Nilpotent linear transformations and the solvability of power-associative nilalgebras☆

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Abstract

We prove some results about nilpotent linear transformations. As an application we solve some cases of Albert's problem on the solvability of nilalgebras. More precisely, we prove the following results: commutative power-associative nilalgebras of dimension n and nilindex $n-1$ or $n-2$ are solvable; commutative power-associative nilalgebras of dimension 7 are solvable.

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1. Introduction

All algebras considered in this paper are not necessarily associative algebras over a field K.

Let A be an algebra. We denote by $\langle a_1, \ldots, a_n\rangle$ the vector space generated by $a_1, \ldots, a_n \in A$. Let U be a subspace of A. We define inductively the following powers of U: $U^1 = U$, $U^n = U^{n-1}U + U^{n-2}U^2 + \cdots + UU^{n-1}$; $U^{(0)} = U$, $U^{(n)} = (U^{(n-1)})^2$. We say that U is *nilpotent* (respectively, *solvable*) when $U^k = 0$ (respectively, $U^{(k)} = 0$) for some k. When U is nilpotent the smallest k such that $U^k = 0$ is called the *index of nilpotency* of U. Analogously, we define the *index of solvability* of U. Clearly, if U is nilpotent then U is solvable.

In a series of papers, Gerstenhaber [6–8] studied nilalgebras and nilpotent linear transformations. In particular, Gerstenhaber gave a sufficient condition for a vector space of nilpotent linear transformations to be a nilpotent algebra. In this paper we consider nilpotent linear transformations on vectors spaces of dimension 1, 2 and 3. As an application of our results, we solve some cases of the problem on solvability of nilalgebras described below.

An algebra A is *power-associative* in case the subalgebra generated by each element of A is associative. For any algebra the (right) *powers* of an element x in A are defined by $x^1 = x$, $x^{n+1} = x^n x$. If A is power-associative then $x^i x^j = x^{i+j}$. An element x in a power-associative algebra A is called *nilpotent* if there exists a k such that $x^k = 0$. The *index of nilpotency* for such an element x is the smallest k such that $x^k = 0$. A power-associative algebra is called a *nilalgebra* if each element is nilpotent. When there is a bound on the indices of nilpotency, the *nilindex* of the algebra is the smallest k such that $x^k = 0$ for all x in A.

The following problem has been open since 1972 (see [1,13], [11–p. 205]).

Albert's Problem. Is every finite-dimensional commutative power-associative nilalgebra solvable?

It is known that this problem has a positive answer when the algebra has dimension $\leqslant 6$ (see [3,4,9]), and dimension n and nilindex n or $n+1$ (see [5]). See also [2]. In this paper we prove that Albert's problem has positive answer for dimension 7, and for algebras of dimension n and nilindex $n-1$ or $n-2$.

2. Nilpotent linear transformations

Let V be a vector space over a field K. We denote by $L(V)$ the set of all linear transformations on V. The set $L(V)$ is a vector space.

Assume that $\dim(V) = n$. Let Ω be a vector subspace of $L(V)$. Assume that R is nilpotent for any $R \in \Omega$. Gerstenhaber [8] proved that: $\dim(\Omega) \leqslant n(n-1)/2$; if $\dim(\Omega) = n(n-1)/2$ then Ω is a nilpotent algebra with index of nilpotency n.

In this section we prove Gerstenhaber's result for $n = 1, 2$ and obtain a more specific result for $n = 3$.

Proposition 1. *Let V be a vector space of dimension 1 over a field K. If $T \in L(V)$ is nilpotent then $T = 0$.*

Proof. The result is clear since $L(V) \cong K$ as K-algebras. $\square$

Theorem 2. *Let V be a vector space of dimension 2 over a field K and let Ω be a vector subspace of $L(V)$. Assume that R is nilpotent for any $R \in \Omega$. Then, if R and S are any elements in Ω, we have $R \cdot S = 0$.*

Proof. If R is a nonzero element in Ω, we may assume that the matrix $[R]$ associated with R is in Jordan canonical form. Let S be any other element of Ω. We have:

$$[R] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad [S] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $T = uR + S$ must be nilpotent for all choices of the scalar u, both the trace and the determinant of $[T]$ must be zero. That is, $a + d = 0$ and $-a^2 - bc - uc = 0$. Since this last equation holds for all u in K, we must have $c = 0$. Consequently, we must also have that $a = 0$. It follows that $d = 0$ as well. Thus $[S]$ is strictly upper triangular. This means that any element of Ω will be represented by a strictly upper triangular 2×2 matrix. The product of any two strictly upper triangular 2×2 matrices is zero. $\square$

Theorem 3. *Let V be a vector space of dimension 3 over a field K and let Ω be a vector subspace of $L(V)$. Assume that R is nilpotent for any $R \in \Omega$. Then either*

(i) $\Omega \cdot \Omega \cdot \Omega = 0$ *or,*
(ii) *The dimension of Ω is 2 and we can choose a basis of V such that $\{P, Q\}$ is a basis of Ω with*

$$[P] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad [Q] = \begin{pmatrix} \varepsilon & 0 & \varepsilon^3 \\ 1 & -2\varepsilon & 3\varepsilon^2 \\ 0 & -1 & \varepsilon \end{pmatrix}.$$

In this case, if R, S, and $R \cdot S$ are all in Ω, then $R = 0$ or $S = 0$.

Proof. Since the dimension of V is 3, the linear transformations in Ω are represented by 3×3 matrices. Since each linear transformation in Ω is nilpotent, its characteristic polynomial must be x^3. We will consider two cases.

Case one. Suppose first that for all A in Ω, we have $A^2 = 0$. Suppose that $A \neq 0$ and B are elements in Ω. Assuming $[A]$ to be in Jordan canonical form, we may suppose that

$$[A] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad [B] = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Since $A^2 = B^2 = (A + B)^2 = 0$, we get $[AB + BA] = 0$. From this we obtain $d = f = g = a + e = 0$. Furthermore,

$$[B] = \begin{pmatrix} a & b & c \\ 0 & -a & 0 \\ 0 & h & i \end{pmatrix} \quad \text{and} \quad [B]^2 = \begin{pmatrix} a^2 & ch & ac + ci \\ 0 & a^2 & 0 \\ 0 & -ah + hi & i^2 \end{pmatrix}.$$

We conclude that $a = i = 0$. Therefore $[A]$ and $[B]$ as well as all other elements of Ω belong to the family of matrices of type

$$\begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & * & 0 \end{pmatrix}.$$

In this case $\Omega \cdot \Omega \cdot \Omega = 0$ which is the possibility contained in part (i).

