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Adapted hyperbolic polygons and symplectic representations for group actions on Riemann surfaces

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ABSTRACT

We prove that given a finite group G together with a set of fixed geometric generators, there is a family of special hyperbolic polygons that uniformize the Riemann surfaces admitting the action of G with the given geometric generators. From these special polygons, we obtain geometric information for the action: a basis for the homology group of surfaces, its intersection matrix, and the action of the given generators of G on this basis. We then use the Frobenius algorithm to obtain a symplectic representation $\mathcal{G}$ of G corresponding to this action. The fixed point set of $\mathcal{G}$ in the Siegel upper half-space corresponds to a component of the singular locus of the moduli space of principally polarized abelian varieties.

We also describe an implementation of the algorithm using the open source computer algebra system SAGE.

1. Introduction

For $g \geq 4$, the singular locus $\mathrm{Sing}(\mathfrak{M}_g)$ of the moduli space $\mathfrak{M}_g$ of compact Riemann surfaces of genus g is the set of (isomorphism classes of) Riemann surfaces of genus g with nontrivial automorphisms ([16]); similarly, the singular locus $\mathrm{Sing}(\mathfrak{A}_g)$ of the moduli space of principally polarized abelian varieties of dimension g is the set of (isomorphism classes of) principally polarized abelian varieties with nontrivial group of automorphisms ([15]). By Torelli's theorem, we will consider $\mathfrak{M}_g \subset \mathfrak{A}_g$ by associating to a Riemann surface M its Jacobian variety JM. Since every automorphism of M induces an automorphism of JM, $\mathrm{Sing}(\mathfrak{M}_g) \subset \mathrm{Sing}(\mathfrak{A}_g)$.

There is a complete description of the components of $\mathrm{Sing}(\mathfrak{M}_g)$ inside $\mathfrak{M}_g$, see [3], but much less is known about $\mathrm{Sing}(\mathfrak{A}_g)$; for instance, there is nothing similar to the familiar Hurwitz upper bound $84(g-1)$ for the order of a group acting on a Riemann surface of genus g in the case of a group acting on a principally polarized abelian variety of dimension g.

If G is the group of automorphisms of a compact Riemann surface M of genus g, then G acts on the homology $H_1(M, \mathbb{Z})$ of M, and by choosing a symplectic basis for $H_1(M, \mathbb{Z})$, one obtains a symplectic representation of G, in $\mathrm{Sp}(2g, \mathbb{Z})$. Since the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on the Siegel upper half space of complex symmetric $g \times g$ matrices with positive definite imaginary part, the set of matrices fixed under the symplectic action for G will contain a period matrix for M, but also, in general, many other period matrices for principally polarized abelian varieties admitting the same group action; that is, the set of fixed period matrices will be a family contained in $\mathrm{Sing}(\mathfrak{A}_g)$. The advantage of knowing that a given finite symplectic group comes from an action of the group of a Riemann surface is that this ensures that its set of fixed matrices is nonempty.

As we mentioned earlier, the components of $\mathrm{Sing}(\mathfrak{M}_g)$ inside $\mathfrak{M}_g$ have been described; however, even for *large* automorphism groups, a standard term used to denote groups of order larger than $4(g-1)$ acting on Riemann surfaces

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of genus g (see [14, Section 1.3] and [1, Lemma 3.18]), computer searches for such groups have been carried out and lists have been published ([14], [1]), but no algebraic equations nor period matrices are known for most of these curves.

It is the main goal of this paper to provide an algorithmic method to construct a symplectic representation for the action of a group G on a Riemann surface M of genus $g \geq 2$ such that the quotient surface M/G has genus zero. This symplectic representation will be the same for all the Riemann surfaces admitting the same geometric action, defined in terms of fixing a generating vector for G; see Sections 5–6.1. Note that a large group necessarily satisfies our condition for the genus zero quotient, and that once the symplectic representation is obtained, the period matrices for the full family of principally polarized abelian varieties admitting the same G-action may be found.

In her paper [11], Linda Keen found a uniquely determined fundamental polygon for every finitely generated Fuchsian group together with a certain "standard"system of generators: a *canonical polygon*.

In particular, such a canonical polygon P for a compact Riemann surface M of genus g (and a standard system of generators) always has $4g$ sides, corresponding to $2g$ geodesics of M whose homotopy classes generate its fundamental group. If the group G of conformal automorphisms of M is nontrivial, then the action of G on this set of geodesics is far from evident. However, the fundamental polygon she constructs for the case of a sphere with punctures will be our starting point.

On the other hand, in a series of papers by Jane Gilman (see [7] and [8]) a conformal automorphism h of prime order acting on a compact Riemann surface M is considered. It is shown that there exists an *adapted basis* for the homotopy (and the first homology) of M, in the sense that it reflects the action of h in an optimal way ([8, Def.3.1] or [7, Cor.3.3]). The intersection matrix for this basis is computed, and a unique normal form for the corresponding symplectic representation of h is found; this normal form depends only on a tuple of integers, the *conformal invariants* of h.

In this paper we follow along the line of these results, in the sense that we will start from a given finite group G and a marked set of generators for G, and end up with a symplectic representation for G, obtained from the construction of a family of special fundamental polygons for all the compact Riemann surfaces M on which G acts with quotient M/G of genus zero as prescribed by the given set of generators.

However, our construction differs from the results of Gilman ([8]), not only in the form but in the purpose: we are not looking for a basis of $H_1(M, \mathbb{Z})$ such that the action of the group on it is in a particularly nice form (see [8, Def.3.1] or [7, Cor.3.3]). We look for polygons from which it is possible (1) to describe a homology basis formed by (classes of) curves on its boundary, and (2) to compute its intersection matrix and the action of the group on this basis from the polygon itself and the given generators for the group acting on the surface.

This is perhaps the main difference in both procedures: Gilman's method starts from a presentation for the Fuchsian group T that uniformizes the quotient surface $M/\langle h\rangle$. Her proof of the existence of an adapted basis for M uses the Schreier–Reidemeister rewriting process, by finding a standard presentation (with $2g$ generators) for the subgroup Γ that uniformizes the surface M from the given presentation for T. The intersection matrix for this basis depends on the conformal invariants of the conjugacy class of h in the mapping-class group of M.

We do not study if her process can be generalized to any group, because our goal is to present a different method to find a symplectic representation of a group acting on a (compact) Riemann surface, which works for any group of conformal automorphisms of a compact Riemann surface M such that M/G has genus zero, the point being that for a general group G the standard canonical polygon for M, with $4g$ sides, does not reflect the action of G in a useful way.

In our approach, the original data is a finite group G, together with a fixed set of generators satisfying certain conditions, that depend on the geometry of the action (see Definition 2.1). These data replace the conformal invariants available in the case of a cyclic group of prime order. The Fuchsian group Γ uniformizing M may be recovered from the data for the group G, as the normal closure in the group Fuchsian T uniformizing M/G of the elements of T corresponding to the relations in the presentation of G satisfied by its given generators; this description for Γ is completely different from its standard presentation with $2g$ generators.

Once we prove the existence of the required polygons, and from them find the homology basis, its intersection matrix, and the action of the given generators for the group on this basis, we use the Frobenius algorithm, which is analogous to finding an orthogonal basis for symmetric forms, to modify our basis to a symplectic one. By conjugating by the change of basis matrix the matrices for the generators obtained previously, we obtain the sought symplectic representation of the group action.

The work is structured as follows: in Section 2 the definitions, notations and framework are established. The new method we present in this paper is described in the following four sections, each being a step toward the goal of finding a symplectic representation. The steps are as follows.

- Step one (Section 3): construction of an *adapted fundamental polygon* for a Fuchsian group T (see Definition 3.3) together with fixed generators $t_1, \ldots t_{k+1}$ satisfying certain presentation (see Eq. (2.1)).
- Step two (Section 4): construction of an *adapted hyperbolic polygon* (see Definition 4.2) for any surface M of genus $g \geq 2$ with a large group of conformal automorphisms G given in terms of an (admissible) generating vector $\mathbf{c} = (c_1, \ldots, c_{k+1})$ for the action of G (see Definitions 2.1 and 2.2).
- Step three (Section 5): the adapted hyperbolic polygon constructed on step two is a special fundamental polygon for the Fuchsian group uniformizing the surface, in the sense that a homology basis for $H_1(M, \mathbb{Z})$ may be found that is formed by sides on the boundary of it, and the action of G on it maybe represented.

  In this step, we find such a homology basis and the corresponding representation of G on it.

- Step four (Section 6): we present a method to compute the intersection matrix of the basis found on step three, from the polygon and the action of G on it.
- Step five (Section 6.1): we use the Frobenius algorithm to symplectify the intersection matrix obtained in step four, and use this basis change to find the symplectic representation for G corresponding to its action on M given by the generating vector $\mathbf{c}$.

