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FINITE PRODUCTS OF REGULARIZED PRODUCTS

Francisco Diaz y Diaz and Eduardo Friedman

Abstract. The product $(\widehat{\prod}_m a_m)\cdot(\widehat{\prod}_m b_m)$ of two regularized products is not in general equal to the regularized product $\widehat{\prod}_m (a_m \cdot b_m)$. We consider the discrepancy F, defined by

$$\exp(F) := \frac{\widehat{\prod}_m (a_m \cdot b_m)}{(\widehat{\prod}_m a_m)\cdot(\widehat{\prod}_m b_m)}.$$

When the terms a_m and b_m depend on parameters, we show in certain cases that F is a polynomial in these parameters.

1. Introduction

The regularized product $\widehat{\prod}_m a_m$ of a countable set $\{a_1, a_2, \dots\}$ of non-zero complex numbers is defined as

$$\widehat{\prod_m} a_m := \exp\big(-f'(0)\big),$$

where we assume that $f(s) := \sum_m a_m^{-s}$ converges in some right half-plane and has a meromorphic continuation to the s-plane which is regular at $s = 0$, so that its derivative f' can be evaluated there. Several authors [KW] [Mi] have found examples where

$$\Big(\widehat{\prod_m} a_m\Big)\cdot\Big(\widehat{\prod_m} b_m\Big) = \widehat{\prod_m}(a_m \cdot b_m),$$

but Mizuno [Mi] has pointed out that this does not hold in general.[1] For instance [FR, eq. (3.10)], if z_i and τ_i $(i = 1, 2)$ are positive real numbers, then

$$\frac{\widehat{\prod}_{m=0}^{\infty}(m\tau_1 + z_1)\cdot(m\tau_2 + z_2)}{\big(\widehat{\prod}_{m=0}^{\infty}(m\tau_1 + z_1)\big)\cdot\big(\widehat{\prod}_{m=0}^{\infty}(m\tau_2 + z_2)\big)} = \exp\bigg(\frac{1}{2}\big(\frac{z_1}{\tau_1} - \frac{z_2}{\tau_2}\big)\log\big(\frac{\tau_1}{\tau_2}\big)\bigg).$$

A more complicated example was obtained by Mizuno [Mi, p. 157], namely,

$$\frac{\widehat{\prod}_{l,m=0}^{\infty}(l\tau_1 + m\eta_1 + z_1)\cdot(l\tau_2 + m\eta_2 + z_2)}{\big(\widehat{\prod}_{l,m=0}^{\infty}(l\tau_1 + m\eta_1 + z_1)\big)\cdot\big(\widehat{\prod}_{l,m=0}^{\infty}(l\tau_2 + m\eta_2 + z_2)\big)} =: \exp(F), \tag{1}$$

where

$$F = \frac{\tau_1\eta_2 - \tau_2\eta_1}{4}\bigg(\frac{\log\big(\frac{\eta_2}{\eta_1}\big)}{\eta_1\eta_2}B_2\big(\frac{z_2\eta_1 - z_1\eta_2}{\tau_2\eta_1 - \tau_1\eta_2}\big) - \frac{\log\big(\frac{\tau_2}{\tau_1}\big)}{\tau_1\tau_2}B_2\big(\frac{z_2\tau_1 - z_1\tau_2}{\tau_1\eta_2 - \tau_2\eta_1}\big)\bigg), \tag{2}$$

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[1] We always assume that logarithm branches are chosen so that $(a_m b_m)^{-s} = a_m^{-s} b_m^{-s}$.

$B_2(x) := x^2 - x + \frac{1}{6}$, all parameters z_i, τ_i and η_i are again assumed real and positive, and $\tau_1\eta_2 - \tau_2\eta_1 \neq 0$.

