STATIONARY PROCESSES WHOSE FILTRATIONS ARE STANDARD

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We study the standard property of the natural filtration associated to a 0–1 valued stationary process. In our main result we show that if the process has summable memory decay, then the associated filtration is standard. We prove it by coupling techniques. For a process whose associated filtration is standard, we construct a product type filtration extending it, based upon the usual couplings and the Vershik's criterion for standardness.

1. Introduction and notation. Let $(X_n : n \leq 0)$ be a $\{0,1\}$-valued stationary process and $\mathcal{F}^X = (\mathcal{F}_n^X : n \leq 0)$ be its natural filtration, so $\mathcal{F}_n^X = \sigma(X_m; m \leq n)$.

Definition 1. A filtration $\mathcal{F}$ is *standard* if it can be immersed on a filtration of diffusive product type (see [6, 7, 8, 15, 16]).

A necessary condition for $\mathcal{F}$ to be standard is that its tail $\mathcal{F}_{-\infty} = \bigcap_{n \leq 0} \mathcal{F}_n$ is trivial. But, as is shown by a counterexample in [15, 16], this condition is not sufficient.

In our main result we show that if $(X_n : n \leq 0)$ has (a slightly weaker condition than) summable memory decay, then $\mathcal{F}^X$ is standard. This is done in Theorem 3 of Section 3. For the proof, we construct explicitly a filtration $\mathcal{G} = (\mathcal{G}_n : n \leq 0)$, where $\mathcal{F}^X$ is immersed, and further, we show it is of diffusive product type. That is, there exists a sequence of i.i.d. uniform r.v.'s $(W_n : n \leq 0)$ such that $\mathcal{G} = \mathcal{F}^W$.

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To be more precise, let $\Sigma=\{0,1\}^{-\mathbb{N}}$ be endowed with the law of $(X_n:n\leq 0)$. Let $(V_n:n\leq 0)$ be a sequence of i.i.d. r.v.'s uniformly distributed on $[0,1]$, independent of $\mathcal{F}^X$. We endow $[0,1]^{-\mathbb{N}}$ with the law of $(V_n:n\leq 0)$ and we fix the probability space $(\Omega,\mathcal{A},\mathbb{P})$ as the product of above spaces, so $\mathbb{P}$ is the product of the laws of $(X_n:n\leq 0)$ and $(V_n:n\leq 0)$. On the other hand, the filtration $\mathcal{G}=(\mathcal{G}_n:n\leq 0)$ is given by $\mathcal{G}_n=\sigma(X_m,V_m:m\leq n)$. Clearly, $\mathcal{F}^X$ is immersed in $\mathcal{G}$ (see [6]). The above mentioned sequence $(W_n:n\leq 0)$ is constructed in Section 2.

The class of processes with summable memory decay has been studied in relation with regenerative representations and perfect simulation algorithms, in particular, see [2, 3, 5, 9]. Gibbs measures with Hölder potentials on fullshifts are examples of measures with summable memory decay (see [1, 13]); a rich discussion and a detailed list of relevant references on this class of measures can be found in [3, 9].

In Section 4 we assume $\mathcal{F}^X$ is standard and we construct an explicit diffusive product type extension $\mathcal{F}^U$ of $\mathcal{F}^X$.

2. An independent sequence. Let $n\leq 0$. We define $f_n=\mathbb{P}(X_n=0|\mathcal{F}^X_{n-1})$ and

$$W_n=f_nV_n\mathbf{1}(X_n=0)+(1-(1-f_n)V_n)\mathbf{1}(X_n=1), \tag{1}$$

where $\mathbf{1}(X_n=i)$ denotes the characteristic function of the event $\{X_n=i\}$, for $i=0,1$.

LEMMA 2. *$(W_n:n\leq 0)$ is a sequence of i.i.d. r.v.'s uniformly distributed in $[0,1]$. Moreover, for all $n\leq 0$, W_n is independent of $\mathcal{G}_{n-1}$, $\mathcal{G}_{n-1}\vee\sigma(W_n)=\mathcal{G}_n$, and $\mathcal{F}^X_{n-1}\vee\sigma(W_n)=\mathcal{F}^X_n\vee\sigma(V_n)$.*

PROOF. First recall the following relation. Let f, V and Z be real bounded measurable functions and $\mathcal{B}$ be a sub σ-field such that f is $\mathcal{B}$-measurable and V is independent of $\mathcal{B}\vee\sigma(Z)$. Then, for any Borel real bounded function h, it holds $\mathbb{E}(h(fV)Z|\mathcal{B})(\omega)=\mathbb{E}(Z|\mathcal{B})(\omega)\int h(f(\omega)v)\,dF_V(v)$ a.s. in ω, where F_V is the distribution function of V.

