Free surface instability in a confined suspension jet

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Abstract

A suspension confined between two close parallel plates is studied in the Stokesian regime. The use of boundary integral equations and the lubrication approximation allows to compute the hydrodynamic forces acting on the particles. The forces are long ranged and depend on the orientation of the relative position and velocity of particles. This tensorial character predicts an "antidrag" that is observed in experiments. The effect of the computed hydrodynamic forces is studied in the dynamics of a jet of particles falling by a gravitational field, which shows a surface instability similar to the Kelvin–Helmholtz one. A theoretical model, based on hydrodynamic-like equations, is able to predict the instability that is produced by the interaction of the long-range forces and the free surface.

Suspensions, that is, solid particles immersed in a fluid—such as sand in water or dust in air—have been extensively studied in physics and engineering science. Their applications are wide, and a few examples are fluidized beds, pneumatic flow or sedimentation. Recent experiments show a wide variety of phenomena, for example the break-up of a blob of particles [1–3], the pattern formation in a rotating suspension [4], and a suspension jet [3]. All of these show the importance of a microscopic description of suspensions. Confined suspensions between two parallel

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plates, in quasi-2D (Q2D) geometries, have been extensively studied because they present many peculiar features and allow for a detailed experimental characterization of the microstructure. The pair correlation function and the structure factor reveal that particles tend to be much more in contact than in a 2D hard disk fluid [6]. The study of diffusion in counterflow stabilized suspensions shows anomalous diffusion [6] and in the case of Brownian motion an "antidrag" hydrodynamic interaction is observed, which decays as the inverse of the relative distance squared [7].

To model suspensions theoretically, different approaches have been employed. From a more phenomenological approach, suspensions are modeled as equivalent fluids or as if the fluid and solid phases constitute a mixture of two miscible fluids. In these models [8], the theoretical description is not closed and empirical parameters are necessary, like the effective viscosity. When the microstructure is taken into account, the dynamics of the individual particles is considered by the influence of the fluid. To compute the forces the fluid exerts on the particles, different methods have been used. More commonly, the fluid phase is described as a continuum, with its dynamics modeled by the hydrodynamic equations [9,10]. The Stokesian Dynamics [10] is a theory for slow flow, where the time scale of particles is bigger than the fluid ones. Using the Faxen law and multipole expansions, it is possible to obtain the particle velocities as a function of the forces acting on the particles.

In this article we study the dynamics of a Q2D confined suspension. In particular, we consider cylindrical particles in a thin cell in an intermediate regime, where the fluid time scale is lower than for the particles, like in Stokesian Dynamics. The choice of cylinders allows to compute explicitly both the long- and short-range forces but the results are applicable also to spherical particles of diameter comparable to the plates separation.

The presence of long-range forces in suspensions is ubiquitous and their treatment is complex. The study of cylindrical particles in Q2D geometry helps to understand the macroscopic effects of the long-range interactions. An important feature that is extracted from this analysis is that the mean force acting on a particle surrounded by an homogeneous medium vanishes, but in the presence of abrupt changes on the concentration of the suspension, they can produce instabilities.

We describe the evolution of a jet of suspended particles driven by the gravitational force. We consider a system of N solid particles that move through an incompressible Newtonian fluid of viscosity η. The fluid is confined between two parallel plates separated at a distance $2d$ in the z direction, being infinite in the other two directions. The particles are restricted to move in the plane and for simplicity we consider cylindrical particles of height L (slightly smaller than $2d$), radius σ, and mass m. The particles are thin, i.e., $2d \ll \sigma$.

