A Dirichelet–Neumann m-point BVP with a p-Laplacian-like operator☆

M. García-Huidobro[a], Chaitan P. Gupta[b,*], R. Manásevich[c]

[a] *Departamento de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Correo 22, Santiago, Chile*
[b] *Department of Mathematics, 084 University of Nevada, Reno Reno, NV 89557 USA*
[c] *Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático, Universidad de Chile, Casilla 170, Correo 3 Santiago, Chile*

Abstract

Let ϕ, θ be odd increasing homeomorphisms from $\mathbf{R}$ onto $\mathbf{R}$ satisfying $\phi(0) = \theta(0) = 0$, and let $f : [a, b] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a function satisfying Carath*é*odory's conditions. Let $\alpha_i \in \mathbf{R}$, $\xi_i \in (a, b)$, $i=1, \ldots, m-2$, $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$ be given. We are interested in the problem of existence of solutions for the m-point boundary value problem:

$$(\phi(u'))' = f(t, u, u'), \quad t \in (a, b),$$
$$u(a) = 0, \quad \theta(u'(b)) = \sum_{i=1}^{m-2} \alpha_i \theta(u'(\xi_i))$$

in the resonance and non-resonance cases. We say that this problem is at *resonance* if the associated problem

$$(\phi(u'))' = 0, \quad t \in (a, b),$$
$$u(a) = 0, \quad \theta(u'(b)) = \sum_{i=1}^{m-2} \alpha_i \theta(u'(\xi_i))$$

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* Corresponding author.

E-mail addresses: mgarcia@mat.puc.cl (M. García-Huidobro), gupta@unr.edu (C.P. Gupta), manasevi@dim.uchile.cl (R. Manásevich).

has a non-trivial solutions. This is the case if and only if $\sum_{i=1}^{m-2}\alpha_i=1$. Our results use topological degree methods. Interestingly enough in the non-resonance case, i.e., when $\sum_{i=1}^{m-2}\alpha_i\neq 1$ the sign of degree for the relevant operator depends on whether $\sum_{i=1}^{m-2}\alpha_i>1$ or $\sum_{i=1}^{m-2}\alpha_i<1$.

Keywords: p-Laplacian; Boundary value problem; Dirichelet-Neumann; Resonance; Non-resonance; Odd increasing homeomorphism from R onto R; Deformation lemma; Leray-Schauder degree; Brouwer degree

1. Introduction

In this paper, we consider the boundary value problem:

$$\begin{aligned}&(\phi(u'))'=f(t,u,u'),\quad t\in(a,b),\\ &u(a)=0,\quad \theta(u'(b))=\sum_{i=1}^{m-2}\alpha_i\theta(u'(\xi_i)),\end{aligned}\tag{1.1}$$

where ϕ, θ are odd increasing homeomorphisms from $\mathbf{R}$ onto $\mathbf{R}$ with $\phi(0)=\theta(0)=0$ and the function $f:[a,b]\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ is Carathéodory. Also $\alpha_i\in\mathbf{R}$, $\xi_i\in(a,b)$, for $i=1,2,\dots,m-2$, are given numbers that satisfy $a<\xi_1<\xi_2<\cdots<\xi_{m-2}<b$.

We say that (1.1) is at *resonance*, if the associated multi-point boundary value problem

$$\begin{aligned}&(\phi(u'))'=0,\quad a<t<b,\\ &u(a)=0,\quad \theta(u'(b))=\sum_{i=1}^{m-2}\alpha_i\theta(u'(\xi_i))\end{aligned}\tag{1.2}$$

has a non-trivial solution.

We are interested here in the problem of existence of solutions for the *m*-point boundary value problem (1.1) in the resonance and in the non-resonance cases.

The study of multi-point boundary value problems in the case $\phi(u)=\theta(u)\equiv u$ was initiated by Il'in and Moiseev in [14,15] and has been the subject of many papers, see for example [2,3,7–13,16]. A three-point boundary value problem for the linear operator and the nonlinear boundary conditions has been dealt in [19].

More recently multi-point boundary value problems containing the *p*-Laplace operator or the more general operator $-(\phi(u'))'$ of problem (1.1), complemented with linear boundary conditions, have been studied in [1,4,6,17] and elsewhere.

In [5], using topological degree arguments, and [18], using upper and lower solutions, multi-point boundary value problems containing the operator $-(\phi(u'))'$ and the nonlinear boundary conditions are studied. The problem considered in [5] presents the feature that it is always at resonance because of the boundary conditions imposed.

Problem (1.1) is of a different nature concerning resonance, since it does not have the property present in [5]. In this case the problem will be at resonance if and only if $\sum_{i=1}^{m-2}\alpha_i=1$, having $u(t)=\rho(t-a)$ as a non-trivial solution, where $\rho\in\mathbf{R}$ is an arbitrary constant.

Our aim in this paper is to obtain existence of solutions for problem (1.1), where the nonlinear homeomorphisms ϕ and θ are in general different, by using topological degree arguments. Thus, in Section 2, we first derive a key deformation lemma that applies to the situation when problem (1.1) is at resonance, no sign restrictions on the numbers α_i, $i = 1, \dots, m-2$, are needed in this lemma. Furthermore, it is important to notice that conditions (ii) and (iii) of that lemma are the same as if the function ϕ that generates the differential operator and the function θ that appears in the boundary conditions were the linear functions, i.e., as if $\phi(s) = \theta(s) = s$. This is due to the homotopy we were able to obtain and it is an answer to the question of finding the simplest conditions on the function f that ensure the existence of solutions to our problem.

In Section 3 many existence theorems for problem (1.1) are derived from this lemma. Finally in Section 4 we consider problem (1.1) when it is at non-resonance. The crucial point here is to prove that the Leray Schauder degree of a certain operator is different from zero. This is shown to be an explicit consequence of the non-resonance condition, i.e., $\sum_{i=1}^{m-2} \alpha_i \neq 1$. In addition, we obtain the interesting property that the degree of the operator changes sign when $\sum_{i=1}^{m-2} \alpha_i$ goes from being less than one to being greater than one.

We shall denote by $C[a, b]$ (resp. $C^1[a, b]$) the classical space of continuous (resp. continuously differentiable) real-valued functions on the interval $[a, b]$. The norm in $C[a, b]$ is denoted by $|\cdot|_\infty$. Also, we shall denote by $L^1(a, b)$ the space of real-valued (equivalence classes of) functions whose absolute value is Lebesgue integrable on (a, b). The Brouwer and Leray–Schauder degree shall be, respectively, denoted by $\deg_{\mathrm{B}}$ and $\deg_{\mathrm{LS}}$.

2. A deformation lemma for the resonance case

We begin this section by formulating a general deformation lemma for the solvability of the boundary value problem (1.1) in the resonance case.

Let $f^* : [a, b] \times \mathbf{R} \times \mathbf{R} \times [0, 1] \to \mathbf{R}$ be a given function satisfying Carathéodory's conditions, i.e. (i) for all $(s, r, \lambda) \in \mathbf{R} \times \mathbf{R} \times [0, 1]$ the function $f^*(\cdot, s, r, \lambda)$ is measurable on $[a, b]$, (ii) for a.e. $t \in [a, b]$ the function $f^*(t, \cdot, \cdot, \cdot)$ is continuous on $\mathbf{R} \times \mathbf{R} \times [0, 1]$, and (iii) for each $R > 0$ there exists a Lebesgue integrable function $\rho_R : [a, b] \to \mathbf{R}$ such that $|f^*(t, s, r, \lambda)| \leqslant \rho_R(t)$ for a.e. $t \in [a, b]$ and all $(s, r, \lambda) \in \mathbf{R} \times \mathbf{R} \times [0, 1]$ with $|s| \leqslant R$, and $|t| \leqslant R$. We suppose that $f(t, s, r) = f^*(t, s, r, 1)$ is the given function in problem (1.1).

We, now, introduce an operator $\mathfrak{B}(u, \lambda) : C^1[a, b] \times [0, 1] \to \mathbf{R}$ defined for $(u, \lambda) \in C^1[a, b] \times [0, 1]$ by

$$\begin{aligned} \mathfrak{B}(u, \lambda) = \lambda \left(\theta(u'(b)) - \sum_{i=1}^{m-2} \alpha_i \theta(u'(\xi_i)) \right) \\ + (1 - \lambda) \left(\int_a^b f^*(\tau, u(\tau), u'(\tau), \lambda)\, \mathrm{d}\tau \right. \\ \left. - \sum_{i=1}^{m-2} \alpha_i \int_a^{\xi_i} f^*(\tau, u(\tau), u'(\tau), \lambda)\, \mathrm{d}\tau \right). \end{aligned} \tag{2.1}$$

For $\lambda \in [0, 1]$ we consider the family of boundary value problems:

$$\begin{aligned} &(\phi(u'))' = \lambda f^*(t, u, u', \lambda), \quad t \in (a, b), \\ &u(a) = 0, \quad \mathfrak{B}(u, \lambda) = 0. \end{aligned} \tag{2.2}$$

Let $\Omega \subset C^1[a,b]$ be a bounded open set. Let us set for $\rho \in \mathbf{R}$, $i_\rho(t) = \rho(t-a)$, for $t \in [a,b]$, and

$$X = \{i_\rho | \rho \in \mathbf{R}\}.$$

Then X is one-dimensional subspace of $C^1[a,b]$. Defining $i : \mathbf{R} \mapsto X$ by $i(\rho) = i_\rho$ it is clear that i is an isomorphism from $\mathbf{R}$ onto X.

