot of natural question arises from this last result, for example:

1. What happens if the inequalities (2.2) does not hold? Is it still possible to have global solutions?
2. If we work in the hole space $\mathbb{R}^2$, What happens with the condition (2.2)?
3. Is it necessary consider radial initial conditions?

To answer this questions, we consider the follow parabolic system

$$\left.\begin{array}{c} \partial_t u_1 = \mu\Delta u_1 - \chi_1\nabla\cdot(u_1\nabla v)\,, \\ \\ \partial_t u_2 = \Delta u_2 - \chi_2\nabla\cdot(u_2\nabla v)\,, \\ \\ v(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|\,(u_1(y,t)+u_2(y,t))\,dy\,, \\ \\ u_1(x,0) = u_{10} \geq 0\,,\ u_2(x,0) = u_{20} \geq 0\,, \end{array}\right\} \tag{2.3}$$

where $t \geq 0$, u_1 and u_2 are the density for the two different chemotaxis species and v is the chemo-attractant, χ_1 , χ_2 , μ are positive constants and positive initial conditions u_{10} , u_{20} are given. In [15] it was proved that if θ_1 , θ_2 satisfies *any* of the inequalities,

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1+\theta_2)^2 < 0 \, , \theta_1 > \mu\frac{8\pi}{\chi_1} \, , \theta_2 > \frac{8\pi}{\chi_2} \, ,$$

then the solutions of (2.3) have a finite time of existence, in other words the system (2.3) blow-up. For the global existence was proved also in [15] that the inequalities

$$\begin{aligned} \theta_1 + \theta_2 &< \frac{8\pi}{\chi_2} \, , \mu < 1 \, , \\ \theta_1 + \theta_2 &< \frac{8\pi}{\chi_2}\mu \, , \mu > 1 \, , \end{aligned}$$

guarantees global existence. This is a partial result, as we just handed the sharp condition for blow-up, but not global existence in time. The tool used in the first article [15] was the Hardy-Littlewood-Sobolev' inequality for one equation, we used this inequality to bound entropy (2.3) following the method applied in [8], but this does not provide the necessary condition.

To obtain the sharp condition and a threshold curve in [21], we used a Hardy-Litlewood-Sobolev' inequality for System proposed by Shafrir & Wolansky in [49], exposed after. We applied this inequality and incorporate some parameters to conclude that condition give in [54] is sharp.

In the present chapter, our aim is to show a resume the most important step given in [15, 21] to obtain the sharp condition for global existence and Blow-up for system (2.3), which is a generalization of the threshold number $8\pi/\chi$ for the classical parabolic-elliptic Keller-segel system. The curves that give The Sharp condition are

$$\begin{aligned} C_1(\theta_1, \theta_1) &= \tfrac{4\pi\mu\theta_1}{\chi_1} + \tfrac{4\pi\theta_2}{\chi_2} - \tfrac{1}{2}(\theta_1+\theta_2)^2 \, , \\ C_2(\theta_1) &= \tfrac{8\pi\mu}{\chi_1}\theta_1 \, , C_3(\theta_1) = \tfrac{8\pi}{\chi_2}\theta_2 \, , \end{aligned} \tag{2.4}$$

which are summaries in Figure 2.1 ,
One open question is to find out if the blow-up has to be simultaneous or not and also to describe the asymptotic near the blow-up time. A first step in this direction was given in [20], where it was shown that the blow-up has to be simultaneous in the radial case. Should it be the same in the general case?, or, Should it depend on more specific information on the initial data?

A second open question is: What happen when the parabola,

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1+\theta_2)^2 = 0 \, , \tag{2.5}$$

intersects any of the line lines,

$$\theta_1 = \frac{8\pi}{\chi_1} \quad \text{or} \quad \theta_2 = \frac{8\pi}{\chi_2} \, . \tag{2.6}$$

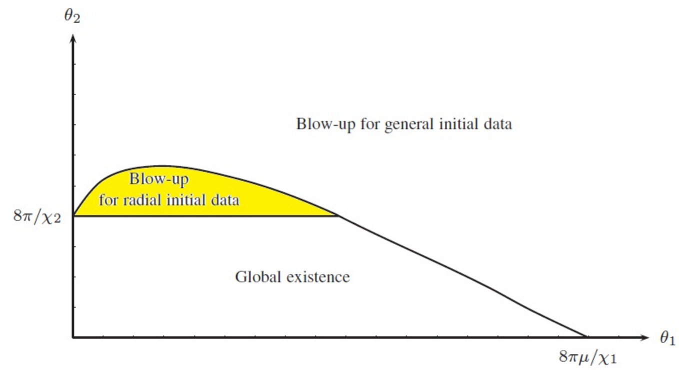

Figure 2.1: Graphic of sharp condition to Global existence and blow-up.

It would be very interesting to study the behavior of the system (2.3) on this lines. One first approximation of this response is given by the study of stationary model, but it has only been a partial result. Another possible future work is the description of the asymptotic behavior in this case seems to require rather different techniques to those used in the one specie case.

2.2 Some notable examples

2.2.1 Blow-up

There exist different definitions to Blow-up of solutions in differential equations. In our case, we consider Blow-up when the existence time for solutions of a differential equation is finite. There are different ways to prove this, here we will show some examples to ODE and PDE.

2.2.2 Blow-up in ODE

We consider the differential equation

$$\left.\begin{array}{rcl} u_t & = & f(u)\,, \\ u(0) & = & u_0\,, \end{array}\right\} \tag{2.7}$$

where $t \to u(t)$ is a function defined over $[0, T]$, with values on $\mathbb{R}$.

Theorem 22 *Let f a continuous, positive function in $[u_0\,, \infty[$ such that*

$$t^* = \int_{u_0}^{+\infty} \frac{ds}{f(s)} < +\infty\,. \tag{2.8}$$

The maximum time of existence to the solutions of (2.7) *is t^*.*

Proof. We obtain from (2.7)

$$\frac{u_t}{f(u)} = 1\,, \tag{2.9}$$

integrating (2.9)

$$\int_0^t \frac{u_\tau}{f(u)} d\tau = t\,. \tag{2.10}$$

We applied the change of variable, $s = u(t)$ in (2.10) and conclude that

$$\int_{u_0}^{u(t)} \frac{ds}{f(s)} ds = t < t^* = \int_{u_0}^{+\infty} \frac{ds}{f(s)} ds\,. \tag{2.11}$$

Further, we have by (2.11)

$$\lim_{t \to t^*} \int_{u_0}^{u(t)} \frac{ds}{f(s)} ds = t^*\,.$$

Therefore, $\lim_{t \to t^*} u(t) = \infty$, since $\epsilon \to \int_{u_0}^{\epsilon} \frac{ds}{f(s)}$ is a bijection. ■

Example 23 *We consider the ODE*

$$\left.\begin{array}{rcl} u_t & = & u^p\,, \\ u(0) & = & u_0\,, \end{array}\right\} \tag{2.12}$$

for $p > 1$ *, we can applied Theorem 22 and obtain* $t^* = \frac{1}{(p+1)u_0^{-(p+1)}}$.

Next, we consider the follow ODE

$$\left.\begin{array}{rcl} u_t & \geq & f(u)\,, \\ u(0) & = & u_0\,. \end{array}\right\} \tag{2.13}$$

Corollary 24 *Let* f *continuous, positive function defined in* $[u_0\,,\, +\infty)$ *such that*

$$t^* = \int_{u_0}^{+\infty} \frac{ds}{f(s)} < +\infty. \tag{2.14}$$

The solution of (2.13) *has maximum existence time less than or equal to* t^*.

Proof. Applied the same argument used in 22. ■

2.2.3 Blow-up in PDE

We consider the PDE

$$\left.\begin{array}{c} u_t - \triangle u = f(u)\,, \Omega \times (0,T)\;, \\ \frac{\partial u}{\partial \vec{n}} = 0\,, \Gamma \times (0,T)\;, \\ u(.,0) = u_0\,, u_0 \geq 0\,, u_0 \in L^2(\Omega)\;. \end{array}\right\} \tag{2.15}$$

Theorem 25 *Let* $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$ *and* f *positive, convex function over* $[u_0, +\infty)$ *such that*

$$t^* = \int_{u_0}^{+\infty} \frac{ds}{f(s)} < +\infty \, . \tag{2.16}$$

Then (2.15) *does not have smooth solution after* t^*.

Proof. Suppose there exists a solution to (2.15) over Ω. Integrating it follows (2.15) over Ω

$$\partial_t \left[\frac{1}{|\Omega|}\int_\Omega u dx\right] - \frac{1}{|\Omega|}\int_\Omega \triangle u dx = \frac{1}{|\Omega|}\int_\Omega f(u)dx \, , \tag{2.17}$$

applying Divergence Theorem in (2.17) we obtain

$$\partial_t \left[\frac{1}{|\Omega|}\int_\Omega u dx\right] - \frac{1}{|\Omega|}\int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} dA = \frac{1}{|\Omega|}\int_\Omega f(u)dx \, . \tag{2.18}$$

By Newmann condition give in (2.15) we have

$$\partial_t \left[\frac{1}{|\Omega|}\int_\Omega u dx\right] = \frac{1}{|\Omega|}\int_\Omega f(u)dx \, . \tag{2.19}$$

Using Jensen' inequality in (2.19) result that

$$\partial_t \left[\frac{1}{|\Omega|}\int_\Omega u dx\right] \geq f\left(\frac{1}{|\Omega|}\int_\Omega u dx\right) . \tag{2.20}$$

Let $v = \frac{1}{|\Omega|}\int_\Omega u dx$ and replacing in (2.20) we obtain

$$\left.\begin{array}{rcl} v_t & \geq & f(v) \, , \\ v(0) & = & \int_\Omega u(x,0)dx > 0 \, . \end{array}\right\} \tag{2.21}$$

Thus, we can applied the Corollary 24 to (2.21) concluding

$$v(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx \, , \tag{2.22}$$

it is defined all most for t^*. We can apply heat equation maximum principle, because

$$u_0 \geq 0 \, , f(u) \geq 0 \, ,$$

then

$$\max_{\bar{\Omega}\times[0,T]} u = \max_{\Gamma_T} u \geq u_0 \, .$$

We know that $u(x,t) \geq 0$, then we have $u(x,t)$ all most to $t = t^*$. ■

Now, we consider another example. Let $p > 1$ and take the follow PDE

$$\left.\begin{array}{c} u_t - \triangle u = u^p \ , \ (x,t) \in \Omega \times (0,T) \ , \\ u(.,0) = u_0 \ , \ u_0 \geq 0 \ , \end{array}\right\} \tag{2.23}$$

where $u(.,t) \in \mathbb{H}_0^1(\Omega)$ for all $t \in (0,T)$. Define ϕ such that,

$$-\triangle \phi = \lambda_1 \phi \text{ on } \Omega \ , \tag{2.24}$$

where $\phi > 0$ in $\mathbb{H}_0^1$ and $\lambda_1 = \min \sigma(-\triangle)$, assume that $\int_\Omega \phi = 1$.

Theorem 26 ***(Caplan Method)*** *If $\int_\Omega u_0 \phi dx$ is large enough. Then* (2.23) *does not have a solution defined for all $t > 0$.*

Proof. Multiplying by ϕ and integrating upon Ω in (2.24), we have

$$\left(\int_\Omega u\phi dx \right)_t - \int_\Omega \triangle u \phi dx = \int_\Omega u^p \phi dx \ . \tag{2.25}$$

Applying integration by parts formula in (2.25), we obtain

$$\left(\int_\Omega u\phi dx \right)_t - \int_\Omega u \triangle \phi dx = \int_\Omega u^p \phi dx \ . \tag{2.26}$$

By (2.24) and Jensen's inequality result that

$$\left(\int_\Omega u\phi dx \right)_t + \lambda_1 \int_\Omega u\phi dx \geq \left(\int_\Omega u\phi dx \right)^p \ . \tag{2.27}$$

Defining the change of variable $v = \int_\Omega u\phi dx$ and replacing in (2.27), we conclude

$$v_t \geq v^p - \lambda_1 v \ . \tag{2.28}$$

By hypothesis v_0 is large enough, then there exists $c > 1$ such that

$$\frac{v^p}{c} \leq v^p - \lambda_1 v > 0 \ ,$$

thus we obtain

$$\int_{v_0}^{+\infty} \frac{c}{s^p} ds \geq \int_{v_0}^{+\infty} \frac{1}{s^p - \lambda_1 s} ds \ . \tag{2.29}$$

Therefore

$$\int_{v_0}^{+\infty} \frac{1}{s^p - \lambda_1 s} ds < +\infty \ ,$$

finally we can applied Corollary 24. ■

The following example show a similar method applied in [8, 15, 21], among others, to obtain Blow-up in the Keller-Segel system.

Theorem 27 *(**Energy Method**) Let u_o such that*

$$E\left(u_0\right) = \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{1}{p+1}\int_\Omega u_0^{p+1} dx < 0\,. \tag{2.30}$$

Then equation (2.23) *does not have a smooth solution defined for all* $t \geq 0$ *.*

Proof. Multiplying (2.23) by u_t and integrating we get

$$\int_\Omega u_t^2 dx = \int_\Omega u_t \triangle u dx + \int_\Omega u_t u^p dx\,, \tag{2.31}$$

then applying integration by parts in (2.31), we conclude

$$-\int_\Omega u_t^2 dx = \left(\frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{u^{p+1}}{p+1} dx\right)_t\,. \tag{2.32}$$

We define the Energy Functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{u^{p+1}}{p+1} dx\,. \tag{2.33}$$

Using the estimate (2.32),

$$\frac{d}{dt}E(u) \leq 0\,,$$

this last inequality implies

$$E(u) \leq E(u^0)$$

and therefore by (2.30) we have $E(u) \leq 0$. In the same way,

$$\begin{aligned}
\left(\frac{1}{2|\Omega|}\int_\Omega u^2 dx\right)_t &= \frac{1}{|\Omega|}\int_\Omega u u_t dx \\
&= \frac{1}{|\Omega|}\int_\Omega u\left(\triangle u + u^p\right) dx \\
&= \frac{1}{|\Omega|}\int_\Omega \left(-|\nabla u|^2 + u^{p+1}\right) dx \\
&= \frac{1}{|\Omega|}\left(-2E - \frac{2}{p+1}\int_\Omega u^{p+1} dx + \int_\Omega u^{p+1} dx\right)\,.
\end{aligned}$$

Calling $v = \frac{1}{2|\Omega|}\int_\Omega u^2 dx$ we have

$$\begin{aligned}
v_t &\geq \frac{1}{|\Omega|}\left(1 - \frac{2}{p+1}\right)\int_\Omega u^{p+1} dx \\
&= \frac{1}{|\Omega|}\left(1 - \frac{2}{p+1}\right)\int_\Omega (u^2)^{\frac{p+1}{2}} dx\,.
\end{aligned}$$

Using Jensen' inequality taking $\frac{p+1}{2} > 1$ result that

$$
\begin{aligned}
v_t &\geq \left(1 - \frac{2}{p+1}\right)\left(\frac{1}{|\Omega|}\int_\Omega u^2 dx\right)^{\frac{p+1}{2}} \\
&= \left(1 - \frac{2}{p+1}\right) v^{\frac{p+1}{2}} .
\end{aligned}
$$

We know that $f(s) = \left(1 - \frac{2}{p+1}\right) s^{\frac{p+1}{2}}$ is a convex function, the result is obtained as consequence of Theorem 25. ■

2.2.4 Blow-up in Chemotaxis Model

In this part, we remember a results obtain to Keller-Segel model, this show the method of moments. We first recall the Keller-Segel model, which is the system of partial equations

$$
\left\{
\begin{array}{rcl}
\frac{\partial u}{\partial t} &=& \nabla(\mu\nabla u) - \nabla.(\chi(u,v)\nabla v) + G(u,v) , \\
\frac{\partial v}{\partial t} &=& F(u,v) ,
\end{array}
\right. \tag{2.34}
$$

for $x \in \Omega$, $\partial\Omega \in C^1$. The biological meaning of variables are:

- $u(x,t)$ density at x in time t
- $v(x,t)$ chemical substance
- μ, r, χ, β positive constants
- $G(u,v)$ fitness function

We assume that there are no deaths or births of individuals, i.e,

$$
G(u,v) = 0 , \tag{2.35}
$$

after a change of variable, we obtain

$$
\left.
\begin{array}{rcl}
\frac{\partial u}{\partial t} &=& \mu\triangle u - \chi\nabla(u\nabla v) , \\
\frac{\partial v}{\partial t} &=& r\triangle u - \mu v + \beta u ,
\end{array}
\right\} \tag{2.36}
$$

with the following initial conditions

$$
\left.
\begin{array}{c}
u(.,0) = u_0 > 0 , \\
v(.,0) = v_0 > 0 , \\
\frac{\partial u}{\partial \vec{n}}(.,t) = 0 , x \in \partial\Omega , \\
\frac{\partial v}{\partial \vec{n}}(.,t) = 0 , x \in \partial\Omega .
\end{array}
\right\} \tag{2.37}
$$

Without loss of generality, we can make a rescaling in time and take $\mu = 1$. Further, we define

$$
\overline{w} = \frac{1}{|\Omega|}\int_\Omega w dx .
$$

Multiplying the second equation in (2.36) by $\frac{1}{r|\Omega|}$ and integrating on Ω , we get

$$\frac{1}{r}\partial_t\overline{v}=\frac{\mu}{r}\overline{v}+\frac{\beta}{r}\overline{u}\,.$$

Divergence Theorem together with condition (2.37) imply

$$\frac{1}{|\Omega|}\int_{\Omega}\triangle v=0\,,$$

now rearranging the terms, we conclude that

$$\frac{1}{r}\left(\partial_t-\mu\right)\overline{v}=\frac{\beta}{r}\overline{u}\,. \tag{2.38}$$

In the same way, multiplying again the second equation of (2.36) by $\frac{1}{r}$, we get

$$\frac{1}{r}\left(\partial_t-\mu\right)v=\triangle u+\frac{\beta}{r}u\,. \tag{2.39}$$

We remember that $\triangle\overline{v}=0$ and integrating on Ω the first equation of (2.36) , we have

$$\begin{aligned}\partial_t\overline{u}&=-\frac{\chi}{|\Omega|}\int_{\Omega}\nabla\left(u\nabla v\right)dx\\&=-\frac{\chi}{|\Omega|}\int_{\partial\Omega}u\nabla v dx\,.\end{aligned}$$

Therefore, by initial conditions

$$\partial_t\overline{u}=0\,.$$

Thus, we obtain mass conservation,

$$\overline{u}(t)=\overline{u}_0\,.$$

Next, denoting $\widetilde{v}=v-\overline{v}$, $\alpha=\frac{\beta}{r}$, hence subtracting (2.38) and (2.39) result

$$\frac{1}{r}\left(\partial_t-\mu\right)\widetilde{v}=\triangle\widetilde{v}+\alpha\left(u-\overline{u_0}\right)\,. \tag{2.40}$$

Assuming $r\gg1$, finally we conclude

$$\triangle\widetilde{v}+\alpha\left(u-\overline{u_0}\right)=0\,. \tag{2.41}$$

In conclusion, we can redefined (2.36)

$$\left.\begin{array}{rcl}\partial_t u&=&\triangle u-\chi\nabla.\left(u\nabla v\right)\,,\\0&=&\triangle v+\alpha\left(u-\overline{u_0}\right)\,.\end{array}\right\} \tag{2.42}$$

Finally, doing the change of variable $v^*=\frac{\widetilde{v}}{\alpha\overline{u_0}}$, $u^*=\frac{\widetilde{u}}{\alpha\overline{u_0}}$, returining to u and v, whereby obtain

$$\left.\begin{array}{c}\partial_t u=\triangle u-\chi\nabla.\left(u\nabla v\right)\,,\\-\triangle v=\alpha\left(u-1\right)\,,\\u(.,t)=u_0\geq0\,,\\v(.,t)=v_0\geq0\,,\\\frac{\partial v}{\partial\overrightarrow{n}}=\frac{\partial u}{\partial\overrightarrow{n}}=0\,,\end{array}\right\} \tag{2.43}$$

which is the simplified Keller-Segel system defined on $\Omega\subseteq\mathbb{R}^2$, $\partial\Omega\in C^1$, under mass conservation hypothesis.

Theorem 28 *Let consider*

$$\frac{\Theta}{2\pi}(8\pi - \chi\Theta) + \frac{1}{2\pi}\chi M(0) < 0 \tag{2.44}$$

where

$$\Theta = \int_{\Omega} u dx = \int_{\Omega} u^0 dx \ , \ M(t) = \int_{\Omega} u(x,t)|x|^2 dx \ ,$$

for $x \in \Omega = B(0, L)$, $u_0 \in C^2(\Omega)$ *and radial symmetric function. Then the solutions to* (2.43) *have finite existence time.*

Remark 29

1. *If* $M(0) \ll 1$, *i.e, the first moment in* 0 *small, then condition* $\frac{8\pi}{\chi} < \Theta$ *is necessary and sufficient to obtain Blow-up. Biologically means close to zero if there is an initial concentration of individuals, this condition persists at time.*
2. *Remember the following formulas:*
   - $\nabla w = w_r \frac{\vec{r}}{r}$
   - $\triangle w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$
   - $m(r) = \int_{B(0,r)} u dx = 2\pi \int_0^r u\overline{r} d\overline{r}$
   - $\frac{\partial m}{\partial r} = 2\pi r u$

Proof. We consider a radial symmetric initial condition, then we have radial symmetric solution to (2.43) obtaining

$$-\frac{1}{r}\frac{\partial}{\partial r}(r v_r) = u - 1 \ , \tag{2.45}$$

integrating

$$-\int_{\Omega} \frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}(\overline{r} v_{\overline{r}}) dx = \int_{\Omega} (u-1) dx \ , \tag{2.46}$$

using the radial formulas recalling in Remark 29, we have

$$-\int_0^r \frac{\partial}{\partial \overline{r}}(\overline{r} v_{\overline{r}}) d\overline{r} = \int_0^r \overline{r}(u-1) d\overline{r} \ , \tag{2.47}$$

it is equivalent to

$$-r v_r = \int_0^r u\overline{r} d\overline{r} - \frac{r^2}{2} \ , \tag{2.48}$$

therefore

$$r\frac{\partial v}{\partial r} = -\frac{1}{2\pi}\left(m - \pi r^2\right) \ . \tag{2.49}$$

On the other hand, from (2.42) we have

$$\partial_t u = \triangle u - \chi\nabla(u\nabla v) \ , \tag{2.50}$$

multiplying (2.50) by $|x|^2$ and integrating

$$\int_\Omega \partial_t |x|^2 u dx = \int_\Omega |x|^2 \triangle u dx - \chi \int_\Omega |x|^2 \nabla \left(u \nabla v\right) dx \, . \tag{2.51}$$

Thus,

$$\begin{aligned}
\partial_t M(t) &= \int_\Omega |x|^2 \triangle u dx - \chi \int_\Omega |x|^2 \nabla \left(u\nabla v\right) dx \\
&= \int_\Omega \triangle |x|^2 u dx + \int_{\partial\Omega} \left(|x|^2 \frac{\partial u}{\partial \overrightarrow{n}} - u \frac{\partial |x|^2}{\partial \overrightarrow{n}} \right) ds + \chi \int_\Omega \nabla |x|^2 \left(u \nabla v\right) dx \\
&= 4\Theta - 2 \int_{\partial\Omega} u x \overrightarrow{n} ds - \chi \left(\int_\Omega \nabla \left(|x|^2 u \nabla v \right) dx - \int_\Omega \nabla \left(|x|^2 \right) u \nabla v dx \right) \\
&= 4\Theta - 2 \int_{\partial\Omega} u |x| ds - \chi \int_{\partial\Omega} |x|^2 u \nabla v . \overrightarrow{n} dx + 2\chi \int_\Omega u \left(x . \nabla v \right) dx \\
&\leq 4\Theta + 2\chi 2\pi \int_0^L u \overline{r} v_{\overline{r}} \overline{r} d\overline{r} \, ,
\end{aligned}$$

using (2.50) we conclude

$$\begin{aligned}
\partial_t M(t) &\leq 4\Theta - 2\chi \int_0^L u \left(m - \pi \overline{r}^2 \right) \overline{r} d\overline{r} \\
&= 4\Theta - 2\chi \left[\int_0^L u m \overline{r} d\overline{r} - \pi \int_0^L u \overline{r}^3 d\overline{r} \right] \\
&= 4\Theta - 2\chi \left[\frac{m}{2\pi} \int_0^L \frac{\partial m}{\partial \overline{r}} d\overline{r} - \frac{1}{2} M \right] \\
&= 4\Theta - 2\chi \left[\frac{1}{4\pi} \Theta^2 - \frac{1}{2} M \right] \, ,
\end{aligned}$$

hence

$$\partial_t M(t) \leq \frac{\Theta\chi}{2\pi} \left(\frac{8\pi}{\chi} - \Theta \right) + \chi M \, . \tag{2.52}$$

M is a decreasing function for all t, since by hypothesis

$$\frac{\Theta\chi}{2\pi} \left(\frac{8\pi}{\chi} - \Theta \right) \leq 0 \, .$$

Integrating in time (2.52), we obtain

$$M(t) \leq \left[\frac{\Theta\chi}{2\pi} \left(\frac{8\pi}{\chi} - \Theta \right) + \chi M(0) \right] t + M(0) \, , \tag{2.53}$$

therefore there exists t_0 such that

$$M(t_0) = 0 \text{ and } M(t) \leq 0 \text{ for all } t > t_0 \, ,$$

consequently,

$$\int_{\Omega} u(x,t)|x|^2 dt = 0 \, , \, t \geq t_0 \, . \tag{2.54}$$

Then (2.54) implies that $u(x,t) = 0$, a.e for all $t > t_0$, therefore $\Theta = 0$ this is a contradiction with the mass conservation. ■

2.3 Hardy-Littlewood-Sobolev' inequality for systems

The most fundamental tool used through this work to prove the sharp condition for existence and Blow-up of solutions to Keller-Segel system (2.3) is the logarithmic Hardy-Littlewood-Sobolev' inequality for systems, which we proceed to recall now. First in [49], it is defined the space of functions,

$$\Gamma_M\left(\mathbb{R}^2\right) = \left\{ \widetilde{\rho} = \left(\widetilde{\rho}_i\right)_{i\in I} : \widetilde{\rho}_i \geq 0, \int_{\mathbb{R}^2} \widetilde{\rho}_i \left|\log \widetilde{\rho}_i\right| dx < \infty, \int_{\mathbb{R}^2} \widetilde{\rho}_i = M_i, \int_{\mathbb{R}^2} \widetilde{\rho}_i \log\left(1+|x|^2\right) < \infty \, , \forall i \in I \right\}$$

where $M = \left(M_i\right)_{i\in I}$ is given. Next, we define the functional $F : \Gamma_M\left(\mathbb{R}^2\right) \to \mathbb{R}$ by

$$F\left[\widetilde{\rho}\right] = \sum_{i\in I} \int_{\mathbb{R}^2} \widetilde{\rho}_i \log \widetilde{\rho}_i dx + \frac{1}{4\pi} \sum_{j,i\in I} a_{i,j} \int_{2} \int_{\mathbb{R}^2} \widetilde{\rho}_i\left(x\right) \log|x-y| \, \widetilde{\rho}_j\left(y\right) dxdy \, ,$$

together with the polynomial condition

$$\Lambda_J\left(M\right) = 8\pi \sum_{i\in J} M_i - \sum_{i,j\in J} a_{ij} M_i M_j \, ,$$

for all $J \neq \varnothing$ and $J \subseteq I$. Then we have the following result,

Theorem 30 ***Hardy-Litlewood-Sobolev's inequality for systems***
Let $A = (a_{ij})$ a symmetric matrix such that $a_{ij} \geq 0$ for all $i, j \in I$ and $M \in \mathbb{R}^n_+$. Then

$$\Lambda_I\left(M\right) = 0 \, , \tag{2.55}$$

together with

$$\begin{aligned} \Lambda_J\left(M\right) &\geq 0 \, , \textit{for all } J \subseteq I \\ if \; \Lambda_J\left(M\right) &= 0 \, , \textit{ for some } J \textit{, then } a_{ii} + \Lambda_{J\setminus\{i\}}\left(M\right) > 0 \, , \textit{for all } i \in J \, , \end{aligned}$$

are necessary and sufficient conditions for the boundlessness from below of F on $\Gamma_M\left(\mathbb{R}^2\right)$. In particular, there exists a minimize ρ of F over $\Gamma_M\left(\mathbb{R}^2\right)$ if and only if

$$\Lambda_I\left(M\right) = 0 \, , \textit{ and } \Lambda_J\left(M\right) > 0 \, , \textit{for all } J \subset I \, .$$

Proof. See [49], Th. 4. ■

2.4 Blow-up and the threshold condition

2.4.1 Blow-up without radial initial condition

Lemma 31 *Let* u_1 , u_2 *and* v *smooth solutions of* (2.3) . *Let define the second moment* $m(t)$ *for the whole population by,*

$$m(t) := \frac{\pi}{\chi_1}\int_{\mathbb{R}^2} u_1 \left|x\right|^2 dx + \frac{\pi}{\chi_2}\int_{\mathbb{R}^2} u_2 \left|x\right|^2 dx\,, \tag{2.56}$$

then we have

$$\frac{d}{dt}m(t) = \frac{4\pi\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1+\theta_2)^2\,. \tag{2.57}$$

Proof. Multiplying the first equation of (2.3) by $|x|^2$ and integrating yields,

$$\partial_t\int_{\mathbb{R}^2} u_1 \left|x\right|^2 dx = \int_{\mathbb{R}^2} \left|x\right|^2 \Delta u_1 dx - \chi_1 \int_{\mathbb{R}^2} \left|x\right|^2 \nabla\cdot(u_1\nabla v)dx\,,$$

then using Green' first identity we obtain

$$\partial_t\int_{\mathbb{R}^2} u_1 \left|x\right|^2 dx = 4\int_{\mathbb{R}^2} u_1 dx + 2\chi_1\int_{\mathbb{R}^2} u_1(x\cdot\nabla v)dx\,.$$

From (2.3), we have that

$$\nabla v = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-y}{\left|x-y\right|^2}\left(u_1(y,t)+u_2(y,t)\right)dy\,.$$

Next, we compute using mass conservation,

$$\partial_t\int_{\mathbb{R}^2} u_1 \left|x\right|^2 dx = 4\theta_1 - \frac{\chi_1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\left(x\cdot\frac{x-y}{\left|x-y\right|^2}u_1(x,t)\left(u_1(y,t)+u_2(y,t)\right)\right)dydx\,. \tag{2.58}$$

Similarly,

$$\partial_t\int_{\mathbb{R}^2} u_2 \left|x\right|^2 dx = 4\theta_2 - \frac{\chi_2}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\left(x\cdot\frac{x-y}{\left|x-y\right|^2}u_2(x,t)\left(u_1(y,t)+u_2(y,t)\right)\right)dydx\,. \tag{2.59}$$

Adding (2.58) and (2.59), it follows

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\pi}{\chi_1}\int_{\mathbb{R}^2} u_1 \left|x\right|^2 dx + \frac{\pi}{\chi_2}\int_{2} u_2 \left|x\right|^2 dx\right)\\ &= \frac{4\pi\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \int_{\mathbb{R}^2\times\mathbb{R}^2} x\cdot\frac{x-y}{\left|x-y\right|^2}\left(u_1(x,t)+u_2(x,t)\right)\left(u_1(y,t)+u_2(y,t)\right)dydx\,.\end{aligned}$$

After, using the symmetry in the variables x and y in the last integral we get,

$$\frac{d}{dt}\left(\frac{\pi}{\chi_1}\int_{\mathbb{R}^2} u_1 \left|x\right|^2 dx + \frac{\pi}{\chi_2}\int_{\mathbb{R}^2} u_2 \left|x\right|^2 dx\right)$$
$$= \frac{4\pi\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2} (x-y)\cdot\frac{x-y}{\left|x-y\right|^2}\left(u_1(x,t)+u_2(x,t)\right)\left(u_1(y,t)+u_2(y,t)\right) dydx$$
$$= \frac{4\pi\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \left(u_1(x,t)+u_2(x,t)\right)\left(u_1(y,t)+u_2(y,t)\right) dydx$$
$$= \frac{4\pi\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1+\theta_2)^2 \, .$$

■

Therefore, if we have that the initial masses θ_1 and θ_2 satisfy,

$$\frac{4\pi\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1+\theta_2)^2 < 0 \tag{2.60}$$

we arrive at the conclusion that $m(t)$ should become negative in finite time, which is impossible since u_1 and u_2 are nonnegative. Consequently, there is a finite blow-up time T^*. In fact being as $\int_{\mathbb{R}^2} u_i \left|x\right|^2 dx \leq \frac{\chi_i}{\pi} m(t)$ both of the density variables u_1 and u_2 blows up. A straightforward generalization of system (2.3) for n species show that in the region

$$4\pi \sum_{i=1}^{n} \frac{\theta_i}{\chi_i} - \frac{1}{2}\left(\sum_{i=1}^{n}\theta_i\right)^2 < 0 \, ,$$

therefore, there is Blow-up for the n density variables.