Case two. We may suppose that there is some P in Ω with Jordan canonical form

$$[P] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose that all of the linear transformations in Ω have upper triangular representation. Since they are nilpotent, they must all be strictly upper triangular. It follows that $\Omega \cdot \Omega \cdot \Omega = 0$.

Now suppose that there is a linear transformation S in Ω whose representation is not upper triangular. Let

$$[S] = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

We must have the characteristic polynomial of $T = uP + S$ to be x^3 for any scalar u. The characteristic polynomial of T is

$$\begin{aligned} C_T(x) = {} & x^3 - (a + e + i)x^2 + (-bd + ae - cg - fh + ai + ei)x \\ & - (d + h)ux + (ceg - bfg - cdh + afh + bdi - aei) \\ & - (bg + fg - ah - di)u - gu^2. \end{aligned}$$

We conclude that: (i) $a + e + i = 0$, (ii) $-bd + ae - cg - fh + ai + ei = 0$, (iii) $d + h = 0$, (iv) $ceg - bfg - cdh + afh + bdi - aei = 0$, (v) $bg + fg - ah - di = 0$, (vi) $g = 0$. Since we assumed that $[S]$ was not upper triangular, the conditions $g = 0$ and $d + h = 0$, given by (vi) and (iii), imply that d must be nonzero. Eq. (v) becomes $d(a - i) = 0$ which implies $a = i$. Using Eq. (i) we have $e = -2a$. Eqs. (ii) and (iv) become (vii) $-3a^2 - bd + df = 0$ and (viii) $2a^3 + abd + cd^2 - adf = 0$. Adding a times (vii) to (viii) gives $a^3 - cd^2 = 0$ so $c = a^3/d^2$. From (vii) we get $f = 3a^2/d + b$.

We obtain that the matrix form for any S in Ω which is not upper triangular must be of the form

$$[S] = \begin{pmatrix} a & b & \frac{a^3}{d^2} \\ d & -2a & b + \frac{3a^2}{d} \\ 0 & -d & a \end{pmatrix}$$

with $d \neq 0$. Letting $\varepsilon = a/d$, we can write

$$[S] = d \begin{pmatrix} \varepsilon & 0 & \varepsilon^3 \\ 1 & -2\varepsilon & 3\varepsilon^2 \\ 0 & -1 & \varepsilon \end{pmatrix} + b[P].$$

Therefore we can take $Q = Q_\varepsilon$ in Ω to be given by its matrix representation

$$[Q] = \begin{pmatrix} \varepsilon & 0 & \varepsilon^3 \\ 1 & -2\varepsilon & 3\varepsilon^2 \\ 0 & -1 & \varepsilon \end{pmatrix}$$

and $S = dQ + bP$.

We now want to show that the dimension of Ω is 2 and that $\{P, Q\}$ is a basis of Ω.

The set Ω determines ε uniquely. This follows since if $Q_{\varepsilon'} \in \Omega$ for some $\varepsilon' \in K$, then $Q_\varepsilon - Q_{\varepsilon'}$ is nilpotent and upper triangular, hence strictly upper triangular, forcing $\varepsilon = \varepsilon'$. Therefore $S \in \langle P, Q \rangle$ for all $S \in \Omega$ whose representation are not upper triangular. On the other hand, if $S \in \Omega$ is represented by an upper triangular matrix, $Q + S \in \Omega$ is not, forcing $S \in \langle P, Q \rangle$ again. We conclude that Ω is two dimensional with basis consisting of P and Q.

Finally, suppose that R, S, and $R \cdot S$ are all elements of Ω. Letting $R = pP + qQ$ and $S = xP + yQ$, then the (3, 1) entry of $[R \cdot S]$ is $-qy$. Since $R \cdot S$ is in Ω, then $qy = 0$. So either $q = 0$ or $y = 0$.

If $q = 0$ then

$$[R \cdot S] = \begin{pmatrix} py & -2\varepsilon py & px + 3\varepsilon^2 py \\ 0 & -py & \varepsilon py \\ 0 & 0 & 0 \end{pmatrix}.$$

If $y = 0$ then

$$[R \cdot S] = \begin{pmatrix} 0 & \varepsilon qx & px \\ 0 & qx & -2\varepsilon qx \\ 0 & 0 & -qx \end{pmatrix}.$$

Since $[R \cdot S]$ is upper triangular and in Ω, it must be a scalar multiple of $[P]$. Thus the diagonal elements and the (1, 3) entry must be zero. In either case we must have $qy = qx = py = px = 0$. Therefore we either get $x = y = 0$ or $p = q = 0$ and so either $[R] = 0$ or $[S] = 0$.

The assertions (i) and (ii) do indeed exclude one another since, for example, $P^2 Q \neq 0$. $\square$

3. Albert's problem

Throughout this section A is a commutative power-associative nilalgebra of finite dimension n and nilindex k over a field K. We assume that K has characteristic zero or sufficiently large. Therefore, from Gerstenhaber [7], the linear operator L_t is nilpotent for any t in A.

3.1. Preliminary results

Proposition 1 and Theorems 2 and 3 are used extensively in this section in the following way. Let V be a subspace of A. The set V is not necessarily a subalgebra. It is just a subspace which means it is closed under addition and scalar multiplication. We let $W = \{x \in A \mid xV \subset V\}$. The set W is called the *stabilizer* of V in A. The set $\Omega = \{L_x \mid x \in W\}$ is a vector space of nilpotent linear transformations. By construction, V is an invariant subspace for all elements of Ω. We let $\overline{\Omega}$ be the set of linear transformations of A/V which are induced by the linear transformations in Ω.

If the dimension of A/V is 1, then Proposition 1 implies that $WA \subset V$.

If the dimension of A/V is 2, then Theorem 2 implies that $W(WA) \subset V$.

If the dimension of A/V is 3, then Theorem 3 implies that either Ω is of a certain form or else $W(W(W(A))) \subset V$.

Proposition 4. *Let A be a commutative power-associative nilalgebra of dimension n. Let B be a $(n-1)$-dimensional subalgebra of A. Then B is an ideal of A.*

Proof. We have to prove that $BA \subset B$. Let $W = \{t \in A \mid tB \subset B\}$. Since B is a subalgebra of A it follows that $B \subset W$. The quotient vector space A/B has dimension 1. Then, by Proposition 1, $WA \subset B$. Therefore $BA \subset B$ since $B \subset W$. □

The following results will be useful.

Proposition 5 [3, Proposition 4]. *Let A be a commutative power-associative nilalgebra of dimension n and nilindex k over a field K with characteristic $\neq 2, 3$. If $k = 1, 2, 3, n, n+1$ then A is nilpotent (thus it is solvable).*

Proposition 6 (Schafer [12, Proposition 2.2, p. 18]). *If an algebra A contains a solvable ideal I, and if A/I is solvable, then A is solvable.*

3.2. Nilindex $n-1$ and $n-2$

Let a be an element of A such that $a^{k-1} \neq 0$ and let $w = \langle a, a^2, \ldots, a^{k-1}\rangle$. Let $W = \{t \in A \mid tw \subset w\}$. Since w is actually a subalgebra, $w \subset W$. Let $\Omega = \{L_t \mid t \in W\}$.

