Discrete solitons and nonlinear surface modes in semi-infinite waveguide arrays

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We discuss the formation of self-trapped localized states near the edge of a semi-infinite array of nonlinear waveguides. We study a crossover from nonlinear surface states to discrete solitons by analyzing the families of odd and even modes centered at different distances from the surface, and reveal the physical mechanism of the nonlinearity-induced stabilization of surface modes.

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Surface modes are a special type of waves localized at an interface between two different media. Surface states have been studied in different fields of physics, including optics [1, 2], where such waves are confined to the interface between periodic and homogeneous dielectric media, and nonlinear dynamics of discrete chains [3]. In periodic systems, staggered modes localized at surfaces are known as Tamm states [4], first found as localized electronic states at the edge of a truncated periodic potential.

Recently it was predicted theoretically and demonstrated experimentally that nonlinear self-trapping of light near the edge of a waveguide array with *self-focusing* nonlinearity can lead to the formation of discrete surface solitons [5, 6]. It was found that the self-trapped surface modes acquire some novel properties different from those of the discrete solitons in infinite lattices: they can only exist above certain power level and for the same amount of power, it is possible to have, in some conditions, up to two surface modes, one stable and the other unstable.

In this Letter, we reveal and explain the physical mechanism of the nonlinearity-induced stabilization of surface modes and their existence above a certain power threshold. In particular, we analyze the families of odd and even modes placed at different distances from the surface, and discuss a crossover between the nonlinear surface states and discrete solitons of a semi-infinite lattice.

We study a semi-infinite array of identical, weakly coupled nonlinear optical waveguides [as shown in the inset of Fig. 1(a)] described by the system of coupled-mode equations [7, 8] for the normalized mode amplitudes E_n,

$$
\begin{aligned}
&i\frac{dE_1}{dz} + \alpha\ E_1 + E_2 + \gamma\ |E_1|^2 E_1 = 0, \\
&i\frac{dE_n}{dz} + \alpha\ E_n + (E_{n+1} + E_{n-1}) + \gamma\ |E_n|^2 E_n = 0,
\end{aligned} \tag{1}
$$

where $n \geq 2$, the propagation coordinate z is normalized to the intersite coupling V, E_n are defined in terms of the actual electric fields $\mathcal{E}_n$ as $E_n = (2V\lambda_0\eta_0/\pi n_0 n_2)^{1/2}\mathcal{E}_n$,

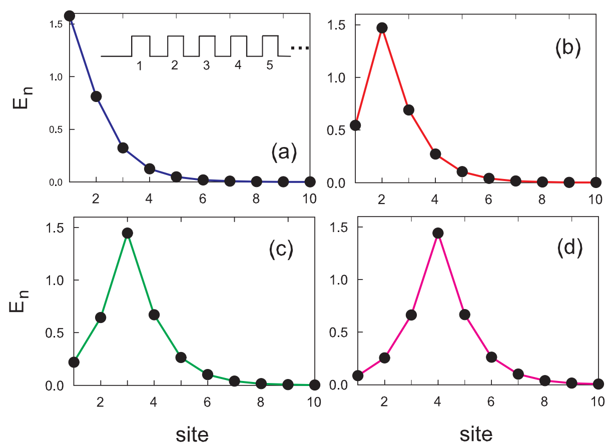

FIG. 1: Examples of surface localized modes at $\beta = 3$ in the array of focusing waveguides ($\gamma = +1$) centered at different distances $d = 0, 1, 2, 3$ from the array edge.

where λ_0 is the free-space wavelength, η_0 is the free-space impedance, α is the normalized linear propagation constant of each waveguide, n_2 and n_0 are nonlinear and linear refractive indices of each waveguide, and $\gamma = \pm 1$ defines focusing or defocusing nonlinearity, respectively.

