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Existence and stability of almost periodic solutions in impulsive neural network models ☆

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ABSTRACT

The existence and global exponential stability of an almost periodic solution of an impulsive neural network model with distributed delays is considered in a matrix setting. The approach transforms the original network into a matrix analysis problem, where a set of sufficient conditions based on spectral radius is presented. A concrete Hopfield model shows the advantages in comparison with a classical norm approach.

1. Introduction

In this article, we will study the existence and stability of an almost periodic solution of an impulsive neural network of n-units and distributed delays, that is described by the system of n-differential equations proposed by Stamov in [1]:

$$\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} \{b_{ij}(t)f_j(x_j(t)) + c_{ij}(t)g_j((k_{ij} * x_j)(t))\} + \gamma_i(t) \quad (1.1)$$

with $i = 1,\ldots,n$ and $t \neq \tau_k$ $(k \in \mathbb{Z})$. The sequence $\{\tau_k\}$ is strictly increasing and unbounded at $\pm\infty$. Moreover, the state x_i experiments at $t = \tau_k$ a jump to $x_i(\tau_k^+) = \lim_{\delta \to 0^+} x_i(\tau_k + \delta)$ that is defined by:

$$\Delta x_i(\tau_k) = x_i(\tau_k^+) - x_i(\tau_k) = \sum_{j=0}^{n} A_{ij}(k)x_j(\tau_k) + I_j(x(\tau_k)) + \mu_i(k). \quad (1.2)$$

The scalars $x_i(t)$ and $x_i(\tau_k)$ denote the state of the ith unit at times $t \neq \tau_k$ and $t = \tau_k$ respectively. Consequently, (1.1) describes the continuous behavior of the ith unit by considering three features:

(a) The leakage rate (also called resistive coefficient) of the ith unit, described by the function $a_i : \mathbb{R} \mapsto \mathbb{R}$, which only takes positive values.
(b) The external inputs on the ith unit, described by $\gamma_i : \mathbb{R} \mapsto \mathbb{R}$.
(c) Its eventual interconnections with other j units of the network, where $j \in \{1,\ldots,n\}\backslash\{i\}$. The strength (synaptic efficiency) of the effect of the jth unit on the ith is given by the scalar function $b_{ij}(\cdot) \geqslant 0$. Moreover, the scalar function $f_j(x_j(t))$ denotes the activation of the jth unit at the time t and usually is described by a Heaviside function or a continuous approximation (see, [2]).

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Nevertheless, it has been pointed out in several works (see, e.g. [3]) that the activation of a unit j can be dependent of its state x_j with time delay, either discrete or continuously distributed. In our case, the scalar function $g_j(\cdot)$ denotes an activation dependent of the total delayed action $(k_{ij} * x_j)(t)$ on all the states $x_j(t)(1 \leqslant j \leqslant n)$, where

$$(k_{ij} * x_j)(t) = \int_0^\infty k_{ij}(r)x_j(t-r)dr$$

and $k_{ij} : \mathbb{R} \mapsto \mathbb{R}$ is a scalar integrable function. The weight of this interconnection is denoted by the nonnegative function $c_{ij} : \mathbb{R} \mapsto \mathbb{R}$.

On the other hand, the impulsive behavior of the ith unit is described by the matrix sequence $A : \mathbb{Z} \mapsto \mathbb{R}^{n\times n}$ (from now on $A_k = A_{ij}(k)$), the vector sequence $\mu_k = \mu_i(k)$ and $I(x(\tau_k))$, with $I : \mathbb{R}^n \mapsto \mathbb{R}^n$.

As it has been pointed out in other works, the impulses defined by (1.2) make possible the modeling of disturbances or can be introduced in order to control the behavior of the networks.

It is interesting to point out that in absence of impulses, (1.3) generalizes classical models: indeed, if $c_{ij}(t) \equiv 0$ then (1.3) becomes a Hopfield neural network (see, e.g. [2,4] and [5]). If $c_{ij}(t) \neq 0$ and $k_{ij}(t) = \delta_{ij}(t-\tau)$ (Dirac's delta function) then (1.3) is a delayed Hopfield neural network (see e.g. [6] and references therein).

For any solution $x(t) = \text{col}\,(x_1(t),\ldots,x_n(t))$ of (1.1) and (1.2), the model can be summarized as follows:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)f(x(t)) + C(t)g((K * x)(t)) + \gamma(t), & t \neq \tau_k, \\ \Delta x(\tau_k) = A_k x(\tau_k) + I(x(\tau_k)) + \mu_k, \end{cases} \tag{1.3}$$

where $A(t) = \text{diag}\{-a_i(t)\}$, $B(t) = \{b_{ij}(t)\}$ and $C(t) = \{c_{ij}(t)\}$ are matrix functions $\mathbb{R} \mapsto \mathbb{R}^{n\times n}$. Moreover, the functions $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfies $\frac{\partial f_i}{\partial x_j} = \frac{\partial g_i}{\partial x_j} = 0$ when $i \neq j$ and A_k satisfies

$$\det(E + A_k) \neq 0 \quad \text{for any} \quad k \in \mathbb{Z}, \quad (E \text{ is the identity matrix}). \tag{1.4}$$

The initial conditions of (1.3) are in $PC = PC(\mathbb{R}_-, \mathbb{R})$, the Banach space of bounded piecewise continuous functions $\varphi : (-\infty, 0] \mapsto \mathbb{R}^n$ with first kind discontinuities at $t = \tau_k$. Throughout this paper, we will assume the existence and uniqueness of a solution $x(t,\varphi)$ of (1.3) with initial condition $\varphi \in PC(\mathbb{R}_-, \mathbb{R}^n)$.

The impulsive linear system associated to (1.3) is

$$\begin{cases} \dot{x}(t) = A(t)x(t) & t \neq \tau_k, \\ \Delta x(\tau_k) = A_k x(\tau_k). \end{cases} \tag{1.5}$$

Let $U_k(t,s)$ ($k \in \mathbb{Z}, t, s \in (\tau_{k-1}, \tau_k] = J_k$) be the Cauchy matrix of (1.5) on J_k. Moreover, (1.4) implies that the solution of (1.5) with initial conditions $x(t_0) = \lim_{t\to t_0^+} x(t) = x_0$ is given by $x(t,t_0,x_0) = X(t,t_0)x_0$ where $X(t,s)$ is defined by:

$$X(t,s) = \begin{cases} U_k(t,s), & t,s \in J_k, \\ U_{k+1}(t,\tau_k^+)\Lambda_k U_k(\tau_k, s), & s \in J_{k-1}, \quad t \in J_k, \\ U_k(t,\tau_k)\Lambda_k^{-1} U_{k+1}(\tau^+, s), & t \in J_{k-1}, \quad s \in J_k, \\ U_{k+1}(t,\tau_k^+)\prod_{j=k}^{i+1} \Lambda_j U_j(\tau_j, \tau_{j-1}^+)\Lambda_i U_i(\tau_i, s), & s \in J_i, \quad t \in J_{k+1}, \quad \tau_i < \tau_k, \\ U_i(t,\tau_i)\prod_{j=i}^{k-1} \Lambda_j^{-1} U_{j+1}(\tau_j^+, \tau_{j+1})\Lambda_k^{-1} U_{k+1}(\tau_k^+, s), & t \in J_i, \quad s \in J_{k+1}, \quad \tau_i < \tau_k, \end{cases}$$

where $\Lambda_i = (E + A_i)$ for any $i \in \mathbb{Z}$.