For each t in W, the subalgebra w is carried into itself by the operator L_t. Therefore L_t induces linear transformation on the quotient vector space A/w. The set of these induced linear transformations on A/w will be called $\overline{\Omega}$. We shall not distinguish L_t in Ω from the linear transformation induced by L_t which is in $\overline{\Omega}$.

In the next three theorems we will use these definitions of a, w, W, Ω and $\overline{\Omega}$.

Theorem 7. *Every commutative power-associative nilalgebra of finite dimension n and nilindex $n-1$ over a field K with characteristic zero or sufficiently large is solvable.*

Proof. In this case $\dim(A/w)=2$. The transformations of A/w in $\overline{\Omega}$ satisfy the hypothesis of Theorem 2 and it follows that $W(WA)\subset w$. Since $w\subset W$, it follows that $(WA)w\subset w$. This implies that $WA\subset W$ whence W is an ideal of A. We remark that w is an ideal of W and that A/W and W/w are algebras of dimension $\leqslant 2$. Therefore, from Proposition 5, it follows that A/W and W/w are solvable. Since w is solvable, it follows from Proposition 6 that W is solvable. Therefore, by Proposition 6 again, A is solvable. □

Theorem 8. *Let A be a commutative power-associative nilalgebra of dimension n and nilindex $n-2$ over a field K with characteristic zero or sufficiently large. Suppose that $\overline{\Omega}$ has a basis $\{P,Q\}$ with*

$$[P]=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},\quad [Q]=\begin{pmatrix}\varepsilon&0&\varepsilon^3\\1&-2\varepsilon&3\varepsilon^2\\0&-1&\varepsilon\end{pmatrix}.$$

Then A is solvable or $W(W(WA))\subset w$.

Proof. Suppose that $L_a=pP+qQ$ and $L_{a^2}=rP+sQ$.

From the first linearization of $z^2z^2=z^4$ we obtain

$$L_{z^3}=4L_{z^2}L_z-2L_zL_zL_z-L_zL_{z^2}.$$

Linearizing this identity we obtain

$$\begin{aligned}L_{z^{i+j+k}}&=L_{z^iz^jz^k}\\&=\frac{1}{3}\big\{4L_{z^{i+j}}L_{z^k}+4L_{z^{i+k}}L_{z^j}+4L_{z^{j+k}}L_{z^i}\\&\qquad-L_{z^i}L_{z^{j+k}}-L_{z^j}L_{z^{i+k}}-L_{z^k}L_{z^{i+j}}\big\}\\&\quad+\frac{1}{3}\big\{-L_{z^i}L_{z^j}L_{z^k}-L_{z^j}L_{z^i}L_{z^k}-L_{z^k}L_{z^j}L_{z^i}\\&\qquad-L_{z^i}L_{z^k}L_{z^j}-L_{z^k}L_{z^i}L_{z^j}-L_{z^j}L_{z^k}L_{z^i}\big\}.\end{aligned}$$

Therefore, using induction, we obtain that L_{a^i} $(i\geqslant 3)$ is in the subring generated by L_a and L_{a^2}.

We compute the matrix for L_{a^3}. The (3, 1) entry of $[L_{a^3}]$ is $-3qs$. Since L_{a^3} is in $\overline{\Omega}$ we must have $qs = 0$. So $q = 0$ or $s = 0$.

If $q = 0$ then

$$[L_{a^3}] = \begin{pmatrix} -ps & 6\varepsilon ps & 3pr - 3\varepsilon^2 ps \\ 0 & 5ps & -9\varepsilon ps \\ 0 & 0 & -4ps \end{pmatrix}.$$

If $s = 0$ then

$$[L_{a^3}] = \begin{pmatrix} 4qr & -9\varepsilon qr & 3pr + 12\varepsilon^2 qr \\ 0 & -5qr & 6\varepsilon qr \\ 0 & 0 & qr \end{pmatrix}.$$

Since $[L_{a^3}]$ is upper triangular and in $\overline{\Omega}$, it must be a scalar multiple of $[P]$. In particular the diagonal and the (1, 3) entries must be zero. In both cases we get $ps = pr = qr = qs = 0$. Therefore we either have $r = s = 0$ or $p = q = 0$ and hence $L_a = 0$ or $L_{a^2} = 0$. We consider separately the three possible cases.

Case 1. Suppose that $L_a = L_{a^2} = 0$. Since L_{a^i} is in the subring generated by L_a and L_{a^2}, we have $L_{a^i} = 0$ for all i. This means that w is an ideal of A. Since A/w is now a nilalgebra of dimension 3, from Proposition 5 we have that A/w is solvable. Since w is also solvable, we obtain from Proposition 6 that A is solvable.

Case 2. Suppose that $L_a = 0$ and $L_{a^2} \neq 0$. Using power-associativity, from the first linearization of the identity $z^2 z^3 = z^5$ we obtain

$$\begin{aligned} &2(ay)((aa)a) + 2(aa)((ay)a) + (aa)((aa)y) \\ &\quad = 2(((ay)a)a)a + (((aa)y)a)a + (((aa)a)y)a + (((aa)a)a)y. \end{aligned}$$

Since $L_a = 0$, all the terms except the last terms on each side of the equal sign are in w. This means that $a^2(a^2 y) \equiv a^4 y \pmod{w}$. In particular, $L_{a^2} L_{a^2} = L_{a^4}$. This implies that $L_{a^2} L_{a^2}$ is in $\overline{\Omega}$. Since L_{a^2} is also in $\overline{\Omega}$, we obtain from Theorem 3 that $L_{a^2} = 0$. This contradicts that $L_{a^2} \neq 0$.

Case 3. Suppose that $L_a \neq 0$ and $L_{a^2} = 0$. Since L_a is nilpotent and $\dim(A/w) = 3$ we must have $L_a^3 = 0$. Since L_{a^i} is in the subring generated by L_a and L_{a^2}, and $L_a^3 = 0$, we have that $L_{a^i} = 0$ for all $i \geqslant 2$. Looking at the matrix of $L_a = pP + qQ$, we see that the matrix of L_a^2 has (3, 1) entry $-q^2$. Therefore $L_a^2 = 0$ would imply $q = 0$, hence $L_a = pP \neq 0$ and then $L_a^2 \neq 0$, a contradiction. This shows that $L_a^2 \neq 0$.

