ON THE CLASSIFICATION OF COMMUTATIVE RIGHT-NILALGEBRAS OF NILINDEX FIVE AND DIMENSION FOUR[1]

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Abstract

Gerstenhaber and Myung in [10] classified all commutative-power associative nilalgebras of dimension 4. In [7] Gerstenhaber and Myung's results are generalized by giving a classification of commutative right-nilalgebras of right-nilindex four and dimension at most four, without assuming power-associativity. In this paper we complete this research and give a classification of commutative right-nilalgebras of right-nilindex five and dimension four, without assuming power-associativity, thus completing the classification of commutative right-nilalgebras of dimension at most four.

Key words: nilpotency, right-nilalgebras, power-associative.

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1 Introduction

The problem of nilpotency in a commutative power-associative algebra is known as *Albert's problem* [1] (1948): Is every commutative finite dimensional power-associative nilalgebra nilpotent?

Suttles [14] in 1972, gave an example of a commutative five dimensional power-associative nilalgebra of nilindex 4 which is solvable and not nilpotent, thus showing Albert's conjecture to be false and forcing a new reformulation of the conjecture. Therefore what is now really known by the Albert conjecture is whether any commutative finite dimensional power associative nilalgebra is solvable.

Gerstenhaber and Myung [10] proved that any commutative and power-associative nilalgebra of nilindex 4 and dimension 4 over a field of characteristic $\neq 2$ is nilpotent. There are many others papers dealing with Albert's problem (see for instance [2], [3], [4], [5], [6], [8], [9], [11] and [12]).

Moreover, in [10] Gerstenhaber and Myung determined the isomorphism classes of commutative power-associative nilalgebras of nilindex four and dimension four. They found one family of algebras parameterized by $F^{\times}/(F^{\times})^2$ and 5 individual algebras, two of them being associative.

More recently, Elduque and Labra in [7] determined the isomorphism classes of commutative right-nilalgebras of nilindex four and dimension four. They do not assume power-associativity. They found 7 individual non isomorphic algebras.

In this paper we complete this study by giving a classification of commutative right-nilalgebras of right-nilindex five and dimension four, over fields of characteristic $\neq 2$ without assuming power-associativity. Extending thus the classification given by Gerstenhaber and Myung.

2 Right-nilpotency

Let A be a nonassociative algebra and let $x \in A$. We define the *right-principal powers of* x by $x^1 = x$ and $x^{n+1} = x^n x$ for all $n \geq 1$. An element $x \in A$ is called *right-nilpotent with right-nilindex* $n > 1$ if $x^n = 0$ and $x^{n-1} \neq 0$. A is called *right-nilalgebra with right-nilindex* n if $x^n = 0$ for all $x \in A$ and there exists some a in A with $a^{n-1} \neq 0$.

The algebra A is called *nilpotent* (respectively, *right-nilpotent*) in case the descending chain of ideals (respectively right ideals) defined by $A^1 = A$, $A^n = \sum_{r+s=n} A^r A^s = \sum_{i=1}^{n-1} A^i A^{n-i}$ for $n > 1$ (or $A^{<1>} = A$ and $A^{<n>} = A^{<n-1>}A$ for $n > 1$) ends up in zero. The smallest r such that $A^r = 0$ (respectively $A^{<r>} = 0$) is called the *index of nilpotency* (respectively *index of right-nilpotency*) of A. Clearly, if A is nilpotent then A is right-nilpotent. Moreover, if A is commutative or anticommutative, then $A^5 = A^3A^2 + A^4A \subseteq A^3A = A^3A = A^{<4>}$. Thus, if $A^{<4>} = 0$, then A is nilpotent with nilpotent index at most 5.

More generally, it is known (see [15, Proposition 1]) that if A is a commutative or anti-commutative algebra, then $A^{2^n} \subseteq A^{<n>}$. Therefore if A is right-nilpotent, it is nilpotent too.

Throughout the paper, all the algebras considered will be commutative and defined over a ground field F of characteristic $\neq 2$.

2.1 Nilindex 5

In what follows A will be a commutative nonassociative algebra that satisfies strictly the identity $x^5 = (((xx)x)x)x = 0$, and contains an element a in A such that $a^4 \neq 0$. This means that A satisfies the identity $x^5 = 0$ and all its linearizations.

The following notation will be used. Given a set S, $\langle S \rangle$ will denote the subspace generated by S, while alg$\langle S \rangle$ will be the subalgebra generated by S.

Theorem 1. *Let A be a commutative algebra of dimension 4 over a field F, char$(F) \neq 2$,which satisfies the identity $x^5 = 0$ strictly and contains elements y with $y^4 \neq 0$. Then A is nilpotent.*

Proof: Extending scalars if necessary to get an infinite ground field, the set $\mathcal{G} = \{x \in A \ \mid \ x^4 \neq 0\} = \{x \in A \ \mid \ \{x, x^2, x^3, x^4\} \text{ is a basis of A } \}$ is Zariski-open and not empty in A, so it is dense and we conclude that $L_x^4 = 0$, for any $x \in \mathcal{G}$ where $L_x(y) = xy$ is the left multiplication operator and, by Zariski density, we have that this is true for any $x \in A$. That is

$$L_x^4 = 0 \ \forall \ x \in A. \tag{1}$$

All references to density will refer to density in Zariski topology (for its definition and main features on not necessarily finite dimensional spaces one may consult [13]).

