

## Research Article

# Maximal Regularity for Flexible Structural Systems in Lebesgue Spaces

Claudio Fernández,<sup>1</sup> Carlos Lizama,<sup>2</sup> and Verónica Poblete<sup>3</sup>

<sup>1</sup> Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Santiago, Chile

<sup>2</sup> Departamento de Matemática, Facultad de Ciencias, Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile

<sup>3</sup> Departamento de Matemática, Facultad de Ciencias, Universidad de Chile, Las Palmeras 3425, Ñuñoa, Santiago, Chile

Correspondence should be addressed to Carlos Lizama, carlos.lizama@usach.cl

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We study abstract equations of the form  $\lambda u'''(t) + u''(t) = c^2 Au(t) + c^2 \mu Au'(t) + f(t)$ ,  $0 < \lambda < \mu$  which is motivated by the study of vibrations of flexible structures possessing internal material damping. We introduce the notion of  $(\alpha; \beta; \gamma)$ -regularized families, which is a particular case of  $(a; k)$ -regularized families, and characterize maximal regularity in  $L^p$ -spaces based on the technique of Fourier multipliers. Finally, an application with the Dirichlet-Laplacian in a bounded smooth domain is given.

## 1. Introduction

During the last few decades, the use of flexible structural systems had steadily increased importance. The study of a flexible aerospace structure involves problems of dynamical system theory governed by partial differential equations.

We consider here the problem of characterizing  $L^p$ -maximal regularity (or well-posedness) for a mathematical model of a flexible space structure like a thin uniform rectangular panel, for example, a solar cell array or a spacecraft with flexible attachments. This problem is motivated by both engineering and mathematical considerations.

The study of *vibrations* of flexible structures possessing internal material damping was first derived by Bose and Gorain [1]. The consideration of external forces leads to more general equations of the form

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Au'(t) + f(t), \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad (1.1)$$

where  $A$  is a closed linear operator acting in a Banach space  $X$  and  $f$  is an  $X$ -valued function. We emphasize that the abstract Cauchy problem associated with (1.1) is in general ill posed; see, for example, [2]. Also it is well known that in order to analyze well-posedness, a direct approach leads to better results than those obtained by a reduction to a first-order equation.

Maximal regularity in Hölder spaces for (1.1) has been recently characterized in [3]. In case  $\alpha = 0$ , there are more literatures. For example, stability of the solution was studied by Gorain in [4]. In [5], Gorain and Bose studied exact controllability and boundary stabilization. More recently, Batkai and Piazzera [6, page 188] have obtained the exact decay rate. We note that well-posedness in Lebesgue spaces in the case of a damped wave equation has been only recently considered by Chill and Srivastava in [7], and in Hölder spaces by Poblete [8]. We note that the class studied in [8] includes equations with delay. In particular, well-posedness of the homogeneous abstract Cauchy problem has been observed in [9] for  $\alpha = 0$  under certain assumptions on  $A$ .

This paper is organized as follows. Section 2, collects results essentially contained in [10] and standard literature on  $R$ -boundedness and maximal regularity (see [11] and [12]). In Section 3 we study, by an operator theoretical method, sufficient conditions for existence of solutions for (1.1). We obtain two results: a description of the solution by means of certain regularized families (Proposition 3.1) and the existence of such families in the particular case of positive self-adjoint operators (Theorem 3.2). In Section 4, we succeed in *characterizing* well-posedness of (1.1) in terms of  $R$ -boundedness of a resolvent set which involves  $A$  (Theorem 4.2). This will be achieved in the Lebesgue spaces  $L^p(\mathbb{R}, X)$ , where  $X$  is a UMD space (see below the definition). The methods to obtain this goal are those incorporated in [13] where a similar problem in case of the first-order abstract Cauchy problem has been studied. Our main result (Theorem 4.2) is a combination of the well-known (and deep) result due to Weis [14] stated in Theorem 2.8 and a direct calculation involving the parameters  $\alpha$ ,  $\beta$ , and  $\gamma$ .

## 2. Preliminaries

Let  $\alpha, \beta, \gamma > 0$  be given. In what follows we denote

$$\begin{aligned} k(t) &= \frac{1}{\alpha} \int_0^t (t-s)e^{-s/\alpha} ds = -\alpha + t + \alpha e^{-t/\alpha}, \quad t \in \mathbb{R}_+, \\ a(t) &= \beta k(t) + \frac{\gamma}{\alpha} \int_0^t e^{-s/\alpha} ds = -(\alpha\beta - \gamma) + \beta t + (\alpha\beta - \gamma)e^{-t/\alpha}, \quad t \in \mathbb{R}_+. \end{aligned} \tag{2.1}$$

In order to give an operator theoretical approach to (1.1) we introduce the following definition.

*Definition 2.1.* Let  $A$  be a closed and linear operator with domain  $D(A)$  defined on a Banach space  $X$ . One calls  $A$  the generator of an  $(\alpha, \beta, \gamma)$ -regularized family  $\{R(t)\}_{t \geq 0} \subset \mathcal{B}(X)$  if the following conditions are satisfied.

- (R1)  $R(t)$  is strongly continuous on  $\mathbb{R}_+$  and  $R(0) = 0$ .
- (R2)  $R(t)D(A) \subset D(A)$  and  $AR(t)x = R(t)Ax$  for all  $x \in D(A), t \geq 0$ .