We can pick a basis of A/w of the form $a(ax)$, ax, x. We will eventually show that $W(W(WA)) \subset w$ by first establishing what W actually contains. We will show that $\dim(W) = \dim(w) + 1$ and that $a(ax)$ is an element of W which is not in w.

We already know that $a^i y \in w$ for all $y \in A$ and $i \geqslant 2$. Therefore, to establish that an element y is in W, we need only show that ya is in w. Since $L_a^3 = 0$, we have $(a(ax))a$ is in w. This shows that $a(ax)$ is in W.

We now establish that the dimension of W/w is one. Suppose that $(c_1 a(ax) + c_2 ax + c_3 x)w \subset w$. Since a is an element of w, this implies $(c_1 a(ax) + c_2 ax +$

$c_3x)a \in w$. Since $a(ax)$, ax, x are linearly independent modulo w, we must have $c_2 = c_3 = 0$. Therefore $a(ax)$ is a basis of W/w.

It follows that $a, a^2, \dots, a^{n-3}, a(ax)$ is a basis of W. Therefore, to obtain that W is a subalgebra of A, it remains to prove that $(a(ax))^2 \in W$.

We first show that $(a^4x)x$ is in w. From the second linearization of $z^4 = z^2z^2$, substituting two y's for two z's we obtain

$$((yy)z)z + 2((yz)y)z + 2((yz)z)y + ((zz)y)y = 2(yy)(zz) + 4(yz)(yz).$$

Now we let $y = a^2$ and $z = x$ to obtain

$$(a^4x)x + 2((a^2x)a^2)x + 2((a^2x)x)a^2 + (x^2a^2)a^2 = 2a^4x^2 + 4(a^2x)^2.$$

The terms on the left ending in a^2 are in w since $L_{a^2} = 0$. Furthermore, $((a^2x)a^2)x$ is in $(wa^2)x \subset w$ since $L_{a^i} = 0$ for all $i \geqslant 2$. On the right-hand side, the terms are in w since $L_{a^4} = 0$ and $(a^2x)^2$ is in $w^2 \subset w$ since $L_{a^2} = 0$. This means that $(a^4x)x$ is in w.

We next show that $(a(ax))^2$ is in W. We use power-associativity of degree 6. From the second linearization of the identity $z^3z^3 = z^6$ we obtain:

$$\begin{aligned}
&2((xx)a)((aa)a) + 4((ax)x)((aa)a) + 4((ax)a)((ax)a)\\
&\quad + ((aa)x)((aa)x) + 4((aa)x)((ax)a)\\
&\quad = ((((xx)a)a)a)a + 2((((ax)x)a)a)a + 2((((ax)a)x)a)a\\
&\qquad + 2((((ax)a)a)x)a + 2((((ax)a)a)a)x + ((((aa)x)x)a)a\\
&\qquad + ((((aa)x)a)x)a + ((((aa)x)a)a)x + ((((aa)a)x)x)a\\
&\qquad + ((((aa)a)x)a)x + ((((aa)a)a)x)x.
\end{aligned}$$

Notice that a^2x is in w and $(ax)a$ is in W. Therefore the terms on the left-hand side are in the set $4((ax)a)^2 + Aa^3 + w^2 + wW$. Hence, the left-hand side is of the form $4((ax)a)^2 + w$. The terms on the right-hand side which end in two a's are in W since $(Aa)a \subset W$. The terms which end with $((\dots)a)x$ or $((\dots)x)a$ must start with three a's and an x. The products $((ax)a)a$, $((aa)x)a$, and $((aa)a)x$ are in w since $L_a^3 = 0$ and $L_{a^i} = 0$ for $i \geqslant 2$. Thus the terms $((\dots)a)x$ or $((\dots)x)a$ are contained in $(wa)x + (wx)a \subset (ax)a + w \subset W$ since $L_{a^i} = 0$ for $i \geqslant 2$. The product $(a^4x)x$ is in w by the previous proof. This proves that $(a(ax))^2$ is in W.

Since W is a subalgebra of A and w is an ideal of W, we have that W/w is an algebra. Also we know that the dimension of W/w is 1. Thus W/w is solvable (by Proposition 5). Therefore $W^2 \subset w$.

Now let W act on the two-dimensional vector space A/W. From Theorem 2 we have that $W(WA) \subset W$. Then it follows that $W(W(WA)) \subset W^2 \subset w$. $\square$

Theorem 9. *Every commutative power-associative nilalgebra of finite dimension n and nilindex $n-2$ over a field K with characteristic zero or sufficiently large is solvable.*

Proof. The element a in now an element whose index of nilpotency is $n-2$. Therefore $\dim(A/w) = 3$. We introduce H as the stabilizer of W in the second portion of the proof and as the stabilizer of $W + W^2$ in the third portion of the proof.

By Theorems 3 and 8, it remains to consider the case where $W(W(WA)) \subset w$. Since $w \subset W$, we have that $W(W(WA)) \subset w$ implies $(W(WA))w \subset w$ which implies $W(WA) \subset W$. This gives $W^3 \subset W$ and $W^3W \subset w$. The proof now splits into three cases.

Case 1. If $W = w$, then from $w(w(wA)) \subset w$ we successively get $w(wA) \subset w$, $wA \subset w$ and $A \subset w$. This is impossible since the dimension of A is n and the dimension of w is $n-3$.

Case 2. Assume that $W \neq w$ and $W^3 \subset w$. This implies that $W^2 \subset W$ and so W is a subalgebra. If H is the stabilizer of W, then $W \subset H$. Since $\dim(A/W) \leqslant 2$ we have $H(HA) \subset W$ and so $HA \subset H$. Thus H is an ideal. The dimension of A/H, H/W, W/w are all $\leqslant 2$ and these algebras are all solvable by Proposition 5. Since w is solvable, we successively get that W, H and A are solvable by Proposition 6.

Case 3. Assume that W^3 is not contained in w. In this case $W + W^2$ is in the stabilizer of $w + W^3$. Since the dimension of $A/(w + W^3) \leqslant 2$ we have $(W + W^2)((W + W^2)A) \subset w + W^3$. In particular $W^2(WA) \subset w + W^3$ and this implies $W^2 W^2 \subset w + W^3 \subset W$. This makes $W + W^2$ a subalgebra and $\dim(A/(W + W^2)) \leqslant 2$. Letting H be the stabilizer of $W + W^2$, we have $H(HA) \subset (W + W^2)$ and so $HA \subset H$. Thus H is an ideal of A. We have $w \subset w + W^3 \subset W + W^2 \subset H \subset A$. Each subalgebra is an ideal of the next larger subalgebra and the dimension of the quotient algebra is $\leqslant 2$. All the quotient algebras are solvable, w is solvable, and therefore A is solvable. □

As a consequence of Proposition 5 and Theorems 7 and 9 we obtain the following result.

