Communications in Algebra®, 36: 132–141, 2008

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870701665248

SEMIPRIMALITY AND NILPOTENCY OF NONASSOCIATIVE RINGS SATISFYING $x(yz) = y(zx)$

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In this article we study nonassociative rings satisfying the polynomial identity $x(yz) = y(zx)$, which we call "cyclic rings." We prove that every semiprime cyclic ring is associative and commutative and that every cyclic right-nilring is solvable. Moreover, we find sufficient conditions for the nilpotency of cyclic right-nilrings and apply these results to obtain sufficient conditions for the nilpotency of cyclic right-nilalgebras.

Key Words: Nonassociative semiprime nilpotent identity algebra.

2000 Mathematics Subject Classification: 17A30; 16N60.

1. INTRODUCTION

We define a *cyclic ring* to be a not necessarily associative ring R that satisfies the *cyclic identity*

$$x(yz) = y(zx) \tag{1}$$

for all x, y, z in R.

Kleinfeld (1995) proves that every prime cyclic ring where $2x = 0$ implies $x = 0$, is a commutative and associative ring. In Sections 2 and 3 of the present article we improve on this result by showing that every semiprime cyclic ring is associative and commutative. We do not assume that $2x = 0$ implies $x = 0$.

In Sections 4 and 5 we study the nilpotency and solvability of cyclic right-nilrings. We prove that every right-nilring is solvable, and that if a ring is

Received September 8, 2006; Revised January 22, 2007. Communicated by I. P. Shestakov.
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right-nilpotent then it is nilpotent. Finally, we use our results to prove that a cyclic right-nilalgebra A is nilpotent when $\dim A \leq 4$ or when A has dimension n and right-nilindex $n+1$. We present examples of right-nilalgebras of dimension 5 which are not nilpotent.

We recall some definitions. A ring R is called a *prime ring* if for every pair of ideals I and J of R, $IJ = 0$ implies that $I = 0$ or $J = 0$. R is called *semiprime* if for every ideal I of R, $I^2 = 0$ implies $I = 0$. It is clear that prime implies semiprime.

For a not necessarily associative nor commutative ring R, we define recursively the followings powers of R and of $x \in R$:

$$R^1 = R \quad \text{and} \quad R^n = \sum_{i+j=n} R^i R^j \quad \text{for } n > 1 \quad (\textit{principal powers}),$$

$$R^{\langle 1 \rangle} = R \quad \text{and} \quad R^{\langle n \rangle} = (R^{\langle n-1 \rangle})R \quad \text{for } n > 1 \quad (\textit{right powers}),$$

$${}^{\langle 1 \rangle}R = R \quad \text{and} \quad {}^{\langle n \rangle}R = R({}^{\langle n-1 \rangle}R) \quad \text{for } n > 1 \quad (\textit{left powers}),$$

$$R^{(1)} = R \quad \text{and} \quad R^{(n)} = (R^{(n-1)})^2 \quad \text{for } n > 1 \quad (\textit{plenary powers}),$$

$$x^{\langle 1 \rangle} = x \quad \text{and} \quad x^{\langle n+1 \rangle} = x^{\langle n \rangle}x \quad \text{for } n > 1 \quad (\textit{right powers}).$$

R is said to be *nilpotent* if for some n, $R^n = 0$, *right-nilpotent (left-nilpotent)* if for some n, $R^{\langle n \rangle} = 0$ (${}^{\langle n \rangle}R = 0$), *solvable* if for some n, $R^{(n)} = 0$, *nil* if for every $x \in R$ the subring of R generated by x is nilpotent, *right-nil* if for every $x \in R$ there is some n such that $x^{\langle n \rangle} = 0$. A right-nilring is said to have *right-nilindex* n if $x^{\langle n \rangle} = 0$ for all elements in the ring and n is the smallest positive integer for which this is true.

As usual, we will denote the associator by $(a, b, c) = (ab)c - a(bc)$, the commutator by $[x, y] = xy - yx$ and the left and right multiplication operators by L_x and R_x, respectively, where $aL_x = xa$ and $aR_x = ax$.

In terms of these operators, the cyclic identity (1) is equivalent to

$$L_y L_x = R_x L_y = R_{xy}. \tag{2}$$

2. IDENTITIES IN CYCLIC RINGS

Lemma 1. *Let R be a cyclic ring. Then the following identities hold in R:*

1. $(xy)(st) = (tx)(sy)$;
2. $(xy)((st)(uv)) = (tx)((sy)(uv))$;
3. $(xy, st, uv) = 0$;
4. $(x, y, st)(uv) = 0$.

