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# Regularity of mild solutions for a class of fractional order differential equations

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ABSTRACT

In this article we show sufficient conditions ensuring the existence and uniqueness of a mild solution to the equation

$$D^{\alpha}u(t) = Au(t) + D^{\alpha-1}f(t,u(t)), \quad 1 < \alpha \leqslant 2, \quad t \in \mathbb{R}, \tag{$*$}$$

in the same space where $f$ belongs. Here $A$ is a sectorial operator defined in a Banach space $X$, $D^{\alpha}$ is the fractional derivative in the Riemann–Liouville sense and $f(\cdot,x) \in \Omega(X)$ for each $x \in X$, where $\Omega(X)$ is a vector-valued subspace of the space of continuous and bounded functions. The subspaces $\Omega(X)$ that we will consider in this article are the space of periodic, almost periodic, almost automorphic and compact almost automorphic vector-valued functions, among others. In particular, we extend and unify recent results established for the equation $(*)$ in the papers Agarwal et al. (2010), Cuevas et al. (2010) and Cuevas and Lizama (2008).



## 1. Introduction

Differential equations involving fractional derivatives have been used to describe a large number of natural phenomena in different areas such as, engineering, physics, economy, and science. For this reason these equations have been studied for many authors, including [4,13,16,22–24,33,36], among others. In particular, the study of abstract semilinear fractional differential equations is of great interest. Some of this papers, studied the existence and uniqueness of solutions with a prescribed qualitative property. For example, in [1,9,10], sufficient conditions have been found for the existence and uniqueness of mild solutions to the equation

$$D^{\alpha}u(t) = Au(t) + D^{\alpha-1}f(t,u(t)),\ t \in \mathbb{R},\ 1 < \alpha \leqslant 2, \tag{1.1}$$

in the vector-valued spaces that consist of almost automorphic, pseudo automorphic or pseudo almost periodic functions, respectively. In all of the above mentioned papers, $A : D(A) \subset X \to X$ is a closed operator of sectorial type $\mu < 0$ with angle $0 \leqslant \theta < \pi(1-\alpha/2)$, and $f : \mathbb{R} \times X \to X$ satisfies a suitable Lipschitz condition. The fractional derivative is understood in the Riemann–Liouville sense [23].

Nevertheless, to the best of our knowledge, existence and uniqueness results of mild solutions for Eq. (1.1) on vector-valued spaces that consist of periodic, pseudo periodic, compact almost automorphic, as well as asymptotic behavior have not been studied in the literature.

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In this article, we attempt to close this gap by means of an unified approach. We will show sufficient conditions to ensure the existence and uniqueness of mild solutions for the abstract semilinear fractional differential Eq. (1.1) in the following classes of vector-valued function spaces: periodic, asymptotically periodic, pseudo periodic, almost periodic, asymptotically almost periodic, pseudo almost periodic, almost automorphic, asymptotically almost automorphic, pseudo almost automorphic, compact almost automorphic, asymptotically compact almost automorphic, pseudo compact almost automorphic, $S$-asymptotically $\omega$-periodic functions, decay functions and mean decay functions. Thus unifying and extending the results appearing in [1,9,10] among others.

For our purpose, we will use some common properties for all the above mentioned categories of vector-valued function spaces (see [27]). They will be summarized in the second section of this paper, together with the definitions and main results that will be used later. In the third section, we extend and unify a key composition theorem appearing in [27, Theorem 4.1] considering, instead of a Lipschitz type condition on the semilinear term $f$, more general and relaxed hypotheses and, consequently, enlarging the number of applications. Then, we will show several types of sufficient conditions on the data $f$ and $A$ to ensure the existence and uniqueness of mild solutions to Eq. (1.1) in each one of the preceding vector-valued function spaces. The proofs of these results will be based on fixed point techniques. We finish this paper with some examples illustrating the feasibility of the abstract results.

## 2. Preliminaries

Let $X$ be a Banach space. We denote

$$BC(X) := \{f : \mathbb{R} \to X : f \text{ is continuous}, \|f\|_\infty := \sup_{t \in \mathbb{R}} \|f(t)\| < \infty\}.$$

Let $P_\omega(X)$ be the space of all vector-valued $\omega$-periodic functions. For the space of almost periodic functions (in the sense of Bohr), we set $AP(X)$ which consists of all functions $f \in BC(X)$ such that for each $\epsilon > 0$ there exists a $\omega > 0$ such that every subinterval of $\mathbb{R}$ of length $\omega$ contains at least one point $\tau$ such that $\|f(t+\tau) - f(t)\|_\infty < \epsilon$. This definition is equivalent to the so-called Bochner's criterion (cf. [32, Theorem 3.1.8]), namely, $f \in AP(X)$ if and only if for every sequence of real numbers $(s'_n)$ there exists a subsequence $(s_n)$ such that $f(\cdot + s_n)$ is uniformly convergent on $\mathbb{R}$. Almost periodic functions are uniformly continuous on $\mathbb{R}$ (cf. [32, Theorem 3.1.4]). The space of compact almost automorphic functions will be denoted by $AA_c(X)$. Recall that a continuous bounded function $f$ belongs to $AA_c(X)$ if and only if for all sequence $(s'_n)$ of real numbers there exists a subsequence $(s_n) \subset (s'_n)$ such that $\lim_{t\to\infty} f(t+s_n) =: \bar{f}(t)$ and $\lim_{t\to\infty} \bar{f}(t-s_n) = f(t)$ uniformly over compact subsets of $\mathbb{R}$. Clearly the function $f$ above is continuous on $\mathbb{R}$. Therefore $f$ is uniformly continuous [18]. In other words compact almost automorphic functions are uniformly continuous on $\mathbb{R}$. The space of almost automorphic functions is defined as follows $AA(X) := \{f \in BC(X) : \text{for all } (s'_n) \text{ there exists } (s_n) \subset (s'_n) \text{ such that} \lim_{t\to\infty} =: \bar{f}(t) \text{ and } \lim_{t\to\infty} \bar{f}(t-s_n) = f(t)\ \forall t \in \mathbb{R}\}$, provided with the norm $\|\cdot\|_\infty$.

Let $\mathcal{F} = \{P_\omega(X), AP(X), AA_c(X), AA(X)\}$ and $\Omega \in \mathcal{F}$. Then we have the following result.

**Theorem 2.1** [27]. *Assume $f, f_1, f_2 \in \Omega$. Then we have*

(i) $f_1 + f_2 \in \Omega$;
(ii) $\lambda f \in \Omega$, *for any scalar* $\lambda$;
(iii) $f_\tau(t) := f(t+\tau) \in \Omega$ *for any* $\tau \in \mathbb{R}$;
(iv) *The range $R_f$ of $f$ is relatively compact in $X$;*
(v) *Let $(g_n) \in \Omega$, such that $g_n \to g$ uniformly on $\mathbb{R}$. Then $g \in \Omega$;*
(vi) *Let $F(t) := \int_0^t f(s)ds$. Then $F \in \Omega$ if and only if $R_f$ is relatively compact in $X$.*

Now we consider the set

$$C_0(X) := \{f \in C(X) : \lim_{|t|\to\infty} \|f(t)\| = 0\},$$

and define the space of asymptotically periodic functions: $AP_\omega(X) := P_\omega(X) \oplus C_0(X)$. Analogously, we define the space of asymptotically almost periodic functions $AAP(X) := AP(X) \oplus C_0(X)$, the space of asymptotically compact almost automorphic functions $AAA_c(X) := AA_c(X) \oplus C_0(X)$ and the space of asymptotically almost automorphic functions $AAA(X) := AA(X) \oplus C_0(X)$.

**Remark 2.2.** We observe that

$$AP_\omega(X) \neq SAP_\omega(X)$$

where $SAP_\omega(X) := \{f \in BC(X) : \exists \omega > 0,\ \|f(t+\omega) - f(t)\| \to 0 \text{ as } t \to \infty\}$. This fact was only recently proved in [21], providing a counterexample to the assertion given in [15, Lemma 2.1]. This way, in general we only have

$$AP_\omega(X) \subset SAP_\omega(X).$$

We will next consider the following set

$$P_0(X) := \{f \in BC(X) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|f(s)\|ds = 0\},$$

and define the following classes of spaces: The space of pseudo-periodic functions $PP_\omega(X) := P_\omega(X) \oplus P_0(X)$, the space of pseudo almost periodic functions $PAP(X) := AP(X) \oplus P_0(X)$, the space of pseudo compact almost automorphic functions $PAA_c(X) := AA_c(X) \oplus P_0(X)$, and the space of pseudo almost automorphic functions $PAA(X) := AA(X) \oplus P_0(X)$. We have the following diagram that summarizes the relation of the different classes of subspaces defined previously.

$$\begin{array}{ccccc} AA(X) & \subset & AAA(X) & \subset & PAA(X) \\ \cup & & \cup & & \cup \\ AA_c(X) & \subset & AAA_c(X) & \subset & PAA_c(X) \\ \cup & & \cup & & \cup \\ AP(X) & \subset & AAP(X) & \subset & PAP(X) \\ \cup & & \cup & & \cup \\ P_\omega(X) & \subset & AP_\omega(X) & \subset & PP_\omega(X) \\ & & \cap & & \\ & & SAP_\omega(X) & & \end{array}$$

In what follows, we denote by $\mathcal{M}(X)$ the set which consists of all the vector-valued function spaces described above. In addition the spaces $C_0(X)$ and $P_0(X)$ will also be treated independently.

