Positive radial solutions to a ‘semilinear’ equation involving the Pucci’s operator

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Abstract

In this article we prove existence of positive radially symmetric solutions for the nonlinear elliptic equation

$$\begin{aligned} \mathscr{M}^{+}_{\lambda,\Lambda}(D^2u) - \gamma u + f(u) = 0 \quad &\text{in} \quad B_R, \\ u = 0 \quad &\text{on} \quad \partial B_R, \end{aligned}$$

where $\mathscr{M}^{+}_{\lambda,\Lambda}$ denotes the Pucci’s extremal operator with parameters $0<\lambda\leqslant\Lambda$ and B_R is the ball of radius R in $\mathbf{R}^N$, $N\geqslant 3$. The result applies to a wide class of nonlinear functions f, including the important model cases: (i) $\gamma = 1$ and $f(s) = s^p$, $1<p<p^{+}_{*}$. (ii) $\gamma = 0$, $f(s) = \alpha s + s^p$, $1<p<p^{+}_{*}$ and $0\leqslant\alpha<\mu^{+}_{1}$. Here p^{+}_{*} is critical exponent for $\mathscr{M}^{+}_{\lambda,\Lambda}$ and μ^{+}_{1} is the first eigenvalue of $\mathscr{M}^{+}_{\lambda,\Lambda}$ in B_R. Analogous results are obtained for the operator $\mathscr{M}^{-}_{\lambda,\Lambda}$.

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1. Introduction

The theory of viscosity solutions provides a very general and flexible theory for the study of a large class of partial differential equations. While originally developed to understand first-order equations, it was successfully extended to cover fully nonlinear second-order elliptic and parabolic equations. Very general existence results are combined with regularity theory to obtain a complete theory. We refer to [2,4] for the basic elements of the theory.

These remarkable general existence results require some structural hypotheses on the fully nonlinear operator, deeply linked to the Perron's method of super and sub-solutions. Essentially the operator has to satisfy proper maximum and comparison principles.

When those structural hypotheses are not satisfied not much is known about existence theory for fully nonlinear operators. In great contrast, for equations with divergence form operators, a vast number of results are known through various different methods including the variational approach and the topological method via degree theory.

In this article we consider the existence of positive solutions to a 'semi-linear' equation involving the Pucci's extremal operators, in which the maximum principle nor the comparison principle hold. Even though these problems have application in areas like financial mathematics [1], our interest is on the theory of equations. The Pucci's extremal operators are perturbations of the usual Laplacian, sharing with it many properties like homogeneity, positivity and comparison properties. However they are not in divergence form, thus deviating in a fundamental manner away from the Laplacian. The Pucci's extremal operators represent an important prototype of fully nonlinear operators, sitting at the center of the theory of regularity.

Our approach to study the existence problem is based on degree theory for compact operators in positive cones. This approach has been successfully applied by many authors to a variety of problems. Of special interest to us is the work of de Figueiredo et al. [8], on which we base our arguments. This approach requires a priori bounds for the solutions, which are obtained via blow up techniques as in the fundamental paper of Gidas and Spruck [9]. The success of this approach rests on Liouville type theorems.

In a recent article [7], the authors studied a Liouville type theorem for the Pucci's operators in the radially symmetric case. We proved the existence of a critical exponent that separates the existence and nonexistence range for power nonlinearities.

We describe our results in a more precise manner next. Let us first recall the definition of the Pucci's extremal operators. Given two parameter $0<\lambda\leqslant\Lambda$, the matrix operators $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$ are defined as follows: if M is a symmetric $N\times N$ matrix

$$\mathcal{M}^+_{\lambda,\Lambda}(M)=\Lambda\sum_{e_i>0}e_i+\lambda\sum_{e_i<0}e_i$$

and

$$\mathcal{M}^{-}_{\lambda,\Lambda}(M) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i,$$

where $e_i = e_i(M)$, $i = 1, \dots, N$, are the eigenvalues of M. The Pucci's operators are obtained applying $\mathcal{M}^{+}_{\lambda,\Lambda}$ or $\mathcal{M}^{-}_{\lambda,\Lambda}$ to the Hessian D^2u of the scalar function u. These two operators have many properties in common, but they are not equivalent. For more details and equivalent definitions see the monograph of Caffarelli and Cabré [2].

We consider the equation

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) + u^p = 0 \quad \text{in } \mathbf{R}^N, \tag{1.1}$$

where $p>1$. For notational simplicity, here and in the rest of the paper, we denote by $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ both operators $\mathcal{M}^{+}_{\lambda,\Lambda}$ and $\mathcal{M}^{-}_{\lambda,\Lambda}$, in such a way that (1.1) represents actually the two corresponding equations. In [7], see also [6], we proved

Theorem 1.1. *Let $N \geqslant 3$. Then there exist numbers $p^{+}_{*}>1$ and $p^{-}_{*}>1$ such that: if $1<p<p^{+}_{*}$ $(1<p<p^{-}_{*})$ then* (1.1) *does not have a radially symmetric C^2 solution.*

The numbers p^{+}_{*} and p^{-}_{*} are called critical exponents for the operators $M^{+}_{\lambda,\Lambda}$ and $M^{-}_{\lambda,\Lambda}$, respectively. When the parameters λ and Λ are equal then $p^{+}_{*} = p^{-}_{*} = p_N = (N+2)/(N-2)$, the usual Sobolev critical exponent, see [3,13]. For other Liouville type theorems we refer the reader to the article by Cutri and Leoni [5]. Notice that in the case $\lambda<\Lambda$, we have $p^{+}_{*}>p_N$ and $p^{-}_{*}<p_N$.

Besides the Liouville type theorem for positive radial solutions we proved in [7], we also obtained an existence result for positive radially symmetric solutions in a ball, when $1<p<p^{\pm}_{*}$. Thus it is natural then to ask for the existence of positive solutions for more general nonlinearities.

