REPRESENTATIONS OF RANK 3 ALGEBRAS

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The class of rank 3 algebras includes the Jordan algebra of a symmetric bilinear form, the trace zero elements of a Jordan algebra of degree 3, pseudo-composition algebras, certain algebras that arise in the study of Riccati differential equations, as well as many other algebras. We investigate the representations of rank 3 algebras and show under some conditions on the eigenspaces of the left multiplication operator determined by an idempotent element that the finite-dimensional irreducible representations are all one-dimensional.

1. INTRODUCTION

In what follows $\mathbb{K}$ will denote an infinite field of characteristic not 2 or 3. A commutative (not necessarily associative) algebra A over $\mathbb{K}$ is said to have *rank 3* if there exists a linear form $\gamma_1 : A \to \mathbb{K}$ and a quadratic form $\gamma_2 : A \to \mathbb{K}$ such that

$$x^3 = \gamma_1(x)x^2 + \gamma_2(x)x \tag{1.1}$$

for all $x \in A$. Here $x^3 = x^2x = xx^2$ for every $x \in A$. We assume in such an algebra that there exists an element y such that y and y^2 are linearly independent; otherwise the rank would be two or smaller.

The class of rank 3 algebras includes many well-known examples such as the Jordan algebras associated to symmetric bilinear forms and the trace zero elements of a degree three Jordan algebra. These Jordan algebras play a prominent role in the study of Lie algebras graded by finite root systems in Benkart and Zelmanov (1996), Allison and Gao (2001) and in the structure of the core of the extended affine Lie algebras (Allison and Gao, 2001; Allison et al., 1997).

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The pseudo-composition algebras investigated in Meyberg and Osborn (1993) are rank 3 algebras which are characterized by the properties that $\gamma_1 = 0$ and $\gamma_2 \neq 0$. Springer (1959) observed that the trace zero elements of a degree 3 Jordan algebra form a pseudo-composition algebra. Such algebras also arise in the study of the Riccati differential equation $\dot{x} = x^2$ (see Walcher, 1994 for example).

Train algebras, introduced by Etherington (1940) as an algebraic framework for treating problems in genetics, are commutative algebras A satisfying the equation $x^t + \beta_1 w(x)x^{t-1} + \cdots + \beta_{t-1}w(x)^{t-1}x = 0$, where $w : A \to \mathbb{K}$ is an algebra homomorphism, $\beta_1, \ldots, \beta_{t-1} \in \mathbb{K}$, and $1 + \beta_1 + \cdots + \beta_{t-1} = 0$. The smallest value of t for which this equation is satisfied identically is called the *rank* of A. Thus, train algebras defined by an equation of the form $x^3 - (1+\delta)w(x)x^2 + \delta w(x)^2 x = 0$, $\delta \in \mathbb{K}$, and Bernstein Jordan algebras, which are special instances of train algebras of rank 3 with $\delta = 0$, provide further examples. Walcher has developed a general theory of these objects showing that most of them can be constructed from a quadratic alternative algebra or from a Clifford algebra and a representation for it. Walcher (1999) is the basic reference for background results in this article.

Here we investigate the representation theory of rank 3 algebras. We suppose that A has an idempotent $e \neq 0$. The left multiplication operator L_e determined by e has eigenvalues $1, \frac{1}{2}$, and $\alpha := \gamma_1(e) - 1$. We will assume that these three eigenvalues are distinct so that L_e is diagonalizable. Then $A = \mathbb{K}e \oplus A_{\frac{1}{2}} \oplus A_\alpha$ is the decomposition into eigenspaces relative to L_e. Any representation $\mu : A \to \mathrm{End}(M)$ decomposes into eigenspaces $M = M_{\frac{1}{2}} \oplus M_\alpha$ relative to $\mu(e)$. A key fact is Proposition 3.11 below, which states that either $A_{\frac{1}{2}} \subseteq \ker \gamma_1$ or $A_{\frac{1}{2}} \subseteq \ker \mu$ for any representation μ of A.

Among the results we establish are the following.

Theorem 1.2. *Assume $\mu : A \to \mathrm{End}(M)$ is a finite-dimensional irreducible representation of a rank 3 algebra $A = \mathbb{K}e \oplus A_{\frac{1}{2}} \oplus A_\alpha$ with $A_{\frac{1}{2}} \subseteq \ker \mu$. Then:*