Therefore, since f_n is $\mathcal{G}_{n-1}$-measurable and V_n is independent from $\mathcal{G}_{n-1}\vee\sigma(X_n)$, for every Borel real bounded measurable function h, it holds

$$\mathbb{E}(h(W_n)|\mathcal{G}_{n-1})=\int_0^1 h(f_nv)\,dv\cdot f_n+\int_0^1 h(1-(1-f_n)v)\,dv\cdot(1-f_n),$$

where we have also used $\mathbb{P}(X_n=0|\mathcal{G}_{n-1})=\mathbb{P}(X_n=0|\mathcal{F}^X_{n-1})$. The changes of variables $y=f_nv$ and $z=1-(1-f_n)v$ yield

$$\mathbb{E}(h(W_n)|\mathcal{G}_{n-1})=\int_0^{f_n} h(y)\,dy+\int_{f_n}^1 h(z)\,dz=\int_0^1 h(v)\,dv.$$

Then W_n is independent of $\mathcal{G}_{n-1}$ and it is uniformly distributed in $[0,1]$. The other statements follow from the equalities

$$X_n = \mathbf{1}(W_n > f_n) \text{ and } V_n = \frac{W_n}{f_n}\mathbf{1}(W_n \le f_n) + \frac{1-W_n}{1-f_n}\mathbf{1}(W_n > f_n). \tag{2}$$ □

Lemma 2 shows that $\mathcal{G}$ is the natural filtration of (X,W) and that $(W_n : n \le 0)$ is a sequence of independent increments for this filtration. Thus, it is direct to prove that $\mathcal{G} = \mathcal{F}^W \Leftrightarrow \mathcal{G}_0 = \mathcal{F}_0^W \Leftrightarrow \mathcal{F}_0^X \subseteq \mathcal{F}_0^W$. Therefore, $\mathcal{F}_0^X \subseteq \mathcal{F}_0^W$ is a sufficient condition for $\mathcal{G} = \mathcal{F}^W$ to be of product type, and thus, for $\mathcal{F}^X$ to be standard.

Now, the condition $\mathcal{F}_0^X \subseteq \mathcal{F}_0^W$ is not always fulfilled, even if the tail σ-field $\mathcal{F}_{-\infty}^X$ is trivial. This is one of the main points in the theory of standardness. A historical reference on this matter, that we ought to the referee, is [11], Section III, paragraph 12. In the next section we exhibit a class of processes verifying $\mathcal{F}_0^X \subseteq \mathcal{F}_0^W$.

3. Stationary processes of summable memory decay are standard. For $N \le K \le 0$, we set $X[N;K] = (X_n : n = N, \dots, K)$ and $X(-\infty;K] = (X_n : n \le K)$. We put $\Sigma^{(K)} = \prod_{n \le K}\{0,1\}$, for every $K \le 0$. A point in $\Sigma^{(K)}$ will be denoted simply by $\mathbf{x}$.

The conditional probability is written $\mathbb{P}(i|\mathbf{x}) = \mathbb{P}(X_0 = i | X(-\infty;-1] = \mathbf{x})$ for $i \in \{0,1\}$, $\mathbf{x} \in \Sigma^{(-1)}$. We assume all the cylinder sets have strictly positive measure and that $\mathbb{P}(i|\mathbf{x}) > 0$ for every $i \in \{0,1\}$, $\mathbf{x} \in \Sigma^{(-1)}$.

For $p \ge 0$, define the following quantity:

$$\gamma_p = 1 - \inf\left\{\frac{\mathbb{P}(i|\mathbf{x})}{\mathbb{P}(i|\mathbf{y})} : i \in \{0,1\}, \mathbf{x},\mathbf{y} \in \Sigma^{(-1)}, \mathbf{x}[-p;-1] = \mathbf{y}[-p;-1]\right\}, \tag{3}$$

where in the case $p = 0$ there is no restriction on the variables $\mathbf{x},\mathbf{y} \in \Sigma^{(-1)}$. The sequence $(\gamma_p : p \ge 0)$ is decreasing and $[0,1]$ valued. This process is said to have complete connections if it verifies $\lim_{p\to\infty}\gamma_p = 0$ (see [9]). Let us show that in this case $\gamma_p \in [0,1)$ for all $p \ge 0$. Simply note that if $\gamma_p < 1$ for some p, then $\gamma_0 < 1$, thus, $\gamma_q < 1$ for all q. Indeed, fix $\mathbf{v} \in \Sigma^{(-p-1)}$. Then for every $\mathbf{x},\mathbf{y} \in \Sigma^{(-1)}$

$$\begin{aligned}\mathbb{P}(i|\mathbf{x}) &\ge (1-\gamma_p)\mathbb{P}(i|\mathbf{v}\mathbf{x}[-p,-1]) \\ &\ge c =: (1-\gamma_p)\inf\{\mathbb{P}(j|\mathbf{v}z) : j \in \{0,1\}, z \in \{0,1\}^p\} > 0,\end{aligned}$$

thus, $\frac{\mathbb{P}(i|\mathbf{x})}{\mathbb{P}(i|\mathbf{y})} \ge c$ from where we deduce $\gamma_0 \le 1 - c$.

If the additional property $\sum_{p\ge 0}\gamma_p < \infty$ holds, the process is said to have summable memory decay. Our next result assumes a weaker condition than summable memory decay.

THEOREM 3. *Assume the process* $(X_n : n \le 0)$ *has complete connections. If*

$$\sum_{\ell=0}^{\infty} \prod_{p=0}^{\ell} (1-\gamma_p) = \infty,$$

then the filtration $\mathcal{F}^X$ *is standard.*

PROOF. First, let us fix a generating r.v. R, that is, such that $\mathcal{F}_0^X = \sigma(R)$. We choose

$$R = \sum_{n \le 0} 3^n X_n, \tag{4}$$

so that, for $n \le 0$, $\{R(\omega) - R(\omega') < 3^n\} = \{X[n;0](\omega) = X[n;0](\omega')\}$. As we pointed out, a sufficient condition ensuring $\mathcal{F}^X$ is standard is that R is $\mathcal{F}_0^W$-measurable. In the sequel, for all $N \le 0$, we will construct a function $F_N : [0,1]^{|N|+1} \to \mathbb{R}$ such that $S_N = F_N(W[N;0])$ converges in probability toward R, and the result will be shown.

