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ON THE MINIMAL SPEED OF FRONT PROPAGATION IN A MODEL OF THE BELOUSOV-ZHABOTINSKY REACTION

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Abstract. In this paper, we answer the question about the existence of the minimal speed of front propagation in a delayed version of the Murray model of the Belousov-Zhabotinsky (BZ) chemical reaction. It is assumed that the key parameter r of this model satisfies $0 < r \leq 1$ that makes it formally monostable. By proving that the set of all admissible speeds of propagation has the form $[c_*, +\infty)$, we show here that the BZ system with $r \in (0,1]$ is actually of the monostable type (in general, c_* is not linearly determined). We also establish the monotonicity of wavefronts and present the principal terms of their asymptotic expansions at infinity (in the critical case $r = 1$ inclusive).

1. Introduction and main results.

In this paper, by using the method of regular super-solutions developed in [22, 23], we prove the existence of the minimal speed of front propagation for a delayed version of the following dimensionless Murray's [19, 20] model of the Belousov-Zhabotinsky (BZ for short) chemical reaction:

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + u(t,x)(1 - u(t,x) - rv(t,x)), \\ v_t(t,x) = \Delta v(t,x) - bu(t,x)v(t,x). \end{cases} \tag{1}$$

The above parameters r, b are positive and the non-negative variables u, v represent the bromous acid and bromide ion concentrations respectively. The traveling fronts $(u,v) = (\phi,\theta)(\nu \cdot x + ct)$ (subjected to the boundary conditions $(\phi,\theta)(-\infty) = (0,1)$, $(\phi,\theta)(+\infty) = (1,0)$, $\phi,\psi \geq 0$), describe the planar waves propagating in a thin layer of reactant mixture filled in a Petri dish [5, 20].

It should be observed that the existence of a real number c_* (the minimal speed) separating the intervals of admissible and non-admissible wave velocities c is one of the primary questions of interest in the study of wave propagation in reaction-diffusion systems. Let us mention just one example here: in the recent work [6], this question was considered for another (acidic nitrate-ferroin) chemical reaction.

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Reaction-diffusion system (1) improves various weaknesses of the original Field and Noyes five-step model for the spatial BZ oscillations [5]. In particular, it is simpler, and it predicts much better the real speed of wave propagation. Indeed, the theoretical speed estimation obtained by Field and Noyes was about 20 times the experimental value [5, pp. 2003-2004]. This discrepancy becomes much lower (about 2 times [19, p. 341]) if we use equations (1). Such a better agreement with real experiments was reached by introducing the tuning parameter r. That is to say, r is not an intrinsic characteristics of the original Field and Noyes system, but it is a new input parameter allowing to incorporate some additional information (the variation of the bromide ion far ahead of the wavefront) about the BZ reaction.

The particular importance of r resides in the fact that the size of this parameter determines whether the system (1) is bistable or monostable. Indeed, if $r \in (0, 1]$ (respectively, $r > 1$) then the non-isolated equilibrium $(u, v) = (0, 1)$ of the system $u' = u(1-u-rv),\ v' = -buv$ is unstable (respectively, non-asymptotically stable). At the same time, the isolated steady state $(u, v) = (1, 0)$ is asymptotically stable for all $r > 0$. Thus we can expect from (1) to have the typical monostability attributes when $r \in (0, 1]$ and the bistability properties when $r > 1$. However, the classical mono/bistability definition [24] does not apply automatically to (1) since the equilibrium $(0, 1)$ is not isolated. Such a degeneracy of $(0, 1)$ also does not permit the use of the recent powerful Liang-Zhao general theory [13] of spreading speeds for abstract monostable evolution systems (where only two steady states are allowed). Neither we can use the recent improvements of [13] by Li and Zhang [12] since in the latter work only *a finite* number of steady states is allowed. Thus the question about the nonlinear character (mono- versus bistable) of system (1) requires a further careful analysis. For $r > 1$, such a study was realised in [23], where the uniqueness of the admissible wave speed $c_\star$ as well as the uniqueness (modulo translation) of the wavefront propagating at the velocity $c_\star$ were established.

Now, if $rb + r \le 1,\ b > 0$, then it was proved in [11, 14, 26] that $c_* = 2\sqrt{1-r}$ is the minimal speed of propagation in the traditional sense of the term and it is linearly determined [7, 9, 10]. In the general case, when $r \in (0, 1)$ and $b > 0$, the existence of the minimal velocity $c_*(\Pi)$ for the waves propagating in special fixed polytopes Π was established in [24, Chapter 8, p. 333]. However, since system (1) possesses a continuum of equilibria, none of these polytopes can cover the whole region admissible for wavefronts, see [24, Fig. 5.1, p. 334]. Furthermore, the most difficult degenerate case $r = 1$ [23] was simply not considered in [8, 11, 14, 24, 25, 26]. Hence, the problem of the existence of the minimal speed of propagation c_* independent on Π for system (1) with $r \in (0, 1],\ b > 0,\ rb + r > 1$, has been remained open and unsolved so far. The main goal of our paper is exactly to present a complete solution to this important problem. In fact, we are going to consider even more general situation by analyzing a delayed version of system (1). The determination of exact/approximate value of c_* is another crucial and difficult issue related to the minimal speed analysis, see [7, 9, 10, 23, 25]. In general, c_* is non-linearly determined, i.e. $c_* > 2\sqrt{1-r}$.

Next, as it was shown in [23], a lower theoretical prediction for propagation speeds in system (1) can be obtained by considering delayed effects during the generation of the bromous acid. For instance, let us consider the following simple delayed model proposed by Wu and Zou in [25] and then studied in [1, 14, 16, 17, 23]

$$\begin{cases} u_t(t, x) = \Delta u(t, x) + u(t, x)(1 - u(t, x) - rv(t - h, x)), \\ v_t(t, x) = \Delta v(t, x) - bu(t, x)v(t, x). \end{cases} \tag{2}$$