Recall that a closed operator $A$ is said to be sectorial of type $\mu \in \mathbb{R}$ and angle $0 \leqslant \theta < \pi/2$ if there exists $M > 0$ such that its resolvent exists outside the sector $\mu + S_\theta := \{\mu + \lambda : \lambda \in \mathbb{C}, |arg(-\lambda)| < \theta\}$ and $\|(\lambda - A)^{-1}\| \leqslant M/|\lambda - \mu|,\ \lambda \notin \mu + S_\theta$. Sectorial operators are well studied in the literature. For a recent reference including several examples and properties we refer the reader to [20]. We also recall the following definition from [5], which has been used extensively for several authors in the treatment of some classes of abstract fractional differential equations (see e.g. [11,12,9,7,2,1]).

**Definition 2.3** [5]. Let $A$ be a closed operator with domain $D(A)$ defined on a Banach space $X$ and $1 < \alpha \leqslant 2$. We say that $A$ is the generator of a $\alpha$-resolvent family if there exists $\mu \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : Re\lambda > \mu\} \subset \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t}S_\alpha(t)xdt,\ Re\lambda > \mu,\ x \in X$. In this case, $S_\alpha(t)$ is called the $\alpha$-resolvent family generated by $A$.

We note that if $A$ is sectorial of type $\mu$ with $0 \leqslant \theta < \pi(1 - \alpha/2)$, then $A$ is the generator of an $\alpha$-resolvent family given by $S_\alpha(t) := 1/(2\pi i)\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda$, where $\gamma$ is a suitable path lying outside the sector $\mu + S_\theta$ (cf. [8]). It is known [8] that if $A$ is a sectorial operator of type $\mu < 0$ for some $M > 0$ and $0 \leqslant \theta < \pi(1 - \alpha/2)$, then there exists $C > 0$ such that

$$\|S_\alpha(t)\|_{B(X)} \leqslant \frac{CM}{1 + |\mu|t^\alpha}, \quad t \geqslant 0. \tag{2.1}$$

We notice that the concept of a $\alpha$- resolvent family as above defined, is closely related to the concept of a resolvent family (see Prüss [34, Chapter 1]). For the scalar case, there is a large bibliography (cf. [17] and references therein). Because of the uniqueness of the Laplace transform, in the border case $\alpha = 1$, the family $S_\alpha(t)$ corresponds to a $C_0$-semigroup, whereas in the case $\alpha = 2$ an $\alpha$-resolvent family corresponds to the concept of a cosine operator family; see Arendt et al. [6] and Fattorini [14]. We note that $\alpha$-resolvent families, as well as resolvent families, are a particular case of $(a,k)$-regularized families [28]. According to [28] an $\alpha$-resolvent family $S_\alpha(t)$ corresponds to a $(1, t^{\alpha-1}/\Gamma(\alpha))$-regularized family.

A characterization of generators of $\alpha$-resolvent families, analogous to the Hille-Yosida Theorem for $C_0$-semigroup, can be directly deduced from [28, Thorem 3.4]. Results on perturbation, approximation, representation as well as ergodic type theorems can be deduced, from the most general context of $(a,k)$-regularized resolvents, (see [28–31]).

We finally recall the following definition.

**Definition 2.4.** [34] A strongly measurable family of operators $\{T(t)\}_{t\geqslant 0} \subset \mathcal{B}(X)$ is called uniformly integrable (or strongly integrable) if $\int_0^\infty \|T(t)\|dt < \infty$. In what follows, we will denote $\|T\| := \int_0^\infty \|T(t)\|dt$ for any uniformly integrable family of such operators $\{T(t)\}_{t\geqslant 0}$.

Observe that in view of (2.1) an $\alpha$-resolvent family is uniformly integrable under the hypothesis that $A$ is sectorial of negative type.

## 3. Regularity under convolution and composition

We will start with the following result on maximal regularity under convolution which is a consequence of [27, Theorem 3.3]:

**Theorem 3.1** [27, Theorem 3.3]. *If $f$ belongs to one of the spaces $\mathcal{M}(X)$, then $w(t) := \int_{-\infty}^t S_\alpha(t-s)f(s)ds$ also belongs to the same space as $f$.*

Concerning the spaces $C_0(X)$ and $P_0(X)$ we have the following result.

**Theorem 3.2.** *Assume that $f \in C_0(X)$ or $f \in P_0(X)$, then $w \in C_0(X)$ or $w \in P_0(X)$ respectively.*

**Proof.** Let $f \in C_0(X)$ and $\epsilon > 0$ be given. There exists $T > 0$ such that $\|f(s)\| < \epsilon$ for all $|s| > T$ and hence we can write

$$w(t) := \int_{-\infty}^{T} S_\alpha(t-s)f(s)ds + \int_{T}^{t} S_\alpha(t-s)f(s)ds. \tag{3.1}$$

Then

$$\|w(t)\| \leqslant \int_{-\infty}^{T} \|S_\alpha(t-s)\|\|f(s)\|ds + \int_{T}^{t} \|S_\alpha(t-s)\|\|f(s)\|ds \leqslant \|f\|_\infty \int_{t-T}^{\infty} \|S_\alpha(v)\|dv + \|S_\alpha\|\epsilon,$$

and using the fact that $S_\alpha$ is integrable we conclude that $w(t) \to 0$ as $t \to \infty$. On the other hand, for $t < -T$ we have

$$\|w(t)\| \leqslant \int_{-\infty}^{t} \|S_\alpha(t-s)\|\|f(s)\|ds \leqslant \epsilon \int_{-\infty}^{-T} \|S_\alpha(t-s)\|ds \leqslant \epsilon\|S_\alpha\|,$$

and we conclude that $w(t) \to 0$ as $t \to -\infty$. It proves that $w \in C_0(X)$.

Let $f \in P_0(X)$. For $R > 0$ we have

$$\frac{1}{2R}\int_{-R}^{R} \|w(t)\|dt \leqslant \frac{1}{2R}\int_{-R}^{R}\left[\int_{-\infty}^{t} \|S_\alpha(t-s)\|\|f(s)\|ds\right]dt \leqslant \frac{1}{2R}\int_{-R}^{R}\left[\int_{0}^{\infty} \|S_\alpha(u)\|\|f(t-s)\|ds\right]dt$$
$$= \int_{0}^{\infty} \|S_\alpha(u)\|\left[\frac{1}{2R}\int_{-R}^{R} \|f(t-s)\|dt\right]ds.$$

Note that the set $P_0(X)$ is translation-invariant. Hence, using the Lebesgue's dominated convergence theorem, we obtain from the above inequality that $\frac{1}{2R}\int_{-R}^{R} \|f(t)\|dt \to 0$ as $R \to \infty$ i.e. $f \in P_0(X)$. The proof is complete. □

It is well known that the study of composition of two functions with special properties plays a key role in discussing the existence of solutions to semilinear equations. Hence, given a bounded function $f \in BC((\mathbb{R} \times X, X))$ and $\phi(\cdot) \in B(X)$, such that $\phi(\cdot)$ belongs to some fixed space $\Omega(X) \subset \mathcal{M}(X)$, we can ask under which conditions on $f$ we have that $f(\cdot, \phi(\cdot))$ belongs to $\Omega(X)$?. To answer this question we need the following notations and results.

We denote

$$C_0(\mathbb{R} \times X, X) = \{h \in C(\mathbb{R} \times X, X) : \lim_{t\to\infty}\|h(t,x)\| = 0 \text{ uniformly for } x \text{ on any compact subset of } X\}.$$

as well as

$$P_0(\mathbb{R} \times X, X) = \{h \in C(\mathbb{R} \times X, X) : f(\cdot, x) \text{ is bounded for all } x \in X \text{ and}$$

$$\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R} \|f(t,x)\|dt = 0 \text{ uniformly in } x \in X\}. \tag{3.2}$$

In what follows, $\Phi$ will denote any of the symbols in the set $\{P_\omega, AP, AA_c, AA\}$. Define

$$\Phi(\mathbb{R} \times X, X) := \{f \in C(\mathbb{R} \times X, X) : f(\cdot, x) \in \Phi(X) \text{ uniformly for all } x \in B \subseteq X,\ B \text{ bounded}\}.$$

Analogously we define the sets $A\Phi(\mathbb{R} \times X, X) = \Phi(\mathbb{R} \times X, X) \oplus C_0(\mathbb{R} \times X, X)$ and $P\Phi(\mathbb{R} \times X, X) = \Phi(\mathbb{R} \times X, X) \oplus P_0(\mathbb{R} \times X, X)$. For example $AAP((\mathbb{R} \times X, X)) = AP(\mathbb{R} \times X, X) \oplus C_0(\mathbb{R} \times X, X)$.

