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Diagonalizability of nonautonomous linear systems with bounded continuous coefficients[☆]

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ABSTRACT

A linear ODE system with bounded continuous coefficients is studied. Provided that the diagonal terms are "separated" and the off-diagonal terms satisfy a smallness threshold, we present some sufficient conditions for the diagonalizability of the system. The results are valid for any couple of Banach subspaces V, W of $BC(\mathbb{R}, \mathbb{C}^n)$ admissible for a family of reduced ($n-1$ dimensional) linear homogeneous systems having exponential dichotomy.
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1. Introduction

The present work establishes conditions ensuring the diagonalizability (or kinematical similarity [5,15] to a diagonal system) of the linear nonautonomous system:

$$x' = A(t)x, \tag{1.1}$$

where $x = \mathrm{col}(x_1, \ldots, x_n) \in \mathbb{C}^n$ and $A(\cdot)$ is a matrix with coefficients $a_{ij}: \mathbb{R} \mapsto \mathbb{C}$, which are in a manifold $W(\mathbb{R}, \mathbb{C}) \subseteq BC(\mathbb{R}, \mathbb{C})$, the Banach space of bounded continuous functions in $\mathbb{R}$ taking values in $\mathbb{C}$.

The system (1.1) is said to be *kinematically similar* to a diagonal system if there exists an invertible and continuously differentiable matrix $Q(\cdot)$ verifying

$$Q'(t) = A(t)Q(t) - Q(t)R(t)$$

such that the change of variables $u = Q^{-1}x$ transforms (1.1) in a diagonal system

$$u' = R(t)u$$

in addition, if $Q(\cdot)$, $Q^{-1}(\cdot)$ and $Q'(\cdot)$ are bounded, $Q(\cdot)$ is denoted as *Lyapunov transformation* (we refer the reader to [18, p. 722] for details).

The diagonalizability of (1.1) is a classical problem in the theory on nonautonomous systems and there exist several works obtaining remarkably elegant conditions. For example (we refer the reader to [3] for a detailed discussion), the full spectrum property of Sacker and Sell [26], Lillo's property [14], n projections Coppel's property [5], the Bylov–Millionshchikov integral separation property [4,17] and its characterization in terms of exponential dichotomy and roughness results by Palmer [23]. In addition, we refer the reader to the works of asymptotic diagonalization of Hsieh et al. [12] and Bodine et al. [2,3], who

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study the diagonalizability of (1.1) for $t \geq t_0$. These results suppose a previous knowledge of an explicit fundamental matrix of (1.1).

On the other hand, in [23] Palmer introduces the α-integral separation property and proves that a family of upper triangular systems is diagonalizable if and only if its diagonal terms are α-integrally separated.

We construct a fundamental matrix for (1.1), which is employed to obtain sufficient conditions ensuring diagonalizability and generalizing our results obtained in [25], where it was assumed that $A(\cdot)$ is an almost periodic matrix [9,30]. In fact, a careful lecture of [25] shows that the almost periodicity does not play a crucial role and the method can be extended to bounded continuous functions or some Banach subspaces $V \subseteq W$ with diagonal terms in V and $a_{ij}(\cdot) \in W$ by introducing some additional properties. Now, considering V and W as "space parameters", we can deduce new results for pseudo almost periodic functions [6] and almost automorphic functions [22] for example.

The first step assumes that the differences of the diagonal terms of $A(t)$ satisfy an α-integral separation property. By using this fact together with a Riccati transformation, the system (1.1) can be decomposed in a set of $(n-1)$-dimensional linear diagonal systems:

$$\xi' = D_k(t)\xi \quad \text{with } k = 1, \ldots, n, \tag{1.2}$$

which admit an exponential dichotomy [5,9]. An explicit characterization of (1.2) in terms of $A(t)$ will be given later.

Similarly, by using the α-integral separation property combined with the Roughness Theorem [5] and a Riccati transformation, the system (1.1) can be decomposed in the family of $(n-1)$-dimensional linear systems:

$$\xi' = [D_k(t) + N_k(t)]\xi \quad \text{with } k = 1, \ldots, n, \tag{1.3}$$

which also admit an exponential dichotomy.

The second step assumes that the off-diagonal terms $a_{ij}(t)$ satisfy some (explicit and implicit) smallness properties and that (1.2)–(1.3) verify admissibility conditions, which will allow us to construct a fundamental matrix in terms of solutions of some nonhomogeneous and nonlinear (quadratic) perturbations of (1.2)–(1.3).

Once that the fundamental matrix is constructed, the diagonalizability problem is addressed. Indeed, we prove that if the off-diagonal smallness conditions for the terms $a_{ij}(t)$ are slightly enforced, then we can construct a Lyapunov transformation [18] by using the solutions of the same nonhomogeneous and quadratic perturbations of (1.2)–(1.3) stated above, obtaining diagonalizability conditions.

We also consider the special (but important and concrete) case of ergodic diagonal coefficients and obtain an alternative way to diagonalization.

The rest of the paper is organized as follows: In Section 2 we introduce some notation and definitions. The Section 3 introduces the assumptions about (1.1). The Section 4 states our main results (diagonalizability conditions), which are proved in the Section 5. The ergodic case is considered in the Section 6.

2. Preliminaries

The notation used throughout the paper is fairly standard. For any $w \in \mathbb{C}$, we will denote their real part, conjugate and module respectively by $\Re(w)$, $\bar{w}$ and $|w|$. Furthermore, given $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we denote the norms:

$$|z|_1 = \sum_{j=1}^{k} |z_j| \quad \text{and} \quad |z|_\infty = \max_{1 \leq j \leq n} |z_j|.$$

The Banach space of bounded continuous functions in $\mathbb{R}$ taking values in $\mathbb{C}^n$ will be denoted by $BC(\mathbb{R}, \mathbb{C}^n) = BC(\mathbb{R}, \mathbb{C}) \times \cdots \times BC(\mathbb{R}, \mathbb{C})$, *i.e.*, n copies of $BC(\mathbb{R}, \mathbb{C})$, and a similar convention will be used for its Banach subspaces $W(\mathbb{R}, \mathbb{C}^n)$.

Given $f = (f_1, \ldots, f_n) \in BC(\mathbb{R}, \mathbb{C}^n)$, let us define $|f_j|_\infty = \sup\{|f_j(t)|: t \in \mathbb{R}\}$. In addition, we denote the norms:

$$\|f\|_1 = \sum_{j=1}^{n} |f_j|_\infty, \quad \text{and} \quad \|f\|_\infty = \max_{1 \leq j \leq n} |f_j|_\infty.$$

Given two functions $f, g \in BC(\mathbb{R}, \mathbb{C}^n)$, we define

$$\langle f(t), g(t) \rangle = \sum_{i=1}^{n} \bar{f}_i(t) g_i(t),$$

and by the Hölder inequality it follows that $|\langle f(t), g(t) \rangle| \leq \|f\|_1 \|g\|_\infty$.

When considering a bounded $n \times n$ matrix function $B(t)$, we denote the induced norms $|B(t)|_1$ and $\|B\|_1$ as follows:

$$|B(t)|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |b_{ij}(t)| \quad \text{and} \quad \|B\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |b_{ij}|_\infty.$$

2.1. Some definitions

For convenience, let us state the following definitions concerning the linear system:

$$z' = C(t)z, \tag{2.1}$$

where $C(\cdot)$ is a matrix with coefficients either in $BC(\mathbb{R},\mathbb{C})$ or in a proper subspace. The fundamental matrix of (2.1) will be denoted by $Z(t)$, without loss of generality, we will assume that $Z(0)=I$.

Definition 1 (*[5,9]*). The system (2.1) admits a (σ,K,P)-exponential dichotomy if there exist constants $\sigma>0$, $K>1$ and a projection P such that the Green function associated with (2.1), G_Z defined by

$$G_Z(t,s)=\begin{cases} Z(t)PZ^{-1}(s) & \text{if } t\geq s,\\ -Z(t)(I-P)Z^{-1}(s) & \text{if } s>t \end{cases}$$

verifies the inequality:

$$|G_Z(t,s)|\leq Ke^{-\sigma|t-s|}.$$

Definition 2 (*[10, p. 438][16, Chapter 5]*). Let $(\mathfrak{B},\mathfrak{D})$ be a pair of Banach subspaces of $BC(\mathbb{R},\mathbb{C}^n)$. The pair $(\mathfrak{B},\mathfrak{D})$ is admissible for $C(\cdot)$ if for any $h\in\mathfrak{B}$, there is a solution $z_h(\cdot)\in\mathfrak{D}$ of the nonhomogeneous system:

$$z'=C(t)z+h(t). \tag{2.2}$$

A direct consequence (see [16, p. 126]) of $(\mathfrak{B},\mathfrak{D})$-admissibility for $C(t)$ is that there exists $c>0$ such that the map $h\in\mathfrak{B}\mapsto z_h\in\mathfrak{D}$ is continuous:

$$\|z_h\|_1\leq c\|h\|_1, \tag{2.3}$$

in addition, if (2.1) admits an exponential dichotomy, we have the uniqueness of the solution $z_h(\cdot)$.

