On spaces of Conradian group orderings

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Abstract

We classify C-orderable groups admitting only finitely many C-orderings. We show that if a C-orderable group has infinitely many C-orderings, then it actually has uncountably many C-orderings, and none of these is isolated in the space of C-orderings. As a relevant example, we carefully study the case of Baumslag-Solitar's group $B(1,2)$. We show that $B(1,2)$ has four C-orderings, each of which is bi-invariant, but its space of left-orderings is homeomorphic to the Cantor set.

Introduction

One of the starting points of the theory of orderable groups is [7], where O. Hölder proved that any Archimedean Abelian ordered group is ordered isomorphic to a subgroup of the additive group of real numbers with the standard ordering. In his seminal work [5], P. Conrad obtained a condition on left-ordered groups which is equivalent to the fact that the conclusion of Hölder's theorem holds 'locally' (see (4) below). Since then, these so-called C-orderings (or Conradian orderings) have played a fundamental role in the theory of left-orderable groups. (See for instance [11, 17].) Recall that a left-invariant (total) ordering $\preceq$ on a group G is said to be *Conradian* if the following four equivalent properties hold (this equivalence will be referred to as the Conrad Theorem, see [1, 5, 9]):

(1) For all $f \succ id$ and $g \succ id$ (for all *positive* f, g, for short), we have $fg^n \succ g$ for some $n \in \mathbb{N}$.

(2) If $1 \prec g \prec f$, then $g^{-1}f^n g \succ f$ for some $n \in \mathbb{N} = \{1, 2, \ldots\}$.

(3) For all positive $g \in G$, the set $S_g = \{f \in G \mid f^n \prec g, \text{ for all } n \in \mathbb{Z}\}$ is a convex subgroup. (By definition, a subset $S \subset G$ is *convex* if whenever $f_1 \prec h \prec f_2$ for some f_1, f_2 in S, we have $h \in S$.)

(4) Given $g \in G$, we denote the maximal (resp. minimal) convex subgroup which does not contain (resp. contains) g by G_g (resp. G^g). For every g, we have that G_g is normal in G^g, and there exists a non-decreasing group homomorphism (to be referred to as the *Conrad homomorphism*) $\tau^g_{\preceq} : G^g \to \mathbb{R}$ whose kernel coincides with G_g. Moreover, this homomorphism is unique up to multiplication by a positive real number.

Recently, two new approaches to this property have been proposed by A. Navas. On the one hand, as it was noticed in [8, 13], in (1) and (2) above one may actually take $n = 2$. The topological counterpart of this is the fact that the space of C-orderings is compact when it is endowed with a natural topology (see §1.1). This leads, for instance, to a new and short proof of the fact that locally indicable groups are C-orderable. (Note that the converse follows from (4).) On the other hand, the dynamical characterization of the Conradian property of [13, 15] leads to applications in the study of the topology of space of group orderings, and to general 'level structure' theorems for left-ordered groups. In this work, this dynamical point of view will be crucial.

Following the first direction above, we focus on the structure of the space of group C-orderings. In particular, we provide complete answers to questions in [13, Question 3.9] and [14, §1.3].

It is known that the space of left-orderings of a group is either finite or uncountable [10, 15]. Although this is no longer true for bi-orderings [3], our first main result shows that this dichotomy persists for C-orderings.

Theorem A. *Let G be a C-orderable group. If G admits infinitely many C-orderings, then it has uncountably many C-orderings. Moreover, none of these is isolated in the space of C-orderings.*

For the second claim of Theorem A, the space of group left-orderings is endowed with the projective topology induced from the discrete one on finite sets, and the subset of C-orderings is endowed with the subspace one (see §1.1 for more details). The space of left-orderings is (Hausdorff, totally disconnected and) compact, and the subset of C-orderings is closed therein. In particular, this implies that the second claim of the statement is stronger than the first.

Our second result concerns groups admitting only finitely many C-orderings, and may be considered as an analogue of Tararin's classification of left-orderable groups admitting finitely many left-orderings [9, Theorem 5.2.1]. For the statement, recall that a series

$$\{id\} = G_0 \lhd G_1 \lhd \ldots \lhd G_{n-1} \lhd G_n = G$$

is said to be *rational* if it is subnormal (*i.e.*, each G_i is normal in G_{i+1}) and each quotient G_{i+1}/G_i is torsion-free rank-1 Abelian.

Theorem B. *Let G be a C-orderable group. If G admits finitely many C-orderings, then G admits a unique (hence normal) rational series. In this series, no quotient G_{i+2}/G_i is Abelian. Conversely, if G is a group admitting a normal rational series*

$$\{id\} = G_0 \lhd G_1 \lhd \ldots \lhd G_{n-1} \lhd G_n = G$$

so that no quotient G_{i+2}/G_i is Abelian, then the number of C-orderings on G equals 2^n.

The proof of Theorem B consists in a non-trivial modification of Tararin's arguments. (Note that the statement of Tararin's theorem is the same as that of Theorem B though changing 'C-orderings' by 'left-orderings', and the condition 'G_{i+2}/G_i non Abelian' by 'G_{i+2}/G_i non bi-orderable'.) First, as observed in [14], if a C-orderable group admits finitely many C-orderings, then it must be solvable. Now the fact that each quotient G_{i+1}/G_i has rank 1 and no quotient G_{i+2}/G_i is Abelian is a consequence of the fact that the space of orderings of higher-rank torsion-free Abelian groups are uncountable (see for example [4, 18]). Finally, we use an extra argument involving the Conrad homomorphism to show that, when the hypotheses are fulfilled, there are only finitely many C-orderings.

Let us point out that Baumslag-Solitar's group $B(1,2) = \langle a, b \mid bab^{-1} = a^2 \rangle$ satisfies the conditions of Theorem B. Therefore, its space of C-orderings is finite. Actually, this space consists of four C-orderings, each of which is bi-invariant (see Proposition 4.1). This example was the starting point of this work, and we provide a direct short argument for this particular case in §4.1. We point out, however, that the space of left-orderings of $B(1,2)$ is uncountable. (Actually, it is homeomorphic to the Cantor set.) In addition, in §4.1 we give a complete description of all left-orderings of $B(1,2)$ which extends to many other left-orderable metabelian groups (see Theorem 4.2).