Shintani [Sh, pp. 204, 206] had earlier considered a related example

$$\frac{\widehat{\prod}_{\mathbf{m}\in\mathbb{N}_0^r} \prod_{j=1}^n L_j(\mathbf{y}+\mathbf{m})}{\prod_{j=1}^n \widehat{\prod}_{\mathbf{m}\in\mathbb{N}_0^r} L_j(\mathbf{y}+\mathbf{m})} =: \mathrm{e}^{F(\mathbf{y})}. \tag{3}$$

Here $L_1, L_2, \dots, L_n$ are n linear forms with positive coefficients in r positive variables $\mathbf{y} = (y_1, \dots, y_r)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the positive integers. Shintani showed that F is a polynomial in $\mathbf{y}$ of degree at most r and used this to study the derivative at $s = 0$ of L-functions $L(s, \chi)$ attached to Hecke characters χ of a totally real number field. In this case L_j is a real embedding of a linear form with coefficients in the number field and $\prod_j L_j$ is the norm form.

Given a regularized product $\widehat{\prod}_m \big(\prod_{j=1}^n a_m^{(j)} \big)$ of terms that factor in some natural way, we may compare it with the product of the individual regularized products $\widehat{\prod}_m a_m^{(j)}$ (when these are defined). Mizuno suggests studying the discrepancy F, defined by

$$\mathrm{e}^F := \frac{\widehat{\prod}_m \prod_{j=1}^n a_m^{(j)}}{\prod_{j=1}^n \widehat{\prod}_m a_m^{(j)}}, \tag{4}$$

as it seems that F is often far simpler than the regularized products themselves. In (1) this is certainly born out since the regularized products involved are essentially Barnes' double Γ-functions. The discrepancy F in (2), on the other hand, is just a polynomial in the constant terms z_i of the regularized product, and a somewhat more complicated function of the coefficients τ_i and η_i of the terms in degree one.

As the few known examples of the discrepancy F only involve products of terms of degree one in m, here we consider products of general polynomials. However, in order to ensure the existence of the regularized products we must make some assumptions on the polynomials. Several authors [Ma] [Ca2] [Sa] [Li] have given conditions on the polynomial P guaranteeing the existence of a meromorphic continuation in s of the Dirichlet series $\sum_{\mathbf{m}\in\mathbb{N}^r} P(\mathbf{m})^{-s}$, or more generally of $\sum_{\mathbf{m}\in\mathbb{N}^r} \varphi(\mathbf{m})P(\mathbf{m})^{-s}$, where $\varphi(\mathbf{m})$ is an arbitrary complex polynomial. We choose Mahler's conditions, as they are simple to state and imply that a meromorphic continuation of the series to the whole s-plane exists and is regular at $s = 0$.

Mahler's hypothesis on P [Ma, p. 385, Klasse A]. $P(\mathbf{x}) = P(x_1, x_2, \dots, x_r) \in \mathbb{C}[\mathbf{x}]$ does not vanish anywhere in the closed real first "octant" $x_i \geq 0$ $(1 \leq i \leq r)$. Its homogeneous part of highest degree $P_{\mathrm{top}}(\mathbf{x})$ is not constant and vanishes nowhere in the closed real first octant, except for $P_{\mathrm{top}}(\mathbf{0}) = 0$.

Notice that if $P_j(\mathbf{x})$ satisfies this assumption for $1 \leq j \leq n$, then so does $\prod_{j=1}^n P_j(\mathbf{x})$. Under Mahler's hypothesis on P_j, we can choose for each j a continuous branch of $\log P_j(\mathbf{x})$ for real $\mathbf{x}$ in the first octant, this choice being unique up to adding a fixed single multiple of $2\pi i$. Having fixed these branches for each j, we define $\log \prod_{j=1}^n P_j(\mathbf{x}) := \sum_{j=1}^n \log P_j(\mathbf{x})$.

Theorem 1. *Let $P_j(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ be n complex polynomials in r variables, all satisfying Mahler's hypothesis above, and define $F = F(P_1, P_2, \ldots, P_n)$ by*

$$\mathrm{e}^F := \frac{\widehat{\prod}_{\mathbf{m}\in\mathbb{N}^r} \prod_{j=1}^n P_j(\mathbf{m})}{\prod_{j=1}^n \widehat{\prod}_{\mathbf{m}\in\mathbb{N}^r} P_j(\mathbf{m})}. \tag{5}$$

Then F is a polynomial of degree at most r in the coefficients of the P_j of non-maximal degree.