In principle, the system is described by the coupled set of Newton's equation for the particles with the forces given produced by the fluid and other external forces, and the Navier–Stokes equations for the fluid with boundary conditions imposed by the particles. This coupled system of equations is too complex and a number of approximations are needed in order to obtain analytic results. If the characteristic time of the particles is much larger than the fluid's one $\tau_f = 4d^2\rho_f/\eta$ and if $2d \ll \sigma$

(both properties that can be easily obtained if the cell is thin), the fluid can be considered as stationary for any given particle configuration and the Hele–Shaw equations can be used to model the fluid [11]:

$$\vec{v}(x,y,z) = \left(1 - \frac{z^2}{d^2}\right)\vec{V}(x,y), \quad \vec{V}(x,y) = -\frac{d^2}{2\eta}\nabla P, \quad \nabla^2 P = 0\,, \tag{1}$$

where $P = P(x,y)$ is the pressure, and the gradients operate in the x–y plane. The three-dimensional velocity profile is valid beyond the viscous boundary layer that has a size comparable to the plate separation. The Laplace equation, together with the boundary conditions given by (1), can be solved using the boundary integral method [12,13] in the limit of a dilute concentration of particles. The resulting pressure and velocity field allow to compute the total force over each particle due to the pressure and viscous contributions

$$\vec{F}_k = -\frac{m}{\tau_p}\vec{u}_k - \frac{m}{8\tau_p}\sum_{i=1}^{N}\mathbb{K}(\vec{R}_{ki})\vec{u}_k\,, \tag{2}$$

where $\tau_p = md/\pi\sigma^2\eta$ is the particle relaxation time, $\vec{R}_{ki} = \vec{R}_i - \vec{R}_k$ is the relative distance between particles, and $\vec{R}_i$ the position of the center of mass of the ith particle. The tensor $\mathbb{K}$ is given by

$$\mathbb{K}(\vec{R}) = (\sigma/R)^2(\mathbb{I} - 2\hat{R}\hat{R})\,, \tag{3}$$

with $\hat{R} = \vec{R}/R$ and $\mathbb{I}$ is the identity tensor. Besides, the net computed torque on each cylinder is null.

These results indicate that there is a viscous drag force acting on a particle, proportional to its own velocity. In addition, there is an effective force between particles that is proportional to the velocities of the partners. This force decays as R^{-2} that in two dimensions corresponds to a long-range force. The interaction force on particle k has a tensorial character and its direction depends both on the direction of the velocity $\vec{u}_i$ and the relative distance $\vec{R}_{ki}$. When $\vec{u}_i$ is parallel to $\vec{R}_{ki}$, the interaction force on k is parallel to $\vec{u}_i$, and if $\vec{u}_i$ is perpendicular to $\vec{R}_{ki}$, the force turns out to be in an opposite direction to $\vec{u}_i$. If spherical particles of radius slightly smaller that d were used instead of cylinders, the force computation is similar. An equivalent expression to Eq. (2) is found, with appropriate geometrical pre-factors. Therefore, a suspension of spheres confined between two plates experience the same long-range interactions and, in particular, the "antidrag" interaction observed in Ref. [7] can be explained by the tensorial character of the force described in this article.

At short interparticle distances, however, the Hele–Shaw approximation cannot be used and the full Navier–Stokes are necessary. When particles are very close such that $|\vec{R}_{ik}| - \sigma \ll \sigma$, the lubrication approximation allows to compute the hydrodynamic forces [14,13]: $\vec{F}_{ik} = -2\alpha(\mathbb{I} - \hat{R}_{ik}\hat{R}_{ik})\vec{u}_{ik}/\sqrt{\varepsilon} - 3\alpha(\vec{u}_{ik}\cdot\hat{R}_{ik})\hat{R}_{ik}/\varepsilon$, where $\alpha = md^2/\tau\sigma^2$, $\vec{u}_{ik} = \vec{u}_i - \vec{u}_k$, and $\varepsilon = (R_{ik} - \sigma)/\sigma$. Note that the lubrication force diverges when two particles come in contact. This forbids hard collisions between them making it unnecessary to include any collisional force into the total force on the cylinders.

For particles at intermediate distances there is no satisfactory results and we consider a mixed model, where the expression used to compute the force on a particle depends on the relative distance with its partners. If the distance R is larger than R_{far} the far field expression (2) is used; if $R < R_{\mathrm{lubr}}$ the lubrication force is used; and finally when $R_{\mathrm{lubr}} < R < R_{\mathrm{far}}$ a linear interpolation between the two expressions is employed.