In order to explain the presence of positive delay h in (2), we would like to invoke the analogy existing between the BZ temporal (i.e. the oregonator [3, 21]) and the BZ spatial oscillator [5]. The both type of oscillations are produced by multi-step reactions containing negative feedback loops built up from several 'elementary reactions' (either 'slow' (M3) with 'fast' (M5) in [3] or 'slow' (R5) + (R6′) in [5]). As it was shown by Epstein and Luo [3] in the case of the oregonator model, the steps (M3)+(M5) can be adequately combined in a single reaction (M3′) if the instantaneous concentration $v(t)$ of the bromide ion is replaced by its delayed concentration $v(t-\tau)$. An important part of their studies is to decide in which of the 'elementary reactions' $v(t)$ may be replaced with $v(t-\tau)$. After investigating two possibilities, Epstein and Luo indicated that one of them is more realistic than the other. Later on, Roussel [21] proposed the third way of incorporating delays in the oregonator. Certainly all this shows the difficulty of choosing convenient 'delayed' variables and estimating the size of delays [3]. Similarly, in the case of Field and Noyes theory [5] of the spatial BZ oscillations, we may replace the 'slow' combination (R5) + (R6′) with a single step describing autocatalytic generation of the bromous acid $HBrO_2$. Then the delayed concentration $u(t-\tau)$ taking into account on the step (R2: $HBrO_2 + Br^- + H^+ \longrightarrow 2HOBr$) generates the following adjustment to the Murray-Field-Noyes model:

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + u(t,x)(1-u(t,x)) - ru(t-h_1,x)v(t,x), \\ v_t(t,x) = \Delta v(t,x) - bu(t-h_2,x)v(t,x). \end{cases} \tag{3}$$

Here the tuning delays $h_1, h_2 \geq 0$ can be chosen different in order to fit better the experimental data. We easily recover (2) from (3) by taking $h_1 = 0,\ h_2 := h$ and $v(t,x) := \tilde{v}(t-h_2,x)$.

In order to obtain equations of the front profiles for system (2), it suffices to plug the wave ansatz $(u,v)(x,t) = (\phi,\theta)(\nu\cdot x + ct)$ into (2):

$$\begin{cases} \phi''(t) - c\phi'(t) + \phi(t)(1-\phi(t) - r\theta(t-ch)) = 0, \\ \theta''(t) - c\theta'(t) - b\phi(t)\theta(t) = 0. \end{cases} \tag{4}$$

The profiles ϕ, θ should also satisfy the following positivity and boundary conditions

$$\phi(t) > 0,\ \theta(t) > 0,\ t \in \mathbb{R},\quad (\phi,\theta)(-\infty) = (0,1),\ (\phi,\theta)(+\infty) = (1,0). \tag{5}$$

In fact, we know from [19, Section 8] that all positive wavefronts to (1) are monotone. On the other hand, Lin and Ruan [15] give an explicit example of two-dimensional diffusive Lotka-Volterra competition system possessing non-monotone fronts. It was also found recently by Fang and Wu [4] that the delayed response may imply the loss of wave's monotonicity in the diffusive Lotka-Volterra competition systems. Therefore, as it was already mentioned in [23], it is a remarkable fact that the inclusion of delay as in (2) does not change the monotone shape of wave profiles:

Theorem 1.1. *Assume that* $r, b > 0$. *If system (2) has a wavefront* $(u,v) = (\phi,\theta)(\nu\cdot x+ct),\ \ \phi > 0,\ \ |\nu| = 1$, *connecting* $(0,1)$ *with* $(1,0)$, *then* $\phi'(t), -\theta'(t) > 0$ *and* $\theta(t), \phi(t) \in (0,1)$ *for all* $t \in \mathbb{R}$.

The above theorem may be considered as one of auxiliary results needed to prove the existence of the minimal speed. In particular, it allows to establish *a priori* asymptotic formulas at infinity for the possible traveling fronts (what is an important ingredient of our approach). Since the deduction of these expansions is not at all straightforward in the degenerate case $r = 1$, we present this part of our studies as a separate statement:

Theorem 1.2. *Let $r = 1$, $b > 0, h \geq 0$ and $(u, v) = (\phi, \theta)$ $(\nu \cdot x + ct)$, $|\nu| = 1$, be a positive wavefront of system (2) connecting $(0, 1)$ with $(1, 0)$. Then, for an appropriate t_1 and $t \to -\infty$, it holds either*

$$(\phi, \phi', \theta, \theta')(t + t_1) = \left(\frac{2c^2 + o(1)}{bt^2}, \frac{-4c^2 + o(1)}{bt^3}, 1 + \frac{2c + o(1)}{t}, \frac{-2c + o(1)}{t^2}\right),$$

or $$(\phi, \phi')(t + t_1) = e^{ct}(1, c)(1 + o(1)), \quad \theta(t + t_1) = 1 + c^{-1}bte^{c(t-ch)}(1 + o(1)).$$

It is natural to expect that (a) all traveling profiles in Theorem 1.2, excepting minimal ones, have an algebraic rate of convergence to 0 at $-\infty$; (b) the minimal waves are decaying exponentially to 0, cf. Theorem 1.3 and [22, Theorem 1.4].

We are in the position now to state our main result concerning the existence of the minimal speed of propagation c_* in the BZ system:

Theorem 1.3. *Let $r \in (0, 1]$, $b > 0, h \geq 0$. Then there is $c_* > 0$ such that (I) for each $c \geq c_*$ system (2) has a positive monotone wavefront $(u, v) = (\phi, \theta)$ $(\nu \cdot x + ct)$, $\phi > 0, |\nu| = 1$, connecting $(0, 1)$ with $(1, 0)$; (II) there does not exist any such wavefront to (2) when $c < c_*$. For each $c > c_*$, there is at least one wavefront with the following asymptotic behavior at $-\infty$: (i) if $r = 1$ then*

$$\begin{aligned} \phi(t) &= \frac{2c^2/b}{t^2} - \frac{8c}{3b}(c^2(1 + h + \frac{1}{b}) - 4)\frac{\ln(-t)}{t^3} + O(\frac{1}{t^3}), \\ \theta(t) &= 1 + \frac{2c}{t} - \frac{4}{3}(c^2(1 + h + \frac{1}{b}) - 4)\frac{\ln(-t)}{t^2} + O(\frac{1}{t^2}),\ t \to -\infty; \end{aligned} \quad (6)$$

(ii) if $r \in (0, 1)$ then, for some $\varepsilon > 0$ and $\lambda := 0.5(c - \sqrt{c^2 - 4(1 - r)})$, it holds

$$\phi(t) = e^{\lambda t} + O(e^{(\lambda+\varepsilon)t}), \quad \theta(t) = 1 - \frac{be^{\lambda(t-ch)}}{1 - r} + O(e^{(\lambda+\varepsilon)t}), \quad t \to -\infty. \quad (7)$$

As we have said, the determination of exact value of c_* presents, in general, a difficult open question. A partial answer was given in [23, Theorem 7], where it was found that $c_* = 2\sqrt{1 - r}$ when $rb\exp(-2h(1 - r)) + r \leq 1$.