Next let us define $F : X \to \mathbf{R}$ by

$$F(i_\rho) = \int_a^b f^*(t, \rho(t-a), \rho, 0)\,\mathrm{d}t - \sum_{i=1}^{m-2} \alpha_i \int_a^{\xi_i} f^*(t, \rho(t-a), \rho, 0)\,\mathrm{d}t$$

and set $\mathscr{F} = F \circ i$. Then $\mathscr{F} : \mathbf{R} \mapsto \mathbf{R}$ is continuous, and is given by

$$\mathscr{F}(\rho) = \int_a^b f^*(t, \rho(t-a), \rho, 0)\,\mathrm{d}t - \sum_{i=1}^{m-2} \alpha_i \int_a^{\xi_i} f^*(t, \rho(t-a), \rho, 0)\,\mathrm{d}t.$$

We have the following lemma.

Lemma 2.1. *Assume that*

(i) *for each* $\lambda \in (0,1)$, *the boundary value problem* (2.2) *has no solution* $u \in \partial\Omega$,
(ii) *equation* $\mathscr{F}(\rho) = 0$ *has no solution for any* ρ *such that* $i_\rho \in \partial\Omega \cap X$, *and*
(iii) *the Brouwer degree* $\deg_{\mathrm{B}}(\mathscr{F}, i^{-1}(\Omega \cap X), 0) \neq 0$.

Then, the boundary value problem (1.1) *has at least one solution in* $\overline{\Omega}$.

Proof. Let us define an operator $\Psi^* : C^1[a,b] \times [0,1] \to C^1[a,b]$ by setting for $(u,\lambda) \in C^1[a,b] \times [0,1]$

$$\Psi^*(u,\lambda)(t) = \int_a^t \phi^{-1}\left[\phi(u'(a)) + \lambda \int_a^s f^*(\tau, u(\tau), u'(\tau), \lambda)\,\mathrm{d}\tau\right]\mathrm{d}s + (t-a)\mathfrak{B}(u,\lambda). \tag{2.3}$$

Since f^* satisfies Carathéodory's conditions, then for $(u,\lambda) \in C^1[a,b] \times [0,1]$ we have $f^*(\cdot, u(\cdot), u'(\cdot), \lambda) \in L^1(a,b)$. Accordingly, the integrand in (2.3) is continuous on $[a,b]$ and the operator Ψ^* is well defined. Furthermore, using standard arguments, one can show that Ψ^* is a completely continuous operator.

Next, for some $\lambda \in (0,1]$, let us suppose that u is a solution to the boundary value problem (2.2). Then by integrating the equation in (2.2) and using that $u(a) = 0$ and $\mathfrak{B}(u,\lambda) = 0$, we see that u satisfies

$$u(t) = \int_a^t \phi^{-1}\left[\phi(u'(a)) + \lambda \int_a^s f^*(\tau, u(\tau), u'(\tau), \lambda)\,\mathrm{d}\tau\right]\mathrm{d}s + (t-a)\mathfrak{B}(u,\lambda) \tag{2.4}$$

for all $t \in [a,b]$, and thus u satisfies

$$u = \Psi^*(u,\lambda). \tag{2.5}$$

Conversely, let us suppose that for some $\lambda \in (0, 1]$, $u \in C^1[a, b]$, satisfies (2.5), equivalently (2.4). From Eq. (2.4) we first see that

$$u(a) = 0$$

and by differentiating that

$$u'(t) = \phi^{-1}\left(\phi(u'(a)) + \lambda \int_a^t f^*(\tau, u(\tau), u'(\tau), \lambda)\, d\tau\right) + \mathfrak{B}(u, \lambda),$$
$$t \in [a, b]. \tag{2.6}$$

Evaluating this equation at $t = a$ we see that

$$\mathfrak{B}(u, \lambda) = 0$$

and thus u satisfies the boundary conditions in problem (2.2). Also, Eq. (2.6) further implies that $\phi(u'(t))$ is absolutely continuous on $[a, b]$ and

$$(\phi(u'(t)))' = \lambda f^*(t, u(t), u'(t), \lambda), \quad \text{a.e. in } (a, b).$$

Thus, u is a solution of problem (2.2). In this form, for $\lambda \in (0, 1]$, we have proved that u is a solution of problem (2.2) if and only if u is a solution of Eq. (2.5), equivalently (2.4).

If, now, there is a function $u \in \partial\Omega$ which is a solution to problem (1.1), then we are done. Accordingly, let us assume that the boundary value problem (1.1) has no solution on $\partial\Omega$. This correspond to saying that problem (2.2), with $\lambda = 1$, does not have a solution on $\partial\Omega$. This combined with assumption (i) of the lemma implies that

$$u \neq \Psi^*(u, \lambda) \quad \text{for all } u \in \partial\Omega \text{ and } \lambda \in (0, 1].$$

We, next, assert that $u \neq \Psi^*(u, 0)$ for all $u \in \partial\Omega$. Indeed, let $u \in \partial\Omega$ be such that $u = \Psi^*(u, 0)$. Then by (2.4)

$$u(t) = (u'(a) + \mathfrak{B}(u, 0))(t - a) \tag{2.7}$$

for all $t \in [a, b]$. Differentiating and evaluating at $t = a$, we find that $\mathfrak{B}(u, 0) = 0$. Hence $u(t) = \rho(t - a) = i_\rho(t)$, where $\rho = u'(a)$ implying that $i_\rho \in \partial\Omega \cap X$. But, since i_ρ must satisfy

$$0 = \mathfrak{B}(i_\rho, 0) = \int_a^b f^*(\tau, \rho(\tau - a), \rho, 0)\, d\tau - \sum_{i=1}^{m-2} \alpha_i \int_a^{\xi_i} f^*(\tau, \rho(\tau - a), \rho, 0)\, d\tau$$
$$= \mathscr{F}(\rho),$$

we obtain a contradiction to assumption (ii) of the lemma. We thus get that

$$u \neq \Psi^*(u, \lambda) \quad \text{for all } u \in \partial\Omega \text{ and } \lambda \in [0, 1].$$

Thus $\deg_{LS}(I - \Psi^*(\cdot, \lambda), \Omega, 0)$ is well defined for all $\lambda \in [0, 1]$. By the homotopy invariance property of Leray–Schauder degree we obtain immediately that

$$\begin{aligned} \deg_{LS}(I - \Psi^*(\cdot, 1), \Omega, 0) &= \deg_{LS}(I - \Psi^*(\cdot, 0), \Omega, 0) \\ &= \deg_{B}(I - \Psi^*(\cdot, 0)|_X, \Omega_0, 0). \end{aligned} \tag{2.8}$$

where $\Omega_0 = \Omega \cap X$. Now since for $v \in X$

$$(I - \Psi^*(\cdot, 0))v = -i_{F(v)},$$

$$\deg_{\mathrm{LS}}(I - \Psi^*(\cdot, 1), \Omega, 0) = \deg_{\mathrm{B}}(-i_{F(\cdot)}, \Omega_0, 0) = -\deg_{\mathrm{B}}(i_{F(\cdot)}, \Omega_0, 0). \tag{2.9}$$

Since, $i^{-1} \circ i_{F(\cdot)} \circ i = \mathscr{F}$, we obtain by using a standard formula in degree theory that

$$\deg_{\mathrm{B}}(i_{F(\cdot)}, \Omega_0, 0)) = \deg_{\mathrm{B}}(\mathscr{F}, i^{-1}(\Omega_0), 0)).$$

Hence, by assumption (iii) of the lemma, it follows that $\deg_{\mathrm{LS}}(I - \Psi^*(\cdot, 1), \Omega, 0) \neq 0$. Thus, the mapping $\Psi \equiv \Psi^*(\cdot, 1) : C^1[a, b] \to C^1[a, b]$ has at least one fixed point in $\overline{\Omega}$ and hence the boundary value problem (1.1) has at least one solution in $\overline{\Omega}$. This completes the proof of the lemma. □

3. Some applications of Lemma 2.1

As in the previous sections ϕ and θ will denote odd increasing homeomorphisms from **R** onto **R** satisfying $\phi(0) = \theta(0) = 0$.