We have implemented these step methods in a computational program in the open source platform SAGE.

In Section 7, we use this symplectic representation for G to describe the locus of principally polarized abelian varieties with the given action of G.

Finally, we want to remark that we consider here M/G being of genus 0, so that we can construct a fundamental polygon for M where all the vertices on its boundary correspond to fixed points of elliptic elements of the Fuchsian group T uniformizing M/G. If the quotient surface has genus greater than 0, the situation is rather different: one needs to also consider hyperbolic elements for T. We will not treat this general case in this work.

2. Framework

We now recall definitions needed in the paper and fix the notation.

An *action* of G (or a G-action) on M is an injective homomorphism $\phi : G \to \mathrm{Aut}(M)$.

A (finite) group G acts on M with *signature*

$$(\gamma; m_1, \dots, m_{k+1}),$$

if γ is the nonnegative integer denoting the genus of the quotient surface M/G, the cover $\pi_G : M \to M/G$ is ramified over $k+1$ points $P_1, \dots, P_{k+1}$, and the m_j are integers at least equal to two, such that π_G is locally m_j to 1 over the corresponding branch point P_j (see [6]). The signature is unique if the m_j are listed in nonincreasing order; we will always assume $m_1 \geq m_2 \geq \dots \geq m_{k+1} \geq 2$.

In this paper we will concentrate on groups which are usually called *large*; that is, groups whose quotient surface has genus zero; this type of groups has proven to be of use, for example in studying moduli space (c.f. [20] and [13]). From now on the action of G on M will be with quotient of genus zero, and the signature $(0; m_1, \dots, m_{k+1})$ of G on M will be abbreviated $(m_1, \dots, m_{k+1})$.

It is known (see for instance [9] or [2, Section 2]) that every action of G on M may be constructed by giving a pair of Fuchsian groups T and Γ, together with an epimorphism $\psi : T \to G$, termed in [10, Section 3] *surface-kernel homomorphism*. In this case, $\Gamma = \ker(\psi)$ is a torsion-free normal subgroup of T, isomorphic to the fundamental group of M, and T has a presentation of the following form.

$$T = \left\langle t_1, t_2, \dots, t_{k+1}; t_1^{m_1} = \dots = t_{k+1}^{m_{k+1}} = 1 = \prod_{j=1}^{k+1} t_j \right\rangle . \tag{2.1}$$

We summarize this information in the following associated short sequence, which will be used repeatedly throughout this work.

$$1 \longrightarrow \Gamma \longrightarrow T \xrightarrow{\psi} G \longrightarrow 1. \tag{2.2}$$

Our starting point will be a finite group G and adequate generators for it. Note that we are not assuming any action of the group at this point. The following definition was introduced in [2].

Definition 2.1. Let G be a finite group and consider $k+1$ integers m_j such that $m_1 \geq m_2 \geq \cdots \geq m_{k+1} \geq 2$.

A $k+1$-tuple $(c_1, \dots, c_{k+1})$ of nontrivial elements of G is called an $(m_1, \dots, m_{k+1})$-*generating vector for G* (or surface kernel generators) if the following conditions are satisfied:

(1) G is generated by the elements $\{c_1, \dots, c_{k+1}\}$,
(2) the order of c_j is m_j for each $1 \leq j \leq k+1$, and
(3) $\prod_{j=1}^{k+1} c_j = 1$.

Note that in the case of having a sequence as in (2.2), the generating vector corresponds to the image by the surface-kernel homomorphism ψ of the generators of T satisfying the presentation given in (2.1). Closely related notions were used in [7, Section 5.1] and [19, Proposition 3].

Definition 2.2. A generating vector $(c_1, \dots, c_{k+1})$ for G is *admissible* if

$$\sum_{j=1}^{k+1} \left(1 - \frac{1}{m_j}\right) > 2, \tag{2.3}$$

where m_j is the order of c_j, $j = 1, \dots, k+1$, and if the following number is an integer

$$g = 1 + |G| \left(-1 + \frac{1}{2} \sum_{j=1}^{k+1} \left(1 - \frac{1}{m_j}\right)\right). \tag{2.4}$$

Remark 2.3. Note that Condition (2.3) is equivalent to the following.

(1) $k \geq 2$, and
(2) if $k = 2$, then $(m_1, m_2, m_3) \neq (6, 3, 2), (3, 3, 3), (4, 4, 2), (5, 3, 2), (4, 3, 2), (m, 2, 2), (3, 3, 2)$, and
(3) if $k = 3$, then $(m_1, m_2, m_3, m_4) \neq (2, 2, 2, 2)$.

The conditions are imposed so that G acts on a Riemann surface M of genus g and so that M/G is a hyperbolic manifold. Furthermore, under these conditions the integer g of (2.4) satisfies $g \geq 2$.

From now on we will only consider admissible generating vectors.

Definition 2.4. Given an action of G on M, or equivalently given T, Γ and ψ as before, consider the presentation for T given by (2.1). Setting $c_j = \psi(t_j)$ for $j = 1, \ldots, k+1$, it is clear that then $(c_1, \ldots, c_{k+1})$ is an $(m_1, \ldots, m_{k+1})$-generating vector for G.

Conversely, given a Fuchsian group T presented as in (2.1) and an $(m_1, \ldots, m_{k+1})$-generating vector $(c_1, \ldots, c_{k+1})$ for G, we can define the epimorphism $\psi : T \to G$ by homomorphic extension to T of the assignment $\psi(t_j) = c_j, j = 1, \ldots, k+1$, for its generators; its kernel Γ is a torsion-free subgroup of T, and it defines a compact Riemann surface $M = \Delta/\Gamma$, where Δ is the (complex) unit disk. Furthermore, M has a natural G-action, since T/Γ (isomorphic to G) acts canonically on Γ-orbits of Δ, and the G-action has signature $(m_1, \ldots, m_{k+1})$.

In this situation, we will say that *G acts on M with generating vector* $(c_1, \ldots, c_{k+1})$.

Remark 2.5. Given a finite group G and an admissible generating vector $\mathbf{c} = (c_1, \ldots, c_{k+1})$ for the group, G has a presentation of the form

$$G = \langle c_1, \ldots, c_{k+1} : c_j^{m_j}, c_1 \ldots c_{k+1}, R_i \rangle \tag{2.5}$$

where the R_i denote extra relations among the generators c_j needed to have the presentation on the right hand side define a finite group.

If G acts on a Riemann surface M with generating vector $\mathbf{c}$, then the Fuchsian group Γ uniformizing M is the normal subgroup of T generated by the words in the relations R_i expressed in terms of the generators t_j of T in the presentation (2.1); this follows immediately from the exact sequence (2.2).

This property will be used to recover the group Γ from the fundamental polygon we will construct; see Section 4.1.

Definition 2.6. Let G be a finite group and $\mathbf{c} = (c_1, \ldots, c_{k+1})$ be an admissible generating vector for G. Then $\mathcal{H} = \mathcal{H}(G, \mathbf{c})$ will denote the set of compact Riemann surfaces of genus g admitting a G-action with generating vector $\mathbf{c}$.

Remark 2.7. Riemann existence type theorems can be used to prove the existence of a Riemann surface with the action of G with a given signature [2, Proposition 2.1]. But our work will show that there is a family of fundamental polygons representing all Riemann surfaces with G-action and a fixed admissible generating vector; see Remark 4.1.

Remark 2.8. In order to compute the dimension of $\mathcal{H}(G, \mathbf{c})$, we look at a representative point M in T_g, the Teichmüller space of genus g. Then, considering G (originally a subgroup of $\mathrm{Aut}(M)$) as a subgroup of the mapping class group acting on T_g, it is known that G fixes a submanifold R_G of T_g of dimension $n-3$, where n is the number of branch points of $M \longrightarrow M/G$ (see [5]). Therefore, the complex dimension of $\mathcal{H}(G, \mathbf{c})$ is $k-2$ for a G-action with generating vector $\mathbf{c} = (c_1, \ldots, c_{k+1})$.

Example 2.9. Generating vectors are relevant.

(1) Let $G = \langle c : c^7 \rangle$ be the cyclic group of order 7. By considering the generating vector

$$\mathbf{c_1} = (c, c^2, c^4)$$

$\mathcal{H}(\mathbb{Z}/7\mathbb{Z}, \mathbf{c_1})$ is the surface of genus three with affine equation $y^7 = x^2(x-1)$ known as the Klein curve; this is the genus three curve with action of $PSL(2, 7)$, fulfilling the Hurwitz bound.

