Bloch approximation in homogenization on bounded domains

C. Conca*, R. Orive†, and M. Vanninathan‡

Abstract

The classical problem of homogenization deals with elliptic operators in periodically oscillating media of small period $\varepsilon > 0$ and the asymptotic behavior of solution u^ε of boundary value problems associated with such operators. In a previous work [5], the authors introduced what is called *Bloch approximation* which provided energy norm approximation for the solution in $\mathbb{R}^N$. This paper continues with the above development and proposes a *modified Bloch approximation*. This function takes into account boundary effects. Its connection with the first order classical correctors is also established with the corresponding error estimate. All the proofs are worked out entirely in the Fourier-Bloch space.

Key words: homogenization, Bloch waves, correctors.

1 Introduction

This paper considers the behavior of solutions of elliptic boundary value problems when the coefficients are periodic with small period $\varepsilon > 0$. In particular, we take the effects of the boundary into account.

The fundamental work of A. Bensoussan, J.L. Lions and G. Papanicolaou [1] presents homogenization results using an approach based on physical space analysis. Now, in this work we are going to take a Fourier point of view and propose accordingly a new way of obtaining the classical correctors in homogenization.

In this direction, we introduced in [5] what we called *Bloch approximation* of the solution of u^ε in the case of the whole space (i.e. without boundaries). It was shown that Bloch approximation contains the classical first and second correctors introduced in [1]. Roughly speaking, this incorporates multiple scale structure of the solution and provides an approximation in the energy norm. The Bloch approximation is given by an elegant oscillatory integral involving the first Bloch wave.

However, the results in a bounded domain do not automatically follow from those in the entire space. Further, the method of obtaining such results in the physical space offers no clue to the construction in the Bloch space. Therefore, we propose a *modified Bloch approximation* such that

*Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, and Centro de Modelamiento Matemático, UMR 2071 CNRS-UChile, Casilla 170/3 - Correo 3, Santiago (Chile).

†Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid (España).

‡IISc-TIFR Mathematics Programme, TIFR Center, P.O. Box 1234, Bangalore - 560012 (India).

the proof of the classical first corrector remains entirely within Bloch space. In this case of a smooth homogenized solution, it is proved that this first corrector term satisfies the classical rate of convergence $\varepsilon^{\frac{1}{2}}$. In its turn, this provides an error estimate of the same order for our modified Bloch approximation. These two issues were further substantiated by the numerical experiments performed by C. Conca, S. Natesan and M. Vanninathan in [4] and [3].

Let us now introduce the problem to be studied in this work. We consider the operator

$$A \stackrel{\text{def}}{=} -\frac{\partial}{\partial y_k}\left(a_{k\ell}(y)\frac{\partial}{\partial y_\ell}\right), \quad y \in \mathbb{R}^N \tag{1.1}$$

where the coefficients satisfy

$$\begin{cases} a_{k\ell} \in L^\infty_\#(Y) \quad \text{where} \quad Y =]0, 2\pi[^N, \text{ i.e., each } a_{k\ell} \text{ is a} \\ Y\text{-periodic bounded measurable function defined on } \mathbb{R}^N, \text{ and} \\ \exists \alpha > 0 \quad \text{such that} \quad a_{k\ell}(y)\eta_k\eta_\ell \geq \alpha|\eta|^2 \quad \forall \eta \in \mathbb{R}^N,\ y \in Y \text{ a.e.,} \\ a_{k\ell} = a_{\ell k} \quad \forall k, \ell = 1, \dots, N. \end{cases} \tag{1.2}$$

For each $\varepsilon > 0$, we consider also the operator A^ε where

$$A^\varepsilon \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_k}\left(a^\varepsilon_{k\ell}(x)\frac{\partial}{\partial x_\ell}\right) \quad \text{with} \quad a^\varepsilon_{k\ell}(x) = a_{k\ell}(\frac{x}{\varepsilon}) \quad x \in \mathbb{R}^N. \tag{1.3}$$

In homogenization theory, it is usual to refer to x and y the slow and the fast variables respectively. They are related by $y = \frac{x}{\varepsilon}$. Associated with A^ε, let us consider the following boundary-value problem

$$A^\varepsilon u^\varepsilon = f \quad \text{in} \quad \Omega, \quad u^\varepsilon \in H^1_0(\Omega), \tag{1.4}$$

which is posed in an arbitrary bounded domain Ω in $\mathbb{R}^N$ and f is a given element in $L^2(\Omega)$. It is classical that the above problem admits a unique solution.

From the classical work [1], it is known that one can associate to A^ε a homogenized operator A^* given by

$$A^* \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_k}\left(q_{k\ell}\frac{\partial}{\partial x_\ell}\right). \tag{1.5}$$

The homogenized coefficients $q_{k\ell}$ are constants and their definition is

$$q_{k\ell} = \frac{1}{|Y|}\int_Y (a_{k\ell} + a_{km}\frac{\partial \chi_\ell}{\partial y_m})dy \stackrel{\text{def}}{=} \mathcal{M}_Y\left(a_{k\ell} + a_{km}\frac{\partial \chi_\ell}{\partial y_m}\right), \tag{1.6}$$

where, for any $k = 1, \dots, N$, χ_k is the unique solution of the cell problem

$$\begin{cases} A\chi_k = \dfrac{\partial a_{k\ell}}{\partial y_\ell} \quad \text{in} \quad Y, \\ \\ \chi_k \in H^1_\#(Y), \quad \mathcal{M}_Y(\chi_k) = 0. \end{cases} \tag{1.7}$$

The theory of homogenization gives the following result: the entire sequence of solutions u^ε of (1.4) converges weakly in $H^1_0(\Omega)$ to the so-called homogenized solution u^* characterized by

$$A^* u^* = f \quad \text{in} \quad \Omega, \quad u^* \in H^1_0(\Omega). \tag{1.8}$$

The problem of the first order correctors is to obtain functions $\widetilde{u}_1^\varepsilon \in H_0^1(\Omega)$ which are easy to construct and at the same time have the following characteristic property

$$\|u^\varepsilon - u^* - \varepsilon u_1^\varepsilon\|_{H_0^1(\Omega)} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{1.9}$$

To obtain such corrector, multiscale expansion method is followed in [1]. The authors consider an asymptotic expansion (with $y = \frac{x}{\varepsilon}$) of the form

$$u^\varepsilon(x) \;=\; u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \cdots , \tag{1.10}$$

where the functions u_i are Y-periodic in the variable y. In particular, they get

$$\begin{aligned} &u_0(x,y) = u^*(x), \\ &u_1(x,y) = \chi_k(y)\tfrac{\partial u^*}{\partial x_k}(x), \end{aligned}$$

where χ_k is the solution of (1.7). Thus, an obvious candidate for the first order corrector is $\varepsilon u_1(x, \frac{x}{\varepsilon})$; however it does not satisfy the boundary condition on $\partial\Omega$ and hence it is natural to introduce cut-off functions m^ε having the following properties

$$\begin{cases} m^\varepsilon \in \mathcal{D}(\Omega), \\ m^\varepsilon(x) = 0 \quad \text{if} \quad \operatorname{dist}(x,\Gamma) \leq \varepsilon, \\ m^\varepsilon(x) = 1 \quad \text{if} \quad \operatorname{dist}(x,\Gamma) \geq 2\varepsilon, \\ \varepsilon^{|\gamma|}|D_x^\gamma m^\varepsilon(x)| \leq c_\gamma \quad \forall \gamma \in \mathbb{Z}_+^N, \end{cases} \tag{1.11}$$

which exist, provided $\partial\Omega$ is smooth enough. Then, the first order correctors can be chosen as

$$u_1^\varepsilon(x) = m^\varepsilon(x)\chi_k(\frac{x}{\varepsilon})\frac{\partial u^*}{\partial x_k}(x). \tag{1.12}$$

Under the hypotheses that $\chi_k \in W^{1,\infty}(Y)$ and $u^* \in H^2(\Omega)$ it is proved in [1] that $\widetilde{u}_1^\varepsilon \in H_0^1(\Omega)$ and it is a first order corrector in the sense that (1.9) is satisfied.

In this work we are going to take a Fourier point of view and propose accordingly a new way of obtaining correctors. A fundamental tool in this process is the *modified Bloch approximation.* With such an approach we obtain in the dual space a first corrector of the solution u^ε and, using Bloch techniques, we prove the convergence of this first corrector to the solution u^ε in the energy norm. We observe further that the modified Bloch approximation has the advantage that spectral methods can be implemented to approximate problems exhibiting multiple scales.

Now, let us briefly summarize how this paper is organized. In Section 2, we give a brief exposition of previous results in homogenization of periodic structures via Bloch decomposition along with the Bloch approximation in the case of the whole space. Section 3 introduces the definition of the modified Bloch approximation taking into account boundary effects and presents its main properties and its connection with the first order correctors. In the fourth section, we establish technical lemmas useful in the sequel. In Section 5, we provide an asymptotic expansion of the modified Bloch approximation which is used in the next section to show that it implicitly contains first order correctors of u^ε. In Section 7 we establish the rate of convergence for the first corrector term. Finally, we conclude with a summary and a brief discussion about the efforts to adapt Bloch waves to the boundary for futures advances in homogenization.