(a) $\mu(e) = \frac{1}{2}I_M$ *(i.e., $M = M_{\frac{1}{2}}$) or* $\mu(e) = \alpha I_M$ *(i.e. $M = M_\alpha$);*
(b) *If $M = M_\alpha$, then* $\dim M = 1$;
(c) *If $M = M_{\frac{1}{2}}$, $\alpha \neq 0$, and $\gamma_2(x) = 0$ for all $x \in A_\alpha$, then* $\dim M = 1$;
(d) *If $M = M_{\frac{1}{2}}$ and $\mathbb{K}$ is algebraically closed, then*

$$\dim M = \begin{cases} 2^{d/2} & \text{if } d \text{ is even} \\ 2^{(d-1)/2} & \text{if } d \text{ is odd,} \end{cases}$$

where $d = \dim(A'_\alpha/\mathrm{rad}(\tau))$. Here $A'_\alpha = \{y' := y - \frac{1}{2}\gamma_1(y)e \mid y \in A_\alpha\}$ and τ is the symmetric bilinear form on A'_α defined by

$$\tau(y', z') := \frac{1}{2}q(y, z) - \frac{1}{4}\vartheta(y, z) + \frac{1}{16}\gamma_1(y)\gamma_1(z),$$

where $q(y, z) = \frac{1}{2}(\gamma_2(y+z) - \gamma_2(y) - \gamma_2(z))$ and ϑ is the symmetric bilinear form given by

$$yz = \frac{1}{2}(\gamma_1(y)z + \gamma_1(z)y) + \vartheta(y, z)e$$

for all $y, z \in A_\alpha$ (as in Remark 2.5 below).

Theorem 1.3. *Assume A is an algebra of rank 3 with an idempotent e such that both γ_1 and γ_2 are 0 on $A_{\frac{1}{2}} \oplus A_\alpha$ and $\alpha \neq 0$. Then the finite-dimensional irreducible representations $\mu : A \to \mathrm{End}(M)$ are one-dimensional and satisfy $A_{\frac{1}{2}} \oplus A_\alpha \subseteq \ker \mu$, and $\mu(e) = \frac{1}{2} I_M$ or $\mu(e) = \alpha I_M$.*

In Section 4, we investigate the case that the rank 3 algebra A has $\gamma_2 = 0$ and relate representations of such algebras to representations of Clifford algebras. In the final section, we give an explicit construction of some indecomposable two-dimensional representations.

Over an algebraically closed field, a rank 3 algebra A having no idempotent must satisfy $x^3 = \gamma_1(x)x^2$ with $\gamma_1(A^2) = 0$ or must be a train algebra with $\delta = \frac{1}{2}$ by Walcher (1994). Thus, except for these special kinds of algebras, rank 3 algebras over an algebraically closed field will possess an idempotent e, and so will fit the considerations of this article when L_e is diagonalizable.

2. PRELIMINARIES

Throughout this work we will assume that A is an algebra over an infinite field $\mathbb{K}$ of characteristic not 2 or 3 satisfying equation (1.1). Associated to the quadratic form $\gamma_2(x)$ is the symmetric bilinear form denoted by q and defined by

$$q(x, y) = \frac{1}{2}(\gamma_2(x + y) - \gamma_2(x) - \gamma_2(y)). \tag{2.1}$$

The following identity from Walcher (1999, R5) relates the values of γ_1, γ_2 and q:

$$2q(x, x^2)\gamma_1(x) - \gamma_2(x^2) = (\gamma_1(x^2) - \gamma_2(x))\gamma_2(x). \tag{2.2}$$

The class defined by (1.1) is too large for our purposes. We impose some additional restrictions; in particular, we suppose that A has an idempotent $e \neq 0$. Note that $\gamma_2(e) = 1 - \gamma_1(e)$ follows from (2.2). Walcher (1999, Prop. 1.3) has shown that the left multiplication operator L_e determined by e satisfies the polynomial identity

$$p(t) = 2t^3 - (1 + 2\gamma_1(e))t^2 + (3\gamma_1(e) - 2)t + 1 - \gamma_1(e), \tag{2.3}$$

so that the eigenvalues of L_e are $1, \frac{1}{2}$, and $\alpha := \gamma_1(e) - 1 = -\gamma_2(e)$. We will assume that these three eigenvalues are distinct so that L_e is diagonalizable.

Proposition 2.4 (Walcher, 1999, Prop. 1.4). *If A is an algebra of rank 3 satisfying (1.1) and e is an idempotent of A such that L_e has distinct eigenvalues, then the following assertions hold:*

(i) $A = \mathbb{K}e \oplus A_{\frac{1}{2}} \oplus A_\alpha$ *where* $A_\lambda = \{x \in A \mid ex = \lambda x\}$ *and* $\lambda = \frac{1}{2}, \alpha$;
(ii) $\gamma_1(y) + 2q(e, y) = 0$ *for all* $y \in A_\lambda, \lambda = \frac{1}{2}, \alpha$;
(iii) *If* $y, z \in A_{\frac{1}{2}}$, *then* $yz - q(y, z)(1 - \alpha)^{-1}e \in A_\alpha$;
(iv) *If* $y \in A_{\frac{1}{2}}$, *and* $z \in A_\alpha$, *then* $yz - \gamma_1(y)z - q(y, z)e \in A_{\frac{1}{2}}$;

(v) *If $y, z \in A_\alpha$, then*

$$yz = \frac{1}{2}(\gamma_1(y)z + \gamma_1(z)y) + q(y,z)\alpha^{-1}e \qquad \text{if } \alpha \neq 0$$

$$yz - \frac{1}{2}(\gamma_1(y)z + \gamma_1(z)y) \in \mathbb{K}e \qquad \text{and} \qquad q(A_\alpha, A_\alpha) = 0 \quad \text{if } \alpha = 0.$$

In particular, $\mathbb{K}e \oplus A_\alpha$ is a subalgebra of A.