Using the nilpotency of the operator L_x, the first linearization of the identity $x^5 = 0$ becomes $((x^2y)x)x + (x^3y)x + x^4y = 0$ for every $x, y \in A$.

Expressing this identity in terms of multiplication operators we obtain:

$$L_x^2 L_{x^2} + L_x L_{x^3} + L_{x^4} = 0 \ \forall \ x \in A. \tag{2}$$

Fix an element $x \in \mathcal{G}$, then for every $y \in A$, there exist $\alpha_{y,x^i} \in F, \ \ i = 1, \cdots, 4$ such that $y = \alpha_{y,x}\, x + \alpha_{y,x^2}\, x^2 + \alpha_{y,x^3}\, x^3 + \alpha_{y,x^4}\, x^4$. The equation (2) is equivalent to the following conditions for every $y \in A$,

$$\alpha_{x^4 y,x} = 0, \tag{3}$$

$$\alpha_{x^4 y,x^2} + \alpha_{x^3 y,x} = 0, \tag{4}$$

$$\alpha_{x^4 y,x^3} + \alpha_{x^3 y,x^2} + \alpha_{x^2 y,x} = 0, \tag{5}$$

$$\alpha_{x^4 y,x^4} + \alpha_{x^3 y,x^3} + \alpha_{x^2 y,x^2} = 0. \tag{6}$$

In particular we have that for $a, b, c, d, e, f \in F$,

$$x^4 x^4 = ax^3 + bx^4, \ \ x^3 x^4 = -ax^2 + cx^3 + dx^4, \ \ x^2 x^4 = -(b+c)x^2 + ex^3 + fx^4$$

If $a = 0$, then the nilpotency of the operator L_{x^4} implies that $b = 0$. Since $x^3(x^4 - dx) = cx^3$, the nilpotency of $L_{x^4 - dx}$ implies that $c = 0$ and $x^3 x^4 = dx^4$. Therefore $d = 0$, as L_{x^3} is nilpotent. In this way we have that $x^3 x^4 = 0 = x^4 x^4$. Now (4) implies that $\alpha_{x^3 x^2,x} = 0 = \alpha_{x^3 x^3,x}$ and then (5) implies that $\alpha_{x^3 x^3,x^2} = 0$. Therefore $x^3 x^3 = gx^3 + hx^4, \ \ g, h \in F$, that is, $x^3(x^3 - hx) = gx^3$ and the nilpotency of $L_{x^3 - hx}$ implies that $g = \alpha_{x^3 x^3,x^3} = 0$. Now (6) implies that $\alpha_{x^2 x^3,x^2} = 0$. Therefore, $I = \langle x^3, x^4 \rangle$ is an ideal of A and A/I is a two-dimensional right-nilalgebra, then A is nilpotent. In particular, we have that $x^2 x^2 \in \langle x^3, x^4 \rangle$ and (5) implies that $\alpha_{x^4 x^2,x^3} = 0$ so $e = 0$ and $\langle x^4 \rangle$ is an ideal of A. Since L_y is nilpotent for every, $y \in A$, we have that $x^4A = 0,\ A\,/\langle x^4\rangle$ is a three-dimensional right-nilalgebra, hence nilpotent, and therefore the whole A is nilpotent.

Now we will see that the case $a \neq 0$ is not possible. If $a \neq 0$, extending scalars if necessary, we can take $a = 1$. The equations (3), (4), (5) and (6) give the following matrices A_i corresponding to $L_{x^i},\ i = 1, \cdots, 4$.

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & d+k-e & b+c & 0 \\ 0 & -(i+f) & -(d+k) & -(b+c) \\ 1 & g & i & e \\ 0 & h & j & f \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & b+c & 1 & 0 \\ 0 & -(d+k) & -(b+2c) & -1 \\ 0 & i & k & c \\ 1 & j & l & d \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -(b+c) & -1 & 0 \\ 0 & e & c & 1 \\ 0 & f & d & b \end{pmatrix}.$$

with $b, c, d, e, f, g, h, i, j, k, l \in F$.

For every $x_1, x_2, x_3, x_4 \in F$, the matrix $A = x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4$ is nilpotent, $A^4 = 0$ and the $(1, 1)$ entry of A^4 is a polynomial $p_{(1,1)}(x_1, x_2, x_3, x_4)$ of degree ≤ 4 which is zero for any value of x_1, x_2, x_3, x_4. Since F is an infinite field, all the coefficient are zero. Using a program of symbolic calculus we obtain that:

The coefficient of $x_1^2x_3x_4$ in $p_{(1,1)}$ is $-c - 2b$, then $c = -2b$.

Replacing this value of c, the coefficient of $x_1x_3x_4^2$ is $f - b^3$, those of $x_2x_3x_4^2$ is $-k + 4b^2$ and those of $x_1^2x_3^2$ is $-d - 2b^2$. Therefore, $f = b^3,\ k = 4b^2$ and $d = -2b^2$.

Substituting the above values, the coefficient of x_3^4 is $-i-2b^3$, those of $x_3^3x_4$ is b^2-e and $i=-2b^2,\ e=b^2$.