It is the main purpose of this article to prove existence theorems for positive radially symmetric solutions for the equation

$$\begin{aligned} \mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) - \gamma u + f(u) = 0 \quad &\text{in} \quad B_R, \\ u = 0 \quad &\text{on} \quad \partial B_R, \end{aligned} \tag{1.2}$$

where B_R is the ball of radius R in $\mathbf{R}^N$ and f is an appropriate nonlinearity. As for Eq. (1.1), Eq. (1.2) represents the two equations corresponding to $\mathcal{M}^{+}_{\lambda,\Lambda}$ and $\mathcal{M}^{-}_{\lambda,\Lambda}$. We observe that when $\lambda = \Lambda = 1$ then $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ simply reduce to the Laplace operator, so (1.2) becomes

$$\begin{aligned} \Delta u - \gamma u + f(u) = 0 \quad &\text{in} \quad B_R, \\ u = 0 \quad &\text{on} \quad \partial B_R. \end{aligned} \tag{1.3}$$

This equation has been studied by many authors, not only in a ball, but on general domains. We refer the reader to the review paper by Lions [10] and the references therein.

Continuing with the description of our results, let us introduce the precise assumptions on our nonlinearity f:

(f0) $f \in C([0,+\infty))$ and is locally Lipschitz.
(f1) $f(s) \geqslant 0$ and there is $1 < p < p_*^{\pm}$ and a constant $C^* > 0$ such that

$$\lim_{s \to +\infty} \frac{f(s)}{s^p} = C^*.$$

(f2) There is a constant $c^* \geqslant 0$ such that $c^* - \gamma < \mu_1^{\pm}$ and

$$\lim_{s \to 0} \frac{f(s)}{s} = c^*,$$

where μ_1^+ (μ_1^-) is the first eigenvalue for $\mathcal{M}^+_{\lambda,\Lambda}$ ($\mathcal{M}^-_{\lambda,\Lambda}$) in B_R. See Theorem 3.1 in Section 3.

The first model problem is $\gamma = 1$ and $f(s) = s^p$, $1 < p < p_*^{\pm}$. The second model problem is $\gamma = 0$ and $f(s) = \alpha s + s^p$, $1 < p < p_*^{\pm}$ and $0 \leqslant \alpha < \mu_1^{\pm}$.

Now we are in a position to state our main theorem

Theorem 1.2. *Assume $N \geqslant 3$ and f satisfies the hypotheses* (f0), (f1) *and* (f2). *Then there exist a positive radially symmetric C^2 solution of* (1.2).

In case of the first model problem, we can extend Theorem 1.2 for positive solutions in $\mathbf{R}^N$. Precisely we have

Theorem 1.3. *Assume $N \geqslant 3$ and $1 < p < p_*^{\pm}$. Then there is a positive radially symmetric C^2 solution of the equation*

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) - u + u^p = 0 \quad in\ \mathbf{R}^N. \tag{1.4}$$

In order to prove or main theorem we use degree theory on positive cones as presented in [8]. A priori bounds for solutions are obtained by blow up method introduced by Gidas and Spruck [7] in combination with the Liouville type Theorem 1.1.

In order to set up our abstract scheme of proof, we have to study existence and regularity of solutions for the equation

$$\begin{aligned} \mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) - \gamma u = g(r) \quad &\text{in} \quad B_R, \\ u = 0 \quad &\text{on} \quad \partial B_R \end{aligned} \tag{1.5}$$

for a given continuous g. Even though this result may be deduced from the general theory of viscosity solutions, we prefer to give a direct proof. Our aim is that all our results are self-contained.

The plan of the paper is the following. In Section 2, we study existence and regularity for Eq. (1.5). In Section 3, we consider the eigenvalue problem for the Pucci's operator, basing our arguments in the Krein–Rutman theorem that is proved by Rabinowitz [14]. With this result we can obtain a more complete existence theorem for Eq. (1.2). In Section 4, we describe the abstract setting in [8] and we prove the necessary a priori bounds that allow to use the abstract theory. We prove here Theorems 1.2 and 1.3.

2. Basic existence theorem

In this section, we study a basic existence theorem upon which we base our arguments to construct a solution to (1.2).

Theorem 2.1. *Let $g: [0, R] \to \mathbf{R}$ be a continuous, nonpositive function and $\gamma \geqslant 0$. Then there exists a unique C^2 positive radial solution to* (1.5).

The proof of this theorem is based on the existence theorem for the initial value problem together with some comparison arguments. Particular attention has to be taken to the regularity of the solution.

Remark 2.1. There is no lose of generality by assuming that $\gamma \geqslant 0$ since, when $\gamma < 0$ we can consider the term γu as part of the function $f(u)$.

In the case of a radially symmetric function, the Pucci's operators have a simple expression. First we note that when $u(x) = \varphi(|x|)$ is a C^2 radially symmetric function then we have

$$D^2u(x) = \frac{\varphi'(|x|)}{|x|} I + \left[\frac{\varphi''(|x|)}{|x|^2} - \frac{\varphi'(|x|)}{|x|^3} \right] X,$$

where I is the $N \times N$ identity matrix and X is the matrix whose entries are $x_i x_j$. Then the eigenvalues of D^2u are $\varphi''(|x|)$, which is simple, and $\varphi'(|x|)/|x|$, which has multiplicity $N - 1$.

In view of this we can give more explicit definition of the Pucci's operators. In the case of $\mathcal{M}^+_{\lambda,\Lambda}$ we define the functions

$$M(s) = \begin{cases} s/\Lambda, & s > 0, \\ s/\lambda & s \leqslant 0 \end{cases} \quad \text{and} \quad m(s) = \begin{cases} \Lambda s, & s > 0, \\ \lambda s, & s \leqslant 0 \end{cases}$$

then we see that u satisfies (1.5) if and only if u satisfies

$$v'' = M\left(-\frac{(N-1)}{r}m(v') + \gamma v + g(r)\right), \tag{2.1}$$

$$v'(0) = 0, \quad v(R) = 0. \tag{2.2}$$

In the case of $\mathcal{M}^-_{\lambda,\Lambda}$ we have a similar situation, just interchanging the roles of λ and Λ.