Remark 2.5. From part (v) of this result we see that there is a symmetric bilinear form ϑ defined on A_α so that

$$yz = \frac{1}{2}(\gamma_1(y)z + \gamma_1(z)y) + \vartheta(y,z)e \tag{2.6}$$

for all $y, z \in A_\alpha$, and $\vartheta(y,z) = q(y,z)\alpha^{-1}$ when $\alpha \neq 0$.

Lemma 2.7. *Assume A is a rank 3 algebra as in Proposition 2.4 with $A_\alpha \subseteq \ker \gamma_2$. Then either $A_\alpha \subseteq \ker \gamma_1$ or there exists an idempotent $f \in A$ such that the multiplication operator L_f has eigenvalues $1, \frac{1}{2}, 0$.*

Proof. We may assume $\alpha \neq 0$, as otherwise the idempotent e defining A_α will have eigenvalues $1, \frac{1}{2}, 0$. If $A_\alpha \not\subseteq \ker \gamma_1$, then there is an element $f \in A_\alpha$ with $\gamma_1(f) = 1$. Now by Proposition 2.4 (v), $yz = \frac{1}{2}(\gamma_1(y)z + \gamma_1(z)y)$ for all $y, z \in A_\alpha$. In particular, $f^2 = f$. By (2.3), the eigenvalues of f are $1, \frac{1}{2}$, and $\gamma_1(f) - 1 = 0$. □

3. REPRESENTATIONS

Assume A is a rank 3 algebra over $\mathbb{K}$ satisfying $x^3 = \gamma_1(x)x^2 + \gamma_2(x)x$ for γ_1 a linear form and γ_2 a quadratic form, and let M be a $\mathbb{K}$-vector space. Following Eilenberg (1948), we say that a linear map $\mu : A \to \mathrm{End}(M)$ is a *representation of A* if the split null extension $A \oplus M$ of M, with multiplication defined by $(a+m)(b+n) = ab + \mu(a)n + \mu(b)m$, $a, b \in A$, $m, n \in M$, also satisfies (1.1), where $\gamma_1(a+m) = \gamma_1(a)$ and $\gamma_2(a+m) = \gamma_2(a)$ are the extensions of the given forms on A. The associated symmetric bilinear form satisfies $q(a,m) = 0 = q(m,n)$ for all $a \in A$, $m, n \in M$. It is easy to verify the following result.

Proposition 3.1. *A linear map $\mu : A \to \mathrm{End}(M)$ is a representation of A if and only if for every $a \in A$,*

$$2\mu(a)^2 + \mu(a^2) = 2\gamma_1(a)\mu(a) + \gamma_2(a)I_M. \tag{3.2}$$

The linearization of this equation gives

$$\mu(a)\mu(b) + \mu(b)\mu(a) + \mu(ab) = \gamma_1(a)\mu(b) + \gamma_1(b)\mu(a) + q(a,b)I_M \tag{3.3}$$

for all $a, b \in A$. Now by (3.2),

$$2\mu(e)^2 + \mu(e) = 2\gamma_1(e)\mu(e) + (1 - \gamma_1(e))I_M, \tag{3.4}$$

so that $\mu(e)$ has eigenvalues $\frac{1}{2}$ and $\alpha = \gamma_1(e) - 1 = -\gamma_2(e) = -q(e, e)$ on M. Since $A \oplus M$ also satisfies (1.1), we may deduce the next result from Proposition 3.1.

Proposition 3.5. *For $A \oplus M$ the following relations hold*:

(i) $A_{\frac{1}{2}} M_{\frac{1}{2}} \subseteq M_\alpha$ *and* $A_{\frac{1}{2}} M_\alpha \subseteq M_{\frac{1}{2}}$;
(ii) $A_\alpha M_{\frac{1}{2}} \subseteq M_{\frac{1}{2}}$;
(iii) *For all* $a \in A_\alpha$, $m \in M_\alpha$, $am = \frac{1}{2}\gamma_1(a)m$ *so that* $A_\alpha M_\alpha \subseteq M_\alpha$.