**Theorem 3.3.** *Let $f \in \Phi(\mathbb{R} \times X, X)$ and $\phi \in \Phi(X)$. Assume that $f(t, \cdot)$ is uniformly continuous in each bounded subset $K$ of $X$ uniformly in $t \in \mathbb{R}$, that is: Given $\epsilon > 0$ and $K \subset X$, there exists $\delta > 0$ such that $x$, $y \in K$ and $\|x - y\| \leqslant \delta$ imply that $\|f(t,x) - f(t,y)\| \leqslant \epsilon$ for all $t \in \mathbb{R}$. Then $f(\cdot, \phi(\cdot)) \in \Phi(X)$.*

**Proof.** For almost periodic functions the result is given in [3, Proposition 1], and for almost automorphic functions in [26, Lemma 2.2]. Let $\Phi = P_\omega$ and asumme that $f \in P_\omega(\mathbb{R} \times X, X)$ and $\phi \in P_\omega(X)$. Let $K = \{\phi(t) \in X : t \in \mathbb{R}\}$ then,

$$f(t+\omega, \phi(t+\omega)) - f(t, \phi(t)) = f(t+\omega, \phi(t)) - f(t, \phi(t)).$$

Since $K$ is bounded then $f(t+\omega, \phi(t)) - f(t, \phi(t)) = 0$ and then $f(\cdot, \phi(\cdot)) \in P_\omega(X)$.

Let $\Phi = AA_c$ be given and assume that $f \in AA_c(\mathbb{R} \times X, X)$ and $\phi \in AA_c(X)$. Let $(s''_n)_n$ be a sequence of real numbers. Now, fix a subsequence $(s'_n)_n$ of $(s''_n)_n$ and $\bar{f} \in BC(\mathbb{R} \times X, X)$ so that the pair $\bar{f}, (s'_n)_n$ is associated with $f$ as in the definition of $AA_c(X)$, similarly fix a subsequence $(s_n)_n$ of $(s'_n)_n$ and consider the pair $(s_n)_n, \overline{\phi}$ associated $\phi$. Let $K \subset \mathbb{R}$ be an arbitrary compact subset and let $\epsilon > 0$. Since the set $\{\phi(t) : t \in \mathbb{R}\}$ is relatively compact, there exist points $x_i \in X, i = 1, \ldots, n_0$ such that for each $t \in \mathbb{R}$ and $i(t) \in \{1, 2, \ldots, n_0\}$

$$\|\phi(t) - x_{i(t)}\| \leqslant \delta \text{ and } \|\overline{\phi}(t) - x_{i(t)}\| \leqslant \delta.$$

Let $N_\epsilon$ be a natural number such that $\|f(t+s_n, x_i) - \overline{f}(t, x_i)\| \leqslant \epsilon$ for all $i = 1, \ldots, n_0$, and $\|\phi(t+s_n) - \overline{\phi}(t)\| < \delta$ for all $t \in K$ (where $\delta$ is given by the uniform continuity of $f$), whenever $n \geqslant N_\epsilon$. In view of the above, for each $t \in K$, and $n \geqslant N_\epsilon$,

$$\begin{aligned}\|f(t+s_n, \phi(t+s_n)) - \overline{f}(t, \overline{\phi}(t))\| &\leqslant \|f(t+s_n, \phi(t+s_n)) - f(t+s_n, \overline{\phi}(t))\| + \|f(t+s_n, \overline{\phi}(t)) - f(t+s_n, x_{i(t)})\| \\ &+ \|f(t+s_n, x_{i(t)}) - \overline{f}(t, x_{i(t)})\| + \|\overline{f}(t, x_{i(t)}) - \overline{f}(t, \overline{\phi}(t))\| \leqslant 4\epsilon.\end{aligned}$$

This proves that the pair $(s_n)_n, \overline{f}(\cdot, \overline{\phi}(\cdot))$ is associated to the function $f(\cdot, \phi(\cdot))$ and then $f(\cdot, \phi(\cdot)) \in AA_c(X)$. □

**Remark 3.4.** We note that the above theorem is also valid for the case of the space $SAP_\omega(X)$. Indeed, assume that $f \in SAP_\omega(\mathbb{R} \times X, X)$ and $\phi \in SAP_\omega(X)$, It follows from hypotheses that, given $\epsilon > 0$ and $K = \{\phi(t) \in X : t \in \mathbb{R}\}$ there exists $\delta > 0$ such that for all $x,\ y \in K$ and $\|x - y\| < \delta$ imply that $\|f(t,x) - f(t,y)\| < \epsilon/2$ for all $t \in \mathbb{R}$. We consider $N > 0$ such that $\|\phi(t+\omega) - \phi(t)\| \leqslant \delta$ and $\|f(t+\omega, x) - f(t,x)\| \leqslant \epsilon/2$ for all $t \geqslant N$ and $x \in K$. Consequently

$$\|f(t+\omega, \phi(t+\omega)) - f(t, \phi(t))\| \leqslant \|f(t+\omega, \phi(t+\omega)) - f(t+\omega, \phi(t))\| + \|f(t+\omega, \phi(t)) - f(t, \phi(t))\| \leqslant \epsilon,$$

for all $t \geqslant N_\epsilon$. This proves that $\|f(t+\omega, \phi(t+\omega)) - f(t, \phi(t))\| \to 0$ as $t \to \infty$ and then $f(\cdot, \phi(\cdot)) \in SAP_\omega(X)$.

**Remark 3.5.** The condition that $f$ is uniformly continuous in each bounded subset of $X$ uniformly in $t \in \mathbb{R}$ is not necessary in the case when $\Phi(X) = P_\omega(X)$ or $\Phi(X) = AP(X)$.

The unified proof of the following result follows the lines of [26, Theorem 2.3] for the case $AAA(X)$ (see also [1, Lemma 2.9] in case $AAP(X)$).

**Theorem 3.6.** *Let $f \in A\Phi(\mathbb{R} \times X, X)$ be such that $f := f_1 + f_2$ where $f_1 \in \Phi(\mathbb{R} \times X, X)$ and $f_2 \in C_0(\mathbb{R} \times X, X)$. Assume that $f_1(t,\cdot)$ and $f(t,\cdot)$ are uniformly continuous in each bounded subset of $X$ uniformly in $t \in \mathbb{R}$. If $\phi \in A\Phi(X)$ then $f(\cdot, \phi(\cdot)) \in A\Phi(X)$.*

**Proof.** Let $f \in A\Phi(\mathbb{R} \times X, X)$ and $\phi \in A\Phi(X)$ be given. We have by definition that $f = f_1 + f_2$ where $f_1 \in \Phi(\mathbb{R} \times X, X),\ f_2 \in C_0(\mathbb{R} \times X, X)$ and $\phi = \phi_1 + \phi_2$ where $\phi_1 \in \Phi(X),\ \phi_2 \in C_0(X)$. Now we decompose $f$ as follows

$$f(t, \phi(t)) = f_1(t, \phi_1(t)) + f(t, \phi(t)) - f_1(t, \phi_1(t)) = f_1(t, \phi_1(t)) + f(t, \phi(t)) - f(t, \phi_1(t)) + f_2(t, \phi_1(t)).$$

Using the fact that $f_1(t,\cdot)$ is uniformly continuous in each bounded subset of $X$ uniformly in $t$ and Theorem 3.3, we obtain $f_1(\cdot, \phi_1(\cdot)) \in \Phi(X)$. Now, set $F(t) := f(t, \phi(t)) - f(t, \phi_1(t)) + f_2(t, \phi_1(t))$. Since $\phi$ and $\phi_1$ are bounded, we can choose a bounded subset $K \subset X$ such that $\phi(\mathbb{R}) \cup \phi_1(\mathbb{R}) \subset K$. It follows from hypotheses that, given $\epsilon/2 > 0$, there exists $\delta := \delta_{\epsilon,K}$ such that $x,\ y \in K$ and $\|x - y\| \leqslant \delta$ imply that

$$\|f(t,x) - f(t,y)\| < \epsilon/2.$$

Since the function $\phi_2$ belongs to $C_0(X)$ and $f_2 \in C_0(\mathbb{R} \times X, X)$, there exists $t_0 > 0$ depending on $\delta$ such that $\|\phi(t) - \phi_1(t)\| < \delta$ and $\|f_2(t, \phi_1(t))\| < \epsilon/2$ for all $|t| > t_0$. Then

$$\|F(t)\| \leqslant \|f(t, \phi(t)) - f(t, \phi_1(t))\| + \|f_2(t, \phi_1(t))\| \leqslant \epsilon/2 + \epsilon/2 = \epsilon$$

for all $|t| > s_0$, i.e. $F \in C_0(X)$. Using the fact that $f_1(\cdot, \phi_1(\cdot)) \in \Phi(X)$ we conclude that $f(\cdot, \phi(\cdot)) \in A\Phi(X)$. □

For our next results we will need the following Lemma.

**Lemma 3.7** [26, Lemma 2.1]. *Let $f \in BC(X)$ be given. Then $f \in P_0(X)$ if and only if for any $\epsilon > 0$,*

$$\lim_{T \to \infty} \frac{1}{2T} mes(M_{T,\epsilon}(f)) = 0,$$

*where $mes(\cdot)$ denotes the Lebesgue measure and $M_{T,\epsilon}(f) := \{t \in [-T, T] : \|f(t)\| > \epsilon\}$.*

The main theorem in case of functions with the pseudo property is the following result. The idea of the proof follows [25] and [26, Theorem 2.4] for the cases $PAP(X),\ PAA(X)$ respectively.

