Relative spinor class fields: A counterexample

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Abstract. It is known that classes of indefinite quadratic forms in a genus are classified by the Galois group of a spinor class field [4]. Hsia has proved the existence of a representation field F with the property that a lattice in the genus represents a fixed given lattice if and only if the corresponding element of the Galois group is trivial on F. Spinor class fields can also be used to classify conjugacy classes of maximal orders in a central simple algebra. In [1] we left open the issue of whether for every fixed given non-maximal order $\mathfrak{H}$ in a central simple division algebra there exists a representation field L with the property that $\mathfrak{H}$ embeds into a given maximal order if and only if the corresponding element of the Galois group is trivial on L. In this work we give a negative answer to this question for central simple division algebras of dimension $\geq 3^2$. The case of non-division algebras is also treated by replacing the phrase *embeds into* by *is contained in a conjugate of.* As a byproduct of the techniques used in this paper we compute the representation field of an Eichler order in a quaternion algebra.

Keywords. Spinor class fields, maximal orders, central simple algebras.

1. Introduction. Let $\mathfrak{A}$ denote an n^2-dimensional central simple algebra over a number field k. Assume that $\mathfrak{A}$ satisfies the following condition:

> **Condition E:** Either $n > 2$ or $n = 2$ and the Eichler condition is satisfied, i.e., there exists an archimedean place $\wp$ of k such that $\mathfrak{A}_\wp$ is not the (unique) quaternion division $\mathbb{R}$-algebra.

Condition E is immediate if k is non-totally real. When condition E is satisfied, then the conjugacy classes of maximal orders in $\mathfrak{A}$ are in correspondence with the elements of the Galois group of an abelian extension $\Sigma_{\mathfrak{A}}/k$, unramified at finite

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places, that can be explicitly described in terms of the ramification of $\mathfrak{A}/k$ [1]. The field $\Sigma_{\mathfrak{A}}$ is called the spinor class field of maximal orders of $\mathfrak{A}$. This correspondence is best seen as a map that associates an element $\rho(\mathfrak{D},\mathfrak{D}')$ in the Galois group to every pair $(\mathfrak{D},\mathfrak{D}')$ of maximal orders and satisfies $\rho(\mathfrak{D},\mathfrak{D}'')=\rho(\mathfrak{D},\mathfrak{D}')\rho(\mathfrak{D}',\mathfrak{D}'')$. The map ρ is defined on pairs of maximal orders by

$$\rho(\mathfrak{D},\mathfrak{D}')=[\mathcal{N}(\sigma),\Sigma_{\mathfrak{A}}/k],$$

where $\mathcal{N}$ denotes the reduced norm on the adelic ring $\mathfrak{A}_{\mathbb{A}}$, the element $\sigma\in\mathfrak{A}_{\mathbb{A}}$ satisfies

$$\mathfrak{D}'=\sigma\mathfrak{D}\sigma^{-1}, \tag{1}$$

and $a\mapsto[a,\Sigma_{\mathfrak{A}}/k]$ denotes the Artin map on ideles. Then, if $\mathfrak{D}$ is fixed, the correspondence $\mathfrak{D}'\mapsto\rho(\mathfrak{D},\mathfrak{D}')$ is a bijection ([1, p. 2029]). Let $\mathfrak{H}$ be an arbitrary suborder of $\mathfrak{D}$. If the sub-set

$$\{\rho(\mathfrak{D},\mathfrak{D}')|\mathfrak{H}\text{ is contained in a conjugate of }\mathfrak{D}'\}\subseteq\mathrm{Gal}(\Sigma_{\mathfrak{A}}/k)$$

is a subgroup, the corresponding fixed field is called the representation field $\Sigma(\mathfrak{H})$. It does not depend on the choice of the order $\mathfrak{D}$ containing $\mathfrak{H}$. Note that if $\mathfrak{H}$ spans a simple subalgebra of $\mathfrak{A}$, any embedding extends to a conjugation by Skolem-Noether Theorem [5]. It follows that the phrase "is contained in a conjugate of" above can be replaced by "embeds into". This is the case for all orders considered here. It is always the case if $\mathfrak{A}$ is a division algebra.