Let us consider the sequences $(V_n : n \le 0)$ and $(W_n : n \le 0)$ introduced in Sections 1 and 2, so

$$X_n = \mathbf{1}(W_n > \mathbb{P}(0|X(-\infty; n-1])). \tag{5}$$

For all $N \le 0$, let us construct an approximation $(\widehat{X}_n^{(N)} : n \le 0)$ of the process. Before N, we put (arbitrarily) $\widehat{X}_n^{(N)} = 0$ for $n < N$, and for $n \in \{N, \dots, 0\}$, the evolution of $\widehat{X}^{(N)}$ is governed by the recurrence

$$\widehat{X}_n^{(N)} = \mathbf{1}(W_n > \mathbb{P}(0|\widehat{X}^{(N)}(-\infty; n-1])). \tag{6}$$

We define $S_N = \sum_{n \le 0} 3^n \widehat{X}_n^{(N)}$, then S_N is a function of $W[N;0]$. To prove the theorem, it is enough to show convergence in probability of S_N toward R. For that purpose, fix $\varepsilon > 0$ and K a positive integer such that $3^{-K} < \varepsilon$. For N smaller than $-K$, one has

$$\mathbb{P}(|S_N - R| > \varepsilon) \le \mathbb{P}(|S_N - R| \ge 3^{-K}) = \mathbb{P}(\widehat{X}^{(N)}[-K;0] \ne X[-K;0]).$$

Therefore, the result will follow once we prove

$$\lim_{N \to -\infty} \mathbb{P}(\widehat{X}^{(N)}[-K;0] \ne X[-K;0]) = 0. \tag{7}$$

The proof relies on ingredients that have been developed in [2], as well as in [5], in alternative shapes. For $i \in \{0,1\}$, set

$$a_0(i) = \inf\{\mathbb{P}(i|\mathbf{x}) : \mathbf{x} \in \Sigma^{(-1)}\}, \tag{8}$$

$$a_p(i|z) = \inf\{\mathbb{P}(i|\mathbf{x}) : \mathbf{x} \in \Sigma^{(-1)}, \mathbf{x}[-p;-1] = z\} \qquad \text{for } p \ge 1,\ z \in \{0,1\}^p. \tag{9}$$

Notice that, for all $p \geq 0$, $z \in \{0,1\}^p$ and $\mathbf{x} \in \Sigma^{(-1)}$, with $\mathbf{x}[-p;-1] = z$, it holds

$$a_p(0|z) + a_p(1|z) \geq (1-\gamma_p)\mathbb{P}(0|\mathbf{x}) + (1-\gamma_p)\mathbb{P}(1|\mathbf{x}) \geq (1-\gamma_p) \tag{10}$$

[for $p = 0$, it simply reads $a_0(0) + a_0(1) \geq 1 - \gamma_0$].

Let $(Z_q : q \geq 0)$ be a Markov chain, taking values in $\mathbb{N}$, with initial value $Z_0 = 0$ and with transition probabilities

$$p_{i,i+1} = 1 - \gamma_i, \qquad p_{i,0} = \gamma_i, \qquad p_{i,j} = 0 \qquad \text{in other cases.}$$

The hypothesis of the theorem is equivalent to the transience or null recurrence of this chain. Thus,

$$\lim_{q\to\infty} P(Z_q \leq K) = 0.$$

To prove (7), and therefore the theorem, is enough to prove the inequality

$$\mathbb{P}(\widehat{X}^{(N)}[-K;0] \neq X[-K;0]) \leq P(Z_{-N} \leq K).$$

For the rest of the proof, we follow the simplification made by the referee to our original proof. The referee introduced for $n \in \{N,\ldots,0\}$ the random variable $L_n^{(N)} = \max\{l \in \mathbb{N} : \widehat{X}^{(N)}[n-l+1;n] = X[n-l+1;n]\}$. Notice that $\{L_0^{(N)} \leq K\} = \{\widehat{X}^{(N)}[-K;0] \neq X[-K;0]\}$.

For $n \in \{N+1,\ldots,0\}$, it follows from the definition of $L^{(N)}$, (5) and (6) that

$$\begin{aligned}\{L_{n-1}^{(N)} = l, L_n^{(N)} = l+1\} \supseteq \{&L_{n-1}^{(N)} = l, W_n < a_l(0|X[n-l;n-1])\} \\ &\cup \{L_{n-1}^{(N)} = l, W_n > 1 - a_l(1|X[n-l;n-1])\}.\end{aligned}$$

Thus, on the set $\{L_{n-1}^{(N)} = l\}$ we have the inequality

$$\mathbb{P}(L_n^{(N)} = l+1|\mathcal{G}_{n-1}) \geq a_l(0|X[n-l;n-1]) + a_l(1|X[n-l;n-1]) \geq 1 - \gamma_l,$$

which proves that

$$\mathbb{P}(L_n^{(N)} = L_{n-1}^{(N)} + 1|\mathcal{G}_{n-1}) \geq 1 - \gamma_{L_{n-1}^{(N)}}.$$

Now, let us prove by induction on $n \in \{N,\ldots,0\}$ that $L_n^{(N)} \geq Z_{n-N}$ in law, namely,

$$\mathbb{P}(L_n^{(N)} > M) \geq \mathbb{P}(Z_{n-N} > M) \qquad \text{for all } M \in \mathbb{N}. \tag{11}$$