**Theorem 3.8.** *Let $f \in P\Phi(\mathbb{R} \times X, X)$ is that $f := f_1 + f_2$ where $f_1 \in \Phi(\mathbb{R} \times X, X)$ and $f_2 \in P_0(\mathbb{R} \times X, X)$. Assume that $f$ satisfy the following conditions:*

(i) *$\{f(t,y) : t \in \mathbb{R}, y \in K\}$ is bounded for every bounded subset $K \subset X$.*
(ii) *$f_1(t,\cdot), f(t,\cdot)$ is uniformly continuous on any bounded subset $K \subset X$ uniformly in $t \in \mathbb{R}$.*

*If $\phi \in P\Phi(X)$, then $f(\cdot, \phi(\cdot)) \in P\Phi(X)$.*

**Proof.** Since $f \in P\Phi(\mathbb{R} \times X, X)$ and $\phi \in P\Phi(X)$, we have by definition that $f = f_1 + f_2$ and $\phi = \phi_1 + \phi_2$ where $f_1 \in \Phi(\mathbb{R} \times X, X)$, $f_2 \in P_0(\mathbb{R} \times X, X)$, $\phi_1 \in \Phi(X)$ and $\phi_2 \in P_0(X)$. Now we decompose $f$ as follows

$$f(t,\phi(t)) = f_1(t,\phi_1(t)) + f(t,\phi(t)) - f_1(t,\phi_1(t)) = f_1(t,\phi_1(t)) + f(t,\phi(t)) - f(t,\phi_1(t)) + f_2(t,\phi_1(t)).$$

Using (ii) and Theorem 3.3 we obtain $f_1(\cdot,\phi_1(\cdot)) \in \Phi(X)$. Now, set $F(t) := f(t,\phi(t)) - f(t,\phi_1(t))$. Next, we prove that $F \in P_0(X)$. Indeed, clearly by (i), $F \in BC(X)$; Then by Lemma 3.7, it is sufficient to show that

$$\lim_{T\to\infty} \frac{1}{2T} mes(M_{T,\epsilon}(F)) = 0.$$

Since $\phi$ and $\phi_1$ are bounded, we can choose a bounded subset $K \subset X$ such that $\phi(\mathbb{R}), \phi_1(\mathbb{R}) \subset K$. It follows from hypotheses that, given $\epsilon > 0$, there exists $\delta := \delta_{\epsilon,K}$ such that $x,\ y \in K$ and $\|x - y\| \leqslant \delta$ imply that $\|f(t,x) - f(t,y)\| < \epsilon$ for all $t \in \mathbb{R}$. Then we have

$$\lim_{T\to\infty} \frac{1}{2T} mes(M_{T,\epsilon}(F)) \leqslant \lim_{T\to\infty} \frac{1}{2T} mes(M_{T,\delta}(\phi_2)).$$

Since $\phi_2 \in P_0(X)$, Lemma 3.7 yields for the above $\delta$ that

$$\lim_{T\to\infty} \frac{1}{2T} mes(M_{T,\delta}(\phi_2)) = 0.$$

This shows that $F \in P_0(X)$. Next we show that $f_2(\cdot,\phi_1(\cdot)) \in P_0(X)$. Since $f_2(\cdot,\phi_1(\cdot))$ is continuous in $[-T,T]$, it is uniformly continuous in $[-T,T]$. Set $I := \phi_1([-T,T])$. Since it is compact in $\mathbb{R}$, one can find finite open balls $O_k,\ k = 1,2,\ldots,m$, with center in $x_k$ and radius $\delta$ small enough such that $I \subset \bigcup_{k=1}^m O_k$ and

$$\|f_2(t,\phi_1(t)) - f_2(t,x_k)\| \leqslant \epsilon/2, \quad \phi_1(t) \in O_k,\ t \in [-T,T]. \tag{3.3}$$

The set $B_k := \{t \in [-T,T] :\ \phi_1(t) \in O_k\}$ is open in $[-T,T] = \bigcup_{k=1}^m B_k$. Let $E_1 = B_1$, $E_k = B_k \setminus \bigcup_{j=1}^{k-1} B_j$ $(2 \leqslant k \leqslant m)$ be given. Then $E_j \cap E_i = \emptyset$, when $i \neq j, 1 \leqslant i,j \leqslant m$. Clearly,

$$\{t \in [-T,T] : \|f_2(t,\phi_1(t))\| \geqslant \epsilon\} \subset \bigcup_{k=1}^m \{t \in E_k : \|f_2(t,\phi_1(t)) - f_2(t,x_k)\| + \|f_2(t,x_k)\| \geqslant \epsilon\} \subset \bigcup_{k=1}^m \{t \in E_k$$
$$: \|f_2(t,\phi_1(t)) - f_2(t,x_k)\| \geqslant \epsilon/2\} \cup \{t \in E_k : \|f_2(t,x_k)\| \geqslant \epsilon/2\}.$$

It follows from (3.3) that the sets $\{\|f_2(t,\phi_1(t)) - f_2(t,x_k)\| \geqslant \epsilon/2\}$ are empty for all $1 \leqslant k \leqslant m$. Therefore,

$$\frac{1}{2T} mes(M_{T,\epsilon}(f_2(\cdot,\phi_1(\cdot)))) \leqslant \frac{1}{2T} \sum_{k=1}^m mes(M_{T,\epsilon/2}(f_2(\cdot,x_k)).$$

Since $f_2(\cdot,x_k) \in P_0(X)$, we obtain

$$\lim_{T\to\infty} \frac{1}{2T} mes(M_{T,\epsilon/2})(f_2(\cdot,x_k)) = 0$$

and hence we have that

$$\lim_{T\to\infty} \frac{1}{2T} mes(M_{T,\epsilon})(f_2(\cdot,\phi_1(\cdot))) = 0,$$

that is $f_2(\cdot,\phi_1(\cdot)) \in P_0(X)$, which ends the proof. □

An immediate corollary of the above results corresponds to a slight extension of the composition theorem recently stated in [27].

**Corollary 3.9.** *Let $\Omega(X) \in \mathcal{M}(X)$ and $f \in \Omega(\mathbb{R} \times X, X)$ be given and fixed. Assume that there exists a constant $L_f > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leqslant L_f \|x - y\|$$

*for all $t \in \mathbb{R}$ and $x,y \in X$. Let $\phi \in \Omega(X)$. Then $f(\cdot,\phi(\cdot)) \in \Omega(X)$.*

**Remark 3.10.** Recall that in [27] the integro-differential equation

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t,u(t)), \quad t \geqslant 0$$

is considered. The above corollary provides a direct extension of the existence results on mild solutions for the cited integro-differential equation to the new classes of spaces $AA_c(X), AAA_c(X)$ and $PAA_c(X)$ (see [27, Theorems 4.3, 4.5 and 4.7]).

The following results corresponds to the cases $C_0(X)$ and $P_0(X)$, which are considered individually.

**Theorem 3.11.** *Let $f \in C_0(\mathbb{R} \times X, X)$ be given and such that $f(t, \cdot)$ is uniformly continuous in each bounded subset $K$ of $X$ uniformly in $t \in \mathbb{R}$. If $\phi \in C_0(X)$, then $f(\cdot, \phi(\cdot)) \in C_0(X)$.*

**Proof.** Let $f \in C_0(\mathbb{R} \times X)$ and $\phi \in C_0(X)$. Let $K = \{\phi(s) \in X : s \in \mathbb{R}\} \cup \{0\}$. We note that by hypothesis, given $\epsilon/2 > 0$, there exists $\delta := \delta_{\epsilon,K}$ such that $x, \ y \in K$ and $\|x - y\| \leqslant \delta$ imply that

$$\|f(t,x) - f(t,y)\| < \epsilon/2.$$

Since the function $\phi$ belongs to $C_0(X)$ and $f \in C_0(\mathbb{R} \times X, X)$ there exists $s_0$ depending of $\delta$ such that $\|\phi(s)\| < \delta$ and $\|f(s,0)\| < \epsilon/2$ for all $|s| > s_0$. We can write

$$f(s,\phi(s)) = f(s,0) + f(s,\phi(s)) - f(s,0),$$

and estimate

$$\|f(s,\phi(s))\| = \|f(s,0)\| + \|f(s,\phi(s)) - f(s,0)\| \leqslant \epsilon/2 + \epsilon/2 = \epsilon$$

for all $|s| > s_0$, i.e. $f(s,\phi(s)) \in C_0(X)$. □

**Theorem 3.12.** *Assume that $f \in P_0(\mathbb{R} \times X, X)$ satisfy the following conditions:*

(i) *$\{f(t,y) : t \in \mathbb{R}, y \in K\}$ is bounded for every bounded subset $K \subset X$;*
(ii) *$f(t, \cdot)$ is uniformly continuous on any bounded subset $K \subset X$ uniformly in $t \in \mathbb{R}$.*

*If $\phi \in P_0(X)$, then $f(\cdot, \phi(\cdot)) \in P_0(X)$.*

**Proof.** Let $f \in P_0(\mathbb{R} \times X, X)$ and $\phi \in P_0(X)$. We note that by (i), the composition theorem and Lemma 3.7 it suffices to show that for all $\epsilon > 0$

$$\lim_{T\to\infty} \frac{1}{2T} mes(M_{T,\epsilon}(f(\cdot, \phi(\cdot)))) = 0.$$