Finally, let us say several words about the organisation of this paper. The front monotonicity is proved in Section 2. Section 3.1 presents the main theoretical tool of our research: regular super-solutions [22, 23]. The use of this tool requires a priori asymptotics of front profiles, this information is obtained in Section 3.2. Then the demonstration of Theorem 1.3 is completed in Section 3.3.

2. **Proof of Theorem 1.1.**

As we know from Section 1, the profiles ϕ, θ of traveling fronts to (2) satisfy system (4), (5). The substitution $\theta(t - ch) = 1 - \psi(t)$ transforms this system into

$$\begin{cases} \phi''(t) - c\phi'(t) + \phi(t)(1 - r - \phi(t) + r\psi(t)) = 0, \\ \psi''(t) - c\psi'(t) + b\phi(t - ch)(1 - \psi(t)) = 0, \\ \phi > 0, \psi < 1,\ \phi(-\infty) = \psi(-\infty) = 0,\ \phi(+\infty) = \psi(+\infty) = 1. \end{cases} \quad (8)$$

Fix some $B \leq -(1 + r + b)$ and consider operators

$$\mathcal{F}_1(\phi, \psi)(t) = \phi(t)(1 - r - B - \phi(t) + r\psi(t)), \mathcal{F}_2(\phi, \psi)(t) = b\phi(t - ch)(1 - \psi(t)) - B\psi(t).$$

Let $z_1 < 0 < z_2$ be the roots of the equation $z^2 - cz + B = 0$. Then every bounded solution (ϕ, ψ) of differential equations in (8) meets the system of integral equations

$$\phi(t) = \mathcal{N}_1(\phi, \psi, c)(t),\ \psi(t) = \mathcal{N}_2(\phi, \psi, c)(t), \quad \text{where} \quad (9)$$

$$\mathcal{N}_j(\phi,\psi,c)(t) = \frac{1}{z_2 - z_1}\left(\int_{-\infty}^{t} e^{z_1(t-s)}\mathcal{F}_j(\phi,\psi)(s)ds + \int_{t}^{+\infty} e^{z_2(t-s)}\mathcal{F}_j(\phi,\psi)(s)ds\right).$$

Conversely, each positive strictly monotone bounded solution (ϕ, ψ) of system (9) yields a wavefront for (8).

Lemma 2.1. *Assume that $\phi(t), \psi(t)$ satisfy differential equations in (8) and the finite limits $\phi(\pm\infty)$ and $\psi(\pm\infty)$ exist. Then $\phi'(\pm\infty) = \psi'(\pm\infty) = 0$.*

Proof. Since bounded solution (ϕ, ψ) of system (8) satisfies integral equations (9), we find that

$$\phi'(t) = \mathcal{M}_1(\phi,\psi)(t),\ \ \psi'(t) = \mathcal{M}_2(\phi,\psi)(t), \quad \text{where}$$

$$\mathcal{M}_j(\phi,\psi)(t) = \frac{1}{z_2 - z_1}\left(\int_{-\infty}^{t} z_1 e^{z_1(t-s)}\mathcal{F}_j(\phi,\psi)(s)ds + \int_{t}^{+\infty} z_2 e^{z_2(t-s)}\mathcal{F}_j(\phi,\psi)(s)ds\right).$$

In consequence, $|\phi'(t)| \leq |\phi|_\infty(1 - r - B + r|\psi|_\infty)/(z_2 - z_1)$, $|\psi'(t)| \leq (b|\phi|_\infty - B|\psi|_\infty)/(z_2 - z_1)$, $t \in \mathbb{R}$, where $|\phi|_\infty := \sup_{s\in\mathbb{R}} |\phi(s)|$. This means that ϕ, ψ, ϕ', ψ' are uniformly continuous on $\mathbb{R}$. By the Barbalat lemma (see e.g. [25, Lemma 2.3]), we obtain that $\phi'(\pm\infty) = \psi'(\pm\infty) = 0$. □

Corollary 1. *Each admissible wavefront speed c must be positive: $c > 0$.*

Proof. It suffices to integrate the second equation of (8) from $-\infty$ to $+\infty$. □

Next, we present a proof of Theorem 1.1. It will be divided in three steps.

Proof. (a) We start by establishing the exact upper and lower bounds for ψ. First, observe that $\psi(+\infty) = 1$, $\psi(-\infty) = 0$ so that $\psi(t') > 1$ [or $\psi(t') < 0$] for some $t' \in \mathbb{R}$ implies that ψ attains its local maximum [minimum, respectively] at some s where $\psi'(s) = 0$, $\psi''(s) \leq 0$, $\psi(s) > 1$ [$\psi'(s) = 0$, $\psi''(s) \geq 0$, $\psi(s) < 0$, respectively]. However, due to the positivity of ϕ this contradicts to the second equation of (8).

Suppose now that $\psi'(s) = 0$ at some point s where $\psi(s) \in [0, 1)$. Then we get $\psi''(s) < 0$ so that s is a local maximum point. Since $\psi(+\infty) = 1$, function ψ attains its non-negative local minimum at some $s' > s$ where $\psi'(s') = 0$, $\psi''(s') \geq 0$, $\psi(s') \in [0, 1)$. This again contradicts to the second equation of (8).

(b) Next, suppose that $\phi(t') > 1$ for some $t' \in \mathbb{R}$. Then there exists a local maximum point s where $\phi'(s) = 0$, $\phi(s) > 1$, $\phi''(s) \leq 0$. Since, by (a), $\psi(s) \in (0, 1]$, we get a contradiction with the first equation of (8).