Proof. 1. Direct calculations using the cyclic identity (1) give

$$(xy)(st) = s(t(xy)) = s(y(tx)) = (tx)(sy).$$

2. Using the cyclic identity (1) and part 1 we obtain

$$(xy)((st)(uv)) = (uv)((xy)(st)) = (uv)((tx)(sy)) = (tx)((sy)(uv)).$$

3. We use parts 1 and 2 and the cyclic identity to get

$$\begin{aligned}(xy)((st)(uv)) &\overset{1}{=} (xy)((vs)(ut)) \overset{2}{=} (sx)((vy)(ut)) \overset{1}{=} (sx)((tv)(uy))\\ &\overset{2}{=} (vs)((tx)(uy)) \overset{1}{=} (vs)((yt)(ux)) \overset{c}{=} (vs)(u(x(yt)))\\ &\overset{c}{=} u((x(yt))(vs)) \overset{c}{=} u(v(s(x(yt)))) \overset{c}{=} (s(x(yt)))(uv)\\ &\overset{c}{=} ((yt)(sx))(uv) \overset{1}{=} ((xy)(st))(uv).\end{aligned}$$

4. We use parts 1–3 and the cyclic identity to obtain:

$$\begin{aligned}((xy)(st))(uv) &\overset{3}{=} (xy)((st)(uv)) \overset{1}{=} (xy)((vs)(ut)) \overset{2}{=} (ys)((vx)(ut))\\ &\overset{1}{=} (ys)((xt)(uv)) \overset{3}{=} ((ys)(xt))(uv) \overset{1}{=} ((st)(xy))(uv)\\ &\overset{c}{=} (x(y(st)))(uv).\end{aligned}$$

□

Corollary 2. *Let R be a cyclic ring. Then R^2 is associative.*

Proof. This follows from part 3 of Lemma 1, since every element of R^2 is a finite sum of products of elements in R. □

Lemma 3. *Let R be a cyclic ring.*

1. *If R satisfies the identity $(x, y, st) = 0$, then R also satisfies the identities $[xy, st] = 0$ and $(x, y, z)(st) = 0$.*
2. *If R is associative, then R satisfies the identity $[x, y](st) = 0$.*

Proof. 1. Using the hypothesis and the cyclic identity (1) we get

$$(xy)(st) = x(y(st)) = (st)(xy).$$

This proves $[xy, st] = 0$. Using it, we obtain

$$\begin{aligned}((xy)z)(st) &= (xy)(z(st)) = (z(st))(xy) = x(y(z(st)))\\ &= x((st)(yz)) = x((yz)(st)) = (x(yz))(st).\end{aligned}$$

This proves part 1.

2. Since R is associative, it satisfies the identity $(x, y, st) = 0$. We use part 1 to get

$$\begin{aligned}(xy)(st) &= (st)(xy) = y((st)x) = y(s(tx))\\ &= y(x(st)) = (yx)(st).\end{aligned}$$

This proves part 2. □

3. SEMIPRIME CYCLIC RINGS

Lemma 4. *Let R be a cyclic ring and let $I = \{x \in R \mid xR^2 = 0\}$. Then I is an ideal of R.*

Proof. Since it is clear that I is closed under addition we only need to show that if $x \in I$ and $y \in R$ then xy and yx are in I. In fact, let $x \in I$, $y, s, t \in R$. Then, using the cyclic identity we get

$$(xy)(st) = s(t(xy)) = s(x(yt)) = 0$$

and

$$(yx)(st) = s(t(yx)) = s(x(ty)) = 0.$$

This proves Lemma 4. □

Theorem 5. *Let R be a semiprime cyclic ring. Then R is associative and commutative.*

Proof. Let I be as in Lemma 4 and let $N = I \cap R^2$. By Lemma 4, N is an ideal of R. Now, $N^2 \subseteq NR^2 = 0$ and given that R is semiprime, we know that $N = 0$. By Lemma 1 part 4, $(x, y, st) \in N$ for any choice of $x, y, s, t \in R$. Hence R satisfies the identity $(x, y, st) = 0$. Lemma 3 shows that $(x, y, z) \in N$ so that R is associative. Finally, we use Lemma 3 again to show that R is commutative. □

4. NILPOTENCY IN CYCLIC RINGS

Theorem 6. *Let R be a cyclic right-nilring of index n without elements of additive order $\leq n$. Then R^2 is nilpotent and R is solvable.*