In other words, if we decompose $P_j(\mathbf{x}) = P_{j,\mathrm{top}}(\mathbf{x}) + \sum_{I_j} a_{I_j}\mathbf{x}^{I_j}$, where the degree $|I_j|$ of the multi-indices I_j is strictly less than the degree of P_j, then F is a polynomial of degree at most r in the a_{I_j} $(1 \le j \le n)$. We note that the logarithm branch for F in (5) is clear, as it is obtained directly from the Dirichlet series defining the regularized products (see (8)).

Shintani and Mizuno's examples cited above show that F in (5) is indeed not a polynomial in the coefficients of the top-degree forms $P_{j,\mathrm{top}}$. Our proof, based on [FR, §3], sheds no light on the dependence of F on these coefficients and yields surprisingly little about F. Rather than compute F explicitly, we show that it is a polynomial by proving the vanishing of all sufficiently high-order partial derivatives.

A direct corollary of Theorem 1 is a generalization to higher-degree polynomials of Shintani's result on products of linear forms (3).

Corollary 2. *Fix n polynomials $P_j(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ in r variables as above. For real $y_i \ge 0$ $(1 \le i \le r)$, let $F(\mathbf{y}) = F(y_1, \ldots, y_r)$ be defined by*

$$\mathrm{e}^{F(\mathbf{y})} := \frac{\widehat{\prod}_{\mathbf{m}\in\mathbb{N}^r} \prod_{j=1}^n P_j(\mathbf{y}+\mathbf{m})}{\prod_{j=1}^n \widehat{\prod}_{\mathbf{m}\in\mathbb{N}^r} P_j(\mathbf{y}+\mathbf{m})}.$$

Then F is a polynomial in $\mathbf{y} = (y_1, \ldots, y_r)$ of degree at most r.

This follows since $P_j(\mathbf{y}+\mathbf{x})$ has the same top-degree form in $\mathbf{x}$ as $P_j(\mathbf{x})$.

We can also treat the Hurwitz form of a regularized product [JL, p. 1].

Corollary 3. *Fix n real polynomials $P_j(\mathbf{x})$ in r variables, all satisfying Mahler's hypothesis and having non-negative coefficients. For real $z_j \ge 0$ $(1 \le j \le n)$, let*

$$\mathrm{e}^{F(\mathbf{z})} := \frac{\widehat{\prod}_{\mathbf{m}\in\mathbb{N}^r} \prod_{j=1}^n \big(z_j + P_j(\mathbf{m})\big)}{\prod_{j=1}^n \widehat{\prod}_{\mathbf{m}\in\mathbb{N}^r} \big(z_j + P_j(\mathbf{m})\big)}.$$

Then F is a polynomial in $\mathbf{z} = (z_1, \ldots, z_n)$ of degree at most r.

We will give in §2 a sharper bound on the degree of $F(\mathbf{z})$. In particular, when $\deg(P_j) > r$ $(1 \le j \le n)$, we will show that F depends on the polynomials P_j, but not on $\mathbf{z}$.

In §2 we state and prove a slightly more general form of Theorem 1, where we allow polynomial powers $\varphi(\mathbf{m})$ and show that F in

$$\mathrm{e}^F := \frac{\widehat{\prod}_{\mathbf{m}\in\mathbb{N}^r} \big(\prod_{j=1}^n P_j(\mathbf{m})\big)^{\varphi(\mathbf{m})}}{\prod_{j=1}^n \widehat{\prod}_{\mathbf{m}\in\mathbb{N}^r} \big(P_j(\mathbf{m})^{\varphi(\mathbf{m})}\big)}$$

also satisfies the conclusion of Theorem 1, with the bound r on the degree of the polynomial replaced by $r + \deg(\varphi)$. In §3 we list some formal properties of F.