On the other hand, if we consider the generating vector

$$\mathbf{c_2} = (c, c, c^5)$$

then $\mathcal{H}(\mathbb{Z}/7\mathbb{Z}, \mathbf{c_2})$ is the hyperelliptic surface of genus three with affine equation $y^7 = x(x-1)$. Its full automorphism group is the cyclic group of order fourteen.

To see these examples in detail (and the general case of cyclic groups of prime order p with signature $(0; p, p, p)$) see [17].

(2) Take G as S_5, the symmetric group in five letters. By considering the generating vector

$$\mathbf{c_1} = ((1, 2, 3, 4, 5), (2, 5, 4, 3), (1, 5)),$$

$\mathcal{H}(S_5, \mathbf{c_1})$ consists of a unique surface of genus four called Bring's curve (see [18]).

However, considering the generating vector

$$\mathbf{c_2} = ((1, 2)(3, 4, 5), (1, 2, 3, 4), (1, 3)(4, 5)),$$

$\mathcal{H}(S_5, \mathbf{c_2})$ consists of a unique surface of genus six called Wiman's curve (c.f. [21]).

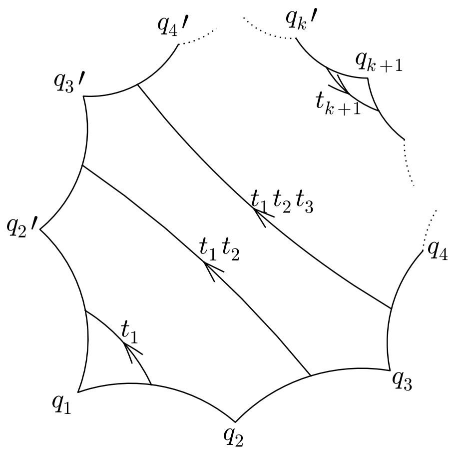

Fig. 1. Canonical fundamental polygon Q for T.

3. Step one: a fundamental polygon for the quotient

In this section we consider a Fuchsian group T uniformizing the Riemann sphere $\widehat{\mathbb{C}}$ with $k+1$ branch points, together with fixed generators $t_1, \ldots, t_{k+1}$ satisfying the presentation given by (2.1). From this information we now construct a suitable fundamental polygon F for T.

We first recall the following definition and proposition from [11]. A hyperbolic polygon that is also a fundamental region for a Fuchsian group T is called *strictly convex* if all its interior angles are strictly less than π, except possibly for those at vertices which are fixed points of elliptic transformations in T of order two.

Proposition 3.1. *[11] Let k be an integer larger than or equal to 3, and let T be a Fuchsian group uniformizing $\widehat{\mathbb{C}}$ with signature $(0; m_1, \ldots, m_{k+1})$, where the m_j satisfy condition (2.3). Fix an ordered sequence of generators $t_1, \ldots, t_{k+1}$ for T satisfying the relations in (2.1).*

Then there exists a strictly convex fundamental polygon Q for T with corresponding side pairing as follows; see Fig. 1.

(1) *The boundary of Q is composed by $2k$ consecutive geodesic segments $\{s_1, s_2, \ldots, s_k, s'_k, s'_{k-1}, \ldots, s'_1\}$, with*

$$t_1 t_2 \ldots t_j(s_j) = s'_j \quad \textit{for } 1 \le j \le k;$$

(2) *the corresponding vertices $\{q_1, q_2, \ldots, q_k, q_{k+1}, q'_k, q'_{k-1}, \ldots, q'_2\}$ satisfy*

$$t_j(q_j) = q_j \quad \textit{for } 1 \le j \le k+1;$$

(3) *the angle at q_1 is $2\pi/m_1$, the angle at q_{k+1} is $2\pi/m_{k+1}$, and the sum of the angles at q_i and q'_i is $2\pi/m_i$ for $i = 2, \ldots, k$.*

We now modify this fundamental polygon Q to one more suitable to our needs. Note that Q has $2k$ sides, and all its vertices are elliptic fixed points: q_j is the fixed point of t_j, for j in $\{1, \ldots, k+1\}$, and $q'_i = (t_1 \ldots t_{i-1})(q_i)$ is the fixed point of the conjugate of t_i by $(t_1 \ldots t_{i-1})^{-1}$, for i in $\{2, \ldots, k\}$. Our polygon will still have $2k$ sides, but its vertices will be the fixed points of all the t_j and some conjugates of t_{k+1}, as shown next.

Proposition 3.2. *Let k be an integer larger or equal to 3, and let T be a Fuchsian group uniformizing $\widehat{\mathbb{C}}$ with signature $(0; m_1, \ldots, m_{k+1})$, where the m_j satisfy the condition (2.3). Fix an ordered sequence of generators $t_1, \ldots, t_{k+1}$ for T satisfying the relations in (2.1).*

Then there exists a fundamental polygon F for T with corresponding side pairing as follows; see Fig. 2.

(1) *The boundary of F is composed by $2k$ consecutive geodesic segments $\{s_1, s_2, \ldots, s_{2k}\}$, with*

$$t_j(s_{2j-2}) = s_{2j-1} \quad \textit{for } 2 \le j \le k, \quad \textit{and} \quad t_1(s_{2k}) = s_1;$$

(2) *the corresponding vertices $\{q_1, q^1_{k+1}, q_2, q^2_{k+1}, \ldots, q_k, q^k_{k+1}\}$ satisfy*

$$t_j(q_j) = q_j \quad \textit{for } 1 \le j \le k, \quad \textit{and} \quad t_{k+1}(q^k_{k+1}) = q^k_{k+1};$$

(3) *the interior angle at each q_j is $\frac{2\pi}{m_j}$ for $1 \le j \le k$, and the sum of the interior angles θ_l at q^l_{k+1} is $\displaystyle\sum_{l=1}^{k} \theta_l = \frac{2\pi}{m_{k+1}}$.*

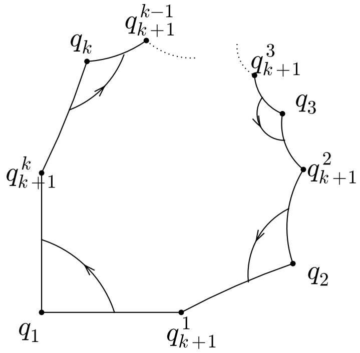

Fig. 2. Adapted fundamental polygon F for T.

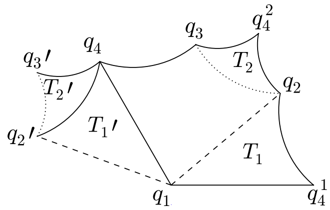

Fig. 3. Modifying Q to obtain F (with $k = 3$).

Proof. Given T and $t_1, \ldots, t_{k+1}$ satisfying the hypotheses, consider the corresponding polygon Q from Lemma 3.1.

For j in $\{1, \ldots, k\}$ set

$$q_{k+1}^{j} = (t_1 t_2 \ldots t_j)^{-1}(q_{k+1});$$

note that $q_{k+1}^{k} = (t_1 \ldots t_k)^{-1}(q_{k+1}) = t_{k+1}(q_{k+1}) = q_{k+1}$.

The following hyperbolic triangles (given by their vertices) are congruent in pairs by elements of T:

$$\begin{aligned}
T_1' &= (q_1, q_{k+1}, q_2') & T_1 &= (q_1, q_{k+1,1}, q_2)\\
T_2' &= (q_2', q_{k+1}, q_3') & T_2 &= (q_2, q_{k+1,2}, q_3)\\
&\vdots & &\vdots\\
T_{k-1}' &= (q_{k-1}', q_{k+1}, q_k') & T_{k-1} &= (q_{k-1}, q_{k+1,k-1}, q_k).
\end{aligned}$$

It is clear that

$$F = \left(Q \setminus \bigcup_{j=1}^{k-1} T_j'\right) \cup \bigcup_{j=1}^{k-1} T_j$$

satisfies the required conditions (Fig. 3). □

Definition 3.3. We may always assume that the fundamental polygon F constructed in Proposition 3.2, has the vertex q_1 at the origin, and the next vertex q_{k+1}^{1} on the positive real axis, by conjugating T and its given generators by an automorphism of the unit disk. This unique F will be called an *adapted fundamental polygon* for T and its fixed set of generators.

Definition 3.4. For the excluded case $k = 2$ in Proposition 3.2, F is the unique hyperbolic quadrilateral with one vertex at the origin and angle $2\pi/m_1$ there, the second vertex on the positive real axis and angle π/m_3, the third vertex in the upper half-plane with angle $2\pi/m_2$ and the fourth vertex with angle π/m_3.

Example 3.5. Consider Example 2.9(1). The polygon F for the corresponding quotient surfaces for both actions by the cyclic group of order seven is exactly the same (see Fig. 4).

Remark 3.6. The converse to Proposition 3.2 also holds, by Poincaré's theorem.