Finally substituting the new values, the coefficient of $x_1x_3^3$ is $-2b^5-bj-1$ and $b\neq 0$. On the other hand, the coefficient of $x_1x_2^3$ is b^8-b^4g, so $g=b^4$. But the coefficient of x_2^4 is b^2 then $b=0$. A contradiction. This finish the proof of the Theorem. $\square$

Remark 1. *In the proof of the above Theorem it was proved that* $x^4A=0$ *and that* $x^2x^2\in\ \langle x^3,x^4\rangle$ *which is an ideal of* A*. Using (6) this fact prove that* $x^2x^3\in\ \langle x^4\rangle$ *and the multiplication table of* A *in the basis* $\{x,x^2,x^3,x^4\}$ *is:*

	x	x^2	x^3	x^4
x	x^2	x^3	x^4	0
x^2	x^3	$\epsilon x^3+\alpha x^4$	βx^4	0
x^3	x^4	βx^4	γx^4	0
x^4	0	0	0	0

for suitable scalars $\epsilon,\alpha,\beta,\gamma\in F$*. Moreover, scaling* x *if necessary,* ϵ *may be taken to be either* 0 *or* 1*, with* $\epsilon=0$ *if and only if* $(A^2A^2)A=0$*.*

Remark 2. *Correa, Hentzel and Peresi in [2] prove the above Theorem in* $char(F)\neq 2,3$ *using the complete linearization of the identity* $x^5=0$ *and evaluating there several elements of* A*. The proof that we give is valid in* $char(F)=3$ *and use elementary linear algebra instead of the complete linearization.*

3 Classification

Our aim in this section is to classify the algebras in Remark 1.

CASE $\epsilon = 0$. Here we have that $A^2A^2 \subseteq A^{<4>} = A^4$.

Table I

	x	x^2	x^3	x^4
x	x^2	x^3	x^4	0
x^2	x^3	αx^4	βx^4	0
x^3	x^4	βx^4	γx^4	0
x^4	0	0	0	0

Denote by $A^0(\alpha, \beta, \gamma)$ the algebra with multiplication given by Table I.

Let $y \in \mathcal{G}\,(= \{z \in A : z^4 \neq 0\})$ then $y = \mu x + \nu x^2 + \rho x^3 + \delta x^4$, with $\mu \neq 0$.

$y^2 = \mu^2 x^2 + 2\mu\nu x^3 + (2\mu\rho + \alpha\nu^2 + 2\nu\rho\beta + \rho^2\gamma)x^4$,

$y^3 = \mu^3 x^3 + (2\mu^2\nu + \mu^2\rho\beta + \mu^2\nu\alpha + 2\mu\nu^2\beta + 2\mu\nu\rho\gamma)x^4$,

$y^4 = (\mu^4 + \mu^3\nu\beta + \mu^3\rho\gamma)x^4 = \mu^3(\mu + \beta\nu + \gamma\rho)x^4$,

$y^2y^2 = (\mu^4\alpha + 4\mu^3\nu\beta + 4\mu^2\nu^2\gamma)x^4 = \mu^2(\mu^2\alpha + 4\mu\nu\beta + 4\nu^2\gamma)x^4$,

$y^2y^3 = (\mu^5\beta + 2\mu^4\nu\gamma)x^4 = \mu^4(\mu\beta + 2\nu\gamma)x^4$,

$y^3y^3 = \mu^6\gamma x^4$.

We have three sub-cases:

Sub-case (i): $A^2A^3 = 0$. Then $\beta = \gamma = 0$. The multiplication table depends on α. We have that $y^4 = \mu^4 x^4$ and $y^2y^2 = \mu^4\alpha x^4 = \alpha y^4$. Therefore:

$$A^0(\alpha, 0, 0) \simeq A^0(\alpha', 0, 0) \;\Leftrightarrow\; \alpha = \alpha'.$$

Moreover $A^0(\alpha,0,0)$ is associative if and only if $\alpha=1$. Note that $A^0(\alpha,0,0)$ satisfies $z^2z^2=\alpha z^4$ for any $z\in A$.

Sub-case (ii): $A^3A^3=0\neq A^2A^3$. Then $\gamma=0\neq\beta$.

In this case we have

$y^4=\mu^3(\mu+\beta\nu)x^4,\ y^2y^2=\mu^3(\mu\alpha+4\nu\beta),\ y^2y^3=\mu^5\beta x^4,\ y^3y^3=0.$

Since $y^4\neq 0$ then $\mu\neq 0\neq\mu+\beta\nu$, and

$$y^2y^2=\alpha' y^4,\ \ \alpha'=\frac{\mu\alpha+4\nu\beta}{\mu+\nu\beta},$$

$$y^2y^3=\beta' y^4,\ \ \beta'=\frac{\mu^2\beta}{\mu+\nu\beta}.$$

If we take $\nu=\frac{\mu(\mu\beta-1)}{\beta}$, we get $\beta'=1$ and $\alpha'=\frac{\alpha-4+4\mu\beta}{\mu\beta}$.

Two possibilities appear:

(a) $\alpha\neq 4$, then taking $\mu=\frac{4-\alpha}{\beta}$, we have the algebra $A^0(0,1,0)$.

(b) $\alpha=4$, and we have the algebra $A^0(4,1,0)$. This algebra satisfies $y^2y^2=4y^4$, for any $y\in\mathcal{G}$, since $y^2y^2=\mu^3(4\mu+4\nu\beta)x^4=4\mu^3(\mu+\nu\beta)x^4=4y^4$ and, since $\mathcal{G}$ is Zariski open, $A^0(4,1,0)$ satisfies $y^2y^2=4y^4$ for any $y\in A$.