1.1. Preliminaries

First, we settle some matters of terminology and notation:

Definition 1. Given $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and the bounded vector function $f = (f_1, f_2, \ldots, f_n) : \mathbb{R} \to \mathbb{R}^n$ with $|f_i|_\infty = \sup_{t\in\mathbb{R}}\{|f_i(t)|\}$, we define the vectors

$$|x| = (|x_1|,\ldots,|x_n|) \in \mathbb{R}_+^n \text{ and } |f|_\infty = (|f_1|_\infty, |f_2|_\infty, \ldots, |f_n|_\infty) \in \mathbb{R}_+^n. \tag{1.6}$$

For $x, y \in \mathbb{R}^n$, $x = \text{col}\ (x_1,\ldots,x_n)$, $y = \text{col}\ (y_1,\ldots,y_n)$ real vectors, the inequality $x \leqslant y$ means $x_i \leqslant y_i$ for every $i \in \{1,\ldots,n\} \subset \mathbb{N}$, i.e. $y - x \in \mathbb{R}_+^n$. Similarly, for a $n \times n$ matrix $A = (a_{ij})$, define $|A| = (|a_{ij}|)_{ij}$ and $|A|_\infty = (|a_{ij}|_\infty)_{ij}$. Also, $A \geqslant 0$ if $a_{ij} \geqslant 0$ for all i, j. Moreover, for $\|\cdot\|_\bullet$ any norm on the vector space of $p \times p$ matrices we will denote by $\rho(A) = \lim_{n\to\infty} \|A^n\|_\bullet^{1/n}$, the spectral radius of A. The spectral radius is independent of the norm $\|\cdot\|_\bullet$ and $\rho(A) \leqslant \|A\|_\bullet$.

Positive matrices and spectral radius have been used in several systems related to (1.3) in impulsive and nonimpulsive contexts, (see, [7–15]), this method have also been employed to studying Lotka–Volterra systems in [16]. Componentwise techniques have been used in differential and difference equations by [17,18].

1.2. Relation with other models

There exists an impressive and extended literature devoted to the mathematical modeling of delayed neural networks. In this context, some models related to (1.3) and its results are summarized in the following tables (the first contains non impulsive models and the second contains the impulsive ones) and references therein:

Solution	Asympt. behavior	Delay	References
Equilibrium point	Unif. asymp. stab.	Time-varying	[19]
Periodic	Global exp. stab.	Discrete	[8,20]
Periodic	Global exp. stab.	Time-varying	[11]
Almost periodic	Global exp. stab.	Time-varying	[21]
Almost periodic	Global attr.	Distributed	[22–24]
Almost periodic	Global exp. stab.	Distributed	[10,17]
Arbitrary	Dissipativity	Distributed	[7,9]

Solution	Asympt. behavior	Delay	References
Equilibrium point	Asympt. stab.	Distributed	[25]
Equilibrium point	Global exp. stab.	Distributed	[12]
Equilibrium point	Global exp. stab.	Time-varying	[15,26]
Equilibrium point	Global exp. stab.	Discrete	[13,14]
Almost periodic	Global exp. stab.	Discrete	[27]
Almost periodic	Global exp. stab.	Distributed	[1]
Almost periodic	Global exp. stab.	Time-varying	[28]

The existence results are obtained by using Banach fixed point results and the exponential stability results are obtained along two main directions. On one hand, there is the Lyapunov–Riccati approach developed in [12,17]. On the other hand, one can use the Gronwall–Bellman inequality [1,28].

1.3. Novelty of this work and outline

The main contribution of this paper lies in the simplicity in which the existence and exponential stability of an almost periodic solution is addressed: (i) The stability definitions and estimations are vectorial. (ii) Our estimations are easy to check and the constants are independent of the initial conditions. (iii) The existence and stability conditions are given in terms of spectral radius of explicit matrices and are better than conditions obtained by using classic norms (see, Section 4). (iv) Some byproducts of our methods are the uniform stability of the almost periodic solution, the point dissipativity of the system and continuity with respect to initial conditions. These last topics are studied in the context of vectorial inequalities and are interesting by its own right by noticing the few number of related results as it is shown in the tables above.

This paper is organized as follows: in Section 2, we present a sufficient condition ensuring the existence and uniqueness of an almost periodic solution. Moreover, we prove that this solution is uniformly stable and that the system is point dissipative. In Section 3 we present a sufficient condition ensuring global exponential stability for the solution described above. An illustrative example is shown in Section 4.

2. Existence of almost periodic solutions

2.1. Definitions

As the solutions of impulsive differential systems are piecewise continuous (see, e.g. [29]), the classical definition of Bohr almost periodicity (see, e.g. [30]) has been extended to an impulsive framework.

Definition 2 [29, p.183]. Let $\{\xi_k\}$ with $k \in \mathbb{Z}$ a sequence in $\mathbb{R}^n$, an integer ℓ is called an ε-almost period of $\{\xi_k\}$ if

$$\|\xi_{k+\ell} - \xi_k\| < \varepsilon \tag{2.1}$$

for any $k \in \mathbb{Z}$ (here $\|\cdot\|$ is a norm in $\mathbb{R}^n$). Hence, $\{\xi_k\}$ is almost periodic if for any $\varepsilon > 0$ there exists a relatively dense set of its ε–almost periods, i.e. there exists $N(\varepsilon) \in \mathbb{N}$ such that there is at least one number $\ell \in [k, k+N] \cap \mathbb{Z}$ satisfying (2.1).

Definition 3 [29, p.195]. Given a sequence $\{\xi_k\}$ we define a family of sequences by $\xi_k^j = \xi_{k+j} - \xi_k$ $(j \in \mathbb{Z})$. We say that the family $\{\xi_k^j\}$ is equipotentially almost periodic if for an arbitrary $\varepsilon > 0$ there exists a relatively dense set of ε-almost periods that are common to all the sequences $\{\xi_k^j\}$.