There exist several results devoted to $(\mathfrak{D},\mathfrak{D})$ admissibility for $C(t)$: we refer the reader to [13,19–21,26] for a general view. In this work, we will be interested in the following result relating exponential dichotomy and admissibility properties.

Proposition 1. *If (2.1) admits a (σ,K,P)-exponential dichotomy on $\mathbb{R}$ and $h(\cdot)\in BC(\mathbb{R},\mathbb{C}^n)$, then the pair (BC,BC) is admissible for $C(t)$. In particular,*

$$z_h(t)=\int_{-\infty}^{\infty}G_Z(t,s)h(s)\,ds \tag{2.4}$$

is the unique bounded continuous solution of (2.2) and it follows that the map $h\mapsto z_h$ is continuous and

$$\|z_h\|_1\leq 2K\|h\|_1/\sigma. \tag{2.5}$$

Proof. A direct computation shows that

$$z_h(t)=\int_{-\infty}^{t}Z(t)PZ^{-1}(s)h(s)\,ds-\int_{t}^{\infty}Z(t)(I-P)Z^{-1}(s)h(s)\,ds$$

is a solution of (2.2). Moreover, by Definition 1, we can deduce that

$$\begin{aligned}|z_h(t)| &\leq \|h\|_1\left(\int_{-\infty}^{t}Ke^{-\sigma(t-s)}\,ds+\int_{t}^{\infty}Ke^{-\sigma(s-t)}\,ds\right)\\ &\leq 2K\|h\|_1/\sigma\end{aligned}$$

and the boundedness follows since h is bounded. As the homogeneous system has the exponential dichotomy, we know that the trivial solution is the unique bounded solution of (2.1), which implies the uniqueness of the bounded solution $z_h(\cdot)$ and the result follows. □

This result can also be extended for some subspaces of $BC(\mathbb{R},\mathbb{C}^n)$.

Definition 3. A bounded function $f:\mathbb{R}\to\mathbb{C}^n$ is:

(a) Almost periodic, i.e., $f\in\mathcal{AP}(\mathbb{R},\mathbb{C}^n)$ [9,11,30], if for any sequence $\{\tilde{s}_n\}$, there exist a subsequence $\{s_n\}$ and a function $g:\mathbb{R}\to\mathbb{C}^n$ such that $f(t+s_n)$ converges uniformly to $g(t)$.

(b) Almost automorphic, i.e., $f\in\mathcal{AA}(\mathbb{R},\mathbb{C}^n)$ [22], if for any sequence $\{\tilde{s}_n\}$, there exist a subsequence $\{s_n\}$ and a function $g:\mathbb{R}\to\mathbb{C}^n$ such that the following limits exist pointwise:

$$\lim_{n\to+\infty}f(t+s_n)=g(t)\quad\text{and}\quad\lim_{n\to+\infty}g(t-s_n)=f(t).$$

(c) Pseudo almost periodic, i.e., $f\in\mathcal{PAP}(\mathbb{R},\mathbb{C}^n)$ [6], if f has the decomposition:

$$f(t)=f_0(t)+\phi(t),\quad\text{with } f_0\in\mathcal{AP}(\mathbb{R},\mathbb{C}^n)\quad\text{and}\quad\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|\phi(r)|\,dr=0.$$

It is well known that $\mathcal{AP}(\mathbb{R},\mathbb{C}^n)$, $\mathcal{AA}(\mathbb{R},\mathbb{C}^n)$ and $\mathcal{PAP}(\mathbb{R},\mathbb{C}^n)$ are Banach subspaces of $BC(\mathbb{R},\mathbb{C}^n)$ and satisfy

$$\mathcal{AP}(\mathbb{R},\mathbb{C}^n)\subset\mathcal{AA}(\mathbb{R},\mathbb{C}^n)\quad\text{and}\quad\mathcal{AP}(\mathbb{R},\mathbb{C}^n)\subset\mathcal{PAP}(\mathbb{R},\mathbb{C}^n).$$

Remark 1. When (2.1) admits a (σ,K,P)-exponential dichotomy on $\mathbb{R}$, we can obtain other results by using Proposition 1 with the spaces stated above:

(i) If $C(\cdot)$ has almost periodic coefficients, it is well known that the pairs $(\mathcal{AP},\mathcal{AP})$ and $(\mathcal{PAP},\mathcal{PAP})$ are admissible for $C(t)$. See [9, Theorem 7.7] and [29, Theorem 2.3] for details.
(ii) If $C(\cdot)$ is constant, it is well known that the pair $(\mathcal{AP},\mathcal{AP})$ and $(\mathcal{PAP},\mathcal{PAP})$ are admissible for $C(t)$. See [9, Theorem 5.11] and [28, Theorem 2.1] for details. In addition, it will be proved that – under additional assumptions – the pair $(\mathcal{AA},\mathcal{AA})$ is admissible.

Definition 4. A Banach subspace $W(\mathbb{R},\mathbb{C}^n)\subset BC(\mathbb{R},\mathbb{C}^n)$ will be called scalar invariant if

(a) a function $\phi=(\phi_1,\ldots,\phi_n)\in W(\mathbb{R},\mathbb{C}^n)$ if and only if $\phi_i\in W(\mathbb{R},\mathbb{C})$ for any $i=1,\ldots,n$;
(b) $W(\mathbb{R},\mathbb{C})$ is a Banach algebra over $\mathbb{C}$.

Remark 2. Notice that $\mathcal{AP}(\mathbb{R},\mathbb{C}^n)$ is scalar invariant. Indeed, the properties (a) and (b) are ensured by Theorems 1.9 and 2.9 from [9]. Similar results can be obtained for $\mathcal{AA}(\mathbb{R},\mathbb{C}^n)$ and $\mathcal{PAP}(\mathbb{R},\mathbb{C}^n)$.

2.2. Auxiliary notation

Let $M_{nm}(\mathbb{C})$ be the set of matrices with n rows, m columns and coefficients in $\mathbb{C}$. The case $n=m$ will be denoted by $M_n(\mathbb{C})$, with identity I_n or I when no ambiguity is possible.

Given the matrices $A(\cdot)$ of (1.1) and I_n, let us recall some notation, which will play a prominent role in our study:

We construct the row vector $A_{kk+1}(t)\in\mathbb{C}^{n-1}$ by deleting the element (k,k) of the k-th row of $A(t)$. Hence, for any $k\in\{1,\ldots,n\}$ mod n, it follows that

$$A_{kk+1}(t)=(a_{k1}(t),a_{k2}(t),\ldots,a_{kk-1}(t),a_{kk+1}(t),\ldots,a_{kn}(t)).$$

Note that the i-th component of $A_{kk+1}(t)$ is defined by

$$\big(A_{kk+1}(t)\big)_i=a_{ki}(t)\quad\text{if } i<k,\quad\text{and}\quad\big(A_{kk+1}(t)\big)_i=a_{ki+1}(t)\quad\text{if } i\ge k.$$

We construct the column vector $A_{k+1k}(t)\in\mathbb{C}^{n-1}$ by deleting the element (k,k) of the k-th column of $A(t)$. Hence, for any $k\in\{1,\ldots,n\}$ mod n, it follows that

$$A_{k+1k}(t)=\operatorname{col}(a_{1k}(t),a_{2k}(t),\ldots,a_{k-1k}(t),a_{k+1k}(t),\ldots,a_{nk}(t)).$$

Hence, the i-th component of $A_{k+1k}(t)$ is defined by

$$\big(A_{k+1k}(t)\big)_i=a_{ik}(t)\quad\text{if } i<k\quad\text{and}\quad\big(A_{k+1k}(t)\big)_i=a_{i+1k}(t)\quad\text{if } i\ge k. \tag{2.6}$$

We construct the matrices $A_{k+1k+1}(t)\in M_{n-1}(\mathbb{C})$ by deleting the k-th column and the k-th file of $A(t)$. In consequence, the (i,j) entry of the matrix $A_{k+1k+1}(t)$ is defined by

$$\big(A_{k+1k+1}(t)\big)_{ij}=\begin{cases}a_{ij}(t) & \text{if } 1\le i,j\le k-1,\\ a_{i+1j}(t) & \text{if } 1\le j\le k-1,\ k\le i\le n-1,\\ a_{ij+1}(t) & \text{if } 1\le i<k-1,\ k\le j\le n-1,\\ a_{i+1j+1}(t) & \text{if } k\le i,j\le n.\end{cases} \tag{2.7}$$

The matrices $L_k\in M_{nn-1}(\mathbb{R})$ are constructed by deleting the k-th column of I_n. Therefore, the L_{ij} element of L_k is defined by

$$L_{ij}=\begin{cases}1 & \text{if } i=j<k \text{ and/or } i=j+1>k,\\ 0 & \text{otherwise.}\end{cases}$$