This work corroborates a general principle concerning C-orderings. On the one hand, these are sufficiently rigid in that they allow deducing structure theorems for the underlying group (*e.g.*, local indicability). However, they are still sufficiently malleable in that, starting with a C-ordering on a group, one may create very many C-orderings, which turn out to be different from the original one (see Example 1.1) with the only exception of the pathological cases described in Theorem B.

1 Preliminaries

1.1 Spaces of group orderings

Given a left-orderable group G (of arbitrary cardinality), we denote the set of all left-orderings on G by $\mathcal{LO}(G)$. This set has a natural topology: a basis of neighborhoods of $\preceq$ in $\mathcal{LO}(G)$ is the family of the sets $U_{g_1,\ldots,g_k}$ of all left-orderings $\preceq'$ on G which coincide with $\preceq$ on $\{g_1,\ldots,g_k\}$, where $\{g_1,\ldots,g_k\}$ runs over all finite subsets of G. Another basis is given by the sets $V_{f_1,\ldots,f_k}$ of all left-orderings $\preceq'$ on G such that all the f_i are $\preceq'$-positive, where $\{f_1,\ldots,f_k\}$ runs over all finite subsets of $\preceq$-positive elements of G. Endowed with this topology, $\mathcal{LO}(G)$ is Hausdorff and totally disconnected, and by (an easy application of) the Tychonov Theorem, it is compact (see for instance [13, §2.1]). The (perhaps empty) subspaces $\mathcal{BO}(G)$ and $\mathcal{CO}(G)$ of bi-orderings and C-orderings on G are, respectively, closed inside $\mathcal{LO}(G)$, hence compact.

If G is countable, then this topology is metrizable: given an exhaustion $G_0 \subset G_1 \subset \ldots$ of G by finite sets, for different $\preceq$ and $\preceq'$, we may define $dist(\preceq, \preceq') = 1/2^n$, where n is the first integer such that $\preceq$ and $\preceq'$ do not coincide on G_n. If G is finitely generated, we may take G_n as the ball of radius n with respect to a fixed finite system of generators.

Example 1.1. In the case of Conradian orderings, there is a natural way to generate new C-orderings starting with a given one. This procedure is useful for approximating a given C-ordering if the series of convex subgroups is long enough (see §2). Let $\preceq$ be a C-ordering, and let

$$\{id\} = G^{id} \subset \ldots \subset G_g \lhd G^g \subset \ldots \subset G$$

be the (perhaps infinite) series of $\preceq$-convex subgroups. Taking any $g \in G \setminus \{id\}$, we may obtain a different C-ordering $\preceq_g$ by 'flipping' the ordering on the quotient G^g/G_g. More precisely, given $f \in G$, we define $f \succ_g id$ if one of the following (mutually excluding) conditions holds:

– $f \succ id$ and $f \notin G^g$,
– $f \succ id$ and $f \in G_g$,
– $f \prec id$ and $f \in G^g \setminus G_g$.

Clearly, this is a total ordering. To see that it is left-invariant, we need to check that the product of any two $\preceq_g$-positive elements h_1, h_2 is still $\preceq_g$-positive. This is obvious if $h_1 = h_2$. Now if $1 \prec_g h_1 \prec_g h_2$, then it is easy to check that both $h_1 h_2$ and $h_2 h_1$ belong to $G^{h_2} \setminus G_{h_2}$. Therefore, the $\preceq$-*signs* of $h_1 h_2$ and $h_2 h_1$ are the same as that of h_1, which implies that $h_1 h_2$ and $h_2 h_1$ are $\preceq_g$-positive.

Finally, to see that $\preceq_g$ is Conradian, it suffices to show that $id \prec_g h_1 \preceq_g h_2$ implies $h_1^{-1} h_2 h_1 \succ_g id$ and $h_2^{-1} h_1 h_2^2 \succ_g id$. The first inequality follows from $h_1^{-1} h_2 \succeq_g id$ and $h_1 \succ_g id$ just using the fact that the product of two positive elements is still positive. For the second inequality, note that h_1 and h_2 commute modulo G_{h_2}. Therefore, $h_2^{-1} h_1 h_2^2 G_{h_2} = h_2 G_{h_2}$, which implies that $h_2^{-1} h_1 h_2^2 \succ_g id$.

For applications of the technique of the preceding example to the problem of approximation of group orderings, see [20].

1.2 From ordered representations to Conradian orderings

We begin by recalling an old theorem due to P. Cohn, M. Zaitseva, and P. Conrad (see [9, Theorem 3.4.1]):

Theorem 1.2. *A group G is left-orderable if and only if it embeds in the group of (order-preserving) automorphisms of a totally ordered set.*

Both implications of this theorem are easy. In one direction, note that a left-ordered group acts on itself by order preserving automorphisms, namely left translations. Conversely, to create a left-ordering on a group G of automorphisms of a totally ordered set $(\Omega, \leq)$, we construct the what is called *induced ordering* from the action as follows. Fix a well-order $\leq^*$ on the elements of Ω, and, for every $f \in G$, let $w_f = \min_{\leq^*}\{w \in \Omega \mid f(w) \neq w\}$. Then we define an ordering $\preceq$ on G by letting $f \succ id$ if and only if $f(w_f) > w_f$. It is not hard to check that this order relation is a (total) left-ordering on G.

In what follows, we will need an important definition which was introduced in [15]. Let G be a group acting by order preserving bijections on a totally ordered space $(\Omega, \leq)$. A *crossing* for the action of G on Ω is a 5-uple (f, g, u, v, w) where f, g (resp. u, v, w) belong to G (resp. Ω) and satisfy:

i) $u < w < v$.

ii) For every $n \in \mathbb{N}$, we have $g^n u < v$ and $f^n v > u$.

iii) There exist M, N in $\mathbb{N}$ such that $f^N v < w < g^M u$.

The reason why this definition is so important is because it actually characterizes the C-orderings, as is shown in [15, Theorem 1.4]. We quote the theorem below.

Theorem 1.3. *A left-ordering $\preceq$ on G is Conradian if and only if the action of G by left translations on itself admits no crossing (when taking $(\Omega, \leq) = (G, \preceq)$).*

The following crucial lemma is essentially proved in [13] in the case of countable groups, but the proof therein rests upon very specific issues about the so-called *dynamical realization* of an ordered group. Here we give a general algebraic proof.