Definition 4 [29, p.201]. A piecewise continuous function $\varphi : \mathbb{R} \mapsto \mathbb{R}^n$, with first kind discontinuity points defined by an unbounded and strictly increasing sequence $\{\xi_k\}_{k\in\mathbb{Z}}$ is almost periodic (i.e. $\mathcal{AP}(\mathbb{R})$) if:

(i) The set of sequences $\left\{\xi_k^j\right\}$, $\xi_k^j = \xi_{k+j} - \xi_k$, $(j = 0, \pm 1, \pm 2, \ldots)$ are equipotentially almost periodic.
(ii) For any $\varepsilon > 0$, there exists a positive number $\delta(\varepsilon) > 0$ such that if the points t' and t'' belong to the same interval of continuity and $|t' - t''| < \delta$, then $\|\varphi(t') - \varphi(t'')\| < \varepsilon$.
(iii) For any $\varepsilon > 0$ there exists a relatively dense set Γ of ε-almost periods such that if $\tau \in \Gamma$, then $\|\varphi(t+\tau) - \varphi(t)\| < \varepsilon$ for all $t \in \mathbb{R}$ which satisfy the condition $|t - \xi_i| > \varepsilon$ $(i = 0, \pm 1, \pm 2, \ldots)$.

In addition, we introduce some definitions tailored for the componentwise setting presented in the sub Section 1.1:

Definition 5. The system (1.3) is point dissipative if there exists a positive vector $r > 0$ such that for any initial condition $\psi \in PC$ satisfying $\sup_{s\in(-\infty,0]}|\psi(s)| < r$, there exists a positive vector $v > 0$ satisfying $\sup_{t\in[0,+\infty)}|x(t,\psi)| < v$.

Definition 6. The solution $x(t,\varphi^*)$ of (1.3) is uniformly stable if given a vector $\varepsilon > 0$, there exist a vector $\delta(\varepsilon) > 0$ such that for any solution $x(t,\psi)$ with initial condition $\psi \in PC(\mathbb{R}_-, \mathbb{R}^n)$ satisfying $0 < \sup_{r\,\in(-\infty,0]}|\psi(r) - \varphi^*(r)| < \delta$ implies

$$\sup_{t\in[0,+\infty)} |x(t,\varphi^*) - x(t,\psi)| < \varepsilon.$$

Definition 7. The solution $x(t,\varphi^*)$ of (1.3) is globally exponentially stable if for any solution $x(t,\psi)$ with initial condition $\psi \in PC(\mathbb{R}_-, \mathbb{R}^n)$, there exist a scalar $\alpha > 0$ and a matrix $\mathcal{C} > E$ satisfying

$$|x(t,\varphi^*) - x(t,\psi)| < \mathcal{C}|\varphi^* - \psi|e^{-\alpha t} \quad t \geqslant 0.$$

2.2. Assumptions

We assume that the following assumptions on (1.3) hold:

(AP1) The matrices functions $A(t)$, $B(t)$, $C(t)$ are Bohr almost periodic.
(AP2) The vectorial function $\gamma(\cdot)$ is Bohr almost periodic.
(AP3) A_k and γ_k are almost periodic.
(AP4) The sequence of impulses $\{\tau_k\}_k$ is of type

$$\tau_k = kT + \alpha_k, \quad T \in \mathbb{R}, \tag{2.2}$$

where α_k is an almost periodic sequence satisfying $2\ sup_{k\in\mathbb{Z}}|\alpha_k| = 2\alpha < T$.

(L1) The functions f and g satisfy the inequalities

$$|f(u) - f(v)| \leqslant \mathcal{L}^f|u - v| \quad \text{and} \quad |g(u) - g(v)| \leqslant \mathcal{L}^g|u - v|,$$

where $\mathcal{L}^f = diag\left(\mathcal{L}_1^f, \ldots, \mathcal{L}_n^f\right) > 0$ and $\mathcal{L}^g = diag(\mathcal{L}_1^g, \ldots, \mathcal{L}_n^g) > 0$.

(L2) I satisfies the vectorial Lipschitz condition

$$|I(x) - I(y)| \leqslant \mathcal{L}^I|x - y|, \quad \text{with} \quad \mathcal{L}^I = diag(\mathcal{L}_1^I, \ldots, \mathcal{L}_n^I) > 0.$$

(S1) There exist $\lambda > 0$ and a nonnegative matrix $M \geqslant E$ such that the matrix $X(t,s)$ satisfies the inequality

$$|X(t,s)| \leqslant Me^{-\lambda(t-s)} \quad \text{with} \quad t \geqslant s.$$

(K1) $\int_0^\infty |k_{ij}(s)|\,ds = \tilde{k}_{ij}$ for $i, j = 1, \ldots, n$.

(S1) is a stability condition of (1.5) and can be viewed as a componentwise version of a stability condition considered in [1,27,29]. On the other hand, it has been proved in [31, Cor.5] that every sequence arising an equipotentially almost periodic family can be represented by (2.2). In addition $T > 2\alpha$ implies

$$\inf_{k\in\mathbb{Z}} \tau_k^1 = \inf_{k\in\mathbb{Z}} \tau_{k+1} - \tau_k \geqslant T - 2\alpha = \theta > 0. \tag{2.3}$$

Finally, other consequence of **(AP4)** is the following result (see e.g. [29, Ch.4]):

Lemma 1. *Let $i(s,t)$ be the number of terms of the sequence $\{\tau_k\}$ in the interval (s,t). If $t-s > \theta = T-2\alpha$ and* **(AP4)** *holds, then it follows that*

$$\frac{i(s,t)}{t-s} < \frac{2}{\theta} = \frac{2}{T-2\alpha}. \tag{2.4}$$

2.3. Main result

In order to shorten the statement of the Theorem, we will define the following matrices:

$$\widehat{A} = \{\widehat{a}_{ij}\} \quad \text{with} \quad \widehat{a}_{ij} = \sup_{t\in\mathbb{R}} |a_{ij}(t)| \quad \text{and} \quad \widetilde{K} = \{\tilde{k}_{ij}\},$$

the constant matrices $\widehat{B}$ and $\widehat{C}$ are defined in a similar way. Now, we can define

$$D = |\widehat{B}|\mathcal{L}^f + |\widehat{C}|\mathcal{L}^g\widetilde{K} \tag{2.5}$$

and by using **(S1)**, we can prove that the matrices

$$\Omega_{D,\mathcal{L}^I}(t) = \int_{-\infty}^{t} |X(t,s)|D\,ds + \sum_{\tau_k<t} |X(t,\tau_k)|\mathcal{L}^I \tag{2.6}$$

and

$$\Lambda = \sup_{t\in\mathbb{R}} \Omega_{D,\mathcal{L}^I}(t) \tag{2.7}$$

are well defined.