Proof. By Corollary 2 R^2 is associative, so it is an associative nilring of index at most n. By the Ivanov–Dubnov–Nagata–Higman theorem (Formanek, 1990), R^2 is nilpotent and therefore R^2 is solvable. Thus R is also solvable. □

Lemma 7. *Let R be a cyclic ring. Then, for every $n \geq 2$:*

1. $R^n = RR^{n-1} + R^{n-1}R$;
2. $RR^{n-1} = {}^{\langle n\rangle}R$;
3. ${}^{\langle 2n\rangle}R = (R^2)^n$;
4. $R^{2n} \subseteq R^{\langle n\rangle}$.

Proof. We proceed by induction on n. For $n = 2$, all four statements are easy to verify. Now we show the inductive step for part 1. We assume that $n \geq 2$ and that $R^k = RR^{k-1} + R^{k-1}R$ for $2 \leq k \leq n$. We want to show that $R^{n+1} = RR^n + R^nR$. Using the inductive hypothesis and the cyclic identity we obtain

$$R^{n+1} = \sum_{k=1}^{n} R^k R^{n+1-k} = RR^n + R^nR + \sum_{k=2}^{n-1} R^k(RR^{n-k} + R^{n-k}R)$$

$$= RR^n + R^nR + \sum_{k=2}^{n-1} R(R^{n-k}R^k + R^kR^{n-k})$$

$$= RR^n + R^nR + R\sum_{k=2}^{n-1}(R^{n-k}R^k + R^kR^{n-k}) \subseteq RR^n + R^nR + RR^n.$$

The reverse inclusion is direct, so this proves the inductive step.

For part 2, we assume that $n \geq 2$ and that $RR^{n-1} = {}^{\langle n\rangle}R$. We want to show that $RR^n = {}^{\langle n+1\rangle}R$. Using part 1, the cyclic identity and the inductive hypothesis we obtain

$$RR^n = R(RR^{n-1} + R^{n-1}R) = R(RR^{n-1}) = R({}^{\langle n\rangle}R) = {}^{\langle n+1\rangle}R.$$

For part 3, we assume that $n \geq 2$ and that ${}^{\langle 2n\rangle}R = (R^2)^n$. We want to show that ${}^{\langle 2n+2\rangle}R = (R^2)^{n+1}$. Using the inductive hypothesis, the cyclic identity and recalling that R^2 is associative, we get

$${}^{\langle 2n+2\rangle}R = R(R({}^{\langle 2n\rangle}R)) = R(R(R^2)^n) = (R^2)^n(R^2) = (R^2)^{n+1}.$$

For part 4, we assume $n \geq 2$ and $R^{2n} \subseteq R^{\langle n\rangle}$. We want to show that $R^{2n+2} \subseteq R^{\langle n+1\rangle}$. Using now parts 1–3 and then the inductive hypothesis we get

$$R^{2n+2} = RR^{2n+1} + R^{2n+1}R = {}^{\langle 2n+2\rangle}R + R^{2n+1}R \subseteq R^{\langle n+1\rangle} + R^{2n}R$$
$$\subseteq R^{\langle n+1\rangle} + R^{\langle n\rangle}R \subseteq R^{\langle n+1\rangle}.$$

This completes the proof of the lemma. □

Theorem 8. *Let R be a cyclic ring. If R is right-nilpotent, then R is nilpotent.*

Proof. This follows directly from Lemma 7 part 4. □

Lemma 9. *If R satisfies $x^{\langle n\rangle} = 0$ and its first linearization (see Zhevlakov et al., 1982), then for every x in R, L_x, and R_x are nilpotent operators. More precisely $L_x^n = 0$ and $R_x^{2n-1} = 0$.*

Proof. We will first show that for every $k \geq 2$, $L_x^k = R_{x^{\langle k\rangle}}$. For $k = 2$ this is direct from (2). Now we assume $L_x^k = R_{x^{\langle k\rangle}}$. Using (2) again we get

$$L_x^{k+1} = L_x^kL_x = R_{x^{\langle k\rangle}}L_x = R_{x^{\langle k+1\rangle}}.$$

This proves our claim. In particular we obtain that $L_x^n = R_{x^{\langle n\rangle}} = 0$.