(R3) The following equation holds:

$$R(t)x = k(t)x + \int_0^t a(t-s)R(s)Ax \, ds \quad (2.2)$$

for all  $x \in D(A)$ ,  $t \geq 0$ . In this case,  $R(t)$  is called the  $(\alpha, \beta, \gamma)$ -regularized family generated by  $A$ .

*Remark 2.2.* It is proved in [10], in the more general context of  $(a, k)$ -regularized families, that an operator  $A$  is the generator of an  $(\alpha, \beta, \gamma)$ -regularized family if and only if there exists  $\omega \geq 0$  and a strongly continuous function  $R : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$  such that  $\{(\lambda^2 + \alpha\lambda^3)/(\beta + \gamma\lambda) : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$  and

$$H(\lambda)x := \frac{1}{\beta + \gamma\lambda} \left( \frac{\lambda^2 + \alpha\lambda^3}{\beta + \gamma\lambda} - A \right)^{-1} x = \int_0^\infty e^{-\lambda t} R(t)x \, dt, \quad \operatorname{Re} \lambda > \omega, \quad x \in X. \quad (2.3)$$

Because of the uniqueness of the Laplace transform, we note that an  $(\alpha, \beta, \gamma)$ -regularized family corresponds to an  $(a, k)$ -regularized family studied in [10]. In fact, we have

$$\hat{a}(\lambda) = \frac{\beta + \gamma\lambda}{\lambda^2 + \alpha\lambda^3}, \quad \hat{k}(\lambda) = \frac{1}{\lambda^2 + \alpha\lambda^3}, \quad \forall \operatorname{Re} \lambda > \omega. \quad (2.4)$$

As in the situation of  $C_0$ -semigroups, we have diverse relations of an  $(\alpha, \beta, \gamma)$ -regularized family and its generator. The following result is a direct consequence of [10, Proposition 3.1 and Lemma 2.2].

**Proposition 2.3.** *Let  $R(t)$  be an  $(\alpha, \beta, \gamma)$ -regularized family on  $X$  with generator  $A$ . Then the following hold.*

- (a) *For all  $x \in D(A)$  one has  $R(\cdot)x \in C^3(\mathbb{R}_+; X)$ .*
- (b) *Let  $x \in X$  and  $t \geq 0$ . Then  $\int_0^t a(t-s)R(s)x \, ds \in D(A)$  and*

$$R(t)x = k(t)x + A \int_0^t a(t-s)R(s)x \, ds. \quad (2.5)$$

Results on perturbation, approximation, asymptotic behavior, representation, as well as ergodic-type theorems for  $(\alpha, \beta, \gamma)$ -regularized families can be also deduced from the more general context of  $(a, k)$ -regularized families (see [10, 15–18]).

We will need the following results on Laplace transform (see [19, Theorem 2.5.1 and Corollary 2.5.2] for a detailed proof).

**Lemma 2.4.** *Suppose that  $q : \mathbb{C}_+ \rightarrow \mathbb{C}$  is holomorphic and satisfies  $\sup_{\operatorname{Re} \lambda > 0} |\lambda q(\lambda)| < \infty$  and let  $b > 0$ . Then there exists  $f \in C(\mathbb{R}_+)$  with  $\sup_{t > 0} |e^{-\omega t} t^{-b} f(t)| < \infty$  such that  $q(\lambda) = \lambda^b \int_0^\infty e^{-\lambda t} f(t) dt$  for all  $\operatorname{Re} \lambda > 0$ .*

**Lemma 2.5.** *Suppose that  $q : \mathbb{C}_+ \rightarrow \mathbb{C}$  is holomorphic and satisfies  $|\lambda q(\lambda)| + |\lambda^2 q'(\lambda)| \leq M$  for all  $\operatorname{Re} \lambda > 0$ . Then there exists a bounded function  $f \in C(\mathbb{R}_+)$  such that  $q(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt$  for all  $\operatorname{Re} \lambda > 0$ .*

We introduce the means

$$\|(x_1, \dots, x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1, 1\}^n} \left\| \sum_{j=1}^n \epsilon_j x_j \right\| \quad (2.6)$$

for  $x_1, \dots, x_n \in X$ .

*Definition 2.6.* Let  $X, Y$  be Banach spaces. A subset  $\mathcal{T}$  of  $\mathcal{B}(X, Y)$  is called  $R$ -bounded if there exists a constant  $c \geq 0$  such that

$$\|(T_1 x_1, \dots, T_n x_n)\|_R \leq c \|(x_1, \dots, x_n)\|_R \quad (2.7)$$

for all  $T_1, \dots, T_n \in \mathcal{T}, x_1, \dots, x_n \in X, n \in \mathbb{N}$ . The least  $c$  such that (2.7) is satisfied is called the  $R$ -bound of  $\mathcal{T}$  and is denoted as  $R(\mathcal{T})$ .

The notion of  $R$ -boundedness was implicitly introduced and used by Bourgain [20] and later on also by Zimmermann [21]. Explicitly it is due to Berkson and Gillespie [22] and to Clément et al. [23].

$R$ -boundedness clearly implies boundedness. If  $X = Y$ , the notion of  $R$ -boundedness is strictly stronger than boundedness unless the underlying space is isomorphic to a Hilbert space [24, Proposition 1.17]. Some useful criteria for  $R$ -boundedness are provided in [11, 24].

*Remark 2.7.* (a) Let  $\mathcal{S}, \mathcal{T} \subset \mathcal{B}(X, Y)$  be  $R$ -bounded sets, then  $\mathcal{S} + \mathcal{T} := \{S + T : S \in \mathcal{S}, T \in \mathcal{T}\}$  is  $R$ -bounded.