2 Survey of previous results

The basic tools of our approach are the Bloch waves associated with the differential operator A introduced in (1.1). The Bloch waves are defined as follows:

Let us consider the following spectral problem parameterized by $\eta \in \mathbb{R}^N$: To find $\lambda = \lambda(\eta) \in \mathbb{R}$ and $\psi = \psi(y;\eta)$ (not identically zero) such that

$$(2.1) \qquad \left\{ \begin{array}{l} A\psi(\cdot;\eta) = \lambda(\eta)\psi(\cdot;\eta) \quad \text{in} \quad \mathbb{R}^N, \\ \quad \psi(\cdot;\eta) \text{ is } (\eta;Y)\text{-}\textit{periodic}, \text{ i.e.,} \\ \psi(y+2\pi m;\eta) = e^{2\pi i m\cdot\eta}\psi(y;\eta) \quad \forall m \in \mathbb{Z}^N,\ y \in \mathbb{R}^N. \end{array} \right.$$

Noting that the problem (2.1) is $\mathbb{Z}^N$-translation invariant with respect to η, we can restrict η to the *dual cell* $Y' = [-\frac{1}{2}, \frac{1}{2}[^N$. Seeking the solution $\psi(y;\eta)$ in the form $e^{iy\cdot\eta}\phi(y,\eta)$, ϕ being Y-periodic in the variable y, one can prove (see [6]) that the above spectral problem admits a discrete sequence of eigenvalues with the following properties:

$$\left\{ \begin{array}{l} 0 \le \lambda_1(\eta) \le \cdots \le \lambda_m(\eta) \le \cdots \to \infty, \\ \forall m \ge 1,\ \lambda_m(\eta) \text{ is a Lipschitz function of } \eta \in Y'. \end{array} \right.$$

The corresponding eigenfunctions denoted by $\psi_m(\cdot;\eta)$ and $\phi_m(\cdot;\eta)$ can be chosen to constitute orthonormal bases in the spaces of all $L^2_{loc}(\mathbb{R}^N)$-functions which are $(\eta;Y)$-periodic and Y-periodic respectively. The functions $\psi_m(\cdot;\eta)$ and $\phi_m(\cdot;\eta)$ (referred to as *Bloch waves*) introduced above enable us to describe the spectral resolution of A (an unbounded self-adjoint operator in $L^2(\mathbb{R}^N)$) in the orthogonal basis $\{e^{iy\cdot\eta}\phi_m(y;\eta) | m \ge 1, \eta \in Y'\}$. To obtain the spectral resolution of A^ε let us introduce Bloch waves at the ε-scale:

$$\lambda_m^\varepsilon(\xi) = \varepsilon^{-2}\lambda_m(\eta), \quad \phi_m^\varepsilon(x;\xi) = \phi_m(y;\eta), \quad \psi_m^\varepsilon(x;\xi) = \psi_m(y;\eta),$$

where the variables (x,ξ) and (y,η) are related by $y = \frac{x}{\varepsilon}$ and $\eta = \varepsilon\xi$. Observe that $\phi_m^\varepsilon(x;\xi)$ is εY-periodic (in x) and $\varepsilon^{-1}Y'$ periodic with respect to ξ. In the same manner, $\psi_m^\varepsilon(\cdot;\xi)$ is $(\varepsilon\xi;\varepsilon Y)$ periodic because of the relation $\psi_m^\varepsilon(x;\xi) = e^{ix\cdot\xi}\phi_m^\varepsilon(x;\xi)$. Note that the dual cell at ε-scale is $\varepsilon^{-1}Y'$ and hence we take ξ to vary in $\varepsilon^{-1}Y'$ in the sequel. With the above notations, we can state the fundamental result concerning the spectral resolution of A^ε.

Theorem 2.1 *Let $g \in L^2(\mathbb{R}^N)$. The m^{th} Bloch coefficient of g at the ε-scale is defined as follows:*

$$(B_m^\varepsilon g)(\xi) = \int_{\mathbb{R}^N} g(x) e^{-ix\cdot\xi} \bar{\phi}_m^\varepsilon(x;\xi)dx \quad \forall m \ge 1,\ \xi \in \varepsilon^{-1}Y'.$$

Then the following inverse formula holds:

$$g(x) = \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} (B_m^\varepsilon g)(\xi) e^{ix\cdot\xi} \phi_m^\varepsilon(x;\xi)d\xi.$$

Further, we have Parseval's identity:

$$\int_{\mathbb{R}^N} |g(x)|^2 dx = \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} |(B_m^\varepsilon g)(\xi)|^2 d\xi.$$

Finally, for all g in the domain of A^ε, we get

$$A^\varepsilon g(x) = \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} \lambda_m^\varepsilon(\xi)(B_m^\varepsilon g)(\xi)e^{ix\cdot\xi}\phi_m^\varepsilon(x;\xi)d\xi. \qquad \blacksquare$$

Using the above theorem, the classical homogenization result was deduced in [7]. To this end, the following results were established and applied.

Proposition 2.2 *We assume that $a_{k\ell}$ satisfy (1.2). Then there exists $\delta > 0$ such that the first eigenvalue $\lambda_1(\eta)$ is an analytic function on $B_\delta = \{\eta \mid |\eta| < \delta\}$, and there is a choice of the first eigenvector $\phi_1(y;\eta)$ satisfying*

$$\begin{cases} \eta \to \phi_1(\cdot;\eta) \in H^1_\#(Y) \text{ is analytic on } B_\delta, \\ \phi_1(y;0) = |Y|^{-1/2}. \end{cases}$$

Moreover, we have the relations

$$\lambda_1(0) = 0, \quad D_k\lambda_1(0) \overset{\text{def}}{=} \frac{\partial\lambda_1}{\partial\eta_k}(0) = 0 \quad \forall k = 1, ..., N,$$
$$\frac{1}{2}D^2_{k\ell}\lambda_1(0) \overset{\text{def}}{=} \frac{1}{2}\frac{\partial^2\lambda_1}{\partial\eta_k\partial\eta_\ell}(0) = q_{k\ell} \quad \forall k, \ell = 1, ..., N,$$

and there exist constants c and $\widetilde{c}$ such that

$$c|\eta|^2 \le \lambda_1(\eta) \le \widetilde{c}|\eta|^2 \quad \forall \eta \in Y', \tag{2.2}$$
$$0 < \lambda_2^{(N)} \le \lambda_m(\eta) \quad \forall m \ge 2,\ \eta \in Y', \tag{2.3}$$

where $\lambda_2^{(N)}$ is the second eigenvalue of the spectral problem for A in the cell Y with Neumann boundary conditions on ∂Y. $\blacksquare$

Let us recall the main steps of the homogenization result deduced in [7] in the case of the whole space $\mathbb{R}^N$. The first one consists of considering a sequence $u^\varepsilon \in H^1(\mathbb{R}^N)$ satisfying

$$\begin{cases} A^\varepsilon u^\varepsilon = f & \text{in } \mathbb{R}^N, \\ u^\varepsilon \rightharpoonup u^* & \text{in } H^1(\mathbb{R}^N)\text{-weak}, \\ u^\varepsilon \to u^* & \text{in } L^2(\mathbb{R}^N)\text{-strong}. \end{cases} \tag{2.4}$$

We can express the equation $A^\varepsilon u^\varepsilon = f$ in $\mathbb{R}^N$ in the equivalent form

$$\lambda_m^\varepsilon(\xi)(B_m^\varepsilon u^\varepsilon)(\xi) = (B_m^\varepsilon f)(\xi) \quad \forall m \ge 1,\ \xi \in \varepsilon^{-1}Y'. \tag{2.5}$$

In the homogenization process, one can neglect all the relations for $m \ge 2$ (see [7] and [9]). In fact, we have the following proposition taken from [5].

Proposition 2.3 *For all $v \in H^1(\mathbb{R}^N)$, we have*

$$\int_{\varepsilon^{-1}Y'} \sum_{m=2}^{\infty} |B_m^\varepsilon v(\xi)|^2 d\xi \le c\varepsilon^2 \|\nabla v\|^2_{L^2(\mathbb{R}^N)}. \qquad \blacksquare$$

Thus we can concentrate our attention only on the relation corresponding to the first Bloch wave:

$$\lambda_1^\varepsilon(\xi)(B_1^\varepsilon u^\varepsilon)(\xi) = (B_1^\varepsilon f)(\xi) \quad \forall \xi \in \varepsilon^{-1}Y'. \tag{2.6}$$

With the aim of passing to the limit in (2.6) as $\varepsilon \to 0$, it was proved in [7] that the first Bloch transform is an approximation to Fourier transform. This result is naturally expected from the above result of regularity on the Bloch spectrum and the fact that $\phi_1^\varepsilon(x;\xi) \to (2\pi)^{-N/2}$, as $\varepsilon \to 0$, $\forall \xi \in \mathbb{R}^N$. More precisely, we have the next result whose proof can be found in [5].