To prove the nilpotency of R_x, we first show that $R_x^kL_y = L_yR_{x^{\langle k\rangle}}$ for every $k \geq 2$. For $k = 2$, we have

$$R_x^2L_y = R_xR_xL_y = R_xL_yL_x = L_yL_xL_x = L_yR_{x^2}.$$

Now we assume $R_x^k L_y = L_y R_{x^{\langle k\rangle}}$. Then

$$R_x^{k+1} L_y = R_x^k R_x L_y = R_x^k L_y L_x = L_y R_{x^{\langle k\rangle}} L_x = L_y R_{x^{\langle k+1\rangle}}.$$

This proves our claim. Now we see that the first linearization of $x^{\langle n\rangle} = 0$ can be written in terms of operators as

$$R_x^{n-1} + \sum_{k=1}^{n-1} L_{x^{\langle k\rangle}} R_x^{n-1-k} = 0.$$

Multiplying this last identity by R_x^n on the left side we get:

$$\begin{aligned} 0 &= R_x^{2n-1} + \sum_{k=1}^{n-1} R_x^n L_{x^{\langle k\rangle}} R_x^{n-1-k} \\ &= R_x^{2n-1} + \sum_{k=1}^{n-1} L_{x^{\langle k\rangle}} R_{x^{\langle n\rangle}} R_x^{n-1-k} = R_x^{2n-1}. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 10. *Let R be a cyclic right-nilring and let $x \in R$. Then any power of x can be written as a single x tapped some number of times by x on the left and then some other number of times by x on the right. In other words, xL^nR^m are all the powers of x we may get.*

Proof. For small exponents this is clear since $x = xL^0R^0$, $x^2 = xL = xR$ (notice that one may be able to write the same element in more than one way). Let us carefully multiply two of these expressions, $u = xL^jR^k$, $v = xL^nR^m$. In the case that $m + n = 0$ it is clear that $v = x$ and that

$$uv = xL^jR^{k+1}.$$

We claim that in the remaining cases (when v has degree at least 2), then

$$uv = xL^{j+k+n+m+1}$$

and we will prove this by induction on $j + k + n + m$ or on the total degree of the product involved. We consider first the case where $m \neq 0$

$$\begin{aligned} uv &= (xL^jR^k)((xL^nR^{m-1})x) = x((xL^jR^k)(xL^nR^{m-1})) \\ &= x(xL^{j+k+n+m}) = xL^{j+k+n+m+1}. \end{aligned}$$

Now, when $m = 0$, the $n \neq 0$ and

$$\begin{aligned} uv &= (xL^jR^k)(x(xL^{n-1})) = x((xL^{n-1})(xL^jR^k)) \\ &= x(xL^{j+k+n+m}) = xL^{j+k+n+m+1}. \end{aligned}$$

This finishes the proof of the lemma. □

Corollary 11. *Let R be a cyclic right-nilring and let $x \in R$. Then the subring $\langle x \rangle \subseteq R$ generated by x is nilpotent. In other words, R is nil.*

Proof. We will assume $x^{\langle n \rangle} = 0$. By Lemma 10, any element $u \in \langle x \rangle^{3n}$ can be written as a linear combination of elements $u = xL^j R^k$ where $j + k \geq 3n - 1$, so either $j \geq n$ or $k \geq 2n - 1$. By Lemma 9 we get $u = 0$ and therefore $\langle x \rangle$ is nilpotent. □

5. NILPOTENCY IN CYCLIC ALGEBRAS

In this section we obtain some results on the nilpotency of cyclic right-nilalgebras. We will prove that if $\dim A = n$, then A is nilpotent in the following cases: (1) when $n \leq 4$ and, (2) when the right-nilindex of A is $n + 1$. These conditions can not be improved in any obvious way as is shown by the following examples.

Example 1. Let A be an algebra with basis x_1, x_2, x_3, x_4, x_5 and the following nonzero products of basis elements

$$x_2x_1 = x_3 \qquad x_4x_2 = x_3 \qquad x_5x_1 = -x_3 \qquad x_3x_1 = x_4 \qquad x_3x_2 = x_5.$$

A is a cyclic right-nilalgebra of dimension 5 and right-nilindex 4, which is not nilpotent. In fact, notice that $A^2 = \langle x_3, x_4, x_5 \rangle$ and $AA^2 = 0$ so that A is clearly cyclic since it satisfies the stronger identity $x(yz) = 0$. We can also see that $A^2A = A^2$ so A is not right-nilpotent. Straightforward calculation of the fourth power of an arbitrary element in A shows that A has right-nilindex 4.