Theorem 1. *If* **(AP1)–(AP4)**, **(L1)–(L2)**, **(S1)**, **(K1)** *are satisfied and $\rho(\Lambda) < 1$, then (1.3) has one solution $x^*(\cdot) \in \mathcal{AP}(\mathbb{R})$.*

Proof. Let $\phi(t) \in \mathcal{AP}(\mathbb{R})$ an arbitrary almost periodic function with first kind discontinuities defined by the sequence $\{\tau_k\}$. We build the auxiliary system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)f(\phi(t)) + C(t)g((K*\phi)(t)) + \gamma(t), & t \neq \tau_k, \\ \triangle x(\tau_k) = A_k x(\tau_k) + I(\phi(\tau_k)) + \mu_k, & t = \tau_k. \end{cases} \tag{2.8}$$

As $\phi(t) \in \mathcal{AP}(\mathbb{R}^n)$, it is easy to prove that $(K*\phi) \in \mathcal{AP}(\mathbb{R}^n)$. In addition, by **(L1)** we have that f and g are uniformly continuous and Theorem 77 from [29] implies that $f(\phi(t)),\ g(\phi(t)) \in \mathcal{AP}(\mathbb{R})$ and

$$\Gamma_\phi(t) = B(t)f(\phi(t)) + C(t)g((K*\phi)(t)) + \gamma(t) \in \mathcal{AP}(\mathbb{R}).$$

Moreover, (2.3) combined with Lemma 37 from [29] imply that $\phi(\tau_k)$ is an almost periodic sequence. Now, by using **(S1)** and Theorem 81 from [29], it follows that

$$(\mathcal{S}\phi)(t) = \int_{-\infty}^{t} X(t,s)\Gamma_\phi(s)\,ds + \sum_{\tau_k<t} X(t,\tau_k)\big[I(\phi(\tau_k)) + \mu_k\big]$$

is an almost periodic solution of (2.8). Hence, we have defined an application $\mathcal{S} : \mathcal{AP}(\mathbb{R}) \mapsto \mathcal{AP}(\mathbb{R})$ and the Theorem follows if we prove that $\mathcal{S}$ has a fixed point.

Let ψ, φ two functions in $\mathcal{AP}(\mathbb{R})$. Notice that

$$(\mathcal{S}\psi)(t) - (\mathcal{S}\varphi)(t) = H_1(t,\psi,\varphi) + H_2(t,\psi,\varphi), \tag{2.9}$$

where $H_i(t,\psi,\varphi)$ $(i = 1,2)$ are defined by

$$H_1(t,\psi,\varphi) = \int_{-\infty}^{t} X(t,s)\big[\Gamma_\psi(s) - \Gamma_\varphi(s)\big]\,ds \quad \text{and}$$
$$H_2(t,\psi,\varphi) = \sum_{\tau_k<t} X(t,\tau_k)[I(\psi(\tau_k)) - I(\varphi(\tau_k))].$$

Now, $|\cdot|$ will denote either the module of a vector or the module of a matrix (see Sub Section 1.1). By using **(L1)** and **(K1)**, we can deduce

$$|\Gamma_\psi(s) - \Gamma_\varphi(s)| \leqslant |B(s)|\mathcal{L}^f|\psi(s) - \varphi(s)| + |C(s)|\mathcal{L}^g|(K*\psi)(s) - (K*\varphi)(s)|.$$
$$\leqslant |B(s)|\mathcal{L}^f|\psi(s) - \varphi(s)| + |C(s)|\mathcal{L}^g\widetilde{K} \sup_{r\in(-\infty,s]} |\psi(r) - \varphi(r)|.$$

Hence, it follows that

$$|H_1(t,\psi,\varphi)| \leqslant \left\{\int_{-\infty}^{t} |X(t,s)|\Big[|\widehat{B}|\mathcal{L}^f + |\widehat{C}|\mathcal{L}^g\widetilde{K}\Big]\,ds\right\} \sup_{t\in\mathbb{R}} |\psi(t) - \varphi(t)|.$$

On the other hand, it can be proved that:

$$|H_2(t,\psi,\varphi)| \leqslant \left\{\sum_{\tau_k<t}|X(t,\tau_k)|\mathcal{L}^I\right\}\sup_{t\in\mathbb{R}}|\psi(t)-\varphi(t)|.$$

Finally, (2.9) combined with the estimations for $H_i(t,\cdot,\cdot)$ imply

$$|\mathcal{S}\psi-\mathcal{S}\varphi|_\infty \leqslant \Lambda|\psi-\varphi|_\infty$$

and the existence of a unique fixed point follows by the contraction principle of Banach [32, Th. 9.1.2.] because $\rho(\Lambda)<1$.

Remark 1. This result extends and simplifies to an impulsive framework a similar existence and uniqueness result obtained in [17, Th.1]. On the other hand, by following the lines of this proof, we could made an alternative result by using vector norms and induced matrix norms, obtaining an inequality of type

$$\|\mathcal{S}\psi-\mathcal{S}\varphi\|_\star \leqslant \|\Lambda\|_\bullet\|\psi-\varphi\|_\star,$$

where $\|\cdot\|_\star$ and $\|\cdot\|_\bullet$ are respectively a vector norm and its induced matrix norm.

Nevertheless, we point out that $\rho(\Lambda)\leqslant\|\Lambda\|_\bullet$ implies that we can obtain better conditions with a componentwise approach. An example is shown in Section 4.

Corollary 1. *If the constants α, T and λ defined in* **(AP4)** *and* **(S1)** *are such that*

$$\rho(\widetilde{\Lambda})<1, \quad \text{with} \quad \widetilde{\Lambda}=\frac{1}{\lambda}MD+\frac{e^{2\lambda\alpha}}{1-e^{-\lambda T}}M\mathcal{L}^I \tag{2.10}$$

then, the system (1.3) has one solution $x^*(\cdot)\in\mathcal{AP}(\mathbb{R})$.

Proof. By **(S1)**, it follows that for $t\in(\tau_k,\tau_{k+1})$:

$$\Omega_{D,\mathcal{L}^I}(t)\leqslant\int_{-\infty}^t e^{-\lambda(t-s)}ds MD+\sum_{j=-\infty}^{k}e^{-\lambda(t-\tau_j)}M\mathcal{L}^I,$$

now by using $t\geqslant\tau_k$ combined with **(AP4)**, implies $t-\tau_{k-j}\geqslant jT-2\alpha$ for $j\geqslant 0$ and we can conclude that $\Omega_{D,\mathcal{L}^I}(t)\leqslant\widetilde{\Lambda}$. Hence, it follows that $\Lambda\leqslant\widetilde{\Lambda}$ and the Corollary is consequence of Theorem 1.

We can relax the assumption **(S1)** by.

(**S**1′) There exist $\lambda_i>0$ $(i=1,\ldots,n)$ and a nonnegative matrix $M\geqslant E$ such that the matrix $X(t,s)$ satisfies the inequality

$$|X(t,s)|\leqslant \mathrm{diag}(e^{-\lambda_1(t-s)},\ldots,e^{-\lambda_n(t-s)})M, \quad t\geqslant s$$

and prove the following result:

Corollary 2. *If (**S**1′) is satisfied and the constants α, T and λ_i defined in* **(AP4)** *and (**S**1′) are such that $\rho(\mathcal{D}_1MD+\mathcal{D}_2M\mathcal{L}^I)<1$ with*

$$\mathcal{D}_1=\mathrm{diag}\left\{\frac{1}{\lambda_1},\ldots,\frac{1}{\lambda_n}\right\} \quad \text{and} \quad \mathcal{D}_2=\mathrm{diag}\left\{\frac{e^{2\lambda_i\alpha}}{1-e^{-\lambda_i T}}\right\}_{i=1}^n$$

then, the system (1.3) has one solution $x^*(\cdot)\in\mathcal{AP}(\mathbb{R})$.