Corollary 10 (Correa et al. [3,4]). *Let A be a commutative power-associative nilalgebra of dimension $\leqslant 6$ over a field of characteristic $\neq 2, 3, 5$. Then A is solvable.*

Remark. The results stated in Theorems 7 and 9 were obtained also independently by Gutiérrez Fernández [10]. Furthermore, it is proved in [10] that the result stated in Corollary 10 is true for commutative nilalgebras which are not necessary power-associative.

3.3. Dimension 7

In this subsection we prove the following result.

Theorem 11. *Let A be a commutative power-associative nilalgebra of dimension 7 over a field K of characteristic zero or sufficiently large. Then A is solvable.*

By Proposition 5 and Theorems 7 and 9 the result is true when the nilindex of A is 1, 2, 3, 5, 6, 7 and 8. Therefore it remains to prove that A is solvable when the nilindex of A is 4.

Throughout the rest of this subsection A is a commutative power-associative nilalgebra of nilindex 4. By Lemma 2 of [4] $L_z^5 = 0$ for all $z \in A$.

Lemma 12 [3, Lemma 6]. *Any commutative power-associative nilalgebra of nilindex 4 over a field of characteristic $\neq 2$ satisfies the following identities*:

$$2((yx)x)x + (x^2y)x + x^3y = 0, \tag{1}$$
$$(yx)x^2 = 0, \tag{2}$$
$$2(yx)(zx) + (yz)x^2 = 0, \tag{3}$$
$$2(((yx)x)x)x + x^3(yx) = 0, \tag{4}$$
$$(yx^2)x^3 = 0, \tag{5}$$
$$(yx^3)x^2 = 0. \tag{6}$$

Lemma 13. *Let A be a commutative power-associative nilalgebra of dimension 7 and nilindex 4 over a field of characteristic $\neq 2$. Then A satisfies the following identities*:

$$(yx)x^3 = 0, \tag{7}$$
$$x^3(yz) = -(zx^2)(yx) - 2((zx)x)(yx), \tag{8}$$
$$(yx^2)x^2 = 0, \tag{9}$$
$$A^2x^2 = (Ax)^2, \tag{10}$$
$$A^3x^3 \subset ((A^2x)x)(Ax) + (Ax)^3. \tag{11}$$

Proof. Identity (7) holds if $x^3 = 0$. Let $x \in A$ with $x^3 \neq 0$ and $X = \langle x, x^2, x^3 \rangle$. Let y be an arbitrary element of A. Let A/X be the quotient vector space of A by X. Since $XL_x \subset X$ the linear map $\overline{L_x} : A/X \to A/X$, given by $(y + X)\overline{L_x} = yL_x + X$, is well-defined. Since $L_x^5 = 0$ we have $(\overline{L_x})^5 = 0$. Since A/X has dimension 4 we have $(\overline{L_x})^4 = 0$. This implies that $yL_x^4 \in X$. Thus $yL_x^4 = \alpha x + \beta x^2 + \delta x^3$ $(\alpha, \beta, \delta \in K)$. Since $L_x^5 = 0$ we obtain $\alpha x^2 + \beta x^3 = yL_x^5 = 0$ and then $\alpha = 0$ and $\beta = 0$. It follows that $yL_x^4 = \delta x^3$. On the other hand, by (4), $(yx)x^3 = -2yL_x^4$. We obtain then that $(yx)x^3 = \lambda x^3$ $(\lambda \in K)$. Since $L_{yx}^5 = 0$ we get $\lambda^5 x^3 = x^3 L_{yx}^5 = 0$. Therefore $\lambda = 0$ and we obtain $(yx)x^3 = 0$. Therefore (7) is an identity of A.

Linearizing (7) we obtain (8). Replacing z by x^2 in (3) and using (7) we get (9). From (3) we get (10). Letting $z \in A^2$ and $y \in A$ in (8) and using (10) we get (11). □

The linearization of (3) is

$$(xy)(zw) + (yz)(xw) + (zx)(yw) = 0. \tag{12}$$

By (12) we have $((yx)(zx))x^3 = -((yx)x^2)((zx)x) - ((zx)x^2)((yx)x)$. Therefore, by (2), we have

$$((yx)(zx))x^3 = 0. \tag{13}$$

From now on we assume that A has dimension 7. Since $(yx)(zx) = -1/2(yz)x^2$ by (3) we have, by (9),

$$((yx)(zx))x^2 = 0. \tag{14}$$

Since $((yx)x)^2 = -1/2(yx)^2x^2 = 1/4(y^2x^2)x^2$ by (3) we have, by (9),

$$((yx)x)^2 = 0. \tag{15}$$

From (8) we obtain $2y^2x^3 = -2(yx^2)(yx) - 4((yx)x)(yx)$. By (12) we have $y^2x^3 = -2(yx^2)(yx)$. Therefore, subtracting the second from the first, we obtain

$$y^2x^3 = -4((yx)x)(yx). \tag{16}$$

Since the kernel of L_{x^3} plays a pivotal role in the proof of Theorem 11, we display a number of useful properties. Identities (2), (5), (7) and (9) imply

$$Ax + Ax^2 \subset \mathrm{Ker}(L_{x^2}) \cap \mathrm{Ker}(L_{x^3}). \tag{17}$$

Applying (13) and (14) we also obtain

$$(Ax)^2 \subset \mathrm{Ker}(L_{x^2}) \cap \mathrm{Ker}(L_{x^3}). \tag{18}$$

Identity (8) yields

$$A\,\mathrm{Ker}(L_x) \subset \mathrm{Ker}(L_{x^3}). \tag{19}$$

In particular, since $x^4 = 0$, we have $x^3 \in \mathrm{Ker}(L_x)$, hence

$$Ax^3 \subset \mathrm{Ker}(L_{x^3}). \tag{20}$$

Replacing z by zx in (8) and applying (2), we deduce

$$((zx)x)x = 0 \quad \Rightarrow \quad A(zx) \subset \mathrm{Ker}(L_{x^3}). \tag{21}$$

From now on we let x be an element of A such that $x^3 \neq 0$ and $X = \langle x, x^2, x^3\rangle$. Thus $L_x \neq 0$ and $L_x^2 \neq 0$. Using (4) and (7) we obtain $L_x^4 = 0$. Therefore the minimal polynomial of L_x is t^4 or t^3. Let J_l denote the $l \times l$ elementary Jordan matrix associated to the eigenvalue 0. Then the possible Jordan canonical forms of L_x are

$$\text{(a)} \begin{bmatrix} J_4 & 0 \\ 0 & J_3 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} J_4 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} J_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(d)} \begin{bmatrix} J_3 & 0 & 0 \\ 0 & J_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{(e)} \begin{bmatrix} J_3 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_2 \end{bmatrix} \quad \text{(f)} \begin{bmatrix} J_3 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(g)} \begin{bmatrix} J_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The basis of A corresponding to each one of these matrices are