We look for stationary modes of the waveguide array in the form $E_n(z) = \exp(i\beta z)E_n$, where β is the nonlinearity-induced shift of the propagation constant. For $\gamma = 0$ we use the ansatz $E_n \sim \sin(nk)$ and obtain the linear spectrum $\beta = \alpha + 2\cos k$, $(0 \leq k \leq \pi)$, and no localized surface modes. The presence of nonlinearity in the model (1) can give rise to *new localized states.* To find those modes, we analyze the stationary equations (1) where, without loss of generality, we scale out the parameter α.

For given β, the system of stationary equations is solved numerically by a multi-dimensional Newton-Raphson scheme. Since we are interested in surface lo-

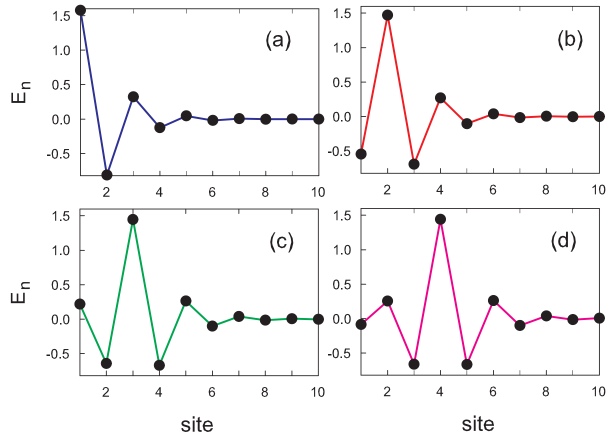

FIG. 2: Examples of localized surface modes at $\beta = -3$ in the array of defocusing waveguides ($\gamma = -1$) located at different distances $d = 0, 1, 2, 3$ from the array edge.

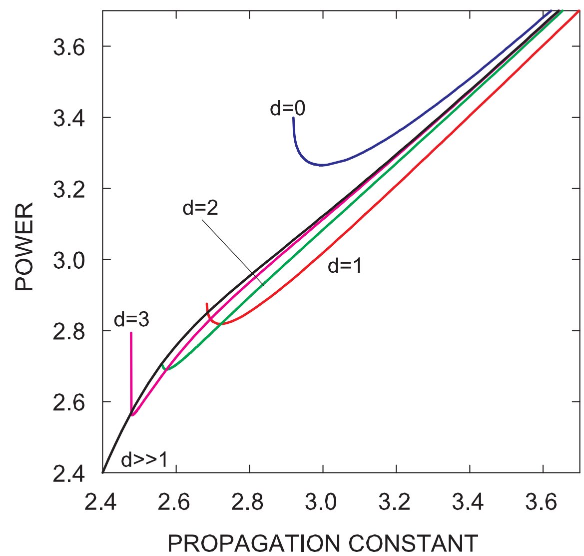

FIG. 3: Normalized power vs. propagation constant β for the surface modes shown in Fig. 1 located at different distances $d = 0, 1, 2, 3$ from the surface. Black curve corresponds to the discrete soliton in an infinite array.

calized modes, we look for the states with maxima near the surface that decay quickly away from the array edge. Similar to an infinite array, these states could be centered at a waveguide site, or centered between waveguides. In an infinite discrete chain, such modes are known as *odd* and *even* states, respectively. In our calculations, we take $N = 51$ waveguides and explore both focusing and defocusing nonlinearities looking for localized modes below and above the linear spectrum band, $|\beta| < 2$.

Figures 1(a-d) and 2(a-d) show examples of the nonlinear localized states centered at different sites near the surface, for both focusing ($\gamma = +1$, $\beta = 3$) and defocusing ($\gamma = -1$, $\beta = -3$) nonlinearities, respectively. The surface state centered at the site $n = 1$ and shown in Fig. 1(a) was predicted earlier by Markis *et al.* [5]. The existence of *multiple localized states* near the surface and their stability are important characteristics of an interplay between nonlinearity and discreteness of the array, on one hand, and the surface created by the lattice truncation, on the other. In both the cases, the states (b,c) describe a crossover regime between the modes (a) with the maximum amplitude at the surface and the modes (d) which are weakly affected by the presence of the surface.