Lemma 1.4. *If a faithful action of a group G by automorphisms of an ordered set Ω has no crossing, then any induced ordering on G is Conradian.*

Proof. Suppose that the ordering $\preceq$ on G induced from some well-order $\leq^*$ on Ω is not Conradian. Then there are $\preceq$-positive elements f, g in G such that $fg^n \prec g$, for every $n \in \mathbb{N}$. This easily implies $f \prec g$. Let $\bar{w} = \min_{\leq^*}\{w_f,\, w_g\}$. We claim that $(fg, fg^2, \bar{w}, g(\bar{w}), fg^2(\bar{w}))$ is a crossing (see Figure 1). Indeed, the inequalities $id \prec f \prec g$ imply that $\bar{w} = w_g \leq^* w_f$ and $g(\bar{w}) > \bar{w}$. Moreover $f(\bar{w}) \geq \bar{w}$, which together with $fg^n \prec g$ yield $\bar{w} < fg^2(\bar{w}) < g(\bar{w})$, hence condition $i)$ of the definition of crossing is satisfied. Note that the preceding argument actually shows that $fg^n(\bar{w}) < g(\bar{w})$, for all $n \in \mathbb{N}$. Thus $fg^2fg^2(\bar{w}) < fg^3(\bar{w}) < g(\bar{w})$. A straightforward induction argument shows that $(fg^2)^n(\bar{w}) < g(\bar{w})$, for all $n \in \mathbb{N}$, which proves the first part of condition $ii)$. For the second part, from $g(\bar{w}) > \bar{w}$ and $f(\bar{w}) \geq \bar{w}$ we conclude that $\bar{w} < (fg)^n(g(\bar{w}))$. Condition $iii)$ follows because $\bar{w} < fg^2(\bar{w})$ implies $fg^2(\bar{w}) < fg^2(fg^2(\bar{w})) = (fg^2)^2(\bar{w})$, and $fg^2(\bar{w}) < g(\bar{w})$ implies $(fg)^2(g(\bar{w})) = fg(fg^2(\bar{w})) < fg(g(\bar{w})) = fg^2(\bar{w})$. $\square$

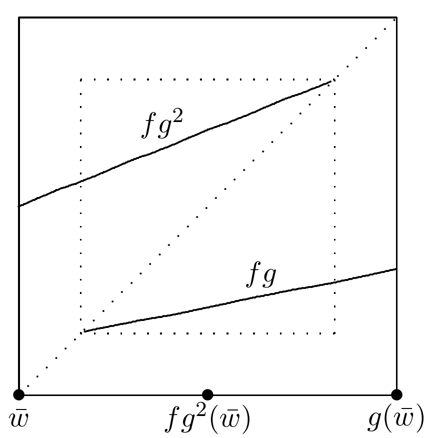

Figure 1: The crossing

Note that if we let w_0 be the first element (w.r.t. $\leq^*$) of Ω, then the stabilizer of w_0 is $\preceq$-convex. Indeed, if $id \prec g \prec f$, with $f(w_0) = w_0$, then $w_0 <^* w_f \leq^* w_g$, and thus $g(w_0) = w_0$. Actually, it is not hard to see that the same argument shows the following.

Proposition 1.5. *Let Ω be a set endowed with a well-order $\leq^*$. If a group G acts faithfully on Ω preserving a total order on it, then there exists a left-ordering on G for which the stabilizer G_{Ω_0} of any initial segment Ω_0 of Ω (w.r.t. $\leq^*$) is convex. Moreover, if the action has no crossing, then this ordering is Conradian.*

Example 1.6. A very useful example of an action without crossings is the action by left translations on the set of left-cosets of any subgroup H which is convex with respect to a C-ordering $\preceq$ on G. Indeed, it is not hard to see that, due to the convexity of H, the order

$\preceq$ induces a total order $\preceq_H$ on the set of left-cosets G/H. Moreover, $\preceq_H$ is $G-$invariant. Now suppose that (f, g, uH, vH, wH) is a crossing for the action. Since $w_1H \prec_H w_2H$ implies $w_1 \prec w_2$, for all w_1, w_2 in G, we have that (f, g, u, v, w) is actually a crossing for the action by left translations of G on itself. Nevertheless, this contradicts Theorem 1.3.

The following is an application of the preceding example. For the statement, we will say that a subgroup H of a group G is *C-relatively convex* if there exists a C-ordering on G for which H is convex.

Lemma 1.7. *For every C-orderable group, the intersection of any family of C-relatively convex subgroups is C-relatively convex.*

Proof. We consider the action of G by left multiplications on each coordinate of the set $\Omega = \prod_\alpha G/H_\alpha$, where $(G/H_\alpha, \preceq_{H_\alpha})$ is the ($G-$invariant ordered) set of left-cosets of the C-relatively convex subgroup H_α. Putting the (left) lexicographic order on Ω and using Example 1.6, it is not hard to see that this action has no crossing. Moreover, since $\{id\}$ is C-convex, the action is faithful.

Now consider an arbitrary family $\Omega_0 \subset \{H_\alpha\}_\alpha$ of C-relatively convex subgroups of G, and let $\leq^*$ be a well-order on Ω for which Ω_0 is an initial segment. For the induced ordering $\preceq$ on G, it follows from Proposition 1.5 that the stabilizer $G_{\Omega_0} = \bigcap_{H \in \Omega_0} H$ is $\preceq$-convex. Moreover, Lemma 1.4 implies that $\preceq$ is a C-ordering, thus concluding the proof. □

We close this section with a simple lemma that we will need later and which may be left as an exercise to the reader (see also [9, Lemma 5.2.3]).

Lemma 1.8. *Let G be a torsion-free Abelian group. Then G admits only finitely many C-orderings if and only if G has rank 1.*

2 Proof of Theorem B

2.1 On groups with finitely many C-orderings

Let G be a C-orderable group admitting only finitely many C-orderings. Obviously, each of these orderings must be isolated in $\mathcal{CO}(G)$. We claim that, in general, if $\preceq$ is an isolated C-ordering, then the series of $\preceq$-convex subgroups

$$\{id\} = G^{id} \subset \ldots \subset G_g \lhd G^g \subset \ldots \subset G$$

is finite. Indeed, let $\{f_1, \ldots, f_n\} \subset G$ be a set of $\preceq$-positive elements such that $V_{f_1,\ldots,f_n}$ consists only of $\preceq$. If the series above is infinite, then there exists a $g \in G$ so that no f_i belongs to $G^g \setminus G_g$. This implies that the flipped ordering $\preceq_g$ is Conradian and different from $\preceq$. However, every f_i is still $\preceq_g$-positive (*c.f.*, Example 1.1), which is impossible because $V_{f_1,\ldots,f_n} = \{\preceq\}$.