3. Principal assumptions and statement of results

The matrices introduced above allow us to define

$$A_k(t)=A_{k+1k+1}(t)-a_{kk}(t)I_{n-1}\quad\text{for any } k\in\{1,\ldots,n\} \text{ mod } n,$$

which can be decomposed on its diagonal and off-diagonal terms as follows:

$$A_k(t)=D_k(t)+N_k(t), \tag{3.1}$$

with diagonal terms described by

$$D_k(t)=\operatorname{Diag}\{a_{ii}(t)-a_{kk}(t)\}\quad\text{with } i\in\{1,\ldots,n\}\setminus\{k\}. \tag{3.2}$$

Moreover, observe that

$$\|N_k\|_1 = \max_{j\in\{1,\ldots,n\}\setminus\{k\}} \sum_{i\neq k}^{n} \|a_{ij}\|_\infty. \tag{3.3}$$

3.1. Diagonal terms properties

We make the following assumption:

(A) If $i \neq j$, then there exist $\alpha > 0$ and $\beta \geq 0$ such that either:

$$\Re \int_s^t \{a_{ii}(r) - a_{jj}(r)\}\,dr > -\beta + \alpha(t-s) \quad \text{or} \quad \Re \int_s^t \{a_{ii}(r) - a_{jj}(r)\}\,dr < \beta - \alpha(t-s)$$

for $t \geq s$.

By appropriate reordering, it is easy to verify that (A) says that the functions $a_{ii}(t) - a_{jj}(t)$ $(i \neq j)$ are α-integrally separated in the sense of Palmer (see [23, Section 3] for details). Furthermore, an important consequence of (A) is the following result.

Lemma 1. *Assume that* (A) *holds, then the diagonal systems*

$$u' = D_k(t)u, \quad k = 1\ldots, n \tag{3.4}$$

admit an (α, e^β, P_k)*-exponential dichotomy with* $P_k = \mathrm{Diag}\{p_{ii}^k\}$ *defined by*

$$p_{ii}^k = \begin{cases} 1 & \text{if } \Re \displaystyle\int_s^t \{a_{ii}(r) - a_{kk}(r)\}\,dr \leq \beta - \alpha(t-s), \\ 0 & \text{if } \Re \displaystyle\int_s^t \{a_{ii}(r) - a_{kk}(r)\}\,dr \geq -\beta + \alpha(t-s) \end{cases}$$

when $t \geq s$ *and* $i \in \{1, \ldots, n\} \setminus \{k\}$.

Proof. The fundamental matrix of (3.4) will be denoted by

$$Z_k(t) = \mathrm{Diag}\left\{\exp\left(\int_0^t a_{ii}(r) - a_{kk}(r)\,dr\right)\right\}_{i\neq k}.$$

In addition, note that $Z_k(t)P_kZ_k^{-1}(s)$ and $Z_k(t)(I-P_k)Z_k^{-1}(s)$ are diagonal matrices.

If $t \geq s$, by (A), we can deduce that the diagonal terms of $Z_k(t)P_kZ_k^{-1}(s)$ are $\exp\big(\int_s^t \{a_{ii}(r) - a_{kk}(r)\}\,dr\big)$ if $\Re\int_s^t \{a_{ii}(r) - a_{kk}(r)\}\,dr \leq \beta - \alpha(t-s)$ or 0 if $\Re\int_s^t \{a_{ii}(r) - a_{kk}(r)\}\,dr \geq -\beta + \alpha(t-s)$.

In consequence, it is straightforward to verify that

$$|Z_k(t)P_kZ_k^{-1}(s)| \leq e^\beta e^{-\alpha(t-s)} \quad \text{if } t \geq s.$$

On the other hand, if $t < s$, we can deduce that the diagonal terms of $Z_k(t)(I-P_k)Z_k^{-1}(s)$ are either $\exp\big(-\int_t^s \{a_{ii}(r) - a_{kk}(r)\}\,dr\big)$ if $\Re\int_s^t \{a_{ii}(r) - a_{kk}(r)\}\,dr \leq -\beta + \alpha(t-s)$ or 0 otherwise. As before, we obtain that

$$|Z_k(t)(I-P_k)Z_k^{-1}(s)| \leq e^\beta e^{-\alpha(s-t)} \quad \text{if } s > t$$

and the lemma follows. □

The previous result allows us to define the functions:

$$u_0^{(k)}(t) = \int_{-\infty}^{\infty} G_{Z_k}(t,s)A_{k+1k}(s)\,ds, \quad k = 1, \ldots, n, \tag{3.5}$$

where $G_{Z_k}(t,s)$ is the Green function corresponding to $Z_k(t)$.

3.2. Explicit off-diagonal term properties

Proposition 2 (*Roughness Property*)**.** *Assume that* (A) *holds. If*

$$\delta_k = \sup_{t\in\mathbb{R}} |N_k(t)|_1 < \frac{\alpha}{4e^{2\beta}}, \quad \textit{for any } k = 1, \ldots, n, \tag{3.6}$$

then the systems

$$u' = A_k(t)u = [D_k(t) + N_k(t)]u, \quad k = 1\ldots, n, \tag{3.7}$$

admit an $(\tilde{\alpha}, M, Q_k)$-exponential dichotomy, where

$$\tilde{\alpha} = \alpha - 2e^{\beta}\delta_k, \qquad M = (5/2)e^{2\beta}, \tag{3.8}$$

and the projection Q_k has the same nullspace as P_k.

Proof. See Proposition 1 from [5, p. 34] or Theorem 4.12 from [29]. □

The fundamental matrices of (3.7) and its Green functions are respectively denoted by $X_k(t)$ and $G_{X_k}(t,s)$. These functions allow us to define

$$x_0^{(k)}(t) = \int_{-\infty}^{\infty} G_{X_k}(t,s)A_{k+1k}(s)\,ds, \quad k = 1,\ldots,n. \tag{3.9}$$

3.3. Implicit off-diagonal properties

If the roughness property (3.6) is satisfied, let us define the additional property:

(B1) Let $\rho > 0$, such that the off-diagonal coefficients of $A(t)$ satisfy

$$\|A_{kk+1}\|_\infty < \frac{\tilde{\alpha}\rho}{4M(\|x_0^{(k)}\|_1 + \rho)^2} \quad \text{and} \quad \|A_{k+1k}\|_1 < \frac{\tilde{\alpha}\rho}{2M} \quad k = 1,\ldots,n,$$

where $\tilde{\alpha}$ and M are stated in (3.8) and $x_0^{(k)}(\cdot)$ are defined by (3.9).

If the roughness property (3.6) is not satisfied, let us define the property.

(C1) Let $\rho > 0$, such that the off-diagonal coefficients of $A(t)$ satisfy

$$\|A_{kk+1}\|_\infty + \frac{\|N_k\|_1}{2(\|u_0^{(k)}\|_1 + \rho)} < \frac{\alpha\rho}{4e^{\beta}(\|u_0^{(k)}\|_1 + \rho)^2} \quad \text{and} \quad \|A_{k+1k}\|_1 < \frac{\alpha\rho}{2e^{\beta}},$$

where $u_0^{(k)}(\cdot)$ $(k = 1,\ldots,n)$ are defined by (3.5).

Properties (B1) and (C1) can be interpreted as smallness thresholds for the off-diagonal terms.

3.4. Admissibility conditions

Let us consider the Banach spaces $V(\mathbb{R},\mathbb{C}^n) \subseteq W(\mathbb{R},\mathbb{C}^n) \subseteq BC(\mathbb{R},\mathbb{C}^n)$ such that

(B2) the pair (W, W) is admissible for $A_k(t)$ (for any $k = 1,\ldots,n$);
(C2) the diagonal terms of $A(\cdot)$ satisfy $a_{ii}(\cdot) \in V(\mathbb{R},\mathbb{C}^n)$ and the pair (W, W) is admissible for $D_k(t)$ (for any $k = 1,\ldots,n$).

4. Diagonalizability results and its consequences

4.1. Main results

Let us consider the Banach space $W(\mathbb{R},\mathbb{C}^n) \subset BC(\mathbb{R},\mathbb{C}^n)$.