For $n = N$, this is obvious because $Z_0 = 0$. Assuming the inequality holds for a given $n \leq -1$, we get

$$\mathbb{P}(L_{n+1}^{(N)} > M) = \mathbb{P}(L_n^{(N)} \geq M, L_{n+1}^{(N)} = L_n^{(N)} + 1)$$

$$\begin{aligned}
&\geq \mathbb{E}(\mathbf{1}(L_n^{(N)} \geq M)(1-\gamma_{L_n^{(N)}}))\\
&\geq \mathbb{E}(\mathbf{1}(Z_{n-N} \geq M)(1-\gamma_{Z_{n-N}}))\\
&= \mathbb{P}(Z_{n-N} \geq M, Z_{n-N+1} = Z_{n-N}+1)\\
&= \mathbb{P}(Z_{n-N+1} > M).
\end{aligned}$$

Here we have used that $L_n^{(N)} \geq Z_{n-N}$, in law, and that the function $l \to \mathbf{1}(l \geq M)(1-\gamma_l)$ is increasing. The theorem is finally obtained by taking $n=0$ in (11). □

REMARK 4. We notice that if $\gamma_p = 0$ for some $p \geq 1$, the process $((X_{n-p+1}, \ldots, X_n) : n \leq 0)$ is a Markov chain and Theorem 3 is well known (see [12]). When $p=0$, the result is trivial because $(X_n : n \leq 0)$ are independent.

4. A product type filtration assuming standardness. In this section we assume $\mathcal{F}^X$ is standard. As stated, we will construct a diffusive product type extension of $\mathcal{F}^X$. We consider the sequences $(V_n : n \leq 0)$ and $(W_n : n \leq 0)$ introduced in Sections 1 and 2, and the filtration $\mathcal{G} = (\mathcal{G}_n : n \leq 0)$ defined by $\mathcal{G}_n = \sigma(X_m, V_m : m \leq n)$. For a notational purpose, if Z and Z' are random elements, we denote by $\mathcal{L}(Z)$ the probability distribution of Z and by $\mathcal{L}(Z|Z'=z')$ its conditional law with respect to the event $\{Z'=z'\}$.

Let ρ_0 be a metric in Σ, consider the following sequence $(\rho_{|n|} : n \leq 0)$ defined recursively, for $n \leq -1$ and $\mathbf{x}, \mathbf{y} \in \Sigma$, by

$$\begin{aligned}
&\rho_{|n|}(\mathbf{x},\mathbf{y})\\
&\quad = \inf\{\mathbb{E}_\Lambda(\rho_{|n|-1}(\mathbf{x}(-\infty;n]\xi 0^{|n|-1},\\
&\qquad\qquad \mathbf{y}(-\infty;n]\eta 0^{|n|-1})) : \Lambda \in \mathcal{J}(\mathbf{x}(-\infty;n], \mathbf{y}(-\infty;n])\},
\end{aligned} \tag{12}$$

where, for every $\mathbf{z}, \mathbf{w} \in \Sigma$, $\mathcal{J}(\mathbf{z},\mathbf{w})$ is the set of couplings of ξ and η whose marginals satisfy $\mathcal{L}(\xi) = \mathcal{L}(X_{n+1}|X(-\infty;n] = \mathbf{z})$ and $\mathcal{L}(\eta) = \mathcal{L}(X_{n+1}|X(-\infty;n] = \mathbf{w})$. We have put $0^{|n|-1} = \underbrace{0\ldots0}_{|n|-1 \text{ times}}$, but instead of $0^{|n|-1}$, any other fixed choice can also be taken.

If $\mathcal{F}^X$ is standard, it satisfies Vershik criterion (see [15, 16]): for all initial metric ρ_0,

$$\lim_{p\to\infty} \alpha_p(\rho_0) = 0 \qquad \text{where } \alpha_p(\rho_0) = \int_{\Sigma\times\Sigma} \rho_p(\mathbf{x},\mathbf{y})\, d\mathbb{P}(\mathbf{x})\, d\mathbb{P}(\mathbf{y}) \quad \text{for } p \geq 0. \tag{13}$$

From the cosiness property introduced in [14] (see also [6, 7, 10]), it suffices to verify (13) for the following well-defined metric $\rho_0(\mathbf{x},\mathbf{y}) = |R(\mathbf{x}) - R(\mathbf{y})|$, for a generating function R. We point out that, in the case of stationary processes, this property will also follow from our construction. We fix R as in (4), and our construction will depend on this arbitrary choice.

From its definition, $\rho_{|n|}(\mathbf{x},\mathbf{y})$ does not depend on $(\mathbf{x}[n+1;0],\mathbf{y}[n+1;0])$, so, since the process is stationary, we get $\alpha_{|n|}(\rho_0)=\int_{\Sigma\times\Sigma}\widetilde{\rho}_{|n|}(\mathbf{x},\mathbf{y})\,d\mathbb{P}(\mathbf{x})\,d\mathbb{P}(\mathbf{y})$, where we set $\widetilde{\rho}_{|n|}(\mathbf{x},\mathbf{y})=\rho_{|n|}(\mathbf{x}0^{|n|},\mathbf{y}0^{|n|})$.