To analyze the linear stability of each nonlinear stationary state found numerically, we introduce a weak perturbation as $E_n(z) = E_n + [u_n(z) + iv_n(z)]\exp(i\beta z)$, and obtain linear evolution equations for u_n and v_n, that can be expressed in a compact form by defining the real vectors $\delta\mathbf{U}\{u_n\}$ and $\delta\mathbf{V} = \{v_n\}$, and real matrices $\mathbf{A} = \{A_{nm}\} = \{\delta_{n,m+1} + \delta_{n,m-1} + (-\beta + 3\gamma|E_n|^2)\ \delta_{n,m}\}$ and $\mathbf{B} = \{B_{nm}\} = \{\delta_{n,m+1} + \delta_{n,m-1} + (-\beta + \gamma|E_n|^2)\ \delta_{n,m}\}$. With these definitions, the combined linear equations can be written in the form, $\delta\ddot{\mathbf{U}} + \mathbf{BA}\ \delta\mathbf{U} = 0$, $\delta\ddot{\mathbf{V}} + \mathbf{AB}\ \delta\mathbf{V} = 0$, where the dot stands for the derivative in z. Therefore, linear stability of nonlinear localized modes is defined by the eigenvalue spectra of the matrices $\mathbf{AB}$ and $\mathbf{BA}$. If any of the real eigenvalues is negative, the corresponding nonlinear stationary solution is unstable; otherwise, the solution is stable. Results of this analysis are consistent with the so-called Vakhitov-Kolokolov stability criterion of nonlinear localized modes, and the solitons determined by the slope of the power dependence $P = \sum_n |E_n|^2$, i.e. the states with $dP/d\beta < 0$ for $\beta > 0$ or $dP/d\beta > 0$ for $\beta < 0$, should be unstable.

Figure 3 shows the power P of the localized surface states vs. the propagation constant for the modes in the focusing waveguides shown in Figs. 1(a-d), and the corresponding curves for the modes of the defocusing waveguides are mirror images. Direct numerical simulations and stability analysis confirm the validity of the Vakhitov-Kolokolov stability criterion; the instability region decreases as the center of the localized mode gets shifted away from the array edge.

Similarly, we have also found even localized modes, akin to the modes found earlier for a semi-infinite nonlinear lattice [3], and verified that all in-phase even modes, for the focusing nonlinearity, and out-of-phase odd modes, for defocusing nonlinearity are all unstable, similar to the case of an infinite array.

In order to get a deeper insight into the physics of the nonlinear stabilization of the surface modes, we calculate the effective energy of the mode $H = -\sum_n (E_n E_{n+1}^* + E_n^* E_{n+1}) - \frac{1}{2}\sum_n |E_n|^4$ as a function of the distance of the collective coordinate of the mode $X = P^{-1}\sum_n n|E_n|^2$ from the surface, similar to the case of a defect [9]. We apply a constraint method and start from the solution centered at the site $\bar{n}$ for given values of β and P. Our goal is to obtain all intermediate solutions between the odd and even stationary configurations for the same

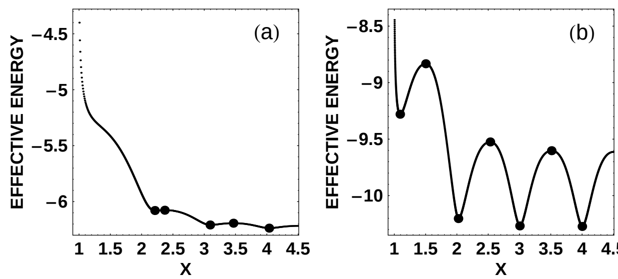

FIG. 4: Effective energy of surface modes vs. coordinate X near the edge of the array: (a) below ($P = 2.85$) and (b) above ($P = 4.05$) threshold. Black dots correspond to the stationary solutions found without constraint.