In [1] we computed the representation field when $\mathfrak{H}$ is the ring of integers of a maximal abelian subfield L. Under some technical conditions on $\mathfrak{A}$ we proved that the representation field is $\Sigma_{\mathfrak{A}}\cap L$. In [2] we determined the spinor class field of a suborder of maximal rank with generators x and y satisfying $x^n,y^n\in\mathcal{O}_k$ and $xy=\eta yx$ where η is a primitive root of unity. In either case we proved the existence of a representation field. This theory has an analog on the theory of spinor class fields for quadratic forms, where the representation fields are known to exist [3]. In [1], we did not know whether the representation field actually existed for any order. In this paper we present a counterexample. In fact, we prove the following result:

Theorem 1. *Let $\mathfrak{A}$ be a central simple algebra of dimension at least 3^2 over the number field k. The representation field exists for every order $\mathfrak{H}$ in $\mathfrak{A}$ if and only if the extension $\Sigma_{\mathfrak{A}}/k$ has exponent 2.*

For instance, if $\mathfrak{A}=\mathbb{M}_n(k)$ where the class group of k has exponent m, the representation field always exists if and only if g.c.d.$(m,n)\in\{1,2\}$. In the counterexamples we provide to prove the necessity, the embedded suborder is a generalized Eichler order $\mathfrak{H}=\mathfrak{D}\cap\mathfrak{D}'$, where $\mathfrak{D}$ and $\mathfrak{D}'$ are maximal orders. As a byproduct of our work on Eichler orders we have the following result:

Theorem 2. *Let $\mathfrak{H}=\mathfrak{D}\cap\mathfrak{D}'$ be an Eichler order in the quaternion algebra $\mathfrak{A}$. Assume that Eichler condition is satisfied. The representation field F of $\mathfrak{H}$ is the maximal subfield of $\Sigma_{\mathfrak{A}}$ that splits completely at the places $\wp$ for which $\mathfrak{D}_\wp\neq\mathfrak{D}'_\wp$.*

2. Eichler orders and embeddings. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be maximal orders in the central simple k-algebra $\mathfrak{A}$. At any local place $\wp$ there exist an element $h_\wp \in \mathfrak{A}_\wp$ such that $\mathfrak{D}'_\wp = h_\wp \mathfrak{D}_\wp h_\wp^{-1}$. The local algebra $\mathfrak{A}_\wp$ is a matrix algebra $\mathbb{M}_f(E)$ where E is a division algebra with uniformizing parameter u and maximal order $\mathfrak{E}$. Without loss of generality we may assume $\mathfrak{D}_\wp = \mathbb{M}_f(\mathfrak{E})$. There exist matrices $p, q \in \mathfrak{D}_\wp^*$ and $d = \mathrm{diag}(u^{t_1}, \dots, u^{t_f})$, where $t_1 \leq \cdots \leq t_f$, such that $h_\wp = pdq$. Then any conjugate to the local order

$$\mathfrak{H} = \mathfrak{D}_\wp \cap \mathfrak{D}'_\wp = p(\mathfrak{D}_\wp \cap d\mathfrak{D}_\wp d^{-1})p^{-1}$$

is said to be a local Eichler order of type $(t_1, \dots, t_f)$. We write $[t_1, \dots, t_f]_\wp$ for the set of such orders. Then clearly $[t_1, \dots, t_f]_\wp = [t_1 + t, \dots, t_f + t]_\wp$ for any integer t, so we can always assume $t_1 = 0$. Furthermore, the relation

$$\mathfrak{D}_\wp \cap d\mathfrak{D}_\wp d^{-1} = d(\mathfrak{D}_\wp \cap d^{-1}\mathfrak{D}_\wp d)d^{-1}$$

Shows that $[t_1, \dots, t_f]_\wp = [-t_f, \dots, -t_1]_\wp$.