Proposition 3.6. *If A has rank* 3 *and* $\mu : A \to \mathrm{End}(M)$ *is a representation of A, then for all $a \in A$,*

$$4\mu(a)^3 - 4\gamma_1(a)\mu(a)^2 + (\gamma_1(a^2) - 3\gamma_2(a))\mu(a) + q(a, a^2)I_M = 0. \tag{3.7}$$

Proof. Start with the relation $2\mu(a)^2 + \mu(a^2) = 2\gamma_1(a)\mu(a) + \gamma_2(a)I_M$, and multiply it on the left by $\mu(a)$ to obtain

$$2\mu(a)^3 + \mu(a)\mu(a^2) = 2\gamma_1(a)\mu(a)^2 + \gamma_2(a)\mu(a). \tag{3.8}$$

Similarly, multiply on the right by $\mu(a)$ and subtract the result from (3.8) to get

$$\mu(a)\mu(a^2) = \mu(a^2)\mu(a). \tag{3.9}$$

Using this identity in equation (3.3), we have for all $a \in A$,

$$2\mu(a)\mu(a^2) + \mu(a^3) = \gamma_1(a)\mu(a^2) + \gamma_1(a^2)\mu(a) + q(a, a^2)I_M. \tag{3.10}$$

Now replace a^3 by $\gamma_1(a)a^2 + \gamma_2(a)a$ in this expression and use (3.8) to obtain (3.7). $\square$

Proposition 3.11. *Assume A is a rank* 3 *algebra. Then either* $A_{\frac{1}{2}} \subseteq \ker\gamma_1$ *or* $A_{\frac{1}{2}} \subseteq \ker\mu$ *for any representation* $\mu : A \to \mathrm{End}(M)$.

Proof. We have from (3.2), $2\mu(a)^2 + \mu(a^2) - \gamma_2(a)I_M = 2\gamma_1(a)\mu(a)$. When $a \in A_{\frac{1}{2}}$, then by Proposition 3.5 the left-hand side maps $M_{\frac{1}{2}}$ to $M_{\frac{1}{2}}$, and M_α to M_α, while the right-hand side sends $M_{\frac{1}{2}}$ to M_α and M_α to $M_{\frac{1}{2}}$. Therefore, $\gamma_1(a)\mu(a) = 0$ for all $a \in A_{\frac{1}{2}}$.

If $A_{\frac{1}{2}} \not\subseteq \ker\gamma_1$, then $\gamma_1(b) \neq 0$ and $\mu(b) = 0$ for some $b \in A_{\frac{1}{2}}$. But then for any $c \in A_{\frac{1}{2}}$,

$$0 = \gamma_1(b + c)\mu(b + c) = \gamma_1(b)\mu(c),$$

so that $\mu(c) = 0$. $\square$

Now we tackle the proofs of the theorems in the introduction. Recall that a representation $\mu : A \to \mathrm{End}(M)$ is *irreducible* if $M \neq 0$ and there is no proper subspace of M which is invariant under all the transformations $\mu(a)$, $a \in A$, and it is *r-dimensional* if $\dim M = r$.

Proof of Theorem 1.2. Throughout we assume $\mu : A \to \mathrm{End}(M)$ is a finite-dimensional irreducible representation of a rank 3 algebra $A = \mathbb{K}e \oplus A_{\frac{1}{2}} \oplus A_\alpha$ with $A_{\frac{1}{2}} \subseteq \ker \mu$.

(a) Because $\mu(A) = \mathbb{K}\mu(e) + \mu(A_\alpha)$, it follows from Proposition 3.5 that $\mu(A)$ leaves invariant the spaces $M_{\frac{1}{2}}$ and M_α. The irreducibility of M forces $M = M_{\frac{1}{2}}$ or $M = M_\alpha$ to hold.

(b) By (iii) of Proposition 3.5, $\mu(a)m = \frac{1}{2}\gamma_1(a)m$ for all $m \in M_\alpha$, $a \in A_\alpha$. Thus when $M = M_\alpha$ and $A_{\frac{1}{2}} \subseteq \ker \mu$, $\mathbb{K}m$ is invariant under $\mu(A)$ for any $m \in M_\alpha$. By irreducibility, $M = \mathbb{K}m$ for $m \neq 0$.

(c) Suppose now that $M = M_{\frac{1}{2}}$. Because $A_{\frac{1}{2}} \subseteq \ker \mu$ and $\mu(e) = \frac{1}{2}I_M$, the representation is completely determined by the action of A_α. Applying μ to the expression for yz in (2.6), we have that

$$\mu(yz) = \frac{1}{2}(\gamma_1(y)\mu(z) + \gamma_1(z)\mu(y)) + \frac{1}{2}\vartheta(y, z)I_M \tag{3.12}$$

for all $y, z \in A_\alpha$, (where $\vartheta(y, z) = q(y, z)\alpha^{-1}$ if $\alpha \neq 0$). Combining that with (3.3) shows that

$$\mu(y)\mu(z) + \mu(z)\mu(y) = \frac{1}{2}(\gamma_1(y)\mu(z) + \gamma_1(z)\mu(y)) + (q(y, z) - \frac{1}{2}\vartheta(y, z))I_M \tag{3.13}$$

for all $y, z \in A_\alpha$.

Suppose $\alpha \neq 0$ and $\gamma_2(y) = 0$ for all $y \in A_\alpha$. Then $q(y, z) = 0$, and hence $\vartheta(y, z) = 0$, for all $y, z \in A_\alpha$. Relation (3.13) becomes

$$\mu(y)\mu(z) + \mu(z)\mu(y) = 1/2(\gamma_1(y)\mu(z) + \gamma_1(z)\mu(y)) \tag{3.14}$$

for all $y, z \in A_\alpha$.