2.4.2 Blow-up in radial case

Consider u_{10} , u_{20} radial initial conditions. Let us define the accumulative mass variables for system (2.3),

$$M_i(r,t) := \int_{D(0,r)} u_i(x,t) = 2\pi \int_0^r u_i(\rho,t)\rho d\rho \, ,$$

for $i = 1, 2$. We are going to prove that if enough species are concentrate at the origin (i.e we have a small initial moment) then any of the inequalities

$$\theta_1 < \frac{8\pi}{\chi}\mu \, , \theta_2 < \frac{8\pi}{\chi_2} \, ,$$

implies Blow-up. Multiplying the first equation of (2.3) by $|x|^2$ and integrating on $\mathbb{R}^2$

$$\frac{d}{dt}\int_{\mathbb{R}^2} u_1|x|^2 dx = \mu\int_{\mathbb{R}^2} \triangle u_1|x|^2 dx - \chi_1\int_{\mathbb{R}^2} \nabla\cdot(u_1\nabla v)dx \, , \tag{2.61}$$

then using the first Green' identity in (2.61) , we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} u_1|x|^2dx = \mu\int_{\mathbb{R}^2} \triangle u_1|x|^2dx - \chi_1\int_{\mathbb{R}^2} \nabla\cdot(u_1\nabla v)dx\ . \tag{2.62}$$

In cylindrical coordinates, the equation for chemical substance is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + u_1 + u_2 = 0\ , \tag{2.63}$$

multiplying by r and integrating upon $(0,r)$ (2.63) , we have

$$\begin{aligned} r\frac{\partial v}{\partial r} &= -\int_0^r \rho u_1 d\rho - \int_0^r \rho u_2 d\rho \\ &= -\frac{1}{2\pi}\int_{D(0,r)} u_1(x,t) - \frac{1}{2\pi}\int_{D(0,r)} u_2(x,t) \\ &= -\frac{M_1+M_2}{2\pi}\ . \end{aligned}$$

Therefore,

$$\frac{\partial v}{\partial r} = -\frac{M_1+M_2}{2\pi r}\ . \tag{2.64}$$

Using (2.64) and the general identity for radial functions

$$x\cdot\nabla\phi(x) = r\frac{\partial\phi}{\partial r}\ , \tag{2.65}$$

we obtain the following calculation

$$\begin{aligned} \int_{\mathbb{R}^2} u_1\left(x\cdot\nabla v\right)dx &= 2\pi\int_0^{+\infty} u_1\rho\frac{\partial v}{\partial\rho}\rho d\rho \\ &= -2\pi\int_0^{+\infty} u_1\frac{M_1+M_2}{2\pi}\rho d\rho \\ &= -\int_0^{+\infty} u_1\left(M_1+M_2\right)\rho d\rho \\ &\le -\int_0^{+\infty} u_1 M_1\rho_1 d\rho \\ &= -\frac{1}{4\pi}\theta_1\ . \end{aligned}$$

From this last estimation and (2.62), we conclude

$$\begin{aligned} \frac{dm}{dt} &\le 4\mu\int_{\mathbb{R}^2} u_1 dx + 2\chi_1\left(-\frac{1}{4\pi}\theta_1^2\right) \\ &= 4\theta_1\mu\left(1-\frac{\chi_1\theta_1}{8\pi\mu}\right)\ . \end{aligned}$$

Applying a similar argument, we prove the other inequality. Finally, we obtain the finite time of existence using the same argument for Blow-up in non radial case.

2.5 Global Existence

2.5.1 Energy and Entropy estimates

Let us proceed formally to find a free energy functional in our system. First, we rewrite the equation for u_1 in (2.3) making the following estimations,

$$\begin{aligned}\partial_t u_1 &= \mu\triangle u_1 - \chi_1\nabla.\left(u_1\nabla v\right)\\ &= \nabla.\left[\mu\nabla u_1 - \chi_1(u_1\nabla v)\right]\\ &= \nabla.\left[u_1\left(\mu\nabla\log(u_1) - \chi_1\nabla v\right)\right]\ ,\end{aligned}$$

therefore

$$\partial_t u_1 = \nabla\cdot u_1\nabla\left(\mu\log u_1 - \chi_1 v\right)\ . \tag{2.66}$$

Next, we multiply both sides of (2.66) by $\mu\log u_1 - \chi_1 v$ and integrate to obtain,

$$\int_{\mathbb{R}^2} u_{1t}\left(\mu\log u_1 - \chi_1 v\right)dx = \int_{\mathbb{R}^2}\left(\mu\log u_1 - \chi_1 v\right)\nabla\cdot\left[u_1\nabla\left(\mu\log u_1 - \chi_1 v\right)\right]dx\ . \tag{2.67}$$

Then, using mass conservation and assuming the necessaries conditions to applied integration by parts formula, we see that (2.67) is equivalent to

$$\frac{d}{dt}\int_{\mathbb{R}^2}\mu u_1\log u_1 dx - \chi_1\int_{\mathbb{R}^2} u_{1t} v dx = -\int_{\mathbb{R}^2} u_1\left|\nabla\left(\mu\log u_1 - \chi_1 v\right)\right|^2 dx\ . \tag{2.68}$$

Similarly, it holds that

$$\frac{d}{dt}\int_{\mathbb{R}^2} u_2\log u_2 dx - \chi_2\int_{\mathbb{R}^2} u_{2t} v dx = -\int_{\mathbb{R}^2} u_2\left|\nabla\left(\log u_2 - \chi_2 v\right)\right|^2 dx\ . \tag{2.69}$$

Now, multiplying by $\frac{1}{\chi_1}$ (2.68) and by $\frac{1}{\chi_2}$ (2.69), adding these results we get that

$$\begin{aligned}&\frac{d}{dt}\left\{\int_{\mathbb{R}^2}\frac{\mu}{\chi_1}u_1\log u_1 dx + \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2\log u_2 dx\right\} - \int_{\mathbb{R}^2}\left(u_{1t} + u_{2t}\right)v dx\\ &= -\int_{\mathbb{R}^2} u_1\left|\nabla\left(\mu\log u_1 - \chi_1 v\right)\right|^2 dx - \int_{\mathbb{R}^2} u_2\left|\nabla\left(\log u_2 - \chi_2 v\right)\right|^2 dx\ .\end{aligned} \tag{2.70}$$

Notice that by initial assumptions,

$$\int_{\mathbb{R}^2}\left(u_{1t} + u_{2t}\right)v dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(u_1 + u_2)v dx\ . \tag{2.71}$$

As conclusion, we deduce from (2.70) and (2.71) that

$$\frac{d}{dt}\left\{\int_{\mathbb{R}^2}\frac{\mu}{\chi_1}u_1\log u_1 dx + \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2\log u_2 dx - \frac{1}{2}\int_{\mathbb{R}^2}(u_1 + u_2)v dx\right\} \le 0\ . \tag{2.72}$$

The equality (2.72) motivate us to define the free energy functional for system (2.3) as

$$E(t) := \frac{\mu}{\chi_1}\int_{\mathbb{R}^2} u_1 \log u_1 dx + \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2 \log u_2 dx - \frac{1}{2}\int_{\mathbb{R}^2} u_1 v dx - \frac{1}{2}\int_{\mathbb{R}^2} u_2 v dx \,. \tag{2.73}$$

In order to validate our estimations, we suppose that,

$$u_1, u_2 \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^2((0,T); H^1(\mathbb{R}^2))\,,$$

$$u_1(1+|x|^2)\,,\, u_2(1+|x|^2)\,,\, u_1\log u_1\,,\, u_2 \log u_2 \text{ are bounded in } L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))\,,$$

Additionally,

$$\nabla\sqrt{u_1}\,,\, \nabla\sqrt{u_2} \in L^1_{loc}(\mathbb{R}^+\,,\, L^1(\mathbb{R}^2))$$

and

$$\nabla v \in L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R}^2)\,.$$

Then, we have that

$$\frac{d}{dt}E(t) = -\frac{1}{\chi_1}\int_{\mathbb{R}^2} u_1 \left|\mu\nabla \log u_1 - \nabla\chi_1 v\right|^2 dx - \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2 \left|\nabla \log u_2 - \nabla\chi_2 v\right|^2 dx \le 0\,. \tag{2.74}$$

As a consequence of (2.74) and the Hardy-Littlewood-Sobolev' inequality [8, 15] was obtained in [15] a first non optimal result about entropy bound, which is summarized in the following theorem,

Theorem 32 *If u_1 and u_2 are positive solutions of* (2.3) *on the interval* $[0,T)$ *and* $\chi_1 \le \chi_2$ *then we have the following Entropy estimates*

- *if $\mu > 1$ then*

  $$\left(1 - \frac{M\chi_2}{8\pi}\right)\int_0^T \int_{\mathbb{R}^2} \left(\frac{1}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{1}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right) dxdt \le C_T$$

  where C_T is a constant depending on T and $M = \theta_1 + \theta_2$.

- *If $\mu \le 1$ then*

  $$\left(1 - \frac{M\chi_2}{8\pi\mu}\right)\int_0^T \int_{\mathbb{R}^2} \left(\frac{1}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{1}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right) dxdt \le \overline{C}_T$$

  where $\overline{C}_T$ is a constant depending on T and $M = \theta_1 + \theta_2$.

Proof. To more details about this preliminary result to see [[15],Theorem 1] attached in the second part of this thesis. ■

Theorem 32 gives bounds for the entropy which is the key tool for the proof of global existence for system (2.3), but it is not sharp condition. In order to improve an optimal result, it would be desirable to use the HLS inequality for systems developed by I. Shafrir & G. Wolansky in [49].

However, as we will show after, a direct application of this tool in our system do not give the expect optimal result that we are looking for. We will prove how an adequate introduction of some auxiliary parameters in (2.74) allows us to improve the sharp result of global existence published in [21], mainly, we will show that if θ_1, θ_2 satisfy,

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1+\theta_2)^2 \geq 0, \;\; \theta_1 < \mu\frac{8\pi}{\chi_1} \;,\; \theta_2 < \frac{8\pi}{\chi_2} \;,$$

then exists global solution in time. No kind of radial symmetry initial condition is assumed.

The first result of this subsection gives us bounds for the entropy functional. We achieve our aim through an appropriate use of the HLS inequality for systems, Th. 30. The main idea of the proof read as follows: Given that a direct application of the HLS inequality would allows us to get bounds *only on a curve* of the $\theta_1\theta_2-$plane for the entropy $\int_{\mathbb{R}^2} u_i(x,t)\log u_i(x,t)dx$, $i=1,2$, we introduce some parameters before apply the HLS inequality. Then we can 'move', 'shrink' and 'dilate' this family of curves, in such a way, that the full region (2.79) is swept and therefore, obtaining an estimation for (2.80) in this region.

We suppose in this part that,

$$\left.\begin{array}{c} u_{10} \;,\, u_{20} \in L^1(\mathbb{R}^2, (1+|x|^2)dx) \;, \\ u_{10}\log u_{10} \;,\, u_{20}\log u_{20} \in L^1(\mathbb{R}^2, dx) \;. \end{array}\right\} \tag{2.75}$$

Lemma 33 (Lower bound for the entropy functional) *Consider a non-negative weak solution of* (2.3) *, such that* $u_i(1+|x|^2)$, $i=1,2$ *are bounded in* $L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$. *Then we have,*

$$\int_{\mathbb{R}^2} u_i\left(x,t\right)\log u_i\left(x,t\right) \geq M\log M - M\log\left[\pi\left(1+t\right)\right] - C \;,\; i=1,2 \;.$$

Proof. In the following C will denote a generic constant. We have from subsection (2.4),

$$\frac{d}{dt}\int_{\mathbb{R}^2}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)|x|^2\,dx = \frac{4\theta_1}{\chi_1}\mu + \frac{4\theta_2}{\chi_2} - \frac{1}{2\pi}\left(\theta_1+\theta_2\right)^2 \;. \tag{2.76}$$

We define

$$n = \frac{\mu}{\chi_1}u_1 + \frac{1}{\chi_2}u_2 \;,$$

$$K = \frac{4\theta_1}{\chi_1}\mu + \frac{4\theta_2}{\chi_2} - \frac{1}{2\pi}\left(\theta_1+\theta_2\right)^2 \;.$$

Thus we obtain

$$\int_{\mathbb{R}^2} n(x,t)\,|x|^2\,dx = Kt + \int_{\mathbb{R}^2} n(x,0)\,|x|^2\,dx \leq C(1+t) \;, \tag{2.77}$$

where $C = \max\left\{K, \int\limits_{\mathbb{R}^2} n(x,0)\,|x|^2\,dx\right\}$. From the inequality $u_i \le Cn$, where $i = 1,2$ and from (2.77) we deduce that

$$\int\limits_{\mathbb{R}^2} u_i(x,t)\,|x|^2\,dx \le C(1+t)\ ,\ i = 1,2\ .$$

Using the same idea presented in [8, Lemma 2.5] we observe that

$$\begin{aligned} \int\limits_{\mathbb{R}^2} u_i\left(x,t\right)\log u_i\left(x,t\right) &\ge \frac{1}{1+t}\int\limits_{\mathbb{R}^2} u_i\left(x,t\right)|x|^2 - C + \int\limits_{\mathbb{R}^2} u_i\left(x,t\right)\log u_i\left(x,t\right) \\ &= \int\limits_{\mathbb{R}^2} u_i\left(x,t\right)\log\left[\frac{u_i(x,t)}{e^{-\frac{|x|^2}{1+t}}}\right] - C\ . \end{aligned} \tag{2.78}$$

Let us now define the variable α as follow

$$\alpha\left(x,t\right) = \frac{1}{\pi\left(1+t\right)}\exp\left(-\frac{|x|^2}{1+t}\right)\ .$$

We obtain then from (2.78) that

$$\begin{aligned} \int\limits_{\mathbb{R}^2} u_i\left(x,t\right)\log u_i\left(x,t\right) &\ge \int\limits_{\mathbb{R}^2} u_i\left(x,t\right)\log\left[\frac{u_i\left(x,t\right)}{\mu\left(x,t\right)}\right]dx - M\log\left[\pi\left(1+t\right)\right] - C \\ &= \int\limits_{\mathbb{R}^2} \frac{u_i\left(x,t\right)}{\mu\left(x,t\right)}\log\left[\frac{u_i\left(x,t\right)}{\alpha\left(x,t\right)}\right]\alpha\left(x,t\right)dx - M\log\left[\pi\left(1+t\right)\right] - C \end{aligned}$$

where $M = \frac{\mu}{\chi_1}\theta_1 + \frac{1}{\chi_2}\theta_2$. Therefore, by Jensen's inequality we get from (2.78) that

$$\int\limits_{\mathbb{R}^2} u_i\left(x,t\right)\log u_i\left(x,t\right) \ge M\log M - M\log\left[\pi\left(1+t\right)\right] - C\ .$$

■

Theorem 34 (Upper bound for the entropy functional) *Consider a non-negative weak solution of (2.3), such that* $u_i\left(1+|x|^2\right)$, $u_i\log u_i$, $i = 1,2$ *are bounded in* $L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$. *If* θ_1, θ_2 *satisfy,*

$$\theta_1 < \frac{8\pi}{\chi_1}\mu\ ,\ \theta_2 < \frac{8\pi}{\chi_2}\ ,\ 8\pi\left(\frac{\theta_1}{\chi_1}\mu + \frac{\theta_2}{\chi_2}\right) - \left(\theta_1+\theta_2\right)^2 > 0\ . \tag{2.79}$$

Then

$$\int_{\mathbb{R}^2} u_i(x,t)\log u_i(x,t)dx \le C\ , \tag{2.80}$$

where $i = 1,2$ *and* C *is a constant depending only on the parameters* θ_1, θ_2, μ, χ_1, χ_2 *and* $E(0)$.

Proof. By (2.73) for all $t > 0$, we have that

$$E(t) \leq E(0) \ .$$

Consequently, we have the computation

$$\begin{aligned}
&\frac{\mu}{\chi_1}\int\limits_{\mathbb{R}^2} u_1(x,t)\log u_1(x,t)\,dx + \frac{1}{\chi_2}\int\limits_{\mathbb{R}^2} u_2(x,t)\log u_2(x,t)\,dx \\
\leq\ & E(0) - \frac{1}{4\pi}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} u_1(x,t)\,u_1(y,t)\log|x-y|\,dxdy - \frac{1}{4\pi}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} u_1(x,t)\,u_2(y,t)\log|x-y|\,dxdy \\
& -\frac{1}{4\pi}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} u_2(x,t)\,u_1(y,t)\log|x-y|\,dxdy - \frac{1}{4\pi}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} u_2(x,t)\,u_2(y,t)\log|x-y|\,dxdy \ .
\end{aligned}$$

Considering the two positives parameters a and b such that

$$a > \chi_1 \ , b > \chi_2 \tag{2.81}$$

and including them in the last inequality we obtain

$$\begin{aligned}
&\frac{\mu}{\chi_1}\int\limits_{\mathbb{R}^2} u_1(x,t)\log u_1(x,t)\,dx + \frac{1}{\chi_2}\int\limits_{\mathbb{R}^2} u_2(x,t)\log u_2(x,t)\,dx \\
\leq\ & E(0) - \frac{a^2}{\mu^2 4\pi}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} \frac{\mu u_1(x,t)}{a}\frac{\mu u_1(y,t)}{a}\log|x-y|\,dxdy \\
& -\frac{ab}{\mu 4\pi}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} \frac{\mu u_1(x,t)}{a}\frac{u_2(y,t)}{b}\log|x-y|\,dxdy \\
& -\frac{ab}{\mu 4\pi}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} \frac{u_2(x,t)}{b}\frac{\mu u_1(y,t)}{a}\log|x-y|\,dxdy \\
& -\frac{b^2}{4\pi}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} \frac{u_2(x,t)}{b}\frac{u_2(y,t)}{b}\log|x-y|\,dxdy \ .
\end{aligned} \tag{2.82}$$

By doing so, we can apply now the HLS inequality for systems (Th.30) to the functions $\frac{\mu u_1}{a}$ and $\frac{u_2}{b}$ on identity (2.82) getting that,

$$\begin{aligned}
&\frac{\mu}{\chi_1}\int\limits_{\mathbb{R}^2} u_1(x,t)\log u_1(x,t) + \frac{1}{\chi_2}\int\limits_{\mathbb{R}^2} u_2(x,t)\log u_2(x,t) \\
\leq\ & E(0) - C + \int\limits_{\mathbb{R}^2} \mu\frac{u_1(x,t)}{a}\log\left(\mu\frac{u_1(x,t)}{a}\right)dx + \int\limits_{\mathbb{R}^2} \frac{u_2(x,t)}{b}\log\left(\frac{u_2(x,t)}{b}\right)dx \ ,
\end{aligned}$$

where the conditions for the existence of the constant C given by Th.30 are

$$\begin{aligned}\Lambda_{\{1\}}(M) &= 8\pi\mu\frac{\theta_1}{a} - a^2\left(\frac{\theta_1}{a}\right)^2 \geq 0\,,\\ \Lambda_{\{2\}}(M) &= 8\pi\frac{\theta_2}{b} - b^2\left(\frac{\theta_2}{b}\right)^2 \geq 0\,,\\ \Lambda_{\{1,2\}}(M) &= 8\pi\left(\mu\frac{\theta_1}{a} + \frac{\theta_2}{b}\right) - (a^2\frac{\theta_1}{a}\frac{\theta_1}{a} + ab\frac{\theta_1}{a}\frac{\theta_2}{b} + b^2\frac{\theta_2}{b}\frac{\theta_2}{b}) = 0\,.\end{aligned}$$

Equivalently,

$$\theta_1 \leq \mu\frac{8\pi}{a}\,, \theta_2 \leq \frac{8\pi}{b}\,, 8\pi\left(\mu\frac{\theta_1}{a} + \frac{\theta_2}{b}\right) - (\theta_1+\theta_2)^2 = 0\,. \tag{2.83}$$

Notice that we have proved that conditions (2.83) imply,

$$\begin{aligned}&\mu\left(\frac{1}{\chi_1} - \frac{1}{a}\right)\int_{\mathbb{R}^2} u_1(x,t)\log u_1(x,t)\,dx + \left(\frac{1}{\chi_2} - \frac{1}{b}\right)\int_{\mathbb{R}^2} u_2(x,t)\log u_2(x,t)\,dx\\ &\leq E(0) - C + \frac{\theta_1\mu}{a}\log\frac{\mu}{a} + \frac{\theta_2}{b}\log\frac{1}{b}\,.\end{aligned} \tag{2.84}$$

We have from Lemma 33 that the functional $\int u_i \log u_i dx$ are bounded lowerly for $i = 1, 2$. In the same way, each coefficient of the entropy functional in (2.84) is positive as long as $a > \chi_1$ and $b > \chi_2$. Then, we take parameters a and b on the intervals (χ_1,∞) and (χ_2, ∞), respectively. We conclude that the estimates (2.80) on region (2.79) are true. ■

2.5.2 Weak Solution

In order, in this part we define the weak solution for (2.3). We start multiplying the first equation of (2.3) by a test function $\psi \in C_0^\infty(\mathbb{R}^2)$ and integrating, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}\psi(x)u_1(x,t)dx\\ =\;& \int_{\mathbb{R}^2}\Delta\psi(x)u_1(x,t)dx\\ &-\frac{\chi_1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\nabla\psi(x)\cdot\frac{x-y}{|x-y|^2}\left(u_1(x,t)u_1(y,t) + u_1(x,t)u_2(y,t)\right)dxdy\\ =\;& \int_{\mathbb{R}^2}\Delta\psi(x)u_1(x,t)dx\\ &-\frac{\chi_1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\nabla\psi(x)\cdot\frac{x-y}{|x-y|^2}\left(u_1(x,t)u_1(y,t) + u_1(x,t)u_2(y,t)\right)dxdy\,.\end{aligned}$$

Notice that,

$$\int_{\mathbb{R}^2\times\mathbb{R}^2}\nabla\psi(x)\cdot\frac{x-y}{|x-y|^2}u_1(x,t)u_2(y,t)dxdy = -\int_{\mathbb{R}^2\times\mathbb{R}^2}\nabla\psi(y)\cdot\frac{x-y}{|x-y|^2}u_1(y,t)u_2(x,t)dxdy\,.$$

Therefore, it follows

$$
\begin{aligned}
&\int \nabla\psi(x)\cdot\frac{x-y}{|x-y|^2}u_1(x,t)u_2(y,t)dxdy\\
&=\frac{1}{2}\int\frac{x-y}{|x-y|^2}\cdot\left(u_1(x,t)u_2(y,t)\nabla\psi(x)-u_1(y,t)u_2(x,t)\nabla\psi(y)\right)dxdy\\
&=\frac{1}{2}\int\frac{x-y}{|x-y|^2}\cdot\left[u_1(y,t)u_2(x,t)\left(\nabla\psi(x)-\nabla\psi(y)\right)-u_1(y,t)u_2(x,t)\nabla\psi(x)+u_1(x,t)u_2(y,t)\nabla\psi(x)\right]dxdy\\
&=\frac{1}{2}\int\frac{x-y}{|x-y|^2}\cdot\left[u_1(y,t)u_2(x,t)\left(\nabla\psi(x)-\nabla\psi(y)\right)-\left\{u_1(y,t)u_2(x,t)-u_1(x,t)u_2(y,t)\right\}\nabla\psi(x)\right]dxdy\,.
\end{aligned}
$$

The expression

$$
\left(\nabla\psi(x)-\nabla\psi(y)\right)\cdot\frac{x-y}{|x-y|^2}\,,
$$

is bounded. If we assume, a priori that u_1 and u_2 are Lipschitz, we observe

$$
\begin{aligned}
&\frac{x-y}{|x-y|^2}\left\{u_1(y,t)u_2(x,t)-u_1(x,t)u_2(y,t)\right\}\nabla\psi(x)\\
&=\left\{\frac{[u_1(y,t)-u_1(x,t)]}{|x-y|}u_2(x,t)+u_1(x,t)\left[\frac{u_2(x,t)-u_2(y,t)}{|x-y|}\right]\right\}\frac{x-y}{|x-y|}\cdot\nabla\psi(x)
\end{aligned}
$$

is bounded too. Then, we define a weak solution for our system as a couple of functions $(u_1,u_2)\in L^\infty(R^+\,,Lip(\mathbb{R}^n))$ which satisfies, for every test functions $\psi\,,\phi\in C_0^\infty(\mathbb{R}^2)$ the equality,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\psi(x)u_1(x,t)dx\\
&=\int_{\mathbb{R}^2}\Delta\psi(x)u_1(x,t)dx\\
&-\frac{\chi_1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\left(\nabla\psi(x)-\nabla\psi(y)\right)\cdot\frac{x-y}{|x-y|^2}\left(u_1(x,t)u_1(y,t)+u_1(y,t)u_2(x,t)\right)dxdy\\
&+\frac{\chi_1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{x-y}{|x-y|^2}\cdot\left\{u_1(y,t)u_2(x,t)-u_1(x,t)u_2(y,t)\right\}\nabla\psi(x)dxdy \qquad ,
\end{aligned}
$$

follow the same argument, the equality for u_2 and ϕ is obtained.

Using this last definition, we can prove that weak solutions are mass conservative. To prove, this we take a test function ψ such that

$$
\psi(r)=\begin{cases}1\,, & r\le 1/2\,,\\ 0\,, & r\ge 1\,.\end{cases}
$$

Thus, we define $\psi_R(x) = \psi(|x|/R)$. Then, there exists some constants C_1 , C_2 and C_3 such that

$$\left| \int_{\mathbb{R}^2} \Delta\psi_R(x) u_1(x,t) dx \right| \leq \frac{C_1}{R^2}$$

$$\left| \frac{\chi_1}{8\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} (\nabla\psi_R(x) - \nabla\psi_R(y)) \cdot \frac{x-y}{|x-y|^2} \left(u_1(x,t)u_1(y,t) + u_1(y,t)u_2(x,t) \right) dxdy \right| \leq \frac{C_2}{R^2}$$

$$\left| \frac{\chi_1}{8\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x-y}{|x-y|^2} \cdot \left\{ u_1(y,t)u_2(x,t) - u_1(x,t)u_2(y,t) \right\} \nabla\psi_R(x) dxdy \right| \leq \frac{C_3}{R}$$

Finally, passing to the limit as $R \to \infty$, we arrive at

$$\frac{d}{dt} \int_{\mathbb{R}^2} u_1(x,t) dx = 0 \, .$$

A similar result follows for u_2 . The conservation of mass for weak solutions, allows us to conclude, using the same techniques, the weak solutions of (2.3) hold identity (2.76) .

