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EXAMPLES OF COMMUTATIVE RIGHT-NILALGEBRAS OVER SMALL FIELDS

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Correa et al. (2003) proved that any commutative right-nilalgebra of nilindex 4 and dimension 4 is nilpotent in characteristic ≠2, 3. They did not assume power-associativity. In this article we will further investigate these algebras without the assumption on the dimension and providing examples in those cases that are not covered in the classification concentrating mostly on algebras generated by one element.

Key Words: Albert's problem; Nilalgebras; Small characteristic.

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1. INTRODUCTION

In a nonassociative algebra A we define right principal powers of an element $x \in A$ recursively as follows:

$$x^1 = x, \qquad x^{n+1} = x^n x.$$

We say that A is *power associative* if for any $x \in A$, the subalgebra generated by x is associative. A is right-nil if there is some $n \in \mathbb{N}$ such that $x^n = 0$ for all $x \in A$. The smallest such n is called the *right-nilindex of A*.

We also define the following descending chains of subalgebras in A:

1. $A^1 \supseteq A^2 \supseteq \dots$, where $A^1 = A$ and $A^n = \sum_{r+s=n} A^r A^s$ for $n > 1$;
2. $A^{\langle 1\rangle} \supseteq A^{\langle 2\rangle} \supseteq \dots$, where $A^{\langle 1\rangle} = A$ and $A^{\langle n\rangle} = A^{\langle n-1\rangle}A$ for $n > 1$;
3. $A^{(1)} \supseteq A^{(2)} \supseteq \dots$, where $A^{(1)} = A$ and $A^{(n)} = (A^{(n-1)})^2$ for $n > 1$.

When one of these chains goes to zero, we say that A is nilpotent, right-nilpotent, or solvable, respectively. It is clear from the definition that nilpotent implies right-nilpotent and right-nilpotent implies solvable.

If the right-nilindex of A is 2, there are two possibilities depending on the characteristic of the underlying field. When char $K \neq 2$ we can easily see that the multiplication is trivial. On the other hand, when char $K = 2$ we construct a power-associative algebra A which is not solvable (Example 3.1).

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If the right-nilindex of A is 3 then we have the following possibilities: When $K \neq \mathbb{F}_2$ then A is a Jordan algebra and therefore power associative (Remark 2.1). In particular, every subalgebra of A generated by one element is associative and nilpotent. When char $K \neq 2$, and A is finite dimensional, then A is nilpotent (see Schafer, 1966). Finally, when $K = \mathbb{F}_2$ we construct an algebra A which is generated as an algebra by one element, A is of infinite dimension and it is not solvable (Example 3.2).

If the right-nilindex of A is 4 and char $K \neq 2$, Elduque and Labra (2006) studied the problem when the dimension of A is at most 4. Here we will study what happens when A is generated by one element. In what follows we will write $|K|$ for the number of elements in the field K. When char $K \neq 2$ and $|K| \geq 3$, then A is nilpotent and dim $A \leq 6$ (see Theorem 2.3). When char $K = 2$, Example 3.3 shows that A may not be power-associative nor solvable. Finally, when $K = \mathbb{F}_3$, we construct an example where A is nilpotent and dim $A = 7$ (Example 3.5).

2. RIGHT-NILINDEX AT MOST FOUR

We will start considering an algebra A with right-nilindex 3. The following is a well-known result.

Remark 2.1. Let A be a commutative K-algebra with right-nilindex 3. If $|K| \geq 3$, then A is a Jordan algebra and therefore power-associative.

Now we will focus on algebras A that are generated by one element and have right-nilindex 4. Correa et al. (2003) show that in a commutative right-nilalgebra A of nilindex 4 over a field of characteristic $\neq 2, 3$ every subalgebra of A generated by a single element is nilpotent of index at most 7. Here we drop the assumptions on the characteristic of the field and we still get the following lemma (see Correa et al., 2003, Theorem 1).