In some of our results we shall assume the following condition, for any $\sigma > 1$, ϕ is such that

$$\limsup_{z\to\infty} \frac{\phi(\sigma z)}{\phi(z)} < \infty. \tag{3.1}$$

If this is the case, we set

$$\alpha(a, b) = \limsup_{z\to\infty} \frac{\phi((b-a)z)}{\phi(z)}. \tag{3.2}$$

Theorem 3.1. *Let $f : [a, b] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ in the boundary value problem* (1.1) *be a continuous function that satisfies the following conditions*:

(i) *there exist non-negative functions $d_1(t)$, $d_2(t)$, and $r(t)$ in $L^1(a, b)$ such that*

$$|f(t, u, v)| \leqslant d_1(t)\phi(|u|) + d_2(t)\phi(|v|) + r(t),$$

for a.e. $t \in [a, b]$ and all $u, v \in \mathbf{R}$,

(ii) *there exist constants $\Lambda \geqslant 0$, $B \geqslant 0$, $A > 0$, and $v_0 > 0$ such that for all v with $|v| > v_0$, all $t \in [a, b]$, and all $u \in \mathbf{R}$, one has*

$$|f(t, u, v)| \geqslant -\Lambda\phi(|u|) + A\phi(|v|) - B,$$

(iii) *there exists an $R > 0$ such that for all ρ, with $|\rho| > R$, either*

$$\rho\left[\int_a^b f(\tau, \rho(\tau - a), \rho)\,\mathrm{d}\tau - \sum_{i=1}^{m-2} \alpha_i \int_a^{\xi_i} f(\tau, \rho(\tau - a), \rho)\,\mathrm{d}\tau\right] > 0$$

or

$$\rho \left[\int_a^b f(\tau, \rho(\tau - a), \rho) \, \mathrm{d}\tau - \sum_{i=1}^{m-2} \alpha_i \int_a^{\xi_i} f(\tau, \rho(\tau - a), \rho) \, \mathrm{d}\tau \right] < 0.$$

Suppose, further, ϕ *satisfies* (3.1), *that*

$$\alpha(a, b) \left(\|d_1\|_{L^1(a,b)} + \frac{\Lambda}{A} \right) + \|d_2\|_{L^1(a,b)} < 1 \tag{3.3}$$

and that the coefficients α_i, $i = 1, \dots, m-2$, *in the boundary conditions of problem* (1.1) *are non-negative with* $\sum_{i=1}^{m-2} \alpha_i = 1$.

Then, the boundary value problem (1.1) *has at least one solution* $u \in C^1[a, b]$.

Proof. First let $\varepsilon > 0$ be such that

$$\gamma_\varepsilon := (\alpha(a, b) + \varepsilon) \left(\|d_1\|_{L^1(a,b)} + \frac{\Lambda}{A} \right) + \|d_2\|_{L^1(a,b)} < 1. \tag{3.4}$$

Next, we see from the definition of $\alpha(a, b)$ that there exists $z_0 > 0$ such that

$$\phi((b - a)z) \leqslant (\alpha(a, b) + \varepsilon)\phi(z) \quad \text{for } z \geqslant z_0. \tag{3.5}$$

We consider the family of boundary value problems (2.2) with $f^*(t, u, v, \lambda) = f(t, u, v)$ for all $(t, u, v, \lambda) \in [a, b] \times \mathbf{R} \times \mathbf{R} \times [0, 1]$, i.e., we consider the family of boundary value problems:

$$\begin{aligned} &(\phi(u'))' = \lambda f(t, u, u'), \quad t \in (a, b), \quad \lambda \in [0, 1], \\ &u(a) = 0, \quad \mathfrak{B}(u, \lambda) = 0. \end{aligned} \tag{3.6}$$

We shall show that the family of boundary value problems (3.6) satisfies the conditions of Lemma 2.1 to conclude that the boundary value problem (1.1) has at least one solution in $C^1[a, b]$.

Let $u \in C^1[a, b]$ be a solution to the boundary value problem (3.6) for some $\lambda \in (0, 1)$. Suppose first $s_0 \in [a, b]$ is such that $|u'(s_0)| \leqslant v_0$, where v_0 is as in assumption (ii). Then, by integrating the equation in (3.6) from s_0 to $t \in [a, b]$, using assumption (i), and the assumption that ϕ is an odd increasing homeomorphism from $\mathbf{R}$ onto $\mathbf{R}$ with $\phi(0) = 0$, we get

$$\begin{aligned} \phi(|u'(t)|) &\leqslant \phi(v_0) + \phi(\|u\|_\infty)\|d_1\|_{L^1(a,b)} + \phi(\|u'\|_\infty)\|d_2\|_{L^1(a,b)} + \|r\|_{L^1(a,b)} \\ &\leqslant \phi(v_0) + \phi((b - a)\|u'\|_\infty)\|d_1\|_{L^1(a,b)} \\ &\quad + \phi(\|u'\|_\infty)\|d_2\|_{L^1(a,b)} + \|r\|_{L^1(a,b)} \end{aligned} \tag{3.7}$$

for all $t \in [a, b]$, since $u(a) = 0$ implies $\|u\|_\infty \leqslant (b - a)\|u'\|_\infty$. It then follows, using (3.5), that either $\|u'\|_\infty \leqslant z_0$ or

$$\begin{aligned} \phi(\|u'\|_\infty) &\leqslant \phi(v_0) + (\alpha(a, b) + \varepsilon)\phi(\|u'\|_\infty)\|d_1\|_{L^1(a,b)} + \phi(\|u'\|_\infty)\|d_2\|_{L^1(a,b)} \\ &\quad + \|r\|_{L^1(a,b)} \leqslant \gamma_\varepsilon \phi(\|u'\|_\infty)\| + \|r\|_{L^1(a,b)} + \phi(v_0). \end{aligned}$$

Thus if $s_0 \in [a, b]$ is such that $|u'(s_0)| \leqslant v_0$, then there is a positive constant C_1 such that

$$\|u'\|_\infty \leqslant C_1. \tag{3.8}$$

Let us, next, suppose that $|u'(t)| > v_0$ for all $t \in [a, b]$. Then, from the boundary condition $\mathfrak{B}(u, \lambda) = 0$, we have that

$$\begin{aligned}\lambda\Bigg[\theta\left[\phi(u'(a)) + \lambda\int_a^b f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\right] \\ - \sum_{i=1}^{m-2}\alpha_i\theta\left[\phi(u'(a)) + \lambda\int_a^{\xi_i} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\right]\Bigg] \\ + (1-\lambda)\Bigg(\int_a^b f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau \\ - \sum_{i=1}^{m-2}\alpha_i\int_a^{\xi_i} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Bigg) = 0.\end{aligned} \tag{3.9}$$

We observe next that since $\sum_{i=1}^{m-2}\alpha_i = 1$, with $\alpha_i \geqslant 0$, $i = 1, 2, \ldots, m-2$, there must exist $\eta_0, \eta_1 \in [a, \xi_{m-2}]$ such that

$$\begin{aligned}\sum_{i=1}^{m-2}\alpha_i\theta\left[\phi(u'(a)) + \lambda\int_a^{\xi_i} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\right] \\ = \theta\left[\phi(u'(a)) + \lambda\int_a^{\eta_0} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\right]\end{aligned} \tag{3.10}$$

and

$$\sum_{i=1}^{m-2}\alpha_i\int_a^{\xi_i} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau = \int_a^{\eta_1} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau. \tag{3.11}$$

Suppose, now, $f(t, u(t), u'(t)) > 0$ for all $t \in [a, b]$. Then, using the fact that θ is an increasing homeomorphism, and Eqs. (3.9)–(3.11), we obtain $0 > 0$, a contradiction. A similar contradiction is obtained if we assume that $f(t, u(t), u'(t)) < 0$ for all $t \in [a, b]$. Hence, there must exist a $\tau_0 \in [a, b]$ such that

$$f(\tau_0, u(\tau_0), u'(\tau_0)) = 0.$$

This and assumption (ii) then gives

$$\phi(|u'(\tau_0)|) \leqslant \frac{B}{A} + \frac{\Lambda}{A}\phi(\|u\|_\infty). \tag{3.12}$$

Next, integrating the equation in (3.6) from τ_0 to $t \in [a, b]$ and using assumption (i) we obtain that

$$\begin{aligned}\phi(|u'(t)|) &\leqslant \phi(|u'(\tau_0)|) + \phi(\|u\|_\infty)\|d_1\|_{L^1(a,b)} \\ &\quad + \phi(\|u'\|_\infty)\|d_2\|_{L^1(a,b)} + \|r\|_{L^1(a,b)} \\ &\leqslant \phi(\|u\|_\infty)\left(\|d_1\|_{L^1(a,b)} + \frac{\Lambda}{A}\right) \\ &\quad + \phi(\|u'\|_\infty)\|d_2\|_{L^1(a,b)} + \|r\|_{L^1(a,b)} + \frac{B}{A} \\ &\leqslant \phi((b-a)\|u'\|_\infty)\left(\|d_1\|_{L^1(a,b)} + \frac{\Lambda}{A}\right) \\ &\quad + \phi(\|u'\|_\infty)\|d_2\|_{L^1(a,b)} + \|r\|_{L^1(a,b)} + \frac{B}{A},\end{aligned} \tag{3.13}$$

using, as before, the fact that $u(a) = 0$ implies $\|u\|_\infty \leqslant (b-a)\|u'\|_\infty$. It now follows, using (3.5), that either $\|u'\|_\infty \leqslant z_0$ or

$$\begin{aligned}\phi(\|u'\|_\infty) &\leqslant \left[(\alpha(a,b)+\varepsilon)\left(\|d_1\|_{L^1(a,b)} + \frac{\Lambda}{A}\right) + \|d_2\|_{L^1(a,b)}\right]\phi(\|u'\|_\infty) \\ &\quad + \|r\|_{L^1(a,b)} + \frac{B}{A} \leqslant \gamma_\varepsilon \phi(\|u'\|_\infty) + \|r\|_{L^1(a,b)} + \frac{B}{A}.\end{aligned} \tag{3.14}$$

This inequality combined with (3.8) implies that in all cases there is a positive constant C such that

$$\|u'\|_\infty \leqslant C.$$

This fact combines in turn with the estimate $\|u\|_\infty \leqslant (b-a)\|u'\|_\infty$ to imply that there exists an $R_0 > R$, where R is as in assumption (iii) such that boundary value problems (3.6) have no solution on the boundary of the ball $B(0, \widetilde{R}) \subset C^1[a, b]$, for every $\widetilde{R} \geqslant R_0$. Accordingly, boundary value problems (3.6) satisfy condition (i) of Lemma 2.1.