We can write

$$f(s,\phi(t)) = f(t,0) + f(t,\phi(t)) - f(t,0).$$

Firstly, we will prove that $f(t,\phi(t)) - f(t,0) \in P_0(X)$. Since $\phi(t)$ is bounded, we can choose $K := \{\phi(t) : t \in \mathbb{R}\} \cup \{0\}$. Under assumption (ii), $f$ is uniformly continuous on the bounded subset $K$. Then, given $\epsilon > 0$, there exists $\delta > 0$ such that $x, \ y \in K$ and $\|x - y\| \leqslant \delta$ imply that

$$\|f(t,x) - f(t,y)\| \leqslant \epsilon \quad \text{for all } t \in \mathbb{R}.$$

Then we have

$$\frac{1}{2T} mes(M_{T,\epsilon}(f(t,\phi(t)) - f(t,0))) \leqslant \frac{1}{2T} mes(M_{T,\delta}(\phi(t))).$$

Since $\phi(t) \in P_0(X)$, [26, Lemma 2.1] yields that for the above $\delta$,

$$\lim_{T\to\infty} \frac{1}{2T} mes(M_{T,\delta}(\phi(t))) = 0,$$

then

$$\lim_{T\to\infty} \frac{1}{2T} mes(M_{T,\epsilon}(f(t,\phi(t)) - f(t,0)) = 0,$$

and this shows that $f(t,\phi(t)) - f(t,0) \in P_0(X)$. Secondly, note that $f(t,0) \in P_0(X)$ and hence from the above $f(t,\phi(t)) - f(t,0) \in P_0(X)$. Therefore $f(t,\phi(t)) \in P_0(X)$. □

We note that if $\Omega(X) \in \mathcal{M}(X) \cup \{C_0(X), P_0(X)\}$ is fixed, then, given $\phi \in \Omega(X)$ and $f \in BC(\mathbb{R} \times X, X)$, sufficient conditions to ensure that $f(\cdot, \phi(\cdot))$ belongs to $\Omega(X)$ should be chosen between the following:

$(H_1)$ $f(t, \cdot)$ is uniformly continuous in each bounded subset of $X$ uniformly in $t \in \mathbb{R}$. More precisely, given $\epsilon > 0$ and $K \subset X$, there exists $\delta > 0$ such that $x, \ y \in K$ and $\|x - y\| \leqslant \delta$ imply that $\|f(t,x) - f(t,y)\| \leqslant \epsilon$.
$(H_2)$ $\{f(t,x) : t \in \mathbb{R}, \ x \in K\}$ is bounded for all bounded subset $K \subset X$.
$(H_3)$ If $f = f_1 + f_2$ where $f_1 \in \Phi(X)$ and $f_2 \in \{C_0(X), P_0\} \setminus \{0\}$, then $f_1(t, \cdot)$ is uniformly continuous in each bounded subset of $X$ uniformly in $t \in \mathbb{R}$.

For example, if $\Omega(X) = AAA(X)$ and $\phi \in AAA(X)$ to ensure that $f(\cdot, \phi(\cdot))$ belongs to $AAA(X)$ we need that $f \in AAA(\mathbb{R} \times X, X)$ and the conditions $(H_1)$ and $(H_3)$. However if $\Phi(X) = PAA(X)$ and $\phi \in PAA(X)$ to ensure that $f(\cdot, \phi(\cdot))$ belongs to $PAA(X)$ we need that $f \in PAA(\mathbb{R} \times X, X)$ and the conditions $(H_1)$, $(H_2)$, $(H_3)$. The whole spectrum of situations is summarized in the following table:

| $\Omega(X)$ | $H_1$ | $H_2$ | $H_3$ |
|---|---|---|---|
| $P_\omega$ | | | |
| $AP$ | | | |
| $A_cA$ | • | | |
| $AA$ | • | | |
| $AP_\omega$ | • | | |
| $AAP$ | • | | |
| $AA_cA$ | • | | • |
| $AAA$ | • | | • |
| $PP_\omega$ | | • | • |
| $PAP$ | | • | • |
| $PA_cA$ | • | • | • |
| $PAA$ | • | • | • |
| $C_0$ | • | | |
| $P_0$ | • | | |

## 4. Existence, uniqueness and regularity of mild solutions

The notion of mild solution that we use in this paper, reads as follows:

**Definition 4.1** [5]. A function $u : \mathbb{R} \to X$ is said to be a mild solution to (1.1) if there exists a strongly continuous family of bounded an linear operators on $X$ such that the function $s \to S_\alpha(t-s)f(s, u(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and

$$u(t) = \int_{-\infty}^{t} S_\alpha(t-s)f(s, u(s))ds \tag{4.1}$$

for all $t \in \mathbb{R}$.

The following theorems are the main results of this work.

**Theorem 4.2.** *Assume that $A$ is sectorial of type $\mu < 0$ with $0 \leqslant \theta < \pi(1 - \alpha/2)$. Let $\Omega(X) \in \mathcal{M}(X) \setminus \{P_\omega(X), AP(X), AA_c(X), AA(X)\} \cup \{C_0(X), P_0(X)\}$ be given. Let $f \in \Omega(\mathbb{R} \times X, X)$ be given and assume that there exists an integrable and bounded function $L_f : \mathbb{R} \to [0, \infty)$ satisfying*

$$\|f(t,x) - f(t,y)\| \leqslant L_f(t)\|x - y\|, \quad \forall x, y \in X, \quad t \geqslant 0. \tag{4.2}$$

*Then Eq. (1.1) has a unique mild solution $u$ which belongs to the same space as $f(\cdot, x)$.*

**Proof.** We define the operator $F : \Omega(X) \to \Omega(X)$ by

$$(F\phi)(t) := \int_{-\infty}^{t} S_\alpha(t-s)f(s, \phi(s))ds \quad t \in \mathbb{R}.$$

Given $\phi \in \Omega(X)$, in view of Corollary 3.9, Theorems 3.11 and 3.12, we have that $s \to f(s, \phi(s))$ belongs to $\Omega(X)$ for $\Omega(X) \in \mathcal{M}(X) \setminus \{P_\omega(X), AP(X), AA(X), AA_c(X)\}$ or $\Omega(X) \in \{C_0(X), P_0(X)\}$ and hence is bounded in $\mathbb{R}$. Since the function $t \to 1/(1 + |\mu|t^\alpha)$ is integrable on $\mathbb{R}_+$ $(1 < \alpha < 2)$, we get that $F\phi$ exists. Now by Theorems 3.1 and 3.2 we obtain that $F\phi \in \Omega(X)$, and hence $F$ is well defined. It suffices to show that the operator $F$ has a unique fixed point in $\Omega(X)$. Let $\phi_1, \ \phi_2 \in \Omega(X)$ be given and define $M_\alpha := \sup_{t \in \mathbb{R}} \|S_\alpha(t)\|$. We have

$$\|F\phi_1(t) - F\phi_2(t)\| \leqslant \int_{-\infty}^{t} M_\alpha L_f(s)\|\phi_1(s) - \phi_2(s)\|ds \leqslant M_\alpha \|L_f\| \|\phi_1 - \phi_2\|_\infty.$$

In general, we get

$$\begin{aligned}\|F^n\phi_1(t) - F^n\phi_2(t)\| &\leqslant \frac{M_\alpha^n}{(n-1)!}\int_0^t L_f(s)\left(\int_0^s L_f(\tau)d\tau\right)^{n-1} ds\|\phi_1 - \phi_2\|_\infty \leqslant \frac{M_\alpha^n}{n!}\left(\int_0^t L_f(s)ds\right)^n \|\phi_1 - \phi_2\|_\infty \\ &\leqslant \frac{M_\alpha^n}{n!}\|L_f\|^n\|\phi_1 - \phi_2\|_\infty.\end{aligned}$$

Since $\frac{(M_\alpha\|L_f\|_1)^n}{n!} < 1$ for $n$ sufficiently large, by the contraction principle, $F$ has a unique fixed point $u \in C_0(X)$. This completes the proof. □

**Remark 4.3.** We must point out that the hypothesis (4.2) cannot be satisfied in case of the spaces $P_\omega(X), AP(X), AA(X)$ or $AA_c(X)$, except in the linear case $f(t,x) = g(t)\ (t \in \mathbb{R})$. Indeed, (4.2) cannot happen when $f(\cdot,x)$ belongs to $P_\omega(X), AP(X), AA(X)$ or $AA_c(X)$ since otherwise $f(t,x) - f(t,y) \in L^1(\mathbb{R}_+)$, leading to a contradiction with the fact that $f(t,x) - f(t,y)$ belongs to $P_\omega(X),\ AP(X),\ AA(X)$ or $AA_c(X)$ respectively.

We can establish the following existence result.