2.5.3 Global Existence of Weak Solutions

Boundlessness of the entropy in the last Theorem, is the main tool that we will use to obtain the following result of global existence.

Theorem 35 (Global Existence of Weak Solutions) *Under assumption* (2.75) *and*

$$8\pi \left(\frac{\theta_1}{\chi_1}\mu + \frac{\theta_2}{\chi_2} \right) - (\theta_1 + \theta_2)^2 > 0 \, , \tag{2.85}$$

$$\theta_1 < \frac{8\pi}{\chi_1}\mu \, , \; \theta_2 < \frac{8\pi}{\chi_2} \, , \tag{2.86}$$

system (2.3) *has a global weak non negative solution such that*

$$(1 + |x|^2 + |\log u_i|)u_i \in L^\infty(0, T \, ; \, L^1(\mathbb{R}^2))$$

and

$$-\frac{1}{\chi_1} \int \int_{[0,T]\times\mathbb{R}^2} u_1 \left| \mu\nabla \log u_1 - \nabla\chi_1 v \right|^2 dx - \frac{1}{\chi_2} \int \int_{[0,T]\times\mathbb{R}^2} u_2 \left| \nabla \log u_2 - \nabla\chi_2 v \right|^2 dx < \infty \, .$$

Before showing the proof, let us first give some explanations on this result. Inequality (2.85) corresponds to the interior of a rotated parabola in the plane $\theta_1\theta_2$. Choosing the parameters μ , χ_1 and χ_2 adequately condition (2.86) may be relevant or can be simply ignored. More precisely we have the following two cases,

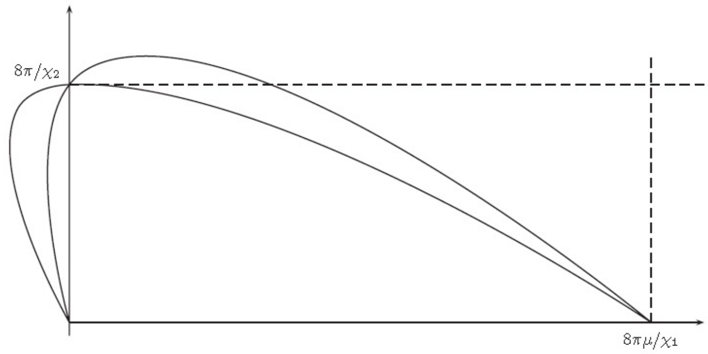

Figure 2.2: Different configurations to parabola (2.87) .

- If the parabola,
$$8\pi\left(\frac{\theta_1}{\chi_1}\mu+\frac{\theta_2}{\chi_2}\right)-(\theta_1+\theta_2)^2=0 \tag{2.87}$$
intersects any of the lines $\theta_1 = 8\pi\mu/\chi_1$ or $\theta_2 = 8\pi/\chi_2$ in the first quadrant of the $\theta_1\theta_2$ plane, (which happens exactly when $\chi_1 < \mu\chi_2/2$ or $\chi_1 > 2\mu\chi_2$) and θ_1, θ_2 satisfies inequalities (2.85) and (2.86) then system (2.3) has a global in time weak solution.

- However, if the parabola (2.87) do not intersect any of the lines $\theta_1 = 8\pi\mu/\chi_1$ or $\theta_2 = 8\pi/\chi_2$ (when $\mu\chi_2/2 \leq \chi_1 \leq 2\mu\chi_2$) in the first quadrant of the $\theta_1\theta_2$ plane, and θ_1, θ_2 satisfies inequality (2.85), then system (2.3) has a global in time weak solution.

This different options are summarized in the Figure 2.2

On the other hand, we point out that all of our results are formally so far. In order to give them rigorousness, we should have a local existence result of smooth solutions. However, we will take another strategy which will allow us to obtain directly global existence in time of weak solutions with the corresponding mathematical rigorousness. In order to prove Th.35 , first, we modify the convolution kernel $k^0(z) = -\frac{1}{2\pi}\log|z|$ in (2.3), by truncating it around zero. This last will allows us to get a regularized version of system (2.3), which is rather easier to work. After proving the existence of global solutions of this last approximate problem, we look for uniform estimates of the solutions and then pass to the limit will give us the result of global existence, we are looking for. After getting this result we recover properties such as mass conservation or the second moment formula by "testing" properly our weak solution. A similar technique was made in the one chemotaxis species case (see [7, 8]), for this last, we only make an sketch to the proof.

Proof (Sketch). For the reader's convenience, we divide the proof in four steps giving special attention where technical difficulties arise in comparison to the single species case.
Step 1. *Regularization of the system.* We define K^ϵ by $K^\epsilon(z) := K^1\left(\frac{z}{\epsilon}\right)$, where K^1 is a radial monotone non-decreasing smooth function satisfying,

$$K^1(z) = \begin{cases} -\frac{1}{2\pi}\log|z| & if \quad |z| \geq 4 \\ 0 & if \quad |z| \leq 1, \end{cases}$$

we also assume that

$$\left|\nabla K^1(z)\right| \leq \frac{1}{2\pi |z|} ,$$

$$K^1(z) \leq -\frac{1}{2\pi}\log|z| \ , \ -\Delta K^1(z) \geq 0 ,$$

for all $z \in \mathbb{R}^2$. Then, we obtain the following regularized version of system (2.3) ,

$$\begin{cases} \partial_t u_1^\epsilon = \Delta u_1^\epsilon - \chi_1 \nabla \cdot (u_1^\epsilon \nabla v^\epsilon) \ , \ t \geq 0 \ , x \in \mathbb{R}^2 \ , \\ \partial_t u_2^\epsilon = \Delta u_2^\epsilon - \chi_2 \nabla \cdot (u_2^\epsilon \nabla v^\epsilon) \ , \\ v^\epsilon = K^\epsilon * (u_1^\epsilon + u_2^\epsilon) \ , \end{cases} \tag{2.88}$$

which we interpret in the distribution sense. Since $K^\varepsilon(z) = K^1(\frac{z}{\varepsilon})$, we also have

$$|\nabla K^\varepsilon(z)| = \frac{1}{\varepsilon}\left|\nabla K\left(\frac{z}{\varepsilon}\right)\right| \leq \frac{1}{\varepsilon}\frac{1}{2\pi |z/\varepsilon|} = \frac{1}{2\pi |z|} . \tag{2.89}$$

The proof of global solutions in $L^2(0,T;H^1(\mathbb{R}^2) \cap C(0,T;L^2(\mathbb{R}^2))$ for system (2.88) with initial data in $L^2(\mathbb{R}^2)$ follows essentially the same lines as in [8, Prop. 2.8] and therefore we omit the proof here.

Step 2. *A prior estimates for the approximate solutions* u_1^ε , u_2^ε *and* v^ε .

Consider a solution $(u_1^\epsilon, u_2^\epsilon)$ of the regularized system. If

$$\theta_1 < \frac{8\pi}{\chi_1}\mu \ , \theta_2 < \frac{8\pi}{\chi_2} \ , 8\pi\left(\frac{\theta_1}{\chi_1}\mu + \frac{\theta_2}{\chi_2}\right) - (\theta_1+\theta_2)^2 \geq 0,$$

then, uniformly as $\epsilon \to 0$, with bounds depending only upon $\int_{\mathbb{R}^2} \left(1+|x|^2\right) u_{i0} dx$ and $\int_{\mathbb{R}^2} u_{i0}\log u_{i0} dx$ with $i = 1, 2$, we have,

(i) The function $(x,t) \to |x|^2 (u_1^\epsilon + u_2^\epsilon)$ is bounded in $L^\infty\left(\mathbb{R}^+_{loc}; L^1(\mathbb{R}^2)\right)$.

(ii) The functions $t \to \int_{\mathbb{R}^2} u_j^\epsilon(x,t)\log u_j^\epsilon(x,t)\, dx$ and $t \to \int_{\mathbb{R}^2} u_j^\epsilon(x,t)\, v^\epsilon(x,t)\, dx$ are bounded for $j = 1, 2$.

(iii) The function $(x,t) \to u_j^\epsilon(x,t)\log\left(u_j^\epsilon(x,t)\right)$ is bounded in $L^\infty\left(\mathbb{R}^+_{loc}; L^1(\mathbb{R}^2)\right)$ for $j = 1, 2$.

(iv) The function $(x,t) \to \nabla\sqrt{u_j^\epsilon(x,t)}$ is bounded in $L^2\left(\mathbb{R}^+_{loc} \times \mathbb{R}^2\right)$ for $j = 1, 2$.

(v) The function $(x,t) \to u_j^\epsilon(x,t)$ is bounded in $L^2\left(\mathbb{R}^+_{loc} \times \mathbb{R}^2\right)$ for $j = 1, 2$.

(vi) The function $(x,t) \to u_j^\epsilon(x,t)\,\Delta v^\epsilon(x,t)$ is bounded in $L^1\left(\mathbb{R}^+_{loc} \times \mathbb{R}^2\right)$ for $j = 1, 2$.

(vii) The function $(x,t) \to \sqrt{u_j^\epsilon(x,t)}\nabla v^\epsilon(x,t)$ is bounded in $L^2\left(\mathbb{R}^+_{loc} \times \mathbb{R}^2\right)$ for $j = 1, 2$.

The proof of estimates (i)-(vii) follows essentially the same steps as in the one species case, therefore, we remit the reader to [8, Lema 2.11]. In addition, we note that from Gagliardo-Nierenberg-Sobolev inequality, for all $p \in [2, \infty)$

$$\|g\|^2_{L^p(\mathbb{R}^2)} \le C^{(p)}_{GNS} \|\nabla g\|^{2-\frac{4}{p}}_{L^2(\mathbb{R}^2)} \|g\|^{\frac{4}{p}}_{L^2(\mathbb{R}^2)}, \text{ for all } g \in H^1\left(\mathbb{R}^2\right) ,$$

taking $g = \sqrt{u_i^\epsilon}$ we obtain,

$$\int_{\mathbb{R}^2} |u_i^\epsilon|^{p/2}\, dx \le \left(C^{(p)}_{GNS}\right)^{\frac{p}{2}} \theta_i \left\|\nabla\sqrt{u_i^\varepsilon}\right\|^{p-2}_{L^2(\mathbb{R}^2)} \tag{2.90}$$

for any $p > 2$. Estimation (iv) along with (2.90) implies that u_i^ϵ is uniformly bounded in $L^q\left(\mathbb{R}^+_{loc} \times \mathbb{R}^2\right)$ for every $q \in [1, \infty)$. Therefore, we have proved the following result

(viii) The function $(x,t) \to u_j^\epsilon(x,t)$ is bounded in $L^p\left(\mathbb{R}^+_{loc} \times \mathbb{R}^2\right)$ for $j = 1, 2, p \ge 1$.

Step 3. *Construction of a strong convergence subsequence in L^p*: To achieve our aim in this step we will apply the Aubin-Lions compactness Lemma.
First, we get a uniform bound on $\|\nabla u_i^\epsilon\|_{L^2_{loc}((\delta,T)\times B_i)}$. We observe that,

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2} |u_1^\epsilon|^2\, dx &= -2\int_{\mathbb{R}^2} |\nabla u_1^\epsilon|^2\, dx + 2\chi_1 \int_{\mathbb{R}^2} u_i^\epsilon \nabla u_i^\epsilon \cdot \nabla v^\epsilon dx \\
&\le -2\int_{\mathbb{R}^2} |\nabla u_1^\epsilon|^2\, dx + 2\chi_1 \left(\int_{\mathbb{R}^2} |\nabla u_i^\epsilon|^2\right)^{1/2} \left(\int_{\mathbb{R}^2} |u_i^\epsilon|^2 |\nabla v^\epsilon|^2\, dx\right)^{1/2} \\
&\le -2\int_{\mathbb{R}^2} |\nabla u_1^\epsilon|^2\, dx + 2\chi_1 \left(\int_{\mathbb{R}^2} |\nabla u_1^\epsilon|^2\right)^{1/2} \left(\int_{\mathbb{R}^2} |u_1^\epsilon|^3\, dx\right)^{1/3} \left(\int_{\mathbb{R}^2} |\nabla v^\epsilon|^6\, dx\right)^{1/6},
\end{aligned}$$

where we have used Holder inequality in the last line. The classical Gagliardo-Nirenberg-Sobolev inequality along with the Calderon-Zigmund inequality allow us to achieve that,

$$\left(\int_{\mathbb{R}^2} |\nabla v^\epsilon|^6\, dx\right)^{1/6} \le C \left(\int_{\mathbb{R}^2} |\Delta v^\epsilon|^{3/2}\, dx\right)^{2/3} . \tag{2.91}$$

From ineq. (2.91) and (2.91) we conclude that,

$$\begin{aligned}
&\frac{d}{dt}\int\limits_{\mathbb{R}^2} |u_1^\epsilon|^2\, dx \\
\leq\ & -2\int\limits_{\mathbb{R}^2} |\nabla u_1^\epsilon|^2\, dx + 2C\chi_1 \left(\int\limits_{\mathbb{R}^2} |\nabla u_1^\epsilon|^2\right)^{1/2} \left(\int\limits_{\mathbb{R}^2} |u_1^\epsilon|^3\, dx\right)^{1/3} \left(\int\limits_{\mathbb{R}^2} |\Delta v^\epsilon|^{3/2}\, dx\right)^{2/3} \\
\leq\ & -2\int\limits_{\mathbb{R}^2} |\nabla u_1^\epsilon|^2\, dx \\
& +2C\chi_1 \left(\int\limits_{\mathbb{R}^2} |\nabla u_1^\epsilon|^2\right)^{1/2} \left(\int\limits_{\mathbb{R}^2} |u_1^\epsilon|^3\, dx\right)^{1/3} \left(\left(\int\limits_{\mathbb{R}^2} |u_1^\epsilon|^{3/2}\, dx\right)^{2/3} + \left(\int\limits_{\mathbb{R}^2} |u_2^\epsilon|^{3/2}\, dx\right)^{2/3}\right) .
\end{aligned}$$

Integrating respect to t and reordering last inequality we obtain now,

$$\begin{aligned}
&2\int_\delta^T \int\limits_{\mathbb{R}^2} |\nabla u_i^\epsilon|^2\, dxdt \\
&- 2C\chi_1 \left\{ \sup_{t\in[\delta,T]} \left(\int_{\mathbb{R}^2} |u_1^\epsilon|^3\, dx\right)^{1/3} \left(\sup_{t\in[\delta,T]} \left(\int\limits_{\mathbb{R}^2} |u_1^\epsilon|^{3/2}\, dx\right)^{2/3} + \sup_{t\in[\delta,T]} \left(\int_{\mathbb{R}^2} |u_2^\epsilon|^{3/2}\, dx\right)^{2/3}\right)\right\} \\
&\cdot \int_\delta^T \left(\int_{\mathbb{R}^2} |\nabla u_1^\epsilon|^2\right)^{1/2} dt + \int_{\mathbb{R}^2} |u_i^\epsilon|^2\, dx - \int_{\mathbb{R}^2} |u_i^\epsilon(x,0)|^2\, dx \leq 0 .
\end{aligned}$$

We observe now that,

$$\int_\delta^T \left(\int\limits_{\mathbb{R}^2} |\nabla u_1^\epsilon|^2\, dx\right)^{1/2} dt \leq T^{1/2} \left(\int_\delta^T \int_{\mathbb{R}^2} |\nabla u_i^\epsilon|^2\, dxdt\right)^{1/2} .$$

Denoting by $X := \|\nabla u_i^\epsilon\|_{L^2_{loc}((\delta,T)\times\mathbb{R}^2)}$ and taking into account viii), we conclude from last inequality that for positive constants a , b and c we have that,

$$aX^2 - bX + c \leq 0 ,$$

in consequence $X := \|\nabla u_i^\epsilon\|_{L^2_{loc}((\delta,T)\times\mathbb{R}^2)}$ is bounded, i.e there exist a constant C such that,

$$\|\nabla u_i^\epsilon\|_{L^2_{loc}((\delta,T)\times\mathbb{R}^2)} \leq C . \tag{2.92}$$

Now, we obtain a bound for $\|du_i^\varepsilon/dt\|_{L^2((0,T);H^{-1}(\mathbb{R}^2))}$. Let $\phi \in H^1(\mathbb{R}^2)$ then we have,

$$\begin{aligned}
\left|\left\langle \frac{du_i^\epsilon}{dt}, \phi\right\rangle\right| &= |\langle \Delta u_i - \nabla\cdot(u_i\nabla v^\epsilon), \phi\rangle| & (2.93)\\
&\leq |\langle \nabla u_i, \nabla\phi\rangle| + |\langle u_i\nabla v^\epsilon, \nabla\phi\rangle| \\
&\leq \|\nabla\phi\|\,\|\nabla u_i\| + \|\nabla\phi\|\,\|u_i\nabla v^\epsilon\| . & (2.94)
\end{aligned}$$

Thus,

$$\left\| \frac{du_i^\epsilon}{dt} \right\|_{H^{-1}(\mathbb{R}^2)} = \sup_{\|\phi\|_{H^1(\mathbb{R}^2)}=1} \left| \left\langle \frac{du_i^\epsilon}{dt}, \phi \right\rangle \right| \le \|\nabla u_i^\varepsilon\|_{L^2(\mathbb{R}^2)} + \|u_i^\varepsilon \nabla\|_{L^2(\mathbb{R}^2)} \le C\,.$$

From the last estimate it follows that,

$$\left\| \frac{du_i^\epsilon}{dt} \right\|_{L^2((\delta,T);H^{-1}(\mathbb{R}^2))} = \left(\int_\delta^T \left\| \frac{du_i^\epsilon}{dt} \right\|^2_{H^{-1}(\mathbb{R}^2)} \right)^{1/2} \le C(T)\,. \tag{2.95}$$

Compactness: In order to apply the **Aubin-Lions' Lemma**, we define the spaces

$$B_0 = H^1(\mathbb{R}^2) \cap \left\{ f| \quad |x|^2 f \in L^1(\mathbb{R}^2) \right\}\,,$$

$$B := L^2(\mathbb{R}^2)$$

and

$$B_1 := B_0'\,.$$

Let $\{f_i\}$ and arbitrary bounded sequence in B, then we have that it is L^2 equi-integrate at infinity (cf.[3, Corollary 5.3.1]) as the following account shows:

$$\begin{aligned} \int_{\{|x|>R\}} f_i^2 dx &\le \frac{1}{R} \int_{\{|x|>R\}} \left(|x|\, f_i^{1/2} \right) f_i^{3/2} dx \le \frac{1}{R} \left(\int_{\{|x|>R\}} |x|^2 f_i dx \right)^{1/2} \left(\int_{\{|x|>R\}} f_i^3 dx \right)^{1/2} \\ &\le \frac{1}{R} \left(\int_{\mathbb{R}^2} |x|^2 f_i dx \right)^{1/2} \left(\int_{\mathbb{R}^2} f^3 dx \right)^{1/2}. \end{aligned}$$

Thus,

$$\lim_{R\to+\infty} \int_{\{|x|>R\}} f_i^2 dx = 0\,, \tag{2.96}$$

uniformly with respect to f_i . From, the Rellich-Kondrakov Theorem (cf.[3, Corollary 5.3.1]) we obtain the compact inclusion,

$$B_0 \hookrightarrow\hookrightarrow B$$

Given that u_i^ε satisfies (2.92) , (2.95) and (2.96) , we can invoke now the Aubin-Lions-Simon theorem to conclude that u_i^ε has a sub sequence which converge strongly in $L^2(0,T,B)$. Therefore, under a subsequence we have that,

$$u_i^\epsilon \to u_i \text{ a.e. in } \mathbb{R}^2 \times (\delta, T]\,. \tag{2.97}$$

We also have proved uniformly boundlessness for $\|u_i^\epsilon\|_{L^p(\mathbb{R}^2)\times[0,T]}$, from this, estimation (2.97) and Vitali' theorem we obtain,

$$u_i^\epsilon \to u_i \;\text{ strongly in } L^p(\mathbb{R}^2 \times [0,T]) \text{ for } p \ge 1\,. \tag{2.98}$$

Step 4. *Pass to the limit*. We pass to the limit in the weak sense to obtain our result of global existence. The most significant technical difficulty to show that u_1, u_2 solved (2.3) arise with the nonlinear terms. In order to prove that

$$u_i^\epsilon \nabla v^\epsilon \rightharpoonup u_i \nabla v, \text{in } D'(\mathbb{R}^+ \times \mathbb{R}^2) \,. \tag{2.99}$$

First, we notice that the expression $u_i^\epsilon |\nabla v^\epsilon|$ is integrate as estimate (vii), along with the following estimate show,

$$\left(\int_{[0,T]\times\mathbb{R}^2} u_i^\epsilon |\nabla v^\epsilon| \, dxdt\right)^2 = \left(\int_{[0,T]\times\mathbb{R}^2} \sqrt{u_i^\epsilon}\sqrt{u_i^\epsilon} |\nabla v^\epsilon| \, dxdt\right)^2$$
$$\le \int_{[0,T]\times\mathbb{R}^2} u_i^\epsilon dxdt \int_{[0,T]\times\mathbb{R}^2} u_i^\epsilon |\nabla v^\epsilon|^2 \, dxdt \le \theta_i T \int_{[0,T]\times\mathbb{R}^2} u_i^\epsilon |\nabla v^\epsilon|^2 \, dxdt \,.$$

It follows that we can interpret $u_i^\epsilon \nabla v^\epsilon$ as an element of $\left(C_0^\infty\left(\mathbb{R}^+ \times \mathbb{R}^2\right)\right)'$ and therefore it has sense its divergence.

In order to prove that $\|\nabla v^\varepsilon\|_{L^r(\mathbb{R}^n)} \le C$ for $r > 2$, we recall the Hardy-Littelwood-Sobolev inequality: For all $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2$ and $0 < \lambda < n$, there exists a constant $C = C(p, q, \lambda) > 0$ such that

$$\left|\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{1}{|x-y|^\lambda} f(x)g(y)dxdy\right| \le C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \,.$$

Taking the supreme over the ball $\|g\|_{L^q(\mathbb{R}^n)} = 1$, on both sides of the last inequality we obtain,

$$\left\|\int_{\mathbb{R}^n} \frac{1}{|x-y|^\lambda} f(x)dx\right\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n)} \le C \|f\|_{L^p(\mathbb{R}^n)} \,. \tag{2.100}$$

In particular,

$$\left\|\int_{\mathbb{R}^n} \frac{1}{|x-y|} f(x)dx\right\|_{L^{\frac{q}{q-1}}(\mathbb{R}^2)} \le C \|f\|_{L^p(\mathbb{R}^2)} \,,$$

where $1 < p, q < \infty$, $1/p + 1/q + 1/2 = 2$. Thus, we have that

$$\|\nabla v^\varepsilon\|_{L^r(\mathbb{R}^n)} = \|\nabla K^\varepsilon * (u_1^\varepsilon + u_2^\varepsilon)\|_{L^r(\mathbb{R}^2)} \tag{2.101}$$
$$\le \left\|\frac{1}{2\pi}\int \frac{1}{|x-y|}(u_1^\varepsilon + u_2^\varepsilon)dx\right\|_{L^r(\mathbb{R}^2)} \tag{2.102}$$
$$\le C\left(\|u_1^\varepsilon\|_{L^p(\mathbb{R}^2)} + \|u_2^\varepsilon\|_{L^p(\mathbb{R}^2)}\right) \le C \,, \tag{2.103}$$

we have used step 2 (viii). From, $r = \frac{q}{q-1}$ and $1/p + 1/q + 1/2 = 2$, we obtain that $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. In addition, $p \in (1, 2)$ implies that $r \in (2, \infty)$. We conclude that (up to a subsequence) $\nabla v^\varepsilon \rightharpoonup h$, where h is in L^r. In order to prove that actually $h = \nabla K * n$ we have to do some extra work yet. With this end in mind, we now propose us to show that,

$$\nabla v^\epsilon \to \nabla v \text{ a.e.} \tag{2.104}$$

We have that,

$$
\begin{aligned}
\nabla v^{\epsilon}-\nabla v &= -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}\left(\left(u_1^{\epsilon}+u_2^{\epsilon}\right)-\left(u_1+u_2\right)\right)(y,t)\,dy \\
&\quad+\int\limits_{|x-y|\le 2\epsilon}\left(\frac{1}{\varepsilon}\nabla K^1\left(\frac{x-y}{\varepsilon}\right)+\frac{|x-y|}{2\pi|x-y|^2}\right)\left(u_1^{\epsilon}+u_2^{\epsilon}\right)(y,t)dy. \quad (2.105)
\end{aligned}
$$

From (2.98) and (2.100) , we deduce that (under a subsequence) the first integral in (2.105) converges to zero a.e. On the other side, estimates (2.89) allows us to conclude that

$$
\begin{aligned}
&\left|\int\limits_{|x-y|\le 2\epsilon}\left(\frac{1}{\varepsilon}\nabla K^1\left(\frac{x-y}{\varepsilon}\right)+\frac{|x-y|}{2\pi|x-y|^2}\right)\left(u_1^{\epsilon}+u_2^{\epsilon}\right)(y,t)dy\right| \\
&\le\int\limits_{|x-y|\le 2\epsilon}\left(\frac{1}{\pi|x-y|}\right)\left(u_1^{\epsilon}+u_2^{\epsilon}\right)(y,t)dy\ .
\end{aligned}
$$

After change of variable to polar coordinates, we observe that last integral converges to 0 as $\varepsilon\to 0$. Therefore, we conclude (2.104) and obtain from [24, Prop. 2.46 (i)] that $\nabla v_{\varepsilon}\rightharpoonup\nabla K*n$ weakly in L^r for $r\ge 2$. Finally we choose conjugate exponents $r=4$ and $p=4/3$ to conclude the convergence (2.98). ■

2.6 Stationary Model

We start recalling the definition for Liouville type system. Let define the follow constants

$$
-\triangle u_i=\mu_i V_i\exp\left(\sum_{j=1}^{n}a_{ij}u_j\right)\ , \tag{2.106}
$$

for $i=1,..,n$. To solve (2.106) we need obtain a pair (μ_i,u_i) such that holds (2.106) and simultaneously satisfy

$$
M_i=\mu_i\int_{\mathbf{R}^2}V_i\exp\left(\sum_{j=1}^{n}a_{ij}u_j\right) \tag{2.107}
$$

this is for all $i=1,..,n$ and $(M_i)_{i=1}^{n}$ are positives constants. In $\mathbf{R}^2$ we define the polynomial condition in $\mathbb{R}^2$

$$
\wedge_J(M)=8\pi\sum_{i\in J}M_i-\sum_{i\in J}\sum_{j\in J}a_{ij}M_iM_j\ , \tag{2.108}
$$

where $J\subseteq I$.

Theorem 36 *Let A a no symmetric positive matrix and assume that*

$$V_i \equiv 1 \tag{2.109}$$

a.e. Then there exists a entire solution (2.106) *that holds* (2.108) *if*

$$A_I(M) = 0 \tag{2.110}$$

and

$$A_J(M) > 0 \tag{2.111}$$

for all $J \subset I$ with $J \neq \Phi$.

