Concentrating solutions in a two-dimensional elliptic problem with exponential Neumann data

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Abstract

We consider the elliptic equation $-\Delta u+u=0$ in a bounded, smooth domain Ω in $\mathbb{R}^2$ subject to the nonlinear Neumann boundary condition $\frac{\partial u}{\partial \nu}=\varepsilon e^u$. Here $\varepsilon>0$ is a small parameter. We prove that any family of solutions u_ε for which $\varepsilon\int_{\partial\Omega} e^u$ is bounded, develops up to subsequences a finite number m of peaks $\xi_i\in\partial\Omega$, in the sense that $\varepsilon e^u \rightharpoonup 2\pi\sum_{k=1}^m \delta_{\xi_i}$ as $\varepsilon\to 0$. Reciprocally, we establish that at least two such families indeed exist for any given $m\geqslant 1$.

Keywords: Nonlinear boundary value problem; Concentrating solutions; Liouville-type equation

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. This paper deals with the analysis of solutions of the boundary value problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \varepsilon e^u & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where ν denotes outer unit normal vector to $\partial\Omega$ and $\varepsilon > 0$ is a small parameter.

Elliptic equations with this type of nonlinear Neumann boundary condition arise in conformal geometry (prescribing Gaussian curvature of the domain and curvature of the boundary), see for instance [8] and references therein, and in corrosion modelling, see [3,6,9]. The Trudinger–Moser and trace inequalities imply the validity of the Sobolev–Orlicz (compact) trace embedding

$$\sup_{u\in H^1(\Omega)\setminus\{0\}} \int_{\partial\Omega} \exp\left(\frac{u}{\|u\|_{H^1}}\right) < +\infty, \tag{2}$$

see [6] for a proof. Note that an extremal of this inequality with $\|u\|_{H^1(\Omega)} = 1$ solves (1) for certain $\varepsilon > 0$. Given a fixed value of the parameter ε, solutions of (1) correspond precisely to critical points in $H^1(\Omega)$ of the free energy functional

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 - \varepsilon \int_{\partial\Omega} e^u. \tag{3}$$

The maximum principle implies that solutions of (1) are automatically positive. Smallness of ε is necessary for existence of a solution as integration against a suitable test function shows. On the other hand, inequality (2) implies that a (unique) local minimizer exists near zero, provided that $\varepsilon > 0$ is sufficiently small. This minimizer represents a “small” solution of Problem (1). The functional is not bounded below, thus suggesting the presence of a second, large solution for $\varepsilon > 0$ small. Compactness of the trace embedding yields the sufficient PS condition for this second solution to exist thanks to the standard mountain pass theorem. In [3,6,9], the following related problem was analyzed:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \varepsilon \sinh u & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Evenness of the associated energy functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \varepsilon \int_{\partial\Omega} \cosh u$$

and the above-mentioned compactness makes Ljusternik–Schnirelmann theory applicable to find that actually infinitely many solutions exist associated to critical values

$$c_1^\varepsilon \leqslant c_2^\varepsilon \leqslant \cdots \leqslant c_k^\varepsilon \leqslant \cdots .$$

For each fixed k it turns out that c_k^ε is bounded above by $C_k \log \frac{1}{\varepsilon}$. This is shown to imply that $\varepsilon \int_{\partial\Omega} \cosh u$ is uniformly bounded for the associated solutions as $\varepsilon \to 0$. Applying similar arguments as those in [6] one can show that the mountain pass solution of (1) has a similar property, namely that $\varepsilon \int_{\partial\Omega} e^{u_\varepsilon}$ remains bounded. Our first result characterizes the asymptotic behavior of families of solutions u_ε with $\varepsilon \int_{\partial\Omega} e^{u_\varepsilon}$ bounded. It turns out that, up to subsequences, there is an integer m, such that $\varepsilon \int_{\partial\Omega} e^{u_\varepsilon} \to 2m\pi$. More precisely, $\varepsilon e^{u_\varepsilon}$ approaches the sum of m Dirac masses at the boundary. The location of these possible points of concentration may be further characterized as critical points of a functional of m points of the boundary which we introduce next: let us consider the Green's function for the Neumann problem

$$\begin{cases} -\Delta_x G(x,y) + G(x,y) = 0, & x \in \Omega, \\ \dfrac{\partial G}{\partial \nu_x}(x,y) = 2\pi \delta_y(x), & x \in \partial\Omega \end{cases} \tag{5}$$

and its regular part

$$H(x,y) = G(x,y) - \log \frac{1}{|x-y|^2}. \tag{6}$$

We define φ_m on $(\partial\Omega)^m$ by

$$\varphi_m(\xi_1, \ldots, \xi_m) = -\left[\sum_{j=1}^m H(\xi_j, \xi_j) + \sum_{i \neq j} G(\xi_i, \xi_j) \right].$$

Theorem 1.1. *Let u_ε be a family of solutions to* (1) *with $\varepsilon \to 0$. If $\varepsilon \int_{\partial\Omega} e^{u_\varepsilon} \leqslant C$ for some constant C independent of ε there exists a subsequence* (*denoted the same way*) *and a finite collection of distinct points $\xi_i \in \partial\Omega$ $i = 1, \ldots, m$ such that*

$$u_\varepsilon \to u^*,$$

where u^* is the solution to

$$\begin{cases} -\Delta u^* + u^* = 0 & \text{in } \Omega, \\ \dfrac{\partial u^*}{\partial \nu} = 2\pi \displaystyle\sum_{i=1}^m \delta_{\xi_i} & \text{on } \partial\Omega. \end{cases}$$

Moreover

$$\varepsilon \int_{\partial\Omega} e^{u_\varepsilon} \to 2\pi m \quad \text{and} \quad \varepsilon e^{u_\varepsilon} \rightharpoonup 2\pi \sum_{i=1}^{m} \delta_{\xi_i},$$

weakly in the sense of Radon measures in $\partial\Omega$, $u_\varepsilon \to u^*$ *in* $L^p(\partial\Omega)$ *and* $L^p(\Omega)$ *for all* $1 \leqslant p < \infty$ *and in* $C^1_{\text{loc}}(\overline{\Omega} - \{\xi_1, \dots, \xi_m\})$. *Additionally* $(\xi_1, \dots, \xi_m)$ *is a critical point of* φ_m, *that is for all* $k = 1, \dots, m$

$$\nabla_{\tau(\xi_k)} H(\xi_k, \xi_k) + \sum_{i \neq k} \nabla_{\tau(\xi_k)} G(\xi_k, \xi_i) = 0, \tag{7}$$

where $\tau(\xi_k)$ *is a tangent vector to* $\partial\Omega$ *at* ξ_k.

We remark that if $\varepsilon \int_{\partial\Omega} e^{u_\varepsilon}$ is unbounded after extracting a subsequence for which $\varepsilon \int_{\partial\Omega} e^{u_\varepsilon} \to \infty$ we have $u_\varepsilon \nearrow \infty$ uniformly in $\overline{\Omega}$.

A natural question is whether families of solutions such as those described in the previous theorem do indeed exist. It can be shown that the mountain pass large solution does correspond to one exhibiting a single spike. However, it is not clear how to set up a Ljusternik–Schnirelmann scheme that predicts the existence of higher-energy solutions, in particular since the functional (3) does not seem to exhibit any useful symmetries. In this paper, we develop a completely different approach to this question which allows us to prove an existence result, which we suspect optimal: given any integer $m \geqslant 1$, there are at least *two* distinct families of solutions u_ε for which $\varepsilon \int_{\partial\Omega} e^{u_\varepsilon} \to 2m\pi$.

Theorem 1.2. *Let* $m \geqslant 1$. *Then for* $\varepsilon > 0$ *sufficiently small there exist two solutions* u_ε *to* (1) *satisfying*

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\partial\Omega} e^{u_\varepsilon} = 2\pi m.$$

The peaks of these two solutions are located near points $\xi_1, \dots, \xi_m \in \partial\Omega$ corresponding to two distinct critical points of φ_m.

We can actually show stronger versions of this result. For instance, if $\partial\Omega$ has more than one component, then pairs of families of m-peak solutions on each component happen to exist. In reality, associated to each *topologically nontrivial* critical point situation associated to φ_m (for instance local maxima or saddle points possibly degenerate), a solution with concentration peaks at a corresponding critical point exists. We elaborate further on these issues at the end of Section 8.

It is important to remark the interesting analogy between these results and those known for the Liouville-type equation

$$\begin{cases} \Delta u + \varepsilon e^u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

Asymptotic behavior of families u_ε of solutions of (8) for which $\varepsilon \int_\Omega e^u_\varepsilon$ remains uniformly bounded is well understood after the works [2,7,10]. It is known that up to subsequences,

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega e^{u_\varepsilon} = 8m\pi \tag{9}$$

for some integer $m \geqslant 1$. More precisely, $\varepsilon e^{u_\varepsilon}$ peaks up as m Dirac masses at points of the domain which correspond to a critical point of a functional similar to φ_m defined from Green's function of $-\Delta$ under Dirichlet boundary condition. The reciprocal question of existence has been addressed among other works in [1,4,5]. In particular in [4], a result is established which may be thought of as an analog of Theorem 2: if the domain is not simply connected, then at least one solution with m peaks exists.

In the rest of this paper we will prove Theorems 1 and 2. Sections 2–8 are devoted to the proof of Theorem 2. Scaling out properly ε around a single point of the boundary leads us formally to the limiting problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial v}{\partial \nu} = e^v & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{10}$$

where $\mathbb{R}^2_+$ denotes the upper half-plane $\{(x_1, x_2) : x_2 > 0\}$ and ν the unit exterior normal to $\partial\mathbb{R}^2_+$.

A family of solutions to (10) is given by

$$w_{\mu,t}(x_1, x_2) = \log \frac{2\mu}{(x_1 - t)^2 + (x_2 + \mu)^2}, \tag{11}$$

where $t \in \mathbb{R}$ and $\mu > 0$ are parameters. It is interesting to point out that the results in [8,11,12] imply that any solution v of (10) which satisfies additionally

$$\int_{\mathbb{R}^2_+} e^v < +\infty$$

must be of the form (11). The solutions predicted in Theorem 2 are constructed using as building blocks these solutions, suitably scaled and projected to make it up to a good order for the boundary condition. Solutions are found as a small additive perturbation of these initial approximations. A linearization procedure leads to a finite-dimensional reduction, where the reduced problem corresponds to that of adjusting variationally the location of the concentration points.

Theorem 1 is established in Section 9. The point concentration behavior of the family is established first, then Pohozaev-type identities in balls around the singularities lead to the desired result.

2. An equation in the upper half-plane

The family of solutions (11) is invariant under translations in the x_1-direction and under the scaling $s \mapsto w(sx) + 2\log s$, $s > 0$. An important property that we will need is the nondegeneracy of these solutions (11) except for the above natural invariances of Eq. (10). Let us define

$$z_0 = \frac{1}{\mu} - 2\,\frac{x_2 + \mu}{x_1^2 + (x_2 + \mu)^2} \tag{12}$$

and

$$z_1 = -2\frac{x_1}{x_1^2 + (x_2 + \mu)^2}. \tag{13}$$

We have the following:

Proposition 2.1. *Any bounded solution of*

$$\begin{cases} \Delta\phi = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial\phi}{\partial\nu} - \dfrac{2\mu}{x_1^2 + \mu^2}\phi = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{14}$$

is a linear combination of z_0 *and* z_1.

Proof. Let ϕ be a solution to (14) and set

$$w(y) = \phi\left(\frac{y}{|y|^2} - (0, \mu)\right).$$

The function w is just the Kelvin transform of ϕ about the point $(0, -\mu)$. The domain of w is the disk $D = B((0, \frac{1}{2\mu}), \frac{1}{2\mu})$ and w is a bounded function that satisfies $\Delta w = 0$ in D,

$$\frac{\partial w}{\partial\nu'} = 2\mu w \quad \text{on } \partial D\backslash\{0\}, \tag{15}$$

where ν' is the exterior unit normal to D. To see this observe that the map $y \mapsto K(y) = \frac{y}{|y|^2} - (0, \mu)$ is anti-conformal (preserves angles and reverses orientation) and maps the normal vector to D to a normal vector to $\partial \mathbb{R}^2_+$. More precisely, if ν' is the exterior unit normal vector to D then

$$\frac{\partial w}{\partial \nu'} = \frac{1}{|y|^2} \frac{\partial \phi}{\partial \nu}.$$

Thus on ∂D

$$\frac{\partial w}{\partial \nu'} = \frac{1}{|y|^2} e^{w_{\mu,0}(K(y))} w$$

and a calculation shows that

$$\frac{1}{|y|^2} e^{w_{\mu,0}(K(y))} = \frac{1}{|y|^2} \frac{2\mu}{\frac{y_1^2}{|y|^4} + \mu^2} = 2\mu.$$

Since w is bounded, by elliptic regularity (15) holds in all ∂D.

By translating in the y_2 direction we can assume that D is the disk centered at the origin with radius $\frac{1}{2\mu}$. We think of w as the real part of an analytic function $\tilde{w}$ and write

$$\tilde{w}(y) = \sum_{k=0}^{\infty} a_k r^k e^{ik\theta}$$

with $y = re^{i\theta}$. Condition (15) is equivalent to

$$\mathrm{Re}\left(\sum_{k=0}^{\infty} a_k(k-1)e^{ik\theta}\right) = 0 \quad \forall \theta$$

and hence $a_0 = 0$, $a_k = 0$ for all $k > 1$. Looking at the real part w of $\tilde{w}$, and recalling that we shifted in the y_2 direction we see that it is a linear combination of

$$y_1 = \frac{x_1}{x_1^2 + (x_2+\mu)^2} \quad \text{and} \quad y_2 - \frac{1}{2\mu} = \frac{x_2+\mu}{x_1^2 + (x_2+\mu)^2} - \frac{1}{2\mu}. \qquad \square$$

3. Ansatz for the solution

We can produce a solution to

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial \nu} = \varepsilon e^u & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

by taking

$$u(x) = w_\mu(x/\varepsilon) - 2\log\varepsilon = \log\frac{2\mu}{x_1^2 + (x_2 + \varepsilon\mu)^2}.$$

Based on this, given $\xi_j \in \partial\Omega$, $\mu_j > 0$ we define

$$u_j(x) = \log\frac{2\mu_j}{|x - \xi_j - \varepsilon\mu_j \nu(\xi_j)|^2}.$$

The choice of ξ_j and μ_j will be made later on.

The ansatz is

$$U(x) = \sum_{j=1}^{m} u_j(x) + H_j^\varepsilon(x), \tag{16}$$

where H_j^ε is a correction term defined as the solution of

$$\begin{cases} -\Delta H_j^\varepsilon + H_j^\varepsilon = -u_j & \text{in } \Omega, \\ \dfrac{\partial H_j^\varepsilon}{\partial \nu} = \varepsilon e^{u_j} - \dfrac{\partial u_j}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Lemma 3.1. *For any* $0 < \alpha < 1$

$$H_j^\varepsilon(x) = H(x, \xi_j) - \log 2\mu_j + O(\varepsilon^\alpha) \tag{18}$$

uniformly in $\overline{\Omega}$, *where H is the regular part of Green's function defined* (6).

We will give the proof of this lemma at the end of the section.

It will be convenient to work with the scaling of u given by

$$v(y) = u(\varepsilon y) + 2\log\varepsilon.$$

If u is a solution of (1) then v satisfies

$$\begin{cases} -\Delta v + \varepsilon^2 v = 2\varepsilon^2 \log \varepsilon & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial v}{\partial \nu} = e^v & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{19}$$

where $\Omega_\varepsilon = \Omega/\varepsilon$. With this scaling u_j becomes

$$v_j(y) = \log \frac{2\mu_j}{|y - \xi_j' - \mu_j \nu(\xi_j')|^2},$$

where $\xi_j' = \xi_j/\varepsilon$ and where we will write ν for the exterior normal unit vector to $\partial\Omega$ and $\partial\Omega_\varepsilon$.