The main step in the proof of Theorem 2.1 is the following

Proposition 2.1. *Assume g is a continuous function. Then there exists a C^2 solution to the initial value problem*

$$v'' = M\left(-\frac{(N-1)}{r}m(v') + \gamma v + g(r)\right), \quad r \in [0, R], \tag{2.3}$$

$$v(0) = d, \quad v'(0) = 0. \tag{2.4}$$

For the proof of this proposition we need a series of lemmas. The first lemma is a regularity result. We observe that the only difficulty with the regularity of the solution to (2.3) and (2.4) may appear at the origin. We have

Lemma 2.1. *Assume $\gamma \geqslant 0$. If $v \in C^1$ is a solution of* (2.3) *and* (2.4) *and g is continuous, then $v \in C^2$.*

Proof. Without lose of generality we can assume that $\gamma = 0$ by considering the term γv as part of the right-hand side. We do the proof just for the operator $\mathcal{M}^+_{\lambda,\Lambda}$, the other is analogous.

The solution v is clearly C^2 in $(0, R]$, so we only need to worry about $r = 0$. Since $v'(0) = 0$, it is sufficient to prove that $v'(r)/r$ converges as $r \to 0$. We split the proof in three cases: (i) $g(0) < 0$, (ii) $g(0) > 0$ and (iii) $g(0) = 0$.

(i) If $g(0) < 0$, then $g(r) < 0$ in $[0, \delta]$, for some $\delta > 0$. Consequently, v satisfies

$$\{v' r^{N-1}\}' = \frac{g(r)}{\lambda} r^{N-1}, \quad r \in (0, \delta].$$

If we integrate from 0 to $r < \delta$ we have

$$\frac{v'(r)}{r} = \frac{\int_0^r g(s) s^{N-1}\, ds}{\lambda r^N},$$

and then

$$\lim_{r \to 0} \frac{v'(r)}{r} = \frac{g(0)}{N\lambda}.$$

(ii) If $g(0)>0$, with the same argument as in (i) we prove that

$$\lim_{r\to 0}\frac{v'(r)}{r}=\frac{g(0)}{\tilde{N}\Lambda},$$

where $\tilde{N}=\lambda(N-1)/\Lambda+1$.

(iii) If $g(0)=0$. We can rewrite Eq. (2.1) in terms of two v-dependent positive bounded functions η and σ as

$$v''(r)=-(N-1)\eta(r)\frac{v'(r)}{r}+\sigma(r)g(r). \tag{2.5}$$

Integrating from 0 to r we get

$$\frac{v'(r)}{r}=\frac{-(N-1)}{r}\int_0^r \eta(s)\frac{v'(s)}{s}\,ds+\frac{1}{r}\int_0^r \sigma(s)g(s)\,ds. \tag{2.6}$$

On the other hand, using that $g(0)=0$ we have that

$$\frac{1}{r}\int_0^r \sigma(s)g(s)\,ds\to 0 \quad \text{as } r\to 0. \tag{2.7}$$

We claim that $v'(r)/r\to 0$ as $r\to 0$. Suppose first that $\lim_{r\to 0}\sup v'(r)/r>0$. From (2.6) and (2.7) we see that it is not possible that $v'(r)>0$ for r small. Suppose then that $v'(r)$ changes sign for r small. Then there is a sequence $\{r_n\}$ such that $r_n\to 0$ and for some $\bar{\varepsilon}>0$

$$\limsup_{n\to\infty}\frac{v'(r_n)}{r_n}=\bar{\varepsilon}.$$

There exists a second sequence $\{\bar{r}_n\}$ with $\bar{r}_n<r_n$ such that $v'(\bar{r}_n)=0$ and $v'(r)>0$ for $r\in(\bar{r}_n,r_n)$. Then we have

$$\frac{v'(\bar{r}_n)}{\bar{r}_n}=0=\frac{-(N-1)}{\bar{r}_n}\int_0^{\bar{r}_n}\eta(s)v'(s)\,ds+\frac{1}{\bar{r}_n}\int_0^{\bar{r}_n}\sigma(s)g(s)\,ds. \tag{2.8}$$

From here, using (2.7), we find that for n large

$$\left|\frac{(N-1)}{\bar{r}_n}\int_0^{\bar{r}_n}\eta(s)\frac{v'(s)}{s}\,ds\right|<\frac{\bar{\varepsilon}}{8}$$

and hence

$$\left|\frac{(N-1)}{r_n}\int_0^{\bar{r}_n}\eta(s)\frac{v'(s)}{s}\,ds\right|<\frac{\bar{\varepsilon}}{8}.$$

On the other hand, from (2.6) and using again (2.7), for n large we have

$$\begin{aligned}\frac{\bar{\varepsilon}}{2}<\frac{v'(r_n)}{r_n} &= \frac{-(N-1)}{r_n}\int_0^{\bar{r}_n}\eta(s)\frac{v'(s)}{s}\,ds-\frac{(N-1)}{r_n}\int_{r_n}^{\bar{r}_n}\eta(s)\frac{v'(s)}{s}\,ds\\ &\quad+\frac{1}{r_n}\int_0^{r_n}\sigma(s)g(s)\,ds\\ &<\frac{\bar{\varepsilon}}{4}-\frac{(N-1)}{r_n}\int_{\bar{r}_n}\eta(s)\frac{v'(s)}{s}\,ds<\frac{\bar{\varepsilon}}{4},\end{aligned}$$

providing a contradiction. A slight modification allows to handle the case $\bar{\varepsilon}=\infty$. With this we conclude the proof since the case $\lim_{r\to 0}\inf v'(r)/r<0$ is similar. $\square$

The next lemma is a compactness result.

Lemma 2.2. *Assume $\{g_n\}$ is a uniformly bounded sequence of continuous functions. If u_n is a solution of* (2.3) *and* (2.4) *with g_n as a right-hand side then there exists $C>0$ such that*

$$\left|\frac{u_n'(r)}{r}\right|\leqslant C \quad \textit{and} \quad |u_n''(r)|<C, \quad \textit{for all } r\in[0,R].$$

Proof. We first claim that if $\{u_n(r_n)\}$ is bounded, with $r_n\in[0,R]$, then $\{u_n'(r_n)/r_n\}$ and $\{u''(r_n)\}$ are bounded. Suppose first that

$$\lim_{n\to+\infty}\frac{u_n'(r_n)}{r_n}=-\infty.$$

From (2.3) and (2.4) and since g_n is uniformly bounded, we have that $u_n''(r_n)\to+\infty$ as $n\to+\infty$.