Proposition 2.4 *Let g^ε and g be in $L^2(\mathbb{R}^N)$. Then*

(i) If $g^\varepsilon \rightharpoonup g$ weakly in $L^2(\mathbb{R}_x^N)$, then $\chi_{\varepsilon^{-1}Y'} B_1^\varepsilon g^\varepsilon \rightharpoonup \widehat{g}$ weakly in $L^2_{loc}(\mathbb{R}_\xi^N)$ provided there is a fixed compact set K such that supp $(g^\varepsilon) \subset K$, $\forall \varepsilon$.

(ii) If $g \in L^2(\mathbb{R}^N)$, we have

$$\chi_{\varepsilon^{-1}Y'}(\xi) B_1^\varepsilon g(\xi) \to \widehat{g}(\xi) \quad \textit{in} \quad L^2(\mathbb{R}_\xi^N). \quad ■$$

Thus, the homogenized equation in the Fourier space

$$q_{k\ell}\xi_k\xi_\ell \widehat{u*}(\xi) = \widehat{f}(\xi) \quad \forall \xi \in \mathbb{R}^N \tag{2.7}$$

is obtained from (2.6) by passing to the limit as $\varepsilon \to 0$. Here, $\widehat{f}$ stands for the classical Fourier transformation of f.

Once the homogenization result in $\mathbb{R}^N$ is established, it is an easy matter to deduce the corresponding result in a bounded domain Ω by localization techniques using cut-off functions (see [7]).

Theorem 2.5 *Let Ω be an arbitrary domain in $\mathbb{R}^N$. We consider a sequence u^ε satisfying (1.4). Then*

$$a_{k\ell}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_\ell} \rightharpoonup q_{k\ell} \frac{\partial u^*}{\partial x_\ell} \quad \textit{in} \quad L^2(\Omega)\textit{-weak}, \quad \forall k = 1, ..., N.$$

In particular, u^* satisfies the homogenized equation (1.8). ■

The next stage of development was the introduction of the *Bloch approximation* of u^ε by the following integral representation:

$$\Theta^\varepsilon(x) \stackrel{\text{def}}{=} \int\limits_{\varepsilon^{-1}Y'} \widehat{u^*}(\xi) e^{ix\cdot\xi} \phi_1^\varepsilon(x;\xi) d\xi, \quad x \in \mathbb{R}^N. \tag{2.8}$$

According to Proposition 2.4, the Fourier transform $\widehat{u*}$ is an approximation of $B_1^\varepsilon u^\varepsilon$, and so heuristically speaking, the Bloch approximation Θ^ε is close to u^ε since higher modes can be neglected. Indeed, this has been rigorously established (loc. cit.) without the hypotheses usually assumed in literature in the justification of correctors.

Theorem 2.6 *Assume that the coefficients $a_{k\ell}$ satisfy (1.2). Let u^ε be the sequence introduced in (2.4). Then if $f \in L^2(\mathbb{R}^N)$, we have*

$$(u^\varepsilon - \Theta^\varepsilon) \to 0 \quad \textit{in} \quad H^1(\mathbb{R}^N). \tag{2.9}$$

Furthermore, we have the estimate

$$\|\nabla(u^\varepsilon - \Theta^\varepsilon)\|_{L^2(\mathbb{R}^N)} \leq c\varepsilon\|f\|_{L^2(\mathbb{R}^N)}. \quad ■ \tag{2.10}$$

Thanks to the above result, we were reduced to expand Θ^ε in terms of ε in order to be able to compare it with the classical correctors for u^ε. To fulfill this task, it is clear from the definition of Θ^ε, that it is necessary to obtain asymptotic expansion of the first Bloch mode $\phi_1^\varepsilon(\cdot;\xi)$. We state now results in this direction and their proofs can be found in [5].

Proposition 2.7 *All odd order derivatives of λ_1 at $\eta = 0$ vanish, i.e.,*

$$D^\beta \lambda_1(0) = 0 \quad \forall \beta \in \mathbb{Z}_+^N,\ |\beta| \ \textit{odd}.$$

Various derivatives of ϕ_1 at $\eta = 0$ can also be calculated, in particular:

$$D_k \phi_1(y;0) \ = \ i\phi_1(y;0)\chi_k(y). \quad \blacksquare$$

Using the above result, we deduce the following one on first order correctors.

Theorem 2.8 *Assume that the hypotheses of Theorem 2.6 hold.*

(i) We have

$$\|\Theta^\varepsilon - u^*\|_{L^2(\mathbb{R}^N)} \le c\varepsilon \|u^*\|_{H^1(\mathbb{R}^N)}.$$

(ii) If $f \in L^2(\mathbb{R}^N)$ and $\chi_k \in W_\#^{1,\infty}(Y)$ where χ_k is the solution of (1.7) and $\chi_k^\varepsilon(x) = \chi_k\left(\frac{x}{\varepsilon}\right)$, then we have

$$\left\| \Theta^\varepsilon - u^* - \varepsilon \chi_k^\varepsilon \frac{\partial u^*}{\partial x_k} \right\|_{H^1(\mathbb{R}^N)} \le c\varepsilon \|f\|_{L^2(\mathbb{R}^N)}. \quad \blacksquare$$

3 Presentation of new results

After discussing the case of the whole space, let us now go back to the case of bounded domains and consider the original problem (1.4). Let us recall that the difficulties of adapting Fourier type techniques in bounded domains are very well-known. Nevertheless, we could prove the homogenization result (namely Theorem 2.5) in the case of bounded domains via localization techniques in the physical space [7]. This shows that certain results on bounded domains can be deduced using the Bloch wave method. We substantiate further this statement by showing how Bloch techniques can be adapted to give the correct definition of the Bloch approximation in bounded domains taking into account the boundary condition. Somewhat surprisingly, this does not involve localization neither in the physical space nor in the momentum space but in the state space. Accordingly, we introduce cut-off functions in $\mathbb{R}$ enjoying the following properties:

$$\begin{cases} \varphi^\varepsilon \in C^1(\mathbb{R}), \quad \varphi^\varepsilon(0) = \dfrac{\partial \varphi^\varepsilon}{\partial t}(0) = 0, \\ \varphi^\varepsilon(u) = u, \quad \text{if} \quad |u| \ge \varepsilon, \\ (\varphi^\varepsilon)' \in W^{1,\infty}(\mathbb{R}), \quad |\varphi^\varepsilon(u)| \le c|u|, \\ |(\varphi^\varepsilon)'(u)| \le c, \quad |(\varphi^\varepsilon)''(u)| \le c\varepsilon^{-1}, \quad \text{for } u \in \mathbb{R}, \text{ a.e.} \end{cases} \tag{3.1}$$

Explicitly, we can take, for example

$$\varphi^\varepsilon(u) = \begin{cases} u & \text{if} \quad |u| \ge \varepsilon, \\ u\, sin(\dfrac{u\pi}{2\varepsilon}) & \text{if} \quad 0 \le u \le \varepsilon, \\ -u\, sin(\dfrac{u\pi}{2\varepsilon}) & \text{if} \quad -\varepsilon \le u \le 0. \end{cases}$$

Next, we define the *modified Bloch approximation* for the Dirichlet boundary value problem as follows:

$$\theta^\varepsilon(x) \stackrel{\text{def}}{=} \int\limits_{\varepsilon^{-1}Y'} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) e^{ix\cdot\xi} \phi_1^\varepsilon(x;\xi) d\xi \qquad \text{for } x \in \Omega, \tag{3.2}$$

where we recall the definition of the Fourier transform used in our work:

$$\widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) = (2\pi)^{-\frac{N}{2}} \int\limits_{\mathbb{R}^N} \varphi^\varepsilon(\widetilde{u}^*(x)) e^{-ix\cdot\xi} d\xi,$$

with $\widetilde{u}^*$ denoting the extension of u^* defined by

$$\widetilde{u}^*(x) = \begin{cases} u^*(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \notin \Omega. \end{cases}$$

Throughout this paper, we use the notation $\widetilde{\cdot}$ to denote the extension by zero outside Ω.

We make now a few comments on the modified Bloch approximation. First of all, it is not difficult to compute this object numerically following the algorithm of [4] and [3]. Secondly, its definition involves only the first Bloch mode which is also the case with $\mathbb{R}^N$. The main difference lies in the fact that it depends on the boundary condition through φ^ε and u^*. At first glance, it may seem strange to introduce nonlinear function of the solution in a linear set-up. However, this introduction is natural if the reader recalls that the solution depends nonlinearly on the boundary. One can also observe in the definition of θ^ε that the values of u^* are modified not only close to the boundary on which u^* vanishes but also at places inside Ω where u^* may be zero. (But, of course, this modification becomes negligible as $\varepsilon \to 0$). This is one of the main differences with the classical expression of first order correctors where values of u^* are taken as such.