Next let

$$\{id\} = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$$

be the series of $\preceq$-convex subgroups of G. According to the Conrad Theorem, every quotient G_i/G_{i-1} embeds into $\mathbb{R}$, and thus it is Abelian. Since every ordering on such a quotient can be extended to an ordering on G (similarly as in Example 1.1), the Abelian quotient G_i/G_{i-1} has only a finite number of orderings. It now follows from Lemma 1.8 that it must have rank 1. Therefore, the series above is rational.

We now show that this series is unique. Suppose

$$\{id\} = H_0 \lhd H_1 \lhd \ldots \lhd H_k = G$$

is another rational series. Since H_{k-1} is C-relatively convex, we conclude that $N = G_{n-1} \cap H_{k-1}$ is $C-$relatively convex by Lemma 1.7. Now G/N is torsion-free Abelian and has only a finite number of orderings, thus it has rank 1. Since convex groups are *isolated*, H_{k-1} and G_{n-1} have the property that $x^r \in G_{n-1}$ (resp. $x^r \in H_{k-1}$) implies $x \in G_{n-1}$ (resp. $x \in H_{k-1}$). This clearly yields $H_{k-1} = G_{n-1}$. Repeating this argument several times, we conclude the uniqueness of the rational series, which is hence normal.

Now we claim that no quotient G_{i+2}/G_i is Abelian. If not, G_{i+2}/G_i would be a rank-2 Abelian group, and so an infinite number of orderings could be defined on it. But since every ordering on this quotient can be extended to a C-ordering on G, this would lead to a contradiction.

2.2 On groups with a normal rational series

In this subsction we prove the converse of Theorem B.

Suppose that G has a normal rational series

$$\{id\} = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$$

such that no quotient G_{i+2}/G_i is Abelian. Clearly, flipping the orderings on the quotients G_{i+1}/G_i we obtain at least 2^n many C-orderings on G. We claim that these are the only possible C-orderings on G. Indeed, let $\preceq$ be a C-ordering on G, and let $a \in G_1$ and $b \in G_2 \setminus G_1$ be two non-commuting elements. Denoting the Conrad homomorphism of the group $\langle a, b \rangle$ endowed with the restriction of $\preceq$ by τ, we have $\tau(a) = \tau(bab^{-1})$. Since G_1 is rank-1 Abelian, we have $bab^{-1} = a^r$ for some rational number $r \neq 1$. Thus $\tau(a) = r\tau(a)$, which implies that $\tau(a) = 0$. Therefore, $a << |b|$, or in other words $a^n \prec |b|$ for every $n \in \mathbb{Z}$, where $|b| = max\{b^{-1}, b\}$. Since G_2/G_1 is rank-1, this actually holds for every $b \neq id$ in $G_2 \setminus G_1$. Thus G_1 is convex in G_2.

Repeating the argument above, though now with G_{i+1}/G_i and G_{i+2}/G_i instead of G_1 and G_2, respectively, we see that the rational series we began with is none other than the series given by the convex subgroups of $\preceq$. Since each G_{i+1}/G_i is rank-1 Abelian, if we choose $b_i \in G_{i+1} \setminus G_i$ for each $i = 0, \ldots, n-1$, then any C-ordering on G is completely determined by the signs of these elements. This shows that G admits precisely 2^n different C-orderings.

3 Proof of Theorem A

Let G be a group admitting a C-ordering $\preceq$ which is isolated in the space of C-orderings. As we have seen at the beginning of §2.1, the series of $\preceq$-convex subgroups must be finite, say

$$\{id\} = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G.$$

Proceeding just as in Example 1.1, any ordering on G_{i+1}/G_i may be extended (preserving the set of positive elements outside of it) to a C-ordering on G. Hence, each quotient must be rank-1 Abelian, so the series above is rational. We claim that this series of $\preceq$-convex subgroups is unique (hence normal) and that no quotient G_{i+2}/G_i is Abelian. In fact, if the series has length 2, then it is normal. Moreover, since no C-ordering on a rank-2 Abelian group is isolated, we have that G_2 is non-Abelian. Then by Theorem B, the series is unique. In the general case, we will use induction on the length of the series. Suppose that every group having an isolated C-ordering whose rational series of convex subgroups

$$\{id\} = H_0 \lhd H_1 \lhd \ldots \lhd H_k$$

has length $k < n$ admits a unique (hence normal) rational series and that no quotient H_{i+2}/H_i is Abelian. Let

$$\{id\} = G_0 \lhd \ldots \lhd G_{n-2} \lhd G_{n-1} \lhd G_n = G$$

be a rational series of length n associated to some isolated C-ordering $\preceq$ on G. Since G_{n-1} is normal in G, for every $g \in G$, the conjugate series

$$\{id\} = G_0^g \lhd \ldots \lhd G_{n-2}^g \lhd G_{n-1}^g = G_{n-1}$$

is also a rational series for G_{n-1}. Since this series is associated to a certain isolated C-ordering, namely the restriction of $\preceq$ to G_{n-1}, we conclude that it is unique by the induction hypothesis. Hence the series must coincide with the original one, or in other words $G_i^g = G_i$. Therefore, the series for G is normal. Moreover, every quotient G_{i+2}/G_i is non-Abelian, because if not then $\preceq$ could be approximated by other C-orderings on G. Thus, by Theorem B, the rational series for G is unique, and G admits only finitely many C-orderings. This completes the proof of Theorem A.

We conclude this section with a short discussion on the structure of bi-orderable groups admitting finitely many C-orderings. To begin with, let us note the following simple

Proposition 3.1. *If a group G has a bi-order that is isolated in the space of C-orderings, then G has finitely many C-orderings, each of which is bi-invariant.*

Proof. The fact that G admits only finitely many C-orderings is direct consequence of Theorem A. Let $\preceq$ be a bi-ordering on G which is isolated in the space of C-orderings, and let $\preceq'$ be any other C-ordering on G. According to the proof of Theorem B, the series of convex subgroups

$$\{id\} = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$$

is the same for both $\preceq$ and $\preceq'$. Moreover, $\preceq'$ is obtained from $\preceq$ by flipping the ordering on some of the quotients G_{i+1}/G_i. Since $\preceq$ is bi-invariant, the set of $\preceq$-positive elements in $G_{i+1} \setminus G_i$ is invariant under conjugacy. Since the flipping procedure interchanges the sets of positive and negative elements (where *negative* means non-trivial and non-positive), the above remains true after flipping. More precisely, the set of $\preceq'$-positive elements in $G_{i+1} \setminus G_i$

is invariant under conjugacy. Since this is true for every index i, this shows that $\preceq'$ is bi-invariant. $\square$

In spite of the fact that the preceding proposition holds for general $n \geq 1$, it only applies for $n = 1, 2$.