If $u_n''(r)>0$ for all $r\in(0,r_n]$, then $u_n'(r_n)>0$, which is impossible. Thus, for all n there exists $\bar{r}_n\in(0,r_n)$ such that $u''(\bar{r}_n)=0$ and $u_n''(r)>0$ for all $r\in(\bar{r}_n,r_n)$. Hence $u'(\bar{r}_n)<u'(r_n)$, which implies that

$$\lim_{n\to+\infty}\frac{u'(\bar{r}_n)}{\bar{r}_n}=-\infty \quad \text{and} \quad u''(\bar{r}_n)=0.$$

This is in contradiction with (2.3). Suppose next that

$$\lim_{n\to+\infty}\frac{u_n'(r_n)}{r_n}=+\infty,$$

then with a similar argument we also get a contradiction. Thus, we have that $\{u_n'(r_n)/r_n\}$ is bounded. Then in view of (2.3) we also see that $\{u_n''(r_n)\}$, proving the claim.

Suppose now that there exists $\{r_n\} \subset [0, R]$ such that

$$\lim_{n \to +\infty} u_n(r_n) = +\infty.$$

Define $v_n(r) = u_n(r)/||u_n||_\infty$. Then $||v_n||_\infty = 1$ and v_n satisfies (2.3) with the right-hand side $g_n/||u_n||_\infty$. Using the claim we just proved we conclude that for a positive constant C

$$\left|\frac{v_n'(r)}{r}\right| < C, \quad |v_n''(r)| < C, \quad \text{for all } r \in [0, R].$$

Then, by Arzela–Ascoli theorem, $v_n \to v$ uniformly in $C^1([0, R])$, up to a sub-sequence. Using Lemma 2.1, we conclude that v is a C^2 solution of (2.3) and (2.4) with $d = 0$ and $g \equiv 0$. This implies that $v \equiv 0$, contradicting $||v||_\infty = 1$. Thus, $\{u_n\}$ is uniformly bounded.

The application of the claim concludes the proof. □

Now we are ready to complete the

Proof of Proposition 2.1. We do the proof just for the operator $\mathcal{M}^+_{\lambda,\Lambda}$. Suppose first that $\gamma d + g(0) < 0$. Consider the initial value problem

$$\{u' r^{N-1}\}' = \frac{(g(r) + \gamma u) r^{N-1}}{\lambda}, \quad u(0) = d, \ u'(0) = 0. \tag{2.9}$$

Using an argument of Ni and Nussbaum [11], we can find a C^2 solution of (2.9). Then, for some $\delta > 0$, u satisfies

$$u'' = M\left(-\frac{N-1}{r} m(u') + g(r) + \gamma u\right), \quad r \in (0, \delta].$$

Next we consider (2.3) with initial value $u(\delta)$ and $u'(\delta)$ at $r = \delta$. From standard theory of ordinary differential equation we find a C^2 solution of this problem for $r \in [\delta, R]$. Thus (2.3) and (2.4) has a C^2 solution in $[0, R]$. The case $\gamma d + g(0) > 0$ is similar.

Now we consider the case $\gamma d + g(0) = 0$. First we use the arguments given above to find a C^2 solution u_n of (2.3) and (2.4) with the right side $g_n(r) = g(r) - 1/n$. Using Lemma 2.1 we find that $u_n \to u$ in $C^1([0, R])$ up to a sub-sequence and, from Lemma 2.2, we conclude that u is a C^2 solution to (2.3) and (2.4). □

Remark 2.2. At this point we cannot guarantee the uniqueness of the solution to (2.3) and (2.4).

In the arguments to follow we need the maximum principle and comparison results for the Pucci's operators. We prove them now in the case of C^2 solutions, since we only need this regular case.

Before continuing let us recall some basic properties of the matrix operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}$. See Lemma 2.10 in [2] for the proof.

Lemma 2.3. *Let M and N be two symmetric matrices then*:

(i) $\mathcal{M}^{+}_{\lambda,\Lambda}(M) + \mathcal{M}^{-}_{\lambda,\Lambda}(N) \leqslant \mathcal{M}^{+}_{\lambda,\Lambda}(M+N) \leqslant \mathcal{M}^{+}_{\lambda,\Lambda}(M) + \mathcal{M}^{-}_{\lambda,\Lambda}(N)$,

(ii) $\mathcal{M}^{-}_{\lambda,\Lambda}(M) + \mathcal{M}^{-}_{\lambda,\Lambda}(N) \leqslant \mathcal{M}^{-}_{\lambda,\Lambda}(M+N) \leqslant \mathcal{M}^{-}_{\lambda,\Lambda}(M) + \mathcal{M}^{+}_{\lambda,\Lambda}(N)$.

Now the maximum and comparison principles

Proposition 2.2. *Let Ω be a bounded domain in $\mathbf{R}^N$.*

(1) *If u is continuous in $\bar{\Omega}$ and u is a C^2 solution of $\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) \leqslant 0$ in Ω, with $u \geqslant 0$ on $\partial\Omega$, then $u \geqslant 0$ in Ω.*

(2) *Let u, v be continuous functions in $\bar{\Omega}$. If u, v are C^2 in Ω and*

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) - \gamma u \leqslant g(x), \quad \text{and} \quad \mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2v) - \gamma v \geqslant g(x)$$

in Ω, with $u \geqslant v$ on $\partial\Omega$. Then $u \geqslant v$ in Ω.

Proof. (1) Let us consider the function $v_\varepsilon = u + \varphi_\varepsilon$, where $\varphi_\varepsilon(x) = \varepsilon(d^2 - |x|^2)$, with d such that $v_\varepsilon \geqslant 0$ on $\partial\Omega$. Since $D^2v_\varepsilon = D^2u - 2\varepsilon I_N$, Lemma 2.3 implies that

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2v_\varepsilon(x)) < 0, \quad \text{for } x \in \Omega.$$

But then v_ε cannot have a minimum. Thus we conclude that $v_\varepsilon \geqslant 0$ in Ω, for all $\varepsilon > 0$. Hence $u \geqslant 0$ in Ω.