Proof. See[14] ■

We have in mind to apply Th.36 in our stationary system problem, thus we get a condition over intial mass. On this way, the first step is reduce the initial stationary system and obtain a Liouville type system Consider the stationary problem to (2.3)

$$\left.\begin{array}{rcl} -\triangle u_1 + \chi_1 \nabla.(u_1 \nabla v) & = & 0\,, \\ -\triangle u_2 + \chi_2 \nabla.(u_2 \nabla v) & = & 0\,, \\ -\triangle v - (u_1 + u_2) & = & 0\,. \end{array}\right\} \tag{2.112}$$

We applied the Schaaf's [46] method to system (2.112), in which the equations are manipulated to obtain a Liouville system, thus we have the follow estimations

$$\nabla.(-\nabla u_1 + \chi_1(u_1 \nabla v)) = 0\,, \tag{2.113}$$

it is equivalent at

$$\nabla[u_1(-\nabla \log u_1 + \chi_1 \nabla v)] = 0\,, \tag{2.114}$$

multiplying (1.89) by $-\log u_1 + \chi_1 v$ and integrating over $\mathbf{R}^2$ we have

$$\int_{\mathbf{R}^2} u_1 \mid \nabla(-\log u_1) + \chi_1 \nabla v \mid^2 dx = 0\,, \tag{2.115}$$

concluding that

$$\mid \nabla(-\log u_1) + \chi_1 \nabla v \mid^2 = 0\,. \tag{2.116}$$

A priori assume $u_1\,(x) > 0$ then

$$\nabla(-\log u_1) + \chi_1 \nabla v = 0\,,$$

therefore

$$\log u_1 - \chi_1 v = K_1\,,$$

as consequence we obtain that

$$u_1 = C_1 \exp(\chi_1 v). \tag{2.117}$$

Using the same estimations, we conclude

$$u_2 = C_2 \exp(\chi_2 v). \tag{2.118}$$

Hence we get,

$$u_i\,(x,t) > 0\,,\, i = 1,2\,.$$

The system (2.112) has been reduced to the equation

$$-\triangle v = C_1 \exp(\chi_1 v) + C_2 \exp(\chi_2 v). \tag{2.119}$$

Next, if we consider the follow system

$$-\triangle v_1 = \frac{C_1}{\alpha} \exp[\alpha\chi_1 v_1 + \beta\chi_1 v_2]\,, \tag{2.120}$$

$$-\triangle v_2 = \frac{C_2}{\beta} \exp[\alpha\chi_2 v_1 + \beta\chi_2 v_2]\,, \tag{2.121}$$

if there exists v_1, v_2 such that solve (2.120)-(2.121), respectively, for all $x \in \mathbb{R}^2$ then

$$v = \alpha v_1 + \beta v_2$$

solve (2.119). We can applied Th.36 to (2.120)-(2.121) and obtain the condition for existence of solutions. We have in (2.120)-(2.121)

$$Q = \begin{pmatrix} \alpha\chi_1 & \beta\chi_1 \\ \alpha\chi_2 & \beta\chi_2 \end{pmatrix}\,.$$

Q should be symmetric matrix α and β, then we have the follow condition

$$\alpha\chi_2 = \beta\chi_1\,. \tag{2.122}$$

We also obtain that

$$\mu_1 = \frac{C_1}{\alpha}\,,\, \mu_2 = \frac{C_2}{\beta}$$

and

$$M_1 = \frac{C_1}{\alpha}\int_{\mathbb{R}^2} \exp\left[\alpha\chi_1 v_1 + \beta\chi_1 v_2\right] dx\,,$$

$$M_2 = \frac{C_2}{\beta}\int_{\mathbb{R}^2} \exp\left[\alpha\chi_2 v_1 + \beta\chi_2 v_2\right] dx\,,$$

we need consider that

$$\int_{\mathbb{R}^2} u_i dx = \theta_i\,,\, i = 1,2$$

therefore

$$C_1 = \frac{\theta_1}{\int_{\mathbf{R}^2} \exp\left[\alpha\chi_1 v_1 + \beta\chi_1 v_2\right]}\,,$$

in the same way we conclude

$$C_2 = \frac{\theta_2}{\int_{\mathbf{R}^2} \exp\left[\alpha\chi_2 v_1 + \beta\chi_2 v_2\right]}\,,$$

thus

$$M_1 = \frac{\theta_1}{\alpha} \, , M_2 = \frac{\theta_2}{\beta} \, ,$$

then there exists a solution to (2.120), if and only if

$$\begin{aligned}
&\Lambda_{\{1\}}(M) = 8\pi \frac{\theta_1}{\alpha} - \alpha\chi_1 \left(\frac{\theta_1}{\alpha}\right)^2 > 0 \, , \\
&\Lambda_{\{2\}}(M) = 8\pi \frac{\theta_2}{\beta} - \beta\chi_2 \left(\frac{\theta_2}{\beta}\right)^2 > 0 \, , \\
&\Lambda_{\{1,2\}}(M) = 8\pi \left(\frac{\theta_1}{\alpha} + \frac{\theta_2}{\beta}\right) - \frac{\chi_1}{\alpha}\theta_1^2 - \frac{\chi_1}{\alpha}\theta_1\theta_2 - \frac{\chi_2}{\beta}\theta_1\theta_2 - \frac{\chi_2}{\beta}\theta_2^2 = 0 \, .
\end{aligned}$$

By hypothesis (2.122), we have that

$$\begin{aligned}
&8\pi \left(\frac{\theta_1}{\alpha} + \frac{\theta_2}{\beta}\right) - \frac{\chi_1}{\alpha}\theta_1^2 - \frac{\chi_1}{\alpha}\theta_1\theta_2 - \frac{\chi_2}{\beta}\theta_1\theta_2 - \frac{\chi_2}{\beta}\theta_2^2 \\
&= 8\pi \left(\frac{\theta_1}{\alpha} + \frac{\theta_2}{\beta}\right) - \frac{\chi_1}{\alpha}\theta_1 (\theta_1 + \theta_2) - \frac{\chi_2}{\beta}\theta_2 (\theta_1 + \theta_2) \\
&= 8\pi \left(\frac{\theta_1}{\alpha} + \frac{\theta_2}{\beta}\right) - \left(\frac{\chi_1}{\alpha}\theta_1 + \frac{\chi_2}{\beta}\theta_2\right)(\theta_1 + \theta_2) \\
&= 8\pi \left(\frac{\theta_1}{\alpha} + \frac{\theta_2}{\beta}\right) - K (\theta_1 + \theta_2)^2 \, .
\end{aligned}$$

Hence the condition to existence of solutions to (2.112) is

$$\frac{8\pi}{K} \left(\frac{\theta_1}{\alpha} + \frac{\theta_2}{\beta}\right) - (\theta_1 + \theta_2)^2 = 0 \tag{2.123}$$

and by condition (2.122)

$$K\alpha = \chi_1 \, , K\beta = \chi_2 \, ,$$

therefore we obtain the follow condition to the existence of a entire solution for (2.120)-(2.121)

$$8\pi \left(\frac{\theta_1}{\chi_1} + \frac{\theta_2}{\chi_2}\right) - (\theta_1 + \theta_2)^2 = 0 \, .$$

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Appendix

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Remarks on the blowup and global existence for a two species chemotactic Keller–Segel system in $\mathbb{R}^2$

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For the Keller–Segel model, it was conjectured by Childress and Percus (1984, Chemotactic collapse in two dimensions. In *Lecture Notes in Biomath*. Vol. 55, Springer, Berlin-Heidelberg-New York, 1984, pp. 61–66) that in a two-dimensional domain there exists a critical number C such that if the initial mass is strictly less than C, then the solution exists globally in time and if it is strictly larger than C blowup happens. For different versions of the Keller–Segel model, the conjecture has essentially been proved. The case of several chemotactic species introduces an additional question: What is the analogue for the critical mass obtained for the single species system? In this paper, we investigate for a two-species model for chemotaxis in $\mathbb{R}^2$ the conditions on the initial data, which determine blowup or global existence in time. Specifically, we find a curve in the plane of masses such that outside of it there is blowup and inside of it global existence in time is proved when the initial masses satisfy a threshold condition. Optimality of this condition is discussed through an analysis in the radial case. Finally, we show in the case of blowup for general data how it is possible to obtain a balance between entropies and prove what species should aggregates first.

Key words: Chemotaxis; Multi-component Keller–Segel model; Blowup of solutions.

1 Introduction

Chemotaxis is one of the simplest mechanisms for the aggregation of species. It makes reference to the aggregation of organisms sensitive to a gradient of a chemical substance. A classical model in chemotaxis was introduced by Keller and Segel [14]. With the cell density $u(x,t)$ and the concentration of the chemical $v(x,t)$ at point x and time t, the Keller–Segel model is

$$\begin{aligned} u_t &= \nabla \cdot (\mu \nabla u - \chi u \nabla v), && x \in \Omega, \quad t > 0, \\ v_t &= \gamma \Delta v - \beta v + \alpha u, && x \in \Omega, \quad t > 0, \end{aligned} \tag{1.1}$$

subject to homogeneous Neumann boundary conditions and positive initial data $u(x,0) = u_0$ and $v(x,0) = v_0$. In this model, χ is the chemotactic sensitivity, γ is the diffusion coefficient of the chemoattractant and μ is the diffusion coefficient of the cell density, β is the rate of consumption and α is the rate of production, all are positive parameters, and $\Omega \subset \mathbb{R}^N$ has smooth boundary $\partial\Omega$. It was conjectured by Childress and Percus [4] that in a two-dimensional domain there exists a critical number C such that if $\int u_0(x)dx < C$ then the solution exists globally in time, and if $\int u_0(x)dx > C$ blowup happens. For different versions of the Keller–Segel model, the conjecture has been essentially proved, finding the critical value $C = 8\pi/\chi$; for a complete review of this topic, we refer the reader to the papers [12, 13] and the references therein, particularly [1, 2, 9, 17, 20].

The case of several chemotactic species introduces an additional question: *What is the analogue for the critical mass obtained for the self-attracting single species system?* This question was already formulated by Wolansky in [21]. The aim of the present paper is to investigate Wolansky's question and to start looking for an answer to his question in case of two species and one diffusive chemoattractant. Our starting point is the following two-species model:

$$\begin{aligned} \partial_t U_1 &= \mu_1^* \Delta U_1 - \chi_1^* \nabla \cdot (U_1 \nabla V), \\ \partial_t U_2 &= \mu_2^* \Delta U_2 - \chi_2^* \nabla \cdot (U_2 \nabla V), \\ \partial_t V(x,t) &= D^* \Delta V + \alpha_1^* U_1 + \alpha_2^* U_2. \end{aligned}$$

Making a dimensional analysis like in Espejo *et al.* [6], Section 2 , it reduces to

$$\begin{aligned} \partial_t u_1 &= \mu \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v), \\ \partial_t u_2 &= \Delta u_2 - \chi_2 \nabla \cdot (u_2 \nabla v), \\ \varepsilon \partial_t v(x,t) &= \Delta v + u_1 + u_2, \end{aligned}$$

where $\varepsilon = \frac{\mu}{D}$. On the other side, assuming that molecular diffusion is much faster than cell diffusion ($\mu \ll D$), it is natural to approximate $\varepsilon \approx 0$ and obtain the following reduced system:

$$\begin{aligned} \partial_t u_1 &= \mu \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v), \\ \partial_t u_2 &= \Delta u_2 - \chi_2 \nabla \cdot (u_2 \nabla v), \\ 0 &= \Delta v + u_1 + u_2. \end{aligned}$$

Another assumption we introduce is the fact that we will study this system in the whole of $\mathbb{R}^2$ because an explicit elementary solution for the Laplace operator is available here; this will allows us to treat general data without any symmetry assumptions. In consequence, we will deal through out this paper with the following simplified two species Keller–Segel model in $\mathbb{R}^2$:

$$\left.\begin{aligned} \partial_t u_1 \quad &= \mu \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v), \\ \partial_t u_2 \quad &= \Delta u_2 - \chi_2 \nabla \cdot (u_2 \nabla v), \\ v(x,t) \quad &= -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| \, (u_1(y,t) + u_2(y,t)) \, dy, \\ u_1(x,0) = u_{10} &\geqslant 0, \; u_2(x,0) = u_{20} \geqslant 0, \end{aligned}\right\} \tag{1.2}$$

where $t \geqslant 0$, u_1 and u_2 are the density variables for the two different chemotaxis species and v is the chemoattractant, χ_1, χ_2, μ are positive constants and positive initial conditions u_{10}, u_{20} are given.

In the first part of this paper, we give a partial answer to Wolansky's question finding a region in the plane of masses in which there is blowup. More precisely, we prove that if the following condition is satisfied:

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1 + \theta_2)^2 < 0, \tag{1.3}$$

where θ_1, θ_2 denote the total initial mass of u_1 and u_2, respectively, then blowup happens. As we will show, model (1.2) preserves both masses, so we have

$$\theta_1 := \int_{\mathbb{R}^2} u_{10}(x)dx = \int_{\mathbb{R}^2} u_1(x,t)dx \qquad \theta_2 := \int_{\mathbb{R}^2} u_{20}(x)dx = \int_{\mathbb{R}^2} u_2(x,t)dx.$$

The proof is based in a suitable adaptation of the moments technique for a multi-species system like equation (1.2). As is proved is Section 2, the curve defined by the equal sign instead of lower than in equation (1.3) determines two regions in the plane of masses where the total moment of system (1.2), defined by

$$m(t) := \frac{\pi}{\chi_1}\int_{\mathbb{R}^2} u_1 |x|^2 \, dx + \frac{\pi}{\chi_2}\int_{\mathbb{R}^2} u_2 |x|^2 \, dx, \tag{1.4}$$

is decreasing or increasing, respectively; it is strictly decreasing in the region defined by equation (1.3) and monotonically increasing otherwise. In this latter case, one could expect to have global existence. However, we show in Section 4 that one can find initial data such that blowup happens and still the total moment is increasing.

At this point, a natural question arises, namely *which is an optimal condition (or region) where global existence is guaranteed?* Inspired from Blanchet *et al.* [1], our strategy to tackle this question consists in proposing a suitable free energy functional for system (1.2). The diffusion coefficient μ, as well as χ_2, plays a key role in the energy functional because both appear in a non-symmetric way in the first two equations in equation (1.2). Precisely, assuming $\chi_1 \leqslant \chi_2$, we prove that any of the following inequalities

$$\int_{\mathbb{R}^2} u_{10} dx + \int_{\mathbb{R}^2} u_{20} dx < \frac{8\pi}{\chi_2}, \qquad \text{if } \mu \geqslant 1,$$
$$\int_{\mathbb{R}^2} u_{10} dx + \int_{\mathbb{R}^2} u_{20} dx < \frac{8\pi}{\chi_2}\mu, \quad \text{if } \mu < 1,$$

yields global existence. The proof uses the energy functional that provides a priori bounds for the entropy of system (1.2). In its turn, these bounds yield the appropriate estimates on which existence is based.

In the second part of this paper, we discuss the optimality of our results concerning global existence and blowup for system (1.2). With this end in mind, we analyse first the radial case. This approach will allow us to show that system (1.2) can blowup even in the region where the total moment $m(t)$ is increasing. Since this seems not to agree completely with the intuition, it suggests us that the moment of one species could probably increase meanwhile the other one is decreasing but in such a way that the total moment $m(t)$ increases. This last statement opens interesting questions, for which a numerical approach would be a first appropriate strategy to consider.

We have also studied in the general case the following question: *Which of the chemotactic species will blowup first in time?* Intuitively, one could at least think that blowup should happen first for the species with the larger chemotactic χ. However, a mathematical proof of this fact can be difficult. We show how the techniques from Jäger and Luckhaus [9] along with the work by Blanchet *et al.* [1] can be used to obtain a balance between entropies and prove what species should blowup first. Precisely, we prove that bounds for the entropy corresponding to the species with the larger chemotactic coefficient yields bounds for the entropy of the other one.

We feel now worth mentioning that Wolansky's question has already been studied by Horstmann [10] and Horstmann and Lucia [11]. These authors first considered the stationary case in a smooth bounded domain with Neumann boundary conditions. Several Lyapunov functionals for different multi-species chemotactic systems are introduced in these papers. These functionals provide tools to study the stationary problem and gives an interesting clue for a possible threshold curve in the case of multi-species. The existence condition they obtained is similar to ours (see, e.g. Theorem 4.2 in [10]); the main difference being the fact that our conditions are both independent of the smallest chemoatractant coefficient. In addition, they consider the blow-up phenomenon in the case $\chi_1 = \chi_2$. In this sense, our results can be seen as a generalisation that allows us to understand the interplay between both chemotaxis coefficients whenever blowup happens. For other general multi-component Keller–Segel models in stationary regime, we refer again the interested reader to Wolansky [21].

The plan of this paper is the following: Section 2 deals with the blow-up existence for system (1.2) for arbitrary positive parameters $\chi_1 \leqslant \chi_2$. Section 3 is devoted to prove global existence in time results. The main novelty is an extension of the free energy functional for the one species Keller–Segel system (cf. [1]) to the case of two species. In Section 4, we improve our results of blowup by considering the radial case. Finally, Section 5 is concerned with the question on which of the species should aggregates first.

2 Blowup for positive arbitrary parameters χ_1, χ_2

Our purpose in this section is to derive sufficient conditions for having blowup for system (1.2). In order to do this, we define adequately the second moment for the whole population, which will allows us to generalise the usual technique of the moments (cf. [1,2,17]) for proving blowup for one chemotaxis species to our system of two species.

Theorem 1 *Let $u_1, u_2, v \in H^2(\mathbb{R}^2)$ be non-negative smooth solutions of eqaution (1.2) such that $u_i(1+|x|^2)$, $i = 1, 2$ are bounded in $L^\infty_{loc}(R^+, L^1(\mathbb{R}^2))$. Let the second moment $m(t)$ for the whole population defined by*

$$m(t) := \frac{\pi}{\chi_1}\int_{\mathbb{R}^2} u_1(x,t)\,|x|^2\,dx + \frac{\pi}{\chi_2}\int_{\mathbb{R}^2} u_2(x,t)\,|x|^2\,dx,$$

then we have

$$\frac{d}{dt}m(t) = \frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1 + \theta_2)^2. \tag{2.1}$$

Proof Multiplying the first equation of equation (1.2) by $|x|^2$ and integrating yields

$$\partial_t \int_{\mathbb{R}^2} u_1 |x|^2 \, dx = \mu \int_{\mathbb{R}^2} |x|^2 \Delta u_1 dx - \chi_1 \int_{\mathbb{R}^2} |x|^2 \nabla \cdot (u_1 \nabla v) dx,$$

then using Greens first identity we obtain

$$\partial_t \int_{\mathbb{R}^2} u_1 |x|^2 \, dx = 4\mu \int_{\mathbb{R}^2} u_1 dx + 2\chi_1 \int_{\mathbb{R}^2} u_1 (x \cdot \nabla v) dx.$$

From equation (1.2), we have that $\nabla v = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} (u_1(y,t) + u_2(y,t)) dy$. Next, we compute using mass conservation

$$\partial_t \int_{\mathbb{R}^2} u_1 |x|^2 \, dx = 4\mu\theta_1 - \frac{\chi_1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left(x \cdot \frac{x-y}{|x-y|^2} u_1(x,t) \left(u_1(y,t) + u_2(y,t)\right) \right) dy dx. \tag{2.2}$$

Similarly,

$$\partial_t \int_{\mathbb{R}^2} u_2 |x|^2 \, dx = 4\theta_2 - \frac{\chi_2}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left(x \cdot \frac{x-y}{|x-y|^2} u_2(x,t) \left(u_1(y,t) + u_2(y,t)\right) \right) dy dx. \tag{2.3}$$

From equations (2.2) and (2.3), it follows that

$$\begin{aligned} &\frac{d}{dt} \left(\frac{\pi}{\chi_1} \int_{\mathbb{R}^2} u_1 |x|^2 \, dx + \frac{\pi}{\chi_2} \int_{\mathbb{R}^2} u_2 |x|^2 \, dx \right) \\ &\quad = \frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left(x \cdot \frac{x-y}{|x-y|^2} \left(u_1(x,t) + u_2(x,t)\right) \left(u_1(y,t) + u_2(y,t)\right) \right) dy dx. \end{aligned}$$

After using the symmetry in the variables x and y in the last integral, we get

$$\begin{aligned} &\frac{d}{dt} \left(\frac{\pi}{\chi_1} \int_{\mathbb{R}^2} u_1 |x|^2 \, dx + \frac{\pi}{\chi_2} \int_{\mathbb{R}^2} u_2 |x|^2 \, dx \right) \\ &\quad = \frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2} \int\limits_{\mathbb{R}^2 \times \mathbb{R}^2} \left((x-y) \cdot \frac{x-y}{|x-y|^2} \left(u_1(x,t) + u_2(x,t)\right) \left(u_1(y,t) + u_2(y,t)\right) \right) dy dx \\ &\quad = \frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2} \int\limits_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\left(u_1(x,t) + u_2(x,t)\right) \left(u_1(y,t) + u_2(y,t)\right) \right) dy dx \\ &\quad = \frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1 + \theta_2)^2. \end{aligned}$$

In consequence, we have proved equation (2.1). □

Therefore, if we have that the initial masses θ_1 and θ_2 satisfy

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1 + \theta_2)^2 < 0, \tag{2.4}$$

we arrive at the conclusion that $m(t)$ should become negative in finite time that is impossible since u_1 and u_2 are non-negative. As a consequence, there is a finite blow-up time T^*. In fact, being as $\int_{\mathbb{R}^2} u_i|x|^2 dx \leqslant \frac{\chi_i}{\pi} m(t)$ both of the density variables u_1 and u_2 blowup. As a consequence, we obtain the following blow-up result.

Theorem 2 *Let $u_1, u_2, v \in H^2(\mathbb{R}^2)$ be non-negative smooth solutions of equation (1.2) such that $u_i(1 + |x|^2)$, $i = 1, 2$ are bounded in $L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^2))$, and let $[0, T^*)$ be the maximal interval of existence. If the inequality*

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1 + \theta_2)^2 < 0,$$

is satisfied then $T^* < \infty$.

Remark Alternatively, one can rewrite the hypotheses on the last theorem following the ideas of the paper from Kurokiba and Ogawa [15] to obtain the same conclusion, changing the hypotheses on u_1, u_2, v by working on the space

$$L^2_s(\mathbb{R}^2) = \{f \in L^1_{loc}(\mathbb{R}^2);\ (1 + |x|^2)^{s/2} f(x) \in L^2(\mathbb{R}^2)\}, \quad s > 1,$$

as the space of solutions for our system.

3 Global existence conditions

Our purpose at this section is to investigate if on the region where the second moment $m(t)$ is increasing

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1 + \theta_2)^2 > 0, \tag{3.1}$$

there exist global bounded solutions u_1 and u_2. In the case of bounded domains and Dirichlet boundary conditions on the chemical concentration, sufficient conditions on the initial masses for having global existence were given by Wolansky [21]. In contrast, we are going to consider $\mathbb{R}^2$ as our domain and discuss throughout this paper the possibility of an optimal region where blow-up phenomena happens. We assume without loss of generality that $\chi_2 \geqslant \chi_1$. In addition, we suppose that

$$\left.\begin{array}{c} \chi_1 \leqslant \chi_2,\ u_{10}, u_{20} \in L^1(\mathbb{R}^2, (1 + |x|^2)dx), \\ u_{10} \log u_{10},\ u_{20} \log u_{20} \in L^1(\mathbb{R}^2, dx). \end{array}\right\} \tag{3.2}$$

Using the entropy method from [5], we will give a partial proof of this showing that in the region

$$\theta_1 + \theta_2 < \frac{8\pi}{\chi_2}, \tag{3.3}$$

there is global existence for system (1.2) when $\mu > 1$; however, we will show that when $\mu \leqslant 1$ then the region of existence also depends on μ. Specifically, we will prove in the last case that the condition for global existence is

$$\theta_1 + \theta_2 < \mu \frac{8\pi}{\chi_2}. \tag{3.4}$$

Then, we will show that equation (3.1) is not enough to guarantee global existence by working with initial radial conditions.

Throughout this paper, we adopt the convention that all of the calculations will exclusively concern locally smooth solutions.

On the basis of [5, 21], we define the free energy for our system (1.2) like

$$E(t) := \frac{\mu}{\chi_1} \int_{\mathbb{R}^2} u_1 \log u_1 dx + \frac{1}{\chi_2} \int_{\mathbb{R}^2} u_2 \log u_2 dx - \frac{1}{2} \int_{\mathbb{R}^2} u_1 v dx - \frac{1}{2} \int_{\mathbb{R}^2} u_2 v dx. \tag{3.5}$$

For one chemotactic species, the free energy functional is a well-known tool and has been introduced for Keller–Segel models and gravitational models (see [2, 8, 18]. For a discussion on the multi-species case (see [21]). In spite of the apparent similarity of equation (3.5) with the energy in the case of one species used in [5], a generalisation for this kind of functionals for n species is not a straightforward task. Specifically, the use of the right coefficient in the entropy plays a fundamental roll in order to obtain the right monotony in the time variable as the proof of the following lemma shows.

Lemma 3 *Let $u_1, u_2 \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2))$ solutions of equation (1.2) such that $u_1(1+|x|^2)$, $u_2(1+|x|^2)$, $u_1 \log u_1$ and $u_2 \log u_2$ are bounded in $L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$, $\nabla\sqrt{u_1}, \nabla\sqrt{u_2} \in L^1_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$ and $\nabla v \in L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R}^2)$. Then,*

$$\frac{d}{dt}E(t) = -\frac{1}{\chi_1} \int_{\mathbb{R}^2} u_1 \left|\mu \nabla \log u_1 - \nabla \chi_1 v\right|^2 dx - \frac{1}{\chi_2} \int_{\mathbb{R}^2} u_2 \left|\nabla \log u_2 - \nabla \chi_2 v\right|^2 dx \leqslant 0. \tag{3.6}$$

Proof Using the equations in equation (1.2) and the Green identities, one gets

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^2} u_1 \log u_1 dx &= \int_{\mathbb{R}^2} (u_{1t} \log u_1 + u_{1t})\, dx = \int_{\mathbb{R}^2} (\log u_1 + 1)\, u_{1t} dx \\
&= \int_{\mathbb{R}^2} (\log u_1 + 1)\, (\mu \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v))\, dx \\
&= \int_{\mathbb{R}^2} (\nabla \log u_1)\, (-\mu \nabla u_1 + \chi_1 u_1 \nabla v)\, dx \\
&= \int_{\mathbb{R}^2} u_1\, (\nabla \log u_1) \left(-\frac{\mu \nabla u_1}{u_1} + \chi_1 \nabla v\right) dx \\
&= \int_{\mathbb{R}^2} -u_1\, (\nabla \log u_1)\, (\mu \nabla \log u_1 - \chi_1 \nabla v)\, dx,
\end{aligned} \tag{3.7}$$

equivalently

$$\frac{d}{dt}\left(\mu\left(1+\frac{\chi_2}{\chi_1}\right)\int_{\mathbb{R}^2} u_1\log u_1 dx\right) = \int_{\mathbb{R}^2} -u_1\left(\mu\nabla\log u_1\right)\left(\nabla\left(\mu\log u_1-\chi_1 v\right)\right)dx$$
$$+\frac{\chi_2}{\chi_1}\int_{\mathbb{R}^2} -u_1\left(\mu\nabla\log u_1\right)\left(\nabla\left(\mu\log u_1-\chi_1 v\right)\right)dx, \tag{3.8}$$

in a similar way

$$\frac{d}{dt}\left(\left(1+\frac{\chi_1}{\chi_2}\right)\int_{\mathbb{R}^2} u_2\log u_2 dx\right) = \int_{\mathbb{R}^2} -u_2\left(\nabla\log u_2\right)\left(\nabla\left(\log u_2-\chi_2 v\right)\right)dx$$
$$+\frac{\chi_1}{\chi_2}\int_{\mathbb{R}^2} -u_2\left(\nabla\log u_2\right)\left(\nabla\left(\log u_2-\chi_2 v\right)\right)dx. \tag{3.9}$$

Now, we calculate the derivative for the potential energy terms in equation (3.5)

$$\begin{aligned}
&\frac{d}{dt}\left(-\frac{\chi_1}{2}\int_{\mathbb{R}^2} u_1 v dx\right)\\
&= -\frac{\chi_1}{2}\int_{\mathbb{R}^2}\left(u_{1t}v+u_1v_t\right)dx = -\frac{\chi_1}{2}\int_{\mathbb{R}^2}\left(u_{1t}v+\left(-\Delta v-u_2\right)v_t\right)dx\\
&= -\frac{\chi_1}{2}\int_{\mathbb{R}^2}\left(u_{1t}v-\left(\Delta v\right)v_t-u_2v_t\right)dx = -\frac{\chi_1}{2}\int_{\mathbb{R}^2}\left(u_{1t}v-v\Delta v_t-u_2v_t\right)dx\\
&= -\frac{\chi_1}{2}\int_{\mathbb{R}^2}\left(u_{1t}v+v\frac{d}{dt}(u_1+u_2)-u_2v_t\right)dx = -\frac{\chi_1}{2}\int_{\mathbb{R}^2}\left(2u_{1t}v+vu_{2t}-u_2v_t\right)dx\\
&= -\frac{\chi_1}{2}\int_{\mathbb{R}^2}\left(2v(\mu\Delta u_1-\chi_1\nabla\cdot(u_1\nabla v))+vu_{2t}-u_2v_t\right)dx\\
&= -\frac{\chi_1}{2}\int_{\mathbb{R}^2} 2\nabla v(-\mu\nabla u_1+\chi_1u_1\nabla v))+vu_{2t}-u_2v_tdx\\
&= -\frac{\chi_1}{2}\int_{\mathbb{R}^2} 2u_1\nabla v(-\mu\nabla\log u_1+\chi_1\nabla v))+vu_{2t}-u_2v_tdx\\
&= \int_{\mathbb{R}^2} u_1\nabla\chi_1 v(\mu\nabla\log u_1-\chi_1\nabla v))-\frac{\chi_1}{2}vu_{2t}+\frac{\chi_1}{2}u_2v_tdx\\
&= \int_{\mathbb{R}^2} u_1\nabla\chi_1 v(\mu\nabla\log u_1-\chi_1\nabla v))-\frac{\chi_1}{2}v(\Delta u_2-\chi_2\nabla\cdot(u_2\nabla v))+\frac{\chi_1}{2}u_2v_tdx\\
&= \int_{\mathbb{R}^2} u_1\nabla\chi_1 v(\mu\nabla\log u_1-\chi_1\nabla v))-\frac{\chi_1}{2}v\nabla\cdot(\nabla u_2-\chi_2u_2\nabla v))+\frac{\chi_1}{2}u_2v_tdx\\
&= \int_{\mathbb{R}^2} u_1\nabla\chi_1 v(\mu\nabla\log u_1-\chi_1\nabla v))+\frac{\chi_1}{2}\nabla v\left(\nabla u_2-\chi_2u_2\nabla v\right))+\frac{\chi_1}{2}u_2v_tdx\\
&= \int_{\mathbb{R}^2} u_1\nabla\chi_1 v(\mu\nabla\log u_1-\chi_1\nabla v))+\frac{\chi_1}{2}u_2\nabla v\left(\nabla\log u_2-\chi_2\nabla v\right))+\frac{\chi_1}{2}u_2v_tdx.
\end{aligned} \tag{3.10}$$