Proposition 3.2. *If a bi-orderable group has finitely many C-orderings, then the number of C-orderings is two or four.*

Proof. Let G be a bi-orderable group having 2^n $C-$orderings for some $n \geq 3$, and let

$$\{id\} = G_0 \lhd G_1 \lhd G_2 \lhd G_3 \unlhd \ldots \unlhd G_n = G$$

be the series of convex subgroups. It is easy to see that given $a \in G_1$, there exist $b \in G_2 \setminus G_1$ and $c \in G_3 \setminus G_2$ such that $bab^{-1} = a^r$ and $cb^q c^{-1} = b^p w$, where $w \in G_1$, and $r, p/q$ are positive rational numbers such that $p/q \neq 1$. Let $t \in \mathbb{Q}$ be positive and such that $cac^{-1} = a^t$. We have

$$a^{r^p} = b^p w a w^{-1} b^{-p} = (cb^q c^{-1}) a (cb^{-q} c^{-1}) = cb^q a^{1/t} b^{-q} c^{-1} = c a^{r^q/t} c^{-1} = a^{r^q}.$$

Since $0 < r \neq 1$, we have that $p = q$, which contradicts the fact that $1 \neq p/q$. $\square$

4 Some non-trivial examples

4.1 The Baumslag-Solitar group

In this subsection, we consider the Baumslag-Solitar group $B(1,2) = \langle a, b \mid bab^{-1} = a^2 \rangle$. We let $\langle\langle a \rangle\rangle$ denote the largest rank-1 Abelian subgroup cointaining a. According to [9, §5.3], this group admits four bi-orderings which are obtained from the series

$$\{id\} \lhd \langle\langle a \rangle\rangle = a^{\mathbb{Z}[\frac{1}{2}]} \lhd \langle a, b \rangle.$$

Here $\mathbb{Z}[\frac{1}{2}]$ denotes the set of dyadic rational numbers, that is,

$$\mathbb{Z}\left[\frac{1}{2}\right] = \left\{ \frac{m}{2^k} \mid m \in \mathbb{Z},\ k \in \mathbb{N} \right\}.$$

Below we give a self-contained proof of the fact that, actually, any C-ordering on $B(1,2)$ coincides with one of these bi-orderings.

Proposition 4.1. *Baumslag-Solitar's group $B(1,2)$ admits only four C-orderings.*

Proof. Let $\preceq$ be a C-ordering on $B(1,2)$. We will determine all $\preceq$-convex subgroups of $B(1,2)$. Without lost of generality, we may assume that $b \succ 1$. Otherwise, we could change $\preceq$ by its 'opposite' ordering $\overline{\preceq}$ (*i.e.*, the one whose positive elements are the inverses of the positive elements of $\preceq$; compare Example 1.1) which has the same convex subgroups.

First we claim that $a << b$ (*i.e.*, $a^n \prec b$ for all $n \in \mathbb{Z}$). Indeed, if we let τ be the (unique up to multiplication by a positive real number) Conrad homomorphism, then we have $\tau(a) = \tau(bab^{-1}) = \tau(a^2) = 2\tau(a)$ which implies that $\tau(a) = 0$. Hence $\tau(a^n) = 0$ for all

$n \in \mathbb{Z}$. Since τ is non-trivial and non-decreasing, we must have $\tau(b) > 0$. Again, from the fact that τ is non-decreasing, we conclude that $a^n \prec b$ as asserted.

Next letting $G_g \subset G^g$ be the convex jump associated to $g \in B(1,2)$, by property (3) in the Introduction we have $a \in G_b \neq B(1,2)$. It follows that an arbitrary $h = a^{n_1} b^{m_1} \cdots a^{n_i} b^{m_i}$ is contained in $ker(\tau)$ if and only if $\sum_k m_k = 0$. It is easy to check that $\sum_k m_k = 0$ holds if and only if $h \in a^{\mathbb{Z}[\frac{1}{2}]}$. This shows that the sequence of $\preceq$-convex subgroups of $B(1,2)$ is

$$\{id\} = G_a \subset G^a = a^{\mathbb{Z}[\frac{1}{2}]} = G_b \subset G^b = B(1,2).$$

Since both G_b and G^b/G_b are torsion-free rank-1 Abelian, we have that $B(1,2)$ admits only four C-orderings. □

We point out that $B(1,2)$ admits infinitely many left-orderings. Indeed, let $\xi \colon B(1,2) \to \mathrm{Homeo}_+(\mathbb{R})$ be the isomorphic imbedding given by $a \colon x \mapsto x+1$ and $b \colon x \mapsto 2x$. We associate to each irrational number ε a left-ordering $\preceq_\varepsilon$ on $B(1,2)$ whose set of positive elements is defined by $\{g \in B(1,2) \mid \xi(g)(\varepsilon) > \varepsilon\}$. (These orderings were introduced by Smirnov in [19].) When ε is rational, the preceding set defines only a partial ordering. However, in this case the stabilizer of the point ε is isomorphic to $\mathbb{Z}$, and hence this partial ordering may be completed to two total left-orderings $\preceq_\varepsilon^+$ and $\preceq_\varepsilon^-$. Here $\preceq_\varepsilon^+$ (resp. $\preceq_\varepsilon^-$) corresponds to the limit of $\preceq_{\varepsilon_n}$ for any sequence of irrational numbers converging to ε by the right (resp. left).

Remark that the opposite orderings (*i.e.,* those of the form $\overline{\preceq}_\varepsilon$) can be obtained the same way as above though now starting with the embedding $a \colon x \mapsto x - 1$, $b \colon x \mapsto 2x$ (and changing ε by $-\varepsilon$). Moreover, as ε tends to $-\infty$ or $+\infty$, the associate orderings converge to bi-orderings. This corroborates a result by Navas (see [13, Proposition 4.1]) according to which no C-ordering is isolated in the space of left-orderings of a group having infinitely many left-orderings.

We next give a complete description of the space of left-orderings of $B(1,2)$.