Theorem 1. *Suppose that $A(\cdot)$ is a matrix with coefficients in a scalar invariant Banach space $W(\mathbb{R},\mathbb{C}^n)$ and (3.6) is satisfied. If (A), (B1) and (B2) hold, then (1.1) has a basis of solutions $\{x_1(t),\ldots,x_n(t)\}$ described by:*

$$x_k(t) = (e_k + L_k v_k(t)) \exp\left(\int_0^t [a_{kk}(s) + \langle \bar{A}_{kk+1}(s), v_k(s)\rangle]\,ds\right), \tag{4.1}$$

where $v_k(\cdot) \in W(\mathbb{R},\mathbb{C}^n)$ is solution of

$$v' = A_k(t)v + A_{k+1k}(t) - \langle \bar{A}_{kk+1}(t), v\rangle v \quad k = 1,\ldots,n \bmod n, \tag{4.2}$$

and satisfies

$$\|v_k - x_0^{(k)}\|_1 < \rho, \tag{4.3}$$

with $x_0^{(k)}(\cdot)$ *defined by* (3.9)*. Moreover, if* (B1) *is satisfied with* $\rho \in (0, 1/2)$*, then there exists a Lyapunov transformation* $Q(t)$ *with coefficients in* $W(\mathbb{R}, \mathbb{C})$ *such that* $Q^{-1}(\cdot)$ *and* $Q'(\cdot)$ *have coefficients in* $W(\mathbb{R}, \mathbb{C})$ *and* (1.1) *is kinematically similar to the diagonal system:*

$$u' = \Phi(t)u \quad \text{with } \Phi(t) = D'(t) = \mathrm{Diag}\{\phi_1(t), \dots, \phi_n(t)\}, \tag{4.4}$$

where $\phi_i(\cdot) \in W(\mathbb{R}, \mathbb{C})$ *is defined by*

$$\phi_i(t) = a_{ii}(t) + \langle \bar{A}_{ii+1}(t), v_i(t)\rangle, \quad i = 1, \dots, n. \tag{4.5}$$

Theorem 2. *Suppose that* $A(\cdot)$ *is a matrix with diagonal coefficients in a Banach space* $V(\mathbb{R}, \mathbb{C}^n)$ *and off-diagonal coefficients in* $W(\mathbb{R}, \mathbb{C}^n)$*, which is scalar invariant. If* (A), (C1), (C2) *hold, then* (1.1) *has a basis of solutions* $\{x_1(t), \dots, x_n(t)\}$ *described by* (4.1)*, where* $v_k(\cdot) \in W(\mathbb{R}, \mathbb{C}^n)$ *and are solutions of* (4.2) *satisfying*

$$\|v_k - u_0^{(k)}\|_1 < \rho, \tag{4.6}$$

with $u_0^{(k)}(\cdot)$ *defined by* (3.5)*. Moreover, if* (C2) *is satisfied with* $\rho \in (0, 1/2)$*, then there exists a Lyapunov transformation* $Q(t)$ *with coefficients in* $W(\mathbb{R}, \mathbb{C})$ *such that* $Q^{-1}(\cdot)$ *and* $Q'(\cdot) \in W$ *have coefficients in* $W(\mathbb{R}, \mathbb{C})$ *and* (1.1) *is kinematically similar to the diagonal system* (4.4)*.*

Theorems 1 and 2 construct a basis of solutions (4.1) for (1.1) by studying a set of Riccati type systems (4.2) having a lower dimension. To the best of our knowledge, this idea has been introduced by Bellman in [1], where some properties of a second order linear equation are deduced by studying a scalar nonlinear equation. To study extended results, we refer the reader to [8,24,27].

4.2. Consequences

By using V and W as "space parameters" together with Theorems 1 and 2, we obtain the following results.

Corollary 1. *Suppose that* $A(\cdot)$ *is an almost periodic matrix. If* (3.6)*,* (A) *and* (B1) *are satisfied with* $\rho \in (0, 1/2)$*, then there exists an almost periodic Lyapunov transformation* $Q(t)$ *such that* $Q^{-1}(\cdot)$ *and* $Q'(\cdot)$ *are almost periodic and* (1.1) *is kinematically similar to the almost periodic diagonal system* (4.4)*.*

Proof. We will use Theorem 1 with $W = \mathcal{AP}(\mathbb{R}, \mathbb{C}^n)$, which is scalar invariant (see *e.g.*, Remark 2). We only have to prove that (B2) is satisfied. Indeed, as Proposition 2 says that (3.1) admits an exponential dichotomy and the functions $t \mapsto A_{k+1k}(t)$ are almost periodic, the $(\mathcal{AP}, \mathcal{AP})$ admissibility follows by Theorem 7.7 from [9]. □

Corollary 2. *Suppose that* $A(\cdot)$ *is a pseudo almost periodic matrix with almost periodic diagonal coefficients. If* (A) *and* (C1) *are satisfied with* $\rho \in (0, 1/2)$*, then there exists a pseudo almost periodic Lyapunov transformation* $Q(t)$ *such that* $Q^{-1}(\cdot)$ *and* $Q'(\cdot)$ *are pseudo almost periodic and* (1.1) *is kinematically similar to the pseudo almost periodic diagonal system* (4.4)*.*

Proof. We will use Theorem 2 with $W = \mathcal{PAP}(\mathbb{R}, \mathbb{C}^n)$ and $V = \mathcal{AP}(\mathbb{R}, \mathbb{C}^n)$. As before, we only need to verify that (C2) is satisfied. As Proposition 2 says that the almost periodic systems (3.4) admit an exponential dichotomy and the functions $t \mapsto A_{k+1k}(t)$ are pseudo almost periodic, the $(\mathcal{PAP}, \mathcal{PAP})$ admissibility follows by Theorem 2.3 from [29]. □

5. Proof of Theorems 1 and 2

5.1. Preparatory results

Observe that the basis (4.1) is described in terms of solutions of the systems (4.2), which are nonlinear perturbations of (3.7) and

$$v' = A_k(t)v + A_{k+1k}(t). \tag{5.1}$$

The core of the proof will be to deduce some properties of the solutions of (4.2) in terms of the solutions of (3.7) and (5.1). In order to do that, let us recall that $V(\mathbb{R}, \mathbb{C}^n) \subseteq W(\mathbb{R}, \mathbb{C}^n)$[1] are Banach subspaces of $BC(\mathbb{R}, \mathbb{C}^n)$ and introduce the auxiliary perturbed systems of (2.1):

$$y' = C(t)y + f(t), \tag{5.2}$$

and

$$z' = C(t)z + f(t) + g(t, z), \tag{5.3}$$

where $f(\cdot) \in W$ and either $C(\cdot) \in W \setminus V$ or $C(\cdot) \in V$.

[1] If no ambiguity is possible, they will be respectively denoted by V and W.

Lemma 2. *Assume that the pair (W, W) is admissible for $C(t)$. In addition, for any perturbation $h(\cdot) \in W$ there exists a unique solution y_h in W, such that*

$$\|y_h\|_1 \leq c\|h\|_1 \quad \text{with } c > 0. \tag{5.4}$$

If $f(\cdot) \in W$ and $g: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following properties:

(i) *For any $\phi \in W(\mathbb{R}, \mathbb{C}^n)$, it follows that $t \to g(t, \phi(t)) \in W(\mathbb{R}, \mathbb{C}^n)$.*
(ii) *$g(t, 0) = 0$ for any $t \in \mathbb{R}$.*
(iii) *There exist $\mathcal{L} > 0$ and $\rho > 0$ such that:*

$$|g(t, y) - g(t, w)|_1 \leq \mathcal{L}|y - w|_1 \quad \text{for any } t \in \mathbb{R} \text{ and } y, w \in \Delta(y_f),$$

where $y_f(\cdot) \in W$ is the unique solution of (5.2) and

$$\Delta(y_f) = \Big\{w \in \mathbb{C}^n : |w - y_f(t)|_1 \leq \rho \text{ for any } t \in \mathbb{R}\Big\}.$$

If $\mathcal{L}$, ρ and c satisfy

$$\mathcal{L} < \frac{\rho}{c(\rho + \|y_f\|_1)}, \quad \text{and} \quad \|f\|_1 < \frac{\rho}{c}, \tag{5.5}$$

then there exists a unique solution $y^*(\cdot) \in W$ of (5.3) satisfying $\|y^* - y_f\|_1 < \rho$.

Proof. Note that given $\varphi(\cdot) \in W_{y_f} = \{y \in W(\mathbb{R}, \mathbb{C}^n): \|y - y_f\|_1 < \rho\}$, assumption (i) implies that $t \mapsto g(t, \varphi(t)) \in W$ and $t \to f(t) + g(t, \varphi(t)) \in W$ since W is a Banach space. Now, by using (W, W) admissibility for $C(t)$, the following map $\mathcal{N}: W \to W$ is well defined:

$$(\mathcal{N}\varphi)(t) = y_{f+g(\cdot,\varphi)} = y_f + y_{g(\cdot,\varphi)},$$

where $y_{f+g(\cdot,\varphi)}$ is the unique solution in W of $y' = C(t)y + f(t) + g(t, \varphi(t))$.

By using this definition, let us denote by $\mathcal{N}0 = y_f$ the unique solution in W of (5.2). In addition, (5.4) together with the right inequality of (5.5) implies that $0 \in \Delta(y_f)$ (*i.e.*, $\|y_f\|_1 < \rho$). Note also that the left-hand side of (5.5) is equivalent to

$$c\mathcal{L} \leq \frac{\rho}{\rho + \|y_f\|_1} < 1, \tag{5.6}$$

we shall need these inequalities later.