(a) $\{y, yx, (yx)x, ((yx)x)x, a, ax, (ax)x\}$ with $(((yx)x)x)x = 0$, $((ax)x)x = 0$.
(b) $\{y, yx, (yx)x, ((yx)x)x, a, ax, b\}$ with $(((yx)x)x)x = 0$, $(ax)x = 0$, $bx = 0$.
(c) $\{y, yx, (yx)x, ((yx)x)x, a, b, c\}$ with $(((yx)x)x)x = 0$, $ax = 0$, $bx = 0$, $cx = 0$.
(d) $\{x, x^2, x^3, a, ax, (ax)x, b\}$ with $((ax)x)x = 0$, $bx = 0$.
(e) $\{x, x^2, x^3, a, ax, b, bx\}$ with $(ax)x = 0$, $(bx)x = 0$.
(f) $\{x, x^2, x^3, a, ax, b, c\}$ with $(ax)x = 0$, $bx = 0$, $cx = 0$.
(g) $\{x, x^2, x^3, a, b, c, d\}$ with $ax = 0$, $bx = 0$, $cx = 0$, $dx = 0$.

When A has a basis of type (c), (e), (f), (g), the proof that A is solvable is the same (modulo minor modifications) as the proof when the algebra has dimension 6 and a basis of type, respectively, (a), (d), (d), (c) in the notation of Section 3.1 of [3].

We prove now that A is solvable when A has a basis of type (a), (b), (d). We present the proof in a series of lemmas.

Lemma 14. *If $A^3\langle x^3\rangle \subset \langle x^3\rangle$ (so, in particular, if $A\langle x^3\rangle \subset \langle x^3\rangle$), then A is solvable.*

Proof. Assume that $A^3\langle x^3\rangle \subset \langle x^3\rangle$. Then $\langle x^3\rangle$ is a solvable ideal of A^3. The quotient $A^3/\langle x^3\rangle$ is a nilalgebra of dimension $\leqslant 6$, so it is solvable by Corollary 10. Therefore A^3 is solvable by Proposition 6. Finally, A^3 solvable implies A solvable. $\square$

Lemma 15. *If $\mathrm{Ker}(L_{x^3})$ is a subalgebra of dimension 6 then A is solvable.*

Proof. Since $\dim(A) = 7$ and $\dim(\mathrm{Ker}(L_{x^3})) = 6$, it follows by Proposition 4 that $\mathrm{Ker}(L_{x^3})$ is an ideal of A. Since $\mathrm{Ker}(L_{x^3})$ and $A/\mathrm{Ker}(L_{x^3})$ are nilalgebras of dimension $\leqslant 6$, they are solvable by Corollary 10. Therefore A is solvable by Proposition 6. $\square$

Lemma 16. *Assume that A has a basis of type* (a). *Then A is solvable.*

Proof. Let

$$x = \alpha_1 y + \alpha_2 yx + \alpha_3 (yx)x + \alpha_4 ((yx)x)x + \alpha_5 a + \alpha_6 ax + \alpha_7 (ax)x \tag{22}$$

($\alpha_i \in K$). Applying L_x^3 to (22) we obtain $\alpha_1 = 0$. Applying L_{x^2} to (22) and using (2) we get $x^3 = \alpha_5 ax^2$. This last equation implies $\alpha_5 \neq 0$ since $x^3 \neq 0$. Therefore

$$ax^2 = \alpha_5^{-1} x^3. \tag{23}$$

Then $(ax^2)x = 0$ since $x^4 = 0$. Therefore by (1) we have $x^3 a = -2((ax)x)x - (ax^2)x = 0$. Therefore

$$a \in \mathrm{Ker}(L_{x^3}). \tag{24}$$

If $yx^3 = 0$ then $Ax^3 = 0$ by (7) and (24). Therefore A is solvable by Lemma 14.

Assume now that $yx^3 \neq 0$. Then $\mathrm{Ker}(L_{x^3}) = \langle yx, (yx)x, ((yx)x)x, a, ax, (ax)x\rangle$ by (7) and (24). It follows by (17) that $\mathrm{Ker}(L_{x^3}) = Ax + Ka$. Therefore, by (18), to conclude that $\mathrm{Ker}(L_{x^3})$ is a subalgebra of A, it remains to prove that $a^2, (tx)a \in \mathrm{Ker}(L_{x^3})$ for all $t \in A$. This in turn follows from (23), (2) and (3) since

$$a^2x^3 = \alpha_5 a^2(ax^2) = 0,$$
$$(a(tx))x^3 = \alpha_5(a(tx))(ax^2) = -\frac{1}{2}\alpha_5 a^2((tx)x^2) = 0.$$

Since $\mathrm{Ker}(L_{x^3})$ is a subalgebra of dimension 6 when $yx^3 \neq 0$, it follows that A is solvable by Lemma 15. □

Lemma 17. *Assume that A has a basis of type* (b). *Then A is solvable.*

Proof. Let

$$x = \alpha_1 y + \alpha_2 yx + \alpha_3(yx)x + \alpha_4((yx)x)x + \alpha_5 a + \alpha_6 ax + \alpha_7 b \tag{25}$$

($\alpha_i \in K$). Applying L_x^3 to (25) we obtain $\alpha_1 = 0$. Applying L_x^2 to (25) we get $x^3 = \alpha_2((yx)x)x$. From this last equation we obtain $\alpha_2 \neq 0$ since $x^3 \neq 0$. Therefore

$$((yx)x)x = \alpha_2^{-1}x^3. \tag{26}$$

We will consider two cases: $\alpha_5 = 0$ and $\alpha_5 \neq 0$.

First case: Assume that $\alpha_5 = 0$. Multiplying (25) by x and using (26) we obtain

$$(yx)x = \alpha_2^{-1}x^2 - \alpha_2^{-2}\alpha_3 x^3. \tag{27}$$

We will prove that $A^3x^3 = 0$. By (11) it is enough to prove that $((Ax)x)(Ax) = 0$ and $(Ax)^3 = 0$.

We have

$$Ax = \langle yx, (yx)x, ((yx)x)x, ax\rangle \subset \langle yx, x^2, x^3, ax\rangle$$

by (26) and (27). Then $(Ax)x \subset \langle x^2, x^3\rangle$ by (27). Therefore, by (2) and (7), $((Ax)x)(Ax) = 0$.