Analogously, we have

$$\begin{aligned}
&\frac{d}{dt}\left(-\frac{\chi_2}{2}\int_{\mathbb{R}^2} u_2 v dx\right)\\
&\quad = \int_{\mathbb{R}^2} u_2\nabla\chi_2 v(\nabla\log u_2-\chi_2\nabla v))+\frac{\chi_2}{2}u_1\nabla v\left(\mu\nabla\log u_1-\chi_1\nabla v\right))+\frac{\chi_2}{2}u_1v_tdx.
\end{aligned} \tag{3.11}$$

Otherwise, we have the following expressions for the potential energy terms in equation (3.5) too:

$$\begin{aligned}
&\frac{d}{dt}\left(-\frac{\chi_1}{2}\int_{\mathbb{R}^2} u_2 v dx\right)\\
&= \int_{\mathbb{R}^2}\left(-\frac{\chi_1}{2}u_2 v_t - \frac{\chi_1}{2}u_{2t}v\right)dx = \int_{\mathbb{R}^2}\left(-\frac{\chi_1}{2}u_2 v_t - \frac{\chi_1}{2}(\Delta u_2 - \chi_2\nabla\cdot(u_2\nabla v))v\right)dx\\
&= \int_{\mathbb{R}^2}\left(-\frac{\chi_1}{2}u_2 v_t + \frac{\chi_1}{2}(\nabla u_2 - \chi_2 u_2\nabla v)\nabla v\right)dx\\
&= \int_{\mathbb{R}^2}\left(-\frac{\chi_1}{2}u_2 v_t + \frac{\chi_1}{2}u_2\nabla v(\nabla\log u_2 - \chi_2\nabla v)\right)dx,
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&\frac{d}{dt}\left(-\frac{\chi_2}{2}\int_{\mathbb{R}^2} u_1 v dx\right) = \int_{\mathbb{R}^2}\left(-\frac{\chi_2}{2}u_1 v_t - \frac{\chi_2}{2}u_{1t}v\right)\\
&dx = \int_{\mathbb{R}^2}\left(-\frac{\chi_2}{2}u_1 v_t + \frac{\chi_2}{2}u_1\nabla v(\mu\nabla\log u_1 - \chi_1\nabla v)\right)dx.
\end{aligned} \tag{3.13}$$

Adding equation (3.7) to equations (3.8)–(3.13), we get

$$\begin{aligned}
&\frac{d}{dt}\ \left(\mu\left(1+\frac{\chi_2}{\chi_1}\right)\int_{\mathbb{R}^2} u_1\log u_1 dx + \left(1+\frac{\chi_1}{\chi_2}\right)\int_{\mathbb{R}^2} u_2\log u_2 dx\right.\\
&\qquad\left.-\frac{\chi_1+\chi_2}{2}\times\int_{\mathbb{R}^2} u_1 v dx - \frac{\chi_1+\chi_2}{2}\int_{\mathbb{R}^2} u_2 v dx\right)\\
&= \int_{\mathbb{R}^2} -u_1\left(\mu\nabla\log u_1 - \nabla\chi_1 v\right)\left(\nabla\left(\mu\log u_1 - \chi_1 v\right)\right)dx\\
&+ \int_{\mathbb{R}^2} -u_2\left(\nabla\log u_2 - \nabla\chi_2 v\right)\left(\nabla\left(\log u_2 - \chi_2 v\right)\right)dx\\
&+\frac{\chi_1}{\chi_2}\int_{\mathbb{R}^2} -u_2\left(\nabla\log u_2 - \chi_2\nabla v\right)\left(\nabla\left(\log u_2 - \chi_2 v\right)\right)dx\\
&+\frac{\chi_2}{\chi_1}\int_{\mathbb{R}^2} -u_1\left(\nabla\mu\log u_1 - \chi_1 v\right)\left(\nabla\left(\log\mu u_1 - \chi_1 v\right)\right)dx,
\end{aligned}$$

or

$$\begin{aligned}
\frac{d}{dt}\left((\chi_1+\chi_2)E(t)\right) = &-\int_{\mathbb{R}^2} u_1\left|\mu\nabla\log u_1 - \nabla\chi_1 v\right|^2 dx - \int_{\mathbb{R}^2} u_2\left|\nabla\log u_2 - \nabla\chi_2 v\right|^2 dx\\
&-\frac{\chi_1}{\chi_2}\int_{\mathbb{R}^2} u_2\left|\nabla\log u_2 - \chi_2\nabla v\right|^2 dx - \frac{\chi_2}{\chi_1}\int_{\mathbb{R}^2} u_1\left|\mu\nabla\log u_1 - \chi_1 v\right|^2 dx.
\end{aligned}$$

In consequence, the energy function (3.5) satisfies equation (3.6). □

Lemma 4 *Let u_1, u_2 positive solutions of equation (1.2) then for $0 \leqslant t \leqslant T$ and a,b positive constants there exists a bound $C_{a,b}(T)$ such that*

$$\int_{au_1+bu_2\leqslant 1}(au_1+bu_2)\left|\log(au_1+bu_2)\right| dx \leqslant C_{a,b}(T). \tag{3.14}$$

Proof We observe first that the function $x \to x|\log x|$ is increasing on the interval $(0, e^{-1})$. On the region

$$\left\{x : au_1(x,t) + bu_2(x,t) \leqslant e^{-1}\right\},$$

we have that

$$\begin{aligned}
&\int_{au_1+bu_2\leqslant e^{-1}} (au_1 + bu_2)\,|\log(au_1 + bu_2)|\,dx \\
&\quad= \int_{au_1+bu_2\leqslant e^{-|x|^2}} (au_1 + bu_2)\,|\log(au_1 + bu_2)|\,dx \\
&\qquad+ \int_{e^{-|x|^2}\leqslant au_1+bu_2\leqslant e^{-1}} (au_1 + bu_2)\,|\log(au_1 + bu_2)|\,dx \\
&\quad\leqslant \int_{au_1+bu_2\leqslant e^{-|x|^2}} e^{-|x|^2}|\log e^{-|x|^2}|dx + \int_{e^{-|x|^2}\leqslant au_1+bu_2\leqslant e^{-1}} (au_1 + bu_2)\,|\log e^{-|x|^2}|dx \\
&\quad= \int_{au_1+bu_2\leqslant e^{-|x|^2}} e^{-|x|^2}\,|x|^2\,dx + \int_{e^{-|x|^2}\leqslant au_1+bu_2\leqslant e^{-1}} (au_1 + bu_2)\,|x|^2\,dx \\
&\quad= \int_{\mathbb{R}^2} e^{-|x|^2}\,|x|^2\,dx + \int_{\mathbb{R}^2} (au_1 + bu_2)\,|x|^2\,dx \\
&\quad= \int_{\mathbb{R}^2} e^{-|x|^2}\,|x|^2\,dx + a\int_{\mathbb{R}^2} u_1\,|x|^2\,dx + b\int_{\mathbb{R}^2} u_2\,|x|^2\,dx. \qquad (3.15)
\end{aligned}$$

Now, from equation (2.1), we have that $m(t)$ is bounded on $[0, T]$ then using that $0 \leqslant \int_{\mathbb{R}^2} u_i|x|^2 dx \leqslant \chi_i m(t)$ we get from equation (3.15) that

$$\begin{aligned}
\int_{au_1+bu_2\leqslant e^{-1}} (au_1 + bu_2)\,|\log(au_1 + bu_2)|\,dx &\leqslant \int_{\mathbb{R}^2} e^{-|x|^2}\,|x|^2\,dx \\
&+a\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^2} u_1\,|x|^2\,dx + b\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^2} u_2\,|x|^2\,dx,
\end{aligned}$$

on the other hand using that $x \to |\log x|$ is decreasing on $(0, 1)$ we get

$$\begin{aligned}
&\int_{e^{-1}<au_1+bu_2<1} (au_1 + bu_2)\,|\log(au_1 + bu_2)|\,dx \\
&\quad\leqslant \int_{e^{-1}<au_1+bu_2<1} (au_1 + bu_2)\,\left|\log e^{-1}\right|\,dx = \int_{e^{-1}<au_1+bu_2<1} (au_1 + bu_2)\,dx \\
&\quad\leqslant \int_{\mathbb{R}^2} (au_1 + bu_2)\,dx = a\theta_1 + b\theta_2.
\end{aligned}$$

In consequence for the constant

$$\begin{aligned}
C_{a,b}(T) := \int_{\mathbb{R}^2} e^{-|x|^2}\,|x|^2\,dx &+ a\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^2} u_1\,|x|^2\,dx \\
&+b\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^2} u_2\,|x|^2\,dx + a\theta_1 + b\theta_2,
\end{aligned}$$

we have equation (3.14). □

Lemma 5 (Logarithmic Hardy–Littlewood–Sobolev inequality, cf. [3]) *Let f be a non-negative function in $L^1(\mathbb{R}^2)$ such that $f\log f$ and $f\log(1+|x|^2)$ belong to $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f dx = M_*$, then*

$$\frac{M_*}{2}\int_{\mathbb{R}^2} f\log f dx + \int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(x)f(y)\log|x-y|\, dxdy \geqslant C(M_*) := \frac{M_*^2}{2}(1+\log\pi+\log M_*).$$

The following theorem gives bounds for the entropy and is the key of the proof of global existence for system (1.2). As we will see that the size of the coefficient of diffusion μ plays a significant role in this result.

Theorem 6 *If u_1 and u_2 are positive solutions of equation (1.2) on the interval of the time $[0,T]$ and $\chi_1 \leqslant \chi_2$ then we have the following entropy estimates:*

(1) *If $\mu > 1$ then*

$$\left(1-\frac{M\chi_2}{8\pi}\right)\int_{\mathbb{R}^2}\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx \leqslant C_T, \tag{3.16}$$

where C_T is a constant depending on T and $M=\theta_1+\theta_2$.

(2) *If $\mu \leqslant 1$ then*

$$\left(1-\frac{M\chi_2}{8\pi\mu}\right)\int_{\mathbb{R}^2}\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx \leqslant \overline{C}_T, \tag{3.17}$$

where $\overline{C}_T$ is a constant depending on T and $M=\theta_1+\theta_2$.

Proof Using $(\chi_1+\chi_2)E(t)\leqslant(\chi_1+\chi_2)E(0)$, we obtain

$$\begin{aligned}
&\mu\left(1+\frac{\chi_2}{\chi_1}\right)\int_{\mathbb{R}^2} u_1(x,t)\log u_1 dx + \left(1+\frac{\chi_1}{\chi_2}\right)\int_{\mathbb{R}^2} u_2(x,t)\log u_2(x,t)dx\\
&\leqslant (\chi_1+\chi_2)\,E(0)+\frac{\chi_1+\chi_2}{2}\int_{\mathbb{R}^2} u_1(x,t)v(x,t)dx+\frac{\chi_1+\chi_2}{2}\int_{\mathbb{R}^2} u_2(x,t)v(x,t)dx\\
&= (\chi_1+\chi_2)\,E(0)+\frac{\chi_1+\chi_2}{2}\int_{\mathbb{R}^2}(u_1(x,t)+u_2(x,t))v(x,t)dx\\
&= (\chi_1+\chi_2)\,E(0)-\frac{\chi_1+\chi_2}{4\pi}\int_{\mathbb{R}^2}(u_1(x,t)+u_2(x,t))\int_{\mathbb{R}^2}\log|x-y|\,(u_1(y,t)+u_2(y,t))\,dydx\\
&= (\chi_1+\chi_2)\,E(0)-\frac{\chi_1+\chi_2}{4\pi}\int\int(u_1(x,t)+u_2(x,t))\,(u_1(y,t)+u_2(y,t))\log|x-y|\,dydx,
\end{aligned}$$

applying now the logarithmic Hardy–Littlewood–Sobolev inequality, we get

$$
\begin{aligned}
&\mu\left(1+\frac{\chi_2}{\chi_1}\right)\int_{\mathbb{R}^2} u_1\log u_1 dx+\left(1+\frac{\chi_1}{\chi_2}\right)\int_{\mathbb{R}^2} u_2\log u_2 dx\\
&\quad\leqslant(\chi_1+\chi_2)\,E(0)+\frac{\chi_1+\chi_2}{4\pi}\left(\frac{M}{2}\int_{\mathbb{R}^2}(u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx-C(M)\right)\\
&\quad=(\chi_1+\chi_2)\,E(0)+\frac{\chi_1+\chi_2}{8\pi}M\int_{\mathbb{R}^2}(u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx-\frac{\chi_1+\chi_2}{4\pi}C(M)\\
&\quad\leqslant(\chi_1+\chi_2)\,E(0)-\frac{\chi_1+\chi_2}{4\pi}C(M)\\
&\qquad+\frac{\chi_1+\chi_2}{8\pi}M\int_{u_1+u_2\leqslant 1}(u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx\\
&\qquad+\frac{\chi_1+\chi_2}{8\pi}M\int_{u_1+u_2>1}(u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx. \qquad (3.18)
\end{aligned}
$$

In order to estimate last integral in equation (3.18), we recall that $\chi_1\leqslant\chi_2$, so we can write that

$$
u_1(x,t)+u_2(x,t)<\chi_2\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\quad(\text{if }\mu>1), \qquad (3.19)
$$

$$
u_1(x,t)+u_2(x,t)<\frac{\chi_2}{\mu}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\quad(\text{if }\mu\leqslant 1). \qquad (3.20)
$$

Taking into account that $f(x)=x\log x$ is increasing in $x\geqslant 1$, using $\mu>1$ and equation (3.19) yields

$$
\begin{aligned}
&\int_{u_1+u_2>1}(u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx\\
&\quad\leqslant\chi_2\int_{u_1+u_2>1}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\chi_2\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx\\
&\quad=\chi_2\int_{u_1+u_2>1}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\chi_2 dx\\
&\qquad+\chi_2\int_{u_1+u_2>1}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx\\
&\quad\leqslant\chi_2\log\chi_2\left(\frac{\mu\theta_1}{\chi_1}+\frac{\theta_2}{\chi_2}\right)\\
&\qquad+\chi_2\int_{u_1+u_2>1}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx. \qquad (3.21)
\end{aligned}
$$

On the other hand, using again that $f(x) = x\log x$ is increasing in $x \geqslant 1$ and choosing now $\mu < 1$ and equation (3.20), we get that

$$\begin{aligned}
&\int_{u_1+u_2>1} (u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx \\
&\leqslant \frac{\chi_2}{\mu}\int_{u_1+u_2>1}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\frac{\chi_2}{\mu}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx \\
&= \frac{\chi_2}{\mu}\int_{u_1+u_2>1}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\frac{\chi_2}{\mu}dx \\
&+\frac{\chi_2}{\mu}\int_{u_1+u_2>1}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx \\
&\leqslant \frac{\chi_2}{\mu}\log\frac{\chi_2}{\mu}\left(\frac{\mu\theta_1}{\chi_1}+\frac{\theta_2}{\chi_2}\right) \\
&+\frac{\chi_2}{\mu}\int_{u_1+u_2>1}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx. \qquad (3.22)
\end{aligned}$$

It follows then from equations (3.18) and (3.21) that

$$\begin{aligned}
&\mu\left(1+\frac{\chi_2}{\chi_1}\right)\int_{\mathbb{R}^2} u_1\log u_1 dx + \left(1+\frac{\chi_1}{\chi_2}\right)\int_{\mathbb{R}^2} u_2\log u_2 dx \\
&\leqslant (\chi_1+\chi_2)\,E(0) - \frac{\chi_1+\chi_2}{4\pi}C(M) + \chi_2\left(\frac{\mu\theta_1}{\chi_1}+\frac{\theta_2}{\chi_2}\right)\frac{\chi_1+\chi_2}{8\pi}M\log\chi_2 \\
&+\frac{\chi_1+\chi_2}{8\pi}M\int_{u_1+u_2\leqslant 1}(u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx \\
&+\frac{\chi_1+\chi_2}{8\pi}M\chi_2\int_{u_1+u_2>1}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx.
\end{aligned}$$

After dividing by $\chi_1+\chi_2$ and simplifying, we get

$$\begin{aligned}
&\frac{\mu}{\chi_1}\int_{\mathbb{R}^2} u_1\log u_1 dx + \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2\log u_2 dx \\
&\leqslant E(0) - \frac{1}{4\pi}C(M) + \chi_2\left(\frac{\mu\theta_1}{\chi_1}+\frac{\theta_2}{\chi_2}\right)\frac{1}{8\pi}M\log\chi_2 \\
&+\frac{1}{8\pi}M\int_{u_1+u_2\leqslant 1}(u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx \\
&+\frac{1}{8\pi}M\chi_2\int_{u_1+u_2>1}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx. \quad (3.23)
\end{aligned}$$

Similarly, for $\mu < 1$, we have that

$$
\begin{aligned}
&\frac{\mu}{\chi_1}\int_{\mathbb{R}^2} u_1 \log u_1 dx + \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2 \log u_2 dx \\
&\quad \leqslant E(0) - \frac{1}{4\pi}C(M) + \chi_2\left(\frac{\theta_1}{\chi_1} + \frac{\theta_2}{\chi_2}\right)\frac{1}{8\pi}M\log\chi_2 \\
&\qquad + \frac{1}{8\pi}M\int_{u_1+u_2\leqslant 1}(u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx \\
&\qquad + \frac{1}{8\pi}M\frac{\chi_2}{\mu}\int_{u_1+u_2>1}(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t))\log(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t))dx. \quad (3.24)
\end{aligned}
$$

Using the convexity of the function $f(x) = x\log x$, we observe that

$$
\frac{1}{\frac{\mu}{\chi_1}+\frac{1}{\chi_2}}\left(\frac{\mu}{\chi_1}u_1 + \frac{1}{\chi_2}u_2\right)\log\left[\frac{1}{\frac{\mu}{\chi_1}+\frac{1}{\chi_2}}\left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right)\right] \leqslant \frac{1}{\frac{\mu}{\chi_1}+\frac{1}{\chi_2}}\left(\frac{\mu}{\chi_1}u_1\log u_1 + \frac{1}{\chi_2}u_2\log u_2\right)
$$

or

$$
\begin{aligned}
&\left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right)\log\left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right) - \left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right)\log\left(\frac{\mu}{\chi_1}+\frac{1}{\chi_2}\right) \\
&\quad \leqslant \frac{\mu}{\chi_1}u_1\log u_1 + \frac{1}{\chi_2}u_2\log u_2. \quad (3.25)
\end{aligned}
$$

In case $\mu > 1$ from equations (3.23) and (3.25), we get

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right)\log\left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right)dx - \int_{\mathbb{R}^2}\left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right)\log\left(\frac{\mu}{\chi_1}+\frac{1}{\chi_2}\right)dx \\
&\quad \leqslant E(0) - \frac{1}{4\pi}C(M) + \chi_2\left(\frac{\mu\theta_1}{\chi_1}+\frac{\theta_2}{\chi_2}\right)\frac{1}{8\pi}M\log\chi_2 \\
&\qquad + \frac{1}{8\pi}M\int_{u_1+u_2\leqslant 1}(u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx \\
&\qquad + \frac{1}{8\pi}M\chi_2\int_{u_1+u_2>1}\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx.
\end{aligned}
$$

After reorganising this inequality, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right)\log\left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right)dx \leqslant E(0) - \frac{1}{4\pi}C(M) \\
&\quad + \chi_2\left(\frac{\theta_1}{\chi_1}+\frac{\theta_2}{\chi_2}\right)\frac{1}{8\pi}M\log\chi_2 + \frac{1}{8\pi}M\int_{u_1+u_2\leqslant 1}(u_1(x,t)+u_2(x,t))\log((u_1(x,t)+u_2(x,t))\,dx \\
&\quad + \frac{1}{8\pi}M\chi_2\int_{\mathbb{R}^2}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx \\
&\quad - \frac{1}{8\pi}M\chi_2\int_{u_1+u_2<1}\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{\mu}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dx.
\end{aligned}
$$

In case $\mu \leqslant 1$, we obtain in a like manner from equations (3.24) and (3.25) the estimative

$$\int_{\mathbb{R}^2} \left(\frac{\mu}{\chi_1}u_1 + \frac{1}{\chi_2}u_2\right) \log\left(\frac{\mu}{\chi_1}u_1 + \frac{1}{\chi_2}u_2\right) dx \leqslant E(0) - \frac{1}{4\pi}C(M)$$
$$+\chi_2\left(\frac{\theta_1}{\chi_1} + \frac{\theta_2}{\chi_2}\right)\frac{1}{8\pi}M\log\chi_2 + \frac{1}{8\pi}M\int_{u_1+u_2\leqslant 1}(u_1(x,t)+u_2(x,t))\log((u_1(x,t)+u_2(x,t))\,dx$$
$$+\frac{1}{8\pi}M\frac{\chi_2}{\mu}\int_{\mathbb{R}^2}\left(\frac{\mu}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{\mu}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right)dx$$
$$-\frac{1}{8\pi}M\frac{\chi_2}{\mu}\int_{u_1+u_2<1}\left(\frac{\mu}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{\mu}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right)dx.$$

In conclusion from the last inequality and taking into account Lemma 4 we find that

$$\left(1 - \frac{M\chi_2}{8\pi}\right)\int_{\mathbb{R}^2}\left(\frac{\mu}{\chi_1}u_1 + \frac{1}{\chi_2}u_2\right)\log\left(\frac{\mu}{\chi_1}u_1 + \frac{1}{\chi_2}u_2\right)dx \leqslant C_T \quad \text{if } \mu > 1, \tag{3.26}$$
$$\left(1 - \frac{M\chi_2}{8\pi\mu}\right)\int_{\mathbb{R}^2}\left(\frac{\mu}{\chi_1}u_1 + \frac{1}{\chi_2}u_2\right)\log\left(\frac{\mu}{\chi_1}u_1 + \frac{1}{\chi_2}u_2\right)dx \leqslant \overline{C}_T \quad \text{if } \mu \leqslant 1, \tag{3.27}$$

where

$$C_T := E(0) - \frac{1}{4\pi}C(M) + \chi_2\left(\frac{\mu\theta_1}{\chi_1} + \frac{\theta_2}{\chi_2}\right)\frac{1}{8\pi}M\log\chi_2$$
$$+\frac{1}{8\pi}M\sup_{0\leqslant t\leqslant T}\int_{u_1+u_2\leqslant 1}(u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx$$
$$-\frac{1}{8\pi}M\chi_2\inf_{0\leqslant t\leqslant T}\int_{u_1+u_2<1}\left(\frac{1}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{1}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right)dx.$$

and

$$\overline{C}_T := E(0) - \frac{1}{4\pi}C(M) + \frac{\chi_2}{\mu}\left(\frac{\theta_1}{\chi_1} + \frac{\theta_2}{\chi_2}\right)\frac{1}{8\pi}M\log\chi_2$$
$$+\frac{1}{8\pi\mu}M\sup_{0\leqslant t\leqslant T}\int_{u_1+u_2\leqslant 1}(u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx$$
$$-\frac{1}{8\pi\mu}M\chi_2\inf_{0\leqslant t\leqslant T}\int_{u_1+u_2<1}\left(\frac{1}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{1}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right)dx.$$

□

Using the techniques from [1], we obtain from Theorem 6 and the identity (2.1) the following existence result for system (1.2).

Theorem 7 *Under hypotheses (3.2), $\mu > 1$ and*

$$M = \theta_1 + \theta_2 < \frac{8\pi}{\chi_2}, \tag{3.28}$$

the system (1.2) has a global smooth non-negative solution such that

$$-\frac{1}{\chi_1}\int_0^\infty\int_{\mathbb{R}^2} u_1\left|\mu\nabla\log u_1-\nabla\chi_1 v\right|^2 dx-\frac{1}{\chi_2}\int_0^\infty\int_{\mathbb{R}^2} u_2\left|\nabla\log u_2-\nabla\chi_2 v\right|^2 dxdt<\infty,$$

$$(1+|x|^2+|\log u_1|)u_1,\ (1+|x|^2+|\log u_2|)u_2\in L^\infty([0,T]\,;L^1(\mathbb{R}^2)).$$

On other hand if $\mu\leqslant 1$, under hypotheses (3.17) and

$$M=\theta_1+\theta_2<\frac{8\pi}{\chi_2}\mu, \tag{3.29}$$

system (1.2) has a global smooth non-negative solution such that

$$-\frac{1}{\chi_1}\int_0^\infty\int_{\mathbb{R}^2} u_1\left|\mu\nabla\log u_1-\nabla\chi_1 v\right|^2 dx-\frac{1}{\chi_2}\int_0^\infty\int_{\mathbb{R}^2} u_2\left|\nabla\log u_2-\nabla\chi_2 v\right|^2 dxdt<\infty,$$

$$(1+|x|^2+|\log u_1|)u_1,\ (1+|x|^2+|\log u_2|)u_2\in L^\infty([0,T]\,;L^1(\mathbb{R}^2)).$$

Proof From Theorem 6 and Lemma 4, we deduce that each of the entropies $\int u_1\log u_1 dx$ and $\int u_2\log u_2$ is upper bounded on $M<\frac{8\pi}{\chi_2}$. We prove the equi-integrability for $(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2)\log(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2)$. From Theorem 6, we have an upper bound for

$$\int_{\mathbb{R}^2}\left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right)\log\left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right).$$

We can also see that $\int(\frac{1}{\chi_1}u_1+\frac{1}{\chi_2}u_2)\log(\frac{1}{\chi_1}u_1+\frac{1}{\chi_2}u_2)dx$ is bounded from below. Let

$$n:=\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2.$$

By identity (2.1), we have that

$$\frac{1}{1+t}\int|x|^2\,n(x,t)dx\leqslant K\qquad\forall t>0.$$

From this, we can mimic the proof of the last part of [1], Lemma 2.5 getting that

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right)\log\left(\frac{\mu}{\chi_1}u_1+\frac{1}{\chi_2}u_2\right)dx\\&\quad\geqslant\left(\frac{\theta_1}{\chi_1}+\frac{\theta_2}{\chi_2}\right)\log\left(\frac{\theta_1}{\chi_1}+\frac{\theta_2}{\chi_2}\right)-\frac{1}{1+t}\left(K+\frac{1}{\pi}\left(\frac{\theta_1}{\chi_1}+\frac{\theta_2}{\chi_2}\right)\right).\end{aligned}$$

Regularisation and proof of space time compactness are now essentially the same as in [1] with small modifications. □

4 Blowup under increasing total moment

Condition (3.29) for global existence seems to agree with the intuition in the sense that it can be paraphrase as 'the smaller the coefficient of diffusion the smaller the region of global existence'. However, in case $\mu \geqslant 1$, condition (3.29) does not depends on μ making an interesting contrast. Our purpose in this section is to research the optimality of condition (3.28) to guarantee global existence in time for system (1.2). With this end in mind, we will consider from now on only initial radial conditions u_{10}, u_{20}, v_0. Let us define the cumulative mass variables for system (1.2) by

$$\begin{aligned} M_1(r,t) &:= \int_{\mathbb{R}^2} u_1(x,t)dx = 2\pi \int_{\mathbb{R}^2} u_1(\rho,t)\rho d\rho, \\ M_2(r,t) &:= \int_{\mathbb{R}^2} u_2(x,t)dx = 2\pi \int_{\mathbb{R}^2} u_2(\rho,t)\rho d\rho \end{aligned} \tag{4.1}$$

We are going to prove that if enough species are concentrate at the origin (i.e. we have a small initial moment) then any of the inequalities

$$\theta_1 > \frac{8\pi}{\chi_1}\mu \text{ (where } \mu \geqslant 1) \tag{4.2}$$

$$\theta_2 > \frac{8\pi}{\chi_2} \tag{4.3}$$

implies blowup.

Theorem 8 *Let u_1, u_2 and v smooth solutions of equation (1.2). Let the second moment $m(t)$ defined by*

$$m_1(t) := \int_{\mathbb{R}^2} u_1 |x|^2 dx, \tag{4.4}$$

then we have

$$\frac{dm}{dt} \leqslant 4\theta_1 \left(1 - \frac{\chi_1\theta_1}{8\pi\mu}\right). \tag{4.5}$$

In a like manner defining

$$\widetilde{m}(t) := \int_{\mathbb{R}^2} u_2 |x|^2 dx, \tag{4.6}$$

then we have

$$\frac{d\widetilde{m}}{dt} \leqslant 4\theta_2 \left(1 - \frac{\chi_2\theta_2}{8\pi}\right). \tag{4.7}$$

Proof Multiplying the first equation of equation (1.2) by $|x|^2$ and integrating yield

$$\frac{d}{dt}\int_{\mathbb{R}^2} u_1 |x|^2 dx = \int_{\mathbb{R}^2} \mu |x|^2 \Delta u_1 dx - \chi_1 \int_{\mathbb{R}^2} |x|^2 \nabla \cdot (u_1 \nabla v) dx,$$

then using Greens first identity, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} u_1 |x|^2 dx = 4\mu \int_{\mathbb{R}^2} u_1 dx + 2\chi_1 \int_{\mathbb{R}^2} u_1 (x \cdot \nabla v) dx. \tag{4.8}$$

In cylindrical coordinates, the equation for the concentration of the chemical is

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + u_1 + u_2,$$

multiplying by r and integrating on $(0,r)$, we find that

$$\begin{aligned} rv_r &= -\int_0^r \rho u_1 d\rho - \int_0^r \rho u_2 d\rho \\ &= -\frac{1}{2\pi}\int_{D(0,r)} \rho u_1 d\rho - \frac{1}{2\pi}\int_{D(0,r)} \rho u_2 d\rho = -\frac{M_1+M_2}{2\pi}. \end{aligned}$$

Therefore,

$$\frac{\partial v}{\partial r} = -\frac{M_1+M_2}{2\pi r}. \tag{4.9}$$

Using now equation (4.9) and the general identity

$$x \cdot \nabla\phi = r\frac{\partial\phi}{\partial r},$$

we obtain that

$$\begin{aligned} &\int_{\mathbb{R}^2} u_1(x\cdot\nabla v)dx \\ &= 2\pi\int_0^\infty u_1\rho\frac{\partial v}{\partial\rho}\rho d\rho = -2\pi\int_0^\infty u_1\left(\frac{M_1+M_2}{2\pi}\right)\rho d\rho \\ &= -\int_0^\infty (M_1+M_2)\,u_1\rho d\rho \\ &\leqslant -\int_0^\infty M_1 u_1 \rho d\rho \\ &= -\frac{1}{4\pi}\theta_1^2. \end{aligned} \tag{4.10}$$

From equations (4.8) and (4.10), we get

$$\begin{aligned} \frac{dm}{dt} &\leqslant 4\mu\int_{\mathbb{R}^2} u_1 dx + 2\chi_1\left(-\frac{1}{4\pi}\theta_1^2\right) \\ &= 4\theta_1\mu\left(1-\frac{\chi_1\theta_1}{8\pi\mu}\right). \end{aligned}$$

In a similar way, we can prove inequality (4.7). □

As a consequence, if we have that if the masses satisfy equation (4.2) or equation (4.3) then some of the moment variables m or $\widetilde{m}$ will become identically zero in a finite time T^*. It follows that u will become zero too in a finite time contradicting the conservation of the mass. In conclusion $T_{\max} < T^*$and there is blowup for system (1.2).