Let $B = A_\alpha \cap \ker \gamma_1$. Then from (3.14) it follows that $\mu(b)\mu(b') + \mu(b')\mu(b) = 0$ for $b, b' \in B$. Thus, $\mu(B)$ is a weakly closed set of transformations on M in the sense of Jacobson (1962, Chap. 2). Moreover, $\mu(b)^2 = 0$ for all $b \in B$. Therefore by Jacobson (1962, Chap. 2, Thm. 1′), there exists a basis for M such that $\mu(b)$ is strictly upper triangular for all $b \in B$. Consequently, $N := \{m \in M \mid \mu(b)m = 0$ for all $b \in B\} \neq 0$.

Now when $B = A_\alpha$, we must have $M = \mathbb{K}n$ for any nonzero $n \in N$ since M is irreducible, so that M is one-dimensional.

When $B \neq A_\alpha$, we may choose $a \in A_\alpha$ with $\gamma_1(a) = 1$. By (3.2), $2\mu(a)^2 + \mu(a^2) = 2\gamma_1(a)\mu(a) = 2\mu(a)$. But by (3.12), $\mu(a^2) = \mu(a)$ must hold since $\vartheta(a, a) = 0$. Therefore $\mu(a)^2 = \frac{1}{2}\mu(a)$. Taking a nonzero $n \in N$, we see that $\mathbb{K}\mu(a)n$ is invariant under the transformation $\mu(a)$. In addition, if $b \in B$, then $\mu(b)\mu(a)n = -\mu(a)\mu(b)n + \frac{1}{2}\gamma_1(a)\mu(b)n = 0$. By irreducibility, $M = \mathbb{K}\mu(a)n$ and M is one-dimensional.

(d) More generally, under the assumption that $M = M_{\frac{1}{2}}$ (but not requiring $\alpha \neq 0$ or $\gamma_2(x) = 0$ for all $x \in A_\alpha$), suppose that

$$A'_\alpha = \left\{ y' := y - \frac{1}{2}\gamma_1(y)e \,\middle|\, y \in A_\alpha \right\}.$$

Then for $y, z \in A_\alpha$ we have by (3.13) above that

$$\begin{aligned}\mu(y')\mu(z') + \mu(z')\mu(y') &= \mu(y)\mu(z) + \mu(z)\mu(y) \\ &\quad - \frac{1}{2}(\gamma_1(y)\mu(z) + \gamma_1(z)\mu(y)) + \frac{1}{8}\gamma_1(y)\gamma_1(z)I_M \\ &= 2\tau(y', z')I_M, \end{aligned} \tag{3.15}$$

where

$$\tau(y', z') := \frac{1}{2}q(y, z) - \frac{1}{4}\vartheta(y, z) + \frac{1}{16}\gamma_1(y)\gamma_1(z).$$

If $R = \mathrm{rad}(\tau)$ is the radical of the form τ on A'_α, then $\mu(R)$ is a weakly closed set of nilpotent transformations on M by (3.15). As a consequence, $N := \{m \in M \mid \mu(z')m = 0 \text{ for all } z' \in R\}$ is nonzero. Since $\mu(z')\mu(y')m = -\mu(y')\mu(z')m = 0$ for all $m \in N$, $y' \in A'_\alpha$, and $z' \in R$, it follows that N is a nonzero A-invariant subspace of M. By irreducibility, $M = N$.

Let V be a vector space complement of R in A'_α ($A'_\alpha = R \oplus V$). Then the representation μ is completely determined by its restriction $\mu : V \to \mathrm{End}(M)$ to V, and this restriction is injective by (3.15) and the nondegeneracy of τ on V. Indeed, if $\mu(y') = 0$ for some $y' \in V$, then from (3.15) it follows that $y' \in R \cap V = 0$. By (3.15) and the universal property of the Clifford algebra (see for example, Jacobson, 1980, Sec. 4.8 for basic facts about Clifford algebras), there is a representation $\hat{\mu} : \mathscr{C}(V, \tau) \to \mathrm{End}(M)$ extending μ to the Clifford algebra determined by V. This Clifford algebra is semisimple by the nondegeneracy of τ on V.