2. Proof of Theorem 1

We first describe Mahler's results concerning the meromorphic continuation of Dirichlet series of the form

$$Z(s) = Z(s; \log P, \varphi) := \sum_{\mathbf{m}\in\mathbb{N}^r} \frac{\varphi(\mathbf{m})}{P(\mathbf{m})^s}, \tag{5}$$

where φ is an arbitrary complex polynomial in r-variables and P satisfies Mahler's hypothesis (see §1). Since we have assumed that $P(\mathbf{x}) \neq 0$ for all $\mathbf{x} = (x_1, \ldots, x_r)$ in the (real) first octant $x_i \geq 0$ $(1 \leq i \leq r)$, a continuous branch of $\log P(\mathbf{x})$ can be chosen in this simply connected region [Ma, §3]. To define $Z(s)$ we fix such a branch. Note that any two continuous branches differ by a continuous discrete-valued function, and so must differ by a fixed multiple of $2\pi i$.

Mahler showed [Ma, pp. 397–398, Satz II] that the series (5) converges absolutely and uniformly in compact subsets of the right half-plane defined by

$$\mathrm{Re}(s)\deg(P) - \deg(\varphi) > r, \tag{6}$$

and that $Z(s)$ has a meromorphic continuation to all of $\mathbb{C}$, regular at $s = 0$. Mahler's proof also yields that $Z(s)$ is analytic in the coefficients of P in a small enough neighborhood (in coefficient-space) of P. The point here is that a branch of $\log P(\mathbf{x})$ on the entire first octant can be chosen locally analytically in coefficient-space.

We shall need the following computation, readily proved by induction on k.

Lemma. *Write the polynomial P in r variables $\mathbf{x} = (x_1, \ldots x_r)$ as $P(\mathbf{x}) = \sum_I a_I \mathbf{x}^I$, where $I = (I_1, \ldots, I_r)$ runs over distinct multi-indices, $a_I \in \mathbb{C}$ and $\mathbf{x}^I := \prod_{i=1}^r x_i^{I_i}$. Let $\mathcal{D} = \frac{\partial^k}{\partial a_{I^{(1)}} \partial a_{I^{(2)}} \cdots \partial a_{I^{(k)}}}$ be a differential operator consisting of k successive partial derivatives with respect to any sequence $a_{I^{(1)}}, a_{I^{(2)}}, \ldots, a_{I^{(k)}}$ of coefficients of P. Then*

$$\mathcal{D}\big(P(\mathbf{x})^{-s}\big) = (-1)^k \bigg(\prod_{p=0}^{k-1}(s+p)\bigg) \mathbf{x}^{\sum_{p=1}^k I^{(p)}} P(\mathbf{x})^{-s-k}.$$

We now prove the following generalization of Theorem 1, which we state in terms of Dirichlet series rather than regularized products.

Theorem 4. *Let $P_j(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ be n complex polynomials in r variables, all satisfying Mahler's hypothesis above, and let $\varphi(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ be any polynomial in $\mathbf{x}$. Define for $\mathrm{Re}(s) \gg 0$,*

$$f_j(s) := \sum_{\mathbf{m}\in\mathbb{N}^r} \frac{\varphi(\mathbf{m})}{P_j(\mathbf{m})^s} \quad (1 \leq j \leq n), \qquad f_0(s) := \sum_{\mathbf{m}\in\mathbb{N}^r} \frac{\varphi(\mathbf{m})}{\prod_{j=1}^n P_j(\mathbf{m})^s}, \tag{7}$$

and, after analytically continuing the f_j $(0 \leq j \leq n)$,

$$F := -f_0'(0) + \sum_{j=1}^n f_j'(0). \tag{8}$$

Then F is a polynomial of degree at most $r + \deg(\varphi)$ in the coefficients of the P_j of non-maximal degree.

Note that F, and even each $f_j(s)$, depends linearly on φ, so we ignore this dependence in the above theorem.

Proof. We shall actually prove a more precise bound on the degree of F. Namely,

Claim. *Given n non-negative integers $\ell_j < \deg(P_j)$, consider F as a function of just the $a_{I^{(j)}}$ appearing in the $P_j(\mathbf{x})$ $(1 \le j \le n)$ as coefficients of terms $a_{I^{(j)}}\mathbf{x}^{I^{(j)}}$ having degree at most ℓ_j in $\mathbf{x}$. Then, as a polynomial in these $a_{I^{(j)}}$ $(1 \le j \le n)$, the degree of F does not exceed $\max_{1\le j\le n}\left\{\left\lfloor \frac{r+\deg(\varphi)}{\deg(P_j)-\ell_j}\right\rfloor\right\}$.*

Here, $\lfloor t\rfloor$ is the integer such that $\lfloor t\rfloor \le t < \lfloor t\rfloor + 1$.