We will seek a solution v of (19) of the form

$$v = V + \phi,$$

where

$$V(y) = U(\varepsilon y) + 2\log\varepsilon \tag{20}$$

and U is defined by (16). Problem (19) can be stated as to find ϕ a solution to

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial \phi}{\partial \nu} = e^V\phi + N(\phi) + R & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{21}$$

where the “nonlinear term” is

$$N(\phi) = e^V(e^\phi - 1 - \phi) \tag{22}$$

and the “error term” is given by

$$R = e^V - \frac{\partial V}{\partial \nu}. \tag{23}$$

At this point it is convenient to make a choice of the parameters μ_j, the objective being to make the error term small. We claim that if

$$\log 2\mu_j = H(\xi_j, \xi_j) + \sum_{i \neq j} G(\xi_i, \xi_j), \tag{24}$$

then we achieve the following behavior for R: for any $0 < \alpha < 1$ there exists C independent of ε, such that

$$|R(y)| \leqslant C\varepsilon^{\alpha} \sum_{j=1}^{m} \frac{1}{1+|y-\xi_j'|} \quad \forall y \in \Omega_\varepsilon \tag{25}$$

and for $W = e^V$

$$W(y) = \sum_{j=1}^{m} \frac{2\mu_j}{|y-\xi_j'-\mu_j \nu(\xi_j')|^2}(1+\theta_\varepsilon(y)) \quad \forall y \in \Omega_\varepsilon \tag{26}$$

with θ_ε satisfying the following estimate:

$$|\theta_\varepsilon(y)| \leqslant C\varepsilon^{\alpha} + C\varepsilon \sum_{j=1}^{m} |y-\xi_j'| \quad \forall y \in \Omega_\varepsilon.$$

Proof of (26).

$$W(y) = \varepsilon^2 \exp\left(\sum_{i=1}^{m} u_i(\varepsilon y) + H_i^{\varepsilon}(\varepsilon y)\right)$$

$$= \varepsilon^2 \exp\left(\sum_{i=1}^{m}\left(\log \frac{2\mu_i}{\varepsilon^2|y-\xi_i'-\mu_i\nu(\xi_i')|^2} + H_i^{\varepsilon}(\varepsilon y)\right)\right).$$

Let us fix a small constant $\delta > 0$ and consider this expression for $|y-\xi_j'| < \frac{\delta}{\varepsilon}$

$$W(y) = \frac{2\mu_j}{|y-\xi_j'-\mu_j\nu(\xi_j')|^2} \exp\left(H_j^{\varepsilon}(\varepsilon y) + \sum_{i\neq j}^{m}\left[\log\frac{2\mu_i}{\varepsilon^2|y-\xi_i'-\mu_i\nu(\xi_i')|^2}\right.\right.$$

$$\left.\left. + H_i^{\varepsilon}(\varepsilon y)\right]\right).$$

Using (18) and the fact that H is $C^1(\partial\Omega^2)$ we have

$$H_i^{\varepsilon}(\varepsilon y) = H(\varepsilon y, \xi_i) - \log(2\mu_i) + O(\varepsilon^{\alpha}) \qquad \forall y \in \Omega_\varepsilon$$

$$= H(\xi_j, \xi_i) - \log(2\mu_i) + O(\varepsilon^{\alpha}) + O(\varepsilon|y-\xi_j'|) \qquad \forall y \in \Omega_\varepsilon.$$

Hence for $|y-\xi_j'|<\frac{\delta}{\varepsilon}$

$$
\begin{aligned}
&H_j^\varepsilon(\varepsilon y)+\sum_{i\neq j}^m\left(\log\frac{2\mu_i}{\varepsilon^2|y-\xi_i'-\mu_i\nu(\xi_i')|^2}+H_i^\varepsilon(\varepsilon y)\right)\\
&\quad=H(\xi_j,\xi_j)-\log(2\mu_j)+\sum_{i\neq j}^m\left(\log\frac{2\mu_i}{|\xi_j-\xi_i|^2}+H(\xi_j,\xi_i)-\log(2\mu_i)\right)\\
&\qquad+O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_j'|),\\
&\quad=H(\xi_j,\xi_j)-\log(2\mu_j)+\sum_{i\neq j}^m G(\xi_j,\xi_i)+O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_j'|)\\
&\quad=O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_j'|)
\end{aligned}
$$

by the choice of μ_j, cf. (24). Therefore

$$
W(y)=\frac{2\mu_j}{|y-\xi_j'-\mu_j\nu(\xi_j')|^2}(1+O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_j'|))\qquad\forall|y-\xi_j'|<\frac{\delta}{\varepsilon}. \tag{27}
$$

If $|y-\xi_j'|>\frac{\delta}{\varepsilon}$ for all $j=1,\ldots,m$ we have $W=O(\varepsilon^2)$, and this together with (27) implies (26). $\square$

Proof of (25). We defined $R=e^V-\frac{\partial V}{\partial\nu}$ with V given by (20). We need to compute $\frac{\partial V}{\partial\nu}=\varepsilon\frac{\partial U}{\partial\nu}$. But

$$
\frac{\partial U}{\partial\nu}=\sum_{i=1}^m\frac{\partial u_i}{\partial\nu}+\frac{\partial H_i^\varepsilon}{\partial\nu}=\varepsilon\sum_{i=1}^m e^{u_i}=\varepsilon\sum_{i=1}^m\frac{2\mu_i}{|x-\xi_i-\varepsilon\mu_i\nu(\xi_i)|^2}.
$$

Hence

$$
\frac{\partial V}{\partial\nu}(y)=\varepsilon\frac{\partial U}{\partial\nu}(\varepsilon y)=\sum_{i=1}^m\frac{2\mu_i}{|y-\xi_i'-\mu_i\nu(\xi_i')|^2}.
$$

Thus, near ξ_j' by the above computation and (27) we obtain

$$
R(y)=e^V-\frac{\partial V}{\partial\nu}=\frac{2\mu_j}{|y-\xi_j'-\mu_j\nu(\xi_j')|^2}(O(\varepsilon^\alpha)+O(\varepsilon|y-\xi_j'|)),\qquad|y-\xi_j'|<\frac{\delta}{\varepsilon}.
$$

If $|y-\xi_j'|>\frac{\delta}{\varepsilon}$ for all $j=1,\dots,m$ then $e^V=O(\varepsilon^2)$ and $\frac{\partial V}{\partial\nu}=O(\varepsilon^2)$ and (25) follows. $\square$

Proof of Lemma 3.1. The boundary condition satisfied by H_j^ε is

$$\frac{\partial H_j^\varepsilon}{\partial\nu}=\varepsilon e^{u_j}-\frac{\partial u_j}{\partial\nu}=2\frac{\varepsilon\mu_j+(x-\xi_j-\varepsilon\mu_j\nu(\xi_j))\cdot\nu(x)}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}$$
$$=2\varepsilon\,\mu_j\frac{1-\nu(\xi_j)\cdot\nu(x)}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}+2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}.$$

Thus

$$\lim_{\varepsilon\to0}\frac{\partial H_j^\varepsilon}{\partial\nu}(x)=2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}\qquad\forall x\neq\xi_j. \tag{28}$$

The regular part of Green's function satisfies

$$\begin{cases}-\Delta_x H(x,y)+H(x,y)=-\log\dfrac{1}{|x-y|^2}, & x\in\Omega,\\ \dfrac{\partial H}{\partial\nu_x}(x,y)=2\dfrac{(x-y)\cdot\nu(x)}{|x-y|^2}, & x\in\partial\Omega.\end{cases}$$

For the difference $z_\varepsilon(x)=H_j^\varepsilon(x)+\log 2\mu_j-H(x,\xi_j)$ we have

$$\begin{cases}-\Delta z_\varepsilon+z_\varepsilon=-\log\dfrac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}+\log\dfrac{1}{|x-\xi_j|^2} & \text{in }\Omega,\\ \dfrac{\partial z_\varepsilon}{\partial\nu}=\dfrac{\partial H_j^\varepsilon}{\partial\nu}-2\dfrac{(x-y)\cdot\nu(x)}{|x-y|^2} & \text{on }\partial\Omega.\end{cases}$$

We claim that for any $p>1$ there exists $C>0$ such that

$$\left\|\frac{\partial H_j^\varepsilon}{\partial\nu}-2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}\right\|_{L^p(\partial\Omega)}\leqslant C\varepsilon^{1/p}. \tag{29}$$

For this it will be convenient to observe first that

$$|1-\nu(\xi_j)\cdot\nu(x)|\leqslant C|x-\xi_j|^2,\ \ |(x-\xi_j)\cdot\nu(x)|\leqslant C|x-\xi_j|^2\qquad\forall x\in\partial\Omega, \tag{30}$$

which can be proved, for example, assuming that $\xi_j = 0$ and that near the origin $\partial\Omega$ is the graph of a function $G : (-a, a) \to \mathbb{R}$ with $G(0) = G'(0) = 0$. Now

$$\frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2} = 2\varepsilon\mu_j \frac{1-\nu(\xi_j)\cdot\nu(x)}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2} + 2\varepsilon\mu_j \frac{(x-\xi_j)\cdot\nu(x)(2(x-\xi_j)\cdot\nu(\xi_j)-\varepsilon\mu_j)}{|x-\xi_j|^2|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}.$$

By (30)

$$\left|\frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}\right| \leqslant C\varepsilon + C\frac{\varepsilon|2(x-\xi_j)\cdot\nu(\xi_j)-\varepsilon\mu_j|}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}. \tag{31}$$

Fix $\rho > 0$ small. Then

$$\left|\frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}\right| \leqslant C\varepsilon \qquad \forall |x-\xi_j| \geqslant \rho,\ x \in \partial\Omega. \tag{32}$$

Now let $p > 1$. Changing variables $x - \xi_j = \varepsilon y$ we have

$$\begin{aligned}\int_{B_\rho(\xi_j)\cap\partial\Omega} \left|\frac{\varepsilon|2(x-\xi_j)\cdot\nu(\xi_j)-\varepsilon\mu_j|}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}\right|^p dx &= C\varepsilon \int_{B_{\rho/\varepsilon}(0)\cap\partial\Omega_\varepsilon} \left|\frac{2y\cdot\nu(0)-\mu_j}{|y-\mu_j\nu(0)|^2}\right|^p dy \\ &\leqslant C\varepsilon \int_0^{\rho/\varepsilon} \frac{1}{(1+s)^p}\, ds \\ &\leqslant C\varepsilon.\end{aligned}$$

Combining this with (31) and (32) we conclude that (29) holds.

For $p > 1$ let us estimate now

$$\begin{aligned}&\left\|\log\frac{1}{|x-\xi_j|^2} - \log\frac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}\right\|_{L^p(\Omega)}^p \\ &= \int_{B_{10\varepsilon\mu_j}(\xi_j)\cap\Omega} \cdots + \int_{\Omega\setminus B_{10\varepsilon\mu_j}(\xi_j)} \cdots = I_1 + I_2.\end{aligned}$$

For I_1 observe that

$$\int_{B_{10\varepsilon\mu_j}(\xi_j)\cap\Omega} \left|\log\frac{1}{|x-\xi_j|^2}\right|^p dx \leqslant C\int_0^{C\varepsilon} |\log r|^p r\, dr \leqslant C\varepsilon^2 \left(\log\frac{1}{\varepsilon}\right)^p.$$

The same bound is true for the integral of $|\log \frac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}|^p$ in $B_{10\varepsilon\mu_j}(\xi_j)\cap\Omega$. Hence

$$|I_1|\leqslant C\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^p.$$

Let us estimate I_2 as follows:

$$\left|\log\frac{1}{|x-\xi_j|^2}-\log\frac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}\right|\leqslant\sup_{0\leqslant t\leqslant 1}\frac{C\varepsilon}{|x-\xi_j-t\varepsilon\mu_j\nu(\xi_j)|}.$$

But if $|x-\xi_j|\geqslant 10\varepsilon\mu_j$ then $|x-\xi_j|\leqslant C|x-\xi_j-t\varepsilon\mu_j\nu(\xi_j)|$ for any $t\in[0,1]$ as can be seen from $|x-\xi_j|\leqslant|x-\xi_j-t\varepsilon\mu_j\nu(\xi_j)|+\mu_j\varepsilon\leqslant|x-\xi_j-t\varepsilon\mu_j\nu(\xi_j)|+\frac{1}{10}|x-\xi_j|$. Thus

$$\left|\log\frac{1}{|x-\xi_j|^2}-\log\frac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}\right|\leqslant\frac{C\varepsilon}{|x-\xi_j|}.$$

Take $1<p<2$ and integrate

$$|I_2|\leqslant C\varepsilon^p\int_{10\mu\varepsilon}^{D} r^{1-p}\,dr\leqslant C\varepsilon^p,$$

where D is the diameter of Ω. In conclusion, for any $1<p<2$ we have

$$\left\|\log\frac{1}{|x-\xi_j|^2}-\log\frac{1}{|x-\xi_j-\varepsilon\mu_j\nu(\xi_j)|^2}\right\|_{L^p(\Omega)}\leqslant C\varepsilon.$$

By L^p theory

$$\|z_\varepsilon\|_{W^{1+s,p}(\Omega)}\leqslant C\left(\left\|\frac{\partial z_\varepsilon}{\partial\nu}\right\|_{L^p(\partial\Omega)}+\|\Delta z_\varepsilon\|_{L^p(\Omega)}\right)\leqslant C\varepsilon^{1/p}$$

for any $0<s<\frac{1}{p}$. By the Morrey embedding we obtain

$$\|z_\varepsilon\|_{C^\gamma(\overline{\Omega})}\leqslant C\varepsilon^{1/p}$$

for any $0<\gamma<\frac{1}{2}+\frac{1}{p}$. This proves the result (with $\alpha=\frac{1}{p}$). □

Remark. The convergence (28) is not uniform in general because $\frac{\partial H_j^\varepsilon}{\partial \nu}(\xi_j) = 0$ while the function $x \mapsto 2\frac{(x-\xi_j)\cdot\nu(x)}{|x-\xi_j|^2}$ can be extended continuously to ξ_j with a value equal to the curvature of $\partial\Omega$ at ξ_j.

4. Solvability of a linear equation

The main result of this section is the solvability of the following linear problem: given h find ϕ, $c_1, \dots, c_m$, such that

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\phi}{\partial\nu} - W\phi = h + \displaystyle\sum_{j=1}^m c_j\chi_j Z_{1j} & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{1j}\phi = 0 & \forall j = 1,\dots,m, \end{cases} \tag{33}$$

where W is a function on $\partial\Omega_\varepsilon$ that satisfies (26), $h \in L^\infty(\partial\Omega_\varepsilon)$ and Z_{1j}, χ_j are defined as follows: let z_{ij} denote the functions z_0, z_1 defined in (12) and (13) with parameter $\mu = \mu_j$ $(i = 0, 1 \;\; j = 1, \dots, m)$

$$z_{0j} = \frac{1}{\mu_j} - 2\frac{x_2 + \mu_j}{x_1^2 + (x_2 + \mu_j)^2}, \qquad z_{1j} = -2\frac{x_1}{x_1^2 + (x_2 + \mu_j)^2}.$$

Around each point $\xi_j' \in \partial\Omega_\varepsilon$ we consider a smooth change of variables

$$F_j^\varepsilon(y) = \frac{1}{\varepsilon}F_j(\varepsilon y), \tag{34}$$

where $F_j : B_\rho(\xi_j) \to M$ is a diffeomorphism and M an open neighborhood of the origin such that $F(\Omega \cap B_\rho(\xi_j)) = \mathbb{R}^2_+ \cap M$, $F(\partial\Omega \cap B_\rho(\xi_j)) = \partial\mathbb{R}^2_+ \cap M$. We can select F_j so that it preserves area. Define

$$Z_{ij}(y) = z_{ij}(F_j^\varepsilon(y)), \qquad i = 0, 1, \quad j = 1, \dots, m.$$

Next, we choose a large but fixed number R_0 and nonnegative smooth function $\chi : \mathbb{R} \to \mathbb{R}$ so that $\chi(r) = 1$ for $r \leqslant R_0$ and $\chi(r) = 0$ for $r \geqslant R_0 + 1$, $0 \leqslant \chi \leqslant 1$. Then set

$$\chi_j(y) = \chi(|F_j^\varepsilon(y)|).$$

All functions above depend on ε but we omit this dependence in the notation.

Eq. (33) will be solved for $h \in L^\infty(\partial\Omega_\varepsilon)$, but we will be able to estimate the size of the solution in terms of the following norm:

$$\|h\|_{*,\partial\Omega_\varepsilon} = \sup_{y\in\partial\Omega_\varepsilon} \frac{|h(y)|}{\varepsilon + \sum_{j=1}^m (1+|y-\xi_j'|)^{-1-\sigma}}, \tag{35}$$

where we fix $0<\sigma<1$ although the precise choice will be made later on.

Proposition 4.1. *Let $d>0$ and m a positive integer. Then there exist $\varepsilon_0>0$, C such that for any $0<\varepsilon<\varepsilon_0$, any family of points $\xi_1,\dots,\xi_m \in \partial\Omega$ with*

$$|\xi_i - \xi_j| \geqslant d \quad \forall i \neq j \tag{36}$$

and any $h \in L^\infty(\partial\Omega_\varepsilon)$ there is a unique solution $\phi \in L^\infty(\Omega_\varepsilon)$, $c_1,\dots,c_m \in \mathbb{R}$ to (33). *Moreover*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C \log\frac{1}{\varepsilon} \|h\|_{*,\partial\Omega_\varepsilon}.$$

To prove this result we shall study first the linear equation

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi = f & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\phi}{\partial\nu} - W\phi = h & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{37}$$

where W satisfies (26) and f, h are in suitable weighted spaces: we consider for h the norm defined in (35) and for f

$$\|f\|_{**,\Omega_\varepsilon} = \sup_{y\in\Omega_\varepsilon} \frac{|f(y)|}{\varepsilon^2 + \sum_{j=1}^m (1+|y-\xi_j'|)^{-2-\sigma}}.$$

We begin by stating an a priori estimate for solutions of (37) satisfying orthogonality conditions with respect to Z_{0j} and Z_{1j}.