**Theorem 4.4.** *Assume that $A$ is sectorial of type $\mu < 0$ with $0 \leqslant \theta < \pi(1-\alpha/2)$ and let $\Omega(X) \in \mathcal{M}(X) \cup \{C_0(X), P_0(X)\}$ be given. Let $f \in \Omega(\mathbb{R}\times X, X)$ be a function that satisfies the Lipschitz condition (4.2) with $L_f \in BC(\mathbb{R})$. Let $\|L_f\| := \sup_{t\in\mathbb{R}} \int_t^{t+1} L_f(s)ds$. If $CM\|L_f\| < |\mu|^{\frac{1}{\alpha}}\alpha\frac{\sin(\pi/\alpha)}{\pi}$, where $C$ and $M$ are the constants given in (2.1), then Eq. (1.1) has a unique mild solution $u \in \Omega(X)$.*

**Proof.** We define the operator $F : \Omega(X) \to \Omega(X)$ by

$$(F\phi)(t) := \int_{-\infty}^{t} S_\alpha(t-s)f(s,\phi(s))ds \quad t \in \mathbb{R}.$$

Given $\phi \in \Omega(X)$, in view of Corollary 3.9, and Theorems 3.11, 3.12, we have that $s \to f(s,\phi(s))$ belongs to $\Omega(X)$, and hence it is bounded in $\mathbb{R}$. Since the function $t \to 1/(1+|\mu|t^\alpha)$ is integrable on $\mathbb{R}_+$ $(1 < \alpha < 2)$, we get that $F\phi$ exists. Now by [27, Theorem 3.3], we obtain that $F\phi \in \Omega(X)$, and hence $F$ is well defined. For $\phi_1, \phi_2 \in \Omega(X)$ we have the following estimate

$$\begin{aligned}\|F\phi_1(t) - F\phi_2(t)\| &\leqslant CM\int_{-\infty}^{t}\frac{L_f(s)}{1+|\mu|(t-s)^\alpha}\|\phi_1(s)-\phi_2(s)\|ds \leqslant CM\left(\sum_{m=0}^{\infty}\int_{t-(m+1)}^{t-m}\frac{L_f(s)}{1+|\mu|(t-s)^\alpha}ds\right)\|\phi_1-\phi_2\|_\infty \\ &\leqslant CM\left(\sum_{m=0}^{\infty}\frac{1}{1+|\mu|(m)^\alpha}\int_{t-(m+1)}^{t-m}L_f(s)ds\right)\|\phi_1-\phi_2\|_\infty \leqslant CM\left(\sum_{m=0}^{\infty}\frac{1}{1+|\mu|(m)^\alpha}\right)\|L_f\|_s\|\phi_1-\phi_2\|_\infty,\end{aligned}$$

which finishes the proof. □

**Corollary 4.5.** *Assume that $A$ is sectorial of type $\mu < 0$ with $0 \leqslant \theta < \pi(1-\alpha/2)$. Let $\Omega(X) \in \mathcal{M}(X) \cup \{C_0(X), P_0(X)\}$ be given. Assume $f \in \Omega(\mathbb{R}\times X, X)$ satisfies the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leqslant L_f\|x-y\| \quad x,y \in X,\ \ t \geqslant 0.$$

*If $CML_f < |\mu|^{\frac{1}{\alpha}}\alpha\frac{\sin(\pi/\alpha)}{\pi}$, then Eq. (1.1) has a unique mild solution $u \in \Omega(X)$.*

**Remark 4.6.** From the last Theorem we recover [1, Theorem 3.3] and [1, Corollary 3.4] for the case of pseudo-almost automorphic functions. The remaining cases are new.

A slightly different condition is established in the next result.

**Theorem 4.7.** *Assume that $A$ is sectorial of type $\mu < 0$ with $0 \leqslant \theta < \pi(1-\alpha/2)$ and $\Omega(X) \in \mathcal{M} \setminus \{P_\omega(X), AP(X), AA_c(X), AA(X)\} \cup \{C_0(X), P_0(X)\}$ be given. Let $f \in \Omega(\mathbb{R}\times X, X)$ be such that satisfy the Lipschitz condition (4.2) and the integral $\int_{-\infty}^{t} L_f(s)ds$ exists for all $t \in \mathbb{R}$. Then Eq. (1.1) has a unique mild solution $u \in \Omega(X)$.*

**Proof.** Define a new norm $|||\phi||| := \sup_{t\in\mathbb{R}}\{v(t)\|\phi(t)\|\}$, where $v(t) := e^{-k\int_{-\infty}^{t}L_f(s)ds}$ and $k$ is a fixed positive constant greater than $M_\alpha := \sup_{t\in\mathbb{R}}\|S_\alpha(t)\|$. Let $\phi_1,\ \phi_2$ be in $\Omega(X)$, then we have

$$\begin{aligned}v(t)\|F(\phi_1)(t) - (F\phi_2)(t)\| &= v(t)\left\|\int_{-\infty}^{t} S_\alpha(t-s)[f(s,\phi_1(s)) - f(s,\phi_2(s))]ds\right\| \leqslant M_\alpha\int_{-\infty}^{t} v(t)L_f(s)\|\phi_1(s)-\phi_2(s)\|ds \\ &\leqslant M_\alpha|||\phi_1-\phi_2|||\int_{-\infty}^{t} v(t)v(s)^{-1}L(s)ds = \frac{M_\alpha}{k}|||\phi_1-\phi_2|||\int_{-\infty}^{t} ke^{k\int_t^s L_f(\tau)d\tau}L_f(s)ds \\ &= \frac{M_\alpha}{k}|||\phi_1-\phi_2|||\int_{-\infty}^{t}\frac{d}{ds}\left(e^{k\int_t^s L_f(\tau)d\tau}\right)ds = \frac{M_\alpha}{k}|||\phi_1-\phi_2|||\left[1 - e^{-k\int_{-\infty}^{t}L_f(\tau)d\tau}\right] \leqslant \frac{M_\alpha}{k}|||\phi_1-\phi_2|||.\end{aligned}$$

Hence, since $\frac{M_\alpha}{k} < 1,\ F$ has a unique fixed point $u \in \Omega(X)$. □

**Remark 4.8.** The previous theorem extends [9, Theorem 3.4] stated previously only in case of the space of pseudo almost automorphic functions and [10, Theorem 3.3] stated in case of the space of almost automorphic functions.

Next we will study the existence of mild solutions of Eq. (1.1) that belongs to $\Omega(X)$, when the function $f$ is not Lipschitz continuous. To avoid the Lipschitz conditions considered in the previous results, we need to assume that $f$ satisfies appropriate compactness conditions.

We begin by introducing the following assumption.

(**B**) There exists a continuous nondecreasing function $W : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(t,x)\| \leqslant W(\|x\|)$$

for all $t \in \mathbb{R}$ and $x \in X$.

Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(t) \geqslant 1$ for all $t \in \mathbb{R}$, and $h(t) \to \infty$ as $|t| \to \infty$. We consider the space $C_h(X) = \left\{ u \in C(\mathbb{R}, X) : \lim_{|t|\to\infty} \frac{u(t)}{h(t)} = 0 \right\}$ endowed with the norm $\|u\|_h := \sup_{t\in\mathbb{R}} \frac{\|u(t)\|}{h(t)}$. In our next result we will need the following Lemma.

**Lemma 4.9** [21]. *A subset $K \subset C_h(X)$ is a relatively compact set if it verifies the following conditions:*

(a) *The set $K(t) := \{u(t) : u \in K\}$ is relatively compact in $X$ for each $t \in \mathbb{R}$;*
(b) *The set $K$ is equicontinuous;*
(c) *For each $\epsilon > 0$ there exists $L > 0$ such that $\|u(t)\| < \epsilon h(t)$ for all $u \in K$ and all $|t| > L$.*

The next theorem is based on the Leray–Schauder alternative theorem [19, Theorem 6.5.4], which read as follows:

**Lemma 4.10.** [19] *Let $D$ be a closed convex subset of a Banach space $X$ such that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then either the set $\{x \in D : x = \lambda F(x), 0 < \lambda < 1\}$ is unbounded or the map $F$ has a fixed point in $D$.*

**Theorem 4.11.** *Let $\Omega(X) \in \mathcal{M}(X) \setminus \{PP_\omega(X), PAP(X), PAA_c(X), PAA(X), P_0(X)\}$ be given. Assume that $A$ is sectorial of type $\mu < 0$ with $0 \leqslant \theta < \pi(1 - \alpha/2)$. Let $f \in \Omega(\mathbb{R} \times X, X)$ be such that satisfy the assumption (**B**), and the following conditions:*

(1) *$f(t,x)$ is uniformly continuous in any bounded subset $K \subset X$ uniformly in $t \in \mathbb{R}$.*
(2) *For each $r \geqslant 0$, the function $t \to \int_{-\infty}^{t} \frac{W(rh(s))}{1+|\omega|(t-s)^\alpha} ds$ is included in $BC(\mathbb{R})$. We set*

$$\beta(r) := CM \left\| \int_{-\infty}^{(\cdot)} \frac{W(rh(s))}{1 + |\omega|((\cdot) - s)^\alpha} ds \right\|_h$$

(3) *For each $\epsilon > 0$ there is $\delta > 0$ such that for every $u,\ v \in C_h(X), \|u - v\|_h < \delta$ implies that $\int_{-\infty}^{t} \frac{\|f(t,u(s)) - f(t,v(s))\|}{1+|\omega|(t-s)^\alpha} ds \leqslant \epsilon$ for all $t \in \mathbb{R}$.*
(4) $\liminf_{\epsilon\to\infty} \frac{\epsilon}{\beta(\epsilon)} > 1$.
(5) *For all $a,\ b \in \mathbb{R},\ a < b$, and $r > 0$, the set $\{f(s, h(s)x) : a \leqslant s \leqslant b, x \in X, \|x\| \leqslant r\}$ is relatively compact in $X$.*

*Then Eq. (1.1) has a mild solution $u$ that belongs to $\Omega(X)$.*

**Proof.** We follow the argument given in [21, Theorem 4.9]. First, we define the operator $F$ on $C_h(X)$ as in the proof of Theorem 4.2. Next, we show that $F$ has a fixed point in $\Omega(X)$. In order to do that, we will divide the proof in several steps.