Now, properties (ii)–(iii) combined with $0 \in \Delta(y_f)$ and (5.4) imply that

$$|(\mathcal{N}\varphi)(t) - (\mathcal{N}0)(t)|_1 = \|y_{f+g(\cdot,\varphi)} - y_f\|_1 \leq c\|g(\cdot, \varphi)\|_1 \leq c\mathcal{L}\|\varphi\|_1.$$

By multiplying (5.6) by $\|y_f\|_1 < \rho$, we can deduce that $\|\mathcal{N}\varphi - y_f\|_1 < \rho$, which implies that $\mathcal{N}: W_{y_f} \to W_{y_f}$ is well defined.

Given $\varphi_1(\cdot)$ and $\varphi_2(\cdot) \in W_{y_f}$, it can be proved that

$$\|\mathcal{N}\varphi_1 - \mathcal{N}\varphi_2\|_1 \leq \|g(\cdot, \varphi_1) - g(\cdot, \varphi_2)\|_1 \leq c\mathcal{L}\|\varphi_1 - \varphi_2\|_1,$$

and (5.6) implies that $\mathcal{N}: W_{y_f} \to W_{y_f}$ is a contractive map. Finally, by the Banach contraction principle, it follows that $\mathcal{N}$ has a unique fixed point $y^*(\cdot) \in W_{y_f}$, which is a solution of (5.3) and the lemma follows. □

Corollary 3. *Assume that (2.1) has a (σ, K, P)-exponential dichotomy on $\mathbb{R}$ and the pair (W, W) is admissible for $C(t)$.*

If $f(\cdot) \in W$ and $g: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following properties:

(i) *For any $\phi \in W(\mathbb{R}, \mathbb{C}^n)$, it follows that $t \to g(t, \phi(t)) \in W(\mathbb{R}, \mathbb{C}^n)$.*
(ii) *$g(t, 0) = 0$ for any $t \in \mathbb{R}$.*
(iii) *There exist $\mathcal{L} > 0$ and $\rho > 0$ such that*

$$|g(t, y) - g(t, w)|_1 \leq \mathcal{L}|y - w|_1 \quad \text{for any } t \in \mathbb{R} \text{ and } y, w \in \Delta(y_f),$$

where $y_f(\cdot) \in W$ is the unique solution of (5.2) and

$$\Delta(y_f) = \Big\{w \in \mathbb{C}^n : |w - y_f(t)|_1 \leq \rho \text{ for any } t \in \mathbb{R}\Big\}.$$

If $\mathcal{L}$, σ, ρ and K satisfy

$$\mathcal{L} < \frac{\sigma\rho}{2K(\rho + \|y_f\|_1)}, \quad \text{and} \quad \|f\|_1 < \frac{\sigma\rho}{2K}, \tag{5.7}$$

then there exists a unique solution $z(\cdot) \in W_{y_f}$ of (5.3).

Proof. By exponential dichotomy together with (2.4) and Proposition 1, it follows that (5.4) is verified with $c = 2K/\sigma$. □

5.2. Proof of Theorem 1

The proof has four steps: firstly, it can be proved (step 1) by elementary (but involved and articulated) computations that the functions $x_k(\cdot)$ defined in (4.1) are solutions of the system (1.1). Secondly, we shall prove (step 2) that (4.2) satisfy the properties stated in Corollary 3 for any $k = 1, \ldots, n$ mod n. In the third step, we will prove that the set of functions $x_k(\cdot)$ is linearly independent. Finally, we will derive the similarity to a diagonal system in the fourth step.

Step 1: The following lemma has been proved in [7].

Lemma 3. *The functions $\{x_k(t)\}_{k=1}^n$ defined by (4.1) are solutions of (1.1).*

Step 2: By Proposition 2, the systems (3.7) admit an $(\tilde{\alpha}, M, Q_k)$-exponential dichotomy. In addition, $A_{k+1k}(t) \in W$ and (B2) ensures (W, W) admissibility for $A_k(t)$. Now, let us consider the function

$$g(t, u) = -\langle \bar{A}_{k+1k}(t), u\rangle u.$$

Notice that $g(\cdot, \cdot)$ satisfies property (i) since $W \subset BC(\mathbb{R}, \mathbb{C}^n)$ is scalar invariant. The property (ii) is satisfied straightforwardly. In order to verify the property (iii), let us define the sets:

$$\Delta(x_0^{(k)}) = \left\{ v \in \mathbb{C}^{n-1} \colon |v - x_0^{(k)}(t)|_1 \le \rho \text{ for any } t \in \mathbb{R} \right\},$$

where $x_0^{(k)}(\cdot)$ are defined by (3.9).

By using Hölder inequality together with

$$\langle \bar{A}_{kk+1}, u_1\rangle u_1 - \langle \bar{A}_{kk+1}, u_2\rangle u_2 = \langle \bar{A}_{kk+1}, u_1\rangle (u_1 - u_2) + \langle \bar{A}_{kk+1}, u_1 - u_2\rangle u_2,$$

it follows that $g(t, u)$ satisfies

$$|g(t, u_1) - g(t, u_2)|_1 \le 2\|A_{kk+1}\|_\infty (\rho + \|x_0^{(k)}\|_1)|u_1 - u_2|_1$$

for any $u_i \in \Delta(x_0^{(k)})$. Hence, (iii) of Corollary 3 is satisfied with the following constant:

$$\mathcal{L}_k = 2\|A_{k+1k}\|_\infty (\rho + \|x_0^{(k)}\|_1).$$

Notice that (B1) implies that inequality (5.7) of Corollary 3 is satisfied. Hence, for every $k \in \{1, \ldots, n\}$ the systems (4.2) have a unique solution $v_k(\cdot) \in W$ satisfying (4.3). In addition, by using Proposition 2 together with (3.9) and the right inequality of (B1), we can deduce that

$$\|v_k\|_1 \le \rho + \|x_0^{(k)}\|_1 \le \rho + \frac{2M}{\tilde{\alpha}}\|A_{k+1k}\|_1 < 2\rho. \tag{5.8}$$

It will be useful later to observe that (5.8) and (B1) imply that

$$\|A_{ii+1}\|_\infty \|v_i\|_1 \le \|A_{ii+1}\|_\infty (\rho + \|x_0^{(i)}\|_1) = \frac{\mathcal{L}_i}{2} < \frac{\tilde{\alpha}\rho}{4M(\|x_0^{(i)}\| + \rho)} < \frac{\tilde{\alpha}}{4M} \tag{5.9}$$

for any $i = 1, \ldots, n$ (mod n).

Step 3: Finally, we will prove that $\{x_i(t)\}_{i=1}^n$ is linearly independent; this will be made by contradiction: assume that there exist scalars $\lambda_i \neq 0$ such that

$$\lambda_1 x_1(t) + \cdots + \lambda_n x_n(t) = 0.$$

We can suppose that $\lambda_1 \neq 0$, by using (4.1) we have

$$e_1 + L_1 v_1(t) = -\sum_{j=2}^{n} \frac{\lambda_j}{\lambda_1}(e_j + L_j v_j(t))e^{G_j(t)}, \tag{5.10}$$

with $G_j \colon \mathbb{R} \to \mathbb{R}$ defined as follows:

$$G_j(t) = \int_0^t \{a_{jj}(s) - a_{11}(s)\}\, ds + \int_0^t \{\langle \bar{A}_{jj+1}(s), v_j(s)\rangle - \langle \bar{A}_{12}(s), v_1(s)\rangle\}\, ds.$$

Notice that Hölder inequality implies that

$$|\langle \bar{A}_{jj+1}(s), v_j(s)\rangle - \langle \bar{A}_{12}(s), v_1(s)\rangle| \le \|A_{jj+1}\|_\infty \|v_j\|_1 + \|A_{12}\|_\infty \|v_1\|_1. \tag{5.11}$$

Now, by using (5.9) combined with (5.11) and the inequalities $\tilde{\alpha} < \alpha$ and $M > 1$ (see Eq. (3.8) for details), we can deduce the estimation:

$$|\langle \bar{A}_{jj+1}(s), v_j(s)\rangle - \langle \bar{A}_{12}(s), v_1(s)\rangle| < \tilde{\alpha}/(2M) < \alpha/2 \quad \text{for any } s \in \mathbb{R}. \tag{5.12}$$

If $t > 0$, the assumption (A) implies that either

$$\Re \int_0^t \{a_{jj}(s) - a_{11}(s)\}\, ds > -\beta + \alpha t \quad \text{or} \quad \Re \int_0^t \{a_{jj}(s) - a_{11}(s)\}\, ds < \beta - \alpha t,$$

which allow us to define the sets:

$$\Gamma_+ = \left\{ 2 \le i \le n \colon \Re \int_0^t \{a_{ii}(s) - a_{11}(s)\}\, ds > -\beta + \alpha t \right\},$$

and

$$\Gamma_- = \left\{ 2 \le i \le n \colon \Re \int_0^t \{a_{ii}(s) - a_{11}(s)\}\, ds < \beta - \alpha t \right\}.$$

By using this fact combined with (5.12) it follows that either (for $t \ge 0$):

$$\Re G_j(t) > -\beta + \frac{\alpha}{2} t \quad \text{if } j \in \Gamma_+ \quad \text{or} \quad \Re G_j(t) < \beta - \frac{\alpha}{2} t \quad \text{if } j \in \Gamma_-.$$

Hence, (5.10) can be written as follows:

$$e_1 + L_1 v_1(t) + \sum_{j \in \Gamma_-} \frac{\lambda_j}{\lambda_1} (e_j + L_j v_j(t)) e^{G_j(t)} = - \sum_{j \in \Gamma_+} \frac{\lambda_j}{\lambda_1} (e_j + L_j v_j(t)) e^{G_j(t)}.$$

By using boundedness of $v_1(\cdot)$ and definition of Γ_- and Γ_+, it follows that the left-hand side is bounded on $\mathbb{R}_+$ whereas the right-hand side is unbounded, obtaining a contradiction. This demonstrates linear independence.