Figure 1 illustrates the regions of global existence and blowup giving by equations (2.4), (3.28), (4.2) and (4.3).

5 What species blowup first?

We return to system (1.2) and formulate us now the following question: *What of the chemotactic species will blowup first*? In the radial case, it happens to be simultaneously

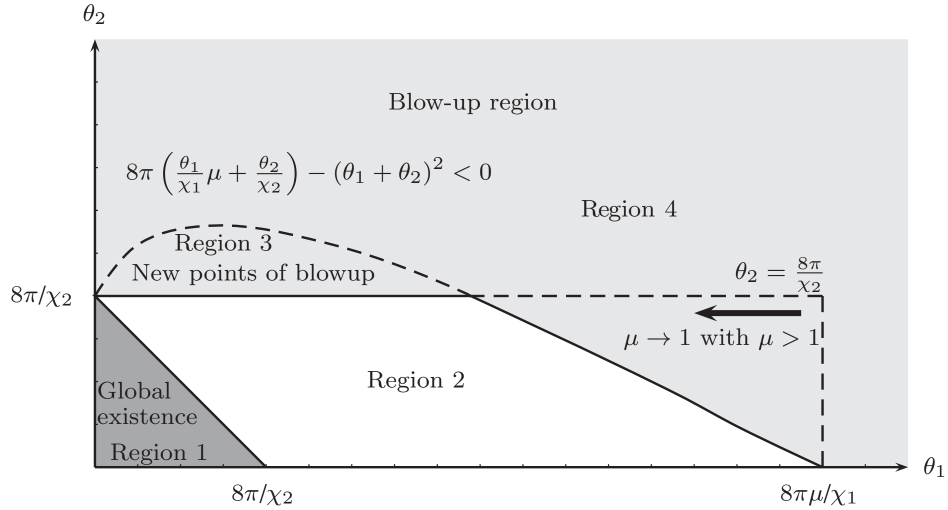

FIGURE 1. Regions of global existence and blowup.

(cf. [6]). An answer for the general case is unknown. Intuitively, one could at least think that blowup should happen first for the species with the bigger chemotactic coefficient χ. However, a mathematical proof of this fact can be difficult. Next lemma will allow us to prove this assertion for masses θ_1 and θ_2 that are relatively 'close' to our threshold curve

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1+\theta_2)^2 = 0.$$

In order to simplify the exposition, we set $\mu = 1$. We point out that all the results below are made before a possible time of blowup happens. Therefore, no problems with a prolongation of the solution after a blowup need to be approached.

Lemma 9 *If u_1 and u_2 are solutions of equation (1.2) then we have estimate*

$$\left(\frac{1}{\chi_1} - \frac{M}{8\pi}\right)\int_{\mathbb{R}^2} u_1 \log u_1 dx + \left(\frac{1}{\chi_2} - \frac{M}{8\pi}\right)\int_{\mathbb{R}^2} u_2 \log u_2 dx \leqslant E(0) - \frac{C(M)}{4\pi} + \frac{1}{8\pi} M(\theta_1+\theta_2)\log 2. \tag{5.1}$$

Proof From equation (3.18) in the proof of Theorem 6, we know that

$$\begin{aligned}&\left(1+\frac{\chi_2}{\chi_1}\right)\int_{\mathbb{R}^2} u_1 \log u_1 dx + \left(1+\frac{\chi_1}{\chi_2}\right)\int_{\mathbb{R}^2} u_2 \log u_2 dx \\ &\leqslant E(0) + \frac{\chi_1+\chi_2}{8\pi} M \int_{\mathbb{R}^2} (u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t))dx - \frac{\chi_1+\chi_2}{4\pi} C(M). \end{aligned} \tag{5.2}$$

The convexity of the function $f(x) = x\log x$ yields

$$(u_1(x,t)+u_2(x,t))\log(u_1(x,t)+u_2(x,t)) \leqslant u_1 \log 2u_1 + u_2 \log u_2. \tag{5.3}$$

Also, from equations (5.2) and (5.3), it follows that

$$\left(1+\frac{\chi_2}{\chi_1}\right)\int_{\mathbb{R}^2} u_1 \log u_1 dx + \left(1+\frac{\chi_1}{\chi_2}\right)\int_{\mathbb{R}^2} u_2 \log u_2 dx$$
$$\leqslant E(0) - \frac{\chi_1+\chi_2}{4\pi}C(M)$$
$$+\frac{\chi_1+\chi_2}{8\pi}M\int_{\mathbb{R}^2}(u_1(x,t)\log 2u_1(x,t) + u_2(x,t)\log 2u_2(x,t))\,dx$$
$$= E(0) - \frac{\chi_1+\chi_2}{4\pi}C(M)$$
$$+\frac{\chi_1+\chi_2}{8\pi}M\int_{\mathbb{R}^2}(u_1(x,t)\,(\log 2+\log u_1(x,t)) + u_2(x,t)\,(\log 2+\log u_2(x,t)))\,dx$$
$$= E(0) - \frac{\chi_1+\chi_2}{4\pi}C(M)$$
$$+\frac{\chi_1+\chi_2}{8\pi}M\int_{\mathbb{R}^2}(u_1(x,t)\log 2 + u_1(x,t)\log u_1(x,t) + u_2(x,t)\log 2 + u_2(x,t)\log u_2(x,t))\,dx$$
$$= E(0) - \frac{\chi_1+\chi_2}{4\pi}C(M)$$
$$+\frac{\chi_1+\chi_2}{8\pi}M\int_{\mathbb{R}^2}(u_1(x,t)+u_2(x,t))\log 2 + u_1(x,t)\log u_1(x,t) + u_2(x,t)\log u_2(x,t)dx$$
$$= E(0) - \frac{\chi_1+\chi_2}{4\pi}C(M) + \frac{\chi_1+\chi_2}{8\pi}M\,(\theta_1+\theta_2)\log 2$$
$$+\frac{\chi_1+\chi_2}{8\pi}M\int_{\mathbb{R}^2}(u_1(x,t)\log u_1(x,t) + u_2(x,t)\log u_2(x,t))\,dx$$
$$= E(0) - \frac{\chi_1+\chi_2}{4\pi}C(M) + \frac{\chi_1+\chi_2}{8\pi}M\,(\theta_1+\theta_2)\log 2$$
$$+\frac{\chi_1+\chi_2}{8\pi}M\int_{\mathbb{R}^2}(u_1(x,t)\log u_1(x,t) + u_2(x,t)\log u_2(x,t))\,dx.$$

In consequence,

$$\left(\frac{1}{\chi_1}-\frac{M}{8\pi}\right)\int_{\mathbb{R}^2} u_1\log u_1 dx + \left(\frac{1}{\chi_2}-\frac{M}{8\pi}\right)\int_{\mathbb{R}^2} u_2\log u_2 dx \leqslant \frac{E(0)}{\chi_1+\chi_2} - \frac{1}{4\pi}C(M)$$
$$+\frac{1}{8\pi}M\,(\theta_1+\theta_2)\log 2.$$

Equivalently,

$$\left(\frac{1}{\chi_1}-\frac{M}{8\pi}\right)\int_{\mathbb{R}^2} u_1\log u_1 dx + \left(\frac{1}{\chi_2}-\frac{M}{8\pi}\right)\int_{\mathbb{R}^2} u_2\log u_2 dx \leqslant E(0) - \frac{C(M)}{4\pi} + \frac{1}{8\pi}M\,(\theta_1+\theta_2)\log 2.$$

□

Theorem 10 *For system (1.2) suppose that*

$$\frac{4\pi\theta_1}{\chi_1}+\frac{4\pi\theta_2}{\chi_2}-\frac{1}{2}(\theta_1+\theta_2)^2<0,\ \ \frac{1}{\chi_1}-\frac{M}{8\pi}>0,\ \ \frac{1}{\chi_2}-\frac{M}{8\pi}<0.$$

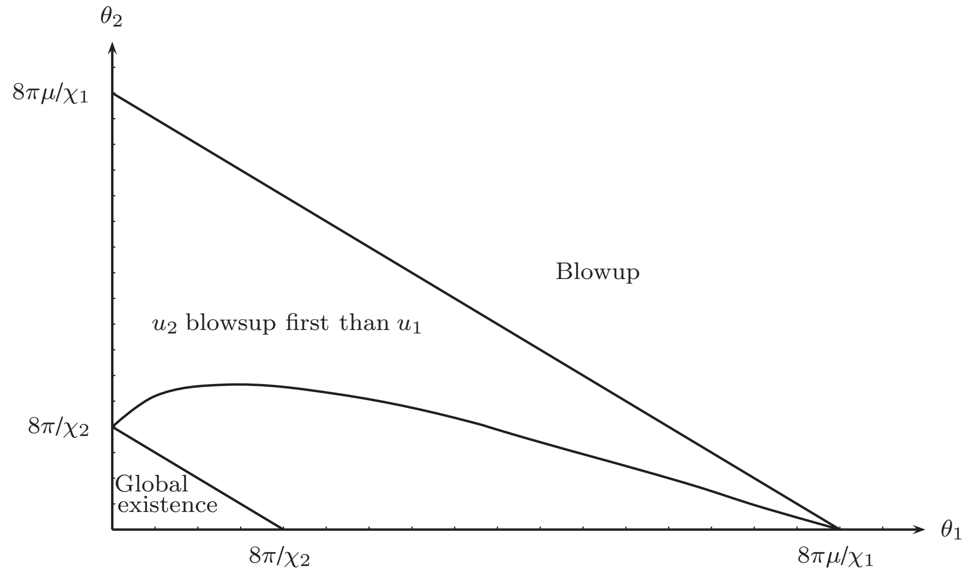

FIGURE 2. Region where u_2 blowup first than u_1.

If u_1 blowup at $t = T_1$ and u_2 blowup at $t = T_2$, then $T_2 \leqslant T_1$.

Proof Let T be a constant such that $T < T_2$. Then, we have $\sup_{0\leqslant t\leqslant T} \|u_2\|_\infty < \infty$, and therefore,

$$\left|\int_{\mathbb{R}^2} u_2 \log u_2 dx\right| \leqslant \int_{\mathbb{R}^2} u_2 \left|\log u_2\right| dx \leqslant \int_{u_2<1} u_2 \left|\log u_2\right| dx + \int_{u_2>1} u_2 \left|\log u_2\right| dx.$$

The first integral is bounded by Lemma 4. For the second one, we observe that

$$\int_{u_2>1} u_2 \left|\log u_2\right| dx = \int_{u_2>1} u_2 \log u_2 dx \leqslant \log \|u_2\|_\infty \int_{u_2>1} u_2 dx$$
$$\leqslant \log \|u_2\|_\infty \int_{\mathbb{R}^2} u_2 dx = \log \|u_2\|_\infty \theta_1.$$

In consequence, $\int_{\mathbb{R}^2} u_2 \log u_2 dx$ is finite for $t \in [0, T]$.
Let us now show that entropy of u_1 is bounded too. A lower estimate for $\int_{\mathbb{R}^2} u_1 \log u_1 dx$ of the form

$$C(1+t) \leqslant \int_{\mathbb{R}^2} u_1 \log u_1 dx \tag{5.4}$$

can be found using the same argument like in Theorem 6. From equations (5.1) and (5.4),

$$C(1+t) \leqslant \int_{\mathbb{R}^2} u_1 \log u_1 dx \leqslant \frac{1}{\frac{1}{\chi_2} - \frac{M}{8\pi}} \left(E(0) - \frac{C(M)}{4\pi} - \left(\frac{1}{\chi_1} - \frac{M}{8\pi}\right) \int_{\mathbb{R}^2} u_2 \log u_2 dx\right) < \infty. \tag{5.5}$$

Using equation (5.5) and the technique from [9], we will show that the L^p norms of u_1 are bounded for every $p > 1$.

From equation (1.2), we have that

$$\partial_t u_1 = \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v),. \tag{5.6}$$

Let $k \geqslant 0$. Multiplying equation (5.6) by $(u_1-k)_+^{p-1}$ $(p>1)$ and integrating,

$$\int_{\mathbb{R}^2} (u_1-k)_+^{p-1}\,\partial_t\,(u_1-k)\,dx = \int_{\mathbb{R}^2} (u_1-k)_+^{p-1}\,\Delta u_1 dx - \chi_1 \int_{\mathbb{R}^2} (u_1-k)_+^{p-1}\,\nabla\cdot(u_1\nabla v)dx.$$

Applying integration by parts, we obtain

$$\int_{\mathbb{R}^2} (u_1-k)_+^{p-1}\,\partial_t u_1 = -\int_{\mathbb{R}^2} \nabla\,(u_1-k)_+^{p-1}\cdot\nabla u_1 dx - \chi_1 \int_{\mathbb{R}^2} (u_1-k)_+^{p-1}\,\nabla\cdot(u_1\nabla v)dx,$$

or

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} (u_1-k)_+^{p}\,dx = -\int_{\mathbb{R}^2} \nabla\,(u_1-k)_+^{p-1}\cdot\nabla(u_1-k)dx - \chi_1 \int_{\mathbb{R}^2} (u_1-k)_+^{p-1}\,\nabla\cdot(u_1\nabla v)dx,$$

equivalently

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} (u_1-k)_+^{p}dx = -\int_{\mathbb{R}^2} \nabla(u_1-k)_+^{p-1}\cdot\nabla(u_1-k)dx - \chi_1\int_{\mathbb{R}^2}(u_1-k)_+^{p-1}\nabla\cdot((u_1-k)\nabla v)dx$$
$$-k\chi_1\int_{\mathbb{R}^2}(u_1-k)_+^{p-1}\,\nabla\cdot\nabla v dx = I_1+I_2+I_3. \tag{5.7}$$

We can now follow some similar techniques like in [1, 5] and [9] to obtain the following:

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^2} \nabla\,(u_1-k)_+^{p-1}\cdot\nabla(u_1-k)dx = (p-1)\int_{\mathbb{R}^2}(u_1-k)_+^{p-2}\,\nabla\,(u_1-k)_+\cdot\nabla(u_1-k)dx\\
&= (p-1)\int_{\mathbb{R}^2}(u_1-k)_+^{\frac{p-2}{2}}\,\nabla\,(u_1-k)_+\cdot(u_1-k)_+^{\frac{p-2}{2}}\,\nabla(u_1-k)dx\\
&= \frac{2(p-1)}{p}\int_{\mathbb{R}^2}\nabla\,(u_1-k)_+^{\frac{p}{2}}\cdot\nabla(u_1-k)_+^{\frac{p}{2}}dx\\
&= \frac{2(p-1)}{p}\int_{\mathbb{R}^2}\left|\nabla\,(u_1-k)_+^{\frac{p}{2}}\right|^2 dx,\\
I_2 &= \int_{\mathbb{R}^2}(u_1-k)_+^{p-1}\,\nabla\cdot((u_1-k)\nabla v)dx = \int_{\mathbb{R}^2}(u_1-k)_+^{p-1}\,[\nabla(u_1-k)\cdot\nabla v + (u_1-k)\Delta v)]\,dx\\
&= \int_{\mathbb{R}^2}(u_1-k)_+^{p-1}\,\nabla(u_1-k)\cdot\nabla v dx + \int_{\mathbb{R}^2}(u_1-k)_+^{p-1}\,(u_1-k)\Delta v)dx\\
&= \frac{1}{p}\int_{\mathbb{R}^2}\nabla(u_1-k)_+^{p}\cdot\nabla v dx + \int_{\mathbb{R}^2}(u_1-k)_+^{p}\,\Delta v dx\\
&= -\frac{1}{p}\int_{\mathbb{R}^2}(u_1-k)_+^{p}\Delta v dx + \int_{\mathbb{R}^2}(u_1-k)_+^{p}\,\Delta v dx\\
&= -\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^2}(u_1-k)_+^{p}(u_1+u_2)dx\\
&= -\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^2}(u_1-k)_+^{p}\,[(u_1-k)+(u_2+k)]\,dx\\
&= -\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^2}(u_1-k)_+^{p+1}dx - \left(1-\frac{1}{p}\right)\int_{\mathbb{R}^2}(u_1-k)_+^{p}\,(u_2+k)\,dx,
\end{aligned}$$

$$I_3 = \int_{\mathbb{R}^2} (u_1 - k)_+^{p-1} \nabla \cdot \nabla v dx = -\int_{\mathbb{R}^2} (u_1 - k)_+^{p-1} \Delta v dx = -\int_{\mathbb{R}^2} (u_1 - k)_+^{p-1} (u_1 + u_2) dx$$
$$= -\int_{\mathbb{R}^2} (u_1 - k)_+^{p} dx - \int_{\mathbb{R}^2} (u_1 - k)_+^{p-1} (u_2 + k)\, dx.$$

Therefore, equation (5.7) becomes

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} (u_1 - k)_+^p\, dx = &-\frac{2(p-1)}{p}\int_{\mathbb{R}^2} \left|\nabla (u_1 - k)_+^{\frac{p}{2}}\right|^2 dx\\ &-\chi_1\left(-\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^2} (u_1 - k)_+^{p+1} dx\right.\\ &\left.-\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^2} (u_1 - k)_+^{p} (u_2 + k)\, dx\right)\\ &-k\chi_1\left(-\int_{\mathbb{R}^2} (u_1 - k)_+^{p} dx - \int_{\mathbb{R}^2} (u_1 - k)_+^{p-1} (u_2 + k)\, dx\right),\end{aligned}$$

or

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} (u_1 - k)_+^p\, dx = &-\frac{2(p-1)}{p}\int_{\mathbb{R}^2} \left|\nabla (u_1 - k)_+^{\frac{p}{2}}\right|^2 dx\\ &+\chi_1\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^2} (u_1 - k)_+^{p} (u_2 + k)\, dx + k\chi_1\int_{\mathbb{R}^2} (u_1 - k)_+^{p} dx\\ &+k\chi_1\int_{\mathbb{R}^2} (u_1 - k)_+^{p-1} (u_2 + k)\, dx + \chi_1\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^2} (u_1 - k)_+^{p+1} dx.\end{aligned} \tag{5.8}$$

The term $\int_{\mathbb{R}^2}(u_1 - k)_+^{p-1}(u_2 + k)dx$ is estimate as follows:

$$\begin{aligned}\int_{\mathbb{R}^2} (u_1 - k)_+^{p-1} (u_2 + k)\, dx &\leqslant \left(|u_2|_\infty + k\right)\int_{\mathbb{R}^2} (u_1 - k)_+^{p-1} dx\\ &= \left(|u_2|_\infty + k\right)\left(\int_{0\leqslant u_1 - k\leqslant 1} (u_1 - k)_+^{p-1} dx + \int_{u_1 - k > 1} (u_1 - k)_+^{p-1} dx\right)\\ &= \left(|u_2|_\infty + k\right)\left(\int_{0\leqslant u_1 - k\leqslant 1} 1 dx + \int_{u_1 - k > 1} (u_1 - k)_+^{p-1}(u_1 - k) dx\right)\\ &= \left(|u_2|_\infty + k\right)\left(\int_{0\leqslant u_1 - k\leqslant 1} 1 dx + \int_{u_1 - k > 1} (u_1 - k)_+^{p-1}(u_1 - k)_+ dx\right)\\ &= \left(|u_2|_\infty + k\right)\left(\int_{0\leqslant u_1 - k\leqslant 1} \frac{u_1}{k} dx + \int_{u_1 - k > 1} (u_1 - k)_+^{p} dx\right)\\ &= \left(|u_2|_\infty + k\right)\left(\frac{\theta_1}{k} + \int_{\mathbb{R}^2} (u_1 - k)_+^{p} dx\right).\end{aligned}$$

In order to estimate the difference

$$\chi_1\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^2} (u_1 - k)_+^{p+1} dx - \frac{2(p-1)}{p}\int_{\mathbb{R}^2} \left|\nabla (u_1 - k)_+^{\frac{p}{2}}\right|^2 dx, \tag{5.9}$$

we apply the Gagliardo–Nirenberg–Sobolev inequality

$$\|u\|_{p^*} \leqslant C\,\|Du\|_{L^p}, \qquad 1 \leqslant p < n,$$

with $p = 1$, $n = 2$ ($\Rightarrow p^* = 2$). We get then that

$$\int u^2 dx \leqslant C\left(\int |\nabla u|\,dx\right)^2$$

and obtain

$$\begin{aligned}
\int_{\mathbb{R}^2}(u_1-k)_+^{p+1}dx &\leqslant C\left(\int_{\mathbb{R}^2}\left|\nabla(u_1-k)_+^{\frac{p+1}{2}}\right|dx\right)^2\\
&= C\left(\int_{\mathbb{R}^2}\left|\nabla(u_1-k)_+^{\frac{p}{2}(1+\frac{1}{p})}\right|dx\right)^2\\
&= C\left(\int_{\mathbb{R}^2}\left|\nabla\left\{(u_1-k)_+^{\frac{p}{2}}\right\}^{(1+\frac{1}{p})}\right|dx\right)^2\\
&= C\left(\left(1+\frac{1}{p}\right)\int_{\mathbb{R}^2}\left|(u_1-k)_+^{\frac{p}{2}(1+\frac{1}{p}-1)}\,\nabla(u_1-k)_+^{\frac{p}{2}}\right|dx\right)^2\\
&= C\left(1+\frac{1}{p}\right)^2\left(\int_{\mathbb{R}^2}\left|(u_1-k)_+^{\frac{1}{2}}\,\nabla(u_1-k)_+^{\frac{p}{2}}\right|dx\right)^2\\
&= C\left(1+\frac{1}{p}\right)^2\left(\int_{\mathbb{R}^2}\left|(u_1-k)_+^{\frac{1}{2}}\,\nabla(u_1-k)_+^{\frac{p}{2}}\right|dx\right)^2\\
&\leqslant C\left(1+\frac{1}{p}\right)^2\left(\int_{\mathbb{R}^2}(u_1-k)_+\,dx\right)\left(\int_{\mathbb{R}^2}\left|\nabla(u_1-k)_+^{\frac{p}{2}}\right|^2dx\right). \qquad (5.10)
\end{aligned}$$

From equations (5.9) and (5.10),

$$\begin{aligned}
&\chi_1\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^2}(u_1-k)_+^{p+1}dx - \frac{2(p-1)}{p}\int_{\mathbb{R}^2}\left|\nabla(u_1-k)_+^{\frac{p}{2}}\right|^2dx\\
&\quad\leqslant\left(C\chi_1\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\right)^2\left(\int_{\mathbb{R}^2}(u_1-k)_+\,dx\right)-\frac{2(p-1)}{p}\right)\int_{\mathbb{R}^2}\left|\nabla(u_1-k)_+^{\frac{p}{2}}\right|^2dx.
\end{aligned}$$

Also, we get from equation (5.8) that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}(u_1-k)_+^p\,dx &\leqslant\left(\chi_1\left(1-\frac{1}{p}\right)\left(\|u_2\|_\infty+k\right)+k\chi_1\right)\int_{\mathbb{R}^2}(u_1-k)_+^p dx\\
&\quad + k\chi_1\left(\|u_2\|_\infty+k\right)\int_{\mathbb{R}^2}(u_1-k)_+^{p-1}dx\\
&\quad + \left\{\chi_1\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^2}(u_1-k)_+^{p+1}dx-\frac{2(p-1)}{p}\int_{\mathbb{R}^2}\left|\nabla(u_1-k)_+^{\frac{p}{2}}\right|^2dx\right\}\\
&\leqslant\left(\chi_1\left(1-\frac{1}{p}\right)\left(\|u_2\|_\infty+k\right)+k\chi_1\right)\int_{\mathbb{R}^2}(u_1-k)_+^p dx
\end{aligned}$$

$$
\begin{aligned}
&+ k\chi_1 \left(\|u_2\|_\infty + k\right) \left(\|u_2\|_\infty + k\right) \left(\frac{\theta_1}{k} + \int_{\mathbb{R}^2} (u_1 - k)_+^p dx\right) \\
&+ \left(C\chi_1 \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p}\right)^2 \left(\int_{\mathbb{R}^2} (u_1 - k)_+ \, dx\right) - \frac{2(p-1)}{p}\right) \\
&\times \int_{\mathbb{R}^2} \left|\nabla (u_1 - k)_+^{\frac{p}{2}}\right|^2 dx \\
= {}& C_1(p, k, \chi_1, \|u_2\|_\infty) \int_{\mathbb{R}^2} (u_1 - k)_+^p dx \\
&+ C_2(p, k, \chi_1, \|u_2\|_\infty) \\
&+ \left(C\chi_1 \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p}\right)^2 \left(\int_{\mathbb{R}^2} (u_1 - k)_+ \, dx\right) - \frac{2(p-1)}{p}\right) \\
&\times \int_{\mathbb{R}^2} \left|\nabla (u_1 - k)_+^{\frac{p}{2}}\right|^2 dx.
\end{aligned}
$$

We observe here that

$$
\begin{aligned}
\int_{\mathbb{R}^2} (u_1 - k)_+ \, dx &= \int_{u_1 - k > 0} (u_1 - k) \, dx \leqslant \int_{u_1 - k > 0} u_1 dx \\
&\leqslant \int_{u_1 > k} u_1 dx \leqslant \frac{1}{K} \int_{u_1 > k} u_1 \log u_1 dx \leqslant \frac{1}{K} \int_{u_1 > k} |u_1 \log u_1| \, dx.
\end{aligned}
$$

In consequence, we can make the integral $\int_{\mathbb{R}^2} (u_1 - k)_+ dx$ as small as we want uniformly in t, taking k big enough. Summarising for any $p > 1$, we can find k big enough such that

$$
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} (u_1 - k)_+^p \, dx \leqslant C_1(p, k, \chi_1, \|u_2\|_\infty) \int_{\mathbb{R}^2} (u_1 - k)_+^p dx + C_2(p, k, \chi_1, \|u_2\|_\infty). \tag{5.11}
$$

Gronwall inequality shows that $\int_{\mathbb{R}^2} (u_1 - k)_+^p dx$ is finite on $[0, T]$.
Using the inequality

$$
x^p \leqslant \left(\frac{\lambda}{\lambda - 1}\right)^{p-1} (x - 1)^p \quad \text{for any } x \geqslant \lambda > 1,
$$

we can deduce that $\int_{u_1 > k} u_1^p dx$ is finite (for details see [1], p. 20). Now, we use that

$$
\begin{aligned}
\int_{\mathbb{R}^2} u_1^p dx &= \int_{u_1 \leqslant k} u_1^p dx + \int_{u_1 < k} u_1^p dx \\
&\leqslant K^{p-1} \int_{u_1 \leqslant k} u_1 dx + \int_{u_1 > k} u_1^p dx \\
&\leqslant K^{p-1} M + \int_{u_1 < k} u_1^p dx,
\end{aligned}
$$

to conclude that $u_1 \in L^p$ for every $p > 1$.
Using regularity results for PDE´s, we conclude that in fact u_1 is bounded in $\mathbb{R}^2 \times [0, T]$.

□

6 Concluding remarks, open questions and comments on the steady state case

Some generalisations of the Keller–Segel model have been studied, in order to include several chemotactic populations (cf. [7, 15, 16, 21]). In case of blowup, the nature of the domain can play an important role to describe the blow-up phenomena (cf. [8]).

We have proved that the total moment

$$m(t) := \frac{\pi}{\chi_1} \int_{\mathbb{R}^2} u_1 |x|^2 \, dx + \frac{\pi}{\chi_2} \int_{\mathbb{R}^2} u_2 |x|^2 \, dx \tag{6.1}$$

can be increasing and system (1.2) can still blowup. This situation is illustrated by region number 3 in Figure 1. We are not in a position to describe the nature of the blowup in this region. We do not know if one species can blowup and the other one could remain bounded globally in time (non-simultaneous blowup). This feature of the blowup suggests us that the moment of one species could probably increase meanwhile the other one is decreasing, in such a way that the total moment $m(t)$ increases. This opens several interesting questions. In particular, to find sharp conditions for initial data in region 3 of Figure 1 providing a full description of the behaviour of both species' densities.