Next, from assumption (iii), for all ρ, $|\rho| > R$, we have that

$$\mathscr{F}(\rho) := \int_a^b f(\tau, \rho(\tau - a), \rho)\,\mathrm{d}\tau - \sum_{i=1}^{m-2} \alpha_i \int_a^{\xi_i} f(\tau, \rho(\tau - a), \rho)\,\mathrm{d}\tau \tag{3.15}$$

is either strictly positive or strictly negative implying that condition (ii) of Lemma 2.1 is satisfied.

Finally, by assumption (iii) and the continuity of the function $\mathscr{F}$ defined in (3.15), for any fixed $\widetilde{R} > R$, it follows that $\mathscr{F}(\widetilde{R})\mathscr{F}(-\widetilde{R}) < 0$. Hence letting X be the one-dimensional space of $C^1[a, b]$, $X = \{i_\rho \mid \rho \in \mathbf{R}\}$, used in Section 2, we have that the Brouwer degree

$$\deg_{\mathrm{B}}(\mathscr{F}, i^{-1}(B(0, \widetilde{R}) \cap X), 0) = \deg_{\mathrm{B}}(\mathscr{F}, (-\widetilde{R}, \widetilde{R}), 0) \neq 0,$$

and condition (iii) of Lemma 2.1 is also satisfied.

Thus from Lemma 2.1 we conclude that the boundary value problem (1.1) has at least one solution u such that $\|u\|_{C^1[a,b]} < \widetilde{R}$. □

An immediate and simple corollary to this theorem (the proof of which is left to the reader) is given by the following result.

Example 3.2. Let $p > 1$, $q > 1$, and for $i = 1, \dots, m-2$. Let $\alpha_i \geqslant 0$, $\xi_i \in (0,1)$, be given numbers such that $\sum_{i=1}^{m-2} \alpha_i = 1$ and $0 < \xi_1 < \cdots < \xi_{m-2} < 1$.

In addition let $B > 0$, $\Lambda > 0$, and $A > 0$, be given numbers, with

$$\Lambda \left(1 + \frac{1}{A}\right) + A < 1.$$

Then, the boundary value problem

$$\begin{aligned} &(\phi_p(u'))' = \Lambda \phi_p(u) + A\phi_p(u') + B, \ \ t \in (0,1), \\ &u(0) = 0, \quad \phi_q(u'(1)) = \sum_{i=1}^{m-2} \alpha_i \phi_q(u'(\xi_i)), \end{aligned} \tag{3.16}$$

has at least one solution $u \in C^1[0,1]$.

We next consider a variant of the last theorem when $\theta = \phi$ in problem (1.1). Basically we change condition (ii) of the last theorem by a new one. This change allows us to consider functions f which are Carathéodory. Furthermore we will impose no restriction on the sign of $\alpha_i \in \mathbf{R}$. We consider the problem

$$\begin{aligned} &(\phi(u'))' = f(t, u, u'), \quad t \in (a,b), \\ &u(a) = 0, \quad \phi(u'(b)) = \sum_{i=1}^{m-2} \alpha_i \phi(u'(\xi_i)), \end{aligned} \tag{3.17}$$

where for $i = 1, \dots, m-2$, α_i, $\xi_i \in (0,1)$ are given numbers such that $\sum_{i=1}^{m-2} \alpha_i = 1$ and $0 < \xi_1 < \cdots < \xi_{m-2} < 1$. The function ϕ denotes an odd increasing homeomorphisms from $\mathbf{R}$ onto $\mathbf{R}$ satisfying $\phi(0) = 0$. The function f is Carathéodory.

Theorem 3.3. *Suppose in problem* (3.17) *the following conditions are satisfied*:

(i) *there exist non-negative functions $d_1(t)$, $d_2(t)$, and $r(t)$ in $L^1(a,b)$ such that*

$$|f(t,u,v)| \leqslant d_1(t)\phi(|u|) + d_2(t)\phi(|v|) + r(t),$$

for a.e. $t \in [a,b]$ and all u, $v \in \mathbf{R}$,

(ii) *there exists $d > 0$ such that for all $u \in C^1[a,b]$ with $\min_{t\in[a,b]} |u'(t)| > d$*

$$\int_a^b f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau - \sum_{i=1}^{m-2} \alpha_i \int_a^{\xi_i} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau \neq 0. \tag{3.18}$$

(iii) *For every $R > 0$, there exists $|\rho| > R$ such that $\mathscr{F}(\rho)\mathscr{F}(-\rho) < 0$, where*

$$\mathscr{F}(\rho) = \int_a^b f(\tau, \rho(\tau - a), \rho)\,\mathrm{d}\tau - \sum_{i=1}^{m-2} \alpha_i \int_a^{\xi_i} f(\tau, \rho(\tau - a), \rho)\,\mathrm{d}\tau.$$

Then, for $\alpha(a,b)$ *as in* (3.2), *problem* (3.17) *has at least one solution* $u \in C^1[a,b]$ *provided*

$$\alpha(a,b)\|d_1\|_{L^1(a,b)} + \|d_2\|_{L^1(a,b)} < 1. \tag{3.19}$$

Proof. As in the proof of Theorem 3.1 let $\varepsilon > 0$ be such that

$$\gamma_\varepsilon := (\alpha(a,b)+\varepsilon)\|d_1\|_{L^1(a,b)} + \|d_2\|_{L^1(a,b)} < 1 \tag{3.20}$$

so that, using the definition of $\alpha(a,b)$, there exists a $z_0 > 0$ such that

$$\phi((b-a)z) \leqslant (\alpha(a,b)+\varepsilon)\phi(z) \quad \text{for } z \geqslant z_0. \tag{3.21}$$

We consider the family of boundary value problems (2.2) with $f^*(t,u,v,\lambda) = \lambda f(t,u,v)$ and with $\mathfrak{B}(u,\lambda)$, see (2.1), now given by

$$\mathfrak{B}(u, \lambda = \phi(u'(b)) - \sum_{i=1}^{m-2} \alpha_i \phi(u'(\xi_i))).$$

Thus we are led to consider the family of boundary value problems

$$\begin{aligned} &(\phi(u'))' = \lambda f(t,u,u'), \quad t \in (a,b), \quad \lambda \in [0,1], \\ &u(a) = 0, \quad \phi(u'(b)) = \sum_{i=1}^{m-2} \alpha_i \phi(u'(\xi_i)). \end{aligned} \tag{3.22}$$

Let $u \in C^1[a,b]$ be a solution to problem (3.22) for some $\lambda \in (0,1)$. Then, integrating the equation in (3.22) from a to t, we find that

$$\phi(u'(t)) = \phi(u'(a)) + \int_a^t f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau,$$

which combined with the second boundary condition yields

$$\int_a^b f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau - \sum_{i=1}^{m-2} \alpha_i \int_a^{\xi_i} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau = 0.$$

Thus, from (ii), it must be that $\min_{t\in[a,b]} |u'(t)| \leqslant d$. Accordingly, there exists $t_0 \in [a,b]$ such that $|u'(t_0)| \leqslant d$. Then, integrating the equation in (3.22) from t_0 to $t \in [a,b]$, we get

$$\phi(u'(t)) = \phi(u'(t_0)) + \int_{t_0}^t f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau,$$

and hence, by using assumption (i), we obtain

$$\phi(\|u'\|_\infty) \leqslant \phi(d) + \phi(\|u\|_\infty)\|d_1\|_{L^1(a,b)} + \phi(\|u'\|_\infty)\|d_2\|_{L^1(a,b)} + \|r\|_{L^1(a,b)}.$$

Using now the fact that $u(a) = 0$ implies $\|u\|_\infty \leqslant (b-a)\|u'\|_\infty$, the definition of $\alpha(a,b)$, and (3.20) we see that either $\|u'\|_\infty \leqslant z_0$ or

$$\begin{aligned} \phi(\|u'\|_\infty) &\leqslant \phi(d) + ((\alpha(a,b)+\varepsilon)\|d_1\|_{L^1(a,b)} + \|d_2\|_{L^1(a,b)})\phi(\|u'\|_\infty) + \|r\|_{L^1(a,b)} \\ &\leqslant \phi(d) + \gamma_\varepsilon \phi(\|u'\|_\infty) + + \|r\|_{L^1(a,b)}. \end{aligned}$$