Lemma 4.2. *There are $R_0>0$ and $\varepsilon_0>0$ so that for $0<\varepsilon<\varepsilon_0$ and any solution ϕ of* (37) *with the orthogonality conditions*

$$\int_{\Omega_\varepsilon} Z_{ij}\chi_j\,\phi = 0 \qquad \forall i=0,1 \quad \forall j=1,\dots,m, \tag{38}$$

we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C(\|h\|_{*,\partial\Omega_\varepsilon} + \|f\|_{**,\Omega_\varepsilon}),$$

where C is independent of ε.

The idea behind this estimate comes from looking at (37) with $f \equiv 0$, $h \equiv 0$ as $\varepsilon \to 0$ at a fixed distance from one of the points, say ξ'_j. After a translation and a rotation so that Ω_ε converges to the upper half-plane $\mathbb{R}^2_+$ and ξ'_j is located at the origin this equation approaches precisely (14).

For the proof of this lemma we need to construct a suitable barrier.

Lemma 4.3. *For $\varepsilon > 0$ small enough there exist $R_1 > 0$, and*

$$\psi : \Omega_\varepsilon \Big\backslash \bigcup_{j=1}^{m} B_{R_1}(\xi'_j) \to \mathbb{R}$$

smooth and positive so that

$$\begin{aligned}
&-\Delta\psi + \varepsilon^2\psi \geqslant \sum_{j=1}^{m} \frac{1}{|y-\xi'_j|^{2+\sigma}} + \varepsilon^2 && \text{in } \Omega_\varepsilon \Big\backslash \bigcup_{j=1}^{m} B_{R_1}(\xi'_j),\\
&\frac{\partial\psi}{\partial\nu} - W\psi \geqslant \sum_{j=1}^{m} \frac{1}{|y-\xi'_j|^{1+\sigma}} + \varepsilon && \text{on } \partial\Omega_\varepsilon \Big\backslash \bigcup_{j=1}^{m} B_{R_1}(\xi'_j),\\
&\psi > 0 && \text{in } \Omega_\varepsilon \Big\backslash \bigcup_{j=1}^{m} B_{R_1}(\xi'_j),\\
&\psi \geqslant 1 && \text{on } \Omega_\varepsilon \cap \left(\bigcup_{j=1}^{m} \partial B_{R_1}(\xi'_j)\right).
\end{aligned}$$

The constants $R_1 > 0$, $c > 0$ can be chosen independently of ε and ψ is bounded uniformly

$$0 < \psi \leqslant C \quad \text{in } \Omega_\varepsilon \Big\backslash \bigcup_{j=1}^{m} B_{R_1}(\xi'_j) .$$

Proof of Lemma 4.2. We take $R_0 = 2R_1$, R_1 being the constant of Lemma 4.3. Thanks to the barrier ψ of that lemma we deduce that the following maximum principle holds in $\Omega_\varepsilon \setminus \cup_{j=1}^m B_{R_1}(\xi_j')$: if $\phi \in H^1(\Omega_\varepsilon \setminus \cup_{j=1}^m B_{R_1}(\xi_j'))$ satisfies:

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi \geqslant 0 & \text{in } \Omega_\varepsilon \Big\backslash \bigcup_{j=1}^m B_{R_1}(\xi_j'), \\ \dfrac{\partial\phi}{\partial\nu} - W\phi \geqslant 0 & \text{on } \partial\Omega_\varepsilon \Big\backslash \bigcup_{j=1}^m B_{R_1}(\xi_j'), \\ \phi \geqslant 0 & \text{on } \Omega_\varepsilon \cap \left(\bigcup_{j=1}^m \partial B_{R_1}(\xi_j')\right), \end{cases}$$

then $\phi \geqslant 0$ in $\Omega_\varepsilon \setminus \cup_{j=1}^m B_{R_1}(\xi_j')$.

Let f, h be bounded and ϕ a solution to (37) satisfying (38). Following [4] we first claim that $\|\phi\|_{L^\infty(\Omega_\varepsilon)}$ can be controlled in terms of $\|f\|_{**,\Omega_\varepsilon}$, $\|h\|_{*,\partial\Omega_\varepsilon}$ and the following inner norm of ϕ:

$$\|\phi\|_i = \sup_{\Omega_\varepsilon \cap (\cup_{j=1}^m B_{R_1}(\xi_j'))} |\phi|.$$

Indeed, set

$$\widetilde{\phi} = C_1\psi\left(\|\phi\|_i + \|f\|_{**,\Omega_\varepsilon} + \|h\|_{*,\partial\Omega_\varepsilon}\right)$$

with C_1 a constant independent of ε. By the above maximum principle we have $\phi \leqslant \widetilde{\phi}$ and $-\phi \leqslant \widetilde{\phi}$ in $\Omega_\varepsilon \setminus \cup_{j=1}^m B_{R_1}(\xi_j')$. Since ψ is uniformly bounded we deduce

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\left(\|\phi\|_i + \|f\|_{**,\Omega_\varepsilon} + \|h\|_{*,\partial\Omega_\varepsilon}\right) \tag{39}$$

for some constant C independent of ϕ and ε.

We prove the lemma by contradiction. Assume that there exist a sequence $\varepsilon_n \to 0$, points $\xi_1^n, \ldots, \xi_m^n$ on $\partial\Omega$ satisfying (36) and functions ϕ_n, f_n and h_n with $\|\phi_n\|_{L^\infty(\Omega_{\varepsilon_n})} = 1$, $\|f_n\|_{**,\Omega_{\varepsilon_n}} \to 0$, $\|h_n\|_{*,\partial\Omega_{\varepsilon_n}} \to 0$ so that for each n ϕ_n solves (37) and satisfies (38). By (39) we see that $\|\phi_n\|_i$ stays away from zero. For one of the indices, say j, we can assume that $\sup_{B_{R_1}(\xi_j')} |\phi_n| \geqslant c > 0$ for all n. Consider $\hat{\phi}_n(z) = \phi_n(z - \xi_j')$ and let us translate and rotate Ω_{ε_n} so that Ω_{ε_n} approaches the upper half-plane $\mathbb{R}^2_+$ and $\xi_j' = 0$. Then by elliptic estimates $\hat{\phi}_n$ converges uniformly on compact sets to a nontrivial solution of (14). By Proposition 2.1 $\hat{\phi}$ is a linear combination of z_{0j} and z_{1j}. On the other hand, we can take the limit in the orthogonality relations (38), observing that limits of the functions Z_{ij} are just rotations and translations of z_{ij}, and we find $\int_{\mathbb{R}^2_+} \chi\hat{\phi}\, z_{ij} = 0$ for $i = 1, 2$. This contradicts the fact that $\hat{\phi} \not\equiv 0$. $\square$

Proof of Lemma 4.3. We take

$$\psi_{1j}(y) = \frac{(y-\xi_j')\cdot \nu(\xi_j')}{r^{1+\sigma}},$$

where $r = |y - \xi_j' - \mu_j \nu(\xi_j')|$. A computation shows that

$$\Delta\psi_{1j} = O(r^{-2-\sigma}) \qquad \text{in } \Omega_\varepsilon \tag{40}$$

and if $\delta > 0$ is small but fixed and $R_1 > 0$ is large and fixed then

$$\frac{\partial\psi_{1j}}{\partial\nu} \geqslant c r^{-1-\sigma} \qquad \text{for } R_1 < r < \delta/\varepsilon,$$

where $c > 0$ is fixed. To prove the last assertion we may suppose that ξ_j' is at the origin and assume that the normal vector at ξ_j' is $(0,-1)$. Hence

$$\psi_{1j}(y) = -\frac{y_2}{r^{1+\sigma}}.$$

Let us write $\partial\Omega_\varepsilon$ near ξ_j' as the graph $\{(y_1,y_2) : y_2 = G_\varepsilon(y_1)\}$ with $G_\varepsilon(y_1) = \frac{1}{\varepsilon}G(\varepsilon y_1)$ and G a smooth function such that $G(0) = 0$ and $G'(0) = 0$. Fix $\delta > 0$ small. Then for $R_1 < r < \delta/\varepsilon$ we have that r is comparable with y_1, $G_\varepsilon'(y_1) = O(\varepsilon r)$ and $G_\varepsilon(y_1) = O(\varepsilon r^2)$. Then

$$\begin{aligned}
\frac{\partial\psi_{1j}}{\partial\nu} &= \frac{1}{\sqrt{G'(\varepsilon y_1)^2+1}}\left(-(1+\sigma)\frac{G_\varepsilon(y_1)G_\varepsilon'(y_1)y_1}{r^{3+\sigma}} - \frac{1}{r^{1+\sigma}}\right.\\
&\qquad \left. +(1+\sigma)\frac{G_\varepsilon(y_1)^2}{r^{3+\sigma}} + \mu_j(1+\sigma)\frac{G_\varepsilon(y_1)}{r^{3+\sigma}}\right)\\
&= \frac{1}{\sqrt{G'(\varepsilon y_1)^2+1}}\left(-\frac{1}{r^{1+\sigma}} + \frac{O(\varepsilon^2 r^4)}{r^{3+\sigma}} + \frac{O(\delta)}{r^{2+\sigma}}\right) \qquad \text{for } R_1 < r < \delta/\varepsilon\\
&= \frac{1}{\sqrt{O(\delta^2)+1}}\left(-\frac{1}{r^{1+\sigma}} + \frac{O(\delta^2)}{r^{1+\sigma}} + \frac{O(\delta)}{r^{2+\sigma}}\right) \qquad \text{for } R_1 < r < \delta/\varepsilon,
\end{aligned}$$

from where the claim follows by taking δ small enough.

Consider also

$$\psi_{2j}(r) = 1 - \frac{1}{r^\sigma}.$$

Then

$$-\Delta\psi_{2j} = \sigma^2 \frac{1}{r^{2+\sigma}} \tag{41}$$

and proceeding analogously as for ψ_{1j} we find

$$\begin{aligned}\frac{\partial\psi_{2j}}{\partial\nu} &= \frac{\sigma}{r^{2+\sigma}} \frac{1}{\sqrt{G'(\varepsilon y_1)^2+1}} \left(-y_1 G'_\varepsilon(y_1), G_\varepsilon(y_1)+\mu_j\right) \\ &= \frac{\sigma}{r^{2+\sigma}} \frac{1}{\sqrt{O(\delta^2)+1}} O(\varepsilon r^2) \qquad \forall R_1 < r < \delta/\varepsilon \\ &= O\left(\frac{\varepsilon}{r^\sigma}\right) \qquad \forall R_1 < r < \delta/\varepsilon.\end{aligned}$$

Now let

$$\psi_{3j} = \psi_1 + C\psi_{2j}.$$

For C large enough (but independent of ε) using (40) and (41) we have

$$-\Delta\psi_{3j} + \varepsilon^2\psi_{3j} \geqslant \sigma^2 \frac{1}{r^{2+\sigma}} \qquad \forall R_1 < |y-\xi'_j| < \delta/\varepsilon. \tag{42}$$

Now recall that W satisfies (26) and therefore

$$W(y) = O\left(\frac{1}{r^2}\right) \qquad \forall R_1 < r < \frac{\delta}{\varepsilon}.$$

Thus

$$\frac{\partial\psi_{3j}}{\partial\nu} - W\psi_{3j} \geqslant \frac{c}{r^{1+\sigma}} - C\frac{1}{r^2} \geqslant \frac{c'}{r^{1+\sigma}} \qquad \text{for } R_1 < r < \delta/\varepsilon \tag{43}$$

with a constant $c' > 0$ if we choose R_1 larger if necessary.

Let $\eta_j \in C_0^\infty(\mathbb{R}^2)$ be such that $0 \leqslant \eta_j \leq 1$, $\eta_j \equiv 1$ in $\Omega_\varepsilon \cap B_{\delta/(2\varepsilon)}(\xi'_j)$, $\eta_j \equiv 0$ in $\Omega_\varepsilon \backslash B_{\delta/\varepsilon}(\xi'_j)$, $|\nabla\eta_j| \leqslant C\varepsilon$ in Ω_ε, $|\Delta\eta_j| \leqslant C\varepsilon^2$ in Ω_ε. Let $\psi_0(y) = \widetilde{\psi}(\varepsilon y)$ where $\widetilde{\psi}$ is the solution to

$$\begin{cases} -\Delta\widetilde{\psi} + \widetilde{\psi} = 1 & \text{in } \Omega, \\ \dfrac{\partial\widetilde{\psi}}{\partial\nu} = 1 & \text{on } \partial\Omega, \end{cases}$$

so that $-\Delta\psi_0 + \varepsilon^2\psi_0 = \varepsilon^2$ in Ω_ε and $\frac{\partial\psi_0}{\partial\nu} = \varepsilon$ on $\partial\Omega_\varepsilon$. In particular, ψ_0 is uniformly bounded in Ω_ε. The function

$$\psi = \sum_{j=1}^{m} \eta_j \psi_{3j} + C\psi_0$$

with C a sufficiently large constant meets the requirements. Indeed by (42)

$$-\Delta\psi + \varepsilon^2\psi \geqslant \sum_{j=1}^{m} \left(-\Delta\eta_j \psi_{3j} - 2\nabla\eta_j \nabla\psi_{3j} + \eta_j \frac{\sigma^2}{r_j^{2+\sigma}} \right) + C\varepsilon^2,$$

where $r_j = |y - \xi_j'|$ and hence

$$-\Delta\psi + \varepsilon^2\psi \geqslant \frac{\sigma^2}{r_j^{2+\sigma}} + C\varepsilon^2, \qquad R_1 < r_j < \frac{\delta}{2\varepsilon}.$$

By construction we have $|\nabla\psi_{3j}| = O(\frac{1}{r_j^{1+\sigma}})$ and hence, choosing C large we have

$$-\Delta\psi + \varepsilon^2\psi \geqslant O(\varepsilon^2) + O\left(\varepsilon \frac{1}{r_j^{1+\sigma}}\right) + C\varepsilon^2 \geqslant c\varepsilon^2, \qquad \frac{\delta}{2\varepsilon} < r_j < \frac{\delta}{\varepsilon}$$

if ε is small enough, and also

$$-\Delta\psi + \varepsilon^2\psi \geqslant c\frac{1}{r_j^{2+\sigma}}, \qquad \frac{\delta}{2\varepsilon} < r_j < \frac{\delta}{\varepsilon}.$$

Finally, a similar argument using (43) yields

$$\frac{\partial\psi}{\partial\nu} - W\psi \geqslant c\frac{1}{r_j^{1+\sigma}} + c\varepsilon, \qquad R_1 < r_j < \frac{\delta}{\varepsilon}$$

for all $j = 1, \ldots, m$. □

We will establish next an a priori estimate for solutions to problem (37) that satisfy orthogonality conditions with respect to Z_{1j} only.

Lemma 4.4. *For ε sufficiently small, if ϕ solves*

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi = f & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\phi}{\partial\nu} - W\phi = h & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{44}$$

and satisfies

$$\int_{\Omega_\varepsilon} Z_{1j}\chi_j\,\phi = 0 \qquad \forall j = 1,\dots,m, \tag{45}$$

then

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\log\frac{1}{\varepsilon}(\|h\|_{*,\partial\Omega_\varepsilon} + \|f\|_{**,\Omega_\varepsilon}), \tag{46}$$

where C is independent of ε.