2.4. Consequences

The following result will show some stability properties of system (1.3) and its unique almost periodic solution $x^*(\cdot)$.

Lemma 2. *If* **(AP1)–(AP4)**, **(L1)**–**(L2)**, **(S1)** *and* **(K1)** *are satisfied and $\rho(\Lambda)<1$, then:*

(i) *Given two initial conditions φ, $\psi\in PC$ of (1.3), it follows that its solutions $x(\cdot,\varphi)$ and $x(\cdot,\psi)$ satisfy the inequality:*

$$|x(t,\psi)-x(t,\varphi)|\leqslant(E-\Lambda)^{-1}M|\psi-\varphi|_\infty, \quad t\geqslant 0. \tag{2.11}$$

(ii) *For any initial condition $\varphi\in PC$, it follows that*

$$|x(t,\varphi)|\leqslant(E-\Lambda)^{-1}M\left[|\varphi|_\infty+\frac{|\gamma|_\infty}{\lambda}+\frac{e^{2\lambda\alpha}|\mu_k|_\infty}{1-e^{-\lambda T}}\right], \quad t\geqslant 0. \tag{2.12}$$

Proof. Let $y(t) = x(t,\varphi) - x(t,\psi)$ and define $Q(t,y(t))$ as follows

$$Q(t,y(t)) = B(t)\tilde{f}(y(t)) + C(t)\tilde{g}(y(t)), \tag{2.13}$$

where $\tilde{f}(y(t)) = f(y(t) + x(t,\psi)) - f(x(t,\psi))$ and $\tilde{g}(y(t)) = g((K * [y + x(\cdot,\psi)])(t)) - g((K * x(\cdot,\psi))(t))$. Hence, by (1.3) it follows that $y(\cdot)$ satisfies

$$\begin{cases} \dot{y}(t) = A(t)y(t) + Q(t,y(t)), & t \neq \tau_k \\ \triangle y(\tau_k) = A_k y(\tau_k) + h(\tau_k, y(\tau_k)). \end{cases}$$

with $h : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ defined by $h(k,y(\tau_k)) = I(y(\tau_k) + x(\tau_k,\psi)) - I(x(\tau_k,\psi))$.

Now, it can be proved that

$$y(t) = X(t,0^+)y(0) + \int_0^t X(t,s)Q(s,y(s))ds + \sum_{\tau_k<t} X(t,\tau_k)h(k,y(\tau_k)). \tag{2.14}$$

In addition, **(L1)** and **(K1)** imply that

$$|Q(s,y(s))| \leqslant \widehat{B}\mathcal{L}^f|y(s)| + \widehat{C}\mathcal{L}^g\widetilde{K} \sup_{r\in(-\infty,s]} |y(r)| \leqslant D \sup_{r\in(-\infty,s]} |y(r)|$$

and **(L2)** implies the estimation

$$|X(t,\tau_k)||h(k,y(\tau_k))| \leqslant |X(t,\tau_k)|\mathcal{L}^I|y(\tau_k)|.$$

Now, by (2.14) and **(S1)**, for any $t \in \mathbb{R}_+$ we can deduce that

$$|y(t)| \leqslant M \sup_{r\in(-\infty,0]} |\varphi(r) - \psi(r)| + \Lambda \sup_{r\in(-\infty,t]} |y(r)|. \tag{2.15}$$

It is straightforward to verify that (2.15) for $s \in [0,t)$ implies

$$|y(s)| \leqslant M \sup_{r\in(-\infty,0]} |\varphi(r) - \psi(r)| + \Lambda \sup_{r\in(-\infty,t]} |y(r)|.$$

and we conclude that

$$\sup_{r\in(-\infty,t]} |y(r)| \leqslant M \sup_{r\in(-\infty,0]} |\varphi(r) - \psi(r)| + \Lambda \sup_{r\in(-\infty,t]} |y(r)|.$$

As $\rho(\Lambda) < 1$, a well known result of nonnegative matrices (see e.g. [32]) says that $(E - \Lambda)$ is invertible and its inverse is positive. Hence, we can deduce

$$\sup_{r\in(-\infty,t]} |y(r)| \leqslant (E - \Lambda)^{-1}M|\varphi - \psi|_\infty.$$

and the statement (i) follows.

Now, by choosing $\psi \equiv 0$ and inserting in (2.11), we obtain:

$$\sup_{r\in(-\infty,t]} |x(r,\varphi)| \leqslant (E - \Lambda)^{-1}M|\varphi|_\infty + \sup_{r\in[0,t)} |x(r,0)|,$$

and (2.12) follows if we prove that the solution $x(t,0)$ satisfies

$$\sup_{r\in[0,t]} |x(r,0)| \leqslant (E - \Lambda)^{-1}M\left\{\frac{|\gamma|_\infty}{\lambda} + \frac{e^{2\lambda\alpha}|\mu_k|_\infty}{1 - e^{-\lambda T}}\right\}. \tag{2.16}$$

Indeed, it is clear that $x(t,0) = x(t)$ satisfies the integral equation

$$x(t) = \int_0^t X(t,s)\{B(s)f(x(s)) + C(s)g((K * x)(s))\}\, ds + \sum_{\tau_k<t} X(t,\tau_k)I(x(\tau_k)) + \int_0^t X(t,s)\gamma(s)\, ds + \sum_{\tau_k<t} X(t,\tau_k)\mu_k.$$

By following the proof of Corollary 1, we can verify that for any $t > 0$ we have

$$|x(t,0)| \leqslant \Lambda \sup_{u\in[0,t]} |x(u,0)| + \int_0^t |X(t,s)||\gamma(s)|ds + \sum_{\tau_k<t} |X(t,\tau_k)||\mu_k| \leqslant \Lambda \sup_{u\in[0,t]} |x(u,0)| + M\left\{\frac{|\gamma|_\infty}{\lambda} + \frac{e^{2\lambda\alpha}|\mu_k|_\infty}{1 - e^{\lambda T}}\right\}.$$

Finally, (2.16) can be deduced by following the same ideas developed for (2.15).

Some consequences of Lemma 2 are the dependence of solutions of (1.3) with respect to initial conditions and the point dissipativeness of the solutions stated by (2.11) and (2.12). Moreover, another consequence is given by the following result:

Corollary 3. *Under the assumptions of* Theorem 1, *it follows that the almost periodic solution* $x^*(\cdot)$ *of* (1.3) *is uniformly stable.*

Proof. Let $\varepsilon > 0$ be an arbitrary positive vector. By positiveness of $(E-\Lambda)^{-1}M$, it follows that the set $\left\{\delta \in \mathbb{R}^n_+ : (E-\Lambda)^{-1}M\delta \leqslant \varepsilon\right\}$ is non empty. We choose a vector $\delta > 0$ and by using (2.11), we can verify that for any initial condition $\varphi \in PC$ such that $\sup_{r\in(-\infty,0]}|\varphi(r) - x^*(r)| < \delta$ it follows that $\sup_{t\in[0,+\infty)}|x(t,\varphi) - x^*(t)| < \varepsilon$ and the uniform stability of $x^*(t)$ follows.