This implies that there exists an $R_0 > 0$ such that

$$\|u\|_{C^1[a,b]} < R_0.$$

Thus for any $\lambda \in (0, 1)$ problem (3.22) does not have solutions in the boundary of the ball $B(0, R_0) \subset C^1[a, b]$, implying that condition (i) of Lemma 2.1 is satisfied. Finally, we see that assumption (iii) of this theorem implies that conditions (ii) and (iii) of Lemma 2.1 are satisfied, and we conclude that problem (3.17) has at least one solution u such that $\|u\|_{C^1[a,b]} \leqslant R_0$. This completes the proof of the theorem. □

Let next $q \in L^1(a, b)$ and let us define

$$\overline{q} = \max\left\{\frac{1}{b-\eta}\int_\eta^b q(s)\,\mathrm{d}s \;\middle|\; \eta \in [a, \xi_{m-2}]\right\}$$

and

$$\underline{q} = \min\left\{\frac{1}{b-\eta}\int_\eta^b q(s)\,\mathrm{d}s \;\middle|\; \eta \in [a, \xi_{m-2}]\right\}.$$

Our, next, existence theorem concern the multi-point boundary value problem:

$$\begin{aligned} &(\phi(u'))' + f(t, u, u') = q(t), \quad t \in (a, b),\\ &u(a) = 0, \quad \theta(u'(b)) = \sum_{i=1}^{m-2} \alpha_i \theta(u'(\xi_i)). \end{aligned} \tag{3.23}$$

Theorem 3.4. *Let $q \in L^1(a, b)$ and $f : [a, b] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ in the boundary value problem* (3.23) *be a continuous function satisfying the following conditions*:

(i) *there exist non-negative functions d_1, d_2, and r in $L^1(a, b)$ such that*

$$|f(t, u, v)| \leqslant d_1(t)\phi(|u|) + d_2(t)\phi(|v|) + r(t)$$

for all a.e. $t \in [a, b]$ and all $u, v \in \mathbf{R}$,

(ii) *there exists $d > 0$ such that*

$$\begin{aligned} &f(t, u, v) > \overline{q} \text{ for } v \geqslant d,\\ &f(t, u, v) < \underline{q} \text{ for } v \leqslant -d \end{aligned}$$

for $t \in [a, b]$ and all $u \in \mathbf{R}$.

Suppose, further, that ϕ satisfies (3.1), *and*

$$\alpha(a, b)\left(\|d_1\|_{L^1(a,b)} + \frac{\Lambda}{A}\right) + \|d_2\|_{L^1(a,b)} < 1. \tag{3.24}$$

If the coefficients α_i, $i = 1, \ldots, m-2$, in the boundary conditions of problem (1.1) *are non-negative and satisfy $\sum_{i=1}^{m-2} \alpha_i = 1$, then the boundary value problem* (3.23) *has at least one solution $u \in C^1[a, b]$.*

Proof. We shall show that the family of multi-point boundary value problems

$$
\begin{aligned}
&(\phi(u'))' = \lambda(q(t) - f(t,u,u')), \quad t \in (a,b), \ \ \lambda \in [0,1],\\
&u(a) = 0, \quad \mathfrak{B}(u,\lambda) = 0.
\end{aligned}
\tag{3.25}
$$

satisfies the conditions (i), (ii), and (iii) of Lemma 2.1.

As in the proof of Theorem 3.1, let $\varepsilon > 0$ be such that

$$
\gamma_\varepsilon = (\alpha(a,b) + \varepsilon)\left(\|d_1\|_{L^1(a,b)} + \frac{\Lambda}{A}\right) + \|d_2\|_{L^1(a,b)} < 1,
\tag{3.26}
$$

and let $z_0 > 0$ be such that

$$
\phi((b-a)z) \leqslant (\alpha(a,b) + \varepsilon)\phi(z) \quad \text{for } z \geqslant z_0.
\tag{3.27}
$$

Let, now, $u(t)$ be a solution of (3.25) for some $\lambda \in (0,1)$. We claim that there exists a $\widetilde{t} \in [a,b]$ such that

$$
-d \leqslant u'(\widetilde{t}) \leqslant d.
$$

Indeed, we see by integrating the equation in (3.25) on the interval $[a,t]$ and using the boundary conditions in (3.25) that

$$
\begin{aligned}
&\lambda\Bigg[\theta\left[\phi(u'(a)) + \lambda\int_a^b (q(\tau) - f(\tau,u(\tau),u'(\tau)))\,\mathrm{d}\tau\right]\\
&\quad - \sum_{i=1}^{m-2}\alpha_i\theta\left[\phi(u'(a)) + \lambda\int_a^{\xi_i}(q(\tau) - f(\tau,u(\tau),u'(\tau)))\,\mathrm{d}\tau\right]\Bigg]\\
&\quad + (1-\lambda)\Bigg(\int_a^b (q(\tau) - f(\tau,u(\tau),u'(\tau)))\,\mathrm{d}\tau\\
&\quad - \sum_{i=1}^{m-2}\alpha_i\int_a^{\xi_i}(q(\tau) - f(\tau,u(\tau),u'(\tau)))\,\mathrm{d}\tau\Bigg) = 0.
\end{aligned}
\tag{3.28}
$$

Next, using that $\sum_{i=1}^{m-2}\alpha_i = 1$, with $\alpha_i \geqslant 0$, $i = 1,\ldots,m-2$, we obtain as in the proof of Theorem 3.1, that there exist $\eta_0, \eta_1 \in [a, \xi_{m-2}]$ such that

$$
\begin{aligned}
&\sum_{i=1}^{m-2}\alpha_i\theta\left[\phi(u'(a)) + \lambda\int_a^{\xi_i}(q(\tau) - f(\tau,u(\tau),u'(\tau)))\,\mathrm{d}\tau\right]\\
&\quad = \theta\left[\phi(u'(a)) + \lambda\int_a^{\eta_0}(q(\tau) - f(\tau,u(\tau),u'(\tau)))\,\mathrm{d}\tau\right],\\
&\sum_{i=1}^{m-2}\alpha_i\int_a^{\xi_i}(q(\tau) - f(\tau,u(\tau),u'(\tau)))\,\mathrm{d}\tau = \int_a^{\eta_1}(q(\tau) - f(\tau,u(\tau),u'(\tau)))\,\mathrm{d}\tau).
\end{aligned}
\tag{3.29}
$$

Hence, if we assume that $u'(t) > d$ for all $t \in [a, b]$, we arrive at the contradiction $0 < 0$, by using the first part of the assumption (ii) of the theorem, the fact that θ is an increasing homeomorphism and Eqs. (3.28) and (3.29).

Similarly, the assumption $u'(t) < -d$ for all $t \in [a, b]$ leads to the contradiction $0 > 0$. This proves the claim that there exists a $\widetilde{t} \in [a, b]$ such that

$$-d \leqslant u'(\widetilde{t}) \leqslant d.$$

Since assumption (i) of the theorem implies that

$$|q(t) - f(t, u, v)| \leqslant d_1(t)\phi(|u|) + d_2(t)\phi(|v|) + \widetilde{r}(t), \tag{3.30}$$

where $\widetilde{r}(t) = r(t) + q(t)$, by integrating the equation in (3.25) from $\widetilde{t}$ to $t \in [a, b]$ and using estimate (3.30), we obtain

$$\begin{aligned}\phi(\|u'\|_\infty) &\leqslant \phi(d) + \phi(\|u\|_\infty)\|d_1\|_{L^1(a,b)} + \phi(\|u'\|_\infty)\|d_2\|_{L^1(a,b)} + \|\widetilde{r}\|_{L^1(a,b)} \\ &\leqslant \phi(d) + \phi((b-a)\|u'\|_\infty)\|d_1\|_{L^1(a,b)} \\ &\quad + \phi(\|u'\|_\infty)\|d_2\|_{L^1(a,b)} + \|\widetilde{r}\|_{L^1(a,b)}.\end{aligned} \tag{3.31}$$

Hence either $\|u'\|_\infty \leqslant z_0$ or from (3.27) we obtain

$$\begin{aligned}\phi(\|u'\|_\infty) &\leqslant \phi(d) + ((\alpha(a, b) + \varepsilon)\|d_1\|_{L^1(a,b)} + \|d_2\|_{L^1(a,b)})\phi(\|u'\|_\infty) \\ &\quad + \|\widetilde{r}\|_{L^1(a,b)} \leqslant \phi(d) + \gamma_\varepsilon\|u'\|_\infty + \|\widetilde{r}\|_{L^1(a,b)}.\end{aligned} \tag{3.32}$$

Combining this inequality with the fact that $u(a) = 0$ implies the estimate $\|u\|_\infty \leqslant (b-a)\|u'\|_\infty$, and (3.32), we obtain that there exists an $R_0 > d$, d is as in (ii), such that for all $\widetilde{R} > R_0$ the family of boundary value problems (3.25) has no solution on $\partial B(0, \widetilde{R})$ for all $0 < \lambda < 1$, where $B(0, \widetilde{R})$ is the ball with center 0 and radius $\widetilde{R}$ in $C^1[a, b]$. We have thus proved that the family of boundary value problems (3.25) satisfies condition (i) of Lemma 2.1.