Proof. Let ϕ satisfy (44) and (45). We will modify ϕ to satisfy all orthogonality relations in (38) and for this purpose we consider modifications with compact support of the functions Z_{0j}. Let $R > R_0 + 1$ be large and fixed. Set

$$\widehat{Z}_{0j}(y) = \psi Z_{0j}(y),$$

where

$$\psi(y) = \tilde h(|F_j^\varepsilon(y)|), \qquad \tilde h(x) = \frac{\log(\delta/\varepsilon) - \log|x|}{\log(\delta/\varepsilon) - \log R}$$

and F_j^ε is the change of variables defined in (34). Here $\delta > 0$ is a small fixed constant. Note that $\tilde h$ is just the solution to

$$\begin{cases} \Delta\tilde h = 0 & \text{in } B_{\delta/\varepsilon}(0)\setminus\overline{B}_R(0),\\ \tilde h = 1 & |x| = R,\\ \tilde h = 0 & |x| = \delta/\varepsilon. \end{cases}$$

Let $\bar\eta_{1j}$, $\bar\eta_{2j}$ be radial smooth cut-off functions on $\mathbb{R}^2$ so that

$$0\leqslant\bar\eta_{1j}\leqslant 1,\ \ |\nabla\bar\eta_{1j}|\leqslant C \ \text{ in } \mathbb{R}^2,$$
$$\bar\eta_{1j} \equiv 1 \ \text{ in } B_R(0),\ \ \bar\eta_{1j} \equiv 0 \ \text{ in } \mathbb{R}^2\setminus B_{R+1}(0)$$

and

$$\bar\eta_{2j} \equiv 1 \ \text{ in } B_{\frac{\delta}{4\varepsilon}}(0),\ \ \bar\eta_{2j} \equiv 0 \ \text{ in } \mathbb{R}^2\setminus B_{\frac{\delta}{3\varepsilon}}(0),$$
$$0\leqslant\bar\eta_{2j}\leqslant 1,\ \ |\nabla\bar\eta_{2j}|\leqslant C\varepsilon/\delta,\ \ |\nabla^2\bar\eta_{2j}|\leqslant C\varepsilon^2/\delta^2 \ \text{ in } \mathbb{R}^2.$$

Write

$$\eta_{1j}(y) = \bar{\eta}_{1j}(F_j^\varepsilon(y)), \qquad \eta_{2j}(x) = \bar{\eta}_{2j}(F_j^\varepsilon(y)). \tag{47}$$

Now define

$$\widetilde{Z}_{0j} = \eta_{1j} Z_{0j} + (1 - \eta_{1j})\eta_{2j}\widehat{Z}_{0j}.$$

Given ϕ satisfying (44) and (45) let

$$\widetilde{\phi} = \phi + \sum_{j=1}^{m} d_j \widetilde{Z}_{0j}, \qquad \text{where } d_j = -\frac{\int_{\Omega_\varepsilon} Z_{0j}\chi_j\ \phi}{\int_{\Omega_\varepsilon} Z_{0j}^2\chi_j}.$$

Estimate (46) is a direct consequence of

Claim.

$$|d_j| \leqslant C \log\frac{1}{\varepsilon}\left(\|h\|_{*,\partial\Omega_\varepsilon} + \|f\|_{**,\Omega_\varepsilon}\right) \quad \forall j = 1, \ldots, m. \tag{48}$$

We start proving this by observing, using the notation $L = -\Delta + \varepsilon^2$, that

$$L(\widetilde{\phi}) = f + \sum_{j=1}^{m} d_j L(\widetilde{Z}_{0j}) \quad \text{in } \Omega_\varepsilon \tag{49}$$

and

$$\left(\frac{\partial}{\partial\nu} - W\right)\widetilde{\phi} = h + \sum_{j=1}^{m} d_j \left(\frac{\partial}{\partial\nu} - W\right)\widetilde{Z}_{0j} \quad \text{on } \partial\Omega_\varepsilon. \tag{50}$$

Thus by Lemma 4.2 we have

$$\|\widetilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\sum_{j=1}^{m}|d_j|\left(\left\|\left(\frac{\partial}{\partial\nu} - W\right)\widetilde{Z}_{0j}\right\|_{*,\partial\Omega_\varepsilon} + \|L(\widetilde{Z}_{0j})\|_{**,\Omega_\varepsilon}\right)$$

$$+ C\|h\|_{*,\partial\Omega_\varepsilon} + C\|f\|_{**,\Omega_\varepsilon}. \tag{51}$$

Multiplying Eq. (49) by $\widetilde{Z}_{0k}$, integrating by parts and using (50) we find

$$d_k\left[\int_{\Omega_\varepsilon} L(\widetilde{Z}_{0k})\widetilde{Z}_{0k}+\int_{\partial\Omega_\varepsilon}\widetilde{Z}_{0k}\left(\frac{\partial}{\partial\nu}-W\right)\widetilde{Z}_{0k}\right]$$
$$=-\int_{\partial\Omega_\varepsilon} h\widetilde{Z}_{0k}-\int_{\Omega_\varepsilon} f\widetilde{Z}_{0k}+\int_{\partial\Omega_\varepsilon}\widetilde{\phi}\left(\frac{\partial}{\partial\nu}-W\right)\widetilde{Z}_{0k}+\int_{\Omega_\varepsilon}\widetilde{\phi}L(\widetilde{Z}_{0k}).$$

This combined with (51) yields

$$d_k\left[\int_{\Omega_\varepsilon} L(\widetilde{Z}_{0k})\widetilde{Z}_{0k}+\int_{\partial\Omega_\varepsilon}\widetilde{Z}_{0k}\left(\frac{\partial}{\partial\nu}-W\right)\widetilde{Z}_{0k}\right]$$
$$\leqslant C\|h\|_{*,\partial\Omega}+C\|f\|_{**,\Omega_\varepsilon}+\|\widetilde{\phi}\|_{L^\infty}\left\|\left(\frac{\partial}{\partial\nu}-W\right)\widetilde{Z}_{0k}\right\|_{*,\partial\Omega_\varepsilon}$$
$$+\|\widetilde{\phi}\|_{L^\infty}\|L(\widetilde{Z}_{0k})\|_{**,\Omega_\varepsilon}$$
$$\leqslant (C\|h\|_{*,\partial\Omega}+\|f\|_{**,\Omega_\varepsilon})\left(1+\left\|\left(\frac{\partial}{\partial\nu}-W\right)\widetilde{Z}_{0k}\right\|_{*,\partial\Omega_\varepsilon}+\|L(\widetilde{Z}_{0k})\|_{**,\Omega_\varepsilon}\right)$$
$$+C\sum_{j=1}^m|d_j|\left(\left\|\left(\frac{\partial}{\partial\nu}-W\right)\widetilde{Z}_{0j}\right\|^2_{*,\partial\Omega_\varepsilon}+\|L(\widetilde{Z}_{0j})\|^2_{**,\Omega_\varepsilon}\right).$$

We will achieve (48) proving the following estimates: for some constant $C>0$ independent of ε we have

$$\int_{\Omega_\varepsilon} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j}+\int_{\partial\Omega_\varepsilon}\widetilde{Z}_{0j}\left(\frac{\partial}{\partial\nu}-W\right)\widetilde{Z}_{0j}\geqslant\frac{1}{C\log\frac{1}{\varepsilon}},\tag{52}$$

$$\|L(\widetilde{Z}_{0j})\|_{**,\Omega_\varepsilon}\leqslant\frac{C}{\log\frac{1}{\varepsilon}},\tag{53}$$

$$\left\|\left(\frac{\partial}{\partial\nu}-W\right)\widetilde{Z}_{0j}\right\|_{*,\partial\Omega_\varepsilon}\leqslant\frac{C}{\log\frac{1}{\varepsilon}}.\tag{54}$$

Proof of (52). We write

$$\int_{\Omega_\varepsilon} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j}=I_0+I_1+I_2+I_3,$$

where $I_l = \int_{R_l} L(\widetilde{Z}_{0j})\widetilde{Z}_{0j}$ and the regions $R_0, \dots, R_3$ are defined in terms of the change of variables F_j^ε defined in (34) as follows. Write $x = (F_j^\varepsilon)^{-1}(y)$, $r = |x|$ and define the following subsets of Ω_ε:

$$R_0 = (F_j^\varepsilon)^{-1}(\{r < R\} \cap \mathbb{R}^2_+), \quad R_1 = (F_j^\varepsilon)^{-1}(\{R < r < R+1\} \cap \mathbb{R}^2_+),$$

$$R_2 = (F_j^\varepsilon)^{-1}\left(\left\{R+1 < r < \frac{\delta}{4\varepsilon}\right\} \cap \mathbb{R}^2_+\right), \quad R_3 = (F_j^\varepsilon)^{-1}\left(\left\{\frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}\right\} \cap \mathbb{R}^2_+\right).$$

We will prove that

$$I_1 \geqslant \frac{\bar{c}}{\log \frac{\delta}{\varepsilon}}$$

with $\bar{c} > 0$ independent of ε, δ and R while the other terms I_0, I_2, I_3 and $\int_{\partial\Omega_\varepsilon} \widetilde{Z}_{0j}(\frac{\partial}{\partial\nu} - W)\widetilde{Z}_{0j}$ can be made small compared to $\frac{1}{\log\frac{1}{\varepsilon}}$ by choosing $\delta > 0$ small and $R > 0$ large, but fixed independently of ε.

Estimate of I_1: We change variables $x = F_j^\varepsilon(y)$ and recall that this map preserves area, so

$$I_1 = \int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} \widetilde{L}(\widetilde{z}_{0j})\widetilde{z}_{0j},$$

$$\widetilde{z}_{0j}(x) = \widetilde{Z}_{0j}((F_j^\varepsilon)^{-1}(x)) = \bar{\eta}_{1j} z_{0j} + (1-\bar{\eta}_{1j})\tilde{h}(x) z_{0j} \tag{55}$$

and $\widetilde{L}$ is a linear operator, which thanks to the definition $F_j^\varepsilon(y) = \frac{1}{\varepsilon}F(\varepsilon y)$, has the expansion

$$\widetilde{L} = -\Delta + O(\varepsilon|x|)\nabla^2 + O(\varepsilon)\nabla + \varepsilon^2. \tag{56}$$

Therefore

$$I_1 = -\int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} \Delta(\tilde{z}_{0j})\,\tilde{z}_{0j} + O(R\varepsilon).$$

Using that z_{0j} is harmonic and that in the region $R < r < R+1$ we have $\bar{\eta}_{2j} \equiv 1$ we compute

$$\begin{aligned}\Delta\tilde{z}_{0j} &= \Delta(\bar{\eta}_{1j} z_{0j} + (1-\bar{\eta}_{1j})\tilde{h} z_{0j}) = \Delta\bar{\eta}_{1j}(1-\tilde{h}) z_{0j} + 2\nabla\bar{\eta}_{1j}\nabla((1-\tilde{h})z_{0j}) \\ &\quad + (1-\bar{\eta}_{1j})\Delta(\tilde{h} z_{0j}), \qquad R < r < R+1.\end{aligned}$$

Since $\Delta \tilde{h} = 0$, $\Delta z_{0j} = 0$ for the last term in the expression above, we have

$$\Delta(\tilde{h} z_{0j}) = 2\nabla \tilde{h} \nabla z_{0j}, \qquad R < r < R+1.$$

But

$$\frac{\partial z_{0j}}{\partial x_1}(x_1, x_2) = 4\frac{x_1(x_2+\mu_j)}{(x_1^2+(x_2+\mu_j)^2)^2}, \qquad \frac{\partial z_{0j}}{\partial x_2}(x_1, x_2) = 2\frac{(x_2+\mu_j)^2 - x_1^2}{(x_1^2+(x_2+\mu_j)^2)^2}.$$

Thus

$$\nabla \tilde{h} \nabla z_{0j} = -\frac{2}{|x|^2(\log(\delta/\varepsilon) - \log R)} \frac{x_2(x_1^2+(x_2+\mu_j)^2) + 2x_1^2\mu_j}{(x_1^2+(x_2+\mu_j)^2)^2} \leqslant 0,$$

so that

$$\Delta(\bar{\eta}_{1j} z_{0j} + (1-\bar{\eta}_{1j})\tilde{h} z_{0j}) \leqslant \Delta \bar{\eta}_{1j}(1-\tilde{h}) z_{0j} + 2\nabla \bar{\eta}_{1j} \nabla((1-\tilde{h}) z_{0j}),$$
$$R < r < R+1.$$

It follows that

$$I_1 \geqslant -\int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} \Delta \bar{\eta}_{1j}(1-\tilde{h}) z_{0j} \tilde{z}_{0j} + 2\int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} \nabla \bar{\eta}_{1j} \nabla \tilde{h} z_{0j} \tilde{z}_{0j}$$
$$-2\int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} \nabla \bar{\eta}_{1j} \nabla z_{0j}(1-\tilde{h}) \tilde{z}_{0j} + O(R\varepsilon).$$

We integrate by parts the first term on the right-hand side above

$$I_1 \geqslant \int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} \nabla \bar{\eta}_{1j} \nabla \tilde{h} z_{0j} \tilde{z}_{0j} - \int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} \nabla \bar{\eta}_{1j} \nabla z_{0j}(1-\tilde{h}) \tilde{z}_{0j}$$
$$+\int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} \nabla \bar{\eta}_{1j} \nabla \tilde{z}_{0j}(1-\tilde{h}) z_{0j} + O(R\varepsilon), \tag{57}$$

observing that the boundary term on $\partial\mathbb{R}^2_+ \cap \{R < r < R+1\}$ is zero because $\bar{\eta}_{1j}$ is radial.

The second term on the right-hand side above is bounded by

$$\int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} |\nabla \bar{\eta}_{1j} \nabla z_{0j}(1-\tilde{h}) \tilde{z}_{0j}| \leqslant C \int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} |\tilde{h}-1||\nabla z_{0j}|.$$

But in the region $R < r < R+1$ we have $|\tilde{h}-1| \leqslant \frac{C}{\log\frac{\delta}{\varepsilon}}$, and $|\nabla z_{0j}| \leqslant \frac{C}{R^2}$ which yields

$$\int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} |\nabla\bar{\eta}_{1j}\nabla z_{0j}(1-\tilde{h})\tilde{z}_{0j}| \leqslant \frac{C}{R\log\frac{\delta}{\varepsilon}}. \tag{58}$$

The third term on the right-hand side of (57) is similar since in the region $R < r < R+1$ we have $|\nabla\tilde{h}| \leqslant \frac{C}{R\log\frac{\delta}{\varepsilon}}$ and hence

$$\begin{aligned}|\nabla\tilde{z}_{0j}| &= |\nabla(\bar{\eta}_{1j}(1-\tilde{h})z_{0j}) + \tilde{h}z_{0j}|\\ &= |\nabla\bar{\eta}_{1j}(1-\tilde{h})z_{0j} - \bar{\eta}_{1j}\nabla\tilde{h}z_{0j} + \bar{\eta}_{1j}(1-\tilde{h})\nabla z_{0j} + \nabla\tilde{h}z_{0j} + \tilde{h}\nabla z_{0j}|\\ &\leqslant \frac{C}{\log\frac{\delta}{\varepsilon}} + \frac{C}{R^2}, \qquad R < r < R+1.\end{aligned}$$

Integrating

$$\int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} |\nabla\bar{\eta}_{1j}\nabla\tilde{z}_{0j}(1-\tilde{h})z_{0j}| \leqslant \frac{CR}{\log^2\frac{\delta}{\varepsilon}} + \frac{C}{R\log\frac{\delta}{\varepsilon}}. \tag{59}$$

Thus from (57)–(59) we obtain

$$I_1 \geqslant \int_{\{R<r<R+1\}\cap\mathbb{R}^2_+} \nabla\bar{\eta}_{1j}\nabla\tilde{h}z_{0j}\tilde{z}_{0j} + O(R\varepsilon) + O\left(\frac{R}{\log^2\frac{\delta}{\varepsilon}}\right) + O\left(\frac{1}{R\log\frac{\delta}{\varepsilon}}\right).$$

In the first integral above z_{0j} and $\tilde{z}_{0j}$ have a lower bound independent of ε, δ, R and $|\nabla\tilde{h}| = (|x|(\log(\delta/\varepsilon) - \log R))^{-1}$. Hence

$$I_1 \geqslant \frac{\bar{c}}{\log\frac{\delta}{\varepsilon}} + O(R\varepsilon) + O\left(\frac{R}{\log^2\frac{\delta}{\varepsilon}}\right) + O\left(\frac{1}{R\log\frac{\delta}{\varepsilon}}\right) \tag{60}$$

with $\bar{c} > 0$ independent of ε, δ, R.