When $\mathbb{K}$ is algebraically closed and $d = \dim V = \dim(A'_\alpha/\mathrm{rad}(\tau))$,

$$\mathscr{C}(V, \tau) \cong \begin{cases} \mathscr{M}_r(\mathbb{K}) & \text{if } d \text{ is even and } r = 2^{d/2} \\ \mathscr{M}_r(\mathbb{K}) \oplus \mathscr{M}_r(\mathbb{K}) & \text{if } d \text{ is odd and } r = 2^{(d-1)/2}. \end{cases}$$

Because a matrix algebra $\mathscr{M}_r(\mathbb{K})$ has a unique (up to isomorphism) irreducible representation (which has dimension r), the assertion in (d) follows. □

Example. Let $A = \mathbb{K}1 \oplus V$ be a Jordan algebra of a symmetric $\mathbb{K}$-bilinear form $(,)$. Thus the product in A is given by

$$(\alpha + v)(\beta + w) = \alpha\beta + (v, w) + \alpha w + \beta v.$$

It is easy to verify for $x = \alpha + v$ that $x^3 = \gamma_1(x)x^2 + \gamma_2(x)x$, where

$$\gamma_1(x) = 2\alpha \quad \text{and} \quad \gamma_2(x) = (v, v) - \alpha^2.$$

Taking u so that $(u, u) = 1/4$, we see that $e = 1/2 + u$ is an idempotent, and relative to L_e, the algebra A has the decomposition $A = \mathbb{K}e \oplus A_{1/2} \oplus A_0$, where $A_{1/2} = \{v \in V \mid (u, v) = 0\}$, and $A_0 = \mathbb{K}f$ for $f = 1 - e = 1/2 - u$. It is clear that $A_{1/2} \subseteq \ker\gamma_1$. Now suppose that $\mu : A \to \mathrm{End}(M)$ is an irreducible representation with $A_{1/2} \subseteq \ker\mu$. According to Theorem 1.2, if $M = M_0$, then $\dim M = 1$. If $M = M_{1/2}$ and $\mathbb{K}$ is algebraically closed, then $\dim M$ depends on the following.

Set $A'_0 = \{y' = y - 1/2\gamma_1(y)e \mid y \in A_0\}$. Let τ be the symmetric bilinear form on A'_0 defined in (d) of Theorem 1.2. Notice $q(f, f) = 0$ and $\vartheta(f, f) = 0$, as f is an idempotent. Thus, $\tau(f', f') = 1/16\gamma_1(f)\gamma_1(f) = 1/16$. So we see that $A'_0/\mathrm{rad}(\tau)$ is one-dimensional.

Therefore, any irreducible A-module with $A_{1/2} \subseteq \ker\mu$ and $M = M_{1/2}$ will be one-dimensional by Theorem 1.2. Any irreducible A-module with $A_{1/2} \subseteq \ker\mu$ and $M = M_0$ likewise will be one-dimensional.

Proof of Theorem 1.3. Under the hypothesis that γ_1 and γ_2 are 0 on $A_{\frac{1}{2}} \oplus A_\alpha$ and $\alpha \neq 0$, we have for a finite-dimensional irreducible representation $\mu : A \to \mathrm{End}(M)$ that $\mu(a)\mu(b) + \mu(b)\mu(a) = -\mu(ab) \in \mu(B)$ for all $a, b \in B := A_{\frac{1}{2}} \oplus A_\alpha$ by (3.3) and Proposition 2.4. Thus, $\mu(B)$ is a weakly closed set of transformations of M. Moreover, by (3.7), $\mu(a)^3 = 0$ for all $a \in B$. Therefore, there exists a basis for M such that $\mu(a)$ is strictly upper triangular for all $a \in B$. Consequently, $N := \{m \in M \mid \mu(a)m = 0 \text{ for all } a \in B\} \neq 0$. Now $M = M_{\frac{1}{2}} \oplus M_\alpha$, and the relations in Proposition 3.5 show that if $m = m_{\frac{1}{2}} + m_\alpha \in N$, where $m_{\frac{1}{2}} \in M_{\frac{1}{2}}$, $m_\alpha \in M_\alpha$, then $m_{\frac{1}{2}}, m_\alpha \in N$. Thus, if $0 \neq m_{\frac{1}{2}} \in N \cap M_{\frac{1}{2}}$, then by irreducibility $\mathbb{K}m_{\frac{1}{2}} = M$. As a similar result holds if $N \cap M_\alpha \neq 0$, it must be that M is one-dimensional; $B = \ker\mu$; and either $\mu(e) = \frac{1}{2}I_M$ or $\mu(e) = \alpha I_M$. ☐

4. THE $\gamma_2 = 0$ CASE

Here we will discuss representations of rank 3 algebras having an idempotent element e such that L_e has eigenvalue zero. Such algebras are closely connected with Clifford algebras as we will see below. In particular, if A is a rank 3 algebra with $\gamma_2 = 0$, then according to the next result, the possible eigenvalues of any idempotent in A are $1, \frac{1}{2}, 0$.