(2) Consider $w_\varepsilon = u + \varphi_\varepsilon - v$. Then, using Lemma 2.3, we find that

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2w_\varepsilon) - \gamma w_\varepsilon < 0.$$

Then w_ε cannot have a negative minimum. □

Proof of Theorem 2.1. We first prove uniqueness of the initial value problem (2.3) and (2.4). Assume that there are two different solutions u_1 and u_2, both satisfying $u_1(0) = u_2(0) = d$. Then $u_1(R) \neq u_2(R)$, because the contrary and Proposition 2.2 would imply that $u_1 = u_2$. If $u_1(R) > u_2(R)$ then we define $u_{1\varepsilon}(r) = u_1(r) - \varepsilon$. We see that $u_{1\varepsilon}$ is a sub-solution and if ε is small we have $u_{1\varepsilon}(R) > u_2(R)$. But then, by Proposition 2.2 we get that $d - \varepsilon u_{1\varepsilon}(0) \geqslant u_2(0) = d$, which is impossible. We conclude that (2.3) and (2.4) has a unique solution.

Next we find appropriate super and sub-solutions to (1.5). Clearly $\underline{u} \equiv 0$ is a sub-solution. For the super-solution we consider $\bar{u}(r) = \frac{\alpha}{2}(r^2 - R^2)$, with $\alpha < 0$. We have

$$\bar{u}'(r) = \alpha r \quad \text{and} \quad \bar{u}''(r) = \alpha.$$

Thus, $\bar{u}$ is a super-solution to (1.5) if α is such that

$$\lambda\alpha + \lambda(N-1)\alpha - \gamma\left(\frac{\alpha}{2}(r^2 - R^2)\right) \leqslant g(r).$$

Let u_1 and u_2 be the solutions to (2.3) and (2.4) with $u_1(0) = \bar{u}(0) + \varepsilon$ and $u_2 = -\varepsilon$, with $\varepsilon > 0$. Then by Proposition 2.2 we have $u_1(R) > 0$ and $u_2(R) < 0$.

To complete the existence part, we only need to prove that the function $d \to u(d, R)$ is continuous in d, where $u(d, R)$ is the solution to (2.3) and (2.4). But this follows in a standard way using the uniqueness of solutions of the initial value problem.

Finally, the uniqueness part follows from Proposition 2.2. □

Remark 2.3. In the case of a general $g(r) \in C([0, R])$, one can also find sub-solution and super-solutions and then the same proof holds.

3. Eigenvalue problem for $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$

In this section, we study the eigenvalue problem for the Pucci's operators. In the Introduction we described our hypothesis (f2) on the nonlinearity f in terms of the first eigenfunction of the operator. Here we prove that such an eigenvalue is positive and the first eigenfunction is positive also. This eigenvalue problem is studied in the context of radially symmetric functions, but one should also have a similar result in general domains. We will prove the following theorem.

Theorem 3.1. *The eigenvalue problem*

$$\begin{aligned} -\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) &= \mu u \quad \textit{in } B_R \\ u > 0 \text{ in } B_R, \quad u &= 0 \quad \textit{on } \partial B_R, \end{aligned} \tag{3.1}$$

has a solution $(\mu_1^{\pm}, u_1^{\pm})$, *with* $\mu_1^{\pm}$ *and* $u_1^{\pm}$ *positive. Moreover, all positive solutions to* (3.1) *are of the form* $(\mu_1^{\pm}, \alpha u_1^{\pm})$, *with* $\alpha > 0$.

Here Eq. (3.1) represents the two eigenvalue problems, for $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$. The eigenpairs (μ_1^+, u_1^+) and (μ_1^-, u_1^-) correspond to the operators $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$, respectively. For the proof of this theorem we rely on ideas from Rabinowitz [14]. The starting point is the Krein–Rutman theorem, which can be proved using a general result on existence of a one parameter family of fixed points, see [14].

Theorem 3.2. *Let $(E, ||\cdot||)$ be a Banach space and K be a closed cone in E with vertex at 0. Let T: $\mathbf{R}^+ \times K \to K$ be a compact operator such that $T(0, u) = 0$ for all $u \in E$, then there exists an unbounded connected component $\mathscr{C}$ of $\mathbf{R}^+ \times K$ of solutions of $u = T(\mu, u)$ and starting from $(0, 0)$.*

Remark 3.1. Here we denote by $\mathbf{R}^+$ the interval $[0, +\infty)$.

Let us consider the cone of nonnegative continuous functions

$$C_\# = \{w \in C[0, R]/w(R) \geqslant 0,\ w(R) = 0\}$$

and define $\mathscr{L}^\pm: C_\# \to C_\#$ as the inverse of $-\mathscr{M}^\pm_{\lambda,\Lambda}$. The operator $\mathscr{L}^\pm$ is well defined after Theorem 2.1 and it is compact by Lemma 2.2. In the next three lemmas we describe the main properties of the operator $\mathscr{L}^\pm$.

Lemma 3.1. *The operator $\mathscr{L}^\pm$ is monotone, that is if $g_1, g_2 \in C_\#$ such that $g_1 \leqslant g_2$ then $\mathscr{L}^\pm(g_1) \leqslant \mathscr{L}^\pm(g_2)$.*

Proof. Direct from Proposition 2.2. □

Lemma 3.2. *Let $g_1, g_2 \in C_\#$, then*

(a) $\mathscr{L}^-(g_1 + g_2) \geqslant \mathscr{L}^-(g_1) + \mathscr{L}^-(g_2)$
(b) $\mathscr{L}^+(g_1 + g_2) \leqslant \mathscr{L}^+(g_1) + \mathscr{L}^+(g_2)$.