Therefore both algebras are not isomorphic.

Sub-case (iii): $A^3A^3\neq 0$. Then $\gamma\neq 0$.

In this case we have $y^4=\mu^3(\mu+\nu\beta+\rho\gamma)x^4$ and $y^2y^3=\mu^4(\mu\beta+2\nu\gamma)x^4$. Since $y^4\neq 0$ then $0\neq\mu+\beta\nu+\rho\gamma$, and we have $y^2y^3=\beta' y^4$, where $\beta'=\frac{\mu(\mu\beta+2\nu\gamma)}{\mu+\nu\beta+\rho\gamma}$.

Taking $\nu=-\frac{\mu\beta}{2\gamma}$ and ρ such that $\mu+\nu\beta+\rho\gamma\neq 0$, we have $\beta'=0$. Hence we may assume that $\beta=0\neq\gamma$. In this case:

$y^4=\mu^3(\mu+\rho\gamma)x^4,\ y^2y^2=\mu^2(\mu^2\alpha+4\nu^2\gamma)x^4,\ y^2y^3=2\mu^4\nu\gamma x^4,\ y^3y^3=\mu^6\gamma x^4.$

Therefore if $\mu \neq 0 \neq \mu + \gamma\rho,\;\; y^4 \neq 0$ and we obtain that

$$y^2y^2 = \alpha' y^4, \text{ where } \alpha' = \frac{\mu^2\alpha + 4\nu^2\gamma}{\mu(\mu+\rho\gamma)}$$
$$y^2y^3 = \beta' y^4, \text{ where } \beta' = \frac{2\mu\nu\gamma}{\mu+\rho\gamma}$$
$$y^3y^3 = \gamma' y^4, \text{ where } \gamma' = \frac{\mu^3\gamma}{\mu+\rho\gamma}.$$

If we take $\nu = 0,\; \rho = \frac{\mu^3\gamma-\mu}{\gamma}$ this imply that $\beta' = 0, \gamma' = 1$ and

$$\alpha' = \frac{\mu\alpha}{\mu+\rho\gamma} = \frac{\mu\alpha}{\mu^3\gamma} = \frac{\alpha}{\mu^2\gamma}$$

With $\mu = \frac{1}{\eta^2\gamma}$, we have $\alpha' = \alpha\gamma\eta^2 = \theta\eta^2$, with $\theta = \alpha\gamma$ and the algebra is (isomorphic to) $A^0(\theta\eta^2, 0, 1)$, for any $\eta \in F^\times$.

We can observe that for the algebra $A = A^0(\theta, 0, 1)$, the matrix of the bilinear form

$$A^2/A^{<4>} \times A^2/A^{<4>} \quad \longrightarrow \quad A^{<4>} = Fx^4 \simeq F$$

in the basis $\{x^2 + A^{<4>}, x^3 + A^{<4>}\}$ is $\left(\begin{smallmatrix}\theta & 0\\ 0 & 1\end{smallmatrix}\right)$.

Then for this matrix we have that $\theta(F^\times)^2$ is the discriminant of this bilinear form. Therefore, denoting the multiplicative group of F by $F^\times$ we have

$$A^0(\theta, 0, 1) \simeq A^0(\theta', 0, 1) \;\Leftrightarrow\; \theta(F^\times)^2 = \theta'(F^\times)^2.$$

Theorem 2. *Let A be a four dimensional commutative and not power associative algebra satisfying the identity $x^5 = 0$ but not $x^4 = 0$, and such that $(A^2A^2)A = 0$. Then A is isomorphic to one and only one of the following algebras: $A^0(\alpha, 0, 0)$ ($\alpha \in F$), $A^0(0, 1, 0)$, $A^0(4, 1, 0)$, $A^0(0, 0, 1)$, or $A^0(\alpha, 0, 1)$ ($\alpha \in \mathcal{H}$), where $\mathcal{H}$ is a set of representatives of $F^\times/(F^\times)^2$.*

Corollary 1. *Let F be a quadratically closed field. Then any four dimensional commutative and not power-associative algebra satisfying the identity $x^5 = 0$ and with multiplication given by table I is isomorphic to one and only one of the following algebras: $A^0(\alpha, 0, 0)$ ($\alpha \in F$), $A^0(0, 1, 0)$, $A^0(4, 1, 0)$, $A^0(0, 0, 1)$ or $A^0(1, 0, 1)$.*

CASE $\epsilon = 1$. Here we have that $A^2A^2 \subseteq A^3 = A^4$.

Table II

	x	x^2	x^3	x^4
x	x^2	x^3	x^4	0
x^2	x^3	$x^3 + \alpha x^4$	βx^4	0
x^3	x^4	βx^4	γx^4	0
x^4	0	0	0	0

Denote by $A^1(\alpha, \beta, \gamma)$ the algebra with this multiplication table.

Let $y \in A \setminus A^2$ then $y = \mu x + \nu x^2 + \rho x^3 + \delta x^4$, with $\mu \neq 0$.