Next, using again that $\alpha_i \geqslant 0$, for every $i = 1, \ldots, m-2$ and $\sum_{i=1}^{m-2}\alpha_i = 1$, we find that for every $\rho \in \mathbf{R}$, there exists an $\eta_\rho \in [a, \xi_{m-2}]$ such that

$$\begin{aligned}&\rho\left\{\int_a^b (f(\tau, \rho(\tau-a), \rho) - q(\tau))\,\mathrm{d}\tau - \sum_{i=1}^{m-2}\alpha_i\int_a^{\xi_i}(f(\tau, \rho(\tau-a), \rho) - q(\tau))\,\mathrm{d}\tau\right\} \\ &\quad = \rho\left\{\int_a^b (f(\tau, \rho(\tau-a), \rho) - q(\tau))\,\mathrm{d}\tau - \int_a^{\eta_\rho}(f(\tau, \rho(\tau-a), \rho) - q(\tau))\,\mathrm{d}\tau\right\} \\ &\quad = \rho\left\{\int_{\eta_\rho}^b (f(\tau, \rho(\tau-a), \rho) - q(\tau))\,\mathrm{d}\tau\right\}.\end{aligned} \tag{3.33}$$

Assumption (ii) implies that for all $\rho \in \mathbf{R}$, with $|\rho| > d$,

$$\rho\int_{\eta_\rho}^b (f(\tau, \rho(\tau-a), \rho) - q(\tau))\,\mathrm{d}\tau > 0.$$

Hence, from (3.33), we obtain that

$$\rho\left\{\int_a^b (f(\tau,\rho(\tau-a),\rho)-q(\tau))\,\mathrm{d}\tau-\sum_{i=1}^{m-2}\alpha_i\int_a^{\xi_i}(f(\tau,\rho(\tau-a),\rho)-q(\tau))\,\mathrm{d}\tau\right\} > 0. \tag{3.34}$$

Finally, the validity of conditions (ii) and (iii) of Lemma 2.1 can now be obtained from (3.34) by an argument totally similar to the one used in proof of Theorem 3.1. This completes the proof of the theorem. □

Remark 3.5. Theorem 3.4 continues to hold if we replace assumption (ii) by the following: "there exists $d > 0$ such that

$$f(t,u,v) < \underline{q} \quad \text{for } u \geqslant d,$$
$$f(t,u,v) > \overline{q} \quad \text{for } u \leqslant -d$$

for a.e. $t \in [a,b]$ and all $v \in \mathbf{R}$."

In our next existence theorem we are able to relax sublinear assumptions of the type used in (i) of Theorem 3.1. Indeed we will allow superlinear behavior of the function f of problem (1.1) with respect to the variable v. Furthermore we will not need condition (3.1) on ϕ.

Theorem 3.6. *Let* $f : [a,b]\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ *in the boundary value problem* (1.1) *be a continuous function which satisfies the following condition*:
there exists $M > 0$ *such that for all* $|v| > M$, *all* $u \in \mathbf{R}$, *and all* $t \in [a,b]$, one has

$$vf(t,u,v) > 0,$$

If furthermore the coefficients α_i, $i = 1,\dots,m-2$, *in the boundary conditions of problem* (1.1) *are non-negative and satisfy* $\sum_{i=1}^{m-2}\alpha_i = 1$, *then the boundary value problem* (1.1) *has at least one solution* $u \in C^1[a,b]$.

Proof. We consider the family of boundary value problems (2.2) with $f^*(t,u,v,\lambda) = f(t,u,v)$ for all $(t,u,v,\lambda) \in [a,b]\times\mathbf{R}\times\mathbf{R}\times[0,1]$, i.e. we consider the family of problems:

$$(\phi(u'))' = \lambda f(t,u,u'), \quad t \in (a,b), \quad \lambda \in [0,1],$$
$$u(a) = 0, \quad \mathfrak{B}(u,\lambda) = 0. \tag{3.35}$$

We shall show that the family of problems (3.35) satisfies the conditions of Lemma 2.1 to conclude that problem (1.1) has at least one solution in $C^1[a,b]$.

Let u be a solution to problem (3.35) for some $\lambda \in (0, 1)$. We note first that the boundary condition

$$\mathfrak{B}(u, \lambda) = \lambda \left(\theta(u'(b)) - \sum_{i=1}^{m-2} \alpha_i \theta(u'(\xi_i)) \right) + (1-\lambda) \left(\int_a^b f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau - \sum_{i=1}^{m-2} \alpha_i \int_a^{\xi_i} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau \right) = 0,$$

can be written as

$$\lambda^2 \left(\theta(u'(b)) - \sum_{i=1}^{m-2} \alpha_i \theta(u'(\xi_i)) \right) + (1-\lambda) \left(\phi(u'(b)) - \sum_{i=1}^{m-2} \alpha_i \phi(u'(\xi_i)) \right) = 0 \tag{3.36}$$

and hence

$$\left(\theta(u'(b)) - \sum_{i=1}^{m-2} \alpha_i \theta(u'(\xi_i)) \right) \left(\phi(u'(b)) - \sum_{i=1}^{m-2} \alpha_i \phi(u'(\xi_i)) \right) \leqslant 0. \tag{3.37}$$

Then letting $u'(\xi_j) = \max_{i=1,\ldots,m-2} u'(\xi_i)$ and $u'(\xi_k) = \min_{i=1,\ldots,m-2} u'(\xi_i)$, we have that (3.37) implies that

$$u'(\xi_k) \leqslant u'(b) \leqslant u'(\xi_j). \tag{3.38}$$

We claim first that $|u'(a)| \leqslant M$, where M is as in the hypotheses of the theorem. Indeed, let us suppose that $u'(a) > M$. It, then, follows that there exists an $\varepsilon > 0$ such that $u'(t) > M$ for $t \in [0, \varepsilon]$, and hence that

$$(\phi(u'(t)))' = \lambda f(t, u(t), u'(t)) > 0 \quad \text{for } t \in [0, \varepsilon]. \tag{3.39}$$

This implies that $u'(t)$ is strictly increasing on $[0, \varepsilon]$. Let us define

$$t_M := \sup\{t \in [a, b] \mid u' \quad \text{is strictly increasing on } [a, t]\}.$$

Clearly $t_M \geqslant \varepsilon$ and $u'(t_M) > M$. If we assume $t_M < b$, then it is immediate to see that there is a $\varepsilon' > 0$ such that u' is strictly increasing on $[t_M, t_M + \varepsilon']$ and hence on $[a, t_M + \varepsilon']$ contradicting the definition of t_M. Thus $t_M = b$ and u' is strictly increasing on $[a, b]$. But this cannot be in light of the last inequality in (3.38) and it must be that $u'(a) \leqslant M$. Since an entire similar argument (using this time the first inequality in (3.38)) gives that $u'(a) \geqslant -M$, we obtain that $|u'(a)| \leqslant M$.

Next, we claim that $|u'(t)| \leqslant M$ for every $t \in [a, b]$. Indeed, let us now set

$$t_M := \max\{t \in [a, b] | \quad |u'(s)| \leqslant M \text{ for } s \in [a, t]\},$$

which implies that $t_M \geqslant a$. The proof of the claim consists then in showing that $t_M = b$.

Suppose that $t_M < b$, then $|u'(t_M)| = M$. If $u'(t_M) = M$, from the definition of t_M, it follows that $u'(t) > M$ for t near t_M and greater than t_M, which implies (as above) that u' is strictly increasing on $(t_M, b]$. In particular it holds that $M < u'(b)$ and thus from (3.38), we find that

$$M < u'(b) \leqslant u'(\xi_j).$$

But this implies that $\xi_j \in (t_M, b]$ and hence that u' must be strictly increasing on $[\xi_j, b]$, which contradicts the last inequality in (3.38). Similarly, if $u(t_M) = -M$ (using the other part of the inequality (3.38)) we obtain a contradiction. Thus, we must have $t_M = b$ and $|u'(t)| \leqslant M$ for every $t \in [a, b]$, i.e. $\|u'\|_\infty \leqslant M$.

Next, since $u(a) = 0$ implies $\|u\|_\infty \leqslant (b-a)\|u'\|_\infty$, we see that $\|u\|_\infty \leqslant (b-a)M$. Accordingly, if $R > (b-a+1)M$ then for $\lambda \in (0, 1)$ the family of boundary value problems (3.35) has no solution on the boundary of the ball $B(0, R) \subset C^1[a, b]$, and condition (i) of Lemma 2.1 is satisfied.

Finally, since the assumed condition of the theorem implies

$$\rho f(t, \rho(t-a), \rho) > 0 \quad \text{for all } |\rho| > M \text{ and all } t \in [a, b],$$

we have that conditions (ii) and (iii) of Lemma 2.1 can be shown to hold by an argument totally similar to the one used in proof of Theorem 3.1. This completes the proof of the theorem. □

The following example is an immediate and simple corollary to this theorem.

Example 3.7. Let $p > 1$, $q > 1$, and for $i = 1, \ldots, m-2$, let $\alpha_i \geqslant 0$, $\xi_i \in (0, 1)$, be given numbers such that $\sum_{i=1}^{m-2} \alpha_i = 1$ and $0 < \xi_1 < \cdots < \xi_{m-2} < 1$.