Theorem 4.2. *Besides the four bi-orderings previously described, the space of left-orderings of $B(1,2)$ is made up of those of the form $\preceq_\varepsilon$ for $\varepsilon \notin \mathbb{Q}$, those of the form $\preceq_\varepsilon^+$ and $\preceq_\varepsilon^-$ for $\varepsilon \in \mathbb{Q}$, and their opposites. In particular, every non Conradian ordering on $B(1,2)$ can be realized as an induced ordering coming from an affine action of $B(1,2)$ on the real line.*

To prove this theorem, we will use the ideas involved in the following well-known orderability criterion (see [6, Theorem 6.8], [12, §2.2.3], or [13] for further details).

Proposition 4.3. *For a countable infinite group G, the following two properties are equivalent:*
– *G is left-orderable,*
– *G acts faithfully on the real line by orientation preserving homeomorphisms.*

Sketch of proof. The fact that a group of orientation preserving homeomorphisms of the real line is left-orderable is a direct consequence of Theorem 1.2.

For the converse, we construct what is called *the dynamical realization of a left-ordering.* Let $\preceq$ be a left-ordering on G. Fix an enumeration $(g_i)_{i \geq 0}$ of G, and let $t(g_0) = 0$. We shall define an order-preserving map $t : G \to \mathbb{R}$ by induction. Suppose that $t(g_0), t(g_1), \ldots, t(g_i)$

have been already defined. Then if g_{i+1} is greater (resp. smaller) than all $g_0, \ldots, g_i$, we define $t(g_{i+1}) = max\{t(g_0), \ldots, t(g_i)\} + 1$ (resp. $min\{t(g_0), \ldots, t(g_i)\} - 1$). If g_{i+1} is neither greater nor smaller than all $g_0, \ldots, g_i$, then there are $g_n, g_m \in \{g_0, \ldots, g_i\}$ such that $g_n \prec g_{i+1} \prec g_m$ and no g_j is between g_n, g_m for $0 \leq j \leq i$. Then we put $t(g_{i+1}) = (t(g_n) + t(g_m))/2$.

Note that G acts naturally on $t(G)$ by $g(t(g_i)) = t(gg_i)$. It is not difficult to see that this action extends continuously to the closure of $t(G)$. Finally, one can extend the action to the whole real line by declaring the map g to be affine on each interval in the complement of $t(G)$. □

Remark 4.4. As constructed above, the dynamical realization depends not only on the left-ordering $\preceq$, but also on the enumeration $(g_i)_{i \geq 0}$. Nevertheless, it is not hard to check that dynamical realizations associated to different enumerations (but the same ordering) are *topologically conjugate*.[1] Thus, up to topological conjugacy, the dynamical realization depends only on the ordering $\preceq$ of G.

An important property of dynamical realizations is that they do not admit global fixed points (*i.e.,* no point is stabilized by the whole group). Another important property is that $g \succ id$ if and only if $g(t(id)) > t(id)$, which allows us to recover the left-ordering from the dynamical realization.

Proof of Theorem 4.2. Given a left-ordering $\preceq$ on $B(1,2)$ we will consider its dynamical realization. We have the following two cases:

Case 1. The element $a \in B(1,2)$ is cofinal (that is, for every g_1, $g_2 \in B(1,2)$, there are n_1, $n_2 \in \mathbb{Z}$ such that $a^{n_1} \prec g_1$ and $a^{n_2} \succ g_2$).

For the next two claims, recall that for any measure μ on a measurable space X and any measurable function $f : X \to X$, the *push-forward measure* $f_*(\mu)$ is defined by $f_*(\mu)(A) = \mu(f^{-1}(A))$, where $A \subseteq X$ is a measurable subset. Note that $f_*(\mu)$ is trivial if and only if μ is trivial. Moreover, one has $(fg)_*(\mu) = f_*(g_*(\mu))$ for all measurable functions f, g.

Claim 1. The subgroup $\langle\langle\, a \rangle\rangle$ preserves a Radon measure ν (*i.e.,* a measure which is finite on compact sets) on the real line which is unique up to scalar multiplication and has no atoms.

Since a is cofinal and $\langle\langle\, a \rangle\rangle$ is rank-1 Abelian, its action on the real line is *free* (that is, no point is fixed by any non-trivial element). By Hölder's theorem (see [6, Theorem 6.10] or [12, §2.2]), $\langle\langle\, a \rangle\rangle$ is semi-conjugated to a group of translations. More precisely, there exists a non-decreasing, continuous, surjective function $\varphi \colon \mathbb{R} \to \mathbb{R}$ such that, to each $g \in \langle\langle\, a \rangle\rangle$ one may associate a translation parameter c_g so that, for all $x \in \mathbb{R}$,

$$\varphi(g(x)) = \varphi(x) + c_g.$$

Now since the Lebesgue measure *Leb* on the real line is invariant by translations, the *push-backward measure* $\nu = \varphi^*(Leb)$ is invariant by $\langle\langle\, a \rangle\rangle$. (Here $\varphi^*(Leb)$ is defined by $\varphi^*(Leb)(A) = Leb(\varphi(A))$.) Since *Leb* is a Radon measure without atoms, this is also the

[1] Two actions $\phi_1 : G \to \mathrm{Homeo}_+(\mathbb{R})$ and $\phi_2 : G \to \mathrm{Homeo}_+(\mathbb{R})$ are topologically conjugate if there exists $\varphi \in \mathrm{Homeo}_+(\mathbb{R})$ such that $\varphi \circ \phi_1(g) = \phi_2(g) \circ \varphi$ for all $g \in G$.

case for ν. Finally, the uniqueness of ν up to scalar multiple is an easy exercise (see for instance [12, Proposition 2.2.38]).

Claim 2. For some $\lambda \neq 1$, we have $b_*(\nu) = \lambda\nu$.