3. Exponential stability of the almost periodic solution

From now on, we will assume that the following property is satisfied

(K2) There exists a constant $\lambda_\circ \in (0,\lambda)$ such that $K(\cdot)$ satisfies

$$\int_0^\infty |k_{ij}(s)|e^{\lambda_\circ s}\, ds < +\infty.$$

Let us define $X_\sigma(t,s) = e^{\sigma(t-s)}X(t,s)$ with $t \geqslant s$, $\sigma \in (0,\lambda_\circ)$ and the matrices:

$$\widetilde{D} = |\widehat{B}|\mathcal{L}^f + |\widehat{C}|\mathcal{L}^g \int_0^\infty K_\sigma(r)\, dr \quad \text{with} \quad K_\sigma(r) = K(r)e^{\sigma r}. \tag{3.1}$$

By using **(S1)** and the fact that $\sigma \in (0,\lambda_\circ)$, we define the matrices

$$\Omega_{\widetilde{D},\mathcal{L}^I,\lambda-\sigma}(t) = \int_{-\infty}^t |X_\sigma(t,s)|\widetilde{D}\, ds + \sum_{\tau_k<t}|X_\sigma(t,\tau_k)|\mathcal{L}^I, \tag{3.2}$$

$$\Lambda_{\lambda-\sigma} = \sup_{t\in\mathbb{R}} \Omega_{\widetilde{D},\mathcal{L}^I,\lambda-\sigma}(t). \tag{3.3}$$

Theorem 2. *If the assumptions of Theorem 1 and* (**K2**) *are satisfied and for $\sigma \in (0,\lambda_\circ)$ we have $\rho(\Lambda_{\lambda-\sigma}) < 1$, then the unique $\mathcal{AP}(\mathbb{R})$ solution $x^*(t)$ of (1.3) is globally σ–exponentially stable on $[0,+\infty)$. Namely, given any initial condition $\varphi \in PC$, it follows:*

$$|x(t,\varphi) - x^*(t)| \leqslant e^{-\sigma t}(E-\Lambda_{\lambda-\sigma})^{-1}M|\varphi - x^*|_\infty \tag{3.4}$$

Proof. Note that $\rho(\Lambda_{\lambda-\sigma}) < 1$ implies $\rho(\Lambda_\lambda) < 1$ and by Theorem 1 there exists a unique solution $x^*(t) \in \mathcal{AP}(\mathbb{R})$. Let $x(t,\varphi)$ be an arbitrary solution of (1.3) and define $y(t) = x(t,\varphi) - x^*(t)$. As in the proof of Lemma 2, $y(\cdot)$ satisfies (2.14). Let $u(t) = y(t)e^{\sigma t}$ and multiply (2.14) by $e^{\sigma t}$, it follows that $u(\cdot)$ satisfies for $t \geqslant 0$:

$$u(t) = X_\sigma(t,0^+)u(0) + \int_0^t e^{\sigma t}X(t,s)Q(s,y(s))ds + \sum_{\tau_k<t} e^{\sigma t}X(t,\tau_k)h(k,y(\tau_k)).$$

By using (2.13), we can prove that

$$\int_0^t e^{\sigma t}|X(t,s)||Q(s,y(s))|ds \leqslant \mathcal{H}_1(t,y) + \mathcal{H}_2(t,y)$$

with $\mathcal{H}_1$ and $\mathcal{H}_2$ defined respectively by

$$\mathcal{H}_1(t,y) = \int_0^t e^{\sigma(t-s)}|X(t,s)||\widehat{B}|\mathcal{L}^f|y(s)e^{\sigma s}|ds \quad \text{and}$$

$$\mathcal{H}_2(t,y) = \int_0^t e^{\sigma(t-s)}|X(t,s)||\widehat{C}|\mathcal{L}^g e^{\sigma s}\left(\int_0^\infty |K(r)|e^{\sigma r}e^{-\sigma r}|y(s-r)|dr\right)ds.$$

Hence

$$\sum_{i=1}^2 \mathcal{H}_i(t,y) \leqslant \int_0^t |X_\sigma(t,s)|\left\{|\widehat{B}|\mathcal{L}^f|u(s)| + |\widehat{C}|\mathcal{L}^g \int_0^\infty |K_\sigma(r)||u(s-r)|dr\right\}ds \leqslant \int_0^t |X_\sigma(t,s)|\widetilde{D}ds \sup_{r\in(-\infty,t]}|u(r)|.$$

Using this fact combined with $\sigma < \lambda_\circ < \lambda$, implies

$$|u(t)| \leqslant |X_\sigma(t,0)||y(0)| + \left(\int_0^t |X_\sigma(t,s)|\widetilde{D}ds + \sum_{\tau_k<t}|X_\sigma(t,\tau_k)|\mathcal{L}^I\right)\sup_{r\in(-\infty,t]}|u(r)| \leqslant M|y(0)| + \Omega_{\widetilde{D},\mathcal{L}^I,\lambda-\sigma}(t)\sup_{r\in[0,t]}|u(r)|$$

$$\leqslant M|y(0)| + \Lambda_{\lambda-\sigma}\sup_{r\in(-\infty,t]}|u(r)|.$$

By following the lines of the proof of Lemma 2 combined with $\rho(\Lambda_{\lambda-\sigma})<1$ and $u(t)=e^{\sigma t}(x(t,\varphi)-x^{*}(t))$, it can be proved that

$$\sup_{r\in[0,+\infty)} e^{\sigma r}|x(r,\varphi)-x^{*}(r)| \leqslant (E-\Lambda_{\lambda-\sigma})^{-1}M|\varphi-x^{*}|_{\infty}. \tag{3.5}$$

and (3.4) follows straightforwardly.

Remark 2. The assumption **(K2)** plays a key role in our proof and it was employed in [10,12,17] and references therein.