Lemma 2.2. *Let A be a commutative K-algebra with right-nilindex 4. We will assume $|K| \geq 4$ so that we can use all the linearizations of the identity $((xx)x)x = 0$. Let $e_1 \in A$ be an arbitrary element and define $e_2 = e_1e_1$, $e_3 = e_2e_1$, $e_4 = e_2e_2$, $e_{i+1} = e_ie_1$ for $i \geq 4$. Then the e_i satisfy:*

(i) $e_1e_3 = 0$, $e_2e_2 = e_4$, $e_1e_i = e_{i+1}$ *for* $i \geq 1$, $i \neq 3$;
(ii) $e_2e_3 = -e_5$;
(iii) $e_2e_4 = e_3e_3 = e_6$;
(iv) $e_2e_5 = e_3e_4 = e_7$;
(v) $e_3e_5 = e_8$;
(vi) $e_2e_6 = e_2e_7 = e_3e_6 = e_3e_7 = e_4e_4 = e_4e_5 = e_4e_6 = e_4e_7 = e_5e_5 = e_5e_6 = e_5e_7 = e_6e_6 = e_6e_7 = e_7e_7 = 0$;
(vii) $2e_i = 0$ *for all* $i \geq 7$.

Proof. Linearizing the identity $((xx)x)x = 0$, we obtain the following:

$$f(x, y) := 2((yx)x)x + (x^2y)x + x^3y = 0.$$

$$g(x, y) := (y^2x)x + 2((xy)y)x + 2((xy)x)y + (x^2y)y = 0.$$

$$h(x, y, z) := 2[((yz)x)x + ((yx)z)x + ((yx)x)z + ((xz)y)x + ((xz)x)y] + (x^2y)z + (x^2z)y = 0.$$

Part (i) follows from the definition of the e_i using the fact that $e_1^4 = 0$. In what follows we will use these identities without further mentioning it.

$0 = f(e_1, e_2) = 0 + e_5 + e_3e_2$ proving (ii).

Part (iii) follows from

$$0 = f(e_1, e_3) = 0 + (e_2e_3)e_1 + e_3e_3 = -e_6 + e_3e_3.$$
$$0 = g(e_1, e_2) = e_6 - 2e_6 + 0 + e_4e_2.$$

Part (iv) follows from

$$0 = f(e_2, e_1) = 2(e_3e_2)e_2 + e_5e_2 + (e_4e_2)e_1 = -2e_5e_2 + e_5e_2 + e_7 = -e_5e_2 + e_7.$$
$$0 = h(e_1, e_2, e_3) = 2((e_2e_3)e_1)e_1 + 2(e_3e_3)e_1 + 0 + 0 + 0 + e_4e_3 + (e_2e_3)e_2 = -2e_7 + 2e_7 + e_4e_3 - e_5e_2.$$

Part (v) follows from

$$0 = g(e_1, e_3) = ((e_3e_3)e_1)e_1 + 0 + 0 + (e_2e_3)e_3 = e_8 - e_3e_5.$$

Part (vii) follows from

$$0 = f(e_1, e_4) = 2e_7 + (e_2e_4)e_1 + e_3e_4 = 4e_7.$$

Part (vi) follows from

$$e_6e_2 = (e_4e_2)e_2 = e_2^4 = 0.$$
$$0 = h(e_1, e_2, e_4) = 2((e_2e_4)e_1)e_1 + 2(e_3e_4)e_1 + 0 + 2(e_5e_2)e_1 + 2e_6e_2 + e_4e_4 + (e_2e_4)e_2 = e_4e_4.$$
$$0 = f(e_1, e_6) = 2e_9 + (e_2e_6)e_1 + e_3e_6 = e_3e_6.$$
$$0 = h(e_1, e_3, e_4) = 2((e_3e_4)e_1)e_1 + 0 + 0 + 2(e_5e_3)e_1 + 2e_6e_3 + (e_2e_3)e_4 + (e_2e_4)e_3) = -e_5e_4.$$
$$0 = h(e_1, e_2, e_5) = 2((e_2e_5)e_1)e_1 + 2(e_3e_5)e_1 + 0 + 2(e_6e_2)e_1 + 2e_7e_2 + e_4e_5 + (e_2e_5)e_2 = e_7e_2.$$
$$0 = f(e_1, e_7) = 2e_10 + (e_2e_7)e_1 + e_3e_7 = e_3e_7.$$
$$0 = f(e_2, e_4) = 2((e_4e_2)e_2)e_2 + (e_4e_4)e_2 + (e_4e_2)e_4 = e_6e_4.$$
$$0 = h(e_1, e_3, e_5) = 2((e_3e_5)e_1)e_1 + 0 + 0 + 2(e_6e_3)e_1 + 2e_7e_3 + (e_2e_3)e_5 + (e_2e_5)e_3 = -e_5e_5.$$