Let us consider the problem

$$(\phi_p(u'))' = g(t, u')(\phi_p(|u|) + 1), \quad t \in (a, b)$$
$$u(a) = 0, \quad \phi_q(u'(b)) = \sum_{i=1}^{m-2} \alpha_i \phi_q(u'(\xi_i)). \tag{3.40}$$

where $g : \mathbf{R} \mapsto \mathbf{R}$ is a continuous function such that $vg(t, v) > 0$, for all $|v| \geqslant M > 0$ and for all $t \in [a, b]$, and where M is a constant. Then problem (3.40) has at least one solution $u \in C^1[a, b]$.

4. Some results for the non-resonance case

In this section we will consider problem (1.1) in the non-resonance case.

Problem (1.1) is in the non-resonance case if problem (1.2) has only the trivial solution. This holds if and only if the coefficients α_i satisfy $\sum_{i=1}^{m-2} \alpha_i \neq 1$. We assume henceforth the α_i's satisfy this condition and that the homeomorphism ϕ in problem (1.1) satisfies condition (3.1). Notice that we do not assume a sign condition on the α_i's. Also, as before, we set

$$\alpha(a,b) = \limsup_{z\to\infty} \frac{\phi((b-a)z)}{\phi(z)}. \tag{4.1}$$

In addition, we shall assume that for any $0<\sigma<1$ we have

$$\tilde{\alpha}(\sigma) = \limsup_{z\to\infty} \frac{(\phi\circ\theta^{-1})(\sigma z)}{(\phi\circ\theta^{-1})(z)} < 1. \tag{4.2}$$

Let us set

$$\sigma^* = \begin{cases} \min\left\{ \dfrac{\sum_{i=1}^{m-2}\alpha_i^+}{1+\sum_{i=1}^{m-2}\alpha_i^-}, \dfrac{1+\sum_{i=1}^{m-2}\alpha_i^-}{\sum_{i=1}^{m-2}\alpha_i^+} \right\} & \text{if } \sum_{i=1}^{m-2}\alpha_i^+ \neq 0 \\ 0 & \text{if } \sum_{i=1}^{m-2}\alpha_i^+ = 0, \end{cases} \tag{4.3}$$

where $\alpha^+ = \max(\alpha,0)$ and $\alpha^- = \max(-\alpha,0)$. Note that $0\leqslant\sigma^*<1$. The main result of this section is the following theorem.

Theorem 4.1. *Let $f : [a,b]\times\mathbf{R}\times\mathbf{R} \to \mathbf{R}$ be a function satisfying Carathéodory's conditions such that the following condition holds: there exist non-negative functions $d_1(t)$, $d_2(t)$, and $r(t)$ in $L^1(a,b)$ such that*

$$|f(t,u,v)| \leqslant d_1(t)\phi(|u|) + d_2(t)\phi(|v|) + r(t),$$

for a.e. $t\in[a,b]$ and all $u, v\in\mathbf{R}$. Suppose, further,

$$\alpha(a,b)\|d_1\|_{L^1(a,b)} + \|d_2\|_{L^1(a,b)} < 1 - \tilde{\alpha}(\sigma^*), \tag{4.4}$$

where α is as defined in (4.1), *σ^* is as defined in* (4.3) *and $\tilde{\alpha}$ is as defined in* (4.2).

Then, the boundary value problem (1.1) *has at least one solution $u\in C^1[a,b]$.*

We need an a priori estimate in the proof of Theorem 4.1 and present this in the following lemma.

Lemma 4.2. *Let $u\in C^1[a,b]$, be such that $\phi(u'(t))$ is absolutely continuous and satisfies*

$$\theta(u'(b)) = \sum_{i=1}^{m-2} \alpha_i\theta(u'(\xi_i)). \tag{4.5}$$

If $\sum_{i=1}^{m-2}\alpha_i^+ \neq 0$, then for every $\varepsilon>0$ with $\tilde{\alpha}(\sigma^*)+\varepsilon<1$, there is a constant $C_\varepsilon>0$ such that

$$\phi(\|u'\|_\infty) \leqslant \frac{1}{(1-(\tilde{\alpha}(\sigma^*)+\varepsilon))}\|(\phi(u'))'\|_{L^1(a,b)} + C_\varepsilon. \tag{4.6}$$

If now $\sum_{i=1}^{m-2}\alpha_i^+ = 0$, then $u'(\eta_0)=0$ for some $\eta_0\in[a,b]$, and

$$\phi(\|u'\|_\infty) \leqslant \|(\phi(u'))'\|_{L^1(a,b)}. \tag{4.7}$$

Proof. If it is the case that $u'(t)$ is constant for $t \in [a, b]$, then from (4.5) and the fact that $\sum_{i=1}^{m-2} \alpha_i \neq 1$, it is immediate that $u' \equiv 0$. Clearly in this case both (4.6) and (4.7) are satisfied.

Suppose next that $\sum_{i=1}^{m-2} \alpha_i^+ \neq 0$ which implies $\sigma^* \neq 0$. Then from (4.5) we see that

$$\theta(u'(b)) + \sum_{i=1}^{m-2} \alpha_i^- \theta(u'(\xi_i)) = \sum_{i=1}^{m-2} \alpha_i^+ \theta(u'(\xi_i)),$$

and thus from the definition of σ^* and the intermediate value property for continuous functions, we find that there exist η_1, η_2 in $[a, b]$ such that

$$\theta(u'(\eta_1)) = \sigma^* \theta(u'(\eta_2))$$

so that

$$u'(\eta_1) = \theta^{-1}(\sigma^* \theta(u'(\eta_2)))$$

and

$$\phi(u'(\eta_1)) = (\phi \circ \theta^{-1})(\sigma^* \theta(u'(\eta_2))).$$

We, next, use the equation

$$\begin{aligned}\phi(u'(t)) &= \phi(u'(\eta_1)) + \int_{\eta_1}^{t} (\phi(u'))'(s)\, \mathrm{d}s \\ &= (\phi \circ \theta^{-1})(\sigma^* \theta(u'(\eta_2))) + \int_{\eta_1}^{t} (\phi(u'))'(s)\, \mathrm{d}s,\end{aligned}$$

to get

$$\phi(\|u'\|_\infty) \leqslant (\phi \circ \theta^{-1})(\sigma^* \theta(\|u'\|_\infty)) + \|(\phi(u'))'\|_{L^1(a,b)}. \tag{4.8}$$

Now, for σ^* as given in (4.3), let $\varepsilon > 0$ be such that $\tilde{\alpha}(\sigma^*) + \varepsilon < 1$. It follows from the definition of $\tilde{\alpha}(\sigma^*)$ that there exists a constant $\tilde{C}_\varepsilon$ such that for $z \in \mathbf{R}$ we have

$$(\phi \circ \theta^{-1})(\sigma^* |z|) \leqslant (\tilde{\alpha}(\sigma^*) + \varepsilon)(\phi \circ \theta^{-1})(|z|) + \tilde{C}_\varepsilon.$$

We thus get from (4.8) that

$$\phi(\|u'\|_\infty) \leqslant (\tilde{\alpha}(\sigma^*) + \varepsilon)(\phi \circ \theta^{-1})(\theta(\|u'\|_\infty)) + \|(\phi(u'))'\|_{L^1(a,b)} + \tilde{C}_\varepsilon.$$

Hence, we obtain the estimate

$$\phi(\|u'\|_\infty) \leqslant \frac{1}{(1 - (\tilde{\alpha}(\sigma^*) + \varepsilon))} \|(\phi(u'))'\|_{L^1(a,b)} + C_\varepsilon,$$

where we have set $\tilde{C}_\varepsilon / (1 - (\tilde{\alpha}(\sigma^*) + \varepsilon)) = C_\varepsilon$. If $\sum_{i=1}^{m-2} \alpha_i^+ = 0$, then from (4.5)

$$\theta(u'(b)) + \sum_{i=1}^{m-2} \alpha_i^- \theta(u'(\xi_i)) = 0,$$

which implies the existence of $\eta_0 \in [a, b]$ such that $u'(\eta_0) = 0$. Hence we now have

$$\phi(u'(t)) = \int_{\eta_0}^{t} (\phi(u'))'(s)\,\mathrm{d}s,$$

from which it is immediate to see that (4.7) holds. This completes the proof of the lemma. □

Proof of Theorem 4.1. We consider the family of boundary value problems:

$$(\phi(u'))' = \lambda f(t, u, u'), \quad t \in (a, b), \quad \lambda \in [0, 1],$$
$$u(a) = 0, \quad \theta(u'(b)) = \sum_{i=1}^{m-2} \alpha_i \theta(u'(\xi_i)). \tag{4.9}$$

Also, we define an operator $\Psi^* : C^1[a, b] \times [0, 1] \to C^1[a, b]$ by setting for $(u, \lambda) \in C^1[a, b] \times [0, 1]$

$$\Psi^*(u, \lambda)(t) = \int_a^t \phi^{-1}\left[\phi(u'(a)) + \lambda \int_a^s f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\right] \mathrm{d}s$$
$$+ (t - a)\left(\theta(u'(b)) - \sum_{i=1}^{m-2} \alpha_i \theta(u'(\xi_i))\right). \tag{4.10}$$

Following standard arguments, it can be proved that Ψ^* is a completely continuous operator. Furthermore reasoning in the same way as in the proof of Lemma 2.1 it can be proved that u is a solution to the family of boundary value problems (4.9) if and only if u is a fixed point for the operator $\Psi^*(\cdot, \lambda)$, i.e., u satisfies

$$u = \Psi^*(u, \lambda). \tag{4.11}$$

We will show next that there is a constant $R > 0$ independent of $\lambda \in [0, 1]$ such that if u satisfies (4.11) for some $\lambda \in [0, 1]$ then $\|u\|_{C^1[a,b]} < R$. We note first that if u satisfies

$$u = \Psi^*(u, 0), \tag{4.12}$$

then it must be that $u = 0$. Indeed from the definition of Ψ^* or from problem (4.9), it follows that $u(t) = \rho(t - a)$ with $\rho = u'(a) = u'(t)$, for all $t \in [a, b]$. Then from the second boundary condition in (4.9), and taking into account that $\sum_{i=1}^{m-2} \alpha_i \neq 1$, we find that $\rho = 0$, implying that $u(t) = 0$ for all $t \in [a, b]$.