Hence, function $\Phi(t) := (1 - \phi(t))\theta(t) \in [0, 1)$, $t \in \mathbb{R}$, is non-constant and non-negative. Since, by (4), profile $\theta(t) = 1 - \psi(t + ch)$ satisfies

$$\theta''(t) - c\theta'(t) - b\theta(t) = -b\theta(t)(1 - \phi(t)),$$

we deduce that

$$\theta(t) = \frac{b}{\zeta_2 - \zeta_1}\left(\int_{-\infty}^{t} e^{\zeta_1(t-s)}\Phi(s)ds + \int_{t}^{+\infty} e^{\zeta_2(t-s)}\Phi(s)ds\right) > 0,\ t \in \mathbb{R},$$

where $\zeta_1 < 0 < \zeta_2$ are the roots of $z^2 - cz - b = 0$. As a consequence, $\psi(t) \in (0, 1)$, $\psi'(t) > 0$, $t \in \mathbb{R}$.

(c) Finally, assume for a moment that $\phi(s) = 1$ for some s. Then $\phi'(s) = 0$, so that (8) implies $\phi''(s) = r(1 - \psi(s)) > 0$, a contradiction.

Suppose now that $\phi'(s) = 0, \phi(s) \in (0, 1)$ at some point s. First we consider the case when additionally $\phi''(s) = 0$ so that $1 - r - \phi(s) + r\psi(s) = 0$. Differentiating the first equation of system (8), we then find that $\phi'''(s) = -r\psi'(s)\phi(s) < 0$. This implies that $\phi(t) > \phi(s)$ for all $t < s$ close to s. As a consequence, there exists $s' < s$ such that $\phi'(s') = 0, \phi''(s') \le 0,\ 1 > \phi(s') > \phi(s),\ \psi(s') < \psi(s)$. But then

$$0 = 1 - r - \phi(s) + r\psi(s) > 1 - r - \phi(s') + r\psi(s'),$$

that yields a contradiction: $0 = \phi''(s') - c\phi'(s') + \phi(s')(1 - r - \phi(s') + r\psi(s')) < 0$.

Similarly, if $\phi''(s) < 0$ then s is a local maximum point and $1-r-\phi(s)+r\psi(s) > 0$. Since $\phi(+\infty) = 1$, function ϕ attains its positive local minimum at some $s' > s$ where $\phi'(s') = 0,\ \phi''(s') \ge 0,\ 0 < \phi(s') < \phi(s),\ \psi(s) \le \psi(s')$ and therefore

$$\begin{aligned} 0 &< 1 - r - \phi(s) + r\psi(s) < 1 - r - \phi(s') + r\psi(s'), \\ 0 &= \phi''(s') - c\phi'(s') + \phi(s')(1 - r - \phi(s') + r\psi(s')) > 0, \end{aligned}$$

a contradiction.

Finally, let $\phi'(s_1) = 0$ and $\phi''(s_1) > 0$ at some point s_1. Then there exists a local maximum point $s < s_1$ where $\phi'(s) = 0$ and $\phi''(s) \le 0$. However, this possibility was already rejected. □

3. The existence of the minimal speed of propagation.

3.1. Method of regular super-solutions.

Our approach is based on the following auxiliary result from [23]:

Theorem 3.1. *Suppose that for given parameters $b, c > 2\sqrt{1-r}$, $r \in (0, 1]$, $h \ge 0$, system (8) has a regular super-solution $(\psi_+,\ \phi_+)$. Then there exists a monotone wavefront for (2) moving at the velocity c and satisfying (6), (7).*

The regular super-solutions were defined in [23] as follows. Let $\lambda = \lambda(c) < \mu = \mu(c)$ denote the real roots of the characteristic equation $z^2 - cz + (1 - r) = 0$.

Definition 3.2. Assume that continuous and piece-wise C^1-smooth functions $\psi_+,\ \phi_+$ are positive and have positive derivatives in some neighborhoods $\mathcal{O}_1,\ \mathcal{O}_2$ of the sets $(-\infty, t_1],\ (-\infty, t_2]$, respectively. We admit that $(\psi'_+,\ \phi'_+)$ can have a finite set $\mathcal{D} = \{d_1 < d_2 < ... < d_M\},\ d_M < \min\{t_1, t_2\}$, of the discontinuity points and one-sided derivatives of $\psi_+,\ \phi_+$ satisfy $\psi'_+(d_j-) > \psi'_+(d_j+)$, $\phi'_+(d_j-) > \phi'_+(d_j+)$. Suppose also that $\psi_+(-\infty) = \phi_+(-\infty) = 0$, $\psi_+(t_1) = \phi_+(t_2) = 1$, and that $\psi_+,\ \phi_+$ are C^2-smooth in some vicinities of t_1, t_2 and that

D1. For a fixed positive $\nu \in (\lambda, \mu),\ m \in \{0, 1\}$, and some positive constants C_1, ϵ,

$$\psi_+(t) = O(te^{\nu t}),\quad (\phi_+(t), \phi'_+(t), \phi''_+(t)) = C_1(-t)^m e^{\nu t}(1, \nu, \nu^2)(1 + o(1)),\ t \to -\infty.$$

D2. If $t \le \min\{t_1, t_2\},\ t \notin \mathcal{D}$, then

$$\begin{cases} \Lambda_1(\phi_+, \psi_+)(t) := \phi''_+(t) - c\phi'_+(t) + \phi_+(t)(1 - r - \phi_+(t) + r\psi_+(t)) < 0, \\ \Lambda_2(\phi_+, \psi_+)(t) := \psi''_+(t) - c\psi'_+(t) + b\phi_+(t - ch)(1 - \psi_+(t)) < 0. \end{cases} \quad (10)$$

D3. If $t_1 < t_2$ then $\phi''_+(t) - c\phi'_+(t) + \phi_+(t)(1 - \phi_+(t)) < 0,\ t \in [t_1, t_2]$.

D4. If $t_1 > t_2$ then $\psi''_+(t) - c\psi'_+(t) + b\min\{1, \phi_+(t - ch)\}(1 - \psi_+(t)) < 0, t \in [t_2, t_1]$.

We will call such $(\psi_+,\ \phi_+)$ a regular super-solution for (8). Observe that we may suppose that ϕ_+ is defined, strictly increasing and smooth on $[t_2, +\infty)$, this fact is implicitly used in **D4**.

Remark 1. Suppose that ϕ_+, ψ_+ are increasing and that inequalities (10) hold for all $t \le \max\{t_1, t_2\}$. Then it is easy to see that conditions **D3**, **D4** are satisfied automatically.