(b) Let  $\mathcal{T} \subset \mathcal{B}(X, Y)$  and  $\mathcal{S} \subset \mathcal{B}(Y, Z)$  be  $R$ -bounded sets, then  $\mathcal{S} \cdot \mathcal{T} := \{S \cdot T : S \in \mathcal{S}, T \in \mathcal{T}\} \subset \mathcal{B}(X, Z)$  is  $R$ -bounded and

$$R(\mathcal{S} \cdot \mathcal{T}) \leq R(\mathcal{S}) \cdot R(\mathcal{T}). \quad (2.8)$$

(c) Also, each subset  $M \subset \mathcal{B}(X)$  of the form  $M = \{\lambda I : \lambda \in \Omega\}$  is  $R$ -bounded whenever  $\Omega \subset \mathbb{C}$  is bounded.

We recall that those Banach spaces  $X$  for which the Hilbert transform is bounded on  $L^p(\mathbb{R}, X)$ , for some  $p \in (1, \infty)$ , are called UMD spaces. For more information and details on the Hilbert transform and the UMD Banach spaces we refer to [12]. Examples of UMD spaces include Hilbert spaces, Sobolev spaces  $W_p^s(\Omega)$ ,  $1 < p < \infty$  (see [25]), Lebesgue spaces  $L^p(\Omega, \mu)$ ,  $1 < p < \infty$ ,  $L^p(\Omega, \mu; X)$ ,  $1 < p < \infty$ , when  $X$  is a UMD space, and the Schatten-von Neumann classes  $C_p(H)$ ,  $1 < p < \infty$  of operators on Hilbert spaces.

After these preliminaries, we state the following operator-valued Fourier multiplier theorem. It is fundamental in our treatment. A proof can be founded in [11].

**Theorem 2.8.** Suppose that  $X$  is a UMD space and let  $1 < p < \infty$ . Let  $M \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(X))$  be such that the following conditions are satisfied.

- (i) The set  $\{M(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded.
- (ii) The set  $\{\rho M'(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded.

Then the operator  $T$  defined by

$$Tf = \left( M(\cdot) \left[ \widehat{f}(\cdot) \right] \right)^\vee \quad \text{where } f \in \mathcal{S}(X) \quad (2.9)$$

extends to a bounded operator from  $L^p(\mathbb{R}, X)$  to  $L^p(\mathbb{R}, X)$ .

### 3. Existence of Solutions

Let  $\alpha, \beta, \gamma \in (0, \infty)$ . Consider the equation

$$u''(t) + \alpha u'''(t) = \beta Au(t) + \gamma Au'(t) + f(t), \quad (3.1)$$

with initial conditions  $u(0) = u'(0) = u''(0) = 0$ , where  $A$  is the generator of an  $(\alpha, \beta, \gamma)$ -regularized family  $R(t)$ . By a solution of (3.1) we understand a function  $u \in C(\mathbb{R}_+; D(A)) \cap C^3(\mathbb{R}_+; X)$  such that  $u' \in C(\mathbb{R}_+; D(A))$  and verify (3.1).

**Proposition 3.1.** Let  $R(t)$  be an  $(\alpha, \beta, \gamma)$ -regularized family on  $X$  with generator  $A$ . If  $f \in L^1_{\text{loc}}(\mathbb{R}_+, D(A^2))$ , then  $u(t)$  given by

$$u(t) = \int_0^t R(t-s)f(s)ds, \quad t \geq 0 \quad (3.2)$$

is a solution of (3.1).

*Proof.* Given that  $x \in D(A)$ , we obtain from Proposition 2.3 that  $R(\cdot)x$ , and hence  $u$ , is of class  $C^3(\mathbb{R}_+, X)$ . For all  $x \in D(A)$ , we have

$$R'(t)x = \left(1 - e^{-t/\alpha}\right)x + \int_0^t \left[\beta + (\gamma/\alpha - \beta)e^{-(t-s)/\alpha}\right]R(s)Ax ds. \quad (3.3)$$

If  $x \in D(A^2)$ , then  $R'(t)x \in D(A)$ . Moreover,

$$\begin{aligned} R''(t)x &= \frac{1}{\alpha}e^{-t/\alpha}x + \frac{\gamma}{\alpha}R(t)Ax + \int_0^t \left(\frac{\beta}{\alpha} - \frac{\gamma}{\alpha^2}\right)e^{-(t-s)/\alpha}AR(s)x ds, \\ R'''(t)x &= -\frac{1}{\alpha^2}e^{-t/\alpha}x + \frac{\gamma}{\alpha}R'(t)Ax + \frac{\beta}{\alpha}R(t)Ax - \frac{\gamma}{\alpha^2}AR(t)x \\ &\quad + \int_0^t \left(\frac{\gamma}{\alpha^3} - \frac{\beta}{\alpha^2}\right)e^{-(t-s)/\alpha}AR(s)x ds. \end{aligned} \quad (3.4)$$

Since  $f \in L^1_{\text{loc}}(\mathbb{R}_+, D(A^2))$ , from (3.2), we have that  $u(t), u'(t) \in D(A)$  and

$$\begin{aligned} u'(t) &= \int_0^t R'(t-s)f(s)ds, \\ u''(t) &= \int_0^t R''(t-s)f(s)ds, \\ u'''(t) &= R''(0)f(t) + \int_0^t R'''(t-s)f(s)ds. \end{aligned} \quad (3.5)$$