Since $\langle\langle a \rangle\rangle \lhd B(1,2)$, for any $a' \in \langle\langle a \rangle\rangle$ and all measurable $A \subset \mathbb{R}$ we must have $b_*(\nu)(a'(A)) = \nu(b^{-1}a'(A)) = \nu(\bar{a}(b^{-1}(A))) = \nu(b^{-1}(A)) = b_*(\nu)((A))$ for some $\bar{a} \in \langle\langle a \rangle\rangle$. (Actually, $a' = \bar{a}^2$.) Thus $b_*(\nu)$ is a measure that is invariant by $\langle\langle a \rangle\rangle$. The uniqueness of the $\langle\langle a \rangle\rangle$-invariant measure up to scalar factor yields $b_*(\nu) = \lambda\nu$ for some $\lambda > 0$. Assume for a contradiction that λ equals 1. Then the whole group $B(1,2)$ preserves ν. Thus there is a *translation number homomorphism* $\tau_\nu \colon B(1,2) \to \mathbb{R}$ defined by

$$\tau_\nu(g) = \begin{cases} \nu([x, g(x)]) & \text{if } g(x) \geq x, \\ -\nu([g(x), x]) & \text{if } g(x) < x. \end{cases}$$

(one easily checks that this definition does not depend on $x \in \mathbb{R}$). The kernel of τ_ν must contain the commutator subgroup of $B(1,2)$; since $a \in B(1,2)'$, this yields $\tau_\nu(a) = 0$. Nevertheless, this is impossible, since –as is easy to see- the kernel of τ_ν coincides with the set of elements having fixed points on the real line (see for instance [12, §2.2.5]).

By Claims 1 and 2, for each $g \in B(1,2)$ we have $g_*(\nu) = \lambda_g(\nu)$ for some $\lambda_g > 0$. Moreover, $\lambda_a = 1$ and $\lambda_b = \lambda$.

Now, for $x \in \mathbb{R}$, let $F(x) = sgn(x - t(id)) \cdot \nu([t(id), x])$. (Note that $F(t(id)) = 0$.) Semi-conjugating the dynamical realization by F yields a faithful representation $A \colon B(1,2) \to \mathrm{Homeo}_+(\mathbb{R})$ of $B(1,2)$ in the group of (orientation-preserving) affine homeomorphisms of the real line. More precisely, for all $g \in B(1,2)$ and all $x \in \mathbb{R}$ we have

$$F(g(x)) = A_g(F(x))$$

where the affine map A_g is given by

$$A_g(x) \;=\; \frac{1}{\lambda_g}x + \frac{1}{\lambda_g}\nu([t(g^{-1}), t(id)])$$

(here we use the convention $\nu([x,y]) = -\nu([y,x])$ for $x > y$). For instance, if $x > t(id)$ and $g \in B(1,2)$ are such that $g(x) > t(id)$, then

$$F(g(x)) \;=\; \frac{1}{\lambda_g}F(x) + \frac{1}{\lambda_g}\nu([t(g^{-1}), t(id)]).$$

The action A induces a (perhaps partial) left-ordering $\preceq_A$, namely $f \succ_A id$ if and only if $A_g(0) > 0$. Clearly, if the orbit under A of 0 is free (that is, for every non-trivial element $g \in B(1,2)$, we have $A_g(0) \neq 0$), then $\preceq_A$ is total and coincides with $\preceq$ (our original ordering).

Now assume that the orbit of 0 is not free. (This may arise for example when $t(id)$ does not belong to the support of ν). In this case, the stabilizer of 0 under the action A is isomorphic to $\mathbb{Z}$. Therefore, $\preceq$ coincides with either $\preceq_A^+$ or $\preceq_A^-$ (the definition of $\preceq_A^\pm$ is similar to the definition of $\preceq_\varepsilon^\pm$ given above).

Due to the discussion above, we need to determine all possible embeddings of $B(1,2)$ into the affine group.

Lemma 4.5. *Every faithful representation of $B(1,2)$ in the affine group is given by*

$$a \sim \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad b \sim \begin{pmatrix} 2 & \beta \\ 0 & 1 \end{pmatrix}$$

for some $\alpha \neq 0$ and $\beta \in \mathbb{R}$.

Proof. One easily checks that a correspondence as above induces a faithful representation. Conversely, let

$$a \sim \begin{pmatrix} s & \alpha \\ 0 & 1 \end{pmatrix}, \quad b \sim \begin{pmatrix} t & \beta \\ 0 & 1 \end{pmatrix}$$

be a representation. Then the following equality must hold:

$$a^2 \sim \begin{pmatrix} s^2 & s\alpha + \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s & \alpha t - s\beta + \beta \\ 0 & 1 \end{pmatrix} \sim bab^{-1}.$$

Thus $s = 1$, $t = 2$, and since the representation is faithful, $\alpha \neq 0$. □

Let α, β be such that $A_a(x) = x + \alpha$ and $A_b(x) = 2x + \beta$. We claim that if the stabilizer of 0 under A is trivial –which implies in particular that $\beta \neq 0$– , then $\preceq_A$ (and hence $\preceq$) coincides with $\preceq_\varepsilon$ if $\alpha > 0$ (resp. $\overline{\preceq}_\varepsilon$ if $\alpha < 0$), where $\varepsilon = \beta/\alpha$. Indeed, if $\alpha > 0$, then for each $g = b^n a^r \in B(1,2)$ we have $A_g(0) = 2^n r\alpha + (2^n - 1)\beta$. Hence $A_g(0) > 0$ holds if and only if

$$2^n \beta/\alpha + 2^n r > \beta/\alpha.$$

Letting $\varepsilon = \beta/\alpha$, one easily checks that the preceding inequality is equivalent to $g \succ_\varepsilon id$. The claim now follows.

In the case the stabilizer of 0 under A is isomorphic to $\mathbb{Z}$, similar arguments to those given above show that $\preceq$ coincides with either $\preceq_\varepsilon^+$, or $\preceq_\varepsilon^-$, or $\overline{\preceq_\varepsilon^+}$, or $\overline{\preceq_\varepsilon^-}$, where ε again equals β/α.

Case 2. The element $a \in B(1,2)$ is not cofinal.

In this case, for the dynamical realization of $\preceq$, the set of fixed points of a, denoted $Fix(a)$, is non-empty. We claim that $b(Fix(a)) = Fix(a)$. Indeed, given $x \in Fix(a)$, we have

$$a(b(x)) = ab(x) = a^{-1}ba(x) = a^{-1}(b(x))\,.$$

Hence $a^2(b(x)) = b(x)$, which implies that $a(b(x)) = b(x)$ as asserted. Observe that since there is no global fixed point for the dynamical realization, we must have $b(x) \neq x$, for all $x \in Fix(a)$.

Now suppose that $b \succ id$ (otherwise, we may consider the opposite ordering), and let $x_1 = inf\{x \in Fix(a) \mid x > t(id)\}$. We claim that $b(x_1) > x_1$. Suppose not. Then $b(x_1) < x_1$, but since $b(t(id)) = t(b) > t(id)$, we also have $b(x_1) > t(id)$. Therefore, $b(x_1)$ is a fixed point of a situated in $(t(id), x_1)$, which is a contradiction.