$$
\begin{aligned}
0 = h(e_1, e_2, e_7) &= 2((e_2e_7)e_1)e_1 + 2(e_3e_7)e_1 + 0 + 2(e_8e_2)e_1 + 2e_9e_2 + e_4e_7 + (e_2e_7)e_2 \\
&= e_4e_7. \\
0 = f(e_2, e_5) &= 2((e_5e_2)e_2)e_2 + (e_4e_5)e_2 + (e_4e_2)e_5 = e_5e_6. \\
0 = f(e_2, e_6) &= 2((e_6e_2)e_2)e_2 + (e_4e_6)e_2 + (e_4e_2)e_6 = e_6e_6. \\
0 = g(e_1, e_5) &= ((e_5e_5)e_1)e_1 + 2(e_6e_5)e_1 + 2e_7e_5 + (e_2e_5)e_5 = e_7e_5. \\
0 = f(e_2, e_7) &= 2((e_7e_2)e_2)e_2 + (e_4e_7)e_2 + (e_4e_2)e_7 = e_6e_7. \\
0 = h(e_2, e_3, e_7) &= 2((e_3e_7)e_2)e_2 + 2((e_3e_2)e_7)e_2 + 2((e_3e_2)e_2)e_7 \\
&\quad + 2((e_2e_7)e_3)e_2 + 2((e_2e_7)e_2)e_3 + (e_4e_3)e_7 + (e_4e_7)e_3 \\
&= e_7e_7.
\end{aligned}
$$

□

Theorem 2.3. *Let A be a commutative K-algebra with right-nilindex 4 generated as an algebra by $a \in A$. If $\operatorname{char} K \neq 2$ and $|K| \geq 4$, then A is nilpotent and $A = \langle a, a^2, a^3, a^2a^2, a^2a^3, a^3a^3 \rangle$ so that $\dim(A) \leq 6$.*

Proof. This is a direct consequence of the preceding lemma letting $e_1 = a$ and noticing that since $\operatorname{char} K \neq 2$ we have $e_i = 0$ for all $i > 6$. In particular A is nilpotent of index at most 7. □

3. EXAMPLES

Example 3.1. Let A be a commutative algebra over K ($\operatorname{char} K = 2$) with basis $\{a_1, a_2, a_3\}$. The multiplication defined on basis elements has commuting nonzero products $a_1a_2 = a_3, a_2a_3 = a_1, a_1a_3 = a_2$. The resulting algebra is power-associative and has right-nilindex 2 (every square is zero), but it is not solvable since $A = A^2 = (A^2)^2 = \ldots$. We may remark that this constitutes an answer in characteristic 2 to a well-known problem of Albert's which asks, "Is every finite-dimensional commutative power-associative nilalgebra solvable?"

Example 3.2. Let A be an algebra over $K = \mathbb{F}_2$ with basis $\{a_1, a_2, \ldots\}$ and nonzero products defined by

$$
a_ia_j = a_{\frac{i+j}{2}+1}, \qquad \text{if } i + j \text{ is even.}
$$

Then A is generated by one element, A is right-nil with right-nilindex 3, and A is not solvable. We may also notice that A is commutative and not power-associative.