To prove the Claim, let

$$\mathfrak{D} = \mathcal{D}_1\mathcal{D}_2\cdots\mathcal{D}_n, \qquad \mathcal{D}_j = \frac{\partial^{k_j}}{\partial a_{I^{(j,1)}}\partial a_{I^{(j,2)}}\cdots\partial a_{I^{(j,k_j)}}},$$

be the composition of n commuting differential operators $\mathcal{D}_j$, each involving only coefficients $a_{I^{(j,p)}}$ of P_j in degree at most ℓ_j. Our Claim amounts to showing that if the order $k = \sum_{j=1}^n k_j$ of $\mathfrak{D}$ satisfies

$$k > \max_{1\le j\le n}\left\{\frac{r+\deg(\varphi)}{\deg(P_j)-\ell_j}\right\}, \tag{9}$$

then $\mathfrak{D}(F) = 0$ identically. For the remainder of this proof we assume (9).

We first prove the formula for $1 \le j \le n$,

(10)

$$\frac{\partial}{\partial s}\bigg|_{s=0}\mathfrak{D}\big(f_j(s)\big) = \begin{cases} (-1)^{k_j}(k_j-1)!\sum_{\mathbf{m}\in\mathbb{N}^r}\varphi(\mathbf{m})\mathbf{m}^{\sum_{p=1}^{k_j} I^{(j,p)}}P_j(\mathbf{m})^{-k_j} & \text{if } \mathfrak{D} = \mathcal{D}_j;\\ 0, & \text{otherwise,}\end{cases}$$

where we shall presently see that the Dirichlet series on the right converges absolutely.

If $\mathfrak{D} \ne \mathcal{D}_j$, so that $\mathfrak{D}$ involves derivatives with respect to coefficients of some $P_{j'}$ $(j \ne j')$, then $\mathfrak{D}\big(f_j(s)\big) = 0$ since f_j depends only on P_j. In proving (10) we may therefore assume $\mathfrak{D} = \mathcal{D}_j$, so that $k_j = k > \frac{r+\deg(\varphi)}{\deg(P_j)-\ell_j}$.

Take $\mathrm{Re}(s) \gg 0$ and apply the Lemma to obtain

$$\mathcal{D}_j\big(f_j(s)\big) = \left((-1)^{k_j}\prod_{p=0}^{k_j-1}(s+p)\right)\sum_{\mathbf{m}\in\mathbb{N}^r}\varphi(\mathbf{m})\mathbf{m}^{\sum_{p=1}^{k_j} I^{(j,p)}}P_j(\mathbf{m})^{-s-k_j}. \tag{11}$$

To check where the above series converges, note that

$$\begin{aligned}\big(k_j + \mathrm{Re}(s)\big)\deg(P_j) - \deg(\varphi) - \sum_{p=1}^{k_j}|I^{(j,p)}| \\ &\ge -\deg(\varphi) - \ell_j k_j + \big(k_j+\mathrm{Re}(s)\big)\deg(P_j)\\ &= -\deg(\varphi) + k_j\big(\deg(P_j)-\ell_j\big) + \mathrm{Re}(s)\deg(P_j),\end{aligned} \tag{12}$$

since we have assumed that $\mathcal{D}_j$ involves only coefficients of terms x^I of degree $|I|$ at most ℓ_j. By (6) and (12), the right-hand side of (11) converges and gives an analytic function of s, provided

$$k_j\big(\deg(P_j)-\ell_j\big) + \mathrm{Re}(s)\deg(P_j) > r + \deg(\varphi).$$

In particular, since $k_j > \frac{r+\deg(\varphi)}{\deg(P_j)-\ell_j}$, we find that right-hand side of (11) converges absolutely in an open right half-plane containing $s=0$. Thus, although (11) was derived for $\mathrm{Re}(s)$ large enough, by analytic continuation it also holds at $s=0$. We may therefore differentiate both sides of (11) with respect to s and set $s=0$ to obtain (10).