Proof. Let $u_i = \mathscr{L}^-(g_i)$, $i = 1, 2$. Then, using Lemma 2.3, we obtain

$$-\mathscr{M}^-_{\lambda,\Lambda}(D^2u_1 + D^2u_2) \leqslant -\mathscr{M}^-_{\lambda,\Lambda}(D^2u_1) - \mathscr{M}^-_{\lambda,\Lambda}(D^2u_2) = g_1 + g_2,$$

from where the inequality follows taking $\mathscr{L}^-$ on both sides. The case $\mathscr{L}^+$ is analogous. □

Lemma 3.3. *Let $g \in C_\#$ and $u = \mathscr{L}^\pm(g)$. If $g \neq 0$ then $u(r) > 0$ for all $r \in (0, R)$. Moreover $u'(R) < 0$.*

Proof. Since $g(r) \neq 0$, there exists $r^* \in (0, R)$ such that $u(r^*) > 0$. Suppose by contradiction that there exists a $\bar{r} \in (0, R)$ such that $u(\bar{r}) = 0$. Assume, without lose of generality, that $\bar{r} < r^*$ and $u(r) > 0$ if $r \in (\bar{r}, r^*)$. Consider the comparison function

$$v_\varepsilon(r) = \varepsilon(e^{-\alpha(r^* - r)^2} - e^{-\alpha(r^* - \bar{r})^2}).$$

We have that for α large $-\mathscr{M}^\pm_{\lambda,\Lambda}(D^2v_\varepsilon) \leqslant 0$ in the annulus $A = \{r/\bar{r} < r < \tilde{r}\}$, where $\tilde{r} = \bar{r} + (r^* - \bar{r})/2$. Choose now ε such that $v_\varepsilon(\tilde{r}) < u(\tilde{r})$. Then, using Proposition 2.2 we get $v_\varepsilon(r) \leqslant u(r)$ for all $r \in (\bar{r}, \tilde{r})$. Since $v'_\varepsilon(\bar{r}) \neq 0$ we get the contradiction since $u'(\bar{r}) = 0$.

With a similar argument we get $u'(R) < 0$. $\square$

Now we are in a position to prove the existence of eigenvalues.

Proof of Theorem 3.1. . Take $u_0 \in C_\# \setminus \{0\}$, we claim that there exists $M > 0$ such that $M\mathscr{L}^\pm u_0 \geqslant u_0$. Suppose that $\mathscr{L}^\pm u_0 - u_0/M \notin C_\#$ for all $M > 0$. Taking the limit as $M \to +\infty$ we have $\mathscr{L}^\pm u_0 \notin Int(C_\#)$, getting a contradiction with Lemma 3.3.

Define now T_ε: $\mathbf{R}^+ \times C_\# \to C_\#$ as $T_\varepsilon(\mu, u) = \mu\mathscr{L}^\pm(u) + \mu\varepsilon\mathscr{L}^\pm(u_0)$, for $\varepsilon > 0$. From Theorem 3.2, there exists a connected component $\mathscr{C}_\varepsilon$ of solution to $T_\varepsilon(\mu, u) = u$.

We show next that $\mathscr{C}_\varepsilon \subset [0, M] \times C_\#$. In fact, let $(\mu, u) \in \mathscr{C}_\varepsilon$, then

$$u = \mu\mathscr{L}^\pm u + \mu\varepsilon\mathscr{L}^\pm u_0,$$

hence $u \geqslant \mu\varepsilon\mathscr{L}^\pm u_0 \geqslant \frac{\mu}{M}\varepsilon u_0$. If we apply $\mathscr{L}^\pm$ we get

$$\mathscr{L}^\pm u \geqslant \frac{\mu}{M}\varepsilon\mathscr{L}^\pm u_0 \geqslant \frac{\mu}{M^2}\varepsilon u_0.$$

But $u \geqslant \mu\mathscr{L}^\pm u$, then $u \geqslant \left(\frac{\mu}{M}\right)^2 \varepsilon u_0$. By recurrence we get

$$u \geqslant \left(\frac{\mu}{M}\right)^n \varepsilon u_0 \quad \text{for all } n \geqslant 2.$$

This implies that $\mu \leqslant M$ and thus $\mathscr{C}_\varepsilon \subset [0, M] \times C_\#$.

Now we conclude. Since $\mathscr{C}_\varepsilon$ is unbounded there exists μ_ε and u_ε so that $(\mu_\varepsilon, u_\varepsilon) \in \mathscr{C}_\varepsilon$ and $||u_\varepsilon||_\infty = 1$. Then, by the compactness of $\mathscr{L}^\pm$ we find $\mu_1 \in [0, M]$ and u_1 with $||u_1||_\infty = 1$ such that $u_1 = \mu_1\mathscr{L}^\pm u_1$. From here we also deduce that $\mu_1 > 0$.

To complete the proof of Theorem 3.1 we need the following lemma, whose proof is simple and can be seen in [14].

Lemma 3.4. *Let K be a closed cone with nonempty interior and* $y_0 \in int(K)$. *Then for all* $y \notin K$ *there exists a unique number* $\delta_{y_0}(y)$ *such that*:

(i) *if* $\mu \in [0, \delta_{y_0}(y)]$ *then* $y_0 + \mu y \in K$,
(ii) *if* $\mu \geqslant \delta_{y_0}(y)$ *then* $y_0 + \mu y \notin K$.

Moreover, if $y_0 + \mu y \in int(K)$ *then* $\mu < \delta_{y_0}(y)$.

Continuing with the proof of Theorem 3.1, let us consider $u \in C_\#$, $u \neq 0$ and $\mu > 0$ such that $u = \mu\mathscr{L}^\pm u$. Here we split the proof.

Case $\mathscr{L}^-$: Define $\gamma_1 = \delta_{u_1}(-u)$ and $\gamma_2 = \delta_u(-u_1)$ as given by the previous lemma, with $K = C_\#$. Using Lemma 3.2(a) we have that

$$\mathscr{L}^-(u_1 - \gamma_1 u) \leqslant \mathscr{L}^-(u_1) - \gamma_1\mathscr{L}^-(u) = \frac{1}{\mu_1}\left(u_1 - \gamma_1\frac{\mu_1}{\mu}u\right)$$

and similarly

$$\mathscr{L}^{-}(u-\gamma_2 u_1)\leqslant\frac{1}{\mu}\left(u-\gamma_2\frac{\mu}{\mu_1}u_1\right).$$

If $u-\gamma_2 u_1\neq 0$, then $\mathscr{L}^{-}(u-\gamma_2 u_1)\in int(C_\#)$, so $\mu/\mu_1<1$. Since $\mathscr{L}^{-}(u_1-\gamma_1 u)\in C_\#$, then $\mu_1/\mu\leqslant 1$ which is a contradiction. Thus $u=\gamma_2 u_1$.