Hence,

$$\begin{aligned} &u''(t) + \alpha u'''(t) - \beta Au(t) - \gamma Au'(t) \\ &= \int_0^t R''(t-s)f(s)ds + f(t) + \alpha \int_0^t R'''(t-s)f(s)ds \\ &\quad - \beta A \int_0^t R(t-s)f(s)ds - \gamma A \int_0^t R'(t-s)f(s)ds. \end{aligned} \quad (3.6)$$

By the other side, for all  $x \in D(A^2)$ , we obtain

$$\begin{aligned} &R''(t)x + \alpha R'''(t)x - \beta AR(t)x - \gamma AR'(t)x \\ &= \frac{1}{\alpha} e^{-t/\alpha} x + \frac{\gamma}{\alpha} AR(t)x + \int_0^t \left( \frac{\beta}{\alpha} - \frac{\gamma}{\alpha^2} \right) e^{-(t-s)/\alpha} AR(s)x ds - \frac{1}{\alpha} e^{-t/\alpha} x + \gamma AR'(t)x \\ &\quad + \beta AR(t)x - \frac{\gamma}{\alpha} AR(t)x + \int_0^t \left( \frac{\gamma}{\alpha^2} - \frac{\beta}{\alpha} \right) e^{-(t-s)/\alpha} AR(s)x ds - \beta AR(t)x - \gamma AR'(t)x \\ &= 0. \end{aligned} \quad (3.7)$$

Since  $f(t) \in D(A^2)$  and  $A$  is closed, from (3.6) we conclude that  $u(t)$  verify (3.1).  $\square$

The following remarkable result provides a wide class of generators of  $(\alpha, \beta, \gamma)$ -regularized families. In what follows we denote

$$\varphi(\lambda) := \frac{1}{\hat{a}(\lambda)} = \frac{\lambda^2(1 + \alpha\lambda)}{\beta + \gamma\lambda}. \quad (3.8)$$

**Theorem 3.2.** *Let  $-B$  be a positive self-adjoint operator on a Hilbert space  $H$  such that*

$$\alpha\beta \leq \gamma. \quad (3.9)$$

*Then  $B$  is the generator of a bounded  $(\alpha, \beta, \gamma)$ -regularized family on  $H$ .*

*Proof.* Since  $-B$  is a positive self-adjoint operator in  $H$ , the spectrum  $\sigma(B)$  is a subset of the negative real axis and the resolvent operator  $(\mu - B)^{-1}$  is defined at least for all negative non real  $\mu$ . Let  $\lambda \in \mathbb{C}$  such that  $\operatorname{Re} \lambda > 0$ . If  $\operatorname{Im} \varphi(\lambda) \neq 0$ , then clearly  $\varphi(\lambda) \in \rho(B)$ . If  $\operatorname{Im} \varphi(\lambda) = 0$ , then we claim that  $\operatorname{Re} \varphi(\lambda) > 0$ . In fact, for  $\lambda = a + bi \in \mathbb{C}$ ,  $a > 0$ , with a direct computation we obtain

$$\begin{aligned} \operatorname{Re} \varphi(\lambda) &= \frac{(a^2 - b^2)(1 + \alpha a)(\beta + \gamma a) - 2ab^2\alpha(\beta + \gamma a) + \alpha\gamma b^2(a^2 - b^2) + 2ab^2(1 + \alpha a)}{(\beta + \gamma a)^2 + \gamma^2 b^2}, \\ \operatorname{Im} \varphi(\lambda) &= \frac{ab(a^2 - b^2)(\beta + \gamma a) + 2ab(1 + \alpha a)(\beta + \gamma a) - \gamma b(a^2 - b^2)(1 + \alpha a) + 2ab^3\alpha\gamma}{(\beta + \gamma a)^2 + \gamma^2 b^2}. \end{aligned} \quad (3.10)$$

Note that  $\operatorname{Im} \varphi(\lambda) = 0$  if and only if  $b = 0$  or  $\alpha(a^2 - b^2)(\beta + \gamma a) + 2a(1 + \alpha a)(\beta + \gamma a) - \gamma(a^2 - b^2)(1 + \alpha a) + 2ab^2\alpha\gamma = 0$ .

Since  $\alpha\beta \leq \gamma$ , we have that

$$\begin{aligned} &\alpha(a^2 - b^2)(\beta + \gamma a) + 2a(1 + \alpha a)(\beta + \gamma a) - \gamma(a^2 - b^2)(1 + \alpha a) + 2ab^2\alpha\gamma \\ &= 2\alpha\gamma ab^2 + b^2(\gamma - \alpha\beta) + \gamma a^2 + 3\alpha\beta a^2 + 2\beta a + 2\alpha\gamma a^2 \\ &> 0. \end{aligned} \quad (3.11)$$

Hence,  $\operatorname{Im} \varphi(\lambda) = 0$  if and only if  $b = 0$ . Since  $a > 0$ , a direct calculation gives

$$\operatorname{Re} \varphi(\lambda) = \frac{a^2(1 + \alpha a)}{\beta + \gamma a} > 0, \quad (3.12)$$

proving the claim. We conclude that  $\varphi(\lambda) \in \rho(B)$  for all  $\operatorname{Re} \lambda > 0$ . Hence (see Kato [26, Section V.3.5]),

$$\|(\varphi(\lambda) - B)^{-1}\| = \frac{1}{\operatorname{dist}(\varphi(\lambda), \sigma(B))} \quad \forall \operatorname{Re} \lambda > 0. \quad (3.13)$$