Proposition 3.2 may also be proved directly by construction, in which case the (real) moduli for the family of adapted polygons become explicit.

The proof of the following result uses hyperbolic geometry, we will not include it here because it falls out of the scope of this work.

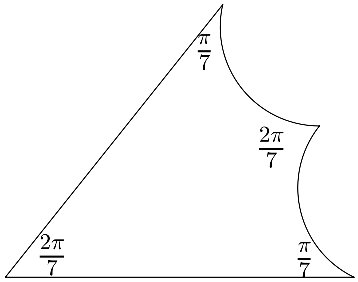

Fig. 4. Adapted polygon F in the triangular case, with signature $(7, 7, 7)$.

Proposition 3.7. *Let k be any integer greater than two and let $m_1, \ldots, m_{k+1}$ be integers greater than or equal to two satisfying the condition in (2.3).*

Then, for each $(\ell_1, \ldots, \ell_k, \theta_1, \ldots \theta_{k-4})$ in an open set in $\mathbb{R}^{2k-4}$ (if k is greater than three; (ℓ_1, ℓ_2) in $\mathbb{R}^2$ if k is three), there exists a hyperbolic polygon with $2k$ consecutive edges of corresponding lengths ℓ_1, $\ell_1, \ldots, \ell_k$, ℓ_k and corresponding interior angles $\frac{2\pi}{m_1}, \theta_1, \frac{2\pi}{m_2}, \theta_2, \ldots, \frac{2\pi}{m_k}, \theta_k$, where θ_{k-3} (if k greater than three), θ_{k-2}, θ_{k-1} and θ_k are uniquely determined and satisfy

$$\sum_{i=1}^{k} \theta_i = \frac{2\pi}{m_{k+1}} \quad .$$

Moreover, for each such polygon, the group T generated by the elliptic elements t_j which pair consecutive edges of lengths ℓ_j, ℓ_j, $j = 1, \ldots, k$, is a Fuchsian group with the given fundamental polygon; furthermore, T uniformizes $\widehat{\mathbb{C}}$ with signature $(0; m_1, \ldots, m_{k+1})$ and has a presentation of the form given in (2.1).

Definition 3.8. We will denote by $\mathcal{F} = \mathcal{F}((0; m_1, \ldots, m_{k+1}))$ the family of hyperbolic polygons constructed in Proposition 3.7. This is the family of adapted fundamental polygons for all Fuchsian groups T uniformizing $\widehat{\mathbb{C}}$ with signature $(0; m_1, \ldots, m_{k+1})$, together with a given set of generators $(t_1, \ldots, t_{k+1})$ satisfying the presentation given in (2.1).

4. Step two: fundamental polygons for the surfaces

In this section, we prove that given G together with an admissible generating vector $\mathbf{c} = (c_1, \ldots, c_{k+1})$ (see Definition 2.2), there exists a family of hyperbolic polygons which uniformize the surfaces admitting this action, and from which the following data may be obtained: a basis for the homology of the corresponding surface, its intersection matrix, and the action of the generators of G on this basis.

We start with a group G together with an admissible generating vector $\mathbf{c} = (c_1, \ldots, c_{k+1})$. Recall from Section 2 that for each $M \in \mathcal{H}(G, \mathbf{c})$ (see Definition 2.6), there exists a Fuchsian group T uniformizing M/G, together with a set of generators $(t_1, \ldots, t_{k+1})$ satisfying the presentation given on (2.1). From Proposition 3.2 we obtain a corresponding adapted polygon F in $\mathcal{F}$ (see Definition 3.8).

A fundamental polygon D for M is obtained by taking $|G|$ images of F under adequate elements of T.

In order to simultaneously choose the suitable images of F and obtain the side pairing for D, we follow the ideas proposed by Dehn, (c.f.[4]).

Consider the group T and its given generators $(t_1, \ldots, t_{k+1})$ satisfying presentation (2.1).

Since

$$t_1 t_2 \ldots t_{k+1} = 1,$$

we may consider the infinite directed Cayley diagram associated to the group T and its generators $t_1, \ldots, t_k$; that is, we forget the generator t_{k+1}.

This is an infinite planar graph whose vertices are the elements of T, and two such vertices g and h are joined by an edge from g to h if there exists j in $\{1, \ldots, k\}$ such that $g = ht_j$; this edge is colored by t_j, and directed from g to h.

Then consider its dual tessellation of the unit disk, given by the images under all elements t of T of the adapted fundamental polygon F under T constructed in the previous section, making F correspond to the identity vertex of the diagram and $t(F)$ to the vertex t.

Now, from this tessellation we choose one copy of F for each element of G, or, equivalently, for each coset representative of T/Γ. The result is, of course, a fundamental set for the action of Γ on M.

The algorithm we use to obtain a subset $S \subseteq T$ corresponding to coset representatives so that the result is actually a polygon is the following.

(1) First include in S the whole orbit $t_1^k(F)$ under the subgroup generated by the generator t_1 of T corresponding to the element $t_1\Gamma$ of G of greatest order; in this way the polygon will start with rotational symmetry at the origin.

(2) Continue by letting t_i be the next generator of T, right-multiply each element $s \in S$ by t_i and append st_i to S in case the corresponding class in G is not yet represented there. Note that we only consider those elements of $s \in S$ which were present before the current iteration of (2).

(3) Repeat step (2) until $|S| = |G|$.

The result is a fundamental polygon D for Γ.

An example is shown in Figs. 8 and 9.

Remark 4.1. The set $\mathcal{D} = \mathcal{D}(G, \mathbf{c})$ of all fundamental polygons D as constructed above parameterizes the set of all normalized Fuchsian groups uniformizing the Riemann surfaces M with the given G-action, marked by the choice $\mathbf{c}$ of generators; that is, we have obtained a covering of $\mathcal{H}(G, \mathbf{c})$, where the lengths of the edges $q_1q_{k+1}^1, q_2q_{k+1}^2, \ldots, q_kq_{k+1}^k$ of F and the angles at the vertices $\{q_{k+1}^1, q_{k+1}^2, \ldots, q_{k+1}^{k-4}\}$ of F are the required $2k-4$ real parameters, for $k \geq 4$ (see Remarks 2.8 and 3.8).

Definition 4.2. Given G and an admissible generating vector $\mathbf{c} = (c_1, \ldots, c_{k+1})$ for G, any D in $\mathcal{D}(G, \mathbf{c})$ will be called an *adapted fundamental polygon* for the action of G.

Remark 4.3. Note that each adapted fundamental polygon D is tessellated by $|G|$ copies of F, labeled by the elements of G.

We will denote the copy corresponding to g in G by F^g, or, sometimes, just by g.

Furthermore, the boundary of D is composed of geodesic arcs, each such arc belonging to one copy of F inside D and one copy of F outside D.

The vertices of D project to ramification points in the surface M; they are naturally grouped into Γ-orbits, that may be determined once the side pairing is deduced; this will be done next.

4.1. The side pairing for D

The side pairing for D is obtained as follows. Let e denote a side of D; then there exist two unique copies of F sharing the side e: F_1 inside D and F_2 outside D.

But there is a unique copy F_3 of F inside D and corresponding to the same coset of T/Γ as F_2: the side of F_3 corresponding to e, which necessarily lies on the boundary of D, is paired to e.

Note that a path along the underlying Cayley graph from one side of D to its paired side represents an element of T that belongs to the kernel of the homomorphism $\psi : \Gamma \to T$ given in (2.2): also a relation in G satisfied by the given generators for G. In this way we recover the group Γ uniformizing M: it is the normal closure in T of the subgroup generated by these paths, choosing one for each pair of sides; see Remark 2.5.

We now illustrate the construction with several examples.

Example 4.4. Continuing with Example 2.9(1) and Example 3.5, the polygons D for the Klein surface and the hyperelliptic one look exactly the same, as seen in Fig. 6; the difference lies in the side pairing, that comes from the relation satisfied by the chosen generators for G.

In the Klein case, the generating vector is (c, c^2, c^4), and the corresponding presentation for the group $G = \mathbb{Z}/7\mathbb{Z}$ is as follows.

$$\mathbb{Z}/7\mathbb{Z} = \langle c_1, c_2 : c_j^7, (c_1c_2)^7, c_2c_1^{-2} \rangle.$$

The group Γ uniformizing the Klein surface is then given by the normal closure in T of $t_2t_1^{-2}$; all the paths along the Cayley diagram joining paired sides are T-conjugate to this one.

Following the Cayley graph depicted on Fig. 5, and looking at the labeling in Fig. 6, the side pairing for the Klein surface is 1–6, 2–11, 3–8, 4–13, 5–10, 7–12, 9–14.