Case $\mathscr{L}^{+}$: Using the previous lemma with $K=-C_\#$ we define $\gamma_1=\delta_{-u_1}(u)$, $\gamma_2=\delta_{-u}(u_1)$. From Lemma 3.2(b) we have then

$$\mathscr{L}^{+}(\gamma_1 u-u_1)\geqslant\frac{1}{\mu_1}\left(\gamma_1\frac{\mu_1}{\mu}u-u_1\right),\quad \mathscr{L}^{+}(\gamma_2 u_1-u)\geqslant\frac{1}{\mu}\left(\gamma_2\frac{\mu}{\mu_1}u_1-u\right).$$

If $-u+\gamma_2 u_1\neq 0$, then $\mathscr{L}^{+}(-u+\gamma_2 u_1)\in int(C_\#)$ so $\mu/\mu_1<1$. Since $\mathscr{L}^{+}(-u_1+\gamma_2 u)\in C_\#$, then $\mu_1/\mu\leqslant 1$ which is a contradiction. Thus $u=\gamma_2 u_1$. □

4. A priori bounds and proof of Theorem 1.2

Our existence theorem will be proved by using the approach of de Figueiredo et al. [8]. It consists in using degree theory for compact operators in cones. This abstract tool is combined with an appropriate a priori bounds and computation of degree.

We start recalling the abstract setting in [8]. Let K be a closed cone with nonempty interior in the Banach space $(E,||\cdot||)$. Let $\Phi: K\to K$ and $F: E\times[0,\infty)\to K$ be compact operators such that $\Phi(0)=0$ and $F(x,0)=\Phi(x)$ for all $x\in E$. Then the following theorem is proved in [12]. See also [8, Proposition 2.1 and Remark 2.1].

Theorem 4.1. *Assume there exist numbers* $0<R_1<R_2$ *and* $T>0$ *such that*:

(i) $x\neq\beta\Phi(x)$ *for all* $0\leqslant\beta\leqslant 1$ *and* $||x||=R_1$,
(ii) $F(x,t)\neq x$ *for all* $||x||=R_2$, $\quad t\in[0,+\infty)$ *and*
(iii) $F(x,t)=x$ *has no solution* $x\in\bar{B}_{R_2}$ *for* $t=T$.

Then Φ *has a fixed point in* $\mathscr{U}$ *where* $\mathscr{U}=\{x\in K/R_1<||x||<R_2\}$.

We note that solving (1.2) is equivalent to find a fixed point of $\Phi: C_\#\to C_\#$ defined as

$$\Phi(u)(r)\overset{\text{def}}{=}\mathscr{L}(f(u(r))),\quad r\in[0,R],$$

where $\mathscr{L}$ is the inverse of $-\mathscr{M}^{\pm}_{\lambda,\Lambda}(D^2\cdot)+\gamma\cdot$. As we mentioned in Section 2 we only need to consider the case $\gamma\geqslant 0$.

By Theorem 2.1 and Lemma 2.2 $\mathscr{L}$ is well defined and compact. We define next the operator F as $F(u,t)(r)=\mathscr{L}(f(u(r)+t))$.

We complete the proof of Theorem 1.2 by proving conditions (i)–(iii) in Theorem 4.1. First a priori bounds

Proposition 4.1. *Let u be a radial C^2 solution of the equation*

$$-\,\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) + \gamma u = f(u+t) \quad in\ B_R,$$
$$u>0\ in\ B_R \quad and\ u=0\ on\ \partial B_R, \tag{4.1}$$

with $t \geqslant 0$. Then there exists a constant C, independent of u, such that

$$||u||_\infty \leqslant C.$$

Proof. We argue by contradiction. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of positive solution to (4.1) such that $||u_n||_\infty \to +\infty$ as $n \to +\infty$. Suppose first that $u_n(r_n) = ||u_n||_\infty$ and $r_n \to 0$, up to sub-sequence.

Let us define

$$v_n(r) = \frac{1}{M_n} u_n(r_n + rM_n^{1-p/2}), \tag{4.2}$$

with $u_n(r_n) = M_n$, then v_n satisfies

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2v_n) + \gamma \frac{u_n}{M_n^p} = \frac{f(u_n+t)}{M_n^p} \quad \text{in } M_n^{p-1/2}(B_R - r_n)$$

and $||v_n||_\infty = 1$. By Lemmas 2.1 and 2.2 we have that, up to a sub-sequence, $v_n \to v$ in $C^1([0,\bar{R}])$, as $n \to +\infty$, where $\bar{R}$ is a fixed number. Note $v \in C^2$.

Since (f1) holds, we have that $f(u_n+t)/M_n^p \to C^* v^p$, and $u_n/M_n^p \to 0$. Thus, as $\bar{R}$ is arbitrary, after a diagonal procedure, we obtain a nontrivial positive v that satisfies

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2v) + C^* v^p = 0 \text{ in } \mathbf{R}^N \quad \text{with } p<p^{\pm}_*. \tag{4.3}$$

But this contradicts Theorem 1.1. Suppose now that $r_n \to r_0 \in (0,R)$, up to sub-sequence. Set $\delta>0$ such that $[r_n-\delta, r_n+\delta] \subset (0,R)$. Let us define now v_n as in (4.2) with $r \in I_n := (-\delta M_n^{p-1/2}, \delta M_n^{p-1/2})$, then v_n satisfies

$$\Lambda v_n'' + \frac{(N-1)}{r + M_n^{p-1/2} r_n} m(v_n') - \frac{\gamma u_n}{M_n^p} = \frac{-f(u_n+t)}{M_n^p} \quad \text{if } v_n'' \geqslant 0,$$
$$\lambda v_n'' + \frac{(N-1)}{r + M_n^{p-1/2} r_n} m(v_n') - \frac{\gamma u_n}{M_n^p} = \frac{-f(u_n+t)}{M_n^p} \quad \text{if } v_n''<0$$

and $||v_n||_\infty = 1$. Since $v_n'(0) = 0$ we can argue as in Lemma 2.2 and find that $v_n \to v$ in $C^1[-a,a]$, where $a>0$ is a fixed number. Also, since (f1) holds, we can argue as before to obtain a nontrivial positive v which satisfies

$$v'' + v^p = 0 \quad \text{in } \mathbf{R} \text{ with } \quad p>1. \tag{4.4}$$

But this is a contradiction. In the case $r_n \to R$, arguing as before we will end up with a nontrivial positive v which satisfies (4.4) in $\mathbf{R}^+$, again a contradiction. $\square$

Our next proposition implies condition (i) in Theorem 4.1.