Lemma 2.1. *If a local Eichler order of type $(t_1, \dots, t_f)$ embeds into a local Eichler order of type $(s_1, \dots, s_f)$, then $\sum_{i>j}(s_i - s_j) \leq \sum_{i>j}(t_i - t_j)$, with equality if and only if both orders coincide.*

Proof. Let μ be the additive Haar measure on $\mathfrak{A}_\wp$ normalized so that $\mu(\mathfrak{D}_\wp) = 1$. Then μ is conjugation invariant ([6, Corollary 3, p. 7]) and $\mu(\mathfrak{H})^{-1} = [\mathfrak{D}_\wp : \mathfrak{H}]$. It suffices therefore to prove that $[\mathfrak{D}_\wp : \mathfrak{H}]$ is a strictly increasing function of $\sum_{i<j}(s_j - s_i)$, where $\mathfrak{H} = \mathfrak{D}_\wp \cap d\mathfrak{D}_\wp d^{-1}$ and $d = \mathrm{diag}(u^{s_1}, \dots, u^{s_f})$. Note that $b \in \mathfrak{H}$ if and only if both b and $d^{-1}bd$ are matrices with coefficients in $\mathfrak{E}$. In particular, if $b = (\beta_{i,j})_{i,j}$ then $b \in \mathfrak{H}$ if and only if for all $i > j$ the coefficient $\beta_{i,j}$ is divisible by $u^{s_i - s_j}$. It follows that $[\mathfrak{D}_\wp : \mathfrak{H}] = |\mathfrak{E}/u\mathfrak{E}|^{\sum_{i>j}(s_i - s_j)}$. □

Proof of Theorem 2. When $n = 2$, every local Eichler order is of type $(0, t)$ for some non-negative integer t. In particular, an Eichler order $\mathfrak{H} = \mathfrak{D} \cap \mathfrak{D}'$ is non-maximal at $\wp$ if and only if $\mathfrak{H}_\wp$ embeds into an order of type $(0, 1)$. We claim that $\mathfrak{H}$ embeds into an order $\mathfrak{D}''$ if and only if

$$\rho(\mathfrak{D}, \mathfrak{D}'') = [\wp_1 \dots \wp_r, \Sigma_{\mathfrak{A}}/k] \tag{2}$$

where $\mathfrak{H}_{\wp_i}$ is non-maximal for $i = 1, \dots, r$, whence the result follows.

To prove the claim, observe that if $\mathfrak{H}$ is conjugate to a sub-order of $\mathfrak{D}''$, then it is contained into $\mathfrak{D} \cap \mathfrak{D}'''$ for some conjugate $\mathfrak{D}'''$ of $\mathfrak{D}''$. Note that $\mathfrak{D}_\wp \neq \mathfrak{D}'''_\wp$ is possible only if $\mathfrak{H}_\wp$ is non-maximal, whence $\rho(\mathfrak{D}, \mathfrak{D}''') = \rho(\mathfrak{D}, \mathfrak{D}'')$ is in the subgroup generated by the set

$$\{ [\wp, \Sigma_{\mathfrak{A}}/k] | \, \mathfrak{H}_\wp \text{ is not maximal} \} .$$

On the other hand, if (2) is satisfied, then we construct an order $\mathfrak{D}'''$ by the following local conditions:

1. $\mathfrak{D}'''_\wp = \mathfrak{D}_\wp$ for $\wp \notin \{\wp_1, \dots, \wp_r\}$.

2. If $\wp = \wp_i$, then $\mathfrak{H}_\wp = \mathfrak{D}'_\wp \cap \mathfrak{D}_\wp$ is not maximal. We write $\mathfrak{D}'_\wp = h\mathfrak{D}_\wp h^{-1}$, where $h = pdq$, and $d = \mathrm{diag}(u^{t_1}, \ldots, u^{t_f})$ as before. Since $\mathfrak{H}_\wp$ is not maximal, at least $t_f \geq 1$. We set $d_0 = \mathrm{diag}(1, \ldots, 1, u)$, $h_0 = pd_0q$, and $\mathfrak{D}'''_\wp = h_0\mathfrak{D}_\wp h_0^{-1}$.