In the hyperelliptic case, the generating vector is (c, c, c^5), and the corresponding presentation for the group $G = \mathbb{Z}/7\mathbb{Z}$ is as follows.

$$\mathbb{Z}/7\mathbb{Z} = \langle c_1, c_2 : c_j^7, (c_1c_2)^7, c_2c_1^{-1} \rangle.$$

Therefore the group Γ uniformizing the hyperelliptic surface is given by the normal closure in T of $t_2t_1^{-1}$, and the side pairing for D is: 1–4, 2–13, 3–6, 5–8, 7–10, 9–12, 11–14.

Here lies another difference with Gilman's method (see [7, Remark 3.7]). In her case, these two automorphisms will have the same matrix representation with respect to an adapted basis. As we have mentioned, we are not adapting the basis to fulfill any condition on the *shape* of the matrix representation for the automorphisms group. We will use the same polygon D to represent the action on the homology (see Section 5) for both situations. The difference between the side pairings will be reflected immediately on the different (nonconjugate) representations we obtain.

The following example illustrates all the previous constructions in the case of the action of the symmetric group S_4 of degree four on genus four. We will carry on this example throughout the rest of the paper.

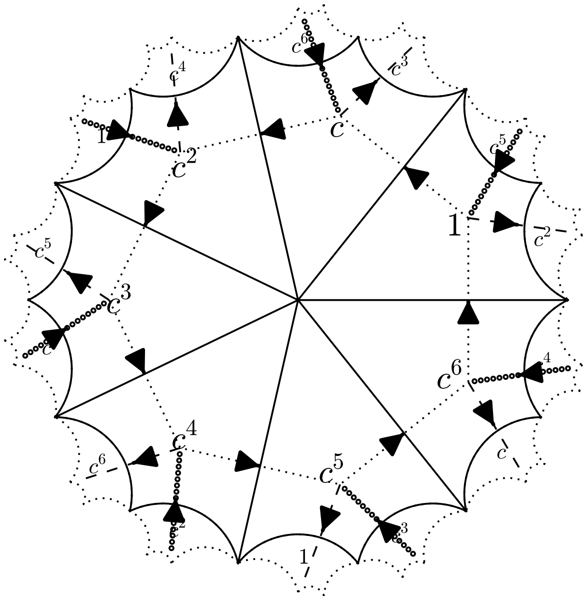

Fig. 5. Tessellation of the unit disk with copies of F obtained from the action of T, and the associated Cayley graph.

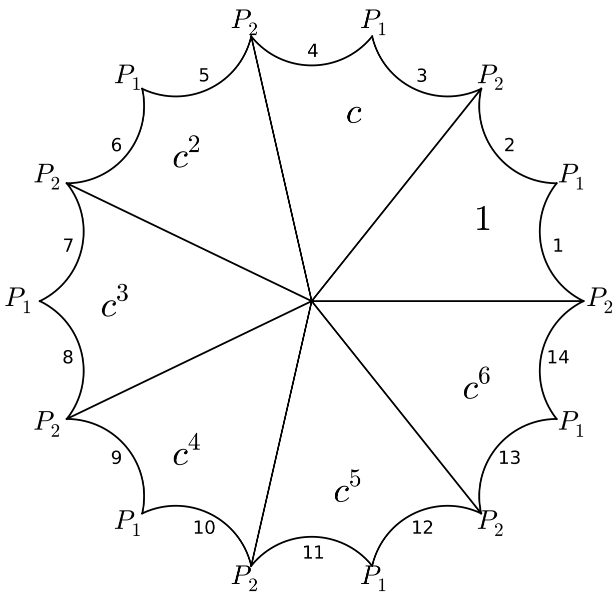

Fig. 6. Adapted fundamental polygon D for the cyclic group of order 7 with signature $(0; 7, 7, 7)$.

Example 4.5. Consider the action of S_4 on a surface S of genus four with generating vector $\mathbf{c} = ((1\,2\,3\,4), (3\,4), (2\,4), (1\,4))$. The signature of the quotient surface S/S_4 is then $(0; 4, 2, 2, 2)$. In this case, T has a presentation of the form

$$T = \langle r, s, t, u : r^4 = s^2 = t^2 = u^2 = rstu = 1\rangle,$$

and it uniformizes the sphere with one branch point of order four and three branch points of order two.

Recall from Section 3 that to obtain the adapted fundamental polygon F for the quotient surface S/S_4 we forget the last generator, and work only with r, s, t.

F is depicted in Fig. 7, where $\theta_1 + \theta_2 + \theta_3 = 2\pi/2$ (corresponding to $m_4 = 2$ in the signature), and the central angle is $2\pi/4$ (corresponding to $m_1 = 4$ in the signature); $m_2 = 2$ and $m_3 = 2$ correspond to the vertices marked with dots.

In Fig. 8 we see, as described in Section 4, the tessellation of the unit disk by T-images of F, indicating a corresponding word $w \in T$ for each copy of F. To clarify the selection process of coset representatives for the elements in $G = T/\Gamma$, we also present a labeling of the polygon where each number from 0 to 23 corresponds to a distinct element of G (see Fig. 9). Solid lines indicate copies of F that will be selected for D, while dotted lines indicate copies of F which are identified with others inside D (this determines the side pairing for D).

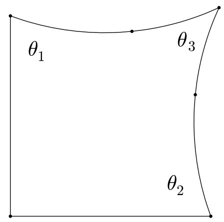

Fig. 7. Adapted polygon F with signature $(0; 4, 2, 2, 2)$.

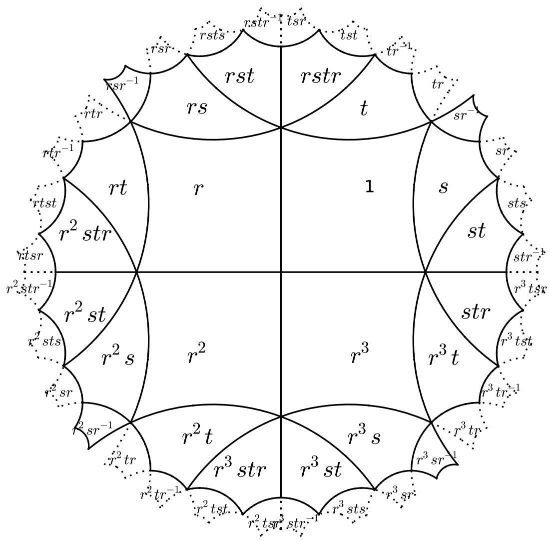

Fig. 8. Tessellation of the unit disk with copies of F obtained from the action of T.

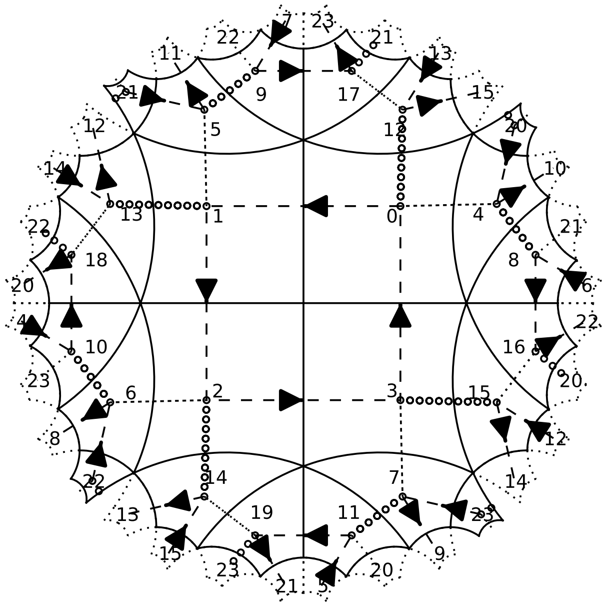

Fig. 9. Tessellation of the unit disk with copies of F obtained from the action of T, and the Cayley graph.

To illustrate the identification of the edges of the resulting polygon D (Fig. 10), we label identified edges with the same number, just changing the sign to illustrate a reversal of orientation. To avoid cluttering the figure, we only present some of the labels, the complete ordered list being

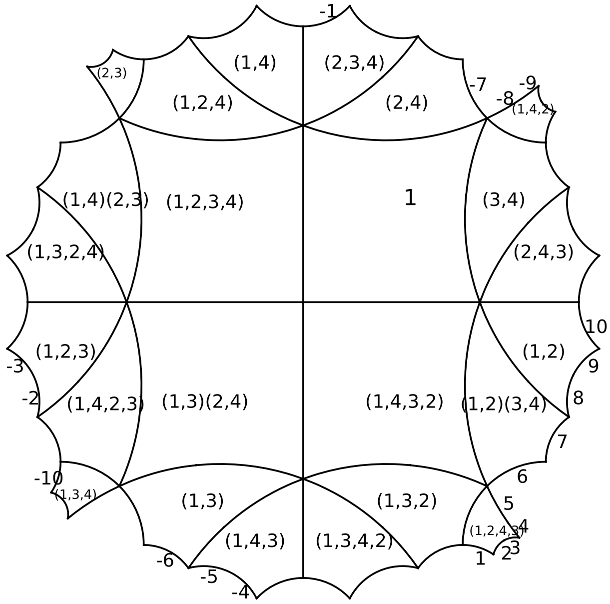

Fig. 10. Side pairing and labeling for the adapted fundamental polygon D.