We now claim that $t(b) > x_1$. Indeed, if not, then we would have $b(t(id)) = t(b) < x_1$. (Note that $t(b)$ cannot be equal to x_1, since x_1 is fixed by a, but $B(1,2)$ acts freely on $t(B(1,2))$.) Since $b(x_1) > x_1$, this would yield $b^{-1}(x_1) \in (t(id), x_1)$. However, since $b^{-1}(x_1)$ belongs to $Fix(a)$, this contradicts the definition of x_1.

We next claim that $b(x_{-1}) \geq x_1$, where $x_{-1} = \sup\{x \in Fix(a) \mid x < t(id)\}$. Indeed, since $b(x_{-1})$ is a fixed point of a, to show this it is enough to show that $b(x_{-1}) > x_{-1}$. This can be easily checked using similar arguments to those above.

We finally claim that $\langle\langle a \rangle\rangle$ is a convex subgroup. First note that, by the definition of the dynamical realization, for every $g \in B(1,2)$ we have $t(g) = g(t(id))$. Then, it follows that for every $g \in \langle\langle a \rangle\rangle$, $t(g) \in (x_{-1}, x_1)$. Now let $m \in \mathbb{Z}$ and r, s in $\mathbb{Z}[\frac{1}{2}]$ be such that $id \prec b^m a^r \prec a^s$. Then we have $t(id) < b^m(t(a^r)) < t(a^s) < x_1$. Since $b(x_{-1}) \geq x_1$, this easily yields $m = 0$, that is, $b^m a^r \in \langle\langle a \rangle\rangle$.

We have thus proved that $\langle\langle a \rangle\rangle$ is a convex (normal) subgroup of $B(1,2)$. Since the quotient $B(1,2)/\langle\langle a \rangle\rangle$ is isomorphic to $\mathbb{Z}$, an almost direct application of the characterization (4) in the Introduction shows that the ordering $\preceq$ is Conradian. This concludes the proof of Theorem 4.2. □

Remark 4.6. It follows from Theorem 4.2 that no ordering is isolated in $\mathcal{LO}(B(1,2))$. Thus this space is homeomorphic to the Cantor set. Moreover, the natural conjugacy action of $B(1,2)$ on $\mathcal{LO}(B(1,2)$ is 'almost' transitive. More precisely, for any irrational ε, the orbit of $\preceq_\varepsilon$ under $B(1,2)$ is dense in the subspace V_a formed by the orderings for which a is positive. This easily follows from the fact that, for all $g \in B(1,2)$, we have $g(\preceq_\varepsilon) = \preceq_{g^{-1}(\varepsilon)}$. The complementary subspace $V_{a^{-1}}$ of $\mathcal{LO}(B(1,2))$ is densely covered by the orbit of $\overline{\preceq}_\varepsilon$.

Remark 4.7. The above method of proof also gives a complete classification –up to topological semiconjugacy– of all actions of $B(1,2)$ by orientation-preserving homeomorphisms of the real line (compare [2, 16]). In particular, all these actions come from left-orderings on the group (compare with the comment before Question 2.3 in [13]).

4.2 Examples of groups with 2^n Conradian orderings but infinitely many left-orderings

The classification of groups having finitely many left-orderings was obtained by Tararin and appears in [9, §5.2]. An example of a group having precisely 2^n orders is $T_n = \mathbb{Z}^n$ endowed with the product rule

$$(b_n, \ldots, b_1) \cdot (b'_n, \ldots, b'_1) = (b_n + b'_n,\, (-1)^{b'_n} b_{n-1} + b'_{n-1}\,, \ldots,\, (-1)^{b'_2} b_1 + b'_1).$$

A presentation for T_n is

$$T_n \cong \langle a_n, \ldots, a_1 \mid R_n \rangle,$$

where the set of relations R_n is

$$a_{i+1} a_i a_{i+1}^{-1} = a_i^{-1} \quad \text{if} \quad i < n, \qquad \text{and} \qquad a_i a_j = a_j a_i \quad \text{if} \quad |i - j| \geq 2.$$

A very simple dynamical argument shows that if a group has finitely many left-orderings, then each of these orderings is Conradian [13, Lemma 3.45]. However, it is natural to ask whether for each $n \geq 2$ there are groups having precisely 2^n Conradian orderings but infinitely many left-orderings. As we have seen in the preceding section, for $n = 2$ this is the case of the Bumslag-Solitar group $B(1,2)$. This holds (with a very similar proof) for many other metabelian left-orderable groups, as for example all Baumslag-Solitar's groups $B(1,\ell)$ for $\ell \geq 2$. It turns out that, in order to construct examples for higher n and having $B(1,\ell)$ as a quotient by a normal convex subgroup, we need to choose an odd integer ℓ. As a concrete example, for $n \geq 3$ we endow $C_n = \mathbb{Z} \times \mathbb{Z}[\frac{1}{3}] \times \mathbb{Z}^n$ with the multiplication

$$\Big(c,\, \frac{m}{3^k}\,, a_n, \ldots, a_1\Big) \cdot \Big(c',\, \frac{m'}{3^{k'}}\,, a'_n, \ldots, a'_1\Big) =$$
$$= \Big(c + c',\; 3^c\frac{m'}{3^{k'}} + \frac{m}{3^k}\;, (-1)^m a'_n + a_n\,, (-1)^{a_n} a'_{n-1} + a_{n-1} \ldots,\; (-1)^{a_2} a'_1 + a_1\Big).$$

Note that the product rule is well defined because if $m/3^k = \bar{m}/3^{\bar{k}}$, then $(-1)^m = (-1)^{\bar{m}}$ (it is here where we use the fact that $\ell = 3$ is odd).

The group C_n admits the presentation

$$C_n \cong \langle c, b, a_n, \ldots, a_1 \mid cbc^{-1} = b^3\,,\ ca_i = a_i c\,,\ ba_n b^{-1} = a_n^{-1}\,,\ ba_i = a_i b \ \text{ if } \ i \neq n\,,\ R_n \rangle.$$

This group satisfies the hypotheses of Theorem B and has exactly 2^{n+2} Conradian orderings. However, it has $B(1,3)$ as a quotient by a normal convex subgroup. Since $B(1,3)$ admits uncountably many left-orderings, the same is true for C_n.

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