(i) For $u \in C_h(X)$, we have that

$$\|Fu(t)\| \leqslant CM \int_{-\infty}^{t} \frac{W(\|u(s)\|)}{1 + |\omega|(t-s)^\alpha} ds \leqslant CM \int_{-\infty}^{t} \frac{W(\|u\|_h h(s))}{1 + |\omega|(t-s)^\alpha} ds. \tag{4.3}$$

It follows from condition $(b)$ that $\lim_{|t|\to\infty} \frac{\|F(t)\|}{h(t)} = 0$ then $F : C_h(X) \to C_h(X)$.

(ii) The map $F$ is continuous. In fact, for $\epsilon > 0$, we take $\delta$ involved in condition (3). If $u, v \in C_h(X)$ and $\|u - v\|_h \leqslant \delta$, then

$$\|Fu(t) - Fv(t)\| \leqslant \left\| \int_{-\infty}^{t} S_\alpha(t-s)(f(s,u(s)) - f(t,v(s)))ds \right\| \leqslant CM \int_{-\infty}^{t} \frac{\|f(t,u(s)) - f(t,v(s))\|}{1 + |\omega|(t-s)^\alpha} ds \leqslant \epsilon,$$

which show the assertion.

(iii) We will show that $F$ is completely continuous. In order to do that, we will use Lemma 4.9. In fact, we set $B_r(X)$ for the closed ball with center at 0 and radius $r$ in a space $X$. Let $V = F(B_r(C_h(X)))$ and $v = F(u)$ for $u \in B_r(C_h(X))$.
Initially, we will prove that $V(t)$ is a relatively compact subset of $X$ for each $t \in \mathbb{R}$. It follows from condition (2) that the function $t \to \frac{W(rh(t-s))}{1+|\omega|(s)^\alpha} ds$ is integrable on $[0, \infty)$. Hence, for $\epsilon > 0$ we can choose $a \geqslant 0$ such that $CM \int_a^\infty \frac{W(rh(s))}{1+|\omega|(t-s)^\alpha} ds \leqslant \epsilon$. Since

$$\left\|\int_a^\infty S_\alpha(s)f(t-s,u(t-s))ds\right\| \leqslant CM\int_a^\infty \frac{W(rh(t-s))}{1+|\omega|(s)^\alpha}ds \leqslant \epsilon$$

we get

$$v(t) \in \overline{ac(\{S_\alpha(s)f(\xi,h(\xi)x : 0 \leqslant s \leqslant a, t-a \leqslant \xi \leqslant t, \|x\| \leqslant r)\})} + B_\epsilon(X),$$

where $c(K)$ denotes the convex hull of $K$. Using that $S_\alpha(\cdot)$ is strongly continuous and the property (5), we infer that $K = \{S_\alpha(s)f(\xi,h(\xi)x : 0 \leqslant s \leqslant a, t-a \leqslant \xi \leqslant t, \|x\| \leqslant r)\}$ is a relatively compact set, and $V(t) \subset \overline{ac(K)} + B_\epsilon(X)$, which establishes our assertion.

We will next show that the set $V$ is equicontinuous. In fact, we can decompose

$$v(t+s) - v(t) = \int_0^s S_\alpha(\xi)f(t+s-\xi,u(t+s-\xi))d\xi + \int_0^a (S_\alpha(\xi+s) - S_\alpha(\xi))f(t-\xi,u(t-\xi))d\xi + \int_a^\infty (S_\alpha(\xi+s) - S_\alpha(\xi))f(t-\xi,u(t-\xi))d\xi.$$

For each $\epsilon > 0$, we can choose $a > 0$ and $\delta_1$ such that

$$\left\|\int_0^s S_\alpha(\xi)f(t+s-\xi,u(t+s-\xi))d\xi + \int_a^\infty (S_\alpha(\xi+s) - S_\alpha(\xi))f(t-\xi,u(t-\xi))d\xi\right\| = CM\left(\int_0^s \frac{W(rh(t+s-\xi))}{1+|\omega|\xi^\alpha}d\xi + 2\int_a^\infty \frac{W(rh(t-\xi))}{1+|\omega|\xi^\alpha}d\xi\right) = \epsilon/2,$$

for $s \leqslant \delta_1$. Moreover, since $\{f(t-\xi,u(t-\xi)) : 0 \leqslant \xi \leqslant a, u \in B_r(C_h(X))\}$ is a relatively compact set and $S_\alpha(\cdot)$ is strongly continuous, we can choose $\delta_2$ such that $\|(S_\alpha(\xi+s) - S_\alpha(\xi))f(t-\xi,u(t-\xi))\| \leqslant \epsilon/2a$ for $s \leqslant \delta_2$. Combining these estimate, we get $\|v(t+s) - v(s)\| \leqslant \epsilon$ for $s$ small enough independent of $u \in B_r(C_h(X))$.

Finally, applying condition (2), we can show that

$$\frac{v(t)}{h(t)} \leqslant \frac{1}{h(t)}\int_{-\infty}^t \frac{W(rh(s))}{1+|\omega|(t-s)^\alpha}ds \to 0, \ |t| \to \infty,$$

and this convergence is independent of $u \in B_r(C_h(X))$. Hence $V$ is a relatively compact set in $C_h(X)$.

(iv) If $u^\lambda(\cdot)$ is a solution of equation $u^\lambda = \lambda F(u^\lambda)$ for some $0 < \lambda < 1$, from the estimate

$$\|u^\lambda(t)\| \leqslant CM\int_{-\infty}^t \frac{W(\|u^\lambda\| rh(s))}{1+|\omega|(t-s)^\alpha}ds \leqslant \beta(\|u^\lambda\|_h)h(t)$$

we get

$$\frac{\|u^\lambda\|_h}{\beta(\|u^\lambda\|_h)} \leqslant 1$$

and, combining it with condition (4), we conclude that the set $\{u^\lambda : u^\lambda = \lambda F(u^\lambda), \lambda \in (0,1)\}$ it is bounded.

(v) Given $\phi \in \Omega(X)$, in view of Theorems 3.3, 3.6 and 3.11, we have that $s \to f(s,\phi(s))$ belongs to $\Omega(X)$, and hence is bounded in $\mathbb{R}$. Since the function $t \to 1/(1+|\mu|t^\alpha)$ is integrable on $\mathbb{R}_+ (1 < \alpha < 2)$, we get that $F\phi$ exists. Now by Lemma 3.2, we obtain that $F\phi \in \Omega(X)$, and hence $F$ is well defined. We can consider $F : \overline{\Omega(X)}^h \to \overline{\Omega(X)}^h$ and using properties (i)–(iii), we deduce that this map is completely continuous. Using the Leray–Shauder alternative theorem [19, Theorem 6.5.4], we infer that $F$ has a fixed point $u \in \overline{\Omega(X)}$. Let $(u_n)_n$ be a sequence in $\Omega(X)$ that converges to $u$ in the norm $\|\cdot\|_h$. It follows from (2) that given $\epsilon > 0$ exists $\delta > 0$, such that if $\|u - v\|_h < \delta$ implies that

$$\|F(u_n)(t) - F(u)(t)\| \leqslant \int_{-\infty}^t \frac{\|f(t,u(s)) - f(t,v(s))\|}{1+|\omega|(t-s)^\alpha}ds \leqslant \epsilon$$

for all $t \in \mathbb{R}$. Choose $N_\delta$ such that $\|u_n - u\| < \delta$ for all $n \geqslant N_\delta$. Then we deduce that $(F(u_n))_n$ converges to $F(u) = u$ uniformly in $\mathbb{R}$. This implies that $u \in \Omega(X)$, and completes the proof. □

**Theorem 4.12.** *Let $\Omega(X) \in \{PP_\omega(X), PAP(X), PA_cA(X), PAA(X), P_0(X)\}$ be given. Assume that $A$ is sectorial of type $\mu < 0$ with $0 \leqslant \theta < \pi(1-\alpha/2)$. Let $f \in \Omega(\mathbb{R}\times X, X)$ be given such that satisfy assumption (**B**), and the following conditions:*

(1) *$f(t,x)$ is uniformly continuous in any bounded subset $K \subset X$ uniformly in $t \in \mathbb{R}$;*
(2) *For each $r \geqslant 0$, the function $t \to \int_{-\infty}^t \frac{W(rh(s))}{1+|\omega|(t-s)^\alpha}ds$ is included in $BC(\mathbb{R})$. We set*

$$\beta(r) := CM\left\|\int_{-\infty}^{(\cdot)} \frac{W(rh(s))}{1+|\omega|((\cdot)-s)^\alpha}ds\right\|_h;$$

(3) *For each* $\epsilon > 0$ *there is* $\delta > 0$ *such that for every* $u,\ v \in C_h(X), \|u - v\|_h < \delta$ *implies that* $\int_{-\infty}^{t} \frac{\|f(t,u(s)) - f(t,v(s))\|}{1+|\omega|(t-s)^{\alpha}} ds \leqslant \epsilon$ *for all* $t \in \mathbb{R}$;
(4) $\liminf_{\epsilon \to \infty} \frac{\epsilon}{\beta(\epsilon)} > 1$;
(5) *For all* $a,\ b \in \mathbb{R},\ a < b$, *and* $r > 0$, *the set* $\{f(s, h(s)x) : a \leqslant s \leqslant b, x \in X, \|x\| \leqslant r\}$ *is relatively compact in X;*
(6) *The set* $\{f(t,x) : t \in \mathbb{R}, x \in K\}$ *is bounded for every bounded* $K \subset X$.
*Then Eq.* (1.1) *has a mild solution u which belongs to* $\Omega(X)$.