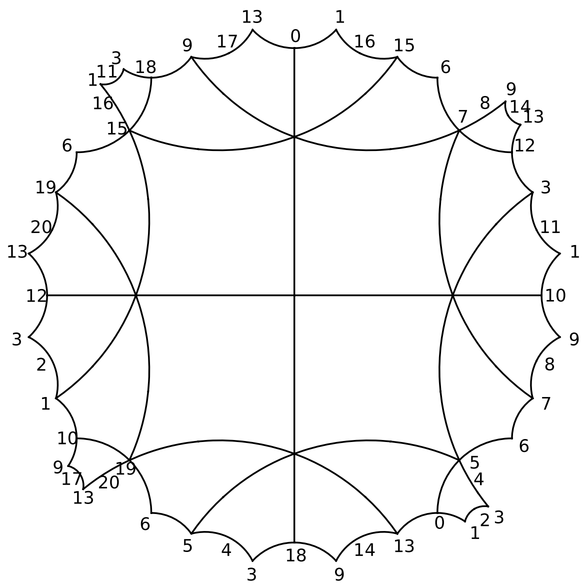

Fig. 11. Vertex labeling for the adapted fundamental polygon D.

$$[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, -9, -8, -7, 18, 19, 20,$$
$$-1, 21, 22, 23, 24, 25, -13, -12, -20, -19, -18, 26, 27, 28, -15, -14, -3, -2,$$
$$-11, -10, -23, -22, -28, -27, -26, -6, -5, -4, -25, -24, -17, -16, -21].$$

We see the adapted fundamental polygon D (see Definition 4.2) for the action of S_4 with signature $(0; 4, 2, 2, 2)$ and generating vector $\mathbf{c} = ((1\,2\,3\,4), (3\,4), (2\,4), (1\,4))$ in Fig. 10.

Fig. 11 depicts the Γ-equivalence classes of vertices in D, where each class is represented by an integer.

5. Step three: the homology basis and its G-action

We now have a hyperbolic polygon D reflecting the given G-action on $M = \Delta/\Gamma$. We will use it to represent the action on a specially suited homology basis for M.

Lemma 5.1. *Let D be a suitable fundamental region for the action of G on $M = \Delta/\Gamma$. Denote by $\beta_1, \dots, \beta_N$ the curves on M corresponding to the sides on the boundary of D, and by $U = \{P_1, \dots, P_v\}$ the set of points in M corresponding to the vertices of the boundary of D.*

Then there exists a basis B of $H_1(M)$ such that every curve γ in B is an integral combination $\gamma = \sum_{j=1}^{N} n_j\beta_j$ of the β's and

$$\delta\left(\sum_{j=1}^{N} n_j\,\beta_j\right) = 0, \tag{5.1}$$

where $\delta : H_1(M, U) \to H_0(U)$ is the natural map on homology.

Proof. It follows from the following exact sequence for relative homology

$$0 \longrightarrow H_1(M) \xrightarrow{j} H_1(M, U) \xrightarrow{\delta} H_0(U) \xrightarrow{i} H_0(M) \longrightarrow 0$$

that $2g + v = N + 1$, where g is the genus of M.

Since j is injective, the dimension of $\mathrm{Im}(j)$ is $2g$; hence the dimension of $\mathrm{Ker}(\delta)$ is also $2g$. The inverse image under j of a basis for $\ker\delta$ will be a basis for $H_1(M)$ with the required properties. □

Note that Eq. (5.1) provides an algorithm to explicitly find a basis B, which is not an *adapted homology basis* in the sense of [7]: it does capture the action, but not necessarily in a *nice way*.

Example 5.2. Let K and H be the Klein and the hyperelliptic surfaces of Example 2.9(1). We will compute a basis for their (first) homology groups, following Lemma 5.1.

For H, consider Fig. 6 and the corresponding side pairing for this surface. Eq. (5.1) is

$$\delta(n_1 \cdot 1 + n_2 \cdot 2 + n_3 \cdot 3 + n_5 \cdot 5 + n_7 \cdot 7 + n_9 \cdot 9 + n_{11} \cdot 11) = 0,$$

from where we obtain one single relation:

$$n_2 = n_1 + n_3 + n_5 + n_7 + n_9 + n_{11}.$$

Hence a basis B_H for $H_1(H, \mathbb{Z})$ is

$$\{\gamma_1 = 1 + 2, \gamma_2 = 3 + 2, \gamma_3 = 5 + 2, \gamma_4 = 7 + 2, \gamma_5 = 9 + 2, \gamma_6 = 11 + 2\}.$$

An analogous procedure for K gives a basis B_K for its first homology group:

$$\{\gamma_1 = 1 + -9, \gamma_2 = 2 + 9, \gamma_3 = 3 + -9, \gamma_4 = 4 + 9, \gamma_5 = 5 + -9, \gamma_6 = 7 + -9\}.$$

In both cases the numbers denote the sides of the polygon D of Fig. 6, remember that the identification of them is not the same in both cases.

Example 5.3. Continuing with Example 4.5, we solve Eq. (5.1) considering the boundary of D from Fig. 10. We obtain the following basis B for the homology

$$\begin{aligned}
\gamma_1 &= 11 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10\\
\gamma_2 &= 13 + -3 + -2 + 12\\
\gamma_3 &= 17 + -9 + -8 + -7 + -6 + -5 + -4 + 14 + 15 + 16\\
\gamma_4 &= 20 + 2 + 3 + 4 + 5 + 6 + 18 + 19\\
\gamma_5 &= 21 + -15 + -14 + -3 + -2 + -1\\
\gamma_6 &= 23 + -9 + -8 + -7 + -6 + -5 + -4 + 14 + 15 + 22\\
\gamma_7 &= 25 + 4 + 5 + 6 + 7 + 8 + 9 + 24\\
\gamma_8 &= 28 + -15 + -14 + 4 + 5 + 6 + 26 + 27,
\end{aligned}$$

where the numbers represent the corresponding sides of D given in Fig. 10 and the complete list of the boundary labeling given there.

5.1. The G-action on the homology basis

Recall from Remark 4.3 that the adapted fundamental polygon D is a union of copies F^g of F for g in G, and therefore the action of any element h in G on D corresponds to a permutation of the copies as follows.

$$h(F^g) = F^{gh}. \tag{5.2}$$

Consider the homology basis B constructed in the previous section, and let γ be an element of B.

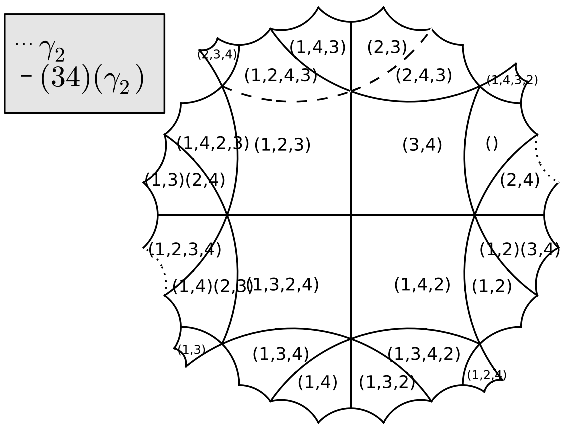

Fig. 12. Action on a curve on the boundary of D.

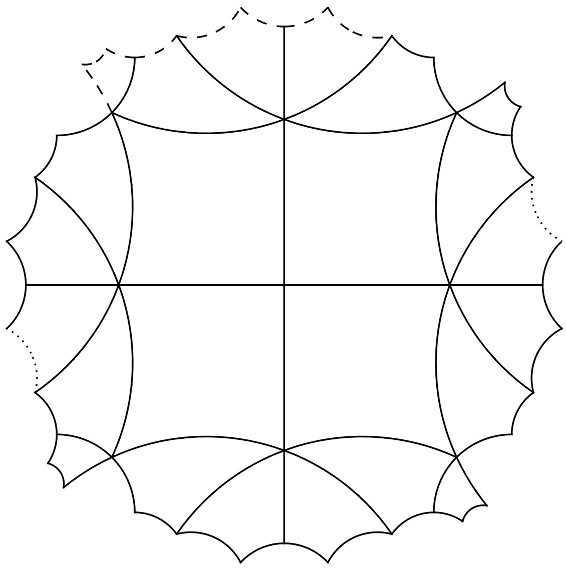

Fig. 13. Action on homology.