Proof. Since $a_i^2 = a_{i+1}$, A is generated by a_1. Let

$$
u = \sum_{i=1}^{n} \alpha_i a_{2i} + \beta_i a_{2i-1}
$$

be an arbitrary element in A. Then

$$
u^2 = \sum_{i=1}^{n} \alpha_i^2 a_{2i+1} + \beta_i^2 a_{2i},
$$

and

$$
\begin{aligned}
u^3 &= \Big(\sum_{i=1}^{n} \alpha_i a_{2i} + \beta_i a_{2i-1}\Big)\Big(\sum_{j=1}^{n} \alpha_j^2 a_{2j+1} + \beta_j^2 a_{2j}\Big) \\
&= \Big(\sum_{i=1}^{n} \alpha_i a_{2i}\Big)\Big(\sum_{j=1}^{n} \beta_j^2 a_{2j}\Big) + \Big(\sum_{i=1}^{n} \beta_i a_{2i-1}\Big)\Big(\sum_{j=1}^{n} \alpha_j^2 a_{2j+1}\Big) \\
&= \Big(\sum_{i=1}^{n} \alpha_i a_{2i}\Big)\Big(\sum_{j=1}^{n} \beta_j^2 a_{2j}\Big) + \Big(\sum_{j=1}^{n} \beta_j a_{2j-1}\Big)\Big(\sum_{i=1}^{n} \alpha_i^2 a_{2i+1}\Big) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \beta_j^2 a_{i+j+1} + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_j \alpha_i^2 a_{i+j+1} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \beta_j (\beta_j + \alpha_i) a_{i+j+1} \\
&= 0.
\end{aligned}
$$

The last equality occurs because in $\mathbb{F}_2$ either $\alpha_i = \beta_j$ or one of the two has to be zero. Therefore we have shown that A is right-nil. On the other hand, $A^2 = \langle a_2, a_3, \dots \rangle \cong A$ so A is not solvable. □

3.1. Nilindex 4

With the following examples we want to show that the hypotheses on the field in Theorem 2.3 are sharp. Let us start by considering what happens when $\operatorname{char} K = 2$.

Example 3.3. Let K be a field with $\operatorname{char} K = 2$ and let A be a commutative K-algebra with basis $\{e_1, e_2, \dots\}$ and multiplication defined by:

(i) $e_1 e_i = e_{i+1}$ for $i \geq 1$, $i \neq 3$, $e_1 e_3 = 0$;
(ii) $e_2 e_2 = e_4$, $e_2 e_3 = e_5$, $e_2 e_4 = e_6$, $e_2 e_5 = e_7$;
(iii) $e_3 e_3 = e_6$, $e_3 e_4 = e_7$, $e_3 e_5 = e_8$;
(iv) $e_8 e_9 = e_1$;
(v)) All other products of basis elements are zero.

Then A is generated by one element, A satisfies the identity $((xx)x)x = 0$, and $A = A^2$ so in particular A is not solvable. We may also notice that A is not power-associative.

Proof. It is immediate from the multiplication table that A is generated as an algebra by e_1. Now let $u \in A$ be an arbitrary element. Then

$$u = \sum_{i=0}^{n} \alpha_i e_i$$

for some $n \in \mathbb{N}$ and $\alpha_i \in K$. We can proceed to calculate u^4:

$$\begin{aligned} u^4 &= ((\alpha_1^2 e_2 + \alpha_2^2 e_4 + \alpha_3^2 e_6)u)u \\ &= (\alpha_1^3 e_3 + \alpha_1\alpha_2^2 e_5 + \alpha_1\alpha_3^2 e_7 + \alpha_1^2\alpha_2 e_4 \\ &\quad + \alpha_2^3 e_6 + \alpha_1^2\alpha_3 e_5 + \alpha_2^2\alpha_3 e_7 + \alpha_1^2\alpha_4 e_6 + \alpha_1^2\alpha_5 e_7)u \\ &= \alpha_1^2\alpha_2^2 e_6 + \alpha_1^2\alpha_3^2 e_8 + \alpha_1^3\alpha_2 e_5 + \alpha_1\alpha_2^3 e_7 + \alpha_1^3\alpha_3 e_6 + \alpha_1\alpha_2^2\alpha_3 e_8 \\ &\quad + \alpha_1^3\alpha_4 e_7 + \alpha_1^3\alpha_5 e_8 + \alpha_1^3\alpha_2 e_5 + \alpha_1\alpha_2^3 e_7 + \alpha_1^2\alpha_2^2 e_6 + \alpha_1^2\alpha_2\alpha_3 e_7 \\ &\quad + \alpha_1^3\alpha_3 e_6 \alpha_1\alpha_2^2\alpha_3 e_8 + \alpha_1^2\alpha_2\alpha_3 e_7 + \alpha_1^2\alpha_3^2 e_8 + \alpha_1^3\alpha_4 e_7 + \alpha_1^3\alpha_5 e_8 \\ &= 0. \end{aligned}$$

It is easy to verify that $A^2 = A$ since $e_1 = e_8 e_9 \in A^2$. □

Example 3.4. We can modify the previous example to obtain a finite dimensional algebra which is not solvable. Let $A = \langle e_1, \ldots, e_9 \rangle$ and modify the multiplication table so that $e_1 e_9 = 0$.