Note that

$$\sup_{\operatorname{Re} \lambda > 0} \left( \frac{|\lambda|^2 + 1}{\operatorname{dist}(\varphi(\lambda), \sigma(B))} \right) < M, \quad (3.14)$$

since  $\operatorname{dist}(\varphi(\lambda), \sigma(B))$  has order  $|\lambda|^2$ . Define  $Q(\lambda) = (1/(\beta + \gamma\lambda))(\varphi(\lambda) - B)^{-1}$ . We have by (3.14) and (3.13) that for all  $\operatorname{Re} \lambda > 0$

$$\|\lambda Q(\lambda)\| = \left\| \frac{\lambda}{(\beta + \gamma\lambda)} (\varphi(\lambda) - B)^{-1} \right\| \leq \frac{|\lambda|}{|\beta + \gamma\lambda|} \frac{1}{\operatorname{dist}(\varphi(\lambda), \sigma(B))} < M. \quad (3.15)$$

On the other hand,

$$\lambda^2 Q'(\lambda) = \frac{-\gamma\lambda}{\beta + \gamma\lambda} [\lambda Q(\lambda)] + [\lambda Q(\lambda)] \left[ \lambda^2 (\varphi(\lambda) - B)^{-1} \right] \left[ \lambda \frac{\widehat{a}(\lambda)'}{\widehat{a}(\lambda)} \right] \frac{1}{\lambda^2 \widehat{a}(\lambda)',} \quad (3.16)$$

where

$$\frac{1}{\lambda^2 \widehat{a}(\lambda)} = \frac{1 + \alpha\lambda}{\beta + \gamma\lambda}, \quad \lambda \frac{\widehat{a}(\lambda)'}{\widehat{a}(\lambda)} = -\frac{2\alpha\gamma\lambda^2 + (\gamma + 3\alpha\beta)\lambda + 2\beta}{(1 + \alpha\lambda)(\beta + \gamma\lambda)} \quad (3.17)$$

and, by (3.14),

$$\left\| \lambda^2 (\varphi(\lambda) - B)^{-1} \right\| \leq \frac{|\lambda^2|}{\text{dist}(\varphi(\lambda), \sigma(B))} < M \quad (3.18)$$

for all  $\text{Re } \lambda > 0$ . We conclude that  $\sup_{\text{Re } \lambda > 0} \|\lambda^2 Q'(\lambda)\| < \infty$ .

By Lemma 2.5 there exists a strongly continuous family  $R(t)$  such that  $\|R(t)\| \leq K$  and  $Q(\lambda) = \widehat{R}(\lambda)$  for  $\text{Re } \lambda > 0$ . In consequence, for all  $\text{Re } \lambda > 0$  we have

$$\widehat{R}(\lambda) = \frac{\varphi(\lambda)}{\lambda^2(1 + \alpha\lambda)} (\varphi(\lambda) - B)^{-1} = \frac{1}{\beta + \gamma\lambda} \left( \frac{\lambda^2 + \alpha\lambda^3}{\beta + \gamma\lambda} - B \right)^{-1}, \quad (3.19)$$

and, by Remark 2.2, it shows that  $R(t)$  is a bounded  $(\alpha, \beta, \gamma)$ -regularized family generated by  $B$ .  $\square$

Since it is a known fact that the Dirichlet-Laplacian operator is a self-adjoint operator on  $L^2(\Omega)$  and  $\sigma(\Delta) \subset (-\infty, 0)$ , we obtain the following corollary.

**Corollary 3.3.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ , and assume that  $\alpha\beta \leq \gamma$ . Then the Dirichlet-Laplacian operator  $\Delta$  with domain  $H^2(\Omega) \cap H_0^1(\Omega)$  is the generator of an  $(\alpha, \beta, \gamma)$ -regularized family on  $X = L^2(\Omega)$ .*

*Remark 3.4.* In Theorem 3.2 the condition  $\alpha\beta \leq \gamma$  is fundamental to have  $\varphi(\lambda) \in \rho(B)$  for all  $\lambda \in \mathbb{C}$  with  $\text{Re } \lambda > 0$ , which is the key in the proof. Figure 1 is the typical situation, where we have mapped by  $\varphi$  the lines  $\text{Re}(\lambda) = 1, 2$ , and  $3$  with  $\alpha = 3, \beta = 1$ , and  $\gamma = 4$ .

Note that in case  $\alpha\beta > \gamma$  it can happen that  $\varphi(\lambda) \in \sigma(B)$ . For example, taking  $\alpha = 1, \beta = 5$ , and  $\gamma = 1$ , we obtain Figure 2 of  $\varphi(\lambda)$  for  $\text{Im}(\lambda) \in \mathbb{R}$  and  $\text{Re}(\lambda) = 1$

#### 4. $L^p$ -Well-Posedness

Having presented preliminary material on  $R$ -boundedness and Fourier multipliers, we will now show how these tools can be used to handle (3.1). Our main result give concrete conditions on the operator  $A$  under which (3.1) has  $L^p$ -maximal regularity.

The definition of  $L^p$ -maximal regularity which we investigate in this section is given as follows.