Corollary 4. *If the constants* α, T *and* λ *are such that* $\rho(\widetilde{\Lambda}_{\sigma-\lambda})<1$, *with*

$$\widetilde{\Lambda}_{\lambda-\sigma}=\frac{1}{\lambda-\sigma}M\widetilde{D}+\frac{e^{2(\lambda-\sigma)\alpha}}{1-e^{-(\lambda-\sigma)T}}M\mathcal{L}^{I},\quad \sigma\in(0,\lambda_{\circ}),$$

then, for any $\sigma\in(0,\lambda_{\circ})$, *the solution* $x^{*}(\cdot)$ *is* σ*–globally exponentially stable.*

Corollary 5. *If* (**S**1′) *is satisfied and the constants* N, λ_i *and* $\xi_i=\lambda_i-\sigma>0$ *are such that* $\rho(\mathcal{D}_1M\widetilde{D}+\mathcal{D}_2M\mathcal{L}^{I})<1$ *with* $\sigma<\lambda_{\circ}<\min\{\lambda_1,\lambda_2\}$ *and*

$$\mathcal{D}_1=diag\left\{\frac{1}{\xi_1},\ldots,\frac{1}{\xi_n}\right\}\quad\text{and}\quad \mathcal{D}_2=diag\left\{\frac{e^{2\xi_i\alpha}}{1-e^{-\xi_iT}}\right\}_{i=1}^{n},$$

then, for any $\sigma\in(0,\lambda_{\circ})$ *the solution* $x^{*}(\cdot)$ *is* σ*–globally exponentially stable.*

Remark 3. A set of conditions for global exponential stability of $\mathcal{AP}(\mathbb{R})$ solutions are given in [1,27] and [28]. These results employ Lemma 26 from [29], which states the existence of a constant $N\in\mathbb{N}$. Nevertheless, this result does not provide an effective method to estimate it. In this sense, the characterization of equipotentially almost periodic sequences stated in **(AP4)** avoids this problem and contributes to obtain conditions easier to check as Corollaries 4 and 5. See also Theorem 3.

4. An explicit impulsive Hopfield neural network

We will study a concrete Hopfield neural network with distributed delays and impulse effects. This model shows how our results are useful and are better than a treatment with a classical norm: let us consider the 2×2 system (1.3) with

$$A(t)=diag\{-a_i\}\quad\text{and}\quad A_k=diag\left\{\alpha_k^{(i)}\right\},\quad i=1,2. \tag{4.1}$$

where $k\in\mathbb{Z}$, $a_i>0$ and $\alpha^{(i)}=\left\{\alpha_k^{(i)}\right\}$ are almost periodic sequences satisfying

$$-1<\inf_{k\in\mathbb{Z}}\alpha_k^{(i)}\leqslant\sup_{k\in\mathbb{Z}}\alpha_k^{(i)}\quad\text{for}\quad i=1,2. \tag{4.2}$$

We consider $B(\cdot)$, $C(\cdot)$, A_k, μ_k and $\gamma(\cdot)$ such that **(AP1)–(AP3)** are satisfied. Moreover, the sequence of impulses is defined in **(AP4)**

$$\tau_k=kT+\upsilon_k,\quad T>2\sup_{k\in\mathbb{Z}}|\upsilon_k|=2|\upsilon|_{\infty}, \tag{4.3}$$

where υ_k is an almost periodic sequence. Finally, the functions $f(\cdot)$, $I(\cdot)$ and $K(\cdot)$ satisfy **(L1)**, **(L2)** and **(K1)** respectively.

Theorem 3. *If the following inequalities are satisfied:*

$$\lambda_i=a_i-\frac{2\ln|1+\alpha^{(i)}|_{\infty}}{T-2|\upsilon|_{\infty}}>0\quad\text{for}\quad i=1,2, \tag{4.4}$$

$$-1<\prod_{i=1}^{2}\left(\frac{d_{ii}}{\lambda_i}+\mathcal{L}_i^{I}\frac{e^{2\lambda_i|\upsilon|_{\infty}}}{1-e^{-T\lambda_i}}\right)-\frac{d_{12}d_{21}}{\lambda_1\lambda_2}<1, \tag{4.5}$$

$$\frac{\sum_{i=1}^{2}\left(\frac{d_{ii}}{\lambda_i}+\mathcal{L}_i^{I}\frac{e^{2\lambda_i|\upsilon|_{\infty}}}{1-e^{-T\lambda_i}}\right)}{1+\prod_{i=1}^{2}\left(\frac{d_{ii}}{\lambda_i}+\mathcal{L}_i^{I}\frac{e^{2\lambda_i|\upsilon|_{\infty}}}{1-e^{-T\lambda_i}}\right)-\frac{d_{12}d_{21}}{\lambda_1\lambda_2}}<1. \tag{4.6}$$

Then, the Hopfield neural network (1.3) *with* (4.1)–(4.3) *has a unique* $\mathcal{AP}(\mathbb{R})$ *solution* $x^{*}(t)$. *In addition, if*

$$\int_0^{\infty}e^{\sigma_{\circ}s}|k_{ij}(s)|<+\infty,\quad\text{with}\quad \sigma_{\circ}\in(0,\min\{\lambda_1,\lambda_2\}), \tag{4.7}$$

then there exists $\sigma_{\circ}^{+}\in(0,\sigma_{\circ})$ *such that* $x^{*}(\cdot)$ *is globally* σ*–exponentially stable with* $\sigma\in(0,\sigma_{\circ}^{+})$.

Proof. Indeed, let $t \in [\tau_n, \tau_{n+1})$ and verify that

$$\int_{-\infty}^{t} |X(t,s)|ds = \sum_{k=1}^{\infty} \int_{\tau_{n-k}}^{\tau_{n-k+1}} |X_{n-k}(t,s)|Dds + \int_{\tau_n}^{t} |X_n(t,s)|Dds \tag{4.8}$$

and for any $s \in [\tau_{n-k}, \tau_{n-k+1})$, $X_{n-k}(t,s)$ is defined by $e^{A(t-s)}\prod_{j=0}^{k-1}(E + A_{n-j})$ when $k \geqslant 1$ and $X_n(t,s) = e^{A(t-s)}$. In particular, it follows that for $k \geqslant 1$:

$$X_{n-k}(t,s) = \begin{bmatrix} e^{-a_1(t-s)}\prod_{j=0}^{k-1}(1+\alpha_{n-j}^{(1)}) & 0 \\ 0 & e^{-a_2(t-s)}\prod_{j=0}^{k-1}\left(1+\alpha_{n-j}^{(2)}\right) \end{bmatrix}.$$

Hence,

$$|X_{n-k}(t,s)| \leqslant \begin{bmatrix} |1+\alpha^{(1)}|_{\infty}^{k} e^{-a_1(t-s)} & 0 \\ 0 & |1+\alpha^{(2)}|_{\infty}^{k} e^{-a_2(t-s)} \end{bmatrix} \tag{4.9}$$

for any $k \geqslant 0$. Moreover, it is straightforward to prove that $i(s,t) = k$. Hence, (4.9) can be written (for any $k \geqslant 0$) as follows:

$$|X_{n-k}(t,s)| \leqslant diag\left\{ e^{\left\{\frac{i(s,t)}{t-s}\ln|1+\alpha^{(1)}|_{\infty} - a_1\right\}(t-s)}, e^{\left\{\frac{i(s,t)}{t-s}\ln|1+\alpha^{(2)}|_{\infty} - a_2\right\}(t-s)} \right\}$$