An interesting remark on Example 1 is that the algebra A^+, defined on the additive group of A with the new multiplication $*$ given by $a * b = (ab + ba)/2$, is isomorphic to a known counterexample given in Suttles (1972) to a conjecture of Albert on the nilpotency of commutative power-associative nilalgebras (see Albert, 1948). Regarding Albert's conjecture, it has been proved that Suttle's counterexample is "best possible" in the sense that, every commutative power associative nilalgebra A of dimension ≤ 4 is nilpotent (Gerstenhaber and Myung, 1975). It has also been proved that A is nilpotent when the dimension is finite equal to n and the nilindex is $\geq n$ (see Correa and Suazo, 1999). It is an open problem to prove that in the remaining (finite dimensional) cases, nilalgebras are solvable.

In our case, cyclic right-nilalgebras are nilpotent when the dimension is n and the right-nilindex is $n + 1$, but not necessarily when the right-nilindex is n as the following example shows.

Example 2. Consider the algebra A with basis a, x_1, x_2, x_3, x_4 and the following nonzero products of basis elements

$$x_1x_1 = x_2 \qquad x_2x_1 = x_3 \qquad x_3x_1 = x_4 \qquad x_3a = x_2 \qquad x_4a = -x_3.$$

A satisfies the identity $x(yz) = 0$, therefore it is cyclic, moreover, A is a right-nilalgebra of dimension 5 and right-nilindex 5, but it is not nilpotent since $(\ldots(((x_2x_1)a)x_1)a\ldots) \neq 0$. In this case A^+ is not power-associative since $x_1^4 = x_4$ but $(x_1x_1)(x_1x_1) = 0$.

In order to obtain our next result, we need the following two technical lemmas:

Lemma 12. *Let A be a cyclic right-nilalgebra and let I be a right-ideal of A. If* $\dim I \leq 2$ *then* $(IA)A = 0$.

Proof. Assume $I \neq 0$. From Lemma 9 we know that for every x in A, the operator R_x is nilpotent. Hence, Ix is properly contained in I.

If $\dim I = 1$, then $Ix = 0$ for every $x \in A$ and therefore $IA = 0$.

Assume now $\dim I = 2$. Since $IA \subseteq I$, we also have $(IA)A \subseteq IA$ so IA is a right-ideal of A. Suppose first that $IA = I$. Then there must exist elements $a, b \in A$ such that $I = Ia + Ib$. Since for every x in A, R_x is nilpotent and $\dim I = 2$, it follows that $(Ix)x = 0$ for every x in A. Therefore, if we multiply $I = Ia + Ib$ on the right side by a, we get $Ia = (Ib)a$. Similarly $Ib = (Ia)b$. Therefore $0 = (I(a+b))(a+b) = (Ia + Ib)(a+b) = I(a+b) = I$, which is a contradiction. Therefore $\dim IA \leq 1$, using the first part we can conclude $(IA)A = 0$. This proves the lemma. □

Lemma 13. *Let A be a cyclic right-nilalgebra and let I be an ideal of A. If the codimension of I is* 1 *and I^2 is properly contained in I, then IA is properly contained in I.*

Proof. $A = I + xF$ for some $x \in A$. Since $IA = I^2 + Ix$, we get $IA/I^2 = Ix/I^2$ is properly contained in I/I^2 and IA is properly contained in I. □

Theorem 14. *Let A be a cyclic right-nilalgebra with dim $A \leq 4$. Then $A^{\langle 5 \rangle} = 0$. In particular, A is nilpotent.*

Proof. Since A is solvable, we know that A^2 is a proper ideal of A. If $\dim A = 3$, then by Lemma 13, A^2A is a right ideal of A of dimension at most 2. Using Lemma 12 for $I = A^2A$ we get $A^{\langle 5 \rangle} = ((A^2A)A)A = 0$. Theorem 8 shows that A is nilpotent. □

Theorem 15. *Let A be a cyclic right-nilalgebra of dimension n and right-nilindex $n+1$. Then A is nilpotent.*

Proof. Let x be an element in A such that $x^{\langle n \rangle} \neq 0$. It is easy to see that $\{x, x^2, \ldots, x^{\langle n \rangle}\}$ are linearly independent so that $A = \langle x \rangle$. From Corollary 11 it follows that A is nilpotent. □

Now we present a cyclic right-nilalgebra which is infinite-dimensional and is not nilpotent.