$y^2 = \mu^2 x^2 + (2\mu\nu + \nu^2)x^3 + (2\mu\rho + \alpha\nu^2 + 2\nu\rho\beta + \rho^2\gamma)x^4$,

$y^3 = \mu^2(\mu+\nu)x^3 + (\mu^2\nu\alpha + \mu(2\mu\nu+\nu^2) + \mu^2\rho\beta + (2\mu\nu+\nu^2)\nu\beta + (2\mu\nu+\nu^2)\rho\gamma)x^4$,

$y^4 = \mu^2(\mu + \nu)[\mu + \nu\beta + \rho\gamma]x^4$

$y^2y^2 = \mu^4x^3 + [\mu^4\alpha + 2\mu^2(2\mu\nu + \nu^2)\beta + (2\mu\nu + \nu^2)^2\gamma]x^4$

$y^2y^3 = [\mu^4(\mu + \nu)\beta + \mu^2(\mu + \nu)(2\mu\nu + \nu^2)\gamma]x^4$,

$y^3y^3 = \mu^4(\mu + \nu)^2\gamma x^4$.

Therefore $y^4 \neq 0$ (that is, $y \in \mathcal{G}$) if and only if $\mu, \mu + \nu, \mu + \nu\beta + \rho\gamma \neq 0$. Putting $\Delta = \mu + \nu\beta + \rho\gamma$ we have that $\Delta \neq 0$ for $y \in \mathcal{G}$. On the other hand

$y^2y^2 - y^3 \in \langle y^4 \rangle = \langle x^4 \rangle$ if and only if $\mu^4 = \mu^2(\mu+\nu)$. Since $\mu \neq 0$, then

$$\mu^2 = \mu+\nu,\ \Delta = \beta\mu^2+(1-\beta)\mu+\rho\gamma,\ \ y^4 = \mu^4\Delta\ x^4, \text{ that is } x^4 = \frac{1}{\mu^4\Delta}y^4. \quad (7)$$

and

$$2\mu\nu+\nu^2 = \nu(\mu+(\mu+\nu)) = (\mu^2-\mu)(\mu+\mu^2) = \mu(\mu-1)\mu(\mu+1) = \mu^2(\mu^2-1). \quad (8)$$

Since $\Delta \neq 0$, using (7) and (8) we have that:

$y^3y^3 = \mu^8\gamma x^4 = \frac{\mu^4\gamma}{\Delta}\ y^4$
$y^2y^3 = (\mu^6\beta + \mu^6(\mu^2-1)\gamma)x^4 = \frac{\mu^2(\beta+(\mu^2-1)\gamma)}{\Delta}\ y^4$
$y^2y^2 = \mu^4x^3 + [\mu^4\alpha + 2\mu^4(\mu^2-1)\beta + (\mu^2(\mu^2-1))^2\gamma]x^4$

On the other hand,

$$\begin{aligned} y^3 = \mu^4x^3 + [&\mu^3(\mu-1)\alpha + \mu^3(\mu^2-1) + \mu^2\rho\beta + \mu^3(\mu^2-1)(\mu-1)\beta \\ &+ \mu^2(\mu^2-1)\rho\gamma])x^4 \\ = \mu^4x^3 + \mu^2&[\mu(\mu-1)\alpha + \mu(\mu^2-1) + (\rho(\beta+(\mu^2-1)\gamma) + \mu(\mu^2-1)(\mu-1)\beta]x^4, \end{aligned}$$

Therefore

$$y^3 = \mu^4x^3 + \mu^2\Gamma x^4 = \mu^4x^3 + \frac{\Gamma}{\mu^2\Delta}y^4,$$

where $\Gamma = \mu(\mu-1)\alpha + \mu(\mu^2-1) + \rho(\beta+(\mu^2-1)\gamma) + \mu(\mu^2-1)(\mu-1)\beta$.

Since $\mu \neq 0$ and $\Delta \neq 0$, we obtain $x^3 = \frac{1}{\mu^4}y^3 - \frac{\Gamma}{\mu^6\Delta}y^4$

Replacing this value in $y^2y^2 = \mu^4x^3 + \mu^4[\alpha + 2(\mu^2-1)\beta + (\mu^2-1)^2\gamma]x^4$ and putting $\Phi = \alpha + 2(\mu^2-1)\beta + (\mu^2-1)^2\gamma$ we obtain

$$y^2y^2 = \mu^4(x^3+\Phi x^4) = y^3 - \frac{\Gamma}{\mu^2\Delta}y^4 + \frac{\Phi}{\Delta}y^4 = y^3 + \frac{\mu^2\Phi - \Gamma}{\mu^2\Delta}\ y^4. \quad (9)$$

Summarizing, if y is any element of A with $y^4 \neq 0$ and $y^2y^2 - y^3 \in \langle y^4 \rangle$, then there are elements $\mu \neq 0$ and ρ in F such that:

$$y^2y^2 = y^3 + \alpha' y^4$$
$$y^2y^3 = \beta' y^4$$
$$y^3y^3 = \gamma' y^4$$

where

$$\alpha' = \frac{\mu^2\Phi - \Gamma}{\mu^2\Delta}, \quad \beta' = \frac{\mu^2\big(\beta + (\mu^2-1)\gamma\big)}{\Delta}, \quad \gamma' = \frac{\mu^4\gamma}{\Delta},$$

with

$$\Delta = \mu + \mu(\mu-1)\beta + \rho\gamma,$$
$$\Phi = \alpha + 2(\mu^2-1)\beta + (\mu^2-1)^2\gamma,$$
$$\Gamma = \mu(\mu^2-1) + \mu(\mu-1)\alpha + (\rho + \mu(\mu^2-1)(\mu-1))\beta + (\mu^2-1)\rho\gamma.$$