A representative for γ in the polygon D is a linear combination of sides of D. Its image on M is a closed curve, starting and ending at the image of one of the P_k in U, where we follow the notation of Lemma 5.1.

The image of γ under one of the fixed generators c_j for G is a nontrivial closed curve in M, and hence its preimage in D is a curve δ contained in D, that starts and ends at vertices in D belonging to the same Γ-orbit. Note that the image curve δ is composed by sides of (some of) the copies F^g in D, determined by the permutation of the original copies corresponding to c_j as in (5.2).

By a homotopy with fixed end points, this image curve δ may be deformed to lie on the boundary of D, and then it may be written as a linear combination of the elements of B.

In this way we associate a $2g \times 2g$ integral matrix to each generator of G, thus obtaining the rational representation for G; that is, the representation of the induced action of G on $H_1(M, \mathbb{Z})$. It is an integral representation of G in dimension $2g$, leaving invariant the intersection matrix J_B for B, to be computed from D in the next section.

We illustrate the process in the following examples.

Example 5.4. Consider Example 4.5 and the homology basis obtained in Example 5.3.

Fig. 12 shows the permutation of the copies of F contained in D corresponding to the second generator of S_4: $(3, 4)$. The second curve γ_2 in the basis B and its image under $(3, 4)$ are also depicted there.

The result of changing the image curve by fixed ends homotopy in D to a curve lying on the boundary of D is shown in Fig. 13. Finally we see that

$$(3\,4)(\gamma_2) = -\gamma_2 + \gamma_5 + \gamma_6 + \gamma_7.$$

Applying the algorithm to our fixed set of generators for G, the following integral representation (with respect to the basis B) is obtained.

$$[c_1]_B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[c_2]_B = \begin{pmatrix} 0 & 0 & -1 & 1 & 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$[c_3]_B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 \end{pmatrix}$$

Example 5.5. Consider the surface K of Example 5.2 and the basis B_K obtained there. The action of the cyclic group G of order 7 on K is given by the generating vector (c, c^2, c^4) (see Example 2.9(1)).

By Fig. 6, the action of c on B_K is given as follows

$$\begin{aligned} c(\gamma_1) &= 3 - 11 = 3 + 2 = \gamma_3 + \gamma_2, \\ c(\gamma_2) &= 4 + 11 = 4 - 2 = \gamma_4 - \gamma_2, \\ c(\gamma_3) &= 5 - 11 = 5 + 2 = \gamma_5 + \gamma_2, \\ c(\gamma_4) &= 6 + 11 = -1 - 2 = -\gamma_1 - \gamma_2, \\ c(\gamma_5) &= 7 - 11 = 7 + 2 = \gamma_6 + \gamma_2, \\ c(\gamma_6) &= 9 - 11 = 9 + 2 = \gamma_2, \end{aligned}$$

where the numbers represent the sides of D as in Fig. 6. The corresponding matrix representation for c on the basis B_K is then

$$[c]_{B_K} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

6. Step four: the intersection matrix for the homology basis

The intersection matrix $J_B = \left(\gamma_i \cdot \gamma_j\right)$ may be found from D, by considering that if two curves in B intersect each other in M, they may be deformed (inside their homotopy class) so they do so only at points in $U = \{P_1, \ldots, P_v\}$, the projection to M of the vertices of D; that is, it is enough to analyze the intersections of the edges of the boundary of D appearing in the basis B at different vertex orbits under Γ.

The procedure is as follows.

(1) Draw a full neighborhood D_j of each vertex P_j in M, including the numbered sides of D that meet at P_j, and their orientation.
(2) On each such picture, draw the curves in the basis B that pass through P_j, including their orientation.

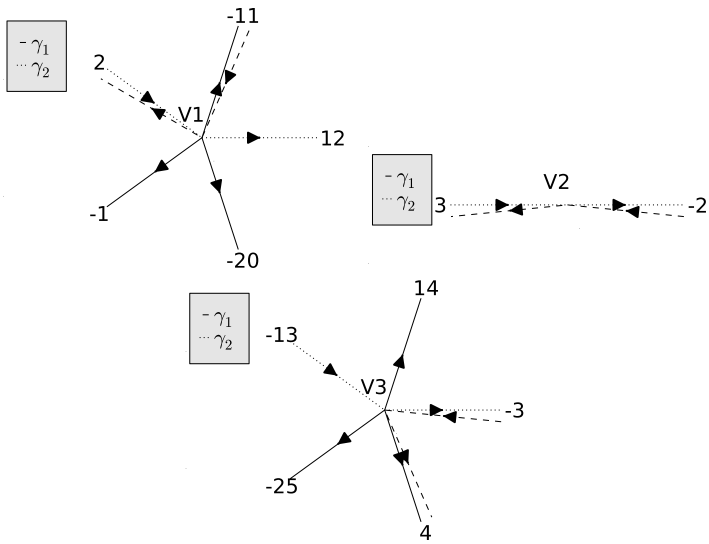

Fig. 14. Zoom in at neighborhoods of three vertices of D.

(3) To compute the intersection number of the two curves γ_i and γ_j in B, for $i < j$, the drawings D_k are modified in the following case: if both curves pass through P_k, and they share a common side e ending or beginning at P_k.

In this case the same situation will hold at some other P_s, and the two drawings D_k and D_s have to be considered *simultaneously*. The curve γ_j is deformed in D_k keeping P_k fixed, moving it counterclockwise away from side e, so that it now lies between the edge e and the next edge in D_k in the counterclockwise direction. Then the corresponding deformation is applied in D_s.

(4) Once all drawings satisfying the condition in (3) have been modified, the intersection numbers are counted by looking at each vertex, as ± 1 according to the given orientations of γ_i and γ_j, and then added to obtain the total intersection number $\gamma_i \cdot \gamma_j$.

Remark 6.1. The basis B will not be a symplectic basis in general; the relation between the intersection matrix J_B and the integral matrices found in Section 5.1 for the chosen generators of G is, of course,

$$[c_j]_B^t J_B \, [c_j]_B = J_B.$$

We illustrate the process with two examples.

Example 6.2. Recall Examples 4.5 and 5.3. To compute the intersection matrix, we need to study the intersection of the curves of B at different vertices of D. In Fig. 14 we show full neighborhoods for the three vertices $V1$, $V2$ and $V3$ of D where γ_1 and γ_2 intersect; these vertices are labeled as 1, 2 and 3 in Fig. 11. γ_1 has been deformed so that it lies in the interior of D, except where it goes through the vertices of ∂D.

Compare Fig. 14 with Fig. 8.

In this example the intersection matrix is given as follows.

$$J_B = \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$$

6.1. *Step five: the symplectic representation for G*

In Section 5 we obtained an integral $2g$-dimensional representation of G in $\mathrm{SL}(2g, \mathbb{Z})$ for the action of G on the basis B, and the intersection matrix J_B for B was found in Section 6.

In order to obtain a symplectic representation for G, we follow a process which is analogous to the one used to find an orthogonal basis for a symmetric form.

Lemma 6.3. *Let E be an alternating non-degenerate bilinear form on a free $\mathbb{Z}$-module L, having values in $\mathbb{Z}$. Then L is an E-orthogonal direct sum $L = [e_1, v_1] \oplus \cdots \oplus [e_n, v_n]$ of 2-dimensional submodules $[e_j, v_j]$, such that $E(e_j, v_j) = d_j$ is an integer > 0 and $d_1|d_2|\dots|d_n$.*

Proof. See [12, Chapter VI, Lemma 1], where a constructive algorithmic proof of the existence of such a basis $\{e_1, \dots, e_n, v_1, \dots, v_n\}$ is given. $\square$

This basis is called a *Frobenius basis* for L with respect to E. It is clear that the intersection matrix J_B found in Section 5 satisfies the hypothesis for Lemma 6.3, hence the corresponding Frobenius basis B_F for L with respect to J_B will be a symplectic one; that is, with intersection matrix

$$J_{\text{can}} = \begin{pmatrix} 0 & I_{g\times g} \\ -I_{g\times g} & 0 \end{pmatrix}.$$

The method from Lemma 6.3 also provides a change of basis matrix P from B to B_F.

To obtain the symplectic representation for G, we conjugate by P the integral representation obtained in Section 5.

Example 6.4. We continue with Example 4.5.

In Example 6.2 we computed the intersection matrix J_B for the basis B found in Example 5.3.

Applying Lemma 6.3 to J_B we obtain the corresponding change of basis matrix P to the symplectic basis B_F, so that $P^t J_B P = J_{\text{can}} = J_{B_F}$. Finally, conjugating by P the integral representation for $G = S_4$ found in Example 5.4 we obtain the required symplectic representation for G as follows.