To state the next formula it will be convenient to call $\mathfrak{D}$ pure if $\mathfrak{D}=\mathcal{D}_j$ for some j $(1\le j\le n)$. Otherwise we will call $\mathfrak{D}$ mixed. We now prove

$$\frac{\partial}{\partial s}\Big|_{s=0}\mathfrak{D}\big(f_0(s)\big)=\begin{cases}0, & \text{if } \mathcal{D} \text{ is mixed;}\\ \frac{\partial}{\partial s}\big|_{s=0}\mathfrak{D}\big(f_j(s)\big) & \text{if } \mathfrak{D}=\mathcal{D}_j.\end{cases} \tag{13}$$

As before, the Lemma gives for $\mathrm{Re}(s)\gg 0$,

(14)

$$\mathfrak{D}\big(f_0(s)\big)=(-1)^k\bigg(\prod_{j=1}^{n}\prod_{p=0}^{k_j-1}(s+p)\bigg)\sum_{\mathbf{m}\in\mathbb{N}^r}\varphi(\mathbf{m})\mathbf{m}^{\sum_{j=1}^n\sum_{p=1}^{k_j}I^{(j,p)}}\prod_{j=1}^{n}P_j(\mathbf{m})^{-s-k_j}.$$

Note that here we have used our choice of branch of $\log\prod_j P_j$.

For convergence of the Dirichlet series (14) we need

$$\sum_{j=1}^{n}\bigg(\big(k_j+\mathrm{Re}(s)\big)\deg(P_j)-\sum_{p=1}^{k_j}|I^{(j,p)}|\bigg)>r+\deg(\varphi).$$

But $|I^{(j,p)}|\le\ell_j$, so for convergence it suffices to ensure that

$$\sum_{j=1}^{n}k_j\big(\deg(P_j)-\ell_j\big)+\mathrm{Re}(s)\sum_{j=1}^{n}\deg(P_j)>r+\deg(\varphi).$$

Setting $s=0$ on the left-hand side above, we calculate

$$\begin{aligned}\sum_{j=1}^{n}k_j\big(\deg(P_j)-\ell_j\big)&\ge\sum_{j=1}^{n}k_j\min_{1\le j\le n}\big\{\deg(P_j)-\ell_j\big\}\\&=k\min_{1\le j\le n}\big\{\deg(P_j)-\ell_j\big\}>r+\deg(\varphi),\end{aligned}$$

by (9). The series in (14) thus converges in an open neighborhood of $s=0$, so (14) holds there.

We can now conclude the proof of (13). If $\mathfrak{D}$ is mixed, then $\prod_{j=1}^n\prod_{p=0}^{k_j-1}(s+p)$ on the right-hand side of (14) includes a factor of s^2. We then have the trivial vanishing $\frac{\partial}{\partial s}\big|_{s=0}\mathfrak{D}\big(f_0(s)\big)=0$. If $\mathfrak{D}=\mathcal{D}_j$ is pure, we have only a single factor of s. From (14) we then find, using $\mathfrak{D}=\mathcal{D}_j$ and $k=k_j$,

$$\frac{\partial}{\partial s}\Big|_{s=0}\mathfrak{D}\big(f_0(s)\big)=(-1)^{k_j}(k_j-1)!\sum_{\mathbf{m}\in\mathbb{N}^r}\varphi(\mathbf{m})\mathbf{m}^{\sum_{p=1}^{k_j}I^{(j,p)}}P_j(\mathbf{m})^{-k_j}=\frac{\partial}{\partial s}\Big|_{s=0}\mathfrak{D}\big(f_j(s)\big),$$

by (10).

Having now established (10) and (13), we deduce

$$\mathfrak{D}(F)=\mathfrak{D}\bigg(-f_0'(s)+\sum_{j=1}^{n}f_j'(s)\bigg)=0,$$

as claimed after (9). □

We can sharpen Corollary 3 by taking all $\ell_j = 0$ in the Claim above.