Theorem 4.1. *Let A be a rank* 3 *algebra with $\gamma_2 = 0$. Then the following hold*:

(i) *A has an idempotent e if and only if $\gamma_1(A^2) \neq 0$;*
(ii) *For every idempotent $e \in A$, $\gamma_1(e) = 1$. Thus, the Peirce decomposition relative to e is given by $A = \mathbb{K}e \oplus A_{\frac{1}{2}} \oplus A_0$ and $\gamma_1(A_0 \oplus A_{\frac{1}{2}}) = 0$. There is a symmetric bilinear form σ on A_0 such that $yz = -4\sigma(y, z)e$ for all $y, z \in A_0$;*
(iii) *Assume $\mu : A \to \mathrm{End}(M)$ is a finite-dimensional irreducible representation of A.*
  (a) *If $\mu(e) = 0$ for some idempotent e of A, then $\dim M = 1$.*
  (b) *Assume $\mu(f) \neq 0$ for any idempotent f, and let $A = \mathbb{K}e \oplus A_{\frac{1}{2}} \oplus A_0$ be the Peirce decomposition of A relative to some idempotent e. Set $d = \dim(A_0/\mathrm{rad}(\sigma))$, where σ is as in* (ii)*. Then if $\mathbb{K}$ is algebraically closed,*

$$\dim M = \begin{cases} 2^{d/2} & \text{if } d \text{ is even} \\ 2^{(d-1)/2} & \text{if } d \text{ is odd.} \end{cases}$$

(*In particular, d is independent of the idempotent chosen.*)

Proof. (i) By Walcher (1999, (R4)),

$$(x^2)^2 = (\gamma_1(x^2) - \gamma_2(x))x^2 + 2\gamma_2(x, x^2)x = \gamma_1(x^2)x^2,$$

so part (i) is an immediate consequence.

(ii) This is (a) of Proposition 5.3 of Walcher (1999).

(iii) Suppose e is an idempotent of A. By part (ii), $A = \mathbb{K}e \oplus A_{\frac{1}{2}} \oplus A_0$, $\gamma_1(e) = 1$, and γ_1 on $A_{\frac{1}{2}} \oplus A_0$ is 0. Let y, z be any two elements of $A_{\frac{1}{2}} \oplus A_0$. Then by (ii) and (3.3), we have $\mu(y)\mu(z) + \mu(z)\mu(y) + \mu(yz) = 0$ for any representation $\mu : A \to \mathrm{End}(M)$ of A. Let σ be as in (ii), and set $R = \mathrm{rad}(\sigma) \subseteq A_0$. Suppose S is a subspace such that $A_0 = R \oplus S$. By Walcher (1999, Cor. 3.4), $B := R \oplus A_{\frac{1}{2}}$ is an ideal of A. Then for $b, b' \in B$, $\mu(b)\mu(b') + \mu(b')\mu(b) = -\mu(bb') \in \mu(B)$. Moreover, $4\mu(b)^3 = 4\gamma_1(b)\mu(b)^2 - (\gamma_1(b^2) - 3\gamma_2(b))\mu(b) - q(b, b^2)I_M = 0$ by (3.7). Thus, $N := \{v \in M \mid \mu(b)v = 0 \text{ for all } b \in B\} \neq 0$. It is easy to see from Proposition 3.5 that $N = N_{\frac{1}{2}} \oplus N_0$ where $N_\lambda = N \cap M_\lambda$ for $\lambda = \frac{1}{2}, 0$. Now for any $a \in A_0$, $v \in N_\lambda$, and $b \in B$, we have

$$\mu(b)\mu(a)v = -\mu(a)\mu(b)v - \mu(ab)v = 0,$$

which implies that $\mu(a)v \in N$. But since $A_0 M_\lambda \subseteq M_\lambda$ for $\lambda = \frac{1}{2}, 0$, we have $\mu(a)v \in N_\lambda$. Thus, if $N_\lambda \neq 0$, $M = N_\lambda$ by irreducibility and $\mu(B) = 0$. It follows from (ii) that $\mu(a)\mu(a') + \mu(a')\mu(a) = -\mu(aa') = 4\sigma(a, a')\mu(e)$ for all $a, a' \in A_0$. If $M = N_0$, then $\mu(e) = 0$, which is impossible by our assumptions. The other option is that $\mu(e) = \frac{1}{2}I_M$ (i.e., $M = N_{\frac{1}{2}}$). Here $\mu(a)\mu(a') + \mu(a')\mu(a) = -\mu(aa') = 2\sigma(a, a')I_M$ for all $a, a' \in A_0$. Note that $\mu(R) = 0$ and $\mu : S \to \mathrm{End}(M)$ is injective by the nondegeneracy of σ on S. Thus, there is a extension $\hat{\mu} : \mathscr{C}(S, \sigma) \to \mathrm{End}(M)$ of the representation μ to the Clifford algebra determined by S and σ. The Clifford algebra $\mathscr{C}(S, \sigma)$ is semisimple since σ is nondegenerate on S. So when $\mathbb{K}$ is algebraically closed and $d = \dim S = \dim(A_0/\mathrm{rad}(\sigma))$,

$$\mathscr{C}(S, \sigma) \cong \begin{cases} \mathscr{M}_r(\mathbb{K}) & \text{if } d \text{ is even and } r = 2^{d/2} \\ \mathscr{M}_r(\mathbb{K}) \oplus \mathscr{M}_r(\mathbb{K}) & \text{if } d \text{ is odd and } r = 2^{(d-1)/2}. \end{cases}$$

This implies that M must have dimension r, as claimed. □

5. INDECOMPOSABLE EXAMPLES

In this section we impose the restriction that γ_1 and γ_2 are 0 on $A_{\frac{1}{2}} \oplus A_\alpha$. This is true, for example, when A is a train algebra of rank 3 or when A satisfies (1.1) with $\gamma_2 = 0$ and $\gamma_1(A^2) \neq 0$. (See Proposition 5.3 of Walcher, 1999 and the results of the previous section.) We give an explicit construction of some indecomposable representations for A in this setting.