We have three sub-cases:

Sub-case (i): $A^2A^3 = 0$. Then $\beta = \gamma = 0$. So we have that

$$\Delta = \mu,\ \Gamma = \mu(\mu-1)\alpha + \mu(\mu^2-1),\ \Phi = \alpha.$$

Then $y^2y^3 = 0$, $y^3y^3 = 0$ and
$y^2y^2 = y^3 + \alpha' y^4$, where $\alpha' = \frac{\mu^2\Phi-\Gamma}{\mu^2\Delta} = \frac{\mu^2\alpha-\mu(\mu-1)\alpha-\mu(\mu^2-1)}{\mu^3} = \frac{\alpha-\mu^2+1}{\mu^2}$, that is,
$(1+\alpha')\mu^2 = 1+\alpha$.

Therefore:

$$A^1(\alpha,0,0) \simeq A^1(\alpha',0,0) \ \Leftrightarrow\ (1+\alpha)(F^\times)^2 = (1+\alpha')(F^\times)^2.$$

Sub-case (ii): $A^3A^3 = 0 \neq A^2A^3$. Then $\gamma = 0 \neq \beta$.

In this case we have

$$\Delta = \beta\mu^2 + (1-\beta)\mu = \mu((\mu-1)\beta+1),\ \Phi = \alpha + 2(\mu^2-1)\beta,$$

$$\Gamma = \mu(\mu-1)\alpha + \mu(\mu^2-1) + \mu(\mu^2-1)(\mu-1)\beta + \rho\beta.$$

Therefore, using (7) and (9) we obtain that

$y^3y^3 = 0, \quad y^2y^2 = y^3 + \alpha' y^4,$ where $\alpha' = \frac{\mu^2\Phi-\Gamma}{\mu^2\Delta}$.
$y^2y^3 = \beta' y^4$, where $\beta' = \frac{\mu^2\beta}{\Delta} = \frac{\mu\beta}{\beta\mu+(1-\beta)}$.

We have two possibilities:

(a) $\beta = 1$. Then $\beta' = 1$ and taking $\mu = 1$ and ρ such that $\Gamma = \mu^2\Phi$ we have that $\alpha' = 0$, so that the algebra is (isomorphic to) $A^1(0,1,0)$.

(b) $\beta \neq 1$. Then taking $\mu = \frac{2(\beta-1)}{\beta}$, we obtain $\beta' = 2$, and then taking ρ such that $\mu^2\Phi = \Gamma$, we have that $\alpha' = 0$, so that the algebra is $A^1(0,2,0)$.

Both algebras are not isomorphic, because the first algebra satisfies $z^2z^2 \cdot z = z^2z^3$, $\forall\, z \in A$ which is not satisfied by the second one. In fact, for $\beta = 1$, if $z = ax + bx^2 + cx^3 + dx^4$, then $z^2 = a^2x^2 + (b^2+2ab)x^3 + 2(ac+bc)x^4$, $z^3 = (a^3+ba^2)x^3 + (ab^2+2a^2b)x^4 + (b^3+2ab^2+a^2c)x^4$, $z^2z^2 = a^4x^3 + 2(a^2b^2 + 2a^3b)x^4$, $(z^2z^2)z = a^5x^4 + ba^4x^4$, $z^2z^3 = a^2(a^3+ba^2)x^4$.

Sub-case (iii): $A^3A^3 \neq 0$. Then $\gamma \neq 0$.

Since $\gamma' = \frac{\mu^4\gamma}{\Delta} = \frac{\mu^4\gamma}{\mu+\mu(\mu-1)\beta+\rho\gamma}$, ρ can be taken so that $\gamma' = 1$, and we have to deal with the algebra $A^1(\alpha,\beta,1)$.

Thus, assume from now on that $\gamma = 1$. Observe that for $y = \mu x + \nu x^2 + \rho x^3 + \delta x^4$ one gets:

$$\begin{aligned}(y^2)^3 &= \mu^4 x^3(\mu^2 x^2 + (2\mu\nu + \nu^2)x^3)\\ &= \big(\mu^6\beta + \mu^4(2\mu\nu + \nu^2)\big)x^4\\ &= \mu^4(\mu^2\beta + 2\mu\nu + \nu^2)x^4,\end{aligned}$$

while $y^3y^3 = \mu^4(\mu+\nu)^2x^4$, so $(y^2)^3 = (y^3)^2$ for any y if and only if $\beta = 1$.

Now, with $\gamma = 1$, we look for elements $y \in \mathcal{G}$ such that $y^2y^2 - y^3 \in \langle\, y^4 \,\rangle$ and $y^3y^3 = y^4$.

This gives $\mu^4 = \Delta$, so $\rho = \mu^4 - \mu - \mu(\mu-1)\beta = \mu(\mu-1)(\mu^2+\mu+1-\beta)$, and

$$\beta' = \frac{\mu^2(\beta + (\mu^2-1))}{\Delta} = \frac{\beta + (\mu^2-1)}{\mu^2} = \frac{\beta-1}{\mu^2} + 1,$$

that is,

$$\beta - 1 = \mu^2(\beta' - 1).$$

Therefore, if $A^1(\alpha,\beta,1)$ is isomorphic to $A^1(\alpha',\beta',1)$, then $(\beta-1)(F^\times)^2 = (\beta'-1)(F^\times)^2$.