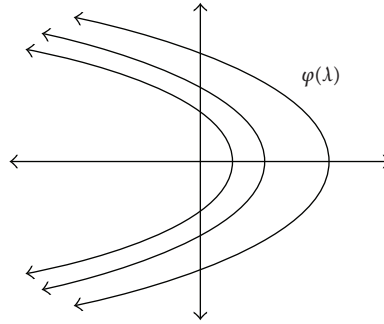


Figure 1

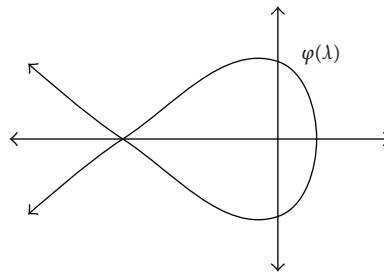


Figure 2

*Definition 4.1.* One says that (3.1) has  $L^p$ -maximal regularity (or is  $L^p$ -well posed) on  $\mathbb{R}_+$  if for each  $f \in L^p(\mathbb{R}_+, X)$  there is a unique function  $u \in W^{3,p}(\mathbb{R}_+, X) \cap W^{1,p}(\mathbb{R}_+, [D(A)]) \cap W^p(\mathbb{R}_+, [D(A)])$  such that (3.1) holds a.e.

The following is the main abstract result of this section. It completely characterizes the maximal regularity of solutions for (3.1) in Lebesgue spaces.

**Theorem 4.2.** *Let  $X$  be a UMD space,  $1 < p < \infty$ , and let  $A$  be the generator of a bounded  $(\alpha, \beta, \gamma)$ -regularized family  $R(t)$ . The following statements are equivalent.*

- (i) Equation (3.1) has  $L^p$ -maximal regularity on  $\mathbb{R}_+$ .
- (ii)  $b(\rho) := -\rho^2((1 + i\alpha\rho)/(\beta + i\gamma\rho)) \in \rho(A)$  for all  $\rho \in \mathbb{R} \setminus \{0\}$  and the set

$$\left\{ \frac{\rho^3}{\beta + i\gamma\rho} R(b(\rho), A) \right\}_{\rho \in \mathbb{R} \setminus \{0\}} \text{ is } R\text{-bounded.} \tag{4.1}$$

*Proof.* (i)  $\Rightarrow$  (ii). By (3.1) and Definition 4.1 together with Proposition 3.1, the convolution operator with kernel

$$K_4(t) := R'''(t)\chi_{(0,\infty)}(t), \quad t \in \mathbb{R}, \tag{4.2}$$

is a bounded operator from  $L^p(\mathbb{R}, X)$  to  $L^p(\mathbb{R}, X)$ . Note that the Fourier transform  $\tilde{R}(\rho)$  exists for  $\rho \neq 0$  because  $R(t)$  is bounded and  $\tilde{R}(\lambda)$  ( $\operatorname{Re} \lambda > 0$ ) can be analytically extended from  $\operatorname{Re} \lambda > 0$  to the imaginary axis. Then the symbol of this convolution operator is given by

$$M(\rho) = \frac{\rho^3}{\beta + i\gamma\rho} R(b(\rho), A), \quad \rho \in \mathbb{R} \setminus \{0\}, \quad (4.3)$$

and the conclusion follows from [11, Proposition 3.17].  
(ii)  $\Rightarrow$  (i). Define  $N(\rho) := (1/(\beta + i\gamma\rho))R(b(\rho), A)$  and

$$N_1(\rho) := AN(\rho). \quad (4.4)$$

We check that the set  $\{N_1(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded.

Since  $(b(\rho) - A)R(b(\rho), A) = I$ , we have that  $AR(b(\rho), A) = b(\rho)R(b(\rho), A) - I$ . Replacing in (4.4)

$$N_1(\rho) = \frac{b(\rho)}{\beta + i\gamma\rho} R(b(\rho), A) - \frac{1}{\beta + i\gamma\rho} I = -\frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \rho^2 N(\rho) - \frac{1}{\beta + i\gamma\rho} I. \quad (4.5)$$

Note that

$$\begin{aligned} \left| \frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \right|^2 &= \frac{1 + \alpha^2\rho^2}{\beta^2 + \gamma^2\rho^2} < \frac{1}{\beta^2} + \frac{\alpha^2}{\gamma^2}, \\ \left| \frac{1}{\beta + i\gamma\rho} \right|^2 &= \frac{1}{\beta^2 + \gamma^2\rho^2} < \frac{1}{\beta^2}. \end{aligned} \quad (4.6)$$

Since the sum of  $R$ -bounded sets is  $R$ -bounded, see [11], we obtain that  $\{N_1(\rho)\}$  is  $R$ -bounded.

We now check that the set  $\{\rho N'_1(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded. With a direct computation, we obtain

$$\begin{aligned} N'_1(\rho) &= b'(\rho)N(\rho) + b(\rho)N'(\rho) + \frac{i\gamma}{(\beta + i\gamma\rho)^2} I \\ &= \frac{2\alpha\gamma}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) - \frac{\gamma + 3\alpha\beta}{(\beta + i\gamma\rho)^2} i\rho^2 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2} \rho N(\rho) \\ &\quad - \frac{i\gamma}{\beta + i\gamma\rho} b(\rho)N(\rho) - \frac{2\alpha\gamma\rho^2 - (\gamma + 3\alpha\beta)i\rho - 2\beta}{\beta + i\gamma\rho} \rho b(\rho)N(\rho)N(\rho) + \frac{i\gamma}{(\beta + i\gamma\rho)^2} I. \end{aligned} \quad (4.7)$$