Corollary 5. *Fix n real polynomials $P_j(\mathbf{x})$ in r variables, all satisfying Mahler's hypothesis and having non-negative coefficients. For* $\mathrm{Re}(z_j) > 0$ $(1 \le j \le n)$ *and* $\mathrm{Re}(s) \gg 0$, *define*

$$f_j(s) := \sum_{\mathbf{m}\in\mathbb{N}^r} \varphi(\mathbf{m})\big(z_j + P_j(\mathbf{m})\big)^{-s}, \quad f_0(s) := \sum_{\mathbf{m}\in\mathbb{N}^r} \varphi(\mathbf{m}) \prod_{j=1}^{n} \big(z_j + P_j(\mathbf{m})\big)^{-s},$$

and,

$$F := -f_0'(0) + \sum_{j=1}^{n} f_j'(0).$$

Then F is a polynomial in the z_j of degree at most $\max\limits_{1\le j\le n} \left\{ \left\lfloor \frac{r+\deg(\varphi)}{\deg(P_j)} \right\rfloor \right\}$.

In particular, F is independent of $\mathbf{z}$ if φ is constant and if all the P_j are of degree at least $r+1$.

We conclude this section with some remarks on Theorem 4.

1. Examination of the proof shows that the conclusion of Theorem 4 still holds if in the definition of $f_j(s)$ $(0 \le j \le n)$ the sum $\sum_{\mathbf{m}\in\mathbb{N}^r} \varphi(\mathbf{m})P_j(\mathbf{m})^{-s}$ is replaced by the integral $\int_{\mathbf{x}\in\mathbb{R}_+^r} \varphi(\mathbf{x})P_j(\mathbf{x})^{-s}\, d\mathbf{x}$. Here $P_0(\mathbf{x}) := \prod_{j=1}^{n} P_j(\mathbf{x})$. The only additional point needed in the proof is Mahler's result [Ma, p. 392, Satz I] showing the convergence and meromorphic continuation of these integrals (still assuming Mahler's hypothesis for each P_j, of course).

2. We have assumed that the polynomials P_j satisfy Mahler's hypothesis, but the formal nature of our proof shows that what matters is that the Dirichlet series $f_j(s)$ is defined, converges absolutely for $\mathrm{Re}(s)\deg(P_j) - \deg(\varphi) > r$, and analytically extends to a function regular at $s = 0$. As we mentioned in §1, other authors have found alternative hypotheses that guarantee this.

Even the polynomial nature of the P_j or φ is not essential, as we could consider series of the form

$$f_j(s) = \sum_{\mathbf{m}} \varphi(\mathbf{m})P_j(\mathbf{m}, \mathbf{a_j})^{-s},$$

where P_j depends on some parameter $\mathbf{a_j}$ ranging over some open subset of some Euclidean space, and $\mathbf{m}$ runs over a countable set. Aside from the obviously necessary convergence for $\mathrm{Re}(s) \gg 0$ and the existence of a meromorphic continuation in s of $f_j(s)$ $(0 \le j \le n)$, what matters in the proof of Theorem 4 is that all sufficiently high-order derivatives $\mathcal{D}_j P_j(\mathbf{m}, \mathbf{a_j})^{-s}$ (taken with respect to $\mathbf{a_j}$) decrease quickly enough with $\mathbf{m}$ for $\sum_{\mathbf{m}} \varphi(\mathbf{m})\mathcal{D}_j P_j(\mathbf{m}, \mathbf{a_j})^{-s}\big|_{s=0}$ to converge.