In this case,

$$\begin{aligned}\Gamma &= \mu(\mu^2-1) + \mu(\mu-1)\alpha + \big(\rho + \mu(\mu^2-1)(\mu-1)\big)\beta + (\mu^2-1)\rho\\ &= \mu(\mu^2-1) + \mu(\mu-1)\alpha + \mu(\mu-1)(2\mu^2+\mu-\beta)\beta\\ &\qquad + (\mu^2-1)\mu(\mu-1)(\mu^2+\mu+1-\beta)\\ &= \mu(\mu-1)\Big(\alpha + \mu + 1 + (2\mu^2+\mu-\beta)\beta + (\mu^2-1)(\mu^2+\mu+1-\beta)\Big).\end{aligned}$$

We are left with two possibilities:

(a) $\beta = 1$. Here

$$
\begin{aligned}
\rho &= \mu(\mu-1)(\mu^2+\mu) = \mu^2(\mu^2-1),\\
\Phi &= \alpha + 2(\mu^2-1) + (\mu^2-1)^2 = \alpha + \mu^4 - 1,\\
\Gamma &= \mu(\mu-1)\big(\alpha + \mu + 1 + 2\mu^2 + \mu - 1 + (\mu^2-1)(\mu^2+\mu)\big)\\
&= \mu(\mu-1)\big(\alpha + 2\mu(\mu+1) + \mu(\mu+1)(\mu^2-1)\big)\\
&= \mu(\mu-1)\big(\alpha + \mu(\mu+1)(\mu^2+1)\big)\\
&= \mu(\mu-1)\alpha + \mu^2(\mu^4-1).
\end{aligned}
$$

Hence $\mu^2\Phi - \Gamma = \mu\alpha$, and thus

$$
\alpha' = \frac{\mu^2\Phi - \Gamma}{\mu^2\Delta} = \frac{\mu\alpha}{\mu^6} = \frac{\alpha}{\mu^5}
$$

and, therefore,

$$
A^1(\alpha,1,1) \simeq A^1(\alpha',1,1) \;\Leftrightarrow\; \alpha(F^\times)^5 = \alpha'(F^\times)^5.
$$

(b) $\beta \neq 1$. Here $A^1(\alpha,\beta,1)$ is isomorphic to $A^1(\alpha',\beta',1)$ if and only if there is a scalar $\mu \in F^\times$ such that $\beta - 1 = \mu^2(\beta'-1)$ and $\alpha' = \frac{\mu^2\Phi-\Gamma}{\mu^6}$, with Φ and Γ as above.

Once $\beta \neq 1$ is fixed, $A^1(\alpha,\beta,1)$ is isomorphic to $A^1(\alpha',\beta,1)$ if and only if $\alpha' = \frac{\mu^2\Phi-\Gamma}{\mu^6}$ for $\mu = \pm 1$, if and only if either $\alpha' = \alpha$ ($\mu = 1$) or $\alpha' = -\alpha + (\beta-1)\beta$ ($\mu = -1$).

In particular, if F is quadratically closed, then one can always take $\beta' = 0$, and then we get the algebras $A^1(\alpha,0,1)$ with

$$
A^1(\alpha,0,1) \simeq A^1(\alpha',0,1) \Leftrightarrow \alpha' = \pm\alpha.
$$

Theorem 3. *Let A be a four dimensional commutative and not associative algebra satisfying the identity $x^5 = 0$ but not $x^4 = 0$, and such that $(A^2A^2)A \neq 0$. Then A is isomorphic to one of the following algebras:*

(i) $A^1(-1, 0, 0)$,

(ii) $A^1(\alpha, 0, 0)$, *with* $\alpha \in -1 + \mathcal{H}$, *where* $\mathcal{H}$ *is a set of representatives of* $F^\times/(F^\times)^2$,

(iii) $A^1(0, 1, 0)$,

(iv) $A^1(0, 2, 0)$,

(v) $A^1(0, 1, 1)$,

(vi) $A^1(\alpha, 1, 1)$, *with* $\alpha \in \mathcal{I}$, *where* $\mathcal{I}$ *is a set of representatives of* $F^\times/(F^\times)^5$,

(vii) $A^1(\alpha, \beta, 1)$, *where* $\beta \in 1 + \mathcal{H}$ *($\mathcal{H}$ as in (ii)).*

Moreover, algebras in different items are not isomorphic, and so are algebras in the same item, with the exception of item (vii), where $A^1(\alpha, \beta, 1)$ is isomorphic to $A^1(\alpha', \beta', 1)$ ($\beta, \beta' \in 1 + \mathcal{H}$) if and only if $\beta = \beta'$ and either $\alpha' = \alpha$ or $\alpha' = -\alpha + (\beta - 1)\beta$.

Corollary 2. *Let F be an algebraically closed field. Then any four dimensional commutative and not power-associative algebra satisfying the identity $x^5 = 0$ and the multiplication given by table II is isomorphic to one and only one of the following algebras: $A^1(0, 0, 0)$, $A^1(-1, 0, 0)$, $A^1(0, 1, 0)$, $A^1(0, 2, 0)$, $A^1(0, 1, 1)$, $A^1(1, 1, 1)$, $A^1(0, 0, 1)$, or $A^1(\alpha, 0, 1)$ for $\alpha \in \mathcal{F}$, where $\mathcal{F}$ is a subset of F satisfying $\mathcal{F} \cup -\mathcal{F} = F^\times$ and $\mathcal{F} \cap -\mathcal{F} = \emptyset$.*

Summarizing, in the case of an algebraically closed field F, Table III display all the non isomorphic four dimensional commutative and not power-associative algebras satisfying the identity $x^5 = 0$, but not satisfying the identity $x^4 = 0$.