Step 4: By denoting the solutions of (4.2) as $v_k(\cdot) = \operatorname{col}(v_{1k}, v_{2k}, \ldots, v_{n-1k})$, we have obtained that (1.1) has a fundamental matrix:

$$X(t) = [I_n + P(t)] e^{D(t)}, \tag{5.13}$$

with $P(t) = \{p_{ij}(t)\}$ and $D(t) = \operatorname{Diag}\{d_i(t)\}$ defined by

$$d_i(t) = \int_0^t \phi_i(s)\, ds \quad \text{and} \quad p_{ij}(t) = \begin{cases} 0 & \text{if } i = j, \\ v_{ij}(t) & \text{if } i = 1, \ldots, j-1, \\ v_{i-1j}(t) & \text{if } i = j+1, \ldots, n. \end{cases}$$

Let us define $Q(t) = I + P(t)$. Now notice that $Q(\cdot)$ has coefficients in $W(\mathbb{R}, \mathbb{C})$ since $v_i(\cdot) \in W(\mathbb{R}, \mathbb{C}^n)$. Furthermore, $Q(\cdot)$ is an invertible matrix. Indeed, from (5.8) combined with $\rho \in (0, 1/2)$ implies that $\|v_k\|_1 < 1$.

As

$$|P(t)|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |p_{ij}(t)| = \max_{1 \le j \le n} |v_j(t)|_1,$$

it follows that $\|P\|_1 < 1$ and $Q(t) = I + P(t)$ is invertible with inverse:

$$Q^{-1}(t) = [I + P(t)]^{-1} = \sum_{j=0}^{\infty} (-1)^j P^j(t),$$

and we can conclude that $Q^{-1}(\cdot)$ has coefficients in $W(\mathbb{R}, \mathbb{C})$ since $W(\mathbb{R}, \mathbb{C}^n)$ is scalar invariant. Now, we obtain

$$Q'(t) = A(t)Q(t) - Q(t)D'(t) = A(t)Q(t) - Q(t)\Phi(t),$$

and it follows that $Q'(\cdot)$ has coefficients in $W(\mathbb{R}, \mathbb{C})$ and $Q(\cdot)$ is a Lyapunov transformation. Finally, it is straightforward to verify that $u = Q^{-1}x$ transforms (1.1) in (4.4). □

5.3. *Proof of Theorem 2*

The proof will follow the same outline than the previous one.

Step 1: By using Lemma 3, it follows that the functions $x_k(\cdot)$ defined in (4.1) are solutions of the system (1.1).

Step 2: We shall prove that (4.2) satisfy the properties stated in Corollary 3 for any $k = 1, \ldots, n \bmod n$. In this case, by using (3.1), these systems will be written as follows:

$$v' = D_k(t)v + A_{k+1k}(t) + N_k(t)v - \langle \bar{A}_{kk+1}(t), v \rangle v \quad k = 1, \ldots, n \bmod n.$$

These systems satisfy the properties of Corollary 3. Indeed, by Lemma 1, the systems (3.4) admit an (α, e^{β}, P_k)-exponential dichotomy. Furthermore, $D_k(t) \in V$ and (C2) implies that (W, W) is admissible for $D_k(t)$.

Now, let us consider the function

$$g(t,u) = N_k(t)u - \langle A_{k+1k}(t), u\rangle u$$

and observe that $g(\cdot,\cdot)$ satisfies property (i) since $W \subset BC(\mathbb{R},\mathbb{C}^n)$ is scalar invariant. The property (ii) is satisfied straightforwardly.

As before, by the Hölder inequality it can be proved that given two vectors $u_i \in \Delta(u_0^{(k)})$, the function $g(t,u)$ satisfies

$$|g(t,u_1) - g(t,u_2)|_1 \le (\|N_k\|_1 + 2\|A_{kk+1}\|_\infty(\rho + \|u_0^{(k)}\|_1))|u_1 - u_2|_1,$$

and (iii) of Corollary 3 is satisfied with $\widehat{\mathcal{L}} = \|N_k\|_1 + 2\|A_{k+1k}\|_\infty(\rho + \|u_0^{(k)}\|_1)$.

Notice that (C1) implies that inequality (5.7) of Corollary 3 is satisfied. Hence, for every $k \in \{1,\dots,n\}$ the systems (4.2) have a unique solution $v_k(\cdot) \in W$ satisfying (4.3). Moreover, we point out that by using Lemma 1 together with (3.5) and the right inequality of (C1), we can deduce that

$$\|v_k\|_1 \le \rho + \|u_0^{(k)}\|_1 < \rho + \frac{2e^\beta}{\alpha}\|A_{k+1k}\|_1 < 2\rho. \tag{5.14}$$

It will be useful later to observe that (5.14) and (C1) imply that

$$\|A_{ii+1}\|_\infty\|v_i\|_1 \le \|A_{ii+1}\|_\infty(\rho + \|u_0^{(i)}\|_1) < \frac{\hat{\mathcal{L}}_i}{2} < \frac{\alpha\rho}{4e^\beta(\|u_0^{(i)}\| + \rho)} < \frac{\alpha}{4e^\beta} \tag{5.15}$$

for any $i = 1,\dots,n \pmod n$.

Steps 3 *and* 4: The proof of the linear independence and similarity is almost a verbatim of the previous one and we left it for the reader. □

6. The case of ergodic coefficients

A function $f(\cdot) \in BC(\mathbb{R},\mathbb{C})$ is said to be *ergodic* (see [30, Chapter 3]) if it has average, *i.e.*, if the following limit exists:

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} f(r)\,dr = \mathcal{M}\{f\}, \tag{6.1}$$

a subspace of $BC(\mathbb{R},\mathbb{C})$ will be called ergodic if its elements are ergodic. The almost periodic and pseudo almost periodic functions are well known ergodic spaces.

When the diagonal terms of $A(\cdot)$ are ergodic, we can obtain more results or reinterpret the previous ones. Throughout this section, it is assumed that

(A′) there exists $\hat{\alpha} > 0$ such that

$$\left|\mathcal{M}\{\Re(a_{ii} - a_{jj})\}\right| \ge \hat{\alpha} \quad \text{for any } i \ne j.$$

Remark 3. It is an easy exercise to verify that assumption (A′) is equivalent to (A) with $\alpha \in (0,\hat{\alpha})$ and $\beta(\alpha) \ge 0$.

Remark 4. A direct consequence of (A′) is that the diagonal systems

$$u' = \mathcal{M}\{D_k\}u, \quad k = 1,\dots,n, \tag{6.2}$$

have eigenvalues outside the strip $|\Re z| \le \hat{\alpha}$. Hence, admit an $(\hat{\alpha}, 1, P_k)$-exponential dichotomy, where $P_k = \mathrm{Diag}\{p_{ii}^k\}$ defined by

$$p_{ii}^k = \begin{cases} 1 & \text{if } \mathcal{M}\{a_{ii}(r) - a_{kk}(r)\} \le -\hat{\alpha}, \\ 0 & \text{if } \mathcal{M}\{a_{ii}(r) - a_{kk}(r)\} \ge \hat{\alpha}. \end{cases}$$

When the off-diagonal terms are in a Banach space $W(\mathbb{R},\mathbb{C}) \subset BC(\mathbb{R},\mathbb{C})$ and the diagonal terms are in an ergodic Banach space $V(\mathbb{R},\mathbb{C}) \subseteq W(\mathbb{R},\mathbb{C})$, we can obtain more results by following the lines of Theorem 2. Firstly, let us introduce the corresponding properties for $A(t)$:

(C1′) Let $\rho > 0$, such that the off-diagonal coefficients of $A(t)$ satisfy

$$\|A_{kk+1}\|_\infty + \frac{\|A_k - \mathcal{M}\{D_k\}\|_1}{2(\|p_0^{(k)}\|_1 + \rho)} < \frac{\hat{\alpha}\rho}{4(\|p_0^{(k)}\|_1 + \rho)^2} \quad \text{and} \quad \|A_{k+1k}\|_1 < \frac{\hat{\alpha}\rho}{2},$$

where

$$p_0^{(k)}(t) = \int_{-\infty}^{\infty} G_{e^{\mathcal{M}\{D_k\}}}(t,s)A_{k+1k}(s)\,ds, \quad \text{with } k = 1,\dots,n.$$

(C2′) The pair (W,W) is admissible for $\mathcal{M}\{D_k\}$ (with $k = 1,\dots,n$).