3. Our proof yields that the values $f_j(0)$, and more generally the values $f_j(-N)$ for each fixed non-negative integer N, are also polynomial functions of the coefficients of P_j of non-maximal degree $(1 \le j \le n)$. To see this it suffices to take $k_j > \frac{N\deg(P_j)+r+\deg\varphi}{\deg(P_j)-\ell_j}$ in (11) and set $s = -N$. Thus, for fixed j, $\ell_j < \deg(P_j)$ and N, considered as a function of just the $a_{I(j)}$ appearing in $P_j(\mathbf{x})$ as coefficients of

terms $a_{I(j)}\mathbf{x}^{I^{(j)}}$ having degree at most ℓ_j, $f_j(-N)$ is a polynomial of degree not exceeding $\left\{\left\lfloor\frac{N\deg(P_j)+r+\deg(\varphi)}{\deg(P_j)-\ell_j}\right\rfloor\right\}$. For a detailed study of $f_j(-N)$ for certain classes of polynomials see [Ca1] [Ca2].

4. The discrepancy F in

$$\mathrm{e}^F := \frac{\widehat{\prod}_m \prod_{j=1}^n a_m^{(j)}}{\prod_{j=1}^n \widehat{\prod}_m a_m^{(j)}}.$$

has the curious property of being unaltered by the omission of any finite number of indices m from all the regularized products.

3. Properties of F

In this brief section we list some formal properties of F in Theorem 4. To make these properties clearer we will write $F_n(P_1,\dots,P_n;\varphi)$ for F. One should bear in mind that F also depends on the branches of $\log P_j$ used.

Proposition 6. *F has the following properties.*

(a) *Symmetry: $F_n(P_1,\dots,P_n;\varphi)$ is independent of the order of the P_j.*
(b) *Vanishing on the diagonal: $F_n(P,P,\dots,P;\varphi)=0$.*
(c) *Reduction of n: For $n\geq 3$,*

$$F_n(P_1,P_2,\dots,P_n;\varphi)=F_{n-1}(P_1\cdot P_2,\dots,P_n;\varphi)+F_2(P_1,P_2;\varphi).$$

(d) *Reduction to two polynomials: For $n\geq 2$,*

$$F_n(P_1,P_2,\dots,P_n;\varphi)=\sum_{j=1}^{n-1}F_2\Big(\prod_{k=1}^{j}P_k,P_{j+1};\varphi\Big).$$

Proof. Property (a) is immediate from the definition of F given in (8). If $P_j=P$ for $1\leq j\leq n$, then $f_0(s)=f_j(ns)$. Property (b) then follows. To prove (c), observe that for $\mathrm{Re}(s)\gg 0$,

(13)

$$\begin{aligned}\sum_{j=1}^n\sum_{\mathbf{m}}\varphi(\mathbf{m})P_j(\mathbf{m})^{-s}-\sum_{\mathbf{m}}\varphi(\mathbf{m})\prod_{j=1}^n P_j(\mathbf{m})^{-s}\\ =\sum_{\mathbf{m}}\varphi(\mathbf{m})\big(P_1(\mathbf{m})P_2(\mathbf{m})\big)^{-s}+\sum_{j=3}^n\sum_{\mathbf{m}}\varphi(\mathbf{m})P_j(\mathbf{m})^{-s}-\sum_{\mathbf{m}}\varphi(\mathbf{m})\prod_{j=1}^n P_j(\mathbf{m})^{-s}\\ +\sum_{j=1}^2\sum_{\mathbf{m}}\varphi(\mathbf{m})P_j(\mathbf{m})^{-s}-\sum_{\mathbf{m}}\varphi(\mathbf{m})\big(P_1(\mathbf{m})P_2(\mathbf{m})\big)^{-s},\end{aligned}$$

where we have used our convention that logarithm branches are always chosen for products so that $\big(P_1(\mathbf{x})P_2(\mathbf{x})\cdots P_k(\mathbf{x})\big)^{-s}=P_1(\mathbf{x})^{-s}P_2(\mathbf{x})^{-s}\cdots P_k(\mathbf{x})^{-s}$ for $\mathbf{x}$ in the first octant. Property (c) follows from (13) by analytic continuation to $s=0$. Property (d) follows from (c) by induction on n. $\square$

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Institut de Mathématiques, Université Bordeaux I, 351 Cours de la Libération, F-33405 Talence cédex, France
E-mail address: diaz@math.u-bordeaux1.fr

Departamento de Matemática, Universidad de Chile, Casilla 653, Santiago 1, Chile
E-mail address: friedman@uchile.cl