Since $\phi_1(\cdot;\eta)$ is analytic for $|\eta| \leq \delta$, we can expand it as in Theorem 2.8 and this gives rise to an asymptotic expansion of the modified Bloch approximation (3.2). The main result in this direction is as follows:

Theorem 3.1 *Let Ω be an open bounded set in $\mathbb{R}^N$ and the modified Bloch approximation θ^ε be defined by (3.2). Then*

(i) θ^ε converges to u^* in $L^2(\Omega)$ as $\varepsilon \to 0$. In fact, we have the estimate

$$\|\theta^\varepsilon - \varphi^\varepsilon(u^*)\|_{L^2(\Omega)} \leq c\varepsilon \|u^*\|_{H^1(\Omega)}.$$

(ii) Under the additional hypotheses that $u^* \in H^2(\Omega)$, $\nabla u^* \in L^4(\Omega)$ and $\chi_k \in W^{1,\infty}(Y)$, we have

$$\theta^\varepsilon - \varphi^\varepsilon(u^*) - \varepsilon\chi_k^\varepsilon \frac{\partial\varphi^\varepsilon(u^*)}{\partial x_k} \to 0 \qquad \text{in } H^1(\Omega),$$

with the notation $\chi_k^\varepsilon(x) = \chi_k\left(\frac{x}{\varepsilon}\right)$. ▪

The connection between the modified Bloch approximation and the corrector property is given by:

Theorem 3.2 *Under the hypotheses that $u^* \in H^2(\Omega)$, $\nabla u^* \in L^4(\Omega)$ and $\chi_k \in W^{1,\infty}(Y)$, we have*

$$\chi_k^\varepsilon \frac{\partial \varphi^\varepsilon(u^*)}{\partial x_k} \in H_0^1(\Omega)$$

and it provides a first corrector in the sense that it satisfies

$$u^\varepsilon - u^* - \varepsilon \chi_k^\varepsilon \frac{\partial \varphi^\varepsilon(u^*)}{\partial x_k} \to 0 \quad \text{in } H_0^1(\Omega). \quad \blacksquare$$

The proof of this theorem is based entirely on a Fourier point of view. In particular, we use Bloch techniques to prove the convergence in the energy method. Putting Theorems 3.1 and 3.2 together, we have easily that

Corollary 3.3 *Under the hypotheses of Theorem 3.2 the modified Bloch approximation approximates the solution in the energy norm:*

$$\|u^\varepsilon - \theta^\varepsilon\|_{H^1(\Omega)} \to 0. \quad \blacksquare$$

Under suitable hypotheses on the homogenized solution and using the maximum principle, we are now in a position to establish a rate of convergence for the first corrector term as well as for the modified Bloch approximation. This error estimate is of order $\varepsilon^{\frac{1}{2}}$, and more precisely, we have

Theorem 3.4 *Let $\chi_k \in W^{1,\infty}(Y)$ and $u^* \in H^2(\Omega)$ the continuous solution of (1.8) with $f \geq 0$ and whenever $N \geq 3$, assume u^* in $W^{2,p}(\Omega)$ with $p = 4(N+1)/(N+4)$. Assume that the boundary of Ω satisfies the standard interior sphere condition at any $x \in \partial\Omega$. Then*

$$\left\| \theta^\varepsilon - \varphi^\varepsilon(u^*) - \varepsilon \chi_k^\varepsilon \frac{\partial \varphi^\varepsilon(u^*)}{\partial x_k} \right\|_{H^1(\Omega)} \leq c\varepsilon^{\frac{1}{2}}. \quad \blacksquare$$

Finally, combining Theorem 3.4 and the classical error estimates of the first corrector term proved in [1] (see pp. 66), we get

Corollary 3.5 *Under the hypotheses of Theorem 3.4 and if $\nabla u^* \in L^\infty(\Omega)$, the modified Bloch approximation approximates the solution as:*

$$\|u^\varepsilon - \theta^\varepsilon\|_{H^1(\Omega)} \leq c\varepsilon^{\frac{1}{2}}. \quad \blacksquare$$

In general, u^* may be zero inside Ω. Thus, to obtain the error estimate of order $\varepsilon^{\frac{1}{2}}$ is as follows:

Remark 3.6 *In the case f changes sign, we decompose u^* in*

$$u^*(x) = u_+^*(x) - u_-^*(x) \qquad x \in \Omega,$$

where $u_+^*(x)$ is the solution of (1.8) with $f_+(x) = \max(0, f(x))$ and $u_-^*(x)$ is the solution of (1.8) with $f_-(x) = \max(0, -f(x))$. Now, we consider the modified Bloch approximation for u_+^ε

$$\theta_+^\varepsilon(x) \quad = \int_{\varepsilon^{-1}Y'} \widehat{\varphi^\varepsilon(\widetilde{u}_+^*)}(\xi) e^{ix\cdot\xi} \phi_1^\varepsilon(x;\xi) d\xi \qquad \text{for } x \in \mathbb{R}^N,$$

and, analogously, θ_-^ε is the modified Bloch approximation of u_-^ε. Then, applying Theorem 3.4 and Corollary 3.5, we establish

$$\|u^\varepsilon - \theta_+^\varepsilon + \theta_-^\varepsilon\|_{H^1(\Omega)} \leq c\varepsilon^{\frac{1}{2}}.$$

4 Preliminary lemmas

Here, we prove a couple of results about the convergence behavior of $\varphi^\varepsilon(\widetilde{u}^*)$.

Lemma 4.1 *Let φ^ε be as in (3.1).*

(i) If $u \in L^2(\Omega)$ then $\varphi^\varepsilon(u) \in L^2(\Omega)$ and we have the estimate

$$\|\varphi^\varepsilon(u) - u\|_{L^2(\Omega)} \leq c\varepsilon.$$

(ii) If $u \in H^1(\Omega)$ (resp. $H^1_0(\Omega)$) then $\varphi^\varepsilon(u) \in H^1(\Omega)$ (resp. $H^1_0(\Omega)$) and

$$\varphi^\varepsilon(u) \to u \quad \text{in } H^1(\Omega) \text{ (resp. } H^1_0(\Omega)) \quad \text{as } \varepsilon \to 0.$$

Proof. First let us prove the estimate in $L^2(\Omega)$. To this end, let us note

$$\varphi^\varepsilon(u) - u = (\varphi^\varepsilon(u) - u)\chi_{\omega^\varepsilon} \quad \text{in } \Omega,$$

where $\chi_{\omega^\varepsilon}$ is the characteristic function of the set

$$\omega^\varepsilon = \{x \in \Omega; \quad |u(x)| \leq \varepsilon\}. \tag{4.1}$$

Hence, using (3.1) we get

$$|\varphi^\varepsilon(u) - u|^2 \leq c|u(x)|^2\chi_{\omega^\varepsilon} \quad x \in \Omega.$$

A simple integration yields the estimate

$$\|\varphi^\varepsilon(u) - u\|_{L^2(\Omega)} \leq c\varepsilon.$$

The next step is to prove the strong convergence in $L^2(\Omega)$ of the first order derivatives. We apply Dominated Convergence Theorem. By Chain Rule, we have

$$\frac{\partial}{\partial x_k}\varphi^\varepsilon(u) = (\varphi^\varepsilon)'(u)\frac{\partial u}{\partial x_k} \quad \text{in } \Omega.$$

Thus, the following relation holds:

$$\frac{\partial}{\partial x_k}(\varphi^\varepsilon(u) - u) = \left((\varphi^\varepsilon)'(u) - 1\right)\frac{\partial u}{\partial x_k}\chi_{\omega^\varepsilon} \quad x \in \Omega,$$

from which we can deduce the uniform bound, namely,

$$\left|\frac{\partial}{\partial x_k}(\varphi^\varepsilon(u) - u)\right|^2 \leq c\left|\frac{\partial u}{\partial x_k}\right|^2.$$

To show the point-wise convergence a.e. in Ω, we introduce the set

$$\omega = \{x \in \Omega; \quad u(x) = 0\} \subset \omega^\varepsilon$$

and use the property that

$$\nabla u(x) = 0 \quad \text{a.e. on } \omega$$

to deduce

$$\frac{\partial u}{\partial x_k}\chi_{\omega^\varepsilon} \to 0 \quad x \in \Omega \text{ a.e.}$$

This completes the proof of Lemma 4.1. ∎

Lemma 4.2 *For $u \in H^2(\Omega)$ with $\nabla u \in L^4(\Omega)$, we have $\varphi^\varepsilon(u) \in H^2(\Omega)$ and*

$$\varepsilon\varphi^\varepsilon(u) \to 0 \quad \text{in } H^2(\Omega).$$

Proof. By virtue of the previous lemma, it is enough to show

$$\varepsilon \frac{\partial^2 \varphi^\varepsilon(u)}{\partial x_i \partial x_j} \to 0 \quad \text{in } L^2(\Omega) \quad \forall i,j = 1,\dots N. \tag{4.2}$$