By (2) and (7)

$$(Ax)^2 \subset \langle (yx)^2, (yx)(ax), (ax)^2\rangle.$$

For any $t \in A$, $((ax)(ax))(tx) = -2((ax)t)((ax)x) = 0$ by (12). Therefore, by (13) and (14), we have

$$(Ax)^3 \subset \langle (yx)^3, (yx)^2(ax), ((yx)(ax))(yx), ((yx)(ax))(ax)\rangle.$$

We now prove that each one of these products is zero. Given any $t \in A$, we obtain

$$((yx)x)(tx) = 0, \quad (((yx)x)x)(tx) = 0 \tag{28}$$

from (2), (7), (26), (27). Hence

$$\begin{aligned}(yx)^2(tx) &= -2((yx)t)((yx)x) \quad (\text{by } (3))\\ &= -2\alpha_2^{-1}((yx)t)x^2 + 2\alpha_2^{-2}\alpha_3((yx)t)x^3 \quad (\text{by } (27))\\ &= 4\alpha_2^{-1}((yx)x)(tx) - 2\alpha_2^{-2}\alpha_3((yx)x^2)(tx)\\ &\quad - 4\alpha_2^{-2}\alpha_3(((yx)x)x)(tx) \quad (\text{by } (3), (8))\\ &= 0 \quad (\text{by } (2), (28))\end{aligned}$$

so, in particular, $(yx)^3 = 0$ and $(yx)^2(ax) = 0$. From (25) we obtain $yx \in \langle x, x^2, x^3, ax, b\rangle$. Therefore $(yx)(ax) \in \langle (ax)(ax), (ax)b\rangle$ by (2) and (7). Then $((yx)(ax))(tx) \in \langle (ax)^2(tx), ((ax)b)(tx)\rangle$, where A being of type (b) implies $(ax)^2(tx) = -2((ax)t)((ax)x) = 0$ by (3) and $((ax)b)(tx) = -(bt)((ax)x) - (t(ax))(bx) = 0$ by (12). Hence $((yx)(ax))(tx) = 0$, which implies $((yx)(ax))(yx) = 0$ and $((yx)(ax))(ax) = 0$. Therefore $(Ax)^3 = 0$.

Therefore, in the case $\alpha_5 = 0$, we have $A^3x^3 = 0$. It follows that A is solvable by Lemma 14.

Second case: Assume that $\alpha_5 \neq 0$. Then

$$A = \langle y, yx, (yx)x, x, x^2, x^3, b\rangle \tag{29}$$

by (25) and (26).

If $(yx)y = 0$ then $((yx)x)(yx) = -1/2((yx)y)x^2 = 0$ by (3). Then by (8) $y^2x^3 = -(yx^2)(yx) - 2((yx)x)(yx) = -(yx^2)(yx)$. By (3) we obtain $y^2x^3 = -2(yx^2)(yx)$. It follows that $(yx^2)(yx) = 0$. Therefore

$$((yx)x)(yx) = 0, \quad (yx^2)(yx) = 0, \tag{30}$$

when $(yx)y = 0$.

Assume that $(yx)y \neq 0$. We have

$$((yx)y)x^3 = -((yx)x^2)(yx) - 2(((yx)x)x)(yx) = 0$$

by (8), (2), (26) and (7). Therefore

$$(yx)y \in \operatorname{Ker}(L_{x^3}). \tag{31}$$

Let

$$(yx)y = \beta_1 y + \beta_2 yx + \beta_3(yx)x + \beta_4 x + \beta_5 x^2 + \beta_6 x^3 + \beta_7 b \tag{32}$$

$(\beta_i \in K)$.

Assume that $\beta_1 \neq 0$. Then $A = \langle (yx)y, yx, (yx)x, x, x^2, x^3, b\rangle$ by (29) and (32). If $b \in \operatorname{Ker}(L_{x^3})$ then $Ax^3 = 0$ by (7) and (31), and this implies that A is solvable by Lemma 14. Assume that $bx^3 \neq 0$. Then

$$\operatorname{Ker}(L_{x^3}) = \langle (yx)y, yx, (yx)x, x, x^2, x^3\rangle.$$

We claim that $\operatorname{Ker}(L_{x^3})$ is a subalgebra of A. Replacing z by yx in (21) we obtain

$$A((yx)x) \subset \operatorname{Ker}(L_{x^3}). \tag{33}$$

By (17), (18) and (20) it therefore sufficies to prove that $((yx)y)^2, ((yx)y)(yx) \in \mathrm{Ker}(L_{x^3})$. We have $((yx)y)^2 = 0$ by (15). Using (8), (2), (26) and (7) we obtain

$$\begin{aligned}(((yx)y)(yx))x^3 &= -((yx)x^2)(((yx)y)x) - 2(((yx)x)x)(((yx)y)x) \\ &= -2\alpha_2^{-1}(((yx)y)x)x^3 = 0.\end{aligned}$$

Therefore $((yx)y)^2$ and $((yx)y)(yx)$ are in $\mathrm{Ker}(L_{x^3})$. Therefore $\mathrm{Ker}(L_{x^3})$ is a subalgebra of A. By Lemma 15, it follows that A is solvable.

Assume that $\beta_1 = 0$ and $\beta_7 \neq 0$. Then multiplying (32) by x^3 we obtain by (7) and (31) that $bx^3 = 0$. If $yx^3 = 0$ then $Ax^3 = 0$. It follows that A is solvable by Lemma 14. If $yx^3 \neq 0$ then $\mathrm{Ker}(L_{x^3}) = \langle yx, (yx)x, x, x^2, x^3, b\rangle$ is a subalgebra of A by (17)–(19) and (33) combined with the relation $b \in \mathrm{Ker}(L_x)$. It follows that A is solvable by Lemma 15.

Assume that $\beta_1 = 0$ and $\beta_7 = 0$. Multiplying (32) by x^2 we obtain $((yx)y)x^2 = \beta_4 x^3$ by (2). Therefore, using (3), we get $((yx)x)(yx) = -1/2((yx)y)x^2 = -\beta_4/2x^3$. Therefore $y^2x^3 = -4((yx)x)(yx) = 2\beta_4 x^3$ by (16). Therefore $\beta_4 = 0$ since $L_{y^2}^5 = 0$. It follows that $y^2x^3 = 0 = ((yx)x)(yx)$, and then $(yx^2)(yx) = -1/2y^2x^3 = 0$ by (3). Therefore relations (30) hold.

Therefore, when $\alpha_5 \neq 0$, A is solvable or relations (30) hold.

Assume that relations (30) hold. We will prove that A is solvable.