Table III

$A^4 = A^{<4>}$,	$A^2A^3 = 0$	$A^0(\alpha, 0, 0)$	$y^2y^2 = \alpha y\ \forall y$
$\dim(A^4) = 1.$	$A^3A^3 = 0 \neq A^2A^3$	$A^0(4, 1, 0)$	$y^2y^2 = 4y^4\ \forall\ y \in A$
$(\epsilon = 0)$,		$A^0(0, 1, 0)$	$\exists\ y,\ y^2y^2 \neq 4y^4$
	$A^3A^3 \neq 0$	$A^0(0, 0, 1)$	$\mathrm{rank}\big(A^2 \times A^2 \to A^4 :$ $(u, v) \mapsto uv\big) = 1$
		$A^0(1, 0, 1)$	$\mathrm{rank}(A^2 \times A^2 \to A^4 :$ $(u, v) \mapsto uv) = 2$
$A^4 = A^3$,	$A^2A^3 = 0$	$A^1(0, 0, 0)$	$\exists y \in A \setminus A^2 : y^2y^2 = y^3$
$\dim(A^4) = 2$,		$A^1(-1, 1, 0)$	$\nexists y \in A \setminus A^2 : y^2y^2 = y^3$
$(\epsilon = 1)$,	$A^3A^3 = 0 \neq A^2A^3$	$A^1(0, 1, 0)$	$y^2y^2 \cdot y = y^2y^3\ \forall\ y \in A$
		$A^1(0, 2, 0)$	$\exists\ y \in A,\ y^2y^2 \cdot y \neq y^2y^3$
	$A^3A^3 \neq 0$	$A^1(0, 1, 1)$	$(y^2)^3 = (y^3)^2\ \forall y$ $\exists z\ z^3z^3 = z^4 \neq 0,\ z^2z^2 = z^3$
		$A^1(1, 1, 1)$	$(y^2)^3 = (y^3)^2\ \forall y$ $\nexists z\ z^3z^3 = z^4 \neq 0,\ z^2z^2 = z^3$
		$A^1(\alpha, 0, 1)$	$\exists y\ (y^2)^3 \neq (y^3)^2$ $A^1(\alpha, 0, 1) \simeq A^1(\alpha', 0, 1)$ $\Leftrightarrow\ \alpha' = \pm\alpha$

REFERENCES

1. Albert, A. A. (1948). On power-associative rings. Trans. Amer. Math. Soc. 64: 552-593.

2. Correa, I., Hentzel, I. R., Peresi, L. A. (2003). On the nilpotency of the commutative right nilalgebras of maximum right nilindex. Int. J. Math. Game Theory and Algebra. 14: 299–305.

3. Correa, I., Suazo, A. (1999). On a class of commutative power-associative nilalgebras. J. of Algebra. 215: 412–417.

4. Correa, Ivan., Peresi, Luiz Antonio. (2001). On the solvability of the five dimensional commutative power-associative nilalgebras. Results Math. 39, no. 1-2: 23–27.

5. Correa, Ivan., Hentzel, Irvin Roy., Peresi, Luiz Antonio. (2003). On the solvability of the commutative power-associative nilalgebras of dimension 6. Linear Algebra Appl. 369: 185–192.

6. Correa, Ivan., Hentzel, Irvin Roy., Julca, Pedro Pablo., Peresi, Luiz Antonio. (2005). Nilpotent linear transformations and the solvability of power-associative nilalgebras. Linear Algebra Appl. 396: 35–53.

7. Elduque, A., Labra, A. (2007). On the classification of commutative right-nilalgebras of dimension at most four. Comm. in Algebra, to appear.

8. Elgueta, Luisa., Suazo, Avelino. (2004). Solvability of commutative

power-associative nilalgebras of nilindex 4 and dimension ≤ 8. Proyecciones. 23, no. 2: 123–129.

9. Elgueta, Luisa., Suazo, Avelino., Gutiérrez Fernández, Juan C. (2005). Nilpotence of a class of commutative power-associative nilalgebras. J. Algebra 291: 492–504.

10. Gerstenhaber, M., Myung, H. C. (1975). On commutative power-associative nilalgebras of low dimension. Proc. Amer. Math. Soc. 48: 29–32.

11. Gutiérrez Fernández, J. C.(2004) On Commutative Power-Associative Nilalgebras. Commun. Algebra. 32 (6): 2243-2250.

12. Gutierrez Fernandez, J. Carlos., Suazo, Avelino. (2005). Commutative power-associative nilalgebras of nilindex 5. Results Math. 47: 296–304.

13. McCrimmon, K. (1967). Generically algebraic algebras. Trans. Amer. Math, Soc. 127: 527-551.

14. Suttles, D. A. (1972). Counterexample to a conjecture of Albert. Notices Amer. Math. Soc. 19: A-566.

15. Zhevlakov, K. A. et als. (1992). "Rings that are nearly associative," Academic Press N. York, London.