Example 3. Let $N = \{x_1, x_2, x_3, \ldots\}$ a countably infinite set of indeterminates and P the set of the words in the letters x_i such that each letter occurs at most once in each word. We say that a word u has length k if it is formed by k letters x_i. Therefore a word has length ≥ 1. Let K be a field and A the set of finite formal sums of words of P and with coefficient in K. We define a noncommutative multiplication in A by:

1. $uv = 0$ if v has length >1, $u = v$ or v is a letter that is in the composition of u;
2. ux is the word obtained adding the letter x at the end of the word u.

A is a cyclic right-nilalgebra of right-nilindex 2 which is not nilpotent since for every n, $(\cdots(((x_1x_2)x_3)x_4)\cdots)x_n \neq 0$.

We recall that an algebra is called *flexible* if it satisfies the flexible identity $(x, y, x) = 0$. Correa (2006) proves that a cyclic flexible finite-dimensional right-nilalgebra is nilpotent. The following theorem improves this result by replacing the hypothesis of finite dimension with nil of bounded index.

Theorem 16. *Let A be a cyclic and flexible right-nilalgebra of right-nilindex n over a field of characteristic* 0 *or greater than n. Then A is nilpotent.*

Proof. From the flexible identity and the linearity of the associator we have

$$\begin{aligned} 0 &= (z + x, y, z + x) = (z, y, z) + (z, y, x) + (x, y, z) + (x, y, x) \\ &= (z, y, x) + (x, y, z), \end{aligned}$$

whence, using the cyclic identity, we get

$$(zy)x = z(yx) - (xy)z + z(xy). \tag{3}$$

We will show by induction on n that $A^{\langle 2n\rangle} \subseteq (A^2)^n$. For $n = 1$ it is trivially true. Now we assume that $A^{\langle 2n\rangle} \subseteq (A^2)^n$ and we will prove $A^{\langle 2n+2\rangle} \subseteq (A^2)^{n+1}$. From (3), $A^{\langle 2n+2\rangle} = (A^{\langle 2n\rangle}A)A \subseteq A^{\langle 2n\rangle}A^2 + A^2A^{\langle 2n\rangle}$ which, using the inductive hypothesis, proves the assertion.

Now we only need to show that A^2 is nilpotent to conclude that A is right-nilpotent and therefore nilpotent by Theorem 8. From Corollary 2, A^2 is associative so we can use the Nagata–Higman Theorem to show that it is nilpotent. □

As a final comment, we would like to mention the use of the computer program Albert (Jacobs et al., 1993) to check conjectures before attempting a formal proof and in general to get a better idea of what results we could expect to be true.

ACKNOWLEDGMENT

Antonio Behn is supported by FONDECYT 1070243. Iván Correa is supported by FONDECYT 1060229. Irvin Roy Hentzel is supported by FONDECYT 7060096 and FIBAS 10-05.

REFERENCES

Albert, A. A. (1948). Power-associative rings. *Trans. Amer. Math. Soc.* 64:552–593.

Correa, I. (2006). Flexible right-nilalgebras satisfying $x(yz) = y(zx)$. *Non-Associative Algebra and Its Applications*. Lecture Notes in Pure and Applied Mathematics. 246:103–106.

Correa, I., Suazo, A. (1999). On a class of commutative power-associative nilalgebras. *J. Algebra* 215:412–417.

Formanek, E. (1990). The Nagata–Higman theorem. *Acta Appl. Math.* 21(1–2):185–192.

Gerstenhaber, M., Myung, H. C. (1975). On commutative power-associative nilalgebras of low dimension. *Proc. Amer. Math. Soc.* 48:29–32.

Jacobs, D. P., Lee, D., Muddana, S. V., Offut, A. J., Prabhu, K., Whiteley, T. (1993). *Albert's User Guide*. Department of Computers Science, Clemson University, Clemson, SC, USA. Available at http://www.cs.clemson.edu/ dpj/albertstuff/albert.html.

Kleinfeld, M. (1995). Rings with $x(yz) = y(zx)$. *Comm. Algebra* 13:5085–5093.

Schafer, R. D. (1966). *An Introduction to Nonassociative Algebras*. New York/London: Academic Press.

Suttles, D. (1972). A counterexample to a conjecture of Albert. *Notices Amer. Math. Soc.* 19:A–556. Abstract 72T-A169.

Zhevlakov, K. A., Slin'ko, A. M., Shestakov, I. P., Shirshov, A. I. (1982). *Rings That are Nearly Associative*. New York/London: Academic Press.