From (29) and (26) it follows that $Ax = \langle yx, (yx)x, x^2, x^3\rangle$ and then $(Ax)x = \langle (yx)x, x^3\rangle$. Therefore, by (2), (7), (15) and (30),

$$((Ax)x)(Ax) = 0.$$

We have $Ax^2 = \langle yx^2, x^3, bx^2\rangle$ by (29) and (2). Then

$$(Ax^2)(Ax) = \langle (yx^2)((yx)x), (bx^2)(yx), (bx^2)((yx)x)\rangle$$

by (5), (7), (9) and (30). By (8) we have $(yx^2)((yx)x) = -((yx)y)x^3 - 2((yx)x)((yx)x)$. Therefore, by (31) and (15), $(yx^2)((yx)x) = 0$. By a linearization of (9) and by (3) we get $(bx^2)(yx) = -(b(yx))x^2 = 2(bx)((yx)x) = 0$. A similar calculation gives $(bx^2)((yx)x) = 0$. Therefore

$$(Ax^2)(Ax) = 0.$$

Since $((Ax)x)(Ax) = 0$ and $(Ax^2)(Ax) = 0$, it follows by (10) and (11) that $A^3x^3 = 0$. Therefore A is solvable by Lemma 14. □

Lemma 18. *Assume that A has a basis of type* (d). *Then A is solvable.*

Proof. Identities (19) combined with the relation $b \in \mathrm{Ker}(L_x)$ and (21) for $z = a$, $z = ax$ yield

$$A(ax) + A((ax)x) + Ab \subset \mathrm{Ker}(L_{x^3}). \tag{34}$$

Assume that $b \in \mathrm{Ker}(L_{x^3})$. If $a \in \mathrm{Ker}(L_{x^3})$ then $\mathrm{Ker}(L_{x^3}) = A$ by (7). In this case, $A\langle x^3\rangle = 0$. Therefore A is solvable by Lemma 14. If $a \notin \mathrm{Ker}(L_{x^3})$ then

$\mathrm{Ker}(L_{x^3}) = \{x, x^2, x^3, ax, (ax)x, b\}$ is a subalgebra of A by (34). It follows that A is solvable by Lemma 15.

For the rest of the proof of this lemma, we assume that $b \notin \mathrm{Ker}(L_{x^3})$.

We have

$$Ax = \langle x^2, x^3, ax, (ax)x\rangle.$$

Therefore

$$(Ax)^2 = \langle (ax)(ax), ((ax)x)(ax)\rangle$$

by (2), (7) and (15). We compute $(Ax)^3$. By (14) we have $((ax)(ax))x^2 = 0$ and $(((ax)x)(ax))x^2 = 0$. By (13) we obtain $((ax)(ax))x^3 = 0$ and $(((ax)x)(ax))x^3 = 0$. By (2) we get $((ax)(ax))((ax)x) = 0$. By (3) we get $(((ax)x)(ax))((ax)x) = -1/2(((ax)x)x)(ax)^2 = 0$. Therefore

$$(Ax)^3 = \langle (ax)^3, (((ax)x)(ax))(ax)\rangle.$$

We have that $(Ax)x = \langle x^3, (ax)x\rangle$. Then

$$((Ax)x)(Ax) = \langle ((ax)x)(ax)\rangle$$

by (2), (7) and (15). Therefore, it follows from (11) that

$$A^3x^3 \subset \langle (ax)^3, ((ax)x)(ax), (((ax)x)(ax))(ax)\rangle.$$

Our aim is to prove that $A^3\langle x^3\rangle \subset \langle x^3\rangle$.

Replacing y by b and z by a in (8) we get

$$(ab)x^3 = 0. \tag{35}$$

Now, replacing y by a and z by b in (8) we get

$$(bx^2)(ax) = 0. \tag{36}$$

Let

$$a^2 = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 a + \alpha_5 ax + \alpha_6 (ax)x + \alpha_7 b \tag{37}$$

($\alpha_i \in K$). Multiplying (37) by x^2 and using (2) we obtain $a^2x^2 = \alpha_1 x^3 + \alpha_4 ax^2 + \alpha_7 bx^2$. Then multiplying this last expression by ax, we get $(a^2x^2)(ax) = \alpha_4(ax^2)(ax)$ by (7) and (36). Since $a^2x^2 = -2(ax)(ax)$ by (3) and $(ax^2)(ax) = -1/2a^2x^3$ by (12), we obtain

$$(ax)^3 = \frac{\alpha_4}{4}a^2x^3. \tag{38}$$

Since A has nilindex 4, we obtain $(a^2a^2)x^3 = a^4x^3 = 0$. Combining (34) with (20) to expand the second factor of $(a^2a^2)x^3 = 0$ by means of (37) we obtain

$$\alpha_4 a^3x^3 = 0. \tag{39}$$

Carrying out an analogous expansion of $(a^2a)x^3$ leads to

$$a^3x^3 = \alpha_4 a^2x^3. \tag{40}$$

Assume that $\alpha_4 \neq 0$. Then $a^2x^3 = 0$ by (39) and (40), and then $(ax)^3 = \alpha_4/4a^2x^3 = 0$ by (38) and $((ax)x)(ax) = -1/4a^2x^3 = 0$ by (16). Therefore $A^3x^3 = 0$.

Assume that $\alpha_4 = 0$. By (38) we have $(ax)^3 = 0$. Let

$$(ax)a = \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 a + \beta_5 ax + \beta_6(ax)x + \beta_7 b \tag{41}$$

($\beta_i \in K$). Using (2), (5), (7), (8) and (35), we obtain from (41) that $(((ax)a)a)x^3 = \beta_4 a^2x^3$. By (1) we get $((ax)a)a = -1/2\{a^3x + (a^2x)a\}$. Therefore $\beta_4 a^2x^3 = -1/2\{(a^3x)x^3 + ((a^2x)a)x^3\}$. By (7) $(a^3x)x^3 = 0$. From (37) we obtain $(a^2x)a)x^3 = 0$ by (2), (5) and (8). Therefore

$$\beta_4 a^2 x^3 = 0. \tag{42}$$

If $\beta_4 \neq 0$ then $a^2x^3 = 0$ by (42). In this case, $A^3x^3 = 0$ as before.

We now assume that $\beta_4 = 0$. From (41) we obtain $((ax)a)x^3 = \beta_7 bx^3$ by (7). Since $((ax)a)x^3 = 0$ by (8) and (2), we obtain $\beta_7 bx^3 = 0$. Since $bx^3 \neq 0$, we obtain $\beta_7 = 0$. From (41) we obtain $((ax)a)x^2 = \beta_1 x^3$ by (2). On the other hand, by (3), $((ax)x)(ax) = -1/2((ax)a)x^2 = -1/2\beta_1 x^3$. It follows that $(((ax)x)(ax))(ax) = -1/2\beta_1 x^3(ax) = 0$ by (7). Therefore $A^3\langle x^3\rangle \subset \langle x^3\rangle$.

Therefore, in any case, we have $A^3\langle x^3\rangle \subset \langle x^3\rangle$. It follows that A is solvable by Lemma 14. $\square$

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