For $\mathbf{x},\mathbf{y}\in\Sigma^{(-1)}$, consider

$$\lambda_m(\mathbf{x},\mathbf{y})=\operatorname{sign}(\widetilde{\rho}_{|m|-1}(\mathbf{x}0,\mathbf{y}0)+\widetilde{\rho}_{|m|-1}(\mathbf{x}1,\mathbf{y}1)-\widetilde{\rho}_{|m|-1}(\mathbf{x}0,\mathbf{y}1)-\widetilde{\rho}_{|m|-1}(\mathbf{x}1,\mathbf{y}0)).$$

A direct computation shows that the following coupling minimizes the expectation $\mathbb{E}_\Lambda(\widetilde{\rho}_{|m|-1}(\mathbf{x}\xi,\mathbf{y}\eta))$:

$\boldsymbol{\xi}\setminus\boldsymbol{\eta}$	**0**	**1**
0	$\mathbb{P}(0\vert\mathbf{x})\wedge\mathbb{P}(0\vert\mathbf{y})$	$(\mathbb{P}(0\vert\mathbf{x})-\mathbb{P}(0\vert\mathbf{y}))^+$
1	$(\mathbb{P}(1\vert\mathbf{x})-\mathbb{P}(1\vert\mathbf{y}))^+$	$\mathbb{P}(1\vert\mathbf{x})\wedge\mathbb{P}(1\vert\mathbf{y})$

if $\lambda_m(\mathbf{x},\mathbf{y})=-1$

and

$\boldsymbol{\xi}\setminus\boldsymbol{\eta}$	**0**	**1**
0	$(\mathbb{P}(0\vert\mathbf{x})-\mathbb{P}(1\vert\mathbf{y}))^+$	$\mathbb{P}(0\vert\mathbf{x})\wedge\mathbb{P}(1\vert\mathbf{y})$
1	$\mathbb{P}(1\vert\mathbf{x})\wedge\mathbb{P}(0\vert\mathbf{y})$	$(\mathbb{P}(1\vert\mathbf{x})-\mathbb{P}(0\vert\mathbf{y}))^+$

if $\lambda_m(\mathbf{x},\mathbf{y})=1$

(see [4], Lemma 5.2, for a similar construction). This coupling is denoted by $\Lambda_m(\cdot,\cdot|\mathbf{x},\mathbf{y})\in\mathcal{J}(\mathbf{x},\mathbf{y})$.

With this notation, we can write $\rho_{|n|}$ in terms of $\rho_{|n|-1}$ by

$$\rho_{|n|}(\mathbf{x},\mathbf{y})=\mathbb{E}_{\Lambda_n(\cdot,\cdot|\mathbf{x},\mathbf{y})}(\rho_{|n|-1}(\mathbf{x}(-\infty;n]\xi0^{|n|-1},\mathbf{y}(-\infty;n]\eta0^{|n|-1})). \tag{14}$$

For each fixed $N\le 0$ and a point $\widehat{\mathbf{x}}^{(N)}\in\Sigma$, we construct an approximation $\widehat{X}^{(N)}[N;0]$ of $X[N;0]$ and a sequence $U^{(N)}[N;0]$ of uniform i.i.d. r.v.'s, defined recursively and such that $\widehat{X}^{(N)}[N;0]$ is measurable with respect to $\sigma(U^{(N)}[N;0])$. This is done inductively starting with $\widehat{X}^{(N)}(-\infty;N-1]=\widehat{\mathbf{x}}^{(N)}(-\infty;N-1]$.

Definition 5. Consider $m\in\{N-1,\dots,-1\}$ and define

$$U^{(N)}_{m+1}=\begin{cases}W_{m+1}, & \text{on } \lambda_m(X(-\infty;m],\widehat{X}^{(N)}(-\infty;m])=-1,\\ 1-W_{m+1}, & \text{on } \lambda_m(X(-\infty;m],\widehat{X}^{(N)}(-\infty;m])=1,\end{cases} \tag{15}$$

and

$$\widehat{X}^{(N)}_{m+1}=\mathbf{1}(U^{(N)}_{m+1}>\mathbb{P}(0|\widehat{X}^{(N)}(-\infty;m])). \tag{16}$$

In the sequel we specify the structure of the sequence and explain how to recover X from $U^{(N)}$. We also study the joint law of X and $\widehat{X}^{(N)}$.

LEMMA 6. *$U^{(N)}[N;0]$ is a sequence of i.i.d. r.v.'s uniformly distributed on $[0,1]$. For all $m \in \{N,\dots,0\}$, $U_m^{(N)}$ is independent of $\mathcal{G}_{m-1}$. Moreover, $\mathcal{G}_{m-1} \vee \sigma(U_m^{(N)}) = \mathcal{G}_m$.*

PROOF. Let $m \in \{N,\dots,0\}$. The law of $U_m^{(N)}$ given $\mathcal{G}_{m-1}$ is the same as the law of W_m given $\mathcal{G}_{m-1}$. Then, the uniform distribution of $U_m^{(N)}$ on $[0,1]$ and the independence between $U_m^{(N)}$ and $\mathcal{G}_{m-1}$ readily follow.