Note that $\mathfrak{D}'''_\wp$ contains $\mathfrak{H}_\wp$ at all places $\wp$ by construction. Then $\mathfrak{H}$ is contained in $\mathfrak{D}'''$. As $\rho(\mathfrak{D}, \mathfrak{D}'') = \rho(\mathfrak{D}, \mathfrak{D}''')$ by construction, it follows that $\mathfrak{D}''$ and $\mathfrak{D}'''$ are conjugate and therefore $\mathfrak{H}$ embeds into $\mathfrak{D}''$. □

3. Proof of Theorem 1.

Proof of the necessity. Assume that the extension $\Sigma_{\mathfrak{A}}/k$ is not of exponent 2. In other words, there exists an element $\sigma \in \mathrm{Gal}(\Sigma_{\mathfrak{A}}/k)$ of order $q > 2$. Let $\wp$ be a finite place of k such that the Frobenius homomorphism at $\wp$ is σ. Now let $\mathfrak{H} = \mathfrak{D} \cap \mathfrak{D}'$, where $\mathfrak{D} = \mathfrak{D}'$ except at $\wp$. We identify $\mathfrak{D}_\wp = \mathbb{M}_f(\mathfrak{E})$ as before and set $\mathfrak{D}'_\wp = d\mathfrak{D}_\wp d^{-1}$, where $d = \mathrm{diag}(1, 1, \ldots, 1, u)$ for a uniformizing parameter u of E. Recall that $\sigma^f = \mathrm{id}$ by definition of $\Sigma_{\mathfrak{A}}$ [1]. By definition, $\mathfrak{H} \subseteq \mathfrak{D}$, $\mathfrak{H} \subseteq \mathfrak{D}'$, and $\rho(\mathfrak{D}, \mathfrak{D}') = \sigma$. It suffices to see that $\mathfrak{H}$ cannot be contained in an order $\mathfrak{D}''$ such that $\rho(\mathfrak{D}, \mathfrak{D}'') = \sigma^{-1}$. Supose this is the case. Certainly $\mathfrak{D}''$ coincide with $\mathfrak{D}$ outside $\wp$, since $\mathfrak{H}$ is maximal there. Set $\mathfrak{D}''_\wp = g\mathfrak{D}_\wp g^{-1}$, where $g = p\,\mathrm{diag}(u^{s_1}, \ldots, u^{s_f})q$, for some $p, q \in \mathfrak{D}^*_\wp$. The order $\mathfrak{H}$ is contained in $\mathfrak{H}'' = \mathfrak{D} \cap \mathfrak{D}''$. Note that $\mathfrak{H}_\wp$ is of type $(0, \ldots, 0, 1)$ and $\mathfrak{H}''_\wp$ is of type $(s_1, \ldots, s_f)$ where

$$\sum_{i<j}(s_j - s_i) = \sum_{i=1}^{f-1} i(f-1-i)(s_{i+1} - s_i) \geq f - 1.$$

It follows from lemma 2.1 that equality must hold, whence $\mathfrak{H} = \mathfrak{H}''$ and we have either $(s_1, \ldots, s_f) = (0, \ldots, 0, 1)$ or $(s_1, \ldots, s_f) = (0, 1, \ldots, 1)$. Since

$$\sigma \neq \sigma^{-1} = \rho(\mathfrak{D}, \mathfrak{D}'') = [\wp, \Sigma_{\mathfrak{A}}/k]^{\sum_{i=1}^f s_i},$$

it follows that $(s_1, \ldots, s_f) = (0, 1, \ldots, 1)$.

Next consider the image $\mathbb{K}$ of $\mathfrak{H}$ in $\mathfrak{D}_\wp/u\mathfrak{D}_\wp$. By definition of $\mathfrak{D}_\wp$ and $\mathfrak{D}'_\wp$ we have that $\mathbb{K}$ is the algebra of matrices in $\mathbb{M}_f(\mathfrak{E}/u\mathfrak{E})$ of the form

$$\begin{pmatrix} a & v \\ 0 & B \end{pmatrix},$$

where B is a block of $f-1$ rows and $f-1$ columns. On the other hand, since $\mathfrak{H} = \mathfrak{D} \cap \mathfrak{D}''$, the algebra $\mathbb{K}$ is conjugate to the algebra of matrices of the form

$$\begin{pmatrix} B & w \\ 0 & a \end{pmatrix},$$

where B is a block of $(f-1) \times (f-1)$. When $f \geq \mathrm{ord}(\sigma) > 2$, these two algebras are not isomorphic, since only the first one has an element P that satisfies the following conditions:

1. $P^2 = P$.