First of all $\varphi^\varepsilon(u) \in H^2(\Omega)$ since $\varphi^\varepsilon \in C^1(\mathbb{R})$, $(\varphi^\varepsilon)' \in W^{1,\infty}(\mathbb{R})$ and $u \in H^2(\Omega)$. Further, by Chain Rule, we have

$$\varepsilon \frac{\partial^2 \varphi^\varepsilon(u)}{\partial x_i \partial x_j}(x) = \varepsilon(\varphi^\varepsilon)'(u(x)) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \varepsilon(\varphi^\varepsilon)''(u(x)) \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x) \qquad \text{in } \Omega. \tag{4.3}$$

The first term on the right side of (4.3) obviously tends to zero in $L^2(\Omega)$. Regarding the second term, we note that it is bounded above by

$$c|\nabla u(x)|^2 \chi_{\omega^\varepsilon}. \tag{4.4}$$

This is because $(\varphi^\varepsilon)'(t) = 1$ if $|t| > \varepsilon$ and $(\varphi^\varepsilon)''(u(x)) = 0$ if $x \notin \omega^\varepsilon$. Since $\nabla u \in L^4(\Omega)$, it follows from our arguments in the last part of the proof of Lemma 4.1, that (4.4) converges to zero in $L^2(\Omega)$. This finishes the proof. ▪

Remark 4.3 *We will use the above lemma with $u = u^*$, the solution of the homogenized equation. Thus, assuming $\partial\Omega$ is smooth and that $f \in L^2(\Omega)$ and $f = div(g)$ with $g \in L^4(\Omega)^N$, it follows from classical regularity results that $u^* \in H^2(\Omega)$ and $\nabla u^* \in L^4(\Omega)^N$. Thus the hypotheses of Lemma 4.2 are satisfied in this particular case.* ▪

Remark 4.4 *It is important to note that if the homogenized solution u^* lies in $H^2(\Omega) \cap H^1_0(\Omega)$ and $\nabla u^* \in L^4(\Omega)^N$ then $\varphi^\varepsilon(u^*)$ is not only in $H^2(\Omega) \cap H^1_0(\Omega)$ but also in $H^2_0(\Omega)$. Hence, $\varphi^\varepsilon(\widetilde{u}^*) \in H^2(\mathbb{R}^N)$.* ▪

Our next result is a generalization of Parseval's identity of Theorem 2.1.

Lemma 4.5 *For $g^\varepsilon \in L^2(\varepsilon^{-1}Y')$ and $\rho \in L^\infty(Y'; H^1_\#(Y))$, we define*

$$G^\varepsilon(x) = \int_{\varepsilon^{-1}Y'} g^\varepsilon(\xi) e^{ix\cdot\xi} \rho(\frac{x}{\varepsilon}; \varepsilon\xi) d\xi, \quad x \in \mathbb{R}^N. \tag{4.5}$$

Then, we have

$$\begin{aligned}
\|G^\varepsilon\|^2_{L^2(\mathbb{R}^N)} &= \int_{\varepsilon^{-1}Y'} |g^\varepsilon(\xi)|^2 \|\rho(\cdot;\varepsilon\xi)\|^2_{L^2(Y)} d\xi, \\
\|\nabla_x G^\varepsilon\|^2_{L^2(\mathbb{R}^N)} &= \int_{\varepsilon^{-1}Y'} |g^\varepsilon(\xi)|^2 \|i\xi\rho(\cdot;\varepsilon\xi) + \varepsilon^{-1}\nabla_y \rho(\cdot;\varepsilon\xi)\|^2_{L^2(Y)^N} d\xi.
\end{aligned}$$

Proof. We expand $\rho(y;\eta)$ as a function of y in the orthonormal basis $\{\phi_m(y;\eta)\}_{m=1}^{\infty}$ where η is a parameter:

$$\rho(y;\eta) = \sum_{m=1}^{\infty} a_m(\eta)\phi_m(y;\eta).$$

Introducing this expression in (4.5), we get

$$G^\varepsilon(x) = \int\limits_{\varepsilon^{-1}Y'} g^\varepsilon(\xi) \sum_{m=1}^{\infty} a_m(\varepsilon\xi) e^{ix\cdot\xi} \phi_m^\varepsilon(x;\xi) d\xi.$$

Applying the Parseval's identity of Theorem 2.1, we get

$$\|G^\varepsilon\|^2_{L^2(\mathbb{R}^N)} = \int\limits_{\varepsilon^{-1}Y'} |g^\varepsilon(\xi)|^2 \sum_{m=1}^{\infty} |a_m(\varepsilon\xi)|^2 d\xi.$$

This completes the proof of the first part of the lemma if we use the Parseval's identity in $L^2(Y)$:

$$\|\rho(\cdot;\eta)\|^2_{L^2(Y)} = \sum_{m=1}^{\infty} |a_m(\eta)|^2 \quad \forall \eta \in Y'.$$

For the second part of the lemma, we differentiate formally $G^\varepsilon(x)$ with respect to x. We obtain

$$\nabla_x G^\varepsilon(x) = \int\limits_{\varepsilon^{-1}Y'} g^\varepsilon(\xi) e^{ix\cdot\xi} \left(i\xi\rho(\frac{x}{\varepsilon};\varepsilon\xi) + \varepsilon^{-1}\nabla_y \rho(\frac{x}{\varepsilon};\varepsilon\xi) \right) d\xi.$$

We remark that the above integral is of the same type as the one analyzed in the first part. This completes the proof. ▪

5 Asymptotic expansion of the modified Bloch approximation

In this section, we are going to prove the Theorem 3.1. To this end, we use the results established in Proposition 2.2 and Proposition 2.7.

First, we show

$$\|\theta^\varepsilon - \varphi^\varepsilon(\widetilde{u}^*)\|_{L^2(\mathbb{R}^N)} \leq c\varepsilon \|u^*\|_{H^1(\Omega)}. \tag{5.1}$$

We use the decomposition

$$\theta^\varepsilon(x) - \varphi^\varepsilon(\widetilde{u}^*(x)) \;=\; u_1^\varepsilon(x) \;+\; u_2^\varepsilon(x) \;+\; u_3^\varepsilon(x). \tag{5.2}$$

where

$$\begin{aligned}
u_1^\varepsilon(x) &= \int\limits_{\varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi)[\phi_1^\varepsilon(x;\xi) - \phi_1^\varepsilon(x;0)] e^{ix\cdot\xi} d\xi, \\
u_2^\varepsilon(x) &= \int\limits_{\varepsilon^{-1}(Y'-B_\delta)} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) \phi_1^\varepsilon(x;\xi) e^{ix\cdot\xi} d\xi, \\
u_3^\varepsilon(x) &= -(2\pi)^{-\frac{N}{2}} \int\limits_{\mathbb{R}^N - \varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) e^{ix\cdot\xi} d\xi.
\end{aligned}$$

Applying Parseval's identity we get:

$$\|u_2^\varepsilon\|^2_{L^2(\mathbb{R}^N)} \leq c \int\limits_{\varepsilon^{-1}(Y'-B_\delta)} |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi \leq c\delta^{-2}\varepsilon^2 \int\limits_{\mathbb{R}^N} |\xi|^2 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi,$$

and

$$\|u_3^\varepsilon\|^2_{L^2(\mathbb{R}^N)} \leq \int\limits_{\mathbb{R}^N-\varepsilon^{-1}B_\delta} |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi \leq c\delta^{-2}\varepsilon^2 \int\limits_{\mathbb{R}^N} |\xi|^2 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi.$$

On the other hand, using Parseval's identity and the estimate

$$\|\phi_1(\cdot;\eta) - \phi(\cdot;0)\|_{H^1(Y)} \leq c|\eta| \quad \text{ for } \eta \in B_\delta,$$

we get

$$\|u_1^\varepsilon\|^2_{L^2(\mathbb{R}^N)} \leq c\varepsilon^2 \int\limits_{\varepsilon^{-1}B_\delta} |\xi|^2 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi.$$

Now, we apply Lemma 4.1 which shows, in particular that $\varphi^\varepsilon(\widetilde{u}^*)$ is bounded in $H^1(\mathbb{R}^N)$ and hence

$$\int\limits_{\mathbb{R}^N} |\xi|^2 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi \leq c\|\widetilde{u}^*\|^2_{H^1(\mathbb{R}^N)} = c\|u^*\|^2_{H^1(\Omega)}.$$

Therefore, (5.1) is proven.