Numerical simulations of particles in a Q2D suspension interacting with the computed forces are performed. Initially, the suspended particles are placed at rest randomly ordered in a rectangle of width $L_x = 90\sigma$ and height $L_y = 600\sigma$, and they fall down due to the action of a gravitational field $-g\hat{y}$. To simulate an infinity jet the vertical direction is periodic, and the force computation uses the minimum image convention [15]. Thus the system initially has a surface that separates the region with suspended particles and the region with pure fluid. As our description is for the dynamics of the suspended particles, and not for the surrounding fluid dynamics that is already solved, it is sensible to call *free surface* the separation between the region with particles and the region empty of them.

The jet consists of $N = 12\,000$ particles. Units are chosen such that the particle diameter σ and the limiting velocity for a single particle $v_\infty = g\tau_p$ are set to one. The gravitational force is $mg = 2.0$. Finally, the values chosen for the cutoffs in the simulations are $R_{\mathrm{lubr}} = 1.3\sigma$ and $R_{\mathrm{far}} = 2.0\sigma$. In Fig. 1, three successive snapshots of the jet are shown. It is observed that the free surfaces become unstable, showing oscillations that grow with time. At the beginning the surface waves are characterized by a short wavelength but later a coarsening process is developed leading to larger structures. Once the size of this structure is comparable to the jet width, interactions between the two surfaces are observed and in-phase surface

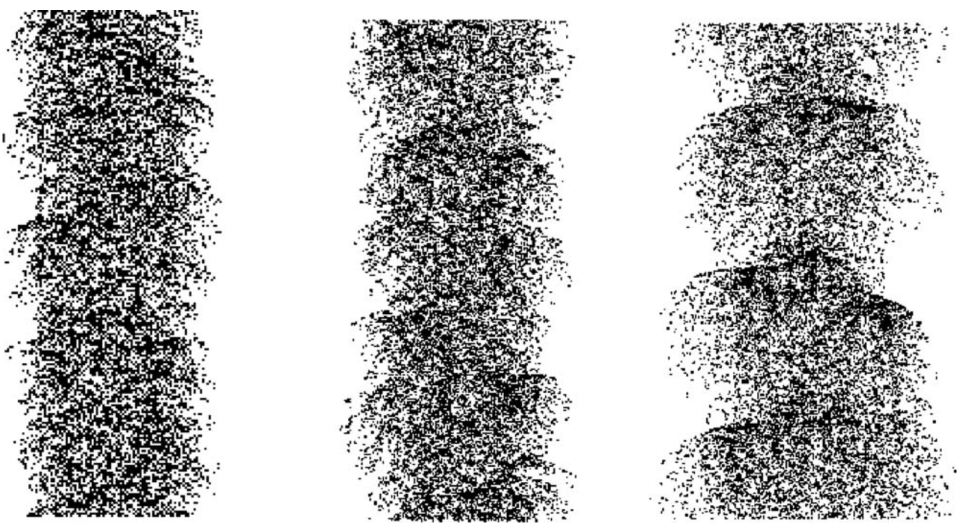

Fig. 1. Numerical simulation of a suspension of $N = 1200$ particles of $\sigma = 1$, under an external force $mg = 2$. From the left to the right: $t = 578$, $t = 788$, and $t = 1388$.

oscillations are obtained. Simulations with other parameters show the same kind of instability, regardless of the intensity of the gravitation acceleration or the initial density.

The observed instability is similar to the one observed when two immiscible fluid in contact move with a relative velocity (Kelvin–Helmholtz instability). To make a quantitative analysis we build a global model, similar to Euler-hydrodynamic equations, for the particle mass and mass current densities, ρ and $\vec{J}$, respectively

$$\frac{\partial \rho(\vec{R})}{\partial t} + \nabla \cdot \vec{J}(\vec{R}) = 0 \,, \tag{4}$$