Proposition 5.1. *Assume A is a rank 3 algebra such that both γ_1 and γ_2 are 0 on $A_{\frac{1}{2}} \oplus A_\alpha$ and $\alpha \neq 0$. If there exists an element $b \in A_\alpha \backslash A_{\frac{1}{2}}^2$ or an element $b \in A_{\frac{1}{2}} \backslash A_\alpha A_{\frac{1}{2}}$, then there is a indecomposable two-dimensional representation $\mu : A \to \mathrm{End}(M)$ of A with $\mu(b) \neq 0$.*

Proof. (1) Assume $b \in A_\alpha \backslash A_{\frac{1}{2}}^2$. Let $Z = A'_\alpha \oplus A_{\frac{1}{2}}$, where A'_α is a subspace of A_α satisfying $A'_\alpha \supseteq A_{\frac{1}{2}}^2$ and $A_\alpha = \mathbb{K}b \oplus A'_\alpha$. If $B := A_{\frac{1}{2}} \oplus A_\alpha$, then $B^2 \subseteq Z$ by Proposition 2.4. Now we assign to the element $a = \zeta e + \eta b + z$ (where $z \in Z$) a 2×2

matrix $\mu(a)$ defined by

$$\mu(a) = \begin{pmatrix} \frac{1}{2}\zeta & \eta \\ & 0 & \frac{1}{2}\zeta \end{pmatrix}.$$

Because $a^2 = \zeta^2 e + 2\alpha\zeta\eta b + z'$ where $z' \in Z$, we have

$$\begin{aligned} &2\mu(a)^2 + \mu(a^2) - 2\gamma_1(a)\mu(a) - \gamma_2(a)I_M \\ &= \begin{pmatrix} \frac{1}{2}\zeta^2 & 2\zeta\eta \\ 0 & \frac{1}{2}\zeta^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\zeta^2 & 2\alpha\zeta\eta \\ 0 & \frac{1}{2}\zeta^2 \end{pmatrix} - 2\zeta\gamma_1(e)\begin{pmatrix} \frac{1}{2}\zeta & \eta \\ 0 & \frac{1}{2}\zeta \end{pmatrix} - \begin{pmatrix} \zeta^2\gamma_2(e) & 0 \\ 0 & \zeta^2\gamma_2(e) \end{pmatrix}. \end{aligned}$$

Now using the relation $\gamma_1(e) = 1 - \gamma_2(e)$ along with $\alpha = \gamma_1(e) - 1 = -\gamma_2(e)$, we see that this expression is zero. Thus, μ determines a representation of A on a two-dimensional space M. If $\{m, n\}$ is the basis of M giving this matrix realization, then $\mathbb{K}m$ is invariant under $\mu(A)$, and it is the unique invariant proper subspace of M. Thus, M is an indecomposable representation of A.

(2) Here we proceed analogously. Assume $b \in A_{\frac{1}{2}} \backslash A_\alpha A_{\frac{1}{2}}$, and let $W = A'_{\frac{1}{2}} \oplus A_\alpha$, where $A_{\frac{1}{2}} \supset A'_{\frac{1}{2}} \supseteq A_\alpha A_{\frac{1}{2}}$ and $A_{\frac{1}{2}} = \mathbb{K}b \oplus A'_{\frac{1}{2}}$. For $a = \zeta e + \eta b + w \in A = \mathbb{K}e \oplus \mathbb{K}b \oplus W$, set

$$\mu(a) = \begin{pmatrix} \frac{1}{2}\zeta & \eta \\ 0 & \alpha\zeta \end{pmatrix}.$$

Then μ determines an indecomposable two-dimensional representation as

$$\begin{aligned} &2\mu(a)^2 + \mu(a^2) - 2\gamma_1(a)\mu(a) - \gamma_2(a)I_M \\ &= \begin{pmatrix} \frac{1}{2}\zeta^2 & \zeta\eta + 2\alpha\zeta\eta \\ 0 & 2\alpha^2\zeta^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\zeta^2 & \zeta\eta \\ 0 & \alpha\zeta^2 \end{pmatrix} - 2\zeta\gamma_1(e)\begin{pmatrix} \frac{1}{2}\zeta & \eta \\ 0 & \alpha\zeta \end{pmatrix} - \begin{pmatrix} \zeta^2\gamma_2(e) & 0 \\ 0 & \zeta^2\gamma_2(e) \end{pmatrix} = 0. \end{aligned}$$

$\square$

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