Now, we are going to prove (ii) of Theorem 3.1. Because of (5.1), it remains to prove that

$$\frac{\partial \theta^\varepsilon}{\partial x_\ell} - \frac{\partial \varphi^\varepsilon(\widetilde{u}^*)}{\partial x_\ell} - \frac{\partial \chi_k}{\partial y_\ell}\frac{\partial \varphi^\varepsilon(\widetilde{u}^*)}{\partial x_k} - \varepsilon\chi_k^\varepsilon \frac{\partial^2 \varphi^\varepsilon(\widetilde{u}^*)}{\partial x_k \partial x_\ell} \to 0 \quad \text{in } L^2(\mathbb{R}^N) \quad \forall k,\ell = 1,\ldots,N. \tag{5.3}$$

A simple application of Lemma 4.2 immediately yields that the fourth term in (5.3) goes to zero in $L^2(\mathbb{R}^N)$. The remaining terms in (5.3) are decomposed individually as follows:

$$\begin{aligned}
\frac{\partial \theta^\varepsilon}{\partial x_\ell} \;=\; & \int\limits_{\varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) i\xi_\ell [\phi_1^\varepsilon(x,\xi) - \phi_1^\varepsilon(x,0)] e^{ix\cdot\xi} d\xi + \int\limits_{\varepsilon^{-1}(Y'-B_\delta)} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) i\xi_\ell \phi_1^\varepsilon(x,\xi) e^{ix\cdot\xi} d\xi + \\
& + (2\pi)^{-\frac{N}{2}} \int\limits_{\varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) i\xi_\ell e^{ix\cdot\xi} d\xi + \varepsilon^{-1} \int\limits_{\varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) i\varepsilon\xi_k (2\pi)^{-\frac{N}{2}} \frac{\partial \chi_k}{\partial y_\ell}(\frac{x}{\varepsilon}) e^{ix\cdot\xi} d\xi + \\
& + \varepsilon^{-1} \int\limits_{\varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) [\frac{\partial \phi_1}{\partial y_\ell}(\frac{x}{\varepsilon};\varepsilon\xi) - i\varepsilon\xi_k (2\pi)^{-\frac{N}{2}} \frac{\partial \chi_k}{\partial y_\ell}(\frac{x}{\varepsilon})] e^{ix\cdot\xi} d\xi + \\
& + \varepsilon^{-1} \int\limits_{\varepsilon^{-1}(Y'-B_\delta)} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) \frac{\partial \phi_1}{\partial y_\ell}(\frac{x}{\varepsilon};\varepsilon\xi) e^{ix\cdot\xi} d\xi,
\end{aligned}$$

$$\frac{\partial \varphi^\varepsilon(\widetilde{u}^*)}{\partial x_\ell} \;=\; (2\pi)^{-\frac{N}{2}} \int\limits_{\varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) i\xi_\ell e^{ix\cdot\xi} d\xi + (2\pi)^{-\frac{N}{2}} \int\limits_{\mathbb{R}^N-\varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) i\xi_\ell e^{ix\cdot\xi} d\xi,$$

$$\frac{\partial \chi_k}{\partial y_\ell}(\frac{x}{\varepsilon})\frac{\partial \varphi^\varepsilon(u^*)}{\partial x_k}(x) = \int\limits_{\varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) i\xi_k (2\pi)^{-\frac{N}{2}}\frac{\partial \chi_k}{\partial y_\ell}(\frac{x}{\varepsilon})e^{ix\cdot\xi}d\xi + \\ + \int\limits_{\mathbb{R}^N-\varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) i\xi_k (2\pi)^{-\frac{N}{2}}\frac{\partial \chi_k}{\partial y_\ell}(\frac{x}{\varepsilon})e^{ix\cdot\xi}d\xi.$$

Putting them together, we reach the decomposition

$$\frac{\partial \theta^\varepsilon}{\partial x_\ell} - \frac{\partial \varphi^\varepsilon(\widetilde{u}^*)}{\partial x_\ell} - \frac{\partial \chi_k^\varepsilon}{\partial y_\ell}\frac{\partial \varphi^\varepsilon(\widetilde{u}^*)}{\partial x_k} = v_1^\varepsilon(x) + v_2^\varepsilon(x) + v_3^\varepsilon(x) + \frac{\partial u_2^\varepsilon}{\partial x_\ell} + \frac{\partial u_3^\varepsilon}{\partial x_\ell}, \tag{5.4}$$

where u_2^ε and u_3^ε were already introduced and where

$$\begin{aligned}
v_1^\varepsilon(x) &= \int\limits_{\varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi) i\xi_\ell \big(\phi_1^\varepsilon(x;\xi) - \phi_1^\varepsilon(x;0)\big)e^{ix\cdot\xi}d\xi,\\
v_2^\varepsilon(x) &= \varepsilon^{-1}\int\limits_{\varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)}(\xi)\big(\frac{\partial \phi_1^\varepsilon}{\partial y_\ell}(x;\xi) - i\varepsilon\xi_k(2\pi)^{-\frac{N}{2}}\frac{\partial \chi_k}{\partial y_\ell}(\frac{x}{\varepsilon})\big)e^{ix\cdot\xi}d\xi,\\
v_3^\varepsilon(x) &= -(2\pi)^{-\frac{N}{2}}\frac{\partial \chi_k^\varepsilon}{\partial y_\ell}(x)\int\limits_{\mathbb{R}^N-\varepsilon^{-1}B_\delta} \widehat{\varphi^\varepsilon(\widetilde{u}^*)} i\xi_k e^{ix\xi}d\xi.
\end{aligned}$$

Applying Lemma 4.5 to the expression defining u_2^ε, we deduce easily

$$\left\|\frac{\partial u_2^\varepsilon}{\partial x_\ell}\right\|^2_{L^2(\mathbb{R}^N)} \le c_\delta \varepsilon^2 \int\limits_{\varepsilon^{-1}(Y'-B_\delta)} |\xi|^4 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi.$$

A more direct application of Parseval's identity yields

$$\begin{aligned}
\left\|\frac{\partial u_3^\varepsilon}{\partial x_\ell}\right\|^2_{L^2(\mathbb{R}^N)} &\le c\delta^{-2}\varepsilon^2 \int\limits_{\mathbb{R}^N-\varepsilon^{-1}B_\delta} |\xi|^4 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi,\\
\|v_3^\varepsilon\|^2_{L^2(\mathbb{R}^N)} &\le c\delta^{-2}\varepsilon^2 \|\chi_k\|^2_{1,\infty} \int\limits_{\mathbb{R}^N-\varepsilon^{-1}B_\delta} |\xi|^4 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi.
\end{aligned}$$

It remains to estimate v_1^ε and v_2^ε. Once more invoking Lemma 4.5, we obtain

$$\begin{aligned}
\|v_1^\varepsilon\|^2_{L^2(\mathbb{R}^N)} &\le c\int\limits_{\varepsilon^{-1}B_\delta} |\xi|^2 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 \|\phi_1(\cdot;\varepsilon\xi) - \phi_1(\cdot;0)\|^2_{L^2(Y)} d\xi,\\
\|v_2^\varepsilon\|^2_{L^2(\mathbb{R}^N)} &\le c\varepsilon^{-2}\int\limits_{\varepsilon^{-1}B_\delta} |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 \left\|\frac{\partial \phi_1}{\partial y_\ell}(\cdot;\varepsilon\xi) - i\varepsilon\xi_k(2\pi)^{-\frac{N}{2}}\frac{\partial \chi_k}{\partial y_\ell}(\cdot)\right\|^2_{L^2(Y)} d\xi.
\end{aligned}$$

Applying Proposition 2.2 and Proposition 2.7, we arrive at

$$\begin{aligned}
\|v_1^\varepsilon\|^2_{L^2(\mathbb{R}^N)} &\le c\varepsilon^2 \int\limits_{\varepsilon^{-1}B_\delta} |\xi|^4 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi,\\
\|v_2^\varepsilon\|^2_{L^2(\mathbb{R}^N)} &\le c\varepsilon^2 \int\limits_{\varepsilon^{-1}B_\delta} |\xi|^4 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi.
\end{aligned}$$

Therefore, we get the following estimate

$$(5.5)\qquad \left\| \frac{\partial \theta^\varepsilon}{\partial x_\ell} - \frac{\partial \varphi^\varepsilon(\widetilde{u}^*)}{\partial x_\ell} - \frac{\partial \chi_k}{\partial y_\ell}\frac{\partial \varphi^\varepsilon(\widetilde{u}^*)}{\partial x_k} - \varepsilon\chi_k^\varepsilon \frac{\partial^2 \varphi^\varepsilon(\widetilde{u}^*)}{\partial x_k \partial x_\ell} \right\|^2_{L^2(\mathbb{R}^N)} \leq c\varepsilon^2 \int_{\mathbb{R}^N} |\xi|^4 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi.$$

According to Lemma 4.2, $\varepsilon\varphi^\varepsilon(\widetilde{u}^*) \to 0$ in $H^2(\mathbb{R}^N)$ and so

$$\varepsilon^2 \int_{\varepsilon^{-1}B_\delta} |\xi|^4 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi \to 0.$$

It then follows from the above estimates that each individual term of the right side of (5.4) converges to zero in $L^2(\mathbb{R}^N)$ and hence (5.3) is proven. This concludes the proof of Theorem 3.1. ∎

6 First order correctors

The aim of this section is to provide a proof of Theorem 3.2 which is concerned with first order correctors for u^ε.