power. We proceed as follows:(i) We calculate an odd stationary mode centered at $\bar{n}$ and obtain all $\{E_n\}$ and the power P, (ii) fix the amplitude at the site $\bar{n}+1$ to be $E_{\bar{n}+1} + \epsilon$, (iii) solve the Newton-Raphson equations for all remaining E_m ($m \neq \bar{n}+1$) with the constraint that the power be kept at P, arriving at an intermediate state centered between $\bar{n}$ and $\bar{n}+1$, and finally (iv) vary ϵ and repeat the procedure until we reach the even configuration, where the amplitudes at the sites $\bar{n}$ and $\bar{n}+1$ coincide.

In Figs. 4(a,b), we show the effective energy of a surface localized mode in a semi-infinite array, $U_{\rm eff}(X) \equiv H(X)$, calculated for two different power values. The extremal points of this curve defined by the condition $dH/dX = 0$ correspond to the stationary localized solutions in the system.

In comparison with an infinite array, the truncation of the waveguide array introduces an effective *repulsive* potential, that is combined with the periodic (Peierls-Nabarro) potential of an infinite waveguide array. As a result, discrete surface modes are possible neither in the linear regime nor in the continuous limit. As we see from Fig. 4(a), for low powers there exists no solution of the equation $dH/dX = 0$ at the surface site $n = 1$; this corresponds to the fact that no surface state is found below the power threshold [5]. However, the modes localized at the sites $n \geq 2$ are still possible.

If the power exceeds the threshold $P = 3.26$, discreteness overcomes a repulsive force of the surface and the surface localized state becomes possible, as shown in Fig. 4(b). The correspondence between the stationary solutions found without constraints (black dots) and the solutions obtained as extremal points using the constraint methods is perfect. As expected, all odd modes are stable compared to even modes, and they all correspond to the condition $dH/dX = 0$.

We also found many other discrete surface modes, including the so-called flat-top surface modes that generalize the corresponding modes of infinite chains [10], and two-soliton bound states or surface twisted modes, which are stable below a certain threshold in the propagation

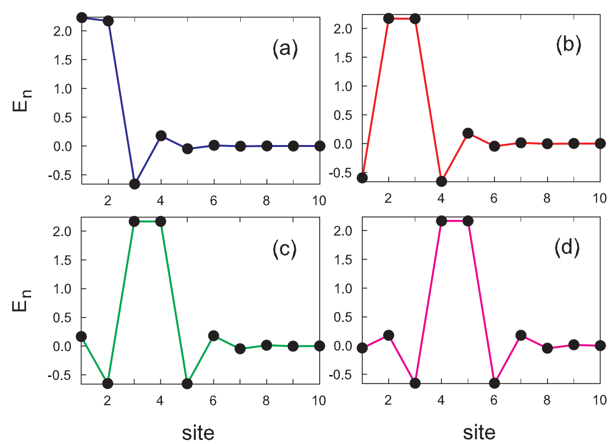

FIG. 5: Examples of stable flat-top localized surface modes at $\beta = -4$ in the array of defocusing waveguides ($\gamma = -1$) centered between different sites near the edge.

constant. Examples of flat-top modes for defocusing nonlinearity are shown in Fig. 5 for $\beta = -4$, and their stability is defined by the Vakhitov-Kolokolov criterion.

In conclusion, we have analyzed different types of nonlinear localized modes near the edge of a semi-infinite waveguide array and revealed the mechanism of the nonlinearity-induced stabilization and power threshold. In addition, we have demonstrated that a similar approach can be applied to other types of nonlinear discrete surface modes, such as flat-top modes and twisted modes, as well as to the case of staggered modes in defocusing waveguides.

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