3.2. **Asymptotic expansions of wavefront profiles.** Here we establish useful formulas describing asymptotic behavior of front profiles both at $+\infty$ and $-\infty$. We start by analyzing more difficult situation when $r = 1$:

$$\begin{cases} \phi''(t) - c\phi'(t) + \phi(t)(\psi(t) - \phi(t)) = 0, \\ \psi''(t) - c\psi'(t) + b\phi(t-ch)(1-\psi(t)) = 0, \\ \phi > 0, \psi < 1, \ \phi(-\infty) = \psi(-\infty) = 0, \ \phi(+\infty) = \psi(+\infty) = 1. \end{cases} \tag{11}$$

Case $r = 1, \ t \to -\infty$. Integrating the second equation of (11) on $(-\infty, t]$, we get

$$\psi'(t) + b\int_{-\infty}^{t} \phi(s-ch)(1-\psi(s))ds = c\psi(t),$$

(here we are using Lemma 2.1) and therefore $\phi \in L_1(\mathbb{R}_-)$ and $(e^{-ct}\psi(t))' < 0$. In particular, $e^{-ct}\psi(t) > \psi(0)$, $t < 0$, so that $\psi(t)$ decays with at most exponential rate as $t \to -\infty$.

Next, the first equation of (11) can be handled through the linear non-autonomous equation

$$y'' - cy' + (\psi(t) - \phi(t))y = 0.$$

Since ψ, ϕ are monotone, the derivative $(\psi - \phi)'$ is Lebesgue integrable on $(-\infty, 0]$ that allows the application of the Levinson asymptotic integration theorem [2]. As a direct consequence of this theorem, we obtain that the above equation has a fundamental system of solutions $y_1(t), y_2(t)$ such that, at $t \to -\infty$,

$$(y_1(t), y_1'(t)) = (1 + o(1), o(1))\omega_-(t), \ (y_2(t), y_2'(t)) = (1 + o(1), c + o(1))\omega_+(t),$$

where

$$\omega_\mp(t) := \exp\left(\int_0^t \frac{2(\psi(s) - \phi(s))ds}{c \pm \sqrt{c^2 - 4(\psi(s) - \phi(s))}}\right).$$

This implies that either

$$\phi'(t) = (c + o(1))\phi(t), \ t \to -\infty, \tag{12}$$

or

$$\phi'(t) = o(1)\phi(t), \ t \to -\infty. \tag{13}$$

In the first case, $\phi(t)$ converges exponentially to 0 at $-\infty$, in the second case, $\phi(t) = (k + o(1))y_1(t)$, $t \to -\infty$, for some $k > 0$, that has the following consequence: $\int_{-\infty}^{0} \psi(s)ds = +\infty$.

Lemma 3.3. *If $r = 1$ then there exists τ_1 such that $\phi''(t) > 0$ for all $t \le \tau_1$. Moreover, if (13) holds then $\phi''(t) = o(1)\phi(t)\psi(t)$ at $-\infty$.*

Proof. Suppose that (12) holds. Then

$$\phi''(t) = c\phi'(t) - \phi(t)(\psi(t) - \phi(t)) = \phi(t)(c^2 + o(1)) > 0, \ t \to -\infty. \tag{14}$$

Now, let assume that ϕ satisfies (13). First, we prove that, for some $\tau_2 < 0$,

$$\phi'(t) < \psi'(t), \ t \le \tau_2.$$

Indeed, since $\zeta(t) = \psi'(t)$ is a bounded solution of

$$\zeta'(t) - c\zeta = -b\phi(t-ch)(1-\psi(t)),$$

we find that

$$\psi'(t) = b\int_t^{+\infty} e^{c(t-s)}\phi(s-ch)(1-\psi(s))ds.$$

Furthermore, (13) yields that $\phi(t-ch) = \phi(t)(1+o(1)),\ t \to -\infty$, so that, by the l'Hôpital rule,

$$\lim_{t\to-\infty} \frac{\int_t^{+\infty} e^{c(t-s)}\phi(s-ch)(1-\psi(s))ds}{\phi(t)} = \lim_{t\to-\infty} \frac{-e^{-ct}\phi(t-ch)(1-\psi(t))}{-ce^{-ct}\phi(t)+e^{-ct}\phi'(t)} = \frac{1}{c}.$$

As a consequence, for some $\tau_3 < 0$,

$$\psi'(t) = \frac{b}{c}\phi(t)(1+o(1)) > \phi'(t),\ t < \tau_3. \tag{15}$$

Therefore, taking into account that $\phi(-\infty) = \psi(-\infty) = 0$, we get

$$\psi(t) = \int_{-\infty}^t \psi'(s)ds > \phi(t) = \int_{-\infty}^t \phi'(s)ds.$$

In fact, $\phi(t) = o(1)\psi(t)$ since we have, as $t \to -\infty$,

$$0 \le \lim \frac{\phi(t)}{\psi(t)} = \lim \frac{\phi'(t)}{\psi'(t)} \le \lim o(1) = 0.$$

Finally, observe that $\xi(t) = \phi''(t)$ satisfies $\xi'(t) - c\xi(t) + F(t) = 0$, where

$$F(t) := \phi'(t)(\psi(t)-\phi(t)) + \phi(t)(\psi'(t)-\phi'(t)) > 0,\ t \le \tau_3.$$

Therefore $\xi'(t) - c\xi(t) < 0,\ t \le \tau_3$, where, without restricting the generality, we may assume that $\xi(\tau_3) = \phi''(\tau_3) \ge 0$ (otherwise $\phi''(t) < 0,\ t < \tau_3$ so that $\phi(-\infty) = -\infty$, a contradiction). But then we get the desired inequality

$$\phi''(t) = \xi(t) > \xi(\tau_3)e^{c(t-\tau_3)} \ge 0,\ t < \tau_3.$$

Additionally, since $\xi(t) = \phi''(t)$ is a bounded function (e.g. see (14)), we find that

$$\phi''(t) = b\int_t^{+\infty} e^{c(t-s)}F(s)ds.$$

Hence, in virtue of the relations $\phi'(t) = o(1)\phi(t),\ \phi(t) = o(1)\psi(t)$, (15), and the l'Hôpital rule, we obtain

$$\lim_{t\to-\infty} \frac{b\int_t^{+\infty} e^{c(t-s)}F(s)ds}{\phi(t)\psi(t)} = \lim_{t\to-\infty} \frac{-be^{-ct}(\phi'(t)(\psi(t)-\phi(t)) + \phi(t)(\psi'(t)-\phi'(t)))}{e^{-ct}(-c\phi(t)\psi(t) + (\psi(t)\phi(t))')}$$

$$= -b\lim_{t\to-\infty} \frac{\phi'(t)/\phi(t) - 2\phi'(t)/\psi(t) + \psi'(t)/\psi(t)}{-c+\phi'(t)/\phi(t)+\psi'(t)/\psi(t)} = 0.$$