To conclude, let us express explicitly X_m in terms of $X(-\infty;m-1]$, $\widehat{X}(-\infty;m-1]$ and $U_m^{(N)}$. From (1) and (15), we get the following:

- if $\lambda_{m-1}(X(-\infty;m-1],\widehat{X}^{(N)}(-\infty;m-1]) = -1$, then $X_m = \mathbf{1}(U_m^{(N)} > \mathbb{P}(0|X(-\infty;m-1]))$,
- if $\lambda_{m-1}(X(-\infty;m-1],\widehat{X}^{(N)}(-\infty;m-1]) = 1$, then $X_m = \mathbf{1}(1-U_m^{(N)} > \mathbb{P}(0|X(-\infty;m-1]))$,

where $\widehat{X}^{(N)}(-\infty;m-1]$ is itself a function of $X(-\infty;m-1]$, $U^{(N)}[N;m-1]$ and $\widehat{\mathbf{x}}^{(N)}(-\infty,N-1]$. $\square$

We observe that $\mathbb{P}(\widehat{X}_m^{(N)} = 0) = \mathbb{P}(0|\widehat{X}^{(N)}(-\infty;m-1])$. Finer relations are given in Lemma 7 below.

Let us write how to recover the whole sequence $X[N;0]$ from $U^{(N)}[N;0]$ and the past. We define a function $G:\{1,-1\}\times[0,1]\times\Sigma \to \{0,1\}$ by

$$G(\lambda,u,\mathbf{x}) = \begin{cases} \mathbf{1}(u > \mathbb{P}(0|\mathbf{x})), & \text{if } \lambda = -1, \\ \mathbf{1}(1-u > \mathbb{P}(0|\mathbf{x})), & \text{if } \lambda = 1. \end{cases}$$

We get $X_m = G(\lambda_{m-1}(X(-\infty;m-1],\widehat{X}^{(N)}(-\infty;m-1]),U_m^{(N)},X(-\infty;m-1])$. Iterating this procedure, we can define functions G_N, such that

$$X[N;0] = G_N(U^{(N)}[N;0],X(-\infty;N-1]). \tag{17}$$

We notice that $\widehat{X}^{(N)}[N;0]$ is a similar function of $U^{(N)}[N;0]$ and $\widehat{\mathbf{x}}^{(N)}(-\infty,N-1]$ (but simpler, in the sense that it does not use λ, or, equivalently, this corresponds to $\lambda_m(\widehat{X}^{(N)}(-\infty;m],\widehat{X}^{(N)}(-\infty;m]) = -1$).

LEMMA 7. *For any sequence $\mathbf{a} \in \Sigma$,*

$$\mathbb{P}(\widehat{X}^{(N)}[N;0] = \mathbf{a}[N;0]) \\ = \mathbb{P}(X[N;0] = \mathbf{a}[N;0] | X(-\infty;N-1] = \widehat{\mathbf{x}}^{(N)}(-\infty;N-1]).$$

For all $m \in \{N,\dots,0\}$, and all $a,b \in \{0,1\}$,

$$\begin{aligned} &\mathbb{P}(X_m = a, \widehat{X}_m^{(N)} = b | \mathcal{G}_{m-1}) \\ &\qquad = \Lambda_{m-1}(a,b|X(-\infty;m-1],\widehat{X}^{(N)}(-\infty;m-1]). \end{aligned} \tag{18}$$

PROOF. Let us write the joint law $\mathcal{L}(X_m, \widehat{X}_m^{(N)}|\mathcal{G}_{m-1})$. Since $\lambda_{m-1}(X(-\infty; m-1], \widehat{X}^{(N)}(-\infty; m-1])$ is $\mathcal{G}_{m-1}$-measurable, we can treat the cases according to the values of this variable. We only check one case, $(a,b) = (0,0)$ and $\lambda_{m-1}(X(-\infty; m-1], \widehat{X}^{(N)}(-\infty; m-1]) = -1$. One has

$$\begin{aligned}
&\mathbb{P}(X_m = 0, \widehat{X}_m^{(N)} = 0|\mathcal{G}_{m-1}) \\
&\quad = \mathbb{P}(W_m \le \mathbb{P}(0|\widehat{X}^{(N)}(-\infty; m-1])|X_m = 0, \mathcal{G}_{m-1})\mathbb{P}(X_m = 0|\mathcal{G}_{m-1}) \\
&\quad = \mathbb{P}(\mathbb{P}(0|X(-\infty; m-1])V_m \le \mathbb{P}(0|\widehat{X}^{(N)}(-\infty; m-1])|X_m = 0, \mathcal{G}_{m-1}) \\
&\qquad \times \mathbb{P}(0|X(-\infty; m-1]) \\
&\quad = \mathbb{P}(0|X(-\infty; m-1]) \wedge \mathbb{P}(0|\widehat{X}^{(N)}(-\infty; m-1]),
\end{aligned}$$

where the last line follows since V_m is a uniform random variable independent of $\mathcal{G}_{m-1} \vee \sigma(X_m)$. $\square$

We define $\widehat{R}^{(N)} = R(\widehat{X}^{(N)}(-\infty; 0])$. Therefore, $\widehat{R}^{(N)}$ is generated by the sequence $U^{(N)}[N; 0]$ and it is independent of $X(-\infty; N-1]$.

LEMMA 8. *The following equality holds:* $\mathbb{E}(|R - \widehat{R}^{(N)}|) = \int_\Sigma \rho_{|N|+1}(\mathbf{x}, \widehat{\mathbf{x}}^{(N)})\, d\mathbb{P}(\mathbf{x})$.