Theorem 3. *Suppose that $A(t)$ is a matrix with diagonal coefficients $a_{ii}(\cdot) \in V(\mathbb{R}, \mathbb{C}^n)$ and off-diagonal coefficients in a scalar invariant Banach space $W(\mathbb{R}, \mathbb{C}^n) \subset BC(\mathbb{R}, \mathbb{C}^n)$. If (A′), (C1′) and (C2′) are satisfied, then (1.1) has a basis of solutions described by (4.1), where $v_k(\cdot) \in W$ are solutions of (4.2) verifying*

$$\|v_k - p_0^{(k)}\|_1 < \rho. \tag{6.3}$$

In addition, if (C1′) is satisfied with $\rho \in (0, 1/2)$, then there exists a Lyapunov transformation $Q(t) \in W$ with $Q^{-1}(\cdot) \in W$ and $Q'(\cdot) \in W$ such that (1.1) is kinematically similar to the diagonal system (4.4) with coefficients in W.

Proof. As in the Proof of Theorem 2, we will follow four steps: firstly, by Lemma 3 we know that the functions described in (4.1) are solutions of (1.1).

Secondly, notice that systems (4.2) can be written as follows:

$$v' = \mathcal{M}\{D_k\}v + A_{k+1k}(t) + [A_k(t) - \mathcal{M}\{D_k\}]v - \langle \bar{A}_{kk+1}(t), v\rangle v \tag{6.4}$$

and we will prove that (6.4) satisfies assumptions of Corollary 3.

Now, let us consider the function

$$g(t, u) = [A_k(t) - \mathcal{M}\{D_k\}]u - \langle A_{k+1k}(t), u\rangle u.$$

In the context of Corollary 3, observe that $g(\cdot, \cdot)$ satisfies property (i) since W is scalar invariant and (ii) is satisfied straightforwardly. As before, by the Hölder inequality it can be proved that given two vectors $u_i \in \Delta(p_0^{(k)})$, the function $g(t, u)$ satisfies $|g(t, u_1) - g(t, u_2)|_1 \le (\|A_k - \mathcal{M}\{D_k\}\|_1 + 2\|A_{kk+1}\|_\infty(\rho + \|p_0^{(k)}\|_1))|u_1 - u_2|_1$ and (iii) of Corollary 3 is satisfied with $\widetilde{\mathcal{L}} = \|A_k - \mathcal{M}\{D_k\}\|_1 + 2\|A_{k+1k}\|_\infty(\rho + \|p_0^{(k)}\|_1)$. In consequence, we have that $v_k(\cdot) \in W$ and (6.3) is satisfied.

Finally, we have to verify that the set (4.1) is linearly independent and kinematically similar to (4.4). Its proof is almost a verbatim of the proof presented before. □

As a consequence, we have the following corollaries.

Corollary 4. *Suppose that $A(\cdot)$ is an almost periodic matrix. If (A′) and (C1′) hold with $\rho \in (0, 1/2)$, then there exists an almost periodic Lyapunov transformation $Q(t)$ such that $Q^{-1}(\cdot)$ and $Q'(\cdot)$ are almost periodic and (1.1) is kinematically similar to the almost periodic diagonal system (4.4).*

Proof. We will use Theorem 3 with $V = W = \mathcal{AP}(\mathbb{R}, \mathbb{C}^n)$, which is ergodic and scalar invariant (see *e.g.*, Remark 2). It remains to prove that (C2′) is satisfied. Indeed, as the diagonal constant systems (6.2) admit an exponential dichotomy, the $(\mathcal{AP}, \mathcal{AP})$ admissibility follows by Theorem 5.11 from [9]. □

Similarly, we obtain the following corollary.

Corollary 5. *Suppose that $A(\cdot)$ is a pseudo almost periodic matrix. If (A′) and (C1′) hold with $\rho \in (0, 1/2)$, then there exists a pseudo almost periodic Lyapunov transformation $Q(t)$ such that $Q^{-1}(\cdot)$ and $Q'(\cdot)$ are pseudo almost periodic and (1.1) is kinematically similar to the pseudo almost periodic diagonal system (4.4).*

Proof. We will use Theorem 3 with $V = W = \mathcal{PAP}(\mathbb{R}, \mathbb{C}^n)$, which is ergodic and scalar invariant (see *e.g.*, Remark 2). It remains to prove that (C2′) is satisfied. Indeed, as the diagonal constant systems (6.2) admit an exponential dichotomy, the $(\mathcal{PAP}, \mathcal{PAP})$ admissibility follows by Theorem 2.1 from [28]. □

Corollary 6. *Suppose that $A(\cdot)$ is an almost automorphic matrix with almost periodic diagonal coefficients. If (A′) and (C1′) hold with $\rho \in (0, 1/2)$, then there exists an almost automorphic Lyapunov transformation $Q(t)$ such that $Q^{-1}(\cdot)$ and $Q'(\cdot)$ are almost automorphic and (1.1) is kinematically similar to the almost automorphic diagonal system (4.4).*

Proof. We use Theorem 3 with $V = \mathcal{AP}(\mathbb{R}, \mathbb{C}^n)$ and $W = \mathcal{AA}(\mathbb{R}, \mathbb{C}^n)$. To finish the proof, we have to verify that (C2′) is satisfied. Indeed, let us consider the equation:

$$x' = \mathcal{M}\{a_{ii} - a_{kk}\}x + h(t), \quad \text{with } \mathcal{M}\{a_{ii} - a_{kk}\} = c \tag{6.5}$$

where h is an almost automorphic function and $\operatorname{Re} c \neq 0$.

We only consider the case $\operatorname{Re} c > 0$ (the other one is left to the reader) and it can be verified that the unique bounded solution is

$$x_0(t) = -\int_t^\infty e^{c(t-s)}h(s)\, ds. \tag{6.6}$$

Let us consider a sequence $\{\tilde{r}_n\}$. By using the fact that $h(\cdot) \in \mathcal{AA}$, there exists a subsequence $\{r_n\} \subseteq \{\tilde{r}_n\}$ such that

$$\lim_{n\to+\infty} h(s + r_n) = \phi(s) \quad \text{and} \quad \lim_{n\to+\infty} \phi(s - r_n) = h(s). \tag{6.7}$$

Observe that

$$x_0(t+r_n) = -\int_{t+r_n}^{\infty} e^{c(t+r_n-s)}h(s)\,ds = -\int_0^{\infty} e^{-cz}h(z+t+r_n)\,dz,$$

and as (6.7) implies that $e^{-cz}h(z+t+r_n) \to e^{-cz}\phi(z+t)$ when $n \to +\infty$, Lebesgue's dominated convergence theorem implies that

$$\lim_{n\to+\infty} x_0(t+r_n) = -\int_t^{\infty} e^{c(t-z)}\phi(z)\,dz = q(t).$$

Now, notice that

$$q(t-r_n) = -\int_{t-r_n}^{\infty} e^{c(t-r_n-z)}\phi(z)\,dz = -\int_0^{\infty} e^{-cs}\phi(s+t-r_n)\,ds$$

and by using again dominated convergence theorem and (6.7), it follows that

$$\lim_{n\to+\infty} q(t-r_n) = -\int_t^{\infty} e^{c(t-s)}h(s)\,ds = x_0(t).$$

In consequence $x_0(\cdot) \in \mathcal{AA}$ since, given a sequence $\{\tilde{r}_n\}$, there exists a subsequence $\{r_n\} \subseteq \{\tilde{r}_n\}$ satisfying (6.7) and

$$\lim_{n\to+\infty} x_0(t+r_n) = q(t) \quad \text{and} \quad \lim_{n\to+\infty} q(t-r_n) = x_0(t). \quad \square \tag{6.8}$$

7. Numerical examples

Example 1. Let us consider the system:

$$x' = \begin{bmatrix} a+\delta_1\sin(\sqrt{2}t) & 0 \\ c(t) & \delta_2\cos(t) \end{bmatrix} x \tag{7.1}$$

where a, δ_1 and δ_2 are positive constants and $c(\cdot) \in W = W(\mathbb{R},\mathbb{C})$ scalar invariant subspace, such that $W(\mathbb{R},\mathbb{C}) \subset BC(\mathbb{R},\mathbb{C})$. In addition, the pair (W,W) is admissible for $a > 0$.