By using Eqs. (2.4) and (4.3), it follows that$|X(t,s)| \leqslant diag\{e^{-\lambda_1(t-s)}, e^{-\lambda_2(t-s)}\}$ and we conclude that (**S**1′) is satisfied with $M = E$ and λ_i defined in (4.4). Now, from $X_\sigma(t,s) = e^{\sigma(t-s)}X(t,s)$ with $\sigma \in (0, \sigma_\circ)$, we obtain

$$|X_\sigma(t,s)| \leqslant \begin{bmatrix} e^{-(\lambda_1-\sigma)(t-s)} & 0 \\ 0 & e^{-(\lambda_2-\sigma)(t-s)} \end{bmatrix}.$$

By (4.7), we have that **(K2)** is satisfied and we can define $\bar{k}_{ij}(\sigma) = \int_0^\infty |k_{ij}(s)|e^{\sigma s}ds$. In addition, let us define the square matrix $\widetilde{D}(\sigma) = \{\tilde{d}_{ij}(\sigma)\}$:

$$\widetilde{D}(\sigma) = \begin{bmatrix} \widehat{b}_{11}\mathcal{L}_1^f + \widehat{c}_{11}\mathcal{L}_1^g\bar{k}_{11}(\sigma) + \widehat{c}_{12}\mathcal{L}_2^g\bar{k}_{21}(\sigma) & \widehat{b}_{12}\mathcal{L}_2^f + \widehat{c}_{11}\mathcal{L}_1^g\bar{k}_{12}(\sigma) + \widehat{c}_{12}\mathcal{L}_2^g\bar{k}_{22}(\sigma) \\ \widehat{b}_{21}\mathcal{L}_1^f + \widehat{c}_{21}\mathcal{L}_1^g\bar{k}_{11}(\sigma) + \widehat{c}_{22}\mathcal{L}_2^g\bar{k}_{21}(\sigma) & \widehat{b}_{22}\mathcal{L}_2^f + \widehat{c}_{21}\mathcal{L}_1^g\bar{k}_{12}(\sigma) + \widehat{c}_{22}\mathcal{L}_2^g\bar{k}_{22}(\sigma) \end{bmatrix}.$$

Notice that $\lim_{\sigma\to 0}\widetilde{D}(\sigma) = D$ because $\lim_{\sigma\to 0}\bar{k}_{ij}(\sigma) = \tilde{k}_{ij}$. Now, let $\xi_i(\sigma) = \lambda_i - \sigma$ and $\xi(\sigma) = (\xi_1(\sigma), \xi_2(\sigma))$. Finally, let $\widetilde{\Lambda}_{\xi(\sigma)} = \mathcal{D}_1\widetilde{D}(\sigma) + \mathcal{D}_2\mathcal{L}^I$, where $\mathcal{D}_i$ are defined in Corollary 5. It is straightforward to prove that

$$\widetilde{\Lambda}_{\xi(\sigma)} = \begin{bmatrix} \frac{\tilde{d}_{11}}{\xi_1} + G(\xi_1)\mathcal{L}_1^I & \frac{\tilde{d}_{12}}{\xi_1} \\ \frac{\tilde{d}_{21}}{\xi_2} & \frac{\tilde{d}_{22}}{\xi_2} + G(\xi_2)\mathcal{L}_2^I \end{bmatrix} \quad \text{with} \quad G(\xi_i) = \frac{e^{2\xi_i|\nu|_\infty}}{1 - e^{-T\xi_i}}.$$

The characteristic polynomial of $\widetilde{\Lambda}_{\xi(\sigma)}$ is given by $z^2 - p(\xi)z + q(\xi)$ with

$$p(\xi) = \sum_{i=1}^{2}\left(\frac{\tilde{d}_{ii}}{\xi_i} + G(\xi_i)\mathcal{L}_i^I\right) \quad \text{and} \quad q(\xi) = \prod_{i=1}^{2}\left(\frac{\tilde{d}_{ii}}{\xi_i} + G(\xi_i)\mathcal{L}_i^I\right) - \frac{\tilde{d}_{12}\tilde{d}_{21}}{\xi_1\xi_2}.$$

Now, by using the Schur–Cohn Theorem combined with $p(\xi(\sigma)) > 0$ for any $\sigma \in [0, \sigma_\circ)$, it follows that $\rho(\widetilde{\Lambda}_{\xi(\sigma)}) < 1$ if and only if the inequalities

$$q^2(\xi(\sigma)) < 1 \quad \text{and} \quad p^2(\xi(\sigma)) < (q(\xi(\sigma)) + 1)^2 \tag{4.10}$$

are verified for any $\sigma \in [0, \min\{\lambda_1, \lambda_2\})$.

On the other hand, Eqs. (4.5) and (4.6) imply that (4.10) follows at $\sigma = 0$, which is equivalent to $\rho(\widetilde{\Lambda}_{\xi(0)}) = \rho(\widetilde{\Lambda}_\lambda) < 1$, with $\lambda = (\lambda_1, \lambda_2)$. Hence, Corollary 2 implies that the impulsive Hopfield neural networks has a unique $\mathcal{AP}(\mathbb{R})$ solution.

In addition, by continuity properties, there exists $\delta > 0$ such that (4.10) and $\rho(\widetilde{\Lambda}_{\xi(\sigma)}) < 1$ are still valid for $\sigma \in [0, \delta)$. Finally, the σ-exponential stability follows by any $\sigma \in (0, \sigma_\circ^+)$ with $\sigma_\circ^+ = \min\{\delta, \sigma_\circ\}$.

As we mentioned in Remark 1, we will see that our results give a better result compared with estimations obtained by using vector norms and matrix induced norms. Indeed, let $p_i(\lambda) = d_{ii}\lambda_i^{-1} + G(\lambda_i)\mathcal{L}_i^I$ ($i = 1, 2$) and notice that

$$\|\Lambda_\lambda\|_1 = \max\left\{p_1 + \frac{d_{12}}{\lambda_1}, p_2 + \frac{d_{21}}{\lambda_2}\right\},$$

where $\|\cdot\|_1$ denotes the row matrix norm.

Now, let us consider matrices $diag\{-a_1(t), -a_2(t)\}$ and A_k such that $\lambda_1 = \lambda_2 = 1$. Moreover, we consider 2×2 matrices D and $\mathcal{L}^I$ such that

$$d_{12} = \frac{1}{2}, \quad d_{21} = \frac{1}{4}, \quad p_1 = \frac{3}{4} \quad \text{and} \quad p_2 = \frac{1}{4}.$$

It follows that $p(\lambda) = 1$ and $q(\lambda) = \frac{1}{16}$, which implies that $\rho(\Lambda_\lambda) < 1$. On the other hand, notice that $\rho(\Lambda_\lambda) < 1 < \|\Lambda_\lambda\|_1 = \frac{5}{4}$. In consequence, our result can be applied but a version inspired in vectorial norms and its matrix induced norms cannot be employed in this case.

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