As per the recipe provided by our earlier work [5], in order to get an expression of the first order corrector, we must seek an expansion of the modified Bloch approximation which is precisely what we have done in Theorem 3.1. Our choice for the first order corrector is thus the following:

$$(6.1)\qquad z^\varepsilon(x) = \varphi^\varepsilon(u^*(x)) + \varepsilon\chi_k^\varepsilon(x)\frac{\partial \varphi^\varepsilon(u^*)}{\partial x_k}(x).$$

Let us begin remarking that z^ε defined by (6.1) indeed belongs to $H_0^1(\Omega)$ under our hypotheses that $u^* \in H^2(\Omega) \cap W_0^{1,4}(\Omega)$ and $\chi_k \in W^{1,\infty}(Y)$. In the statement of Theorem 3.2, we have $\varphi^\varepsilon(u^*)$ in the place of u^* which is perfectly legal according to Lemma 4.1.

The result announced in Theorem 3.2 would follow if we show

$$\|\nabla(\widetilde{u}^\varepsilon - \widetilde{z}^\varepsilon)\|_{L^2(\mathbb{R}^N)} \to 0, \quad \text{as } \varepsilon \to 0,$$

(or) equivalently

$$(6.2)\qquad \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} \lambda_m^\varepsilon(\xi) |B_m^\varepsilon \widetilde{u}^\varepsilon(\xi) - B_m^\varepsilon \widetilde{z}^\varepsilon(\xi)|^2 d\xi \to 0, \quad \text{as } \varepsilon \to 0.$$

In order to prove this, we observe that $\widetilde{u}^\varepsilon$ is the solution of

$$(6.3)\qquad A^\varepsilon \widetilde{u}^\varepsilon + N^\varepsilon \widetilde{u}^\varepsilon = \widetilde{f} \quad \text{in } \mathbb{R}^N, \text{ and } \widetilde{u}^\varepsilon \in H^1(\mathbb{R}^N),$$

where $N^\varepsilon \widetilde{u}^\varepsilon$ is defined as an element of $H^{-1}(\mathbb{R}^N)$ by

$$(6.4)\qquad {}_{H^{-1}(\mathbb{R}^N)}< N^\varepsilon \widetilde{u}^\varepsilon, v >_{H^1(\mathbb{R}^N)} = -{}_{H^{-\frac{1}{2}}(\partial\Omega)}< \frac{\partial u^\varepsilon}{\partial n_{A^\varepsilon}}, v >_{H^{\frac{1}{2}}(\partial\Omega)} \qquad \forall v \in H^1(\mathbb{R}^N).$$

The above equation (6.3) can be written equivalently in terms of Bloch coefficients:

$$(6.5)\qquad \lambda_m^\varepsilon(\xi) B_m^\varepsilon \widetilde{u}^\varepsilon + B_m^\varepsilon N^\varepsilon \widetilde{u}^\varepsilon = B_m^\varepsilon \widetilde{f}, \qquad \forall m \geq 1,\ \xi \in \varepsilon^{-1}Y'.$$

Now, we expand the integral of (6.2)

$$(6.6)\qquad \int\limits_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} \left(B^\varepsilon_m \widetilde{f}(\xi) \overline{B^\varepsilon_m \widetilde{u}^\varepsilon(\xi)} + \lambda^\varepsilon_m(\xi) |B^\varepsilon_m \widetilde{z}^\varepsilon(\xi)|^2 - B^\varepsilon_m \widetilde{f}(\xi) \overline{B^\varepsilon_m \widetilde{z}^\varepsilon(\xi)} - B^\varepsilon_m \widetilde{z}^\varepsilon(\xi) \overline{B^\varepsilon_m \widetilde{f}(\xi)} \right) d\xi$$
$$+ \int\limits_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} \left(-B^\varepsilon_m N^\varepsilon \widetilde{u}^\varepsilon(\xi) \overline{B^\varepsilon_m \widetilde{u}^\varepsilon(\xi)} + B^\varepsilon_m N^\varepsilon \widetilde{u}^\varepsilon(\xi) \overline{B^\varepsilon_m \widetilde{z}^\varepsilon(\xi)} + B^\varepsilon_m \widetilde{z}^\varepsilon(\xi) \overline{B^\varepsilon_m N^\varepsilon \widetilde{u}^\varepsilon(\xi)} \right) d\xi.$$

The last three terms of (6.6) vanish because, in fact, for all $v \in H^1_0(\Omega)$ we have

$$\int\limits_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} B^\varepsilon_m N^\varepsilon \widetilde{u}^\varepsilon(\xi) \overline{B^\varepsilon_m \widetilde{v}(\xi)} d\xi = {}_{H^{-1}(\mathbb{R}^N)}< N^\varepsilon \widetilde{u}^\varepsilon, \widetilde{v} >_{H^1(\mathbb{R}^N)} = 0.$$

Regarding the first term of (6.6), we have to distinguish two cases: $m = 1$ and $m \geq 2$. For $m = 1$, we use the Proposition 2.4. For the case $m \geq 2$, using the Proposition 2.3, we see that the contribution of all higher modes together tends to zero. Thus, the first term converges to

$$\int\limits_{\mathbb{R}^N} \widetilde{f} \widetilde{u}^* dx.$$

The treatment of third and fourth terms of (6.6) is similar and their sum has the following limit since $\widetilde{z}^\varepsilon \to \widetilde{u}^*$ weakly in $H^1(\mathbb{R}^N)$:

$$-2 \int\limits_{\mathbb{R}^N} \widetilde{f} \widetilde{u}^* dx.$$

So, it remains to study the limiting behavior of the second term in the expression (6.6). Below, we prove that

$$(6.7)\qquad \int\limits_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} \lambda^\varepsilon_m(\xi) |B^\varepsilon_m \widetilde{z}^\varepsilon(\xi)|^2 d\xi \to \int\limits_{\mathbb{R}^N} \widetilde{f} \widetilde{u}^* dx \stackrel{\text{def}}{=} (\widetilde{f}, \widetilde{u}^*),$$

with will establish (6.2) and thereby Theorem 3.2. We introduce the notations

$$\begin{aligned} a^\varepsilon(u,v) &= \int\limits_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} \lambda^\varepsilon_m(\xi) B^\varepsilon_m u(\xi) \overline{B^\varepsilon_m v(\xi)} d\xi, \qquad \forall u, v \in H^1(\mathbb{R}^N), \\ a^*(u,v) &= \int\limits_{\mathbb{R}^N} q_{ij} \xi_i \xi_j \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi, \qquad \forall u, v \in H^1(\mathbb{R}^N). \end{aligned}$$

Now, we decompose $a^\varepsilon(\widetilde{z}^\varepsilon)$ and $(\widetilde{f}, \widetilde{u}^*)$:

$$\begin{aligned} a^\varepsilon(\widetilde{z}^\varepsilon) &= a^\varepsilon(\widetilde{z}^\varepsilon, \widetilde{z}^\varepsilon - \theta^\varepsilon) + a^\varepsilon(\widetilde{z}^\varepsilon - \theta^\varepsilon, \theta^\varepsilon) + a^\varepsilon(\theta^\varepsilon), \\ (\widetilde{f}, \widetilde{u}^*) &= a^*(\widetilde{u}^*, \widetilde{u}^* - \varphi^\varepsilon(\widetilde{u}^*)) + a^*(\widetilde{u}^* - \varphi^\varepsilon(\widetilde{u}^*), \varphi^\varepsilon(\widetilde{u}^*)) + a^*(\varphi^\varepsilon(\widetilde{u}^*)), \end{aligned}$$

where θ^ε is the modified Bloch approximation (3.2). Then, thanks to (5.3) and Lemma 4.1 we have $(\widetilde{z}^\varepsilon - \theta^\varepsilon) \to 0$ in $H^1(\mathbb{R}^N)$ and $\varphi^\varepsilon(\widetilde{u}^*) \to \widetilde{u}^*$ in $H^1(\mathbb{R}^N)$. Thus, proving (6.7) is equivalent to showing

$$(6.8)\qquad a^\varepsilon(\theta^\varepsilon) - a^*(\varphi^\varepsilon(u^*)) \to 0 \quad \text{as } \varepsilon \to 0.$$

Now, by definition of θ^ε, we get

$$a^\varepsilon(\theta^\varepsilon) = \int\limits_{\varepsilon^{-1}Y'} \lambda_1^\varepsilon(\xi)|\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi.$$

Therefore, if we decompose

$$\begin{aligned} a^\varepsilon(\theta^\varepsilon) - a^*(\varphi^\varepsilon(\widetilde{u}^*)) &= \int\limits_{\varepsilon^{-1}B_\delta} (\lambda_1^\varepsilon(\xi) - q_{ij}\xi_i\xi_j)|\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi + \int\limits_{\varepsilon^{-1}(Y'-B_\delta)} \lambda_1^\varepsilon(\xi)|\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi + \\ &+ \int\limits_{\mathbb{R}^N - \varepsilon^{-1}B_\delta} q_{ij}\xi_i\xi_j|\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi, \end{aligned}$$

we see that each term on the right side can be estimated using Proposition 2.2 and Proposition 2.7. We finally arrive at

$$|a^\varepsilon(\theta^\varepsilon) - a^*(\varphi^\varepsilon(\widetilde{u}^*))| \leq c\varepsilon^2 \int\limits_{\mathbb{R}^N} |\xi|^4 |\widehat{\varphi^\varepsilon(\widetilde{u}^*)}|^2 d\xi.$$

A direct application of Lemma 4.2 shows that this last term tends to zero. This completes the proof of Theorem 3.2. ■

7 Error estimates

Finally, in this last section we prove Theorem 3.4. To this end, we prove the following estimates of $\varphi^\varepsilon(\widetilde{u}^*)$.