Notice that (A$'$) is satisfied with $\hat{\alpha} \in (0,a)$ since

$$\mathcal{M}\{a_{11}-a_{22}\} = a > \hat{\alpha}. \tag{7.2}$$

The assumption (C1$'$) is satisfied if δ_1 and δ_2 are small enough such that

$$\sup_{-\infty<t<+\infty} |\delta_2\cos(t) - \delta_1\sin(\sqrt{2}t)| < \frac{\hat{\alpha}\rho}{2(\|p_0^{(1)}\|_\infty + \rho)} < \frac{\hat{\alpha}}{2}, \tag{7.3}$$

with $p_0^{(1)}(\cdot) \in W$ defined by

$$p_0^{(1)}(t) = \int_{-\infty}^{t} e^{-a(t-s)}c(s)\,ds,$$

and $c(\cdot)$ satisfies

$$\|c\|_\infty < \frac{1}{2\rho}\min\left\{\hat{\alpha}\rho^2, \frac{\hat{\alpha}}{2} - \sup_{-\infty<t<\infty}|\delta_2\cos(t)-\delta_1\sin(\sqrt{2}t)|\right\}. \tag{7.4}$$

Theorem 3 says that (7.1) has a Cauchy matrix:

$$X(t) = \begin{bmatrix} 1 & 0 \\ v_1^*(t) & 1 \end{bmatrix} \begin{bmatrix} e^{at+\frac{\delta_1(1-\cos(\sqrt{2}t))}{\sqrt{2}}} & 0 \\ 0 & e^{\delta_2\sin(t)} \end{bmatrix},$$

where

$$v_1^*(t) = \int_{-\infty}^{t} e^{\int_s^t \varphi(r)\,dr} c(s)\,ds \quad \text{with } \varphi(t) = -a - \delta_1\sin(\sqrt{2}t) + \delta_2\cos(t)$$

is the unique solution in W satisfying $\|v_1^* - p_0^{(1)}\|_\infty < \rho$ of

$$v_1' = -av_1 + c(t) + \{\delta_2\cos(t) - \delta_1\sin(\sqrt{2}t)\}v_1. \tag{7.5}$$

In addition, by following the lines of the Proof of Lemma 2, we can prove that

$$v_2' = \{a + \delta_1 \sin(\sqrt{2}t) - \delta_2 \cos(t)\}v_2 - c(t)v_2^2 \tag{7.6}$$

has $v_2^*(t) = 0$ as the unique bounded solution, satisfying $\|v_2^*\|_\infty < \rho$.

In addition, if $0 < \rho < 1/2$, the system is kinematically similar to

$$u' = \begin{bmatrix} a + \delta_1 \sin(\sqrt{2}t) & 0 \\ 0 & \delta_2 \cos(t) \end{bmatrix} u. \tag{7.7}$$

Example 2. Now, let us consider (7.1), with $c(\cdot) \in \mathcal{PAP}(\mathbb{R}, \mathbb{C})$ defined by

$$c(t) = \varepsilon\left\{\delta_2 \cos(t) - \delta_1 \sin(\sqrt{2}t) - a + r(t)\right\} \quad \text{with } \mathcal{M}\{|r|\} = 0.$$

Notice that by choosing $\delta_1 > 0$, $\delta_2 > 0$ such that

$$a > \delta_1 + \delta_2, \tag{7.8}$$

then (A) is satisfied with $\beta = 0$ and $\alpha > a - (\delta_1 + \delta_2)$.

Moreover, (C1) is satisfied by choosing $\varepsilon > 0$ such that

$$\varepsilon \sup_{-\infty < t < +\infty} |\delta_2 \cos(t) - \delta_1 \sin(\sqrt{2}t) - a + r(t)| < \frac{\alpha\rho}{2} \min\left\{1, \frac{1}{2\rho^2}\right\}. \tag{7.9}$$

Then, Corollary 2 says that (7.1) has a Cauchy matrix:

$$X(t) = \begin{bmatrix} 1 & 0 \\ v^*(t) & 1 \end{bmatrix} \begin{bmatrix} e^{at + \frac{\delta_1(1-\cos(\sqrt{2}t))}{\sqrt{2}}} & 0 \\ 0 & e^{\delta_2 \sin(t)} \end{bmatrix},$$

where v^* is the solution of (7.5) defined by

$$v^*(t) = \varepsilon\left\{e^{\int_{-\infty}^t \varphi(\tau)\,d\tau} - 1 + \int_{-\infty}^t e^{\int_t^s \varphi(\tau)\,d\tau} r(s)\,ds\right\}$$

and $\varphi(t)$ is defined as in the previous example.

Finally, if $0 < \rho < 1/2$, then the system is kinematically similar to (7.7).

References

[1] R. Bellman, Stability Theory of Differential Equations, McGraw-Hill, New York, 1953.
[2] S. Bodine, A dynamical system result on asymptotic integration of linear differential systems, J. Differential Equations 187 (2003) 1–22.
[3] S. Bodine, R.J. Sacker, A new approach to asymptotic diagonalization of linear differential systems, J. Dynam. Differential Equations 12 (2000) 229–245.
[4] B.F. Bylov, On the reduction of a system of linear equations to diagonal form, Amer. Math. Soc. Transl. 89 (1970) 51–59.
[5] W.A. Coppel, Dichotomies in Stability Theory, in: Lecture Notes in Mathematics, vol. 629, Springer-Verlag, Heidelberg, 1978.
[6] T. Diagana, Pseudo Almost Periodic Functions in Banach Spaces, Nova Science Publishers, New York, 2008.
[7] P. Figueroa, Asymptotic behavior of Poincare's type differential equations of second and third order, M.Sc. Thesis, 2003 (in Spanish).
[8] P. Figueroa, M. Pinto, Riccati equations and nonoscillatory solutions of third order differential equations, Dynam. Systems Appl. 17 (2008) 459–475.
[9] A.M. Fink, Almost Periodic Differential Equations, in: Lecture Notes in Mathematics, vol. 377, Springer-Verlag, Heidelberg, 1974.
[10] P. Hartman, Ordinary Differential Equations, SIAM, 2002.
[11] Y. Hino, T. Naito, N. Van Minh, J.S. Shin, Almost Periodic Solutions of Differential Equations in Banach Spaces, Taylor and Francis, London, 2002.
[12] P.F. Hsieh, F. Xie, On asymptotic diagonalization of linear ordinary differential equations, Dyn. Contin. Discrete Impuls. Syst. 4 (1998) 351–377.
[13] P.E. Kloeden, H.M. Rodrigues, Dynamics of a class of ODEs more general than almost periodic, Nonlinear Anal. 74 (2011) 2695–2719.
[14] J.C. Lillo, Approximate similarity and almost periodic matrices, Proc. Amer. Math. Soc. 12 (1961) 400–407.
[15] L. Markus, Continuous matrices and the stability of differential systems, Math. Z. 62 (1955) 310–319.
[16] J.L. Massera, J.J. Schäffer, Linear Differential Equations and Function Spaces, Academic Press, New York, 1966.
[17] V.M. Millionshchikov, Linear systems of ordinary differential equations, Differential Equations 7 (1971) 297–299.
[18] V.M. Millionshchikov, Lyapunov Transformation, in: Encyclopaedia of Mathematics, Kluwer Academic Publishers, 1997.
[19] S. Murakami, T. Naito, N. Van Minh, Evolution semigroups and sums of commuting operators: a new approach to the admissibility theory of function spaces, J. Differential Equations 164 (2000) 240–285.
[20] T. Naito, N. Van Minh, Evolution semigroups and spectral criteria for almost periodic solutions of periodic evolution equations, J. Differential Equations 152 (1999) 358–376.
[21] T. Naito, N. Van Minh, J.S. Shin, New spectral criteria for almost periodic solutions of evolution equations, Studia Math. 145 (2001) 97–111.
[22] G.M. N'Guérékata, Topics in Almost Automorphy, Springer, New York, 2005.
[23] K.J. Palmer, Exponential dichotomy, integral separation and diagonalizability of linear systems of ordinary differential equations, J. Differential Equations 43 (1982) 184–203.
[24] M. Pinto, A generalization of Poincaré's theorem for recurrence systems, Appl. Anal. 85 (2006) 373–381.
[25] M. Pinto, G. Robledo, Cauchy matrix for linear almost periodic systems and some consequences, Nonlinear Anal. 74 (2011) 5426–5439.
[26] R.J. Sacker, G.R. Sell, A spectral theory for linear differential systems, J. Differential Equations 27 (1978) 320–358.
[27] S. Trofimchuk, M. Pinto, L^p—perturbations of invariant subbundles for linear systems, J. Dynam. Differential Equations 14 (2002) 743–761.
[28] C. Zhang, Pseudo almost periodic solutions of some differential equations, J. Math. Anal. Appl. 181 (1994) 62–76.
[29] C. Zhang, Pseudo almost periodic solutions of some differential equations II, J. Math. Anal. Appl. 192 (1995) 543–561.
[30] C. Zhang, Almost Periodic Type Functions and Ergodicity, Science Press, Beijing, 2003. Kluwer Academic Publishers, Dordrecht.