In the rest of the argument we will assume that $\lambda \in (0, 1]$. Also we will suppose that $\sum_{i=1}^{m-2} \alpha_i^+ \neq 0$ and hence $0 < \sigma^*$ since the proof for the case $\sigma^* = 0$ is simpler.

Let us choose $\varepsilon > 0$ such that $\tilde{\alpha}(\sigma^*) + \varepsilon < 1$ and

$$(\alpha(a, b) + \varepsilon)\|d_1\|_{L^1(a,b)} + \|d_2\|_{L^1(a,b)} < 1 - (\tilde{\alpha}(\sigma^*) + \varepsilon), \tag{4.13}$$

which can be done in view of the assumption (4.4). It then follows from the definition of $\alpha(a, b)$ that there exist a positive constant C_ε^1 such that for all $z \in \mathbf{R}$, we have

$$(\phi((b - a)|z|) \leqslant (\alpha(a, b) + \varepsilon)(\phi(|z|) + C_\varepsilon^1. \tag{4.14}$$

Let u be a solution of boundary value problems (4.9) for some $\lambda \in [0, 1]$. Then $u \in C^1[a, b]$ with $\phi(u'(t))$ absolutely continuous on $[a, b]$ and satisfies

$$\theta(u'(b)) = \sum_{i=1}^{m-2} \alpha_i \theta(u'(\xi_i)).$$

Hence from Lemma 4.2, we have the estimate

$$\phi(\|u'\|_\infty) \leqslant \frac{1}{(1 - (\tilde{\alpha}(\sigma^*) + \varepsilon))} \|(\phi(u'))'\|_{L^1(a,b)} + C_\varepsilon^2, \tag{4.15}$$

where C_ε^2 is a positive constant. Now from our assumptions on the function *f*, the definitions of $\alpha(a, b)$, $\tilde{\alpha}(\sigma^*)$, the choice of ε, C_ε^1, C_ε^2 and (4.15), we find that

$$\begin{aligned}
\|(\phi(u'))'\|_{L^1(a,b)}) &\leqslant \phi(\|u\|_\infty)\|d_1\|_{L^1(a,b)} + \phi(\|u'\|_\infty)\|d_2\|_{L^1(a,b)} + \|r\|_{L^1(a,b)} \\
&\leqslant \phi((b-a)\|u'\|_\infty)\|d_1\|_{L^1(a,b)} + \phi(\|u'\|_\infty)\|d_2\|_{L^1(a,b)} \\
&\quad + \|r\|_{L^1(a,b)} \\
&\leqslant [(\alpha(a, b) + \varepsilon)\|d_1\|_{L^1(a,b)} + \|d_2\|_{L^1(a,b)}]\phi(\|u'\|_\infty) \\
&\quad + \|r\|_{L^1(a,b)} + C_\varepsilon^1\|d_1\|_{L^1(a,b)} \\
&\leqslant \frac{(\alpha(a, b) + \varepsilon)\|d_1\|_{L^1(a,b)} + \|d_2\|_{L^1(a,b)}}{(1 - (\tilde{\alpha}(\sigma^*) + \varepsilon))} \\
&\quad \times \|(\phi(u'))'\|_{L^1(a,b)} + C_\varepsilon,
\end{aligned} \tag{4.16}$$

where

$$C_\varepsilon = \|r\|_{L^1(a,b)} + C_\varepsilon^1\|d_1\|_{L^1(a,b)} + C_\varepsilon^2[(\alpha(a, b) + \varepsilon)\|d_1\|_{L^1(a,b)} + \|d_2\|_{L^1(a,b)}].$$

It, now, follows from (4.13) that there exists a constant $R_0 > 0$, independent of $\lambda \in (0, 1]$ such that if u is a solution of the family of boundary value problems (4.10) then

$$\|(\phi(u'))'\|_{L^1(a,b)}) \leqslant R_0.$$

This, combined with (4.15), and the fact that $u(a) = 0$ implies that $\|u\|_\infty \leqslant (b-a)\|u'\|_\infty$, yield that there exist a constant $R > 0$ such that

$$\|u\|_{C^1[a,b]} < R.$$

This in turn implies that $\deg_{\mathrm{LS}}(I - \Psi^*(\cdot, \lambda), B(0, R), 0)$ is well defined for all $\lambda \in [0, 1]$, where $B(0, R)$ is the ball with center 0 and radius R in $C^1[a, b]$.

In what follows we will use the notation of Section 2. Thus X will denote the one-dimensional subspace of $C^1[a, b]$ given by $X = \{i_\rho \mid \rho \in \mathbf{R}\}$, $i_\rho(t) = (t-a)\rho$ and $i : \mathbf{R} \mapsto X$ is the isomorphism from $\mathbf{R}$ onto X given by $i(\rho) = i_\rho$. Let us define the function $G : \mathbf{R} \mapsto \mathbf{R}$ by

$$G(\rho) = \left(\sum_{i=1}^{m-2} \alpha_i - 1\right)\theta(\rho), \tag{4.17}$$

and note that for $v \in X$, $v(t) = \rho(t-a)$ for some $\rho \in \mathbf{R}$. Now, since

$$(I - \Psi^*(\cdot, 0))(v) = i_{G(\rho)},$$

it is easy to see that

$$G = i^{-1} \circ (I - \Psi^*(\cdot, 0))|_X \circ i,$$

and hence, by the homotopy invariance property of Leray–Schauder degree, it follows that

$$\begin{aligned} &\deg_{\mathrm{LS}}(I - \Psi^*(\cdot, 1), B(0, R), 0) = \deg_{\mathrm{LS}}(I - \Psi^*(\cdot, 0), B(0, R), 0) \\ &\deg_{\mathrm{B}}(I - \Psi^*(\cdot, 0)|_X, X \cap B(0, R), 0) = \deg_{\mathrm{B}}(G, (-R, R), 0). \end{aligned}$$

Thus taking into account (4.17), we obtain the interesting formulas for the degree:

$$\deg_{\mathrm{LS}}(I - \Psi^*(\cdot, 1), B(0, R), 0) = \begin{cases} 1 & \text{if } \sum_{i=1}^{m-2} \alpha_i > 1, \\ -1 & \text{if } \sum_{i=1}^{m-2} \alpha_i < 1. \end{cases}$$

Hence if $\sum_{i=1}^{m-2} \alpha_i \neq 1$ we have that $\deg_{\mathrm{LS}}(I - \Psi^*(\cdot, 1), B(0, R), 0) \neq 0$ and there is a $u \in B(0, R)$ that satisfies

$$u = \Psi^*(\cdot, 1),$$

equivalently u is a solution to the boundary value problem (4.1). This completes the proof of the theorem. □

We have the following simple application of this theorem.

Example 4.3. Let $p > 1$, $q > 1$ and $\xi_i \in (0, 1)$, $i = 1, \dots, m-2$, be given numbers such that and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. Let $\psi : \mathbf{R} \mapsto \mathbf{R}$ be given by

$$\psi(s) = |s|^{p-2} s \log(1 + |s|).$$

Then ψ is an odd increasing homeomorphism of $\mathbf{R}$ onto $\mathbf{R}$, with $\psi(0) = 0$. If

$$2A + \tfrac{3}{2} B + (\sigma^*)^{(p-1)/(q-1)} < 1,$$

then the boundary value problem

$$\begin{aligned} &(\psi(u'))' = At^{-1/2}\psi(u) + Bt^{-1/3}\psi(u') + r(t), \quad t \in (0, 1) \\ &u(0) = 0, \quad \phi_q(u'(1)) = \sum_{i=1}^{m-2} \alpha_i \phi_q(u'(\xi_i)), \end{aligned} \tag{4.18}$$

where $\sum_{i=1}^{m-2} \alpha_i \neq 1$, has a solution $u \in C^1[0, 1]$. Recall that $\phi_q(s) = |s|^{q-2} s$.

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