This proves the last assertion of the lemma. □

Lemma 3.4. *Let $r = 1$ and assume that (13) holds. Then for every $b_* < b < b^*$ there exists $\tau_4 < 0$ such that*

$$\frac{2c^2/b^*}{t^2} < \phi(t) < \frac{2c^2/b_*}{t^2},\ t \le \tau_4. \tag{16}$$

That is $\phi(t) = (1+o(1))2c^2t^{-2}/b$. Additionally, $\phi'(t) = -(1+o(1))4c^2t^{-3}/b$.

Proof. Since $\phi''(t) = o(1)\phi(t)\psi(t),\ \phi(t) = o(1)\psi(t),\ t \to -\infty$, we deduce from (11), (15) that, given $\epsilon_1 \in (0, b/2)$, there is $\tau_5 < 0$ such that, for all $t \le \tau_5$, we have

$$\phi(t)\frac{(b-\epsilon_1)}{c}\int_{-\infty}^t \phi(u)du < c\phi'(t) = \phi(t)\psi(t)(1+o(1)) < \phi(t)\frac{(b+\epsilon_1)}{c}\int_{-\infty}^t \phi(u)du.$$

Integrating these inequalities from $-\infty$ to t, we get, for all $t \le \tau_5$,

$$\frac{(b-\epsilon_1)}{2c^2}(\int_{-\infty}^{t}\phi(u)du)^2 < \phi(t) = (\int_{-\infty}^{t}\phi(s)ds)' < \frac{(b+\epsilon_1)}{2c^2}(\int_{-\infty}^{t}\phi(u)du)^2. \quad (17)$$

After separating integrals and then again integrating from t to $s = \tau_5 > t$, we obtain

$$\frac{(b-\epsilon_1)(s-t)}{2c^2} < (\int_{-\infty}^{t}\phi(s)ds)^{-1} - (\int_{-\infty}^{s}\phi(s)ds)^{-1} < \frac{(b+\epsilon_1)(s-t)}{2c^2},$$

so that

$$\left((\int_{-\infty}^{s}\phi(s)ds)^{-1} + \frac{(b+\epsilon_1)(s-t)}{2c^2}\right)^{-1} < \int_{-\infty}^{t}\phi(s)ds < \frac{2c^2/(b-\epsilon_1)}{s-t}.$$

In this way, we can indicate $\tau_4 < \tau_5$ such that

$$-\frac{2c^2/(b+2\epsilon_1)}{t} < \int_{-\infty}^{t}\phi(s)ds < -\frac{2c^2/(b-2\epsilon_1)}{t}, \ t \le \tau_4.$$

Now, (17) implies the inequalities (16) while the estimation of $\phi'(t)$ at the beginning of the proof assures that $\phi'(t) = -(1+o(1))4c^2t^{-3}/b, \ t \to -\infty$. ☐

Corollary 2. *Let $r = 1$ and assume that (13) holds. Then $\psi(t) = -(1+o(1))2ct^{-1}$ and $\psi'(t) = (1+o(1))2ct^{-2}, \ t \to -\infty$.*

Proof. This is a direct consequence of (15), Lemma 3.4 and relation $\psi(t) = c(1+o(1))\phi'(t)/\phi(t)$, $t \to -\infty$, established in the proof of this lemma. ☐

Cases $r \in (0,1), \ t \to -\infty$ and $r \in (0,1], \ t \to +\infty$. The following result was proved in [23, Lemma 11 and Corollary 12]:

Lemma 3.5. *Let (ϕ,ψ) be a traveling front of (8) and $r \in (0,1)$. Then (a) $c \ge 2\sqrt{1-r}$, (b) there exists finite limit $\lim \phi'(t)/\phi(t) \in \{\lambda,\mu\}$ as $t \to -\infty$, (c) moreover, for some t_1, $m \in \{0,1\}$ and $\nu(c) \in \{\lambda(c),\mu(c)\}$, we have $\psi(t+t_1) = O(te^{\nu(c)t})$, $(\phi(t+t_1),\phi'(t+t_1)) = (-t)^m e^{\nu(c)t}(1,\nu(c))(1+o(1))$.*

Arguing as in the proof Lemma 3.5 (c), we can obtain a similar result for the case $r = 1$:

Lemma 3.6. *Let $r = 1$ and assume (12). Then, for an appropriate $t_1, \delta > 0$, it holds, as $t \to -\infty$,*

$$\psi(t+t_1) = -c^{-1}bte^{c(t-ch)}(1+o(1)), \ (\phi(t+t_1),\phi'(t+t_1)) = e^{ct}(1,c)(1+o(e^{\delta t})).$$

Proof. By (12), both $\phi(t)$ and $\phi'(t)$ decay exponentially at $-\infty$. Then $\psi(t)$ has the same property due to the inequality $\psi(t) < \sqrt{\max\{1,2b\}\phi(t)}, \ t \in \mathbb{R}$, proved in [23, Theorem 6]. Applying now [18, Proposition 7.2] alternately to each equation of (11), we get the above expansions. ☐

Finally, as a straightforward consequence of [23, Lemma 14], we obtain

Lemma 3.7. *The limit $\lim_{t\to+\infty}(\phi'(t))^{-1}(-\phi(t)+\psi(t))$ exists and is finite.*

3.3. **Proof of Theorem 1.3.** The proof of Theorem 1.3 is essentially based on the approach briefly presented in Subsection 3.1. Thus it requires the construction of a regular super-solution. This work will be done in the next two lemmas. Then Theorem 3.1 will be invoked in order to complete the proof of our main result.