Hence

$$\begin{aligned}
\rho N'_1(\rho) &= \frac{2\alpha\gamma\rho}{(\beta+i\gamma\rho)^2}\rho^3N(\rho) - \frac{\gamma+3\alpha\beta}{(\beta+i\gamma\rho)^2}i\rho^3N(\rho) - \frac{2\beta}{(\beta+i\gamma\rho)^2}\rho^2N(\rho) + \frac{i\gamma-\alpha\gamma\rho}{(\beta+i\gamma\rho)^2}\rho^3N(\rho) \\
&\quad + \left(2\alpha\gamma\rho^2 - (\gamma+3\alpha\beta)i\rho - 2\beta\right)\frac{1+i\alpha\rho}{(\beta+i\gamma\rho)^2}\rho^4N(\rho)N(\rho) + \frac{i\gamma\rho}{(\beta+i\gamma\rho)^2}I \\
&= \frac{2\alpha\gamma\rho}{(\beta+i\gamma\rho)^2}\rho^3N(\rho) - \frac{\gamma+3\alpha\beta}{(\beta+i\gamma\rho)^2}i\rho^3N(\rho) - \frac{2\beta}{(\beta+i\gamma\rho)^2}\rho^2N(\rho) + \frac{i\gamma-\alpha\gamma\rho}{(\beta+i\gamma\rho)^2}\rho^3N(\rho) \\
&\quad + 2\alpha\gamma\frac{1+i\alpha\rho}{(\beta+i\gamma\rho)^2}\rho^3N(\rho)\rho^3N(\rho) - (\gamma+3\alpha\beta)\frac{i-\alpha\rho}{(\beta+i\gamma\rho)^2}\rho^3N(\rho)\rho^2N(\rho) \\
&\quad - 2\beta\frac{1+i\alpha\rho}{\beta+i\gamma\rho}\rho^3N(\rho)\rho N(\rho) + \frac{i\gamma\rho}{(\beta+i\gamma\rho)^2}I \\
&= \frac{\alpha\gamma\rho-3\alpha\beta i}{(\beta+i\gamma\rho)^2}\rho^3N(\rho) - \frac{2\beta}{(\beta+i\gamma\rho)^2}\rho^2N(\rho) + 2\alpha\gamma\frac{1+i\alpha\rho}{(\beta+i\gamma\rho)^2}\rho^3N(\rho)\rho^3N(\rho) \\
&\quad - (\gamma+3\alpha\beta)\frac{i-\alpha\rho}{(\beta+i\gamma\rho)^2}\rho^3N(\rho)\rho^2N(\rho) - 2\beta\frac{1+i\alpha\rho}{\beta+i\gamma\rho}\rho^3N(\rho)\rho N(\rho) + \frac{i\gamma\rho}{(\beta+i\gamma\rho)^2}I.
\end{aligned} \tag{4.8}$$

Since the set  $\{\rho^3N(\rho)\}$  is  $R$ -bounded and the complex functions appearing in the above equality are bounded, we obtain the claim from the fact that the sum of  $R$ -bounded sets is again  $R$ -bounded. We employ now Theorem 2.8 to conclude that the operator  $T_1$  defined by

$$T_1f = \left(N_1(\cdot)\left[\widehat{f}(\cdot)\right]\right)^\vee \quad \text{where } f \in \mathcal{S}(X) \tag{4.9}$$

extends to a bounded operator from  $L^p(\mathbb{R}, X)$  to  $L^p(\mathbb{R}, X)$ .

Define

$$N_2(\rho) := \frac{\rho}{\beta+i\gamma\rho}AR(b(\rho), A). \tag{4.10}$$

We will prove that the sets  $\{N_2(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  and  $\{\rho N'_2(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  are  $R$ -bounded.

In fact, note that  $N_2(\rho) = \rho N_1(\rho) = -((1 + i\alpha\rho)/(\beta + i\gamma\rho))\rho^3 N(\rho) - (\rho/(\beta + i\gamma\rho))I$ . Hence the set  $\{N_2(\rho)\}$  is  $R$ -bounded. Moreover, we have

$$\begin{aligned}
\rho N_2'(\rho) &= \rho^2 N_1'(\rho) + \rho N_1(\rho) \\
&= \frac{\alpha\gamma\rho^2 - 3\alpha\beta\rho i}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) + 2\alpha\gamma \frac{\rho + i\alpha\rho^2}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^3 N(\rho) \\
&\quad - (\gamma + 3\alpha\beta) \frac{i - \alpha\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^3 N(\rho) - 2\beta \frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \rho^3 N(\rho) \rho^2 N(\rho) \\
&\quad + \frac{i\gamma\rho^2}{(\beta + i\gamma\rho)^2} I + N_2(\rho),
\end{aligned} \tag{4.11}$$

obtaining that the set  $\{\rho N_2'(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded. By Theorem 2.8 we conclude that the operator  $T_2$  defined by

$$T_2 f = \left( N_2(\cdot) \left[ \widehat{f}(\cdot) \right] \right)^\vee \quad \text{where } f \in \mathcal{S}(X) \tag{4.12}$$

extends to a bounded operator from  $L^p(\mathbb{R}, X)$  to  $L^p(\mathbb{R}, X)$ .