Lemma 7.1 *Assume that $u^* \in H^2(\Omega)$ is continuous solution of (1.8) with $f \geq 0$ (not zero) and whenever $n \geq 3$ $u^* \in W^{2,p}(\Omega)$ with $p = 4(n+1)/(n+4)$. Moreover, assume that the boundary Ω satisfies the standard interior sphere condition at any $x \in \partial\Omega$. Then,*

$$\|\widetilde{u}^* - \varphi^\varepsilon(\widetilde{u}^*)\|_{H^1(\mathbb{R}^N)} \leq c\varepsilon^{\frac{1}{2}} \|u^*\|_{H^2(\Omega)}, \tag{7.1}$$

and

$$\varepsilon\|\nabla^2\varphi^\varepsilon(\widetilde{u}^*)\|^2_{L^2(\mathbb{R}^N)} \leq c. \tag{7.2}$$ ■

Proof. By Lemma 4.1, we only need to prove (7.1) that

$$\left\|\frac{\partial}{\partial x_k}(\widetilde{u}^* - \varphi^\varepsilon(\widetilde{u}^*))\right\|_{L^2(\mathbb{R}^N)} \leq c\varepsilon^{\frac{1}{2}} \|u^*\|_{H^2(\Omega)} \tag{7.3}$$

Since $(\varphi^\varepsilon)'(t) = 1$ if $|t| > \varepsilon$ and is bounded, we get

$$\left\|\frac{\partial}{\partial x_k}(\widetilde{u}^* - \varphi^\varepsilon(\widetilde{u}^*))\right\|^2_{L^2(\mathbb{R}^N)} \leq c \left\|\frac{\partial u^*}{\partial x_k}\right\|_{L^2(\omega^\varepsilon)},$$

with ω^ε defined as in (4.1). Thanks to the strong maximum principle (see Lemma 3.4 in [8]), there exists $c > 0$ such that

$$\omega^\varepsilon \subseteq \{x \in \Omega; \quad \text{dist}(x, \Gamma) \leq c\varepsilon\}.$$

For ν small enough ($\nu \leq \nu_0$), let us denote by S_ν the boundary of the domain defined by $\{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) \geq \nu\}$. By virtue of the imbedding theorem (see [2] pp. 197) we have

$$\int_{S_\nu} |v|^2 \, dS \leq C\|v\|^2_{H^1(\{x\in\Omega \,|\, \mathrm{dist}(x,\partial\Omega)\geq\nu\})} \leq C\|v\|^2_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega).$$

Considering this inequality with $v = \partial_k u^*$ and integrating with respect ν from 0 to $c\varepsilon$, we obtain

$$\left\| \frac{\partial u^*}{\partial x_k} \right\|^2_{L^2(\omega^\varepsilon)} \leq C\varepsilon \|u^*\|^2_{H^2(\Omega)}, \tag{7.4}$$

and we prove (7.3).

Now, we see (7.2). By (3.1) and (4.3), we get

$$\varepsilon \left\| \frac{\partial^2 \varphi^\varepsilon(\widetilde{u}^*)}{\partial x_i \partial x_j} \right\|^2_{L^2(\mathbb{R}^N)} \leq c\varepsilon \left\| \frac{\partial^2 u^*}{\partial x_i \partial x_j} \right\|^2_{L^2(\Omega)} + \varepsilon^{-1} \left\| \frac{\partial u^*}{\partial x_i} \frac{\partial u^*}{\partial x_j} \right\|^2_{L^2(\omega^\varepsilon)}.$$

By imbedding theorems (see [2]) we have

$$\int_{S_\nu} |v|^4 \, dS \leq C\|v\|^4_{W^{1,p}(\{x\in\Omega \,|\, \mathrm{dist}(x,\partial\Omega)\geq\nu\})} \leq C\|v\|^4_{W^{1,p}(\Omega)}, \quad \forall v \in W^{1,p}(\Omega).$$

Considering this inequality with $v = |\nabla u^*|$ and integrating with respect ν from 0 to $c\varepsilon$, we obtain

$$\left\| \frac{\partial u^*}{\partial x_i} \frac{\partial u^*}{\partial x_j} \right\|^2_{L^2(\omega^\varepsilon)} \leq C\varepsilon,$$

and we conclude the proof of (7.2). ▪

Proof of Theorem 3.4. By (i) in Theorem 3.1 we have the estimate in $L^2(\Omega)$. Now, applying Lemma 7.1 in (5.5) we conclude the proof. ▪

Finally, thanks to Lemma 7.1, Corollary 3.5 is proved using the Theorem 3.4 and error estimate (see [1] pp. 66)

$$\|u^\varepsilon - u^* - \varepsilon u_1^\varepsilon\|_{H_0^1(\Omega)} \leq c\varepsilon^{\frac{1}{2}},$$

u_1^ε defined in (1.12) and if $\nabla u^* \in L^\infty(\Omega)^N$.

Conclusions

We have shown a better approximation of the solution (which we have been calling Bloch Approximation in [5]). To take into account boundary effects, we have modified the Bloch approximation. We have established its connection with the first order classical corrector worked out entirely in the Fourier-Bloch space. Now, taking into account that the Bloch approximation has for future advances we thought that it will not be out of context to include the following brief discussion about our on-going efforts to adapt Bloch waves to domains with boundaries. Details will appear in a future publication.

Just as in the case of Fourier analysis in domains with boundary, it will be convenient to proceed systematically in steps to see the effect of boundary on Bloch waves. We start with half-space, then analyze quarter space and so on. It is important to realize that the basic object in the Bloch analysis is the following transform:

$$\widetilde{f}(y;k) = \sum_{\gamma \in \mathbb{Z}^N} f(y + 2\pi\gamma) e^{-ik\cdot(y+2\pi\gamma)}, \qquad k \in Y'.$$

This has been well exploited in [5] in our proposal of Bloch approximation in the entire space. It is therefore natural to seek the corresponding transform which takes into account the presence of boundary, study its properties and apply them to homogenization problems. To this end, our idea is to restrict suitably the translations used in the above expression. For instance, in the case of half-space, translations will be chosen to be tangential to the boundary, more precisely, in $\mathbb{Z}^N_+$. This will produce an object of interest in the half-space. As already demonstrated by our earlier work [5] in the case of the entire space, above efforts will throw more light in the homogenization process in domains with boundary. In particular, it will lead to a better approximation of the solution. We plan to carry out this analysis in a future publication.

Acknowledgements

This research was done thanks to a visit of R. Orive and M. Vanninathan to the Center of Mathematical Modelling and to the Department of Mathematical Engineering of the University of Chile. These authors thank warmly these Institutions. R. Orive was supported by the Fondap's Programme on Mathematical-Mechanics and the grant BFM2002-03345 of the MCYT (Spain). M. Vanninathan was supported by the Fondap's Programme on Mathematical-Mechanics and"Marcel Dassault" Chair.

The authors would like to thank the anonymous referee whose comments allowed us not only to improve the presentation of our results but also to strengthen some of them in the earlier version of the article.

References

[1] A. Bensoussan, J. L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam, 1978.

[2] H. Brezis, *Analyse Fonctionnelle, Théorie et Applications*, Collection Mathématiques Appliquées pour la Maîtrise, Masson, Paris, 1983.

[3] C. Conca and S. Natesan, *Numerical methods for elliptic partial differential equations with rapidly oscillating coefficients*, Comput. Methods Appl. Mech. Engrg., 192 (2003), pp. 47–76.

[4] C. Conca, S. Natesan and M. Vanninathan, *Numerical solution of elliptic partial differential equations by Bloch waves method*, in XVII CEDYA: Congress on Differential Equations and Applications/VII CMA: Congress on Applied Mathematics, Dep. Mat. Apl., Univ. Salamanca, Salamanca, 2001, pp. 63–83.

[5] C. Conca, R. Orive and M. Vanninathan, *Bloch approximation in homogenization and applications*, SIAM J. Math. Anal. **33** (2002), pp. 1166–1198.

[6] C. Conca, J. Planchard and M. Vanninathan, *Fluids and Periodic Structures*, no. **38** in Research in Applied Mathematics, J. Wiley/Masson, New York/Paris, 1995.

[7] C. Conca and M. Vanninathan, *Homogenization of periodic structures via Bloch decomposition*, SIAM J. Appl. Math. **57** (1997), pp. 1639–1659.

[8] D. Gilberg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1977.

[9] F. Santosa and W. W. Symes, *A dispersive effective medium for wave propagation in periodic composites*, SIAM J. Appl. Math. **51** (1991), pp. 984–1005.