$$\frac{\partial \vec{J}(\vec{R})}{\partial t} + \frac{\vec{J}(\vec{R})}{\rho(\vec{R})} \cdot \nabla \vec{J}(\vec{R}) = -\frac{1}{\tau_p}\vec{J}(\vec{R}) - \frac{1}{2m\tau_p}\rho(\vec{R}) \int \mathrm{d}\vec{R}' \,\mathbb{K}(\vec{R}-\vec{R}')\vec{J}(\vec{R}') + \rho(\vec{R})\vec{g} \,, \tag{5}$$

where the average force density over the suspension, produced by the far force contribution (2), was included. The near force contribution can be neglected in this simple model because its effect is to reduce velocity fluctuations, but it does not modify the mean velocity as it only affects the relative velocity. In the presence of a free surface, an additional equation must be added to describe the evolution of the free surface position $\xi(y,t)$. The equation is obtained as in the case of two fluids, imposing continuity between the movement of the suspension and the surface. That is, the mean suspension velocity at the surface must be equal to the surface velocity

$$\frac{\partial \xi}{\partial t} + \left.\frac{J_y}{\rho}\right|_{x=\xi} \frac{\partial \xi}{\partial y} = \left.\frac{J_x}{\rho}\right|_{x=\xi} \,. \tag{6}$$

Assuming that at the beginning the two free surfaces do not interact, we will consider the simple case of a single free surface, limiting a semi-infinite homogeneous suspension in the $x<0$ region, of density ρ_0. Eq. (5) implies that a stationary homogeneous solution exists with $\vec{J}_0 = -\rho_0 g \tau_p \hat{y}/(1+\beta_0/8)$, where $\beta_0 = \pi\sigma^2\rho_0/4m$ is the area fraction of the suspension. Note that the presence of the free surface reduces the value of J_0 that when there is no free surface $\vec{J}_0 = -\rho_0 g \tau_p \hat{y}$. Linear perturbation in Fourier space of Eqs. (4)–(6) is performed around the equilibrium state. Solutions in the form $\exp(\lambda s)$ are looked for, and the eigenvalues λ, for $k \ll 1$ and $\beta_o \ll 1$, in dimensionless units are $\lambda_1 = \mathrm{i}k\cos\phi - k(\beta_o^2/2)|\cos\phi|\cos\phi\sin\phi$, $\lambda_2 = \mathrm{i}k(\beta_o/2)\cos\phi + \mathrm{i}k(\beta_o^2/2)\cos\phi - k(b_o^2/2)|\cos\phi|\cos\phi\sin\phi$, $\lambda_3 = -1 + \beta_o + k(\beta_o/2)|\cos\phi|\cos\phi\sin\phi$, and $\lambda_4 = -1 - \beta_o - k(\beta_o/2)(\mathrm{i}\cos\phi + |\cos\phi|\cos\phi\sin\phi)$, where ϕ is the angle between the wave vector $\vec{k}$ and the $\hat{y}$ direction.

Two of the eigenvalues, λ_3 and λ_4, have negative real parts for small k and therefore correspond to damped motion. However, the real parts of λ_1 and λ_2 are positive for $\pi/2 < \phi < \pi$ and $3\pi/2 < \phi < 2\pi$ and an instability is predicted. The analysis shows that the system becomes unstable for any strength of the gravitational force and, coming back to the original unit, the instability rate is directly proportional to τ_p^{-1}, which is the only quantity with units of time. In the limit $k \ll 1$ the real parts of λ_1 and λ_2 are proportional to the wave vector. It is reasonable to expect that a more detailed model, which includes terms proportional to gradients of $\vec{J}$, the viscous effect produced by the lubrication forces and fluctuations, will produce that for high enough wave vectors; the real part of the eigenvalues becomes negative again. Hence, it is expected that the system is unstable for a range of wave vectors going from zero to a finite value. The absence of a lower limit in k for the instability and the linear proportionality of the growth rate with k explain the coarsening process that is observed in the simulations. A correct description of the coarsening process needs a nonlinear treatment of the instability and it is beyond the scope of the preset work.

In summary, the hydrodynamic-like formulation of the long-range hydrodynamic forces allows to predict that an homogeneous moving medium produces no net force acting on a particle, but a curved free surface does produce a net force. The interaction of the long-range forces with the free surface is responsible for the surface instability that is observed in the simulations.

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