The other assumption made in Theorem 2.3 is on the size of the field ($|K| \geq 4|$) which leaves as the only alternative to consider $K = \mathbb{F}_3$. Although this may not be such a traditional source of examples, in a recent article, Bremner (2007) gives an example of a power associative algebra over the field of three elements which is not strictly power associative, i.e., when we extend the field, the resulting algebra is not power associative.

It is interesting to notice that for Lemma 2.2 we need a field with at least 4 elements to linearize $x^4 = 0$ to get the identity $2((yx)x)x + (x^2y)x + x^3y = 0$. When $K = \mathbb{F}_3$, the remaining linearizations $g(x, y)$ and $h(x, y, z)$ are still valid and we can obtain an additional identity by calculating $(x + y)^4 - (x - y)^4$ to get

$$2((yx)x)x + 2((xy)y)y + (x^2y)x + (y^2x)y + x^3y + y^3x = 0.$$

The main difficulties in working with this last identity arise because it is not homogeneous in each of the variables.

Example 3.5. Let $K = \mathbb{F}_3$ and let A be a commutative algebra with basis $\{e_1, \ldots, e_7\}$ and multiplication defined by:

(i) $e_1 e_i = e_{i+1}$ for $i = 1, 2, 4, 5, 6$;
(ii) $e_2 e_2 = e_4$, $e_2 e_3 = -e_5 - e_7$, $e_2 e_4 = e_6$, $e_3 e_3 = e_6$;
(iii) All other products of basis elements are zero.

Then A satisfies the identity $((xx)x)x = 0$, A is nilpotent of index 8 and $\dim A = 7$.

Proof. Let $u = \alpha_1 e_1 + \ldots \alpha_7 e_7$ be an arbitrary element in A. Then:

$$u^2 = \alpha_1^2 e_2 - \alpha_1\alpha_2 e_3 + \alpha_2^2 e_4 - \alpha_1\alpha_4 e_5 + \alpha_2\alpha_3 e_5 + (\ldots)e_6 + (\ldots)e_7,$$

$$
\begin{aligned}
u^3 &= \alpha_1^3 e_3 + \alpha_1\alpha_2^2 e_5 - \alpha_1^2\alpha_4 e_6 + \alpha_1\alpha_2\alpha_3 e_6 + \alpha_1^2\alpha_2 e_4 + \alpha_1\alpha_2^2 e_5 + \alpha_2^3 e_6 \\
&\quad -\alpha_1^2\alpha_3 e_5 - \alpha_1\alpha_2\alpha_3 e_6 + \alpha_1^2\alpha_4 e_6 + (\ldots)e_7 \\
&= \alpha_1^3 e_3 + \alpha_1^2\alpha_2 e_4 - \alpha_1\alpha_2^2 e_5 - \alpha_1^2\alpha_3 e_5 + \alpha_2^3 e_6 + (\ldots)e_7, \\
u^4 &= \alpha_1^3\alpha_2 e_5 - \alpha_1^2\alpha_2^2 e_6 - \alpha_1^3\alpha_3 e_6 + \alpha_1\alpha_2^3 e_7 - \alpha_1^3\alpha_2 e_5 - \alpha_1^3\alpha_2 e_7 + \alpha_1^2\alpha_2^2 e_6 + \alpha_1^3\alpha_3 e_6 \\
&= (\alpha_1\alpha_2^3 - \alpha_1^3\alpha_2)e_7 \\
&= \alpha_1\alpha_2(\alpha_2^2 - \alpha_1^2)e_7 \\
&= 0.
\end{aligned}
$$

The last equality occurs because in $\mathbb{F}_3$ either $\alpha_1^2 = \alpha_2^2$ or one of the two has to be zero. This way we have shown that A is right-nil with right-nilindex four. □

Remark 3.6. It is worth noticing that these calculations also show that A satisfies the identity $x^4 = 0$ but not strictly, that is, the right-nilindex of A increases if we extend the scalar field and A does not satisfy all the linearizations of the identity $x^4 = 0$.

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