2. $\mathbb{K}P$ is an ideal of dimension 1 over the residue field.

$\square$

Remark 3.1. If the order of σ is bigger than 3, we can give a simpler proof by showing that $\mathfrak{H}$ cannot be contained in an order $\mathfrak{D}''$ such that $\rho(\mathfrak{D}, \mathfrak{D}'') = \sigma^2$.

Proof of the sufficiency. Let $\mathfrak{A}$ be a central simple k-algebra of dimension n^2. Let $\mathfrak{A}_{\mathbb{A}}$ be the algebra of adelic points of $\mathfrak{A}$. For any $\sigma \in \mathfrak{A}_{\mathbb{A}}^*$ we define an idele $\mathcal{N}(\sigma) \in J_k$ by the local relations $\mathcal{N}(\sigma)_\wp = N_\wp(\sigma_\wp)$, where $N_\wp$ is the reduced norm on $\mathfrak{A}_\wp$. Let $\mathfrak{A}_{\mathbb{A}}^*$ act on the set of maximal orders by conjugation and let Γ be the stabilizer of the maximal order $\mathfrak{D}$. It follows from §3 in [1] that the set of spinor genera of maximal orders of $\mathfrak{A}$ is in correspondence with the quotient $J_k/k^*\mathcal{N}(\Gamma)$, and therefore with the Galois group of the corresponding class field $\Sigma_{\mathfrak{A}}$. Since $\mathfrak{A}$ has dimension greater than 2^2, condition E is satisfied, whence spinor genera coincide with conjugacy classes. The orders containing a copy of $\mathfrak{H}$ correspond to the images under $\mathcal{N}$ of the generators for $\mathfrak{D}|\mathfrak{H}$, i.e., the elements $\sigma \in \mathfrak{A}_{\mathbb{A}}^*$ satisfying $\mathfrak{H} \subseteq \sigma\mathfrak{D}\sigma^{-1}$ [1]. It suffices, therefore, to prove that the image $[\mathcal{N}(X), \Sigma_{\mathfrak{A}}/k]$ of the set $X = X_{\mathfrak{D}|\mathfrak{H}}$ of generators is a subgroup of $\mathrm{Gal}(\Sigma_{\mathfrak{A}}/k)$.

Let σ and τ be generators for $\mathfrak{D}|\mathfrak{H}$. By definition, this means $\mathfrak{H} \in \sigma\mathfrak{D}\sigma^*$ and $\mathfrak{H} \in \tau\mathfrak{D}\tau^*$. We define a third adelic element $\lambda \in \mathfrak{A}_{\mathbb{A}}$ as follows:

- if the valuations $v_\wp[N_\wp(\sigma_\wp)]$ and $v_\wp[N_\wp(\tau_\wp)]$ have the same parity, we define $\lambda_\wp = 1$.
- if $v_\wp[N_\wp(\sigma_\wp)]$ is odd and $v_\wp[N_\wp(\tau_\wp)]$ is even, we define $\lambda_\wp = \sigma_\wp$.
- if $v_\wp[N_\wp(\sigma_\wp)]$ is even and $v_\wp[N_\wp(\tau_\wp)]$ is odd, we define $\lambda_\wp = \tau_\wp$.
- if $\wp$ is archimedean, we define $\lambda_\wp = \sigma_\wp\tau_\wp$.

Since the property defining a generator is local, the element λ is a generator. On the other hand, we have

$$\mathcal{N}(\lambda) = \mathcal{N}(\sigma)\mathcal{N}(\tau)ur^2$$

for some ideles $r \in J_k$ and $u \in J_{k,\infty}^+$, where $J_{k,\infty}^+$ is the subgroup of ideles that are positive at real places and units at finite places. Since the extension $\Sigma_{\mathfrak{A}}/k$ is unramified, the subgroup $J_{k,\infty}^+$ has trivial image under the Artin map. By the hypothesis on $\mathrm{Gal}(\Sigma_{\mathfrak{A}}/k)$, also r^2 has trivial image. $\square$

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