Estimate of I_0: By (56) and since $\Delta z_{0j} = 0$ we have

$$\widetilde{L}(\tilde{z}_{0j}) = O(\varepsilon), \qquad r < R \tag{61}$$

and this implies

$$I_0 = O(R\varepsilon). \tag{62}$$

Estimate of I_2: Changing variables as before

$$I_2 = \int_{\{R+1<r<\frac{\delta}{4\varepsilon}\}\cap\mathbb{R}^2_+} \widetilde{L}(\tilde{z}_{0j})\tilde{z}_{0j}.$$

In the region $R+1 < r < \frac{\delta}{4\varepsilon}$ we have $\tilde{z}_{0j} = \tilde{h} z_{0j}$ and therefore

$$|\Delta\tilde{z}_{0j}| = 2|\nabla\tilde{h}\nabla z_{0j}| \leqslant \frac{C}{r^3 \log\frac{\delta}{\varepsilon}}, \qquad R+1 < r < \frac{\delta}{4\varepsilon}.$$

For the other terms we find

$$\begin{aligned} |\nabla^2\tilde{z}_{0j}| &\leqslant |\nabla^2\tilde{h}|z_{0j} + 2|\nabla\tilde{h}\nabla z_{0j}| + \tilde{h}|\nabla^2 z_{0j}| \\ &= O\left(\frac{1}{r^2\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{1}{r^3\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{1}{r^3}\right), \qquad R+1 < r < \frac{\delta}{4\varepsilon}, \end{aligned}$$

so

$$O(\varepsilon|x|)|\nabla^2\tilde{z}_{0j}| = O\left(\frac{\varepsilon}{r\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon}{r^2}\right), \qquad R+1 < r < \frac{\delta}{4\varepsilon}.$$

Also

$$|\nabla\tilde{z}_{0j}| \leqslant |\nabla\tilde{h}|z_{0j} + \tilde{h}|\nabla z_{0j}| = O\left(\frac{1}{r\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{1}{r^2}\right), \qquad R+1 < r < \frac{\delta}{4\varepsilon}.$$

Hence

$$\widetilde{L}(\tilde{z}_{0j}) = O\left(\frac{1}{r^3\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon}{r\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon}{r^2}\right) + \varepsilon^2 z_{0j}, \quad R+1 < r < \frac{\delta}{4\varepsilon}. \tag{63}$$

This yields

$$\begin{aligned} \int_{\{R+1<r<\frac{\delta}{4\varepsilon}\}\cap\mathbb{R}^2_+} \widetilde{L}(\tilde{z}_{0j})\tilde{z}_{0j} &= O\left(\frac{1}{R\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\delta}{\log\frac{\delta}{\varepsilon}}\right) \\ &\quad + O(\varepsilon^2)\int_{\{R+1<r<\frac{\delta}{4\varepsilon}\}\cap\mathbb{R}^2_+} \tilde{z}_{0j}^2. \end{aligned}$$

We estimate the last integral using the fact that in the region $R+1 < r < \frac{\delta}{4\varepsilon}$ $\tilde z_{0j} = \tilde h z_{0j}$ and z_{0j} is bounded, thus

$$\int_{\{R+1<r<\frac{\delta}{4\varepsilon}\}\cap\mathbb{R}^2_+} \tilde z_{0j}^2 \leqslant C \int_{R+1}^{\frac{\delta}{4\varepsilon}} \left(\frac{\log\frac{\delta}{\varepsilon} - \log r}{\log\frac{\delta}{\varepsilon} - \log R}\right)^2 r\,dr \leqslant C\frac{\delta^2}{\varepsilon^2\log^2\frac{\delta}{\varepsilon}}.$$

This and the previous estimate show that

$$I_2 = O\left(\frac{1}{R\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\delta}{\log\frac{\delta}{\varepsilon}}\right). \tag{64}$$

Estimate of I_3: In the region $\frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}$ the definition of $\tilde z_{0j}$ is $\tilde z_{0j} = \bar\eta_{2j}\tilde h z_{0j}$. We will estimate each term of (56) using the facts that $\nabla\bar\eta_{2j} = O(\frac{\varepsilon}{\delta})$, $|\nabla^2\bar\eta_{2j}| = O(\frac{\varepsilon^2}{\delta^2})$ and that in the considered region $\tilde h = O(\frac{1}{\log\frac{\delta}{\varepsilon}})$ which implies also $\tilde z_{0j} = O(\frac{1}{\log\frac{\delta}{\varepsilon}})$. We obtain

$$\begin{aligned}
\Delta\tilde z_{0j} &= \Delta\bar\eta_{2j}\tilde h z_{0j} + 2\nabla\bar\eta_{2j}\nabla(\tilde h z_{0j}) + \bar\eta_{2j}\Delta(\tilde h z_{0j})\\
&= \Delta\bar\eta_{2j}\tilde h z_{0j} + 2\nabla\bar\eta_{2j}\nabla\tilde h z_{0j} + 2\nabla\bar\eta_{2j}\nabla z_{0j}\tilde h + 2\bar\eta_{2j}\nabla\tilde h\nabla z_{0j}\\
&= O\left(\frac{\varepsilon^2}{\delta^2\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon}{r\delta\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon}{r^2\delta\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{1}{r^3\log\frac{\delta}{\varepsilon}}\right)\\
&= O\left(\frac{\varepsilon^2}{\delta^2\log\frac{\delta}{\varepsilon}}\right), \qquad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}.
\end{aligned}$$

Next

$$\nabla^2\tilde z_{0j} = \nabla^2\bar\eta_{2j}\tilde h z_{0j} + 2\nabla\bar\eta_{2j}\nabla(\tilde h z_{0j}) + \bar\eta_{2j}\nabla^2(\tilde h z_{0j}), \qquad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}$$

and by the above computations

$$\begin{aligned}
\nabla^2\tilde z_{0j} &= O\left(\frac{\varepsilon^2}{\delta^2\log\frac{\delta}{\varepsilon}}\right) + \bar\eta_{2j}(\nabla^2\tilde h\tilde z_{0j} + 2\nabla\tilde h\nabla\tilde z_{0j} + \tilde h\nabla^2\tilde z_{0j})\\
&= O\left(\frac{\varepsilon^2}{\delta^2\log\frac{\delta}{\varepsilon}}\right), \qquad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}.
\end{aligned}$$

Similarly

$$\nabla\tilde z_{0j} = \nabla\bar\eta_{2j}\tilde h z_{0j} + \bar\eta_{2j}\nabla\tilde h z_{0j} + \bar\eta_{2j}\tilde h\nabla z_{0j}$$
$$= O\left(\frac{\varepsilon}{\delta\log\frac{\delta}{\varepsilon}}\right), \qquad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}.$$

This shows that

$$\widetilde L(\tilde z_{0j}) = O\left(\frac{\varepsilon^2}{\delta^2\log\frac{\delta}{\varepsilon}}\right), \qquad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}. \tag{65}$$

and integrating

$$I_3 = O\left(\frac{1}{\log^2\frac{\delta}{\varepsilon}}\right). \tag{66}$$

Estimate of $\int_{\partial\Omega_\varepsilon}\widetilde Z_{0j}(\frac{\partial}{\partial\nu} - W)\widetilde Z_{0j}$: We change variables through the map F_j^ε:

$$\int_{\partial\Omega_\varepsilon}\widetilde Z_{0j}\left(\frac{\partial}{\partial\nu} - W\right)\widetilde Z_{0j} = \int_{\partial\mathbb{R}^2_+}\tilde z_{0j}(B(\tilde z_{0j}) - \widetilde W\tilde z_{0j})b(x),$$

where $\tilde z_{0j}$ is defined in (55), $\widetilde W(x) = W((F_j^\varepsilon)^{-1}(x))$ and b is a positive function arising from the change of variables bounded uniformly in ε. B is a differential operator of order one on $\partial\mathbb{R}^2_+$. Rotating Ω_ε so that $\nabla F_j^\varepsilon(\xi'_j) = I$ we find the following expansion for B

$$B = -\frac{\partial}{\partial x_2} + O(\varepsilon|x|)\nabla.$$

Let us estimate first the integral in the region $|x| < R$, where $\tilde z_{0j} = z_{0j}$. Then

$$B(\tilde z_{0j}) = -\frac{\partial z_{0j}}{\partial x_2} + O(\varepsilon), \qquad |x| < R, x\in\partial\mathbb{R}^2_+.$$

On the other hand recall (26), that is

$$W(y) = \frac{2\mu_j}{|y - \xi'_j - \mu_j\nu(\xi'_j)|^2}(1 + O(\varepsilon^\alpha(1+|y|))).$$

Since we have the expansion $(F_j^\varepsilon)^{-1}(x) = \xi_j' + x + O(\varepsilon|x|)$ we find

$$\begin{aligned}\widetilde{W}(x) &= W((F_j^\varepsilon)^{-1}(x)) = W(\xi_j' + x + O(\varepsilon|x|))\\ &= \frac{2\mu_j}{x_1^2+\mu_j^2} + O\left(\frac{\varepsilon^\alpha(1+|x|)}{1+|x|^2}\right), \qquad x=(x_1,0),\, |x|<\frac{\delta}{\varepsilon}.\end{aligned} \tag{67}$$

Thus

$$B(\tilde{z}_{0j}) - \widetilde{W}\tilde{z}_{0j} = O(\varepsilon^\alpha), \qquad x \in \partial\mathbb{R}^2_+,\ \ |x|<R \tag{68}$$

and therefore

$$\int_{\partial\mathbb{R}^2_+\cap\{|x|<R\}} \tilde{z}_{0j}(B(\tilde{z}_{0j}) - \widetilde{W}\tilde{z}_{0j})b(x) = O(R\varepsilon^\alpha).$$

Next, in the region $R<|x|<R+1$ we have

$$\begin{aligned}\nabla\tilde{z}_{0j} &= \nabla(\bar{\eta}_{1j}(1-\tilde{h})z_{0j} + \tilde{h}z_{0j})\\ &= \nabla\bar{\eta}_{1j}(1-\tilde{h})z_{0j} - \bar{\eta}_{1j}\nabla\tilde{h}z_{0j} + \bar{\eta}_{1j}(1-\tilde{h})\nabla z_{0j} + \nabla\tilde{h}z_{0j} + \tilde{h}\nabla z_{0j}\\ &= O\left(\frac{1}{\log\frac{\delta}{\varepsilon}}\right) + \bar{\eta}_{1j}(1-\tilde{h})\nabla z_{0j} + \tilde{h}\nabla z_{0j}.\end{aligned}$$

Since $\tilde{h}$ is radial this implies

$$B(\tilde{z}_{0j}) = -\tilde{h}\frac{\partial z_{0j}}{\partial x_2} + O\left(\frac{1}{R^2\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{R\varepsilon}{\log\frac{\delta}{\varepsilon}}\right), \qquad R<|x|<R+1, x\in\partial\mathbb{R}^2_+.$$

Using (67) we see that

$$B(\tilde{z}_{0j}) - \widetilde{W}\tilde{z}_{0j} = O\left(\frac{1}{R^2\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{R\varepsilon}{\log\frac{\delta}{\varepsilon}}\right), \quad R<|x|<R+1, x\in\partial\mathbb{R}^2_+. \tag{69}$$

It follows that

$$\int_{\partial\mathbb{R}^2_+\cap\{R<|x|<R+1\}} \tilde{z}_{0j}(B(\tilde{z}_{0j}) - \widetilde{W}\tilde{z}_{0j})b(x) = O\left(\frac{1}{R^2\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{R\varepsilon}{\log\frac{\delta}{\varepsilon}}\right).$$

Using the fact that $\tilde{h}$ has zero normal derivative on $\partial\mathbb{R}^2_+$ we deduce

$$\begin{aligned} B(\tilde{h}z_{0j}) &= -\tilde{h}\frac{\partial z_{0j}}{\partial x_2} + O(\varepsilon r)(\nabla\tilde{h}z_{0j} + \tilde{h}\nabla z_{0j}) \\ &= -\tilde{h}\frac{\partial z_{0j}}{\partial x_2} + O\left(\frac{\varepsilon}{\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon}{r}\right), \qquad R+1 < r < \frac{\delta}{\varepsilon}. \end{aligned} \tag{70}$$

On the other hand, using (67) we have

$$B(\tilde{z}_{0j}) - \widetilde{W}\tilde{z}_{0j} = O\left(\frac{\varepsilon}{\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon^\alpha}{r}\right)$$

and we conclude

$$\int_{\partial\mathbb{R}^2_+\cap\{R+1<r<\frac{\delta}{4\varepsilon}\}} \tilde{z}_{0j}(B(\tilde{z}_{0j}) - \widetilde{W}\tilde{z}_{0j})b(x) = O\left(\frac{\delta}{\log\frac{\delta}{\varepsilon}}\right).$$

Finally we consider $\frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}$. Here we have $\widetilde{z}_{0j} = \bar{\eta}_{2j}\tilde{h}z_{0j}$ and $\tilde{h}, z_{0j} = O(\frac{1}{\log\frac{\delta}{\varepsilon}})$, $\nabla\bar{\eta}_{2j} = O(\frac{\varepsilon}{\delta})$. Using these facts, estimate (70) and that $\bar{\eta}_{2j}$ has zero normal derivative we find

$$\begin{aligned} B(\widetilde{z}_{0j}) &= B(\bar{\eta}_{2j})\tilde{h}z_{0j} + \bar{\eta}_{2j}B(\tilde{h}z_{0j}) \\ &= O\left(\frac{\varepsilon^2 r}{\delta\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{1}{r^2}\right) + O\left(\frac{\varepsilon}{\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon}{r}\right), \qquad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}. \end{aligned} \tag{71}$$

Integrating we have

$$\int_{\partial\mathbb{R}^2_+\cap\{\frac{\delta}{4\varepsilon}<r<\frac{\delta}{3\varepsilon}\}} \tilde{z}_{0j}B(\tilde{z}_{0j})b(x) = O\left(\frac{1}{\log^2\frac{\delta}{\varepsilon}}\right).$$

From (67) we have

$$\widetilde{W} = O\left(\frac{\varepsilon^\alpha}{r}\right), \quad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{\varepsilon} \tag{72}$$

and this implies

$$\int_{\partial\mathbb{R}^2_+\cap\{\frac{\delta}{4\varepsilon}<r<\frac{\delta}{3\varepsilon}\}} \widetilde{W}\tilde{z}_{0j}^2 b(x) = O\left(\varepsilon^\alpha\log\frac{\delta}{\varepsilon}\right).$$

Thus

$$\int_{\partial\mathbb{R}^2_+\cap\{\frac{\delta}{4\varepsilon}<r<\frac{\delta}{3\varepsilon}\}} \tilde z_{0j}(B(\tilde z_{0j}) - \widetilde W \tilde z_{0j}) b(x) = O\left(\frac{1}{\log^2\frac{\delta}{\varepsilon}}\right) + O\left(\varepsilon^\alpha \log\frac{\delta}{\varepsilon}\right)$$

and therefore

$$\int_{\partial\Omega_\varepsilon} \widetilde Z_{0j}\left(\frac{\partial}{\partial\nu} - W\right)\widetilde Z_{0j} = O\left(\frac{1}{R^2\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\delta}{\log\frac{\delta}{\varepsilon}}\right). \tag{73}$$

Combining (60), (62), (64), (66) and (73) we obtain

$$\int_{\Omega_\varepsilon} L(\widetilde Z_{0j}) Z_{0j} \geqslant \frac{\bar c}{\log\frac{\delta}{\varepsilon}} + O\left(\frac{1}{R\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\delta}{\log\frac{\delta}{\varepsilon}}\right).$$

Choosing $\delta > 0$ small and $R > 0$ large (fixed independently of ε) we conclude that (52) holds for $\varepsilon > 0$ small enough.

Proof of (53). By (61) we deduce

$$L(\widetilde Z_0) = O(\varepsilon), \qquad r < R. \tag{74}$$

Also (63) implies

$$L(\widetilde Z_{0j}) = O\left(\frac{1}{r^3\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon}{r\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon}{r^2}\right) + \varepsilon^2 \widetilde Z_{0j},$$
$$R+1 < r < \frac{\delta}{4\varepsilon} \tag{75}$$

and from (65) we obtain

$$L(\widetilde Z_{0j}) = O\left(\frac{\varepsilon^2}{\delta^2\log\frac{\delta}{\varepsilon}}\right), \qquad \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon}. \tag{76}$$

Thus, we only need to estimate the size of $L(\widetilde Z_{0j})$ in the region $R < r < R+1$. In this region we have $\widetilde Z_{0j} = \eta_{1j} Z_{0j} + (1-\eta_{1j})\psi Z_{0j}$ and hence

$$\Delta \widetilde Z_{0j} = \Delta\eta_{1j}(1-\psi) Z_{0j} - 2\nabla\eta_{1j}\nabla\psi Z_{0j} + 2\nabla\eta_{1j}\nabla Z_{0j}(1-\psi) + \eta_{1j}\Delta Z_{0j}$$

$$
\begin{aligned}
&+ (1-\eta_{1j})\Delta(\psi Z_{0j}) \\
&= O\left(\frac{1}{\log\frac{\delta}{\varepsilon}}\right) + \eta_{1j}\Delta Z_{0j} + (1-\eta_{1j})\Delta(\psi Z_{0j}), \qquad R < r < R+1.
\end{aligned}
$$

Using the change of variables $x = F_j^{\varepsilon}(y)$ and recalling the definitions of Z_{0j} and ψ we have

$$
\Delta_y Z_{0j} = \Delta_x z_{0j} + O(\varepsilon) = O(\varepsilon), \qquad R < r < R+1
$$

and

$$
\begin{aligned}
\Delta_y(\psi Z_{0j}) &= \Delta_x(\tilde{h} z_{0j}) + O(\varepsilon) \\
&= 2\nabla\tilde{h}\nabla z_{0j} + O(\varepsilon), \qquad R < r < R+1 \\
&= O\left(\frac{1}{\log\frac{\delta}{\varepsilon}}\right) + O(\varepsilon), \qquad R < r < R+1.
\end{aligned}
$$

Thus

$$
L(\widetilde{Z}_{0j}) = O\left(\frac{1}{\log\frac{\delta}{\varepsilon}}\right), \qquad R < r < R+1.
$$

This bounds and (74)–(76) imply (53).