Proposition 4.2. *There is $R_1 > 0$ so that the equation*

$$\begin{aligned} &-M^{\pm}_{\lambda,\Lambda}(D^2u) + \gamma u = \beta f(u) \quad \text{in } B_R \\ &u > 0 \text{ in } B_R, \quad u = 0 \qquad\qquad \text{on } \partial B_R, \end{aligned} \tag{4.5}$$

$\beta \in [0, 1]$, *has no solution u with $0 < \|u\|_\infty < R_1$.*

Proof. We argue by contradiction. Let $\{(u_n, \beta_n)\}_{n\in\mathbb{N}}$ be a sequence of positive solution to (4.5) such that $\|v_n\|_\infty \to 0$ as $n \to +\infty$. Define $v_n = u_n/\|u_n\|_\infty$, then we have that v_n satisfies

$$M^{\pm}(D^2v_n) - \gamma v_n + \beta_n \frac{f(u_n)}{u_n} v_n = 0 \quad \text{in } B_R$$

and $\|v_n\|_\infty = 1$. Using Lemmas 2.1 and 2.2, up to a sub-sequence, we find $v_n \to v$ in $C^1[0, R]$, $\beta_n \to \beta \in [0, 1]$. Moreover $v \in C^2$ and $\|v\|_\infty = 1$. By hypothesis (f2) we then obtain that v satisfies

$$\begin{aligned} &M^{\pm}(D^2v) + (\beta c^* - \gamma)v = 0 \quad \text{in } B_R, \\ &v > 0 \text{ in } B_R, \quad v = 0 \qquad\qquad \text{on } \partial B_R. \end{aligned}$$

If $\beta c^* - \gamma \leqslant 0$ then we get a contradiction with the maximum principle, Proposition 2.2. If $0 < \beta c^* - \gamma$ then by (f2) $\beta c^* - \gamma < \lambda_1^{\pm}$ and we get a contradiction with Theorem 2.1. $\square$

In order to prove condition (iii) in Theorem 4.1 we need a previous lemma.

Lemma 4.1. *Let u be a positive radial solution to*

$$\begin{aligned} &-M^{\pm}_{\lambda,\Lambda}(D^2u) + \gamma u = f(u+t) \quad \text{in } B_R, \\ &u = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial B_R \end{aligned}$$

with t large, then there exist $C > 0$, independent of u, such that $u(0) \geqslant CR^2t^p$.

Proof. After integration, we see that all we have to prove is the existence of $C > 0$ so that

$$-u'(r) \geqslant Crt^p \quad \text{for all } r \in (0, R). \tag{4.6}$$

We have that u satisfies

$$\Lambda u'' + \frac{(N-1)}{r} m(u') - \gamma u = -f(u+t) \quad \text{if } u'' \geqslant 0,$$

$$\lambda u'' + \frac{(N-1)}{r} m(u') - \gamma u = -f(u+t) \quad \text{if } u'' < 0.$$

From (f1) we have that there exists $C > 0$, independent of u, such that $-\gamma u + f(u+t) > Ct^p$, for t large. Let us fix $\bar{r} \in (0, R)$. If $u''(\bar{r}) \geqslant 0$, then $\lambda \frac{(N-1)}{\bar{r}} u'(\bar{r}) \leqslant -Ct^p$, thus $-u'(\bar{r}) \geqslant Ct^p \bar{r}/(\lambda(N-1))$. If $u''(\bar{r}) < 0$, suppose first $u''(r) < 0$ for all $0 \leqslant r < \bar{r}$. In this case we have that u is decreasing and satisfies

$$\{r^{N-1} u'\}' \leqslant \frac{-Ct^p r^{N-1}}{\lambda}. \tag{4.7}$$

Integrating from 0 to $\bar{r}$ we get (4.6) at $\bar{r}$. Suppose now that there exists a first $\hat{r} < \bar{r}$ such that

$$u''(\hat{r}) = 0 \quad \text{and hence} \quad -u'(\hat{r}) \geqslant \frac{Ct^p \hat{r}}{\lambda(N-1)}.$$

From here and since $u'' < 0$ for $r \in I \coloneqq (\hat{r}, \bar{r})$ we have that u is decreasing in I and satisfies (4.7) in I. Integrating (4.7) from $\hat{r}$ to $\bar{r}$ we obtain

$$-u'(\bar{r})\bar{r}^{N-1} \geqslant \frac{Ct^p \hat{r}^{\tilde{N}}}{\lambda(N-1)} + \frac{Ct^p}{\lambda}\left(\frac{\bar{r}^N}{N} - \frac{\hat{r}^N}{N}\right) \geqslant \frac{Ct^p}{\lambda}\frac{\bar{r}^N}{N}. \qquad \square$$

Proposition 4.3. *There exists a constant $T > 0$ so that if* (4.1) *possesses a solution u, then*

$$0 \leqslant t \leqslant T.$$

Proof of Proposition 4.3. We argue by contradiction. Suppose there is a sequence $\{t_n\}_{n \in \mathbb{N}}$ such that $t_n \to +\infty$ as $n \to \infty$, such that for each t_n there exist a solution u_n to (4.1). By Lemma 4.1 we have $||u_n||_\infty \equiv M_n \to +\infty$ as $n \to +\infty$. Since $M_n \geqslant CR^2 t_n^p$ we can argue as in the proof of Proposition 4.1 to reach a contradiction. $\square$

Proof of Theorem 1.2. Propositions 4.1–4.3 gives the conditions (i)–(iii) in Theorem 4.1, from where the result follows. $\square$

Proof of Theorem 1.3. We apply Theorem 1.2 to find a solution of

$$\begin{aligned} &-\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2 u) + u = u^p \quad \text{in } B_{R_n}, \\ &u > 0 \text{ in } B_{R_n}, \quad u = 0 \quad \text{on } \partial B_{R_n}, \end{aligned} \tag{4.8}$$

where $R_n \to \infty$. We find thus a sequence which is uniformly bounded by application a blow up argument as in Proposition 4.1. $\square$

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