For $j = 1, 2, 3$

$$[c_j]_{B_F} = P^{-1}[c_j]_B P = \begin{bmatrix} C_j & 0 \\ 0 & C_j^{-t} \end{bmatrix}$$

where C_j^{-t} is the transposed inverse of the matrix C_j, and each C_j is given by

$$C_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad C_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

7. Applications

For any integer $g \geq 1$, the Siegel upper half-space of degree g is

$$\mathbb{H}_g = \{Z \in M_{g\times g}(\mathbb{C}) : Z^t = Z, \Im Z >> 0\},$$

the set of symmetric complex $g \times g$ matrices with positive definite imaginary part.

The symplectic group $\text{Sp}(2g, \mathbb{Z})$ acts on $\mathbb{H}_g$ as follows,

$$\begin{array}{ccc} \text{Sp}(2g, \mathbb{Z}) \times \mathbb{H}_g & \longrightarrow & \mathbb{H}_g \\ \left(\begin{pmatrix} @A & B \\ C & D \end{pmatrix}, Z\right) & \dashrightarrow & (AZ + B)(CZ + D)^{-1}, \end{array}$$

and the moduli space A_g of principally polarized complex Abelian varieties of dimension g is parameterized by $\mathbb{H}_g/\text{Sp}(2g, \mathbb{Z})$. See for instance [12, Chapter VIII, Section 2].

The symplectic representation of $G = S_4$ obtained in the previous section will be used now to find all the Riemann matrices Z in $\mathbb{H}_4$ fixed under this action, thus explicitly describing a family in the singular locus of $\mathcal{A}_4$, corresponding to the principally polarized abelian varieties with given G-action.

Theorem 7.1. *Consider the symmetric group S_4 of degree four, with generating vector $\mathbf{c} = ((1\,2\,3\,4), (3\,4), (2\,4), (1\,4))$.*

Then there exists a two-dimensional family $\mathcal{A}(G, \mathbf{c})$ of principally polarized abelian varieties of dimension four admitting the given group action. It is given by the matrices Z_{τ_1,τ_2} in $\mathbb{H}_4$ of the following form.

$$Z_{\tau_1,\tau_2} = \begin{bmatrix} \tau_1 & -\tau_2 & -\tau_2 & -\tau_1 + 3\,\tau_2 \\ -\tau_2 & \tau_1 & -\tau_2 & -\tau_1 + 3\,\tau_2 \\ -\tau_2 & -\tau_2 & \tau_1 & -\tau_1 + 3\,\tau_2 \\ -\tau_1 + 3\,\tau_2 & -\tau_1 + 3\,\tau_2 & -\tau_1 + 3\,\tau_2 & 4\,\tau_1 - 12\,\tau_2 \end{bmatrix}$$

Furthermore, $\mathcal{A}(G, \mathbf{c})$ contains the one-dimensional family parameterized by $\mathcal{H}(G, \mathbf{c})$.

Additionally, there are only two Riemann surfaces in $\mathcal{H}(G, \mathbf{c})$ whose automorphism group is larger than S_4; one is Bring's curve, the unique compact Riemann surface of genus four admitting an action by the symmetric group S_5; the other one has as full automorphism group $S_4 \times \mathbb{Z}/3\mathbb{Z}$.

The Riemann matrix for Bring's curve in the above family is $Z_{4\tau_0,\tau_0}$, with τ_0 in $\mathbb{H}_1$ defined modulo $\Gamma_0(5)$ by $j(\tau_0) = -25/2$ and $j(5\tau_0) = -29^3 \cdot 5/2^5$. The Riemann matrix for the other curve is obtained by setting $\tau_1 = \frac{5}{12} i\sqrt{3}$ and $\tau_2 = \frac{1}{12} i$.

Proof. Consider the symplectic representation for G found in Example 6.4. Its set of fixed points in $\mathbb{H}_4$ gives the two-dimensional family.

That $\mathcal{A}(G, \mathbf{c})$ contains the one-dimensional family parameterized by $\mathcal{H}(G, \mathbf{c})$ holds by construction.

The fact that $\mathcal{H}(G, \mathbf{c})$ contains only two Riemann surfaces with automorphism group larger than G, and the description of the larger groups, come from [14].

To compute the Riemann matrices for these two curves, the method is applied again, this time to the larger groups S_5 and $S_4 \times \mathbb{Z}/3\mathbb{Z}$, with their corresponding generating vectors.

The first case was done in [18], by hand calculations at the time, and it provided the results quoted here.

The second case was verified with our program, obtaining the following symplectic representation associated to the extra generator of order three.

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 1 & 2 & 2 & -7 \\ 0 & -1 & 0 & 1 & 2 & 1 & 2 & -7 \\ 0 & 0 & -1 & 1 & 2 & 2 & 1 & -7 \\ 0 & 0 & 0 & -5 & -7 & -7 & -7 & 28 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & -1 & -1 & 4 \end{bmatrix}$$

The determination of its fixed points in $\mathbb{H}_4$ leads to the result. □

8. The routine

We have implemented the method described in this paper using the computer algebra system SAGE, following the procedure outlined in Sections 3 and 4 to implement the graphical aspect.

It is worth mentioning that there was a previous implementation available on the web, developed by C. O'Ryan in his thesis under our direction, which could produce drawings of fundamental regions for particular cases (http://www.geom.uiuc.edu/apps/unifweb). Unfortunately this site is no longer functional.

The algebraic procedures described in Sections 5, 6.1 and 7, have also been implemented as SAGE routines. In this section we briefly outline the inner works of the SAGE routines and we list some of the basic commands needed to produce the examples. A worksheet containing these commands is publicly available at http://www.sagenb.org/home/pub/2756.

Since the polygon D is the dual of the Cayley graph of the group G, we implemented it internally this way. The faces are therefore labeled as group elements. Edges are identified by the face to its left plus the precise generator a_i which glues it to the face on its right. Given this representation, we choose coset representatives as described in Section 4 making all the necessary edge identifications. The next step is to label the vertices lying on the border of the whole polygon taking care to use the same label for identified vertices. Once we have done this, we solve a simple linear system (see Eq. (5.1)) to obtain the basis B. For this basis we compute the action of G and the intersection matrix. Now we have to find the new basis B' for which the intersection matrix is canonical. In order to get a simpler output, we attempt to find G invariant subspaces of dimension g for which all elements have zero intersection. If we can find at least one such subspace then the representation will have a $g \times g$ block of zeros in the lower left corner. If we can find two of them whose sum is the whole space and such that the change of basis lies in $GL(2g, \mathbb{Z})$, then the resulting representation will have blocks of zeros in the upper right and lower left corners.

The problem of finding invariant Riemann matrices involves solving a system of nonlinear equations in many variables. In the particular case when the representation has at least a $g \times g$ block of zeros in the lower left corner, the equations become linear and the program returns a basis for the linear (affine) space of invariant matrices.

To find invariant Riemann matrices for those representations which do not have blocks of zeros, the program creates a generic symmetric matrix and returns the ideal generated by the necessary conditions on the coefficients of the matrix. It is likely though that the equations rapidly become intractable in this case.

Example 8.1. What follows is a list of some of the commands implemented in SAGE to get the results presented in the examples.

```
sage: FundamentalPoligon([4,2,2,2]).draw()
sage: G=SymmetricGroup(4)
sage: P=Poly(G,[4,2,2,2])
sage: P.draw()
```

we obtain Fig. 10 without the labeling.

It may be noticed that we just entered the signature and let the program choose a suitable generating vector for the group.

The list of generators actually used can be found by

```
sage: P.generators
```

The symplectic representation for the set of generators is

```
sage: P.symplectic_group_generators()
```

The matrix which generates the space of invariant Riemann matrices under the action of G is found as follows.

```
sage: m=P.moebius_invariant()
```

It is also possible to obtain further information about the hyperbolic polygon generated internally by the program in SAGE; for instance, the border with identified edges and the set of curves chosen as a basis for the homology.

```
sage: P.border sage: P.loops
```

Example 8.2. To specify the generators to be used, we define the group G and pick a generator:

```
sage: G=CyclicPermutationGroup(7)
sage: x=G.0
```

Now the two possible actions are studied by

```
sage: R1=Poly(G,[x,x,x^5])
sage: R2=Poly(G,[x,x^2,x^4])
```

Remark 8.3. We have also programmed some additional routines, such as:

Given a group G, presented as a group of permutations, and a genus g, it returns all possible signatures for which there is an appropriate generating vector for G acting on a curve of genus g. For example:

```
sage: suggest_signatures(SymmetricGroup(4),3)
```

A generating vector for one of the suggested signatures can be found

```
sage: find_generators(SymmetricGroup(4),[4,4,3])
```

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