As we mentioned in the Introduction, Wolansky's question was already studied by Horstmann [10] and Horstmann and Lucia [11] in the stationary case and for a bounded domain with Neumann boundary conditions. This approach is important because it allows us to have a first idea about the optimal region of existence for system (1.2) in the plane of masses. The main difference between their conclusions and ours is the dependence on the chemoattractant coefficients; our conditions of global existence depend only on χ_2.

Let us now briefly discuss the stationary model associate with system (1.2), which is

$$\begin{aligned} 0 &= \mu \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v) \quad \text{in} \quad \mathbb{R}^2, \\ 0 &= \Delta u_2 - \chi_2 \nabla \cdot (u_2 \nabla v) \quad \text{in} \quad \mathbb{R}^2, \\ 0 &= \Delta v + u_1 + u_2 \quad \text{in} \quad \mathbb{R}^2. \end{aligned} \tag{6.2}$$

Through a variational formulation, Horstmann [11], Theorem 4.2, analyses a system similar to equation (6.2), in a two-dimensional disk and Neumann boundary conditions. As pointed out by this author, it would be interesting to know which are the optimal conditions on the parameters θ_1, θ_2 that allows us to conclude the existence of blowup or global existence in time. When considered in $\mathbb{R}^2$, these problems have interesting relations to Moser–Trudinger-like type inequalities for systems (see, for example the paper from Shafrir and Wolansky [19]. Another interesting question is: *In case of blowup, should this be simultaneous or not?* In the radial symmetrical case, it was shown that it has to be simultaneous (see Espejo *et al.* [6]), in the general, non-radial case the question remains open up to our knowledge.

On the other side, Horstmann [10] studies some blow-up results for a system similar to equation (1.2) on a bounded domain in case $\chi_1 = \chi_2$. The corresponding result of our Theorems 2 and 7 in case $\chi_1 = \chi_2$ and $\mu = 1$ reads as follows: *There exist positive initial data* (u_{10}, u_{20}, v_0) *satisfying*

$$\chi_1 \int_{\mathbb{R}^2} (u_{10} + u_{20}) dx > 8\pi$$

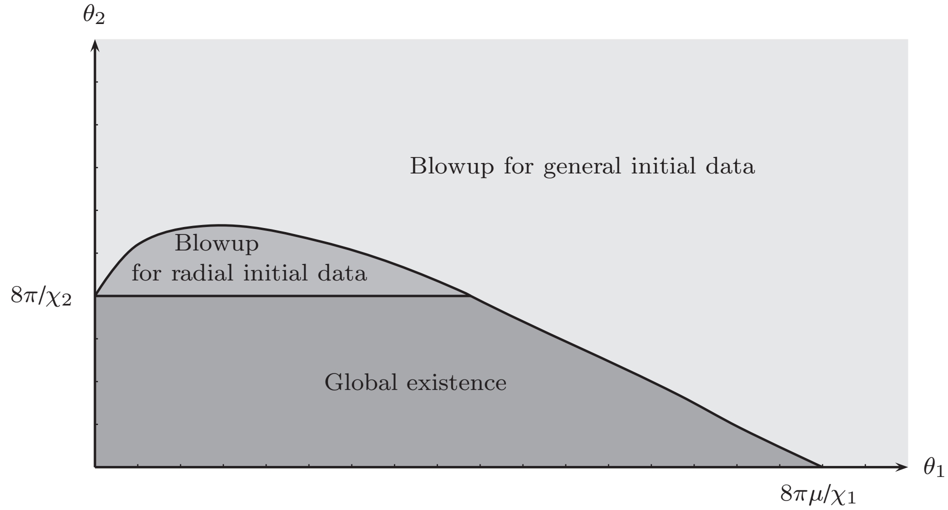

FIGURE 3. In the middle gray region, the time behaviour of the solutions should not only depend on the coordinates (θ_1, θ_2) but on more specific information about u_{10} and u_{20}.

such that

$$\limsup \|u_1(t) + u_2(t)\|_{L^\infty(\Omega)} = \infty \text{ as } t \to T_{\max},$$

where $T_{\max}$ *denotes the maximal time of existence and* $(u_1(t), u_2(t), v(t))$ *is a solution of equation (1.2). However, if*

$$\chi_1 \int_{\mathbb{R}^2} (u_{10} + u_{20}) dx < 8\pi,$$

then the corresponding solution $((u_1(t), u_2(t), v(t))$ *of equation (1.2) exists globally in time and is uniformly bounded.*

This is exactly the analogue to what is stated in Theorem 8.2 of [10]. In this sense, our results can be regarded as generalising the case $\chi_1 = \chi_2$ when working in $\mathbb{R}^2$. As observed by Horstmann, this latter situation is very special, since the whole system can be reduced to the classical one species system. However, if $\chi_1 \neq \chi_2$, there is no way to reduce the multi-species chemotaxis model to a single species model.

Before concluding, let us mention that our results suggest a partial answer to Wolansky's question. Namely, the case of two chemotaxis species in $\mathbb{R}^2$ would have a threshold curve that divide the plane of masses in two regions corresponding to different behaviours of the solutions as illustrated in Figure 3.

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Sharp condition for blow-up and global existence in a two species chemotactic Keller–Segel system in $\mathbb{R}^2$

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For the parabolic–elliptic Keller–Segel system in $\mathbb{R}^2$ it has been proved that if the initial mass is less than $8\pi/\chi$, a global solution exists, and in case the initial mass is larger than $8\pi/\chi$, blow-up happens. The case of several chemotactic species introduces an additional question: What is the analog for the critical mass obtained for the single species system? We find a threshold curve in the two species case that allows us to determine if the system is a blow-up or a global in time solution. No radial symmetry is assumed.

Key words: chemotaxis, multicomponent Keller–Segel model, sharp conditions

1 Introduction

The Keller–Segel model describes the aggregation of living organisms like cells, bacteria or amoebae. This is the simplest mechanism of aggregation. The most famous example in nature for this type of cell motion is the Dictyostelium discoideum or Slime mould; this amoeba was discovered in the first half of the 20th century. The slime mould is a unicellular organism that detect an extracellular signal and transforms it into an intracellular signal. These signal activates oriented cell movement towards a signal, this is an aggregation process. The signal is a chemical secreted by themselves and is called cyclic Adenosine Monophosphate (cAMP).

A classical mathematical model in chemotaxis was introduced by Keller and Segel in 1971 [12]. The Keller–Segel model is as follows:

$$\begin{aligned} u_t &= \nabla\cdot(\mu\nabla u - \chi u\nabla v) \quad && x\in\Omega, \quad t>0,\\ v_t &= \gamma\Delta v - \beta v + \alpha u && x\in\Omega, \quad t>0, \end{aligned} \tag{1}$$

where $u(x,t)$ is the cell density and $v(x,t)$ is the concentration of chemical at point x and time t subject to the homogeneous Neumann boundary conditions and positive initial data $u(x,0)=u_0$ and $v(x,0)=v_0$. In this model, χ is the chemotactic sensitivity, γ is the diffusion coefficient of the chemo-attractant, μ is the diffusion coefficient of cell density, β is the rate of consumption and α is the rate of production, all are positive parameters, and $\Omega \subset \mathbb{R}^N$ has smooth boundary $\partial\Omega$. It was conjectured by Childress and Percus [5] that in a two-dimensional domain there exists a critical number C such that if $\int u_0(x)dx < C$ then the solution exists globally in time, and if $\int u_0(x)dx > C$, then blow-up happens. For different versions of the Keller–Segel model, the conjecture has been essentially proved, finding the critical value $C = 8\pi/\chi$; for a complete review of this topic, we refer readers to [9, 10] and the references therein, and [2, 4, 11, 13, 15].

In the case of several chemotactic species, a new question arises, namely: *Is there a critical curve in the plane of initial masses $\theta_1\theta_2$ delimiting on one side global existence and blow-up on the other side?* This question was previously formulated by Wolansky in [16], and from Theorem 5 of this last paper we readily deduce the following result.

Theorem 1 *Consider the system*

$$\begin{aligned}\partial_t u_1 &= \mu\Delta u_1 - \chi_1\nabla\cdot(u_1\nabla v)\\ \partial_t u_2 &= \Delta u_2 - \chi_2\nabla\cdot(u_2\nabla v)\\ 0 &= \Delta v + u_1 + u_2 - v,\end{aligned}$$

along with Dirichtlet boundary conditions for v and initial radial data: $u_1(0,\cdot)=\varphi$, $u_2(0,\cdot)=\psi$, $v(0,\cdot)=\phi$, with $\varphi,\psi,\phi \geqslant 0$ on the two-dimensional disc of radius 1. Further, let θ_1, θ_2 be the total preserved masses of the chemotactic species. Assume further that

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1+\theta_2)^2 > 0, \quad \theta_1 < 8\pi\mu/\chi_1, \qquad \theta_2 < 8\pi/\chi_2\,. \tag{2}$$

Then for $(u_1(0,\cdot),u_2(0,\cdot)) \in Y_N$ with

$$Y_N = \left\{u_1,u_2 : B(0)\to\mathbb{R}^+ : \int u_i = \theta_i, \qquad \int_{B_1(0)} u_i\log u_i < \infty\right\},$$

there exists a global in time classical solution.

A natural question arises from this last result. What happens if inequalities (2) do not hold? Is it still possible to have global solutions? With regard to this question it is worth recalling here a result from Conca *et al.* [6], who considered the following system in the whole space in two dimensions:

$$\left.\begin{array}{r}\partial_t u_1 = \mu\Delta u_1 - \chi_1\nabla\cdot(u_1\nabla v), \qquad x\in\mathbb{R}^2, \quad t>0\\ \partial_t u_2 = \Delta u_2 - \chi_2\nabla\cdot(u_2\nabla v), \qquad x\in\mathbb{R}^2, \quad t>0\\ v(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|\,(u_1(y,t)+u_2(y,t))\,dy, \quad x\in\mathbb{R}^2, \quad t>0\\ u_1(x,0)=u_{10}\geqslant 0,\ u_2(x,0)=u_{20}\geqslant 0, \quad x\in\mathbb{R}^2, \quad t>0\end{array}\right\}, \tag{3}$$

where u_1 and u_2 are the density variables for two different chemotaxis species and v is the chemoattractant, χ_1,χ_2, μ are positive constants and positive initial conditions u_{10},u_{20} are

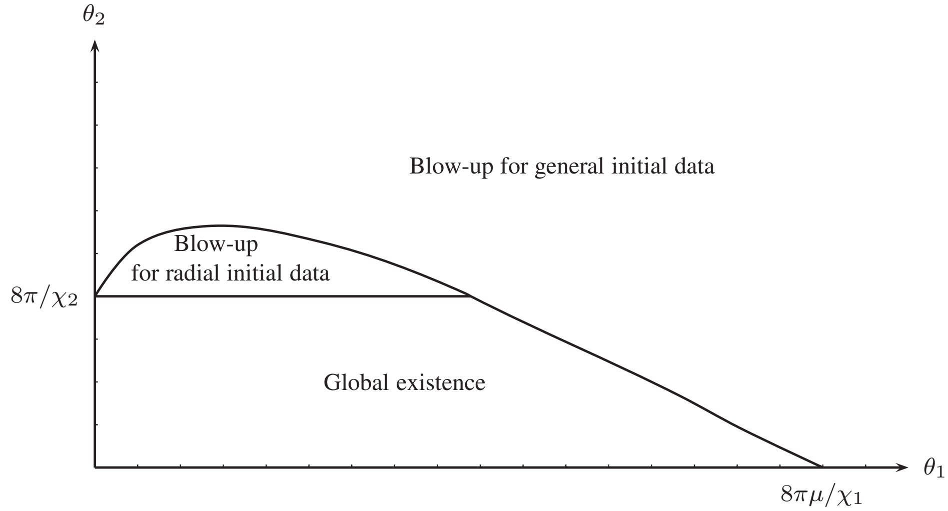

FIGURE 1. Regions of global existence in time and blow-up.

given. In their last paper it was proved that if θ_1, θ_2 satisfy *any* of the inequalities

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1 + \theta_2)^2 < 0, \quad \theta_1 > \mu\frac{8\pi}{\chi_1}, \quad \theta_2 > \frac{8\pi}{\chi_2},$$

then system (3) can blow up. It was also proved in [6] that the inequalities

$$\theta_1 + \theta_2 < \frac{8\pi}{\chi_2}, \quad \mu \geqslant 1$$
$$\theta_1 + \theta_2 < \frac{8\pi}{\chi_2}\mu, \quad \mu < 1$$

guarantee global existence.

In the present paper we aim to give a step further improving the results of global existence from [6] and to prove that even in the *non-radial case*, inequalities (2) guarantee global existence for system (3). In consequence, we give a generalization of the threshold number $8\pi/\chi$ for the classical parabolic–elliptic Keller–Segel system in $\mathbb{R}^2$ to a curve for the two species system. The global existence in time results of the present paper along with the blow-up results from [6] are summarised in Figure 1.

2 Preliminaries

Let us proceed formally to find a free energy functional for our system. First we write the equation for u_1 in (3) in the form

$$\partial_t u_1 = \nabla \cdot u_1 \nabla (\mu \log u_1 - \chi_1 v). \tag{4}$$

Next, we multiply both sides of (4) by $\mu \log u_1 - \chi_1 v$ and integrate to obtain

$$\int_{\mathbb{R}^2} u_{1t} (\mu \log u_1 - \chi_1 v)\, dx = \int_{\mathbb{R}^2} (\mu \log u_1 - \chi_1 v) \nabla \cdot u_1 \nabla (\mu \log u_1 - \chi_1 v)\, dx. \tag{5}$$

Then using mass conservation and integrating by parts, we see that (5) is equivalent to

$$\frac{d}{dt}\int_{\mathbb{R}^2}\mu u_1\log u_1 dx-\chi_1\int_{\mathbb{R}^2}u_{1t}v dx=-\int_{\mathbb{R}^2}u_1\left|\nabla\left(\mu\log u_1-\chi_1 v\right)\right|^2 dx. \tag{6}$$

Similarly,

$$\frac{d}{dt}\int_{\mathbb{R}^2}u_2\log u_2 dx-\chi_2\int_{\mathbb{R}^2}u_{2t}v dx=-\int_{\mathbb{R}^2}u_2\left|\nabla\left(\log u_2-\chi_2 v\right)\right|^2 dx. \tag{7}$$

Now we add $\frac{1}{\chi_1}$ (6) and $\frac{1}{\chi_2}$ (7) to obtain

$$\begin{aligned}&\frac{d}{dt}\left\{\int_{\mathbb{R}^2}\frac{\mu}{\chi_1}u_1\log u_1 dx+\frac{1}{\chi_2}\int_{\mathbb{R}^2}u_2\log u_2 dx\right\}-\int_{\mathbb{R}^2}\left(u_{1t}+u_{2t}\right)v dx\\&\quad=-\int_{\mathbb{R}^2}u_1\left|\nabla\left(\mu\log u_1-\chi_1 v\right)\right|^2 dx-\int_{\mathbb{R}^2}u_2\left|\nabla\left(\log u_2-\chi_2 v\right)\right|^2 dx.\end{aligned} \tag{8}$$

We observe at this point that

$$\begin{aligned}\int_{\mathbb{R}^2}\left(u_{1t}+u_{2t}\right)v dx&=-\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(u_1(x,t)+u_2(x,t)\right)_t\int_{\mathbb{R}^2}\log|x-y|\left(u_1(y,t)+u_2(y,t)\right)dydx\\&=-\frac{1}{4\pi}\frac{d}{dt}\int_{\mathbb{R}^2\times\mathbb{R}^2}\left(u_1(x,t)+u_2(x,t)\right)\left(u_1(y,t)+u_2(y,t)\right)\log|x-y|\,dydx\\&=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(u_1+u_2)v dx.\end{aligned} \tag{9}$$

In conclusion, we deduce from (8) and (9) that

$$\frac{d}{dt}\left\{\int_{\mathbb{R}^2}\frac{\mu}{\chi_1}u_1\log u_1 dx+\frac{1}{\chi_2}\int_{\mathbb{R}^2}u_2\log u_2 dx-\frac{1}{2}\int_{\mathbb{R}^2}(u_1+u_2)v dx\right\}\leqslant 0. \tag{10}$$

Result (10) motivates us to define the free energy functional for system (3) as

$$E(t):=\frac{\mu}{\chi_1}\int_{\mathbb{R}^2}u_1\log u_1 dx+\frac{1}{\chi_2}\int_{\mathbb{R}^2}u_2\log u_2 dx-\frac{1}{2}\int_{\mathbb{R}^2}u_1 v dx-\frac{1}{2}\int_{\mathbb{R}^2}u_2 v dx. \tag{11}$$

In order to give validity to our calculations, we suppose not only that $u_1, u_2 \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^2((0,T); H^1(\mathbb{R}^2))$ but also that $u_1(1+|x|^2)$, $u_2(1+|x|^2)$, $u_1\log u_1$ and $u_2\log u_2$ are bounded in $L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$. In addition, $\nabla\sqrt{u_1}, \nabla\sqrt{u_2} \in L^1_{\text{loc}}(\mathbb{R}^+, L^1(\mathbb{R}^2))$ and $\nabla v \in L^\infty_{\text{loc}}(\mathbb{R}^+\times\mathbb{R}^2)$.

Then we have that

$$\frac{d}{dt}E(t)=-\frac{1}{\chi_1}\int_{\mathbb{R}^2}u_1\left|\mu\nabla\log u_1-\nabla\chi_1 v\right|^2 dx-\frac{1}{\chi_2}\int_{\mathbb{R}^2}u_2\left|\nabla\log u_2-\nabla\chi_2 v\right|^2 dx\leqslant 0. \tag{12}$$

As a consequence of (12) and the Hardy–Littlewood–Sobolev (HLS) inequality [5, 9], the following entropy bound was obtained in [6].

Theorem 2 *If u_1 and u_2 are positive solutions of* (3) *on the interval* $[0,T)$ *and* $\chi_1 \leqslant \chi_2$, *then we have the following entropy estimates:*

- *If* $\mu > 1$, *then*

$$\left(1-\frac{M\chi_2}{8\pi}\right)\int_0^T\int_{\mathbb{R}^2}\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dxdt \leqslant C_T,$$

where C_T *is a constant depending on* T *and* $M = \theta_1 + \theta_2$.

- *If* $\mu \leqslant 1$, *then*

$$\left(1-\frac{M\chi_2}{8\pi\mu}\right)\int_0^T\int_{\mathbb{R}^2}\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)\log\left(\frac{1}{\chi_1}u_1(x,t)+\frac{1}{\chi_2}u_2(x,t)\right)dxdt \leqslant \overline{C}_T,$$

where $\overline{C}_T$ *is a constant depending on* T *and* $M = \theta_1 + \theta_2$.

Theorem 2 gives bounds for the entropy which is a key tool for the proof of global existence for system (3). In order to improve this last result, it would be desirable to use the HLS inequality for systems developed by Shafrir and Wolansky in [14]. However, as we will show in Section 2, a direct application of this tool to our system does not give the optimal result that we are looking for. We will show how an adequate introduction of some auxiliary parameters in (12) allows us to improve the result of global existence obtained in [6], namely we will show that if θ_1, θ_2 satisfy

$$\frac{4\pi\mu\theta_1}{\chi_1}+\frac{4\pi\theta_2}{\chi_2}-\frac{1}{2}(\theta_1+\theta_2)^2>0, \quad \theta_1<\mu\frac{8\pi}{\chi_1}\ , \ \theta_2<\frac{8\pi}{\chi_2}$$

then global solutions in time exist. No kind of radial symmetry is assumed.

The most fundamental tool used through this paper is the logarithmic HLS's inequality for systems, which we proceed to recall now. Following the notation in [14] we define the space

$$\Gamma_M\left(\mathbb{R}^2\right)=\Bigg\{\widetilde{\rho}=(\widetilde{\rho}_i)_{i\in I}\ :\widetilde{\rho}_i\geqslant 0,\int_{\mathbb{R}^2}\widetilde{\rho}_i\left|\log\widetilde{\rho}_i\right|dx<\infty,$$
$$\int_{\mathbb{R}^2}\widetilde{\rho}_i=M_i,\int_{\mathbb{R}^2}\widetilde{\rho}_i\log\left(1+|x|^2\right)<\infty,\forall i\in I\Bigg\},$$

where $M = (M_i)_{i\in I}$ is given. Next we define the functional $F : \Gamma_M(\mathbb{R}^2) \to R$ by

$$F\left[\widetilde{\rho}\right]=\sum_{i\in I}\int_{\mathbb{R}^2}\widetilde{\rho}_i\log\widetilde{\rho}_i dx+\frac{1}{4\pi}\sum_{j,i\in I}a_{i,j}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\widetilde{\rho}_i\left(x\right)\log|x-y|\,\widetilde{\rho}_j\left(y\right)dxdy$$

and the polynomial

$$\Lambda_J\left(M\right)=8\pi\sum_{i\in J}M_i-\sum_{i,j\in J}a_{ij}M_iM_j, \qquad \forall\emptyset\neq J\subseteq I.$$

Then we have the following.

Theorem 3 *Hardy–Litlewood–Sobolev's inequality for systems*
Let $A = (a_{ij})$ *a symmetric matrix such that* $a_{ij} \geqslant 0$ *for all* $i, j \in I$ *and* $M \in \mathbb{R}_+^n$. *Then:* $\Lambda_I(M) = 0$ *and*

$$\Lambda_J(M) \geqslant 0, \quad \textit{for all } J \subseteq I$$
$$\textit{if } \Lambda_J(M) = 0 \textit{ for some } J, \textit{ then } a_{ii} + \Lambda_{J\setminus\{i\}}(M) > 0, \quad \forall i \in J$$

are necessary and sufficient conditions for the boundedness from below of F *on* $\Gamma_M(\mathbb{R}^2)$. *There exists a minimizer* ρ *of* F *over* $\Gamma_M(\mathbb{R}^2)$ *if and only if*

$$\Lambda_I(M) = 0, \quad \textit{and} \quad \Lambda_J(M) > 0, \qquad \textit{for all } J \nsubseteq I$$

Proof See [30, Theorem 4]. □

3 Global existence

The first result of this section gives us bounds for entropy functionals. We achieve our aim through an appropriate use of the HLS inequality for systems, Theorem 3. The main idea of the proof reads as follows: Given that a direct application of the HLS inequality would allow us to get bounds *only on a curve* of the $\theta_1\theta_2$-plane for the entropies $\int_{\mathbb{R}^2} u_i(x,t)\log u_i(x,t)dx$, $i = 1, 2$, we introduce some parameters before applying the HLS inequality. This step will allows us 'to move', 'to shrink' and 'to dilate' this curve in such a way the the full region (18) is swept and therefore obtain estimate (19) in this region.

We suppose throughout this paper that

$$\left.\begin{array}{c} u_{10}, u_{20} \in L^1(\mathbb{R}^2, (1+|x|^2)dx), \\ u_{10}\log u_{10},\ u_{20}\log u_{20} \in L^1(\mathbb{R}^2, dx) \end{array}\right\}. \tag{13}$$

Lemma 4 (Lower bound for the entropy functionals) *Consider a non-negative weak solution of (3) such that* $u_i(1+|x|^2)$, $i = 1, 2$ *are bounded in* $L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$. *Then we have*

$$\int_{\mathbb{R}^2} u_i(x,t)\log u_i(x,t) \geqslant M\log M - M\log[\pi(1+t)] - C, \quad i = 1, 2.$$

Proof In the following, C will denote a generic constant. We have from [6, Theorem 1] that

$$\frac{d}{dt}\int_{\mathbb{R}^2}\left(\frac{\mu}{\chi_1}u_1(x,t) + \frac{1}{\chi_2}u_2(x,t)\right)|x|^2\,dx = \frac{4\theta_1}{\chi_1}\mu + \frac{4\theta_2}{\chi_2} - \frac{1}{2\pi}(\theta_1+\theta_2)^2. \tag{14}$$

We define

$$n := \frac{\mu}{\chi_1}u_1 + \frac{1}{\chi_2}u_2;$$

and

$$K := \frac{4\theta_1}{\chi_1}\mu + \frac{4\theta_2}{\chi_2} - \frac{1}{2\pi}(\theta_1+\theta_2)^2.$$

Thus, we obtain

$$\int_{\mathbb{R}^2} n(x,t)\,|x|^2\,dx = Kt + \int_{\mathbb{R}^2} n(x,0)\,|x|^2\,dx \leqslant C(1+t), \tag{15}$$

where $C := \max\{K, \int_{\mathbb{R}^2} n(x,0)\,|x|^2\,dx\}$. From the inequality $u_i \leqslant Cn$, where $i = 1,2$ and (15) we deduce that

$$\int_{\mathbb{R}^2} u_i(x,t)\,|x|^2\,dx \leqslant C(1+t), \quad i = 1,2.$$

Using the same idea presented in [4, Lemma 2.5], we observe that

$$\begin{aligned}\int_{\mathbb{R}^2} u_i\,(x,t) \log u_i\,(x,t) &\geqslant \tfrac{1}{1+t} \int_{\mathbb{R}^2} u_i\,(x,t)\,|x|^2 - C + \int_{\mathbb{R}^2} u_i\,(x,t) \log u_i\,(x,t) \\ &= \int_{\mathbb{R}^2} u_i\,(x,t) \log \left[\frac{u_i\,(x,t)}{e^{-\frac{|x|^2}{1+t}}}\right] - C.\end{aligned} \tag{16}$$

Let us now define the variable μ as

$$\mu\,(x,t) = \frac{1}{\pi\,(1+t)} \exp\left(-\frac{|x|^2}{1+t}\right).$$

We then obtain from (16) that

$$\begin{aligned}\int_{\mathbb{R}^2} u_i\,(x,t) \log u_i\,(x,t) &\geqslant \int_{\mathbb{R}^2} u_i\,(x,t) \log \left[\frac{u_i\,(x,t)}{\mu\,(x,t)}\right] dx - M \log\left[\pi\,(1+t)\right] - C \\ &= \int_{\mathbb{R}^2} \frac{u_i\,(x,t)}{\mu\,(x,t)} \log \left[\frac{u_i\,(x,t)}{\mu\,(x,t)}\right] \mu\,(x,t)\,dx - M \log\left[\pi\,(1+t)\right] - C,\end{aligned} \tag{17}$$

where $M = \frac{\mu}{\chi_1}\theta_1 + \frac{1}{\chi_2}\theta_2$. Using Jensen's inequality we get from (17) that

$$\int_{\mathbb{R}^2} u_i\,(x,t) \log u_i\,(x,t) \geqslant M \log M - M \log\left[\pi\,(1+t)\right] - C.$$

□

Theorem 5 (Upper bound for entropy functionals) *Consider a non-negative weak solution of (3) such that* $u_i(1+|x|^2)$, $u_i \log u_i$, $i = 1,2$ *are bounded in* $L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$. *If* (θ_1, θ_2) *satisfies*

$$\theta_1 < \frac{8\pi}{\chi_1}\mu; \quad \theta_2 < \frac{8\pi}{\chi_2}; \qquad 8\pi\left(\frac{\theta_1}{\chi_1}\mu + \frac{\theta_2}{\chi_2}\right) - (\theta_1+\theta_2)^2 > 0, \tag{18}$$

then we have

$$\int_{\mathbb{R}^2} u_i(x,t) \log u_i(x,t) dx \leqslant C, \tag{19}$$

where $i = 1,2$ *and* C *is a constant depending only on the parameters* θ_1 *and* θ_2, μ, χ_1, χ_2, $E(0)$.