**Proof.** The proof is similar to the given for the above theorem. Only we need to consider the composition Theorem 3.8 in the last part (v). □

## 5. Examples

We finish the paper with the following examples.

**Example 5.1.** Let $X = L^2[0,\pi]$ and let $\Omega(X) \in \mathcal{M}(X) \setminus \{P_{\omega}(X), AP(X), AA_c(X), AA(X)\} \cup \{C_0(X), P_0(X)\}$ be given. We consider the operator $A$ defined on $X$ by

$$Au = u'' - \tau u, \quad (\tau > 0) \tag{5.1}$$

with domain $D(A) = \{u \in L^2[0,\pi] : u'' \in L^2[0,\pi], u(0) = u(\pi) = 0\}$. It is well known that $\triangle = u''$ is the generator of an analytic semigroup on $L^2[0,\pi]$. Hence $(\tau I - A)$ is sectorial of type $\mu = -\tau$. Then the equation

$$\partial_t^{\alpha} u(t,x) = \partial_x^2 u(t,x) - \tau u(t,x) + \partial_t^{\alpha-1}(a(t) + \beta b(t)\sin(u(t,x))) \quad t \in \mathbb{R}, x \in [0,2\pi] \tag{5.2}$$

can be formulated as the inhomogeneous problem (1.1), where $u(t) = u(t,\cdot)$. Let us consider the nonlinearity $f(t,\phi) = a(t) + \beta b(t)\sin(\phi(s))$ for all $\phi \in X,\ s \in [0,\pi],\ t \in \mathbb{R}, \beta \in \mathbb{R},\ b \in C_0(\mathbb{R})$ and $a \in \Omega(X)$. We observe that $\beta b(\cdot)\sin(\phi(s)) \in C_0(X)$ imply that $f(\cdot,x)$ belongs to $\Omega(X)$ and hence we have that

$$\|f(t,\phi_1) - f(t,\phi_2)\|_2^2 \leqslant \int_0^{\pi} \beta^2 |b(t)|^2 |\sin(\phi_1(s)) - \sin(\phi_2(s))|^2 ds \leqslant \beta^2 |b(t)|^2 \|\phi_1 - \phi_2\|_2^2.$$

In consequence, the fractional differential Eq. (5.2) has a unique mild solution $u \in \Omega(X)$ if either $|b|^2 \in L^1(\mathbb{R})$ (Theorem 4.2) or $\int_{-\infty}^{t} |b(s)|^2 ds$ exist (Theorem 4.7). If we assume that $b \in L^{\infty}(\mathbb{R})$ and $|\beta| < \frac{\alpha \sin(\pi/\alpha)}{CM\|b\|_{\infty}|\mu|^{-1/\alpha}\pi}$, then same conclusion holds, by Corollary 4.5.

**Example 5.2.** Let $X = L^2[0,\pi]$ and let $\Omega(X) \in \{P_{\omega}(X), AP(X), AA_c(X), AA(X)\}$ be given. We consider the problem

$$\partial_t^{\alpha} u(t,x) = \partial_x^2 u(t,x) - \tau u(t,x) + \partial_t^{\alpha-1}(\beta b(t)\sin(u(t,x))) \quad t \in \mathbb{R}, x \in [0,2\pi]. \tag{5.3}$$

As in the above example, consider the nonlinearity $f(t,\phi) = \beta b(t)\sin(\phi(s))$ for all $\phi \in X,\ s \in [0,\pi],\ t \in \mathbb{R},\ \beta \in \mathbb{R}$ and $b(t) \in \Omega(\mathbb{R})$. We also assume that $b \in L^{\infty}(\mathbb{R})$ and $|\beta| < \frac{\alpha \sin(\pi/\alpha)}{CM\|b\|_{\infty}|\mu|^{-1/\alpha}\pi}$. We observe that $\beta b(\cdot)\sin(\phi(s)) \in \Omega(X)$ and then $f(\cdot,x)$ belongs to $\Omega(X)$. Moreover,

$$\|f(t,\phi_1) - f(t,\phi_2)\|_2^2 \leqslant \int_0^{\pi} \beta^2 |b(t)|^2 |\sin(\phi_1(s)) - \sin(\phi_2(s))|^2 ds \leqslant \beta^2 |b(t)|^2 \|\phi_1 - \phi_2\|_2^2.$$

Then, by Corollary 4.5, the problem (5.3) has a unique mild solution which belongs to $\Omega(X)$.

**Example 5.3.** Let $X = L^2[0,\pi]$ and let $\Omega(X) \in \mathcal{M}(X) \setminus \{P_{\omega}(X), AP(X), AA_c(X), AA(X)\} \cup \{C_0(X), P_0(X)\}$ be given. Just as in the above example consider the function $f : \mathbb{R} \times X \to X$ as

$$f(t,\phi)(x) = g(t) + e^{-|t|} \left| \int_0^x \sin\left(\frac{\phi(s) + 1}{\|\phi\|_{L^2} + 2}\right) ds \right|^{\beta} \sin(x),$$

where $\beta \in (0,1)$ and $g \in \Omega(X)$. Then the fractional differential Eq. (1.1) has a mild solution that belongs to $\Omega(X)$. A calculation shows the following two estimates

$$\|f(t,\phi)\|_{L^2} \leqslant \|g(t)\|_{L^2} + e^{-|t|} \frac{\pi^{(1+2\beta)/2}}{2} \quad t \in \mathbb{R},\ \phi \in X$$

and

$$\|f(t,\phi_1) - f(t,\phi_2)\|_{L^2} \leqslant e^{-|t|} 3^{\beta} \pi^{(1+2\beta)/2} \|\phi_1 - \phi_2\|_{L^2}^{\beta} \quad t \in \mathbb{R},\ \phi_1, \phi_2 \in X,$$

which follows from the fact that $f \in \Omega(\mathbb{R} \times X, X)$ is uniformly continuous on bounded sets of $X$ uniformly in $t \in \mathbb{R}$.

It is straightforward to verify that

$$\|f(t,\phi)\|_{L^2} \leqslant \|g\|_\infty + \pi^{(1+2\beta)/2}\left(\frac{\|\phi\|_{L^2}+1}{\|\phi\|_{L^2}+2}\right)^\beta \quad t \in \mathbb{R},\ \phi \in X.$$

Hence, we can define in (**B**) $W(x) = \|g\|_\infty + \pi^{(1+2\beta)/2}\left(\frac{x+1}{x+2}\right)^\beta$ and $h(t) = e^t,\ t \in \mathbb{R}$. Consequently, we see that

$$\int_{-\infty}^t \frac{W(rh(s))}{1+|\mu|(t-s)^\alpha}ds \leqslant \|W\|_\infty \int_0^\infty \frac{1}{1+|\mu|(s)^\alpha}ds < M \quad \text{for all } t \in \mathbb{R}$$

and

$$\int_{-\infty}^t \frac{\|f(s,u(s)) - f(s,v(s))\|_{L^2}}{1+|\mu|(t-s)^\alpha}ds \leqslant \int_{-\infty}^t \frac{e^{-|s|}3^\beta \pi^{(1+2\beta)/2}\|\phi_1(s)-\phi_2(s)\|_{L^2}^\beta}{1+|\mu|(t-s)^\alpha}ds \leqslant \int_{-\infty}^t \frac{3^\beta \pi^{(1+2\beta)/2}}{1+|\mu|(t-s)^\alpha}\|\phi_1-\phi_2\|_h^\beta ds$$
$$\leqslant M'\|\phi_1-\phi_2\|_h^\beta,$$

which means that conditions (2) and (3) of Theorems 4.11, 4.12 are satisfied. An easy computation leads to $\lim_{\epsilon\to\infty}\frac{\epsilon}{h(\epsilon)} > 1$. An argument involving Simon's theorem (see [35, Theorem 1, pages 71–74]) proves that the set $K = \{f(s,h(s)x) : a \leqslant s \leqslant b, x \in X, \|x\| \leqslant r\}$ is relatively compact in $X$. In fact, we can verify the estimate

$$\|f(s,\phi)\|_{L^2} \leqslant \|g\|_\infty + \pi^{(1+2\beta)/2} \quad t \in \mathbb{R},\ \phi \in X.$$

Hence, for $0 \leqslant a_1 < a_2 \leqslant \pi$, the integral $\int_{a_1}^{a_2} f(s,h(s)\phi(s))ds$ is bounded uniformly in $s$ and $\phi$. On the other hand, we can infer the following estimate:

$$|f(s,\phi(s))(x) - f(s,\phi(s))(x')| \leqslant |x-x'|^\beta + \pi^\beta|x-x'|.$$

Therefore

$$\int_0^{\pi-h} |f(s,\phi(s))(x+h) - f(s,\phi(s))(x)|dx \to 0 \quad as\ h \to 0$$

uniformly in $s$ and $\phi$. Finally, Simon's theorem leads to the conclusion that $K$ is relatively compact. Using Theorems 4.11 and 4.12, respectively, we deduce that Eq. (1.1) has a mild solution that belongs to $\Omega(X)$.

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