Proof of (54). By (68) we see that

$$
\frac{\partial \widetilde{Z}_{0j}}{\partial \nu} - W\widetilde{Z}_{0j} = O(\varepsilon), \qquad y \in \partial\Omega_\varepsilon,\ |y| < R.
$$

From (69) we also obtain

$$
\frac{\partial \widetilde{Z}_{0j}}{\partial \nu} - W\widetilde{Z}_{0j} = O\left(\frac{\varepsilon r}{\log\frac{\delta}{\varepsilon}}\right) + O(\varepsilon^\alpha), \qquad y \in \partial\Omega_\varepsilon,\ R < |y| < R+1.
$$

Finally using (70), (72) and (71) we also see that

$$
\begin{aligned}
&\frac{\partial \widetilde{Z}_{0j}}{\partial \nu} - W\widetilde{Z}_{0j} = O\left(\frac{\varepsilon^2 r}{\delta\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{1}{r^2}\right) + O\left(\frac{\varepsilon}{\log\frac{\delta}{\varepsilon}}\right) + O\left(\frac{\varepsilon}{r}\right), \\
&\qquad y \in \partial\Omega_\varepsilon,\ R+1 < |y| < \frac{\delta}{3\varepsilon}.
\end{aligned}
$$

These inequalities readily imply (54). □

Proof of Proposition 4.1. To prove the solvability of (33) we consider first a related problem: that of given $h \in L^\infty(\partial\Omega_\varepsilon)$ find $\phi \in L^\infty(\Omega_\varepsilon)$ and $d_1, \ldots, d_m \in \mathbb{R}$, such that

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi = \sum_{j=1}^m d_j \chi_j Z_{1j} & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\phi}{\partial\nu} - W\phi = h & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{1j}\phi = 0 & \forall\, j = 1, \ldots, m. \end{cases} \tag{77}$$

First we prove that for any $\phi, d_1, \ldots, d_m$ solution to (77) the bound

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C \log\frac{1}{\varepsilon}\|h\|_{*,\partial\Omega_\varepsilon} \tag{78}$$

holds. Indeed, by Lemma 4.4 we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C \log\frac{1}{\varepsilon}\left(\|h\|_{*,\partial\Omega_\varepsilon} + \sum_{j=1}^m |d_j|\right) \tag{79}$$

and therefore it is enough to prove that $|d_j| \leqslant C\|h\|_{*,\partial\Omega_\varepsilon}$.

Let η_{2j} be the cut-off function defined in (47) and multiply Eq. (77) by $\eta_{2k}Z_{1k}$. Integrating by parts we find

$$\begin{aligned} d_k \int_{\Omega_\varepsilon} \chi_k Z_{1k}^2 = &-\int_{\partial\Omega_\varepsilon} h\eta_{2k}Z_{1k} + \int_{\partial\Omega_\varepsilon} \phi\frac{\partial\eta_{2k}}{\partial\nu}Z_{1k} + \int_{\partial\Omega_\varepsilon} \phi\eta_{2k}\left(\frac{\partial Z_{1k}}{\partial\nu} - WZ_{1k}\right) \\ &+ \int_{\Omega_\varepsilon} \phi(-\Delta(\eta_{2k}Z_{1k}) + \varepsilon^2\eta_{2k}Z_{1k}). \end{aligned} \tag{80}$$

But $Z_{1j} = O(\frac{1}{1+r})$ and $\nabla\eta_{2j} = O(\varepsilon)$ so $|\int_{\partial\Omega_\varepsilon} \phi\frac{\partial\eta_{2k}}{\partial\nu}Z_{1k}| \leqslant C\varepsilon\log\frac{1}{\varepsilon}$. Also, using (67) and proceeding similarly as with (68) we obtain

$$\frac{\partial Z_{1j}}{\partial\nu} - WZ_{1j} = O\left(\frac{\varepsilon}{1+r}\right) + O\left(\frac{\varepsilon^\alpha}{1+r^2}\right), \qquad |y| < \tfrac{\delta}{\varepsilon},\ y \in \partial\Omega_\varepsilon$$

and this implies

$$\int_{\partial\Omega_\varepsilon}\left|\frac{\partial Z_{1j}}{\partial\nu} - WZ_{1j}\right| = O(\varepsilon^\alpha). \tag{81}$$

We also compute

$$\Delta(\eta_{2j} Z_{1j}) = \Delta\eta_{2j} Z_{1j} + 2\nabla\eta_{2j}\nabla Z_{1j} + \eta_{2j}\Delta Z_{1j}$$
$$= O\left(\frac{\varepsilon^2}{1+r}\right) + O\left(\frac{\varepsilon}{1+r^2}\right) + \eta_{2j}\Delta Z_{1j}.$$

But $-\Delta Z_{1j} + \varepsilon^2 Z_{1j} = \widetilde{L}(z_{1j})$ where $\widetilde{L}$ is the linear operator (56). Thus

$$-\Delta Z_{1j} + \varepsilon^2 Z_{1j} = O\left(\frac{\varepsilon}{1+r^2}\right) + O\left(\frac{\varepsilon^2}{1+r}\right)$$

and this readily implies

$$\int_{\Omega_\varepsilon} |-\Delta(\eta_{2j} Z_{1j}) + \varepsilon^2 \eta_{2j} Z_{1j}| = O\left(\varepsilon \log\frac{1}{\varepsilon}\right). \tag{82}$$

Combining (80)–(82) we conclude that

$$d_k \int_{\Omega_\varepsilon} \chi_k Z_{1k}^2 \leqslant C\|h\|_{*,\partial\Omega_\varepsilon} + C\varepsilon^\alpha \|\phi\|_{L^\infty(\Omega_\varepsilon)}$$

and this combined with (79) yields

$$|d_k| \leqslant C\left(\|h\|_{*,\partial\Omega_\varepsilon} + C\varepsilon^\alpha \log\frac{1}{\varepsilon}\sum_{j=1}^m |d_j|\right).$$

This implies

$$|d_k| \leqslant C\|h\|_{*,\partial\Omega_\varepsilon} \tag{83}$$

which proves (78).

Now consider the Hilbert space

$$H = \left\{\phi \in H^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} \chi_j Z_{1j}\phi = 0 \quad \forall j = 1,\dots,m\right\}$$

with the norm $\|\phi\|_{H^1}^2 = \int_{\Omega_\varepsilon} |\nabla\phi|^2 + \varepsilon^2\phi^2$. Eq. (77) is equivalent to find $\phi \in H$, such that

$$\int_{\Omega_\varepsilon} (\nabla\phi\nabla\psi + \varepsilon^2\phi\psi) - \int_{\partial\Omega_\varepsilon} W\phi\psi = \int_{\partial\Omega_\varepsilon} h\psi \qquad \forall\psi \in H.$$

By Fredholm's alternative this is equivalent to the uniqueness of solutions to this problem, which is guaranteed by (78).

To show solvability of (33) let $Y_i \in L^\infty(\Omega_\varepsilon)$, $d_{ij} \in \mathbb{R}$ be the solution to (77) with $h = \chi_i Z_{1i}$, that is

$$\begin{cases} -\Delta Y_i + \varepsilon^2 Y_i = \sum_{j=1}^m d_{ij} \chi_j Z_{1j} & \text{in } \Omega_\varepsilon, \\ \frac{\partial Y_i}{\partial \nu} - W Y_i = \chi_i Z_{1i} & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{1j} Y_i = 0 & \forall\, j = 1, \ldots, m. \end{cases} \tag{84}$$

By the previous argument there is a unique $Y_i \in L^\infty(\Omega_\varepsilon)$ solution to this equation, and moreover we have the estimates

$$\|Y_i\|_{L^\infty(\Omega_\varepsilon)} \leqslant C \log \frac{1}{\varepsilon}, \qquad |d_{ij}| \leqslant C \tag{85}$$

for some constant C independent of ε. We shall show that

$$d_{ij} = A\delta_{ij} + O\left(\varepsilon^\alpha \log \frac{1}{\varepsilon}\right), \tag{86}$$

where $A > 0$ is independent of ε and $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$ is Kronecker's delta. Assuming this for a moment, we see that the matrix D with entries d_{ij} is invertible for small ε and $\|D^{-1}\| \leqslant C$ uniformly in ε. Then, given $h \in L^\infty(\partial\Omega_\varepsilon)$ we find ϕ_1, $d_1, \ldots, d_m$ the solution to (77) and define

$$\phi = \phi_1 + \sum_{i=1}^m c_i Y_i,$$

where c_i is such that $\sum_{i=1}^m c_i d_{ij} = -d_j$ $\forall j = 1, \ldots, m$. Then ϕ satisfies (33) and we have the estimate

$$\begin{aligned} \|\phi\|_{L^\infty(\Omega_\varepsilon)} &\leqslant \|\phi_1\|_{L^\infty(\Omega_\varepsilon)} + \log \frac{1}{\varepsilon} \sum_{i=1}^m |c_i| \leqslant C \log \frac{1}{\varepsilon} \|h\|_{*,\partial\Omega_\varepsilon} + \log \frac{1}{\varepsilon} \sum_{i=1}^m |d_i| \\ &\leqslant C \log \frac{1}{\varepsilon} \|h\|_{*,\partial\Omega_\varepsilon}, \end{aligned}$$

by (83).

To prove (86) we multiply (84) by $\eta_{2j} Z_{1j}$ and integrate by parts

$$
\begin{aligned}
d_{ij} \int_{\Omega_\varepsilon} \chi_j Z_{1j}^2 + \delta_{ij} \int_{\partial\Omega_\varepsilon} \chi_j Z_{1j}^2 &= \int_{\partial\Omega_\varepsilon} \left(\frac{\partial Z_{1j}}{\partial \nu} - W Z_{1j} \right) \eta_{2j} Y_i + \int_{\partial\Omega_\varepsilon} \frac{\partial \eta_{2j}}{\partial \nu} Z_{1j} Y_i \\
&\quad + \int_{\Omega_\varepsilon} Y_i (-\Delta(\eta_{2j} Z_{1j}) + \varepsilon^2 \eta_{2j} Z_{1j}) \\
&= O\left(\varepsilon^\alpha \log \frac{1}{\varepsilon} \right),
\end{aligned}
$$

using (81), (82) and (85). □

Remark. A slight modification of the proof above also shows that for any $h \in L^\infty(\partial\Omega_\varepsilon)$ and $f \in L^\infty(\Omega_\varepsilon)$ the equation

$$
\begin{cases}
-\Delta\phi + \varepsilon^2 \phi = f & \text{in } \Omega_\varepsilon, \\
\dfrac{\partial \phi}{\partial \nu} - W\phi = h + \sum_{j=1}^m c_j \chi_j Z_{1j} & \text{on } \partial\Omega_\varepsilon, \\
\int_{\Omega_\varepsilon} \chi_j Z_{1j} \phi = 0 & \forall j = 1, \ldots, m
\end{cases}
$$

has a unique solution ϕ, $c_1, \ldots, c_m$ and that the estimates

$$
\begin{aligned}
\|\phi\|_{L^\infty(\Omega_\varepsilon)} &\leqslant C \log \frac{1}{\varepsilon} (\|h\|_{*,\partial\Omega_\varepsilon} + \|f\|_{**,\Omega_\varepsilon}), \\
|c_j| &\leqslant C(\|h\|_{*,\partial\Omega_\varepsilon} + \|f\|_{**,\Omega_\varepsilon}) \qquad \forall j = 1, \ldots, m
\end{aligned}
$$

hold with C independent of ε.

The result of Proposition 4.1 implies that the unique solution $\phi = T(h)$ of (33) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of all functions h in L^∞ for which $\|h\|_{*,\partial\Omega_\varepsilon} < \infty$, into L^∞.

It is important for later purposes to understand the differentiability of the operator T with respect to the variables ξ_i'. Fix $h \in \mathcal{C}_*$ and let $\phi = T(h)$. We want to compute derivatives of ϕ with respect to, say, ξ_k'. Formally, $Z = \partial_{\xi_k'} \phi$ should satisfy in Ω_ε the equation

$$
-\Delta Z + \varepsilon^2 Z = 0 \qquad \text{in } \Omega_\varepsilon
$$

and on $\partial\Omega_\varepsilon$ the boundary condition

$$
\frac{\partial Z}{\partial \nu} - W Z = -\partial_{\xi_k'}(W)\,\phi + c_k\, \partial_{\xi_k'}(Z_{1k}\chi_k) + \sum_j d_j\, Z_j \chi_j,
$$

where (still formally) $d_j = \partial_{\xi_k'}(c_j)$. The orthogonality conditions now become

$$\int_{\Omega_\varepsilon} Z_{1j}\chi_j Z = 0 \quad \text{if } j \neq k,$$
$$\int_{\Omega_\varepsilon} Z_{1k}\chi_k Z = -\int_{\Omega_\varepsilon} \partial_{\xi_k'}(Z_{1k}\chi_k)\phi.$$

Let us write $\tilde{Z} = Z + b_k\chi_k Z_{1k}$ where

$$b_k \int_{\Omega_\varepsilon} \chi_k^2 |Z_{1k}|^2 \equiv \int_{\Omega_\varepsilon} \phi\, \partial_{\xi_k'}(\chi_k Z_{1k}).$$

Hence $\int_{\Omega_\varepsilon} \tilde{Z}\chi_j Z_{1j} = 0$ for all j,

$$-\Delta\tilde{Z} + \varepsilon^2\tilde{Z} = a \quad \text{in } \Omega_\varepsilon$$

and

$$\frac{\partial\tilde{Z}}{\partial\nu} - W\tilde{Z} = b + \sum_j d_j\, Z_j\chi_j,$$

where

$$a = b_k(-\Delta(\chi_k Z_{1k}) + \varepsilon^2\chi_k Z_{1k})$$

and

$$b = -\partial_{\xi_k'}(W)\,\phi + c_k\,\partial_{\xi_k'}(Z_{1k}\chi_k) + \frac{\partial(\chi_k Z_{1k})}{\partial\nu} - W\chi_k Z_{1k}$$

with

$$\|b\|_{*,\partial\Omega_\varepsilon} \leqslant C\log\frac{1}{\varepsilon}\|h\|_{*,\partial\Omega_\varepsilon}, \qquad \|a\|_{**,\Omega_\varepsilon} \leqslant C\log\frac{1}{\varepsilon}\|h\|_{*,\partial\Omega_\varepsilon}.$$

The remark above gives

$$\|\partial_{\xi_k'}\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\left(\log\frac{1}{\varepsilon}\right)^2 \|h\|_{*,\partial\Omega_\varepsilon}. \tag{87}$$

5. The nonlinear problem

Consider the nonlinear equation

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial\phi}{\partial\nu} - W\phi = R + N(\phi) + \displaystyle\sum_{j=1}^{m} c_j\chi_j Z_{1j} & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon}\chi_j Z_{1j}\phi = 0 & \forall j = 1,\dots,m, \end{cases} \tag{88}$$

where W is as in (26) and N, R are defined in (22) and (23), respectively.

Lemma 5.1. Let $m > 0$, $d > 0$. *Then there exist* $\varepsilon_0 > 0$, $C > 0$, *such that for* $0 < \varepsilon < \varepsilon_0$ *and any* $\xi_1,\dots,\xi_m \in \partial\Omega$ *satisfying*

$$|\xi_i - \xi_j| \geqslant d \qquad \forall i \neq j,$$

the problem (88) *admits a unique solution* ϕ, $c_1,\dots,c_m$ *such that*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\varepsilon^\alpha|\log\varepsilon|, \tag{89}$$

where α *is any number in the interval* $(0,1)$. *Furthermore, the function* $\xi' \to \phi(\xi') \in C(\bar\Omega_\varepsilon)$ *is* C^1 *and*

$$\|D_{\xi'}\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\,\varepsilon^\alpha|\log\varepsilon|^2. \tag{90}$$

Proof. In terms of the operator T defined in the previous section, problem (88) becomes

$$\phi = T(N(\phi) + R) \equiv A(\phi)\,. \tag{91}$$

For a given number $\gamma > 0$, let us consider the region

$$\mathcal{F}_\gamma \equiv \{\phi \in C(\bar\Omega_\varepsilon)\ :\ \|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant \gamma\,\varepsilon^\alpha|\log\varepsilon|\}.$$

From Proposition 4.1, we get

$$\|A(\phi)\|_{L^\infty(\Omega_\varepsilon)} \leqslant C|\log\varepsilon|\left[\|N(\phi)\|_{*,\partial\Omega_\varepsilon} + \|R\|_{*,\partial\Omega_\varepsilon}\right].$$

Estimate (25) implies that $\|R\|_{*,\partial\Omega_\varepsilon} \leqslant C\varepsilon^{\alpha'}$, for any $\alpha' \in (0,1)$. Also, the definition of N in (22) immediately yields $\|N(\phi)\|_{*,\partial\Omega_\varepsilon} \leqslant C\,\|\phi\|^2_{L^\infty(\Omega_\varepsilon)}$. It is also immediate that N

satisfies, for $\phi_1, \phi_2 \in \mathcal{F}_\gamma$,

$$\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega_\varepsilon} \leqslant C\gamma\varepsilon^\alpha|\log\varepsilon|\,\|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)},$$

where C is independent of γ. Hence we get

$$\|A(\phi)\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\varepsilon^\alpha|\log\varepsilon|\left[\gamma^2\,\varepsilon^\alpha|\log\varepsilon|^2 + 1\right],$$

$$\|A(\phi_1) - A(\phi_2)\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\,\gamma\,\varepsilon^\alpha|\log\varepsilon|\,\|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)}\,.$$

It follows that for all sufficiently small ε we get that A is a contraction mapping of $\mathcal{F}_\gamma$, and therefore a unique fixed point of A exists in this region.