Proof From (12) we have that

$$E\,(t) \leqslant E\,(0), \qquad \forall t > 0.$$

In consequence, we have the following estimate:

$$\begin{aligned}
&\frac{\mu}{\chi_1}\int_{\mathbb{R}^2} u_1(x,t)\log u_1(x,t)\,dx + \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2(x,t)\log u_2(x,t)\,dx \\
&\quad \leqslant E(0) - \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u_1(x,t)\,u_1(y,t)\log|x-y|\,dxdy \\
&\quad - \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u_1(x,t)\,u_2(y,t)\log|x-y|\,dxdy \\
&\quad - \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u_2(x,t)\,u_1(y,t)\log|x-y|\,dxdy - \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u_2(x,t)\,u_2(y,t)\log|x-y|\,dxdy.
\end{aligned}$$

We introduce positive parameters a and b in the last inequality in the following way

$$\begin{aligned}
&\frac{\mu}{\chi_1}\int_{\mathbb{R}^2} u_1(x,t)\log u_1(x,t)\,dx + \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2(x,t)\log u_2(x,t)\,dx \\
&\quad \leqslant E(0) - \frac{a^2}{\mu^2 4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\mu u_1(x,t)}{a}\frac{\mu u_1(y,t)}{a}\log|x-y|\,dxdy \\
&\quad - \frac{ab}{\mu 4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\mu u_1(x,t)}{a}\frac{u_2(y,t)}{b}\log|x-y|\,dxdy \\
&\quad - \frac{ab}{\mu 4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{u_2(x,t)}{b}\frac{\mu u_1(y,t)}{a}\log|x-y|\,dxdy \\
&\quad - \frac{b^2}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{u_2(x,t)}{b}\frac{u_2(y,t)}{b}\log|x-y|\,dxdy.
\end{aligned} \tag{20}$$

By doing so, we can now apply the HLS inequality for systems (Theorem 3) to the functions $\mu u_1/a$ and u_2/b in identity (20) getting that

$$\begin{aligned}
&\frac{\mu}{\chi_1}\int_{\mathbb{R}^2} u_1(x,t)\log u_1(x,t) + \frac{1}{\chi_2}\int_{\mathbb{R}^2} u_2(x,t)\log u_2(x,t) \\
&\quad \leqslant E(0) - C + \int_{\mathbb{R}^2} \mu\frac{u_1(x,t)}{a}\log\left(\mu\frac{u_1(x,t)}{a}\right)dx + \int_{\mathbb{R}^2} \frac{u_2(x,t)}{b}\log\left(\frac{u_2(x,t)}{b}\right)dx,
\end{aligned}$$

where the conditions for the existence of the constant C given by Theorem 3 are

$$\begin{aligned}
&\Lambda_{\{1\}}(M) = 8\pi\mu\frac{\theta_1}{a} - a^2\left(\frac{\theta_1}{a}\right)^2 \geqslant 0; \\
&\Lambda_{\{2\}}(M) = 8\pi\frac{\theta_2}{b} - b^2\left(\frac{\theta_2}{b}\right)^2 \geqslant 0; \\
&\Lambda_{\{1,2\}}(M) = 8\pi\left(\mu\frac{\theta_1}{a} + \frac{\theta_2}{b}\right) - \left(a^2\frac{\theta_1}{a}\frac{\theta_1}{a} + 2ab\frac{\theta_1}{a}\frac{\theta_2}{b} + b^2\frac{\theta_2}{b}\frac{\theta_2}{b}\right) = 0.
\end{aligned}$$

Equivalently,

$$\left.\begin{array}{l}
\theta_1 \leqslant \mu\frac{8\pi}{a}, \qquad \theta_2 \leqslant \frac{8\pi}{b} \\
8\pi\left(\mu\frac{\theta_1}{a} + \frac{\theta_2}{b}\right) - (\theta_1+\theta_2)^2 = 0
\end{array}\right\}. \tag{21}$$

In conclusion we have proved that condition (21) implies

$$\mu\left(\frac{1}{\chi_1}-\frac{1}{a}\right)\int_{\mathbb{R}^2} u_1(x,t)\log u_1(x,t)+\left(\frac{1}{\chi_2}-\frac{1}{b}\right)\int_{\mathbb{R}^2} u_2(x,t)\log u_2(x,t) \leqslant E(0)-C+\frac{\theta_1\mu}{a}\log\frac{\mu}{a}+\frac{\theta_2}{b}\log\frac{1}{b}. \tag{22}$$

We have from Lemma 4 that the functionals $\int u_i \log u_i dx$ are bounded below for $i = 1, 2$. On the other hand, each of the coefficients of the entropy functionals in (22) are positive as long as $a > \chi_1$ and $b > \chi_2$. Then we take parameters a and b on the intervals (χ_1,∞) and (χ_2,∞) respectively We conclude that estimate (19) hold on region (18). □

Boundedness of entropies in the last theorem is the main tool that we will use to obtain the following result of global existence.

Theorem 6 (Global existence of weak solutions) *Under assumption (13) and*

$$8\pi\left(\frac{\theta_1}{\chi_1}\mu+\frac{\theta_2}{\chi_2}\right)-(\theta_1+\theta_2)^2>0, \tag{23}$$

$$\theta_1<\frac{8\pi}{\chi_1}\mu; \qquad \theta_2<\frac{8\pi}{\chi_2}, \tag{24}$$

system (3) has a global weak non-negative solution such that

$$(1+|x|^2+|\log u_i|)u_i \in L^\infty(0,T;L^1(\mathbb{R}^2))$$

and

$$-\frac{1}{\chi_1}\int\int_{[0,T]\times\mathbb{R}^2} u_1\left|\mu\nabla\log u_1-\nabla\chi_1 v\right|^2 dx-\frac{1}{\chi_2}\int\int_{[0,T]\times\mathbb{R}^2} u_2\left|\nabla\log u_2-\nabla\chi_2 v\right|^2 dx<\infty.$$

Before giving the proof, let us first give some explanation of this result. Inequality (23) corresponds to the interior of a rotated parabola in the plane $\theta_1\theta_2$. Choosing the parameters μ, χ_1 and χ_2 appropriately, condition (24) may be relevant or can be simply ignored. Next, Figure 2 illustrates the two possible cases:

More precisely we have that,

- if the parabola
$$8\pi\left(\frac{\theta_1}{\chi_1}\mu+\frac{\theta_2}{\chi_2}\right)-(\theta_1+\theta_2)^2=0 \tag{25}$$
intersects either of the lines $\theta_1 = 8\pi\mu/\chi_1$ or $\theta_2 = 8\pi/\chi_2$ in the first quadrant of the $\theta_1\theta_2$ plane (which happens exactly when $\chi_1 < \mu\chi_2/2$ or $\chi_1 > 2\mu\chi_2$), then system (3) has global existence in time weak solutions as long as the initial masses satisfy inequalities (23) together with (24).
- However, if the parabola (25) does not intersect either of the lines $\theta_1 = 8\pi\mu/\chi_1$ or $\theta_2 = 8\pi/\chi_2$ (when $\mu\chi_2/2 \leqslant \chi_1 \leqslant 2\mu\chi_2$) in the first quadrant of the $\theta_1\theta_2$ plane, then inequality (23) is enough to guarantee that system (3) has a global in time weak solution.

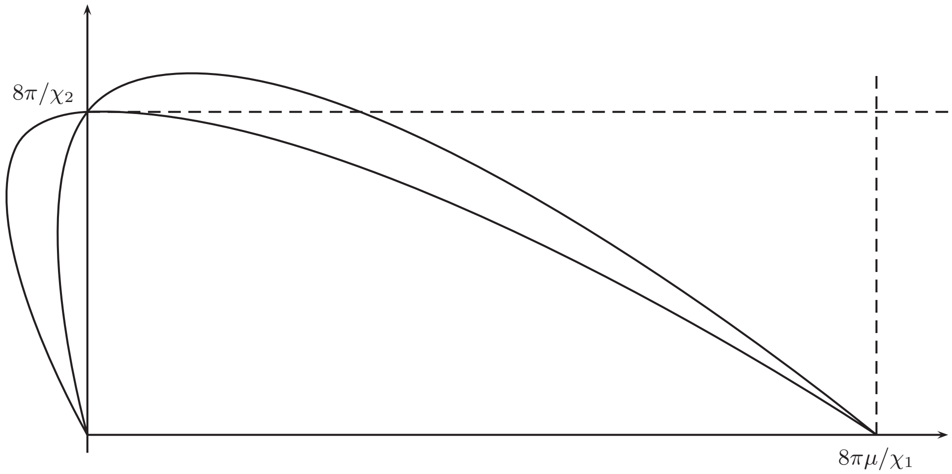

FIGURE 2. Two basic configurations of parabola (25).

On the other hand, we should point out that all of our results are formal so far. In order to make them rigorous, we should have a local existence result of smooth solutions. However, we will take another strategy which will allow us to obtain directly global existence in time of weak solutions with the corresponding mathematical rigour. In order to prove Theorem 6, we first modify the convolution kernel $k^0(z) = -\frac{1}{2\pi}\log|z|$ in (3) by truncating it around zero. This last will allow us to get a regularized version of system (3), which is rather easier to work. After proving the existence of global solutions of this last approximate problem, we look for uniform estimates of solutions and then pass to the limit that will give us the result of global existence we are looking for. After getting this result we recover properties such as mass conservation or the second moment formula by testing properly our weak solution. A similar technique was made in the one chemotaxis species case (see [4, 5]).

Proof (Sketch) For the reader's convenience, we divide the proof into four steps giving special attention where technical difficulties arise in comparison to the single species case.

Step 1. *Regularization of the system.* We define K^ϵ by $K^\epsilon(z) := K^1\left(\frac{z}{\epsilon}\right)$, where K^1 is a radial monotone non-decreasing smooth function satisfying

$$K^1(z) = \begin{cases} -\frac{1}{2\pi}\log|z| & \text{if } |z| \geqslant 4 \\ 0 & \text{if } |z| \leqslant 1 \end{cases}.$$

Assume also that

$$\left|\nabla K^1(z)\right| \leqslant \frac{1}{2\pi|z|}$$

$$K^1(z) \leqslant -\frac{1}{2\pi}\log|z|; \quad -\Delta K^1(z) \geqslant 0; \quad \forall z \in \mathbb{R}^2$$

for any $z \in \mathbb{R}^2$. Then we consider the following regularized version of system (3)

$$\begin{cases} \partial_t u_1^\epsilon = \Delta u_1^\epsilon - \chi_1 \nabla \cdot (u_1^\epsilon \nabla v^\epsilon), & t \geqslant 0, \quad x \in \mathbb{R}^2 \\ \partial_t u_2^\epsilon = \Delta u_2^\epsilon - \chi_2 \nabla \cdot (u_2^\epsilon \nabla v^\epsilon), & t \geqslant 0, \quad x \in \mathbb{R}^2, \\ v^\epsilon = K^\epsilon * \left(u_1^\epsilon + u_2^\epsilon\right), & t \geqslant 0, \quad x \in \mathbb{R}^2 \end{cases} \tag{26}$$

which we interpret in the sense of distributions. Since $K^{\varepsilon}(z) = K^1(\frac{z}{\varepsilon})$, we also have

$$|\nabla K^{\varepsilon}(z)| = \frac{1}{\varepsilon}\left|\nabla K\left(\frac{z}{\varepsilon}\right)\right| \leqslant \frac{1}{\varepsilon}\frac{1}{2\pi\,|z/\varepsilon|} = \frac{1}{2\pi\,|z|}. \tag{27}$$

The proof of global solutions in $L^2(0,T;H^1(\mathbb{R}^2)\cap C\left(0,T;L^2(\mathbb{R}^2)\right))$ for system (26) with initial data in $L^2(\mathbb{R}^2)$ follows essentially the same lines as in [4, Proposition 2.8] and therefore we omit the proof here.

Step 2. A priori *estimates for the approximate solutions* u_1^{ε}, u_2^{ε} *and* v^{ε}.

Consider a solution $\left(u_1^{\epsilon},u_2^{\epsilon}\right)$ of the regularized system. If

$$\theta_1 < \frac{8\pi}{\chi_1}\mu;\quad \theta_2 < \frac{8\pi}{\chi_2};\qquad 8\pi\left(\frac{\theta_1}{\chi_1}\mu + \frac{\theta_2}{\chi_2}\right) - (\theta_1+\theta_2)^2 > 0,$$

then, uniformly as $\epsilon \to 0$, with bounds depending only upon $\int_{\mathbb{R}^2}(1+|x|^2)u_{i0}dx$ and $\int_{\mathbb{R}^2} u_{i0}\log u_{i0}dx$ with $i=1,2$, we have the following estimates:

(i) The function $(x,t)\to |x|^2\left(u_1^{\epsilon}+u_2^{\epsilon}\right)$ is bounded in $L^{\infty}\left(\mathbb{R}^+_{loc};L^1(\mathbb{R}^2)\right)$.

(ii) The functions $t\to \int_{\mathbb{R}^2} u_j^{\epsilon}(x,t)\log u_j^{\epsilon}(x,t)\,dx$ and $t\to\int_{\mathbb{R}^2} u_j^{\epsilon}(x,t)\,v^{\epsilon}(x,t)\,dx$ are bounded for $j=1,2$.

(iii) The function $(x,t)\to u_j^{\epsilon}(x,t)\log\left(u_j^{\epsilon}(x,t)\right)$ is bounded in $L^{\infty}\left(\mathbb{R}^+_{loc};L^1(\mathbb{R}^2)\right)$ for $j=1,2$.

(iv) The function $(x,t)\to\nabla\sqrt{u_j^{\epsilon}(x,t)}$ is bounded in $L^2\left(\mathbb{R}^+_{loc}\times\mathbb{R}^2\right)$ for $j=1,2$.

(v) The function $(x,t)\to u_j^{\epsilon}(x,t)$ is bounded in $L^2\left(\mathbb{R}^+_{loc}\times\mathbb{R}^2\right)$ for $j=1,2$.

(vi) The function $(x,t)\to u_j^{\epsilon}(x,t)\,\Delta v^{\epsilon}(x,t)$ is bounded in $L^1\left(\mathbb{R}^+_{loc}\times\mathbb{R}^2\right)$ for $j=1,2$.

(vii) The function $(x,t)\to\sqrt{u_j^{\epsilon}(x,t)}\nabla v^{\epsilon}(x,t)$ is bounded in $L^2\left(\mathbb{R}^+_{loc}\times\mathbb{R}^2\right)$ for $j=1,2$.

The proof of estimates (i)–(vii) follows essentially the same steps as in the one species case and therefore we refer the reader to [4, Lema 2.11].

As a consequence of estimate (ii), the first two equations of system (3) have the hyper-contractivity property [4, Theorem 3.5], i.e. for any $1<p<\infty$, there exists a continuous function $h_p^j:(0,T)\to\mathbb{R}$ such that $\left\|u_j^{\epsilon}(\cdot,t)\right\|_{L^p(\mathbb{R}^2)}\leqslant h_p^j(t)$, $j=1,2$. Hence, $u_j^{\epsilon}\in L^{\infty}((\delta,T),L^p(\mathbb{R}^2))$, $p\in(1,\infty)$ for any $\delta\in(0,T)$. Therefore, we have the following result:

(viii) The function $(x,t)\to u_j^{\epsilon}(x,t)$ is bounded in $L^{\infty}((\delta,T),L^p(\mathbb{R}^2))$ for $j=1,2$, $p>1$.

Step 3. *Construction of a strong convergence subsequence in* L^p. To achieve our aim in this step we will apply the Aubin–Lions compactness lemma.

First we get a uniform bound on $\|\nabla u_i^\epsilon\|_{L^2_{\mathrm{loc}}((\delta,T)\times\mathbb{R}^2)}$. We observe that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}|u_i^\epsilon|^2\,dx &= -2\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dx + 2\chi_1\int_{\mathbb{R}^2}u_i^\epsilon\nabla u_i^\epsilon\cdot\nabla v^\epsilon dx \leqslant -2\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dx\\ &\quad + 2\chi_1\left(\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\right)^{1/2}\left(\int_{\mathbb{R}^2}|u_i^\epsilon|^2\,|\nabla v^\epsilon|^2\,dx\right)^{1/2} \leqslant -2\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dx\\ &\quad + 2\chi_1\left(\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\right)^{1/2}\left(\int_{\mathbb{R}^2}|u_i^\epsilon|^3\,dx\right)^{1/3}\left(\int_{\mathbb{R}^2}|\nabla v^\epsilon|^6\,dx\right)^{1/6}, \qquad (28)\end{aligned}$$

where we have used the Hölder inequality in the last line. The classical Gagliardo–Nirenberg–Sobolev inequality along with the Calderon–Zigmund inequality allow us to conclude that

$$\left(\int_{\mathbb{R}^2}|\nabla v^\epsilon|^6\,dx\right)^{1/6} \leqslant C\left(\int_{\mathbb{R}^2}|\Delta v^\epsilon|^{3/2}\,dx\right)^{2/3}. \qquad (29)$$

From inequalities (28) and (29) we deduce that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}|u_i^\epsilon|^2\,dx\\ &\leqslant -2\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dx + 2C\chi_1\left(\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\right)^{1/2}\left(\int_{\mathbb{R}^2}|u_i^\epsilon|^3\,dx\right)^{1/3}\left(\int_{\mathbb{R}^2}|\Delta v^\epsilon|^{3/2}\,dx\right)^{2/3}\\ &\leqslant -2\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dx + 2C\chi_1\left(\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\right)^{1/2}\left(\int_{\mathbb{R}^2}|u_i^\epsilon|^3\,dx\right)^{1/3}\\ &\quad\times\left(\left(\int_{\mathbb{R}^2}|u_1^\epsilon|^{3/2}\,dx\right)^{2/3} + \left(\int_{\mathbb{R}^2}|u_2^\epsilon|^{3/2}\,dx\right)^{2/3}\right).\end{aligned}$$

Integrating with respect to t and reordering last inequality, we now obtain

$$\begin{aligned}&2\int_\delta^T\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dxdt - 2C\chi_1\left\{\sup_{t\in[\delta,T]}\left(\int_{\mathbb{R}^2}|u_i^\epsilon|^3\,dx\right)^{1/3}\left(\sup_{t\in[\delta,T]}\left(\int_{\mathbb{R}^2}|u_1^\epsilon|^{3/2}\,dx\right)^{2/3}\right.\right.\\ &\left.\left.+\sup_{t\in[\delta,T]}\left(\int_{\mathbb{R}^2}|u_2^\epsilon|^{3/2}\,dx\right)^{2/3}\right)\right\}\int_\delta^T\left(\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\right)^{1/2}dt + \int_{\mathbb{R}^2}|u_i^\epsilon|^2\,dx - \int_{\mathbb{R}^2}|u_i^\epsilon(x,0)|^2\,dx \leqslant 0.\end{aligned}$$

We observe now that

$$\int_\delta^T\left(\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dx\right)^{1/2}dt \leqslant (T-\delta)^{1/2}\left(\int_0^T\int_{\mathbb{R}^2}|\nabla u_i^\epsilon|^2\,dxdt\right)^{1/2}.$$

Denoting by $X := \|\nabla u_i^\epsilon\|_{L^2_{loc}((\delta,T)\times\mathbb{R}^2)}$ and taking into account (viii), we conclude from last two estimates that for positive constants a, b and c we have that

$$aX^2 - bX + c \leqslant 0,$$

in consequence $X := \|\nabla u_i^\epsilon\|_{L^2_{loc}((\delta,T)\times\mathbb{R}^2)}$ is bounded, i.e there exists a constant C such that

$$\|\nabla u_i^\epsilon\|_{L^2_{loc}((\delta,T)\times\mathbb{R}^2)} \leqslant C. \qquad (30)$$

Now we obtain a bound for $\|du_i^\varepsilon/dt\|_{L^2((\delta,T);H^{-1}(\mathbb{R}^2))}$. First of all we notice that in the middle of the proof of estimation (30) we have proved that

$$\|u_i\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)} \leqslant \left(\int_{\mathbb{R}^2} |u_i^\epsilon|^3\,dx \right)^{1/3} \left(\left(\int_{\mathbb{R}^2} |u_1^\epsilon|^{3/2}\,dx\right)^{2/3} + \left(\int_{\mathbb{R}^2} |u_2^\epsilon|^{3/2}\,dx \right)^{2/3} \right). \quad (31)$$

It follows from the last estimate and (viii) that for some constant C we have

$$\|u_i\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)} \leqslant C. \quad (32)$$

Let $\phi \in H^1(\mathbb{R}^2)$, then we have

$$\begin{aligned}|\langle du_i^\varepsilon/dt, \phi\rangle| &= |\langle \Delta u_i - \nabla\cdot(u_i\nabla v^\epsilon), \phi\rangle| \leqslant |\langle \nabla u_i, \nabla\phi\rangle| + |\langle u_i\nabla v^\epsilon, \nabla\phi\rangle| \\ &\leqslant \|\nabla\phi\|_{L^2(\mathbb{R}^2)}\|\nabla u_i\|_{L^2(\mathbb{R}^2)} + \|\nabla\phi\|_{L^2(\mathbb{R}^2)}\|u_i\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)}. \end{aligned} \quad (33)$$

Thus,

$$\|du_i^\varepsilon/dt\|_{H^{-1}(\mathbb{R}^2)} = \sup_{\|\phi\|_{H^1(\mathbb{R}^2)}=1} |\langle du_i^\varepsilon/dt, \phi\rangle| \leqslant \|\nabla u_i^\varepsilon\|_{L^2(\mathbb{R}^2)} + \|u_i^\varepsilon\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)}.$$

From the last estimate and taking into account (30) and (32), it follows that

$$\|du_i^\varepsilon/dt\|_{L^2((\delta,T);H^{-1}(\mathbb{R}^2))} = \left(\int_\delta^T \|du_i^\varepsilon/dt\|^2_{H^{-1}(\mathbb{R}^2)}\,dt\right)^{1/2} \leqslant C. \quad (34)$$

Compactness: In order to apply the Aubin–Lions Lemma, we define the spaces $B_0 = H^1(\mathbb{R}^2) \cap \{f| \quad |x|^2 f \in L^1(\mathbb{R}^2)\}$, $B := L^2(\mathbb{R}^2)$ and $B_1 := B_0'$. Let $\{f_i\}$ be an arbitrary bounded sequence in B_0, then we have L^2equi-integrability at infinity (cf. [1, Corollary 5.3.1]) as the following account shows:

$$\begin{aligned}\int_{\{|x|>R\}} f_i^2 dx &\leqslant \frac{1}{R}\int_{\{|x|>R\}} \left(|x|\, f_i^{1/2}\right) f_i^{3/2} dx \leqslant \frac{1}{R}\left(\int_{\{|x|>R\}} |x|^2 f_i dx\right)^{1/2}\left(\int_{\{|x|>R\}} f_i^3 dx\right)^{1/2} \\ &\leqslant \frac{1}{R}\left(\int_{\mathbb{R}^2} |x|^2 f_i dx\right)^{1/2}\left(\int_{\mathbb{R}^2} f^3 dx\right)^{1/2}.\end{aligned}$$

Thus,

$$\lim_{R\to+\infty}\int_{\{|x|>R\}} f_i^2 dx = 0 \;\text{ uniformly with respect to } f_i. \quad (35)$$

From the Rellich–Kondrakov Theorem (cf. [1, Corollary 5.3.1] we obtain the compact inclusion

$$B_0 \hookrightarrow\hookrightarrow B.$$

Given that u_i^ε satisfies (30), (34) and (35), we can now invoke the Aubin–Lions–Simon theorem to conclude that u_i^ε has a subsequence that converge strongly in $L^2(\delta, T, B)$. Therefore, up to a subsequence we have that

$$u_i^\epsilon \to u_i \text{ a.e. in } \mathbb{R}^2 \times [\delta, T]. \quad (36)$$

We have also proved uniformly boundedness for $\|u_i^\epsilon\|_{L^p(\mathbb{R}^2)\times[\delta,T]}$, from this, estimation (36) and the Vitali theorem, we obtain

$$u_i^\epsilon \to u_i \text{ strongly in } L^p(\mathbb{R}^2 \times [0,T]) \text{ for } p \geqslant 1. \tag{37}$$

Step 4. *Pass to the limit*. We pass now to the limit in the weak sense to obtain our result of global existence. The most significant technical difficulty to show that u_1, u_2 solved (3) arise with the nonlinear terms. In order to prove that

$$u_i^\epsilon \nabla v^\epsilon \rightharpoonup u_i \nabla v, \text{in } D'(\mathbb{R}^+ \times \mathbb{R}^2), \tag{38}$$

we first notice that the expression $u_i^\epsilon |\nabla v^\epsilon|$ is integrable as estimate (vii) of part 2 along with the following estimate shows

$$\left(\int_{[0,T]\times\mathbb{R}^2} u_i^\epsilon |\nabla v^\epsilon| \, dxdt\right)^2 = \left(\int_{[0,T]\times\mathbb{R}^2} \sqrt{u_i^\epsilon}\sqrt{u_i^\epsilon} |\nabla v^\epsilon| \, dxdt\right)^2$$
$$\leqslant \int_{[0,T]\times\mathbb{R}^2} u_i^\epsilon dxdt \int_{[0,T]\times\mathbb{R}^2} u_i^\epsilon |\nabla v^\epsilon|^2 \, dxdt \leqslant \theta_i T \int_{[0,T]\times\mathbb{R}^2} u_i^\epsilon |\nabla v^\epsilon|^2 \, dxdt.$$

It follows that we can interpret $u_i^\epsilon \nabla v^\epsilon$ as an element of $\left(C_0^\infty\left(\mathbb{R}^+ \times \mathbb{R}^2\right)\right)'$ and therefore its divergence is defined.

In order to prove that $\|\nabla v^\varepsilon\|_{L^r(\mathbb{R}^n)} \leqslant C$ for $r > 2$, we recall the HLS inequality: For all $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 < p, q < \infty$, such that $1/p + 1/q + \lambda/n = 2$ and $0 < \lambda < n$, there exists a constant $C = C(p,q,\lambda) > 0$ such that

$$\left|\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{1}{|x-y|^\lambda} f(x)g(y)dxdy\right| \leqslant C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Taking the supremum over the ball $\|g\|_{L^q(\mathbb{R}^n)} = 1$ on both sides of the last inequality, we obtain

$$\left\|\int_{\mathbb{R}^n} \frac{1}{|x-y|^\lambda} f(x)dx\right\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n)} \leqslant C \|f\|_{L^p(\mathbb{R}^n)}. \tag{39}$$

In particular

$$\left\|\int_{\mathbb{R}^n} \frac{1}{|x-y|} f(x)dx\right\|_{L^{\frac{q}{q-1}}(\mathbb{R}^2)} \leqslant C \|f\|_{L^p(\mathbb{R}^2)} \text{ where } 1 < p, q < \infty, \text{ and } 1/p + 1/q + 1/2 = 2.$$

Thus, we have that

$$\|\nabla v^\varepsilon\|_{L^r(\mathbb{R}^n)} = \|\nabla K^\varepsilon * (u_1^\varepsilon + u_2^\varepsilon)\|_{L^r(\mathbb{R}^n)} \tag{40}$$
$$\leqslant \left\|\frac{1}{2\pi}\int \frac{1}{|x-y|}(u_1^\varepsilon + u_2^\varepsilon)dx\right\|_{L^r(\mathbb{R}^n)} \leqslant C\left(\|u_1^\varepsilon\|_{L^p(\mathbb{R}^2)} + \|u_2^\varepsilon\|_{L^p(\mathbb{R}^2)}\right) \leqslant C, \tag{41}$$

where we have used step 2 (viii). From $r = \frac{q}{q-1}$ and $1/p + 1/q + 1/2 = 2$ we obtain that $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. In addition, $p \in (1,2)$ implies that $r \in (2,\infty)$. We conclude that (up to a subsequence) $\nabla v^{\varepsilon} \rightharpoonup h$, where h is in L^r. In order to prove that actually $h = \nabla K * n$ we have to do some extra work yet. With this end in mind, we now propose to show that

$$\nabla v^{\epsilon} \to \nabla v \qquad a.e. \tag{42}$$

We have that

$$\begin{aligned}\nabla v^{\epsilon} - \nabla v = & -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\left(\left(u_1^{\epsilon}+u_2^{\epsilon}\right)-(u_1+u_2)\right)(y,t)\,dy \\ & + \int_{|x-y|\leqslant 2\epsilon}\left(\frac{1}{\varepsilon}\nabla K^1\left(\frac{x-y}{\varepsilon}\right)+\frac{|x-y|}{2\pi\,|x-y|^2}\right)\left(u_1^{\epsilon}+u_2^{\epsilon}\right)(y,t)dy. \end{aligned} \tag{43}$$

We deduce from (37) and (39) that (up to a subsequence) the first integral in (43) converges to zero *a.e.* On the other hand, estimates (27) allows us to conclude that

$$\begin{aligned} & \left|\int_{|x-y|\leqslant 2\epsilon}\left(\frac{1}{\varepsilon}\nabla K^1\left(\frac{x-y}{\varepsilon}\right)+\frac{|x-y|}{2\pi\,|x-y|^2}\right)\left(u_1^{\epsilon}+u_2^{\epsilon}\right)(y,t)dy\right| \\ & \leqslant \int_{|x-y|\leqslant 2\epsilon}\left(\frac{1}{\pi\,|x-y|}\right)\left(u_1^{\epsilon}+u_2^{\epsilon}\right)(y,t)dy. \end{aligned}$$

Last integral converges to 0 as $\varepsilon \to 0$, therefore we conclude (42).

We therefore obtain from [8, Prop. 2.46 (i)] that $\nabla v_{\varepsilon} \rightharpoonup \nabla K * n$ weakly in L^r for $r \geqslant 2$. Finally, we choose conjugate exponents $r = 4$ and $p = 4/3$ to conclude the convergence (38). □

4 Conclusions and open questions

It has been proved in this paper that system (3) has a threshold curve that determines global existence or blow-up. A more difficult task is to find out if the blow-up has to be simultaneous or not and also to describe the asymptotics near the blow-up time. A first step in this direction was given by Espejo *et al.* in [7], where it was shown that the blow-up has to be simultaneous in the radial case. Should it be the same in the general case? Or should it depend on more specific information on the initial data? With regard to this point it is worth recalling that according to [6] it is possible to have blow-up even in the case that the total moment

$$m(t) := \frac{\pi}{\chi_1}\int_{\mathbb{R}^2} u_1(x,t)\,|x|^2\,dx + \frac{\pi}{\chi_2}\int_{\mathbb{R}^2} u_2(x,t)\,|x|^2\,dx \tag{44}$$

is increasing, that is when we have

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1+\theta_2)^2 > 0.$$

This opens up a new possibility: The density of one chemotactic species could be increasing meanwhile the other decreases. That is to say, the question of a simultaneous blow-up or not as well as a possible collapse mass separation could eventually not only depend on the radial symmetry of the initial data but also on the L^1 size of the initial data.

On the other hand, if the parabola

$$\frac{4\pi\mu\theta_1}{\chi_1} + \frac{4\pi\theta_2}{\chi_2} - \frac{1}{2}(\theta_1 + \theta_2)^2 = 0 \tag{45}$$

intersects any of the lines

$$\theta_1 = \frac{8\pi}{\chi_1} \quad \text{or} \quad \theta_2 = \frac{8\pi}{\chi_2}, \tag{46}$$

it would be very interesting to study the behaviour of system (3) on this lines. Here it is worth recalling that the proof of convergence towards a delta function at $T = \infty$ in the one species case, when total mass is exactly $8\pi/\chi$, uses in a essential way that the second moment is preserved (see, for instance, [3]). In contrast, for the two species case, the rotated parabola (45) can intersect any of the lines (46) and then we obtain threshold lines on which the second moment is not preserved. A description of the asymptotic behaviour in this case seems to require rather different techniques to those used in the one species case.

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