Finally, define

$$N_3(\rho) := \frac{\rho^2}{\beta + i\gamma\rho} R(b(\rho), A) = \rho^2 N(\rho). \tag{4.13}$$

The set  $\{N_3(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded from hypothesis and also note that the set  $\{\rho N_3'(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded, since

$$\begin{aligned}
\rho N_3'(\rho) &= 2\rho^2 N(\rho) - \frac{i\gamma}{\beta + i\gamma\rho} \rho^3 N(\rho) - \frac{2\alpha\gamma}{\beta + i\gamma\rho} \rho^3 N(\rho) \rho^3 N(\rho) \\
&\quad + \frac{\gamma + 3\alpha\beta}{\beta + i\gamma\rho} i \rho^3 N(\rho) \rho^2 N(\rho) + \frac{2\beta}{\beta + i\gamma\rho} \rho N(\rho) \rho^3 N(\rho).
\end{aligned} \tag{4.14}$$

Again by Theorem 2.8 we conclude that the operator  $T_3$  defined by

$$T_3 f = \left( N_3(\cdot) \left[ \widehat{f}(\cdot) \right] \right)^\vee \quad \text{where } f \in \mathcal{S}(X) \tag{4.15}$$

extends to a bounded operator from  $L^p(\mathbb{R}, X)$  to  $L^p(\mathbb{R}, X)$ . From (4.9), (4.12), and (4.15) and since it is clear that (3.1) has  $L^p$ -maximal regularity if the convolution operator with each one of the kernels

$$K_1(t) := AR(t)\chi_{(0,\infty)}(t), \quad K_2(t) := AR'(t)\chi_{(0,\infty)}(t), \quad K_3(t) := R''(t)\chi_{(0,\infty)}(t), \quad t \in \mathbb{R}, \quad (4.16)$$

is a bounded operator from  $L^p(\mathbb{R}, X)$  to  $L^p(\mathbb{R}, X)$  (see [11]), we conclude (i) and the proof is complete.  $\square$

Of course,  $R$ -boundedness in (4.1) can be replaced by boundedness in case  $X = H$  is a Hilbert space.

**Corollary 4.3.** *The solution  $u$  of (3.1), under the conditions given by Theorem 4.2, satisfies the following maximal regularity property:  $u, u' \in L^p(\mathbb{R}_+; [D(A)])$  and  $Au, Au', u'', u''' \in L^p(\mathbb{R}_+; X)$ . Moreover, there exists a constant  $C > 0$  independent of  $f \in L^p(\mathbb{R}_+; X)$  such that*

$$\|u\|_p + \|u'\|_p + \|u''\|_p + \|u'''\|_p + \|Au\|_p + \|Au'\|_p \leq C\|f\|_p. \quad (4.17)$$

The proof follows by the closed-graph theorem.

As an example, we consider for  $A = \Delta$  the vibration equation subject to the action of an external force. Explicitly, we consider

$$\begin{aligned} v_{tt}(t, x) + \lambda v_{ttt}(t, x) &= c^2(\Delta v(t, x) + \mu \Delta v_t(t, x)) + f(t, x) \text{ in } ]0, T] \times \Omega, \\ v(t, x) &= 0 \text{ on } ]0, T] \times \Omega, \\ v(0, x) &= 0 \text{ in } \Omega, \\ v_t(0, x) &= 0 \text{ in } \Omega, \\ v_{tt}(0, x) &= 0 \text{ in } \Omega \end{aligned} \quad (4.18)$$

in a smooth bounded region  $\Omega \subset \mathbb{R}^n$ . Also, we assume that  $f \in L^2(\mathbb{R}; L^2(\mathbb{R}^n))$ . We have the following application in the Hilbert space setting.

**Theorem 4.4.** *Let  $\Omega$  be a bounded domain with smooth boundary  $\partial\Omega$  in  $\mathbb{R}^n$ . Suppose that  $0 < \lambda < \mu$ . Then the initial value problem (4.18) defined on  $L^2(\Omega)$  with Dirichlet boundary conditions has  $L^2$ -maximal regularity on  $\mathbb{R}_+$ .*

*Proof.* Let  $\alpha = \lambda$ ,  $\beta = c^2$ , and  $\gamma = c^2\mu$  and note that  $\alpha\beta < \gamma$  if and only if  $\lambda < \mu$ . By Corollary 3.3 we conclude that  $\Delta$  generates a bounded  $(\alpha, \beta, \gamma)$ -regularized family on  $L^2(\Omega)$ .

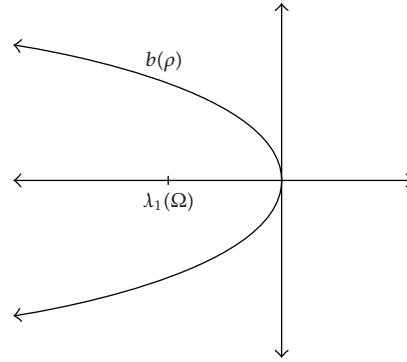


Figure 3

Note that we have  $b(\rho) = -\rho^2((1 + i\alpha\rho)/(\beta + i\gamma\rho)) \in \rho(\Delta)$  and there exists a constant  $C > 0$  such that

$$\begin{aligned} \left\| \frac{\rho^3}{\beta + i\gamma\rho} R(b(\rho), \Delta) \right\| &= \left\| \rho \frac{b(\rho)}{1 + i\alpha\rho} (b(\rho) - \Delta)^{-1} \right\| \\ &= \frac{|\rho|}{|1 + i\alpha\rho|} \frac{|b(\rho)|}{\text{dist}(b(\rho), \lambda_1(\Omega))} \leq C, \end{aligned} \quad (4.19)$$

for all  $\rho \in \mathbb{R}$ . Here  $\lambda_1(\Omega)$  is the first eigenvalue of the Dirichlet-Laplacian. Hence, by Theorem 4.2 the assertion follows.  $\square$

*Remark 4.5.* In Figure 3, we show  $b(\rho)$  in case  $\lambda = 3$ ,  $\mu = 4$ , and  $c^2 = 1$ .

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