Let us now discuss the differentiability of ϕ. Since R depends continuously (in the *-norm) on the m-tuple

$$\xi' = (\xi'_1, \ldots, \xi'_m),$$

the fixed point characterization obviously yields so for the map $\xi' \mapsto \phi$. Then, formally,

$$-\partial_{\xi'_k} N(\phi) = \partial_{\xi'_k} W(e^\phi - \phi - 1) + W[e^\phi - 1]\partial_{\xi'_k}\phi\,.$$

Since $\|\partial_{\xi'_k} W\|_{*,\partial\Omega_\varepsilon}$ is uniformly bounded, we conclude

$$\|\partial_{\xi'_k} N(\phi)\|_{*,\partial\Omega_\varepsilon} \leqslant C\left[\|\phi\|_{L^\infty(\Omega_\varepsilon)} + \|\partial_{\xi'_k}\phi\|_{L^\infty(\Omega_\varepsilon)}\right]\|\phi\|_{L^\infty(\Omega_\varepsilon)}$$

$$\leqslant C\left[\varepsilon^\alpha|\log\varepsilon| + \|\partial_{\xi'_k}\phi\|_{L^\infty(\Omega_\varepsilon)}\right]\varepsilon^\alpha|\log\varepsilon|\,.$$

Also observe that we have

$$\partial_{\xi'_k}\phi = (\partial_{\xi'_k} T)\left(-(N(\phi) + R)\right) + T\left(-\partial_{\xi'_k}\left[N(\phi) + R\right]\right),$$

so that, using (87),

$$\|\partial_{\xi'_k}\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\,|\log\varepsilon|\Big[|\log\varepsilon|\|(N(\phi) + R)\|_{*,\partial\Omega_\varepsilon}$$

$$+ \|\partial_{\xi'_k} N(\phi)\|_{*,\partial\Omega_\varepsilon} + \|\partial_{\xi'_k} R\|_{*,\partial\Omega_\varepsilon})\Big].$$

Since it is also easily checked that $\|\partial_{\xi'_k} R\|_{*,\partial\Omega_\varepsilon} \leqslant C\varepsilon^{\alpha'}$ for any $\alpha' \in (0, 1)$, we conclude from the above computation that

$$\|\partial_{\xi'_k}\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\,\varepsilon^\alpha|\log\varepsilon|^2 \quad \text{for all } k.$$

The above computation can be made rigorous by using the implicit function theorem and the fixed point representation (91) which guarantees C^1 regularity in ξ'. □

6. Variational reduction

In view of Lemma 5.1, given $\xi = (\xi_1, \dots, \xi_m) \in \partial\Omega^m$ satisfying $|\xi_i - \xi_j| \geqslant d \ \forall i \neq j$, we define $\phi(\xi)$ and $c_j(\xi)$ to be the unique solution to (88) satisfying the bound (89).

Given $\xi = (\xi_1, \dots, \xi_m) \in \partial\Omega^m$ we write

$$U(\xi) = \sum_{j=1}^{m} \Big(u_j(x) + H_j^\varepsilon(x) \Big),$$

the ansatz defined in (16). Set

$$F_\varepsilon(\xi) = J_\varepsilon(U(\xi) + \tilde{\phi}(\xi)), \tag{92}$$

where J_ε is the functional defined in (3) and

$$\tilde{\phi}(\xi)(x) = \phi(\xi)\left(\frac{x}{\varepsilon}\right), \quad x \in \Omega. \tag{93}$$

Lemma 6.1. *If $\xi = (\xi_1, \dots, \xi_m) \in (\partial\Omega)^m$ satisfying* (36) *is a critical point of F_ε then $u = U(\xi) + \tilde{\phi}(\xi)$ is a critical point of J_ε, that is, a solution to* (1).

Proof. Let

$$I_\varepsilon(v) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla v|^2 + \varepsilon^2 v^2 - \int_{\partial\Omega_\varepsilon} e^v.$$

Then $F_\varepsilon(\xi) = J_\varepsilon(U(\xi) + \tilde{\phi}(\xi)) = I_\varepsilon(V(\xi') + \phi(\xi'))$, where $\xi' = \xi/\varepsilon$. Therefore

$$\frac{\partial F_\varepsilon}{\partial \xi_k} = \frac{1}{\varepsilon} \frac{\partial I_\varepsilon(V(\xi') + \phi(\xi'))}{\partial \xi'_k} = \frac{1}{\varepsilon} DI_\varepsilon(V(\xi') + \phi(\xi'))\Big[\frac{\partial V(\xi')}{\partial \xi'_k} + \frac{\partial \phi(\xi')}{\partial \xi'_k}\Big].$$

Since $v = V(\xi') + \phi(\xi')$ solves (88)

$$\frac{\partial F_\varepsilon}{\partial \xi_k} = \frac{1}{\varepsilon} \sum_{i=1}^{m} c_i \int_{\partial\Omega_\varepsilon} \chi_i Z_{1i} \Big[\frac{\partial V(\xi')}{\partial \xi'_k} + \frac{\partial \phi(\xi')}{\partial \xi'_k}\Big].$$

Let us assume that $DF(\xi) = 0$. From the previous equation we conclude that

$$\sum_{i=1}^{m} c_i \int_{\partial\Omega_\varepsilon} \chi_i Z_{1i} \Big[\frac{\partial V(\xi')}{\partial \xi'_k} + \frac{\partial \phi(\xi')}{\partial \xi'_k}\Big] = 0 \qquad \forall k = 1, \dots, m.$$

Since $\|\frac{\partial\phi(\xi')}{\partial\xi'_k}\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\varepsilon^\alpha |\log\varepsilon|^2$ and $\frac{\partial V(\xi')}{\partial\xi'_k} = \pm Z_{1k} + o(1)$ where $o(1)$ is in the L^∞ norm, it follows that

$$\sum_{i=1}^{m} c_i \int_{\partial\Omega_\varepsilon} \chi_i Z_{1i} (\pm Z_{1k} + o(1)) = 0 \qquad \forall k = 1, \dots, m,$$

which is a strictly diagonal dominant system. This implies that $c_i=0$ $\forall i=1, \dots, m$. □

In order to solve for critical points of the function F, a key step is its expected closeness to the function $J_\varepsilon(U)$, which we will analyze in the next section.

Lemma 6.2. *The following expansion holds*:

$$F_\varepsilon(\xi) = J_\varepsilon(U) + \theta_\varepsilon(\xi),$$

where

$$|\theta_\varepsilon| + |\nabla\theta_\varepsilon| \to 0,$$

uniformly on points satisfying the constraints (36).

Proof. Let $\tilde\theta_\varepsilon(\xi') = I_\varepsilon(V+\phi) - I_\varepsilon(V)$. In order to get the proof of this lemma, we need to show that

$$|\tilde\theta_\varepsilon| + \varepsilon^{-1}|\nabla_{\xi'}\tilde\theta_\varepsilon| = o(1).$$

Taking into account $DI_\varepsilon(V+\phi)[\phi] = 0$, a Taylor expansion and an integration by parts give

$$\begin{aligned} &I_\varepsilon(V+\phi) - I_\varepsilon(V) \\ &\quad = \int_0^1 D^2 I_\varepsilon(V+t\phi)[\phi]^2\,(1-t)\,dt \\ &\quad = \int_0^1 \left(\int_{\partial\Omega_\varepsilon} [N(\phi)+R]\,\phi + \int_{\partial\Omega_\varepsilon} e^V[1-e^{t\phi}]\,\phi^2\right)(1-t)\,dt, \end{aligned} \tag{94}$$

so we get

$$I_\varepsilon(V+\phi)-I_\varepsilon(V)=\tilde{\theta}_\varepsilon=O(\varepsilon^{2\alpha}|\log\varepsilon|^3)\,.$$

taking into account that $\|\phi\|_{L^\infty(\Omega_\varepsilon)}\leqslant C\varepsilon^\alpha|\log\varepsilon|$. Let us differentiate with respect to ξ_k'

$$\partial_{\xi_k'}[I_\varepsilon(V+\phi)-I_\varepsilon(V)]=\int_0^1\Big(\int_{\partial\Omega_\varepsilon}\partial_{\xi_k'}[(N(\phi)+R)\,\phi]$$
$$+\int_{\partial\Omega_\varepsilon}\partial_{\xi_k'}[e^V[1-e^{t\phi}]\,\phi^2]\Big)(1-t)\,dt\,.$$

Using the fact that $\|\partial_{\xi'}\phi\|_*\leqslant C\,\varepsilon^\alpha|\log\varepsilon|^2$ and the estimates of the previous sections we get

$$\partial_{\xi_{kl}'}[I_\varepsilon(V+\phi)-I_\varepsilon(V)]=\partial_{\xi_{kl}'}\tilde{\theta}_\varepsilon=O(\varepsilon^{2\alpha}|\log\varepsilon|^4)\,.$$

The continuity in ξ of all these expressions is inherited from that of ϕ and its derivatives in ξ in the L^∞ norm. The proof is complete. □

7. Expansion of the energy

Lemma 7.1. *Let* μ_j *be given by* (24). *Then for any* $0<\alpha<1$

$$J_\varepsilon(U)=m(\beta-2\pi+2\pi\log 2)+2\pi m\log\frac{1}{\varepsilon}-\pi\sum_{j=1}^m\left[H(\xi_j,\xi_j)+\sum_{i\neq j}G(\xi_i,\xi_j)\right]$$
$$+O(\varepsilon^\alpha),$$

where

$$\beta=\int_{-\infty}^{\infty}\frac{1}{1+x^2}\log\frac{1}{1+x^2}\,dx.$$

Proof. Define

$$U_j(x)=u_j(x)+H_j^\varepsilon(x),$$

so we may rewrite (16) in equivalent form $U=\sum_{j=1}^m U_j$. Then

$$
\begin{aligned}
J_\varepsilon(U) &= \frac{1}{2}\int_\Omega \left|\sum_{j=1}^m \nabla U_j\right|^2 + \frac{1}{2}\int_\Omega \left(\sum_{j=1}^m U_j\right)^2 - \varepsilon\int_{\partial\Omega} \exp\left(\sum_{j=1}^m U_j\right) \\
&= \sum_{j=1}^m \int_\Omega (|\nabla U_j|^2 + U_j^2) + \sum_{i\neq j}\int_\Omega (\nabla U_i \nabla U_j + U_i U_j) - \varepsilon \int_{\partial\Omega} \exp\left(\sum_{j=1}^m U_j\right) \\
&= I_A + I_B + I_C.
\end{aligned}
$$

Let us analyze the behavior of I_A. We have

$$
\int_\Omega |\nabla U_j|^2 + U_j^2 = \int_\Omega |\nabla u_j|^2 + \int_\Omega u_j^2 + \int_\Omega |\nabla H_j^\varepsilon|^2 + \int_\Omega (H_j^\varepsilon)^2 + 2\int_\Omega \nabla u_j \nabla H_j^\varepsilon \tag{95}
$$

$$
+ 2\int_\Omega u_j H_j^\varepsilon. \tag{96}
$$

Multiplying (17) by H_j^ε yields

$$
\begin{aligned}
\int_\Omega |\nabla H_j^\varepsilon|^2 + (H_j^\varepsilon)^2 &= -\int_\Omega u_j H_j^\varepsilon + \int_{\partial\Omega} \frac{\partial H_j^\varepsilon}{\partial \nu} H_j^\varepsilon \\
&= -\int_\Omega u_j H_j^\varepsilon + \varepsilon\int_{\partial\Omega} e^{u_j} H_j^\varepsilon - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} H_j^\varepsilon
\end{aligned}
$$

and replacing in (95) we obtain

$$
\int_\Omega |\nabla U_j|^2 + U_j^2 = \int_\Omega |\nabla u_j|^2 + \int_\Omega u_j^2 + 2\int_\Omega \nabla u_j \nabla H_j^\varepsilon + \int_\Omega u_j H_j^\varepsilon \tag{97}
$$

$$
+ \varepsilon \int_{\partial\Omega} e^{u_j} H_j^\varepsilon - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} H_j^\varepsilon. \tag{98}
$$

Multiplying (17) by u_j and integrating we find

$$
\int_\Omega u_j^2 + \int_\Omega H_j^\varepsilon u_j = -\int_\Omega \nabla H_j^\varepsilon \nabla u_j + \varepsilon \int_{\partial\Omega} e^{u_j} u_j - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} u_j.
$$

Combining this and (97) we arrive at

$$
\int_\Omega |\nabla U_j|^2 + U_j^2 = \varepsilon \int_{\partial\Omega} e^{u_j}(u_j + H_j^\varepsilon),
$$

where we have used

$$\int_\Omega \nabla u_i \nabla u_j = \int_{\partial\Omega} \frac{\partial u_i}{\partial \nu} u_j, \qquad \int_\Omega \nabla u_i \nabla H_j^\varepsilon = \int_{\partial\Omega} \frac{\partial u_i}{\partial \nu} H_j^\varepsilon \tag{99}$$

with $i = j$. Let us find the asymptotic behavior of the expression:

$$\int_\Omega |\nabla U_j|^2 + U_j^2 = \varepsilon \int_{\partial\Omega} \frac{2\mu_j}{|x-\xi_j-\varepsilon\mu_j \nu(\xi_j)|^2} \left(\log \frac{1}{|x-\xi_j-\varepsilon\mu_j \nu(\xi_j)|^2} \right.$$
$$\left. + H(x,\xi_j) + O(\varepsilon^\alpha) \right).$$

Changing variables $\varepsilon\mu_j y = x - \xi_j$

$$\int_\Omega |\nabla U_j|^2 + U_j^2 = \int_{\partial\Omega_{\varepsilon\mu_j}} \frac{2}{|y-\nu(0)|^2} \left(\log \frac{1}{|y-\nu(0)|^2} + H(\xi_j + \varepsilon\mu_j y, \xi_j) \right.$$
$$\left. - 2\log(\varepsilon\mu_j) \right) + O(\varepsilon^\alpha).$$

But

$$\int_{\partial\Omega_{\varepsilon\mu_j}} \frac{2}{|y-\nu(0)|^2} = \pi + O(\varepsilon), \ \int_{\partial\Omega_{\varepsilon\mu_j}} \frac{2}{|y-\nu(0)|^2} \log \frac{1}{|y-\nu(0)|^2} = \beta + O(\varepsilon^\alpha).$$

and for $0 < \alpha < 1$

$$\int_{\partial\Omega_{\varepsilon\mu_j}} \frac{2}{|y-\nu(0)|^2} (H(\varepsilon\mu_j y, \xi_j) - H(\xi_j,\xi_j)) = \int_{\partial\Omega_{\varepsilon\mu_j}} \frac{2}{|y-\nu(0)|^2} O(\varepsilon^\alpha |y|^\alpha)$$
$$= O(\varepsilon^\alpha).$$

Therefore

$$\int_\Omega |\nabla U_j|^2 + U_j^2 = 2\beta + 2\pi H(\xi_j,\xi_j) - 4\pi \log(\varepsilon\mu_j)$$
$$+ \int_{\partial\Omega_{\varepsilon\mu_j}} \frac{2}{|y-\nu(0)|^2} (H(\varepsilon\mu_j y, \xi_j) - H(\xi_j,\xi_j)) + O(\varepsilon^\alpha)$$
$$= 2\beta + 2\pi H(\xi_j,\xi_j) - 4\pi \log(\varepsilon\mu_j) + O(\varepsilon^\alpha).$$

Thus

$$I_A = m\beta + 2\pi m \log \frac{1}{\varepsilon} + \pi \sum_{j=1}^{m} \Big[H(\xi_j, \xi_j) - 2 \log(\varepsilon \mu_j) \Big] + O(\varepsilon^{\alpha}). \tag{100}$$

We consider now

$$\begin{aligned} I_B &= \sum_{i \neq j}^{m} \int_\Omega \nabla U_i \nabla U_j + U_i U_j \\ &= \sum_{i \neq j}^{m} \int_\Omega \nabla u_i \nabla u_j + 2 \int_\Omega \nabla u_i \nabla H_j^\varepsilon + \int_\Omega \nabla H_i^\varepsilon \nabla H_j^\varepsilon + \int_\Omega u_i u_j + 2 \int_\Omega u_i H_j^\varepsilon \\ &\quad + \int_\Omega H_i^\varepsilon H_j^\varepsilon. \end{aligned}$$