PROOF. We must show $\mathbb{E}(\rho_0(X, \widehat{X}^{(N)})) = \int_\Sigma \rho_{|N|+1}(\mathbf{x}, \widehat{\mathbf{x}}^{(N)})\, d\mathbb{P}(\mathbf{x})$. Notice that $\rho_{|N|+1}$ does not depend on coordinates $\{N, \dots, 0\}$, so

$$\begin{aligned}
&\int_\Sigma \rho_{|N|+1}(\mathbf{x}, \widehat{\mathbf{x}}^{(N)})\, d\mathbb{P}(\mathbf{x}) \\
&\quad = \mathbb{E}(\rho_{|N|+1}(X, \widehat{\mathbf{x}}^{(N)})) \\
&\quad = \mathbb{E}(\rho_{|N|+1}(X(-\infty; N-1]0^{|N|+1}, \widehat{X}^{(N)}(-\infty; N-1]0^{|N|+1})).
\end{aligned}$$

Recall (14), that in our case reads, for $m \le -1$,

$$\begin{aligned}
&\rho_{|m|}(X(-\infty; m]0^{|m|}, \widehat{X}^{(N)}(-\infty; m]0^{|m|}) \\
&\quad = \mathbb{E}_{\Lambda_m(\cdot,\cdot|X(-\infty;m], \widehat{X}^{(N)}(-\infty;m])} \\
&\qquad \times (\rho_{|m|-1}(X(-\infty; m]\xi 0^{|m|-1}, \widehat{X}^{(N)}(-\infty; m]\eta 0^{|m|-1})).
\end{aligned}$$

Then, Lemma 7 shows that, for any measurable function h, it holds:

$$\begin{aligned}
&\mathbb{E}(\mathbb{E}_{\Lambda_m(\cdot,\cdot|X(-\infty;m], \widehat{X}^{(N)}(-\infty;m])}(h(X(-\infty; m]\xi, \widehat{X}^{(N)}(-\infty; m]\eta))) \\
&\quad = \mathbb{E}(h(X(-\infty; m+1], \widehat{X}^{(N)}(-\infty; m+1])).
\end{aligned}$$

Hence,

$$\begin{aligned}&\mathbb{E}(\rho_{|m|}(X(-\infty;m]0^{|m|},\widehat{X}^{(N)}(-\infty;m]0^{|m|}))\\&\qquad=\mathbb{E}(\rho_{|m|-1}(X(-\infty;m+1]0^{|m|-1},\widehat{X}^{(N)}(-\infty;m+1]0^{|m|-1})).\end{aligned}$$

The argument holds for all $m\in\{N-1,\dots,-1\}$ and the lemma is proved. $\square$

R is determined from the whole past up to $N-1$ and the i.i.d. r.v.'s $U^{(N)}[N;0]$. In fact, from (17), $R(X(-\infty;0])=R(X(-\infty;N-1]G_N(U^{(N)}[N;0], X(-\infty;N-1]))$.

The following result is a direct consequence of the martingale theorem, and we skip a detailed proof.

LEMMA 9. *Let $N\le 0$, $\delta>0$, $Z[N;0]$ be a sequence of uniform i.i.d. r.v. independent of $X(-\infty;N-1]$ and H a measurable function such that*

$$X[N;0]=H(Z[N;0],X(-\infty;N-1]).$$

Then, there exists an integer $K=K(N,\delta,H)<N$ and a function $\Phi:[0,1]^{|N|+1}\times\{0,1\}^{N-K}\to\mathbb{R}$, which depends on N,δ,H, that verify

$$\mathbb{P}(|\Phi(Z[N;0],X[K;N-1])-R|>\delta)<\delta.$$

One of the tools we need is given by the following construction. Let us take $\delta>0$ and consider $N=N(\delta)\le 0$ such that $\alpha_{|N|+1}(\rho_0)<\delta$. By Fubini's theorem, we can choose a sequence $\widehat{\mathbf{x}}^{(N)}\in\Sigma$ verifying the following property:

$$\int_\Sigma \rho_{|N|+1}(\mathbf{x},\widehat{\mathbf{x}}^{(N)})\,d\mathbb{P}(\mathbf{x})<\delta. \tag{19}$$

The choice of such $\widehat{\mathbf{x}}^{(N)}$ for each relevant N is arbitrary and will influence our construction. From Lemma 8, we obtain that, for such N and $\widehat{\mathbf{x}}^{(N)}$, the next bound holds:

$$\mathbb{E}(|R-\widehat{R}^{(N)}|)\le\delta.$$

Now we construct a sequence $(U_n:n\le 0)$ of uniform i.i.d. r.v. that will give us a product type filtration such that $\mathcal{F}^X$ is immersed on. Fix a positive sequence $(\delta_j:j\ge 0)$ decreasing to 0.

- Initially, at step 0, we choose N_0 and $\widehat{\mathbf{x}}^{(N_0)}\in\Sigma$ such that $\alpha_{|N_0|+1}(\rho_0)<\delta_0$ and

  $$\int \rho_{|N_0|+1}(\mathbf{x},\widehat{\mathbf{x}}^{(N_0)})\,d\mathbb{P}(\mathbf{x})<\delta_0.$$

  We construct $U^{(N_0)}[N_0;0]$ and $\widehat{X}^{(N_0)}[N_0;0]$ following Definition 5. We put $M_0=1$, $M_1=N_0$ and $H_0=G_{N_0}$, so that $X[M_1;0]=H_0(U^{(N_0)}[M_1;0], X(-\infty;M_1-1])$, see (17). In particular, we have that $\mathbb{E}(|R-\widehat{R}^{(N_0)}|)\le\delta_0$. We finally put $U[N_0;0]=U^{(N_0)}[N_0;0]$.