Lemma 3.8. *Assume that $r \in (0,1]$ and $(\phi(t),\psi(t))$ is a monotone traveling front to system (8). Then for every $c' > c$ there exists $\sigma > 1$ such that, for all $t \in \mathbb{R}$, the pair $(\phi_\sigma, \psi_\sigma) = \sigma(\phi,\psi)$ satisfies*

$$\begin{cases} \phi''_\sigma(t) - c'\phi'_\sigma(t) + \phi_\sigma(t)(1 - r - \phi_\sigma(t) + r\psi_\sigma(t)) < 0, \\ \psi''_\sigma(t) - c'\psi'_\sigma(t) + b\phi_\sigma(t-ch)(1-\psi_\sigma(t)) < 0. \end{cases} \tag{18}$$

Proof. System (18) is equivalent to

$$\begin{cases} \Psi(t) := (\phi'(t))^{-1}\phi(t)(-\phi(t) + r\psi(t)) < (\sigma-1)^{-1}(c'-c), \\ -b(\sigma-1)\phi(t-ch)\psi(t) < (c'-c)\psi'(t). \end{cases} \tag{19}$$

Since $\phi(t) > 0$ for all $t \in \mathbb{R}$, we only have to prove the first inequality of (19), for some appropriate $\sigma > 1$. From Lemma 3.5, we know that $(\phi'(t))^{-1}\phi(t)$ has a finite limit at $t = -\infty$ when $r \in (0,1)$. This implies that $\Psi(-\infty) = 0,\ \Psi(+\infty) = -\infty$ (when $r < 1$) and, for $r = 1$, $\Psi(+\infty) \in \mathbb{R}$ (by Lemma 3.7) with

$$\Psi(t) = (\phi'(t))^{-1}\phi(t)(\psi(t) - \phi(t)) = c - \frac{\phi''(t)}{\phi'(t)} < c,\ t < \tau_1,$$

where τ_1 is defined in Lemma 3.3. Hence, if $\sigma > 1$ is close to 1, then $\Psi(t) < (\sigma-1)^{-1}(c'-c)$ for all $t \in \mathbb{R}$. □

Corollary 3. *Let $r \in (0,1)$ and $c' > c \geq 2\sqrt{1-r}$, σ be as in Lemma 3.8. Then $(\phi_\sigma, \psi_\sigma)$ is a regular super-solution to (8) considered with the speed c'.*

Proof. It is a direct consequence of Lemma 3.8, Remark 1 and Lemma 3.5 (c). Observe here that $\nu(c) \in (\lambda(c'), \mu(c'))$ for $c' > c$ and, due to Lemma 3.5 (c), $\phi''(t+t_1) = \nu^2(c)(-t)^m e^{\nu(c)t}(1+o(1)),\ t \to -\infty$. □

Remark 2. By Lemma 3.6, proof of Corollary 3 also works when $r = 1$ and $\phi(t)$ satisfies (12) (so that $\nu(c) = \mu(c) = c$). However, if $\phi(t)$ satisfies (13), we have to change our arguments. See the next proposition.

Lemma 3.9. *Take $r = 1,\ c' > c$, and let $(\phi_\sigma, \psi_\sigma) = \sigma(\phi,\psi)$ be as in Lemma 3.8. Assume also that (13) holds and set*

$$\phi^+_\sigma(t) := \min\{\phi_\sigma(t), \rho e^{ct}\}, \quad \psi^+_\sigma(t) := \min\{\psi_\sigma(t), \rho D e^{ct}\}, \quad t \in \mathbb{R},$$

where $D := be^{-cc'h}/(c'c - c^2)$. Then, for each sufficiently large $\rho > 0$, the pair $(\phi^+_\sigma, \psi^+_\sigma)$ gives a regular super-solution for system (8) considered with the speed c'.

Proof. Condition **D1** with $\nu = c$ of the definition of regular super-solution is satisfied in the obvious way. The rest of the proof will be broken into several parts:

Claim I. *Equation $\phi_\sigma(t) = \rho e^{ct}$ [respectively, $\psi_\sigma(t) = \rho D e^{ct}$] has a unique solution $d_1 := d_1(\rho)$ [respectively, $d_2 := d_2(\rho)$] for each sufficiently large ρ. Moreover, $\phi'_\sigma(d_1) < \rho c e^{cd_1}$ and $\psi'_\sigma(d_2) < \rho c D e^{cd_2}$.*

Consider, for example, the first equation. Let T_1 be such that

$$\phi'(t) < c\phi(t), \qquad c^2 - c'c + \sigma\psi(t) < 0 \quad \text{for all } t \leq T_1.$$

If we take $\rho > \rho_1 := 2e^{-cT_1}$ then $\rho e^{cT_1} > 2 > \phi_\sigma(T_1)$. Since, for a fixed ρ, it holds $\rho e^{ct} \ll \phi_\sigma(t)$ as $t \to +\infty$, we conclude about the existence of at least one

$d_1 = d_1(\rho) \leq T_1$ solving $\phi_\sigma(t) = \rho e^{ct}$. Additionally, $\phi'_\sigma(d_1) < c\phi_\sigma(d_1) = \rho c e^{cd_1}$ if $\rho > \rho_1$. This assures the uniqueness of d_1.

Claim II. *$d_j(\rho),\ d_1(\rho) - d_2(\rho) \to -\infty$ as $\rho \to \infty$.*

Indeed, in virtue of Lemma 3.4 and Corollary 2 we may suppose that, for $|\delta c_j| \ll c$,

$$\frac{2\sigma(c+\delta c_1)^2}{d_1^2 b} = \phi_\sigma(d_1) = \rho e^{cd_1}, \ -\frac{2\sigma(c+\delta c_2)}{d_2} = \psi_\sigma(d_2) = \rho D e^{cd_2}.$$

Therefore, for some δ close to 0,

$$d_j(\rho) \ < c^{-1}(|\ln D| - \ln\rho), \quad -\frac{(c+\delta)d_2}{bd_1^2} = D^{-1}e^{c(d_1-d_2)}. \tag{20}$$

The first inequality implies that $d_j(\rho) \to -\infty$ when $\rho \to +\infty$. Therefore, if there are a sequence $\rho_j \to +\infty$ and K such that $|d_1(\rho_j) - d_2(\rho_j)| \leq K$, we obtain that $\lim d_2(\rho_j)/d_1(\rho_j) = 1$. But then the second relation in (20) yields a contradiction: $d_1(\rho) - d_2(\rho_j) \to -\infty$. This shows that $|d_1(\rho_j) - d_2(\rho_j)| \to +\infty$ as $j \to +\infty$. Now, suppose that $d_1(\rho_j) - d_2(\rho_j) =: x_j \to +\infty$. In this notation, the second equation of (20) can be written as $(d_2+x)^2 + 2Qe^{-cx}d_2 = 0,\ 2Q := (c+\delta)D/b$. But then

$$d_1(\rho_j) = d_2(\rho_j) + x_j = -Qe^{-cx_j} \pm \sqrt{2x_jQe^{-cx_j} + Q^2e^{-2cx_j}} \to 0,\ j \to \infty,$$

a contradiction proving that $\lim(d_1(\rho) - d_2(\rho)) = -\infty$ as $\rho \to \infty$.

Claim III. *$(\phi_\sigma^+, \psi_\sigma^+)$ is a regular super-solution for each sufficiently large $\rho > 0$.*

By Claim II, we can choose ρ large enough to have $d_1(\rho) + h < d_2(\rho)$. It suffices to check inequalities (10) for each of the intervals $(-\infty, d_1]$, $[d_1, d_2]$, $[d_2, +\infty)$. The case $t \in (-\infty, d_1]$ is an easy one because of

$$\phi''_+(t) - c'\phi'_+(t) + \phi_+(t)(\psi_+(t) - \phi_+(t)) < \rho e^{ct}\{c^2 - c'c + \sigma\psi(t)\} < 0, \quad t \leq d_1(\rho);$$
$$\psi''_+(t) - c'\psi'_+(t) + b\phi_+(t - c'h)(1 - \psi_+(t)) = -bD\rho^2 e^{-cc'h}e^{2ct} < 0, \quad t \leq d_1(\rho) + h.$$

We also have $\Lambda_2(\phi_\sigma^+, \rho D e^{ct}) \leq \Lambda_2(\rho e^{ct}, \rho D e^{ct}) < 0$ for $t \leq d_2(\rho)$, and we can suppose that $|d_2(\rho)|$ is so large that $\Lambda_1(\rho e^{ct}, \rho D e^{ct}) < 0,\ t \leq d_2(\rho)$. Next, if $t \in [d_1, d_2]$ then $\Lambda_1(\phi_\sigma^+, \rho D e^{ct}) \leq \Lambda_1(\phi_\sigma, \psi_\sigma) < 0$, and if $t \geq d_2$, it holds that $\Lambda_j(\phi_\sigma^+, \psi_\sigma^+) \leq \Lambda_j(\phi_\sigma, \psi_\sigma) < 0$. Taking into account Claim I and Remark 1, we complete the proof. □

Proof of Theorem 1.3. Fix $h \geq 0,\ r \in (0, 1]$ and consider

$$\mathcal{C}(h, r) := \{c \geq 0 : \text{ system (2) has a wavefront propagating at the velocity } c\}.$$

By Theorem 4.1 from [8] (if $r \in (0, 1)$) and Theorem 7 from [23] (if $r = 1$), $\mathcal{C}(h, r)$ contains the infinite interval $(2, +\infty)$. Assume now that $c_0 \in \mathcal{C}(h, r),\ c_0 \geq 2\sqrt{1-r}$, and take an arbitrary $c' > c_0$. Then Lemma 3.9 with Remark 2 (for $r = 1$) and Corollary 3 (when $r \in (0, 1)$) assure the existence of a regular super-solution for system (8) taken with the velocity c'. By Theorem 3.1 stated in Section 3.1, there exists a monotone traveling front for (2) propagating at the velocity c' and satisfying (6), (7). As a consequence, for each $c_0 \in \mathcal{C}(h, r)$ we obtain $[c_0, +\infty) \subset \mathcal{C}(h, r)$ so that $\mathcal{C}(h, r)$ is a proper connected unbounded subinterval of $[0, +\infty)$. Set $c_* = \inf \mathcal{C}(h, r)$, then $c_* \in \mathcal{C}(h, r)$ (and therefore $c_* > 0$ due to Corollary 1). Indeed, let $c_j \downarrow c_*$ be a strictly decreasing sequence of velocities and (ϕ_j, ψ_j) be a sequence of corresponding traveling fronts (existing in virtue of the first part of the proof). Since

$$0 = \phi_j(-\infty) + \psi_j(-\infty) < \phi_j(t) + \psi_j(t) < \phi_j(+\infty) + \psi_j(+\infty) = 2$$

and the function $\phi_j(t)+\psi_j(t)$ is increasing in t for each fixed j, we may assume that $\phi_j(0) + \psi_j(0) = 3/2, \quad j = 1, 2, 3, \dots$ Using the standard compactness arguments and then applying the Lebesgue's dominated convergence theorem to the system of integral equations (9):

$$\phi_j(t) = \mathcal{N}_1(\phi_j, \psi_j, c_j)(t),\ \psi_j(t) = \mathcal{N}_2(\phi_j, \psi_j, c_j)(t),$$

we may assume, without restricting the generality, that $\lim_j(\phi_j, \psi_j) = (\hat{\phi}, \hat{\psi})$ uniformly on bounded intervals, where $(\hat{\phi}, \hat{\psi})$ is a monotone solution of (8) with $c = c_*$. Since $(\hat{\phi}, \hat{\psi})(\pm\infty)$ are steady state solutions of (8) and $\hat{\phi}(-\infty) + \hat{\psi}(-\infty) \le \hat{\phi}(0) + \hat{\psi}(0) = 3/2 \le \hat{\phi}(+\infty) + \hat{\psi}(+\infty)$, we find that necessarily

$$\hat{\phi}(-\infty) = 0,\ \hat{\psi}(-\infty) \in [0, 1],\ \hat{\phi}(+\infty) = \hat{\psi}(+\infty) = 1,$$

(if $\hat{\phi}(-\infty) > 0$, then $\hat{\psi}(-\infty) = \hat{\phi}(-\infty) = 1$ and thus $\hat{\phi}(0) + \hat{\psi}(0) = 2$, a contradiction). To finish the proof of the corollary, we have to establish that $\hat{\psi}(-\infty) = 0$. In order to prove this, we can apply the inequality $\psi_j(t) < \sqrt{\max\{1, 2b\}\phi_j(t)},\ t \in \mathbb{R}$, established in Theorem 6 from [23] for $r \in (0, 1]$. We obtain $\hat{\psi}(t) \le \sqrt{\max\{1, 2b\}\hat{\phi}(t)}$, $t \in \mathbf{R}$, so that $\hat{\psi}(-\infty) = 0$. □

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