- Assume at step $j-1$ we have constructed a sequence $U[M_j;0]$ and a function H_{j-1} such that

$$X[M_j;0]=H_{j-1}(U[M_j;0],X(-\infty;M_j-1]). \tag{20}$$

We obtain $K_j<M_j$ and Φ_j by applying Lemma 9 with $N=M_j$, $\delta=\delta_j/2$, $Z[M_j;0]=U[M_j;0]$ and $H=H_{j-1}$. We choose N_j and $\widehat{\mathbf{x}}^{(N_j)}$ such that

$$\begin{gathered}\alpha_{|N_j|+1}(\rho_0)<3^{K_j-M_j+1}\cdot\delta_j/2 \quad \text{and}\\ \int\rho_{|N_j|+1}(\mathbf{x},\widehat{\mathbf{x}}^{(N_j)})\,d\mathbb{P}(\mathbf{x})<3^{K_j-M_j+1}\cdot\delta_j/2.\end{gathered} \tag{21}$$

We set $M_{j+1}=M_j+N_j-1$.
- Applying the construction on the shifted process $(X_{n+M_j-1}:n\le 0)$ and using stationarity, we construct a sequence $U[M_{j+1};M_j-1]$ of uniform i.i.d. r.v., which is independent of $U[M_j;0]$, such that

$$X[M_{j+1};M_j-1]=G_{N_j}(U[M_{j+1};M_j-1],X(-\infty;M_{j+1}-1]). \tag{22}$$

From (20) and (22), we can define a function H_j in terms of G_{N_j} and H_{j-1} such that $X[M_{j+1};0]=H_j(U[M_{j+1};0],X(-\infty;M_{j+1}-1])$.

A repeated use of Lemma 6 in the construction of the blocks $U[M_{j+1};M_j-1]$ gives that $(U_n:n\le 0)$ is a sequence of i.i.d. r.v.'s uniformly distributed in $[0,1]$, so $\mathcal{F}^U$ is a diffusive product type filtration.

Theorem 10. *If $\mathcal{F}^X$ is standard, then $\mathcal{F}^X$ is immersed in the diffusive product type filtration $\mathcal{F}^U$.*

Proof. It is enough to construct a function S such that $R(X(-\infty;0])=S(U(-\infty;0])$. For $j\ge 1$, set $S_j(w)=\Phi_j(U[M_j;0](w),\widehat{X}[K_j;M_j-1](w))$, where $\widehat{X}=\widehat{X}^{(M_{j+1})}$ is the process generated in Definition 5 starting from $\widehat{\mathbf{x}}^{(N_j)}$. This means $\widehat{X}(-\infty;M_{j+1}-1]=\widehat{\mathbf{x}}^{(N_j)}(-\infty;N_j-1]$, where we identify points in $\Sigma^{(M_{j+1}-1)}$ and $\Sigma^{(N_j-1)}$. Therefore, S_j is a function of $U[M_{j+1};0]$ because $\widehat{X}[K_j;M_j-1]$ is a function of $U[M_{j+1};M_j-1]$. It remains to prove that S_j converges in probability to R.

Notice that $X[K_j;M_j-1]=\widehat{X}[K_j;M_j-1]$ implies $S_j=\Phi_j(U[M_j;0],X[K_j;M_j-1])$. Then

$$\mathbb{P}(S_j\neq\Phi_j(U[M_j;0],X[K_j;M_j-1]))\le P(X[K_j;M_j-1]\neq\widehat{X}[K_j;M_j-1]).$$

Recall that $|R(\mathbf{x})-R(\mathbf{y})|<3^{-k}$ implies $\mathbf{x}[-k;0]=\mathbf{y}[-k;0]$, then we get

$$\begin{aligned}&\mathbb{P}(X[K_j;M_j-1]\neq\widehat{X}[K_j;M_j-1])\\&\qquad\le\mathbb{P}(|R(X(-\infty;M_j-1])-R(\widehat{X}(-\infty;M_j-1])|\ge 3^{-(M_j-1-K_j)})\\&\qquad\le 3^{M_j-1-K_j}\mathbb{E}(|R(X(-\infty;M_j-1])-R(\widehat{X}(-\infty;M_j-1])|),\end{aligned}$$

where we have identified Σ and $\Sigma^{(M_j-1)}$. By applying Lemma 8 to the shifted process and in view of the choice of N_j in (21), we find

$$\mathbb{E}(|R(X(-\infty;M_j-1])-R(\widehat{X}(-\infty;M_j-1])|)\le 3^{K_j-M_j+1}\delta_j/2.$$

We have proven $\mathbb{P}(S_j\ne\Phi_j(U[M_j;0],X[K_j;M_j-1]))\le\delta_j/2$. On the other hand, the choice of K_j done in Lemma 9 guarantees that $\mathbb{P}(|\Phi_j(U[M_j;0],X[K_j;M_j-1])-R(X(-\infty,0])|>\delta_j/2)\le\delta_j/2$. Therefore,

$$\begin{aligned}&\mathbb{P}(|S_j-R(X(-\infty,0])|>\delta_j)\\&\quad\le\mathbb{P}(S_j\ne\Phi_j(U[M_j;0],X[K_j;M_j-1]))\\&\qquad+\mathbb{P}(|\Phi_j(U[M_j;0],X[K_j;M_j-1])-R(X(-\infty,0])|>\delta_j/2)\le\delta_j,\end{aligned}$$

then the convergence in probability follows. $\square$

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