Multiplying Eq. (17) by H_i^ε and integrating we find

$$\int_\Omega \nabla H_j^\varepsilon \nabla H_i^\varepsilon + \int_\Omega H_j^\varepsilon H_i^\varepsilon = - \int_\Omega u_j H_i^\varepsilon + \varepsilon \int_{\partial\Omega} e^{u_j} H_i^\varepsilon - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} H_i^\varepsilon.$$

Hence

$$\begin{aligned} I_B = \sum_{i \neq j}^{m} \int_\Omega \nabla u_i \nabla u_j + 2 \int_\Omega \nabla u_i \nabla H_j^\varepsilon + \int_\Omega u_i u_j + \int_\Omega u_i H_j^\varepsilon + \varepsilon \int_{\partial\Omega} e^{u_j} H_i^\varepsilon \\ - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} H_i^\varepsilon. \end{aligned}$$

Multiplication of (17) by u_i and integration by parts yields

$$\int_\Omega u_j u_i + \int_\Omega H_j^\varepsilon u_i = - \int_\Omega \nabla H_j^\varepsilon \nabla u_i + \varepsilon \int_{\partial\Omega} e^{u_j} u_i - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} u_i.$$

Replacing in the expression above and using (99) we find

$$I_B = \frac{\varepsilon}{2} \sum_{i \neq j}^{m} \int_\Omega e^{u_i} (u_j + H_j^\varepsilon).$$

A similar argument as for I_A shows that

$$I_B = \pi \sum_{i \neq j}^{m} G(\xi_i, \xi_j) + O(\varepsilon^\alpha). \tag{101}$$

Regarding the expression I_C we have

$$I_C = -\varepsilon \int_{\partial\Omega} e^{\sum_{j=1}^m U_j} = -\varepsilon \sum_{k=1}^{m} \int_{\partial\Omega} e^{\sum_{j=1}^m u_j + H_j^\varepsilon}.$$

Using the definition of u_j and (18) for each term we have

$$\varepsilon \int_{\partial\Omega} e^{\sum_{j=1}^m u_j + H_j^\varepsilon} = \varepsilon \int_{\partial\Omega} \frac{e^{H(x,\xi_j)+O(\varepsilon^\alpha)}}{|x - \xi_j - \varepsilon\mu_j \nu(\xi_j)|^2} E_j(x),$$

where

$$E_j(x) = \exp\left(\sum_{i \neq j} \log \frac{1}{|x - \xi_i - \varepsilon\mu_i \nu(\xi_i)|^2} + H(x, \xi_i) + O(\varepsilon^\alpha)\right).$$

Changing variables $\varepsilon\mu_j y = x - \xi_j$ we have

$$e^{H(\xi_j + \varepsilon\mu_j y, \xi_j) + O(\varepsilon^\alpha)} = e^{H(\xi_j,\xi_j)} + O(\varepsilon^\alpha |y|^\alpha)$$

and

$$\begin{aligned} E_j(\xi_j + \varepsilon\mu_j y, \xi_j) = \exp\Bigg(&\sum_{i \neq j} \log \frac{1}{|\xi_j - \xi_i + \varepsilon\mu_j y - \varepsilon\mu_i \nu(\xi_i)|^2} \\ &+ H(\xi_j + \varepsilon\mu_j y, \xi_i) + O(\varepsilon^\alpha)\Bigg) \\ = \exp&\left(\sum_{i \neq j} \log \frac{1}{|\xi_i - \xi_j|^2} + H(\xi_j, \xi_i)\right) + O(\varepsilon^\alpha |y|^\alpha) \\ = \exp&\left(\sum_{i \neq j} G(\xi_j . \xi_i)\right) + O(\varepsilon^\alpha |y|^\alpha). \end{aligned}$$

Therefore, by the definition of μ_j in (24)

$$\varepsilon \int_{\partial\Omega} e^{\sum_{j=1}^{m} u_j + H_j^{\varepsilon}} = \frac{1}{\mu_j} \pi e^{H(\xi_j,\xi_j)+\sum_{i\neq j} G(\xi_j,\xi_i)} + O(\varepsilon^{\alpha})$$
$$= 2\pi + O(\varepsilon^{\alpha}).$$

Thus

$$I_C = -2\pi m + O(\varepsilon^{\alpha}). \tag{102}$$

Thanks to (100)–(102) we have

$$J_\varepsilon(U) = (m\beta - 2\pi) + 2\pi m \log \frac{1}{\varepsilon} + \pi \sum_{j=1}^{m} \left[-2\log(\mu_j) + H(\xi_j, \xi_j) + \sum_{i\neq j} G(\xi_i, \xi_j) \right] + O(\varepsilon^{\alpha}).$$

Employing again (24) we have

$$J_\varepsilon(U) = m(\beta - 2\pi) + 2\pi m \log \frac{1}{\varepsilon} + 2\pi m \log 2 - \pi \sum_{j=1}^{m} \left[H(\xi_j, \xi_j) + \sum_{i\neq j} G(\xi_i, \xi_j) \right] + O(\varepsilon^{\alpha}). \qquad \square$$

8. Proof of Theorem 1.2

Let $\hat{\Omega}_m = (\partial\Omega)^m \setminus D$, where D denotes the diagonal. Namely,

$$\hat{\Omega}_m = \{(\xi_1, \ldots, \xi_m) \in (\partial\Omega)^m \ : \ \xi_i \neq \xi_j \ \text{if} \ \ i \neq j\}.$$

Proof. According to Lemma 6.2, the function $U(\xi)+\tilde{\phi}(\xi)$, where U and $\tilde{\phi}$ are defined, respectively, by (16) and (93), is a solution of Problem (1) if we adjust ξ so that it is a critical point of $F_\varepsilon(\xi) = J_\varepsilon(U(\xi)+\tilde{\phi}(\xi))$ defined by (92). This is obviously equivalent to finding a critical point of

$$\tilde{F}_\varepsilon(\xi) = \frac{1}{\pi} \left(F_\varepsilon(\xi) - m\beta + 2\pi m(1 - \log 2) + 2\pi m \log \varepsilon \right).$$

On the other hand, from Lemmas 6.2 and 7.1, we have that for $\xi \in \hat{\Omega}_m$, such that its components satisfy $|\xi_i - \xi_j| \geqslant d$,

$$\tilde{F}_\varepsilon(\xi) = \varphi_m(\xi) + \varepsilon^\alpha \Theta_\varepsilon(\xi), \tag{103}$$

where Θ_ε and $\nabla_\xi \Theta_\varepsilon$ are uniformly bounded in the considered region as $\varepsilon \to 0$.

Given one component $\mathcal{C}_0$ of $\partial\Omega$, let $\Lambda : S^1 \to \mathcal{C}_0$ be a continuous bijective function that parametrizes $\mathcal{C}_0$. We call $\tilde{\Omega}_m$ the region in $\mathcal{C}_0^m \setminus D$, where $|\xi_i - \xi_j| > d$ and we show that φ_m has at least two distinct critical points in $\tilde{\Omega}_m$.

The function φ_m is C^1, bounded from above in $\hat{\Omega}_m$ (and hence in $\tilde{\Omega}_m$) and such that

$$\varphi_m(\xi_1, \dots, \xi_m) \to -\infty \text{ as } |\xi_i - \xi_j| \to 0 \text{ for some } i \neq j.$$

Hence, since d is arbitrarily small, φ_m has an absolute maximum M in $\tilde{\Omega}_m$.

On the other hand, the Ljusternik–Schnirelmann theory is applicable in our setting so that the number of critical points for φ_m can be estimate from below by $\text{cat}(\tilde{\Omega}_m)$, the Ljusternik–Schnirelmann category of $\tilde{\Omega}_m$ relative to $\tilde{\Omega}_m$. Let us recall that $\text{cat}(\tilde{\Omega}_m)$ is the minimal number of closed and contractible in $\tilde{\Omega}_m$ sets whose union covers $\tilde{\Omega}_m$.

Observe that $\text{cat}(\tilde{\Omega}_m) > 1$. Indeed, by contradiction, assume that $\text{cat}(\tilde{\Omega}_m) = 1$. This means that $\tilde{\Omega}_m$ is contractible in itself, namely there exist a point $\xi^0 \in \tilde{\Omega}_m$ and a continuous function $\Gamma : [0, 1] \times \tilde{\Omega}_m \to \tilde{\Omega}_m$, such that, for all $\xi \in \tilde{\Omega}_m$,

$$\Gamma(0, \xi) = \xi, \quad \Gamma(1, \xi) = \xi^0.$$

Define $f : S^1 \to \tilde{\Omega}_m$ to be the continuous function given by

$$f(\xi_1) = (\Lambda(\xi_1), \Lambda(e^{2\pi i \frac{1}{m}} \xi_1), \dots, \Lambda(e^{2\pi i \frac{m-1}{m}} \xi_1)).$$

Let $\eta : [0, 1] \times S^1 \to S^1$ be the well defined continuous map given by

$$\eta(t, \xi_1) = \Lambda^{-1} \circ \pi_1 \circ \Gamma(t, f(\xi_1)),$$

where π_1 denotes the projection on the first component. The function η is a contraction of S^1 to a point and this gives a contradiction.

Thus we conclude that $\text{cat}(\tilde{\Omega}_m) \geqslant 2$, for any $m \geqslant 1$. Hence, if we define

$$\Xi = \{C \subset \tilde{\Omega}_m \ : \ C \text{ closed and } \text{cat}(C) \geqslant 2\}$$

and

$$c = \sup_{C \in \Xi} \inf_{\bar{\xi} \in C} \varphi_m(\bar{\xi}), \tag{104}$$

Ljusternik–Schnirelmann theory gives that c is a critical level.

If $c \neq M$, we conclude that there are at least two distinct critical points for φ_m in $\tilde{\Omega}_m$. If $c = M$, hence (104) implies that there is at least one set C, with $\mathrm{cat}(C) \geqslant 2$, where the function φ_m reaches its absolute maximum. In this case we conclude that there are infinitely many critical points for φ_m in $\tilde{\Omega}_m$.

These critical points persist under small C^0-perturbation of the function. For this reason, from (103) we can conclude that also the function $\tilde{F}_\varepsilon$, which is C^0 close to φ_m in $\tilde{\Omega}_m$, has at least two distinct critical points in $\tilde{\Omega}_m$. Since d is arbitrarily small, $\tilde{F}_\varepsilon$ has at least two critical points in $\hat{\Omega}_m$ and hence problem (1) has at least two distinct solutions. □

Remark 8.1. As mentioned in the introduction, one can get a stronger result than Theorem 1.2 under the assumption that the function φ_m has, in addition to the ones described in the proof of Theorem 1.2, some other critical points in $\hat{\Omega}_m$ with the property of being *topologically nontrivial*, for instance (possibly degenerate) local minima or maxima, or saddle points.

Let us define what we mean by *topologically nontrivial* critical point for φ_m.

Let Σ be an open set compactly contained in $\hat{\Omega}_m$ with smooth boundary. We recall that φ_m *links in* Σ *at critical level* $\mathcal{C}$ *relative to* B *and* B_0 if B and B_0 are closed subsets of $\bar{\Sigma}$ with B connected and $B_0 \subset B$ such that the following conditions hold: let us set Γ to be the class of all maps $\Phi \in C(B, \Sigma)$ with the property that there exists a function $\Psi \in C([0,1] \times B, \Sigma)$, such that

$$\Psi(0, \cdot) = \mathrm{Id}_B, \quad \Psi(1, \cdot) = \Phi, \quad \Psi(t, \cdot)|_{B_0} = \mathrm{Id}_{B_0} \text{ for all } t \in [0,1].$$

We assume

$$\sup_{y \in B_0} \varphi_m(y) < \mathcal{C} \equiv \inf_{\Phi \in \Gamma} \sup_{y \in B} \varphi_m(\Phi(y)) \tag{105}$$

and for all $y \in \partial\Sigma$, such that $\varphi_m(y) = \mathcal{C}$, there exists a vector τ_y tangent to $\partial\Sigma$ at y such that

$$\nabla \varphi_m(y) \cdot \tau_y \neq 0\,. \tag{106}$$

Under these conditions a critical point $\bar{y} \in \Sigma$ of φ_m with $\varphi_m(\bar{y}) = \mathcal{C}$ exists. Not only this: any function C^1 close to φ_m inherits such critical point.

Going back to our problem, Lemma 6.2 and 7.1 yield that, if φ_m has a topologically nontrivial critical point $\xi = (\xi_1, \dots, \xi_m)$ in $\hat{\Omega}_m$ which satisfies (36), then F_ε itself has a critical point $\xi^\varepsilon = (\xi_1^\varepsilon, \dots, \xi_m^\varepsilon)$, close to ξ for ε small, such that

$$\nabla\varphi_m(\xi_1^\varepsilon, \dots, \xi_m^\varepsilon) \to 0, \quad \varphi_m(\xi_1^\varepsilon, \dots, \xi_m^\varepsilon) \to \mathcal{C}.$$

Hence Lemma 6.1 guarantees the existence of a solution u_ε for (1). Furthermore, from the ansatz (16), we get that, as $\varepsilon \to 0$, u_ε remains uniformly bounded on $\Omega \setminus \bigcup_{j=1}^m B_\delta(\xi_i^\varepsilon)$, and

$$\sup_{B_\delta(\xi_i^\varepsilon)} u_\varepsilon \to +\infty$$

for any $\delta > 0$.

9. Blow up behavior as $\varepsilon \to 0$

In this section we give a proof of Theorem 1.1, but before we need a couple of preliminaries.

Consider the linear equation

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on } \partial\Omega \end{cases} \tag{107}$$

with $h \in L^1(\partial\Omega)$.

The next result is a variant of an estimate of Brezis and Merle [2].

Lemma 9.1. *For any $0 < k < \pi$ there is a constant C depending on k and Ω such that for any $h \in L^1(\partial\Omega)$ and u the solution of* (107) *we have*

$$\int_{\partial\Omega} \exp\left(\frac{k\,|u(x)|}{\|h\|_{L^1(\partial\Omega)}}\right) dx \leqslant C.$$

Proof. We have the representation formula

$$u(x) = \int_{\partial\Omega} \overline{G}(x, y) h(y)\, dy, \tag{108}$$

where $\overline{G} = \frac{1}{2\pi} G$ and G is Green's function defined in (5). Hence

$$\int_{\partial\Omega} \exp\left(\frac{k|u(x)|}{\|h\|_{L^1(\partial\Omega)}}\right) dx \leqslant \int_{\partial\Omega} \exp\left(k \int_{\partial\Omega} |\overline{G}(x,y)| \frac{|h(y)|}{\|h\|_{L^1(\partial\Omega)}}\, dy\right) dx.$$

Using Jensen's inequality we find

$$\int_{\partial\Omega} \exp\left(\frac{k|u(x)|}{\|h\|_{L^1(\partial\Omega)}}\right) dx \leqslant \int_{\partial\Omega}\int_{\partial\Omega} \exp\left(k|\overline{G}(x,y)|\right) \frac{|h(y)|}{\|h\|_{L^1(\partial\Omega)}}\, dy\, dx.$$

But $|\overline{G}(x,y)| \leqslant |\log|x-y||/\pi + C$ so

$$\exp(k|\overline{G}(x,y)|) \leqslant \frac{C'}{|x-y|^{k/\pi}}$$

for all x, y since we are in a bounded domain. Therefore

$$\begin{aligned}\int_{\partial\Omega} \exp\left(\frac{k|u(x)|}{\|h\|_{L^1(\partial\Omega)}}\right) dx &\leqslant C' \int_{\partial\Omega}\int_{\partial\Omega} \frac{1}{|x-y|^{k/\pi}}\, ds(x) \frac{|h(y)|}{\|h\|_{L^1(\partial\Omega)}}\, dy \\ &\leqslant C'' \frac{1}{1-k/\pi}. \qquad \square\end{aligned}$$

We also need the following "strong maximum principle".

Lemma 9.2. *There exists a constant $c > 0$ such that for all $h \in L^1(\partial\Omega)$ with $h \geqslant 0$, the solution u of* (107) *satisfies*

$$u(x) \geqslant c \int_{\partial\Omega} h\, ds \qquad \text{a.e. } \Omega$$

Proof. First note that $G \geqslant 0$ and by the classical strong maximum principle, for each $y \in \partial\Omega$ $G(\cdot, y)$ cannot attain its minimum in Ω. Also, by the Hopf lemma if $G(x,y) = 0$ for some $x, y \in \partial\Omega$, $x \neq y$ then the normal derivative $\frac{\partial G}{\partial \nu_x}(x,y)$ is negative, which is impossible. Therefore, for each $y \in \partial\Omega$ we have $G(\cdot, y) > 0$ in $\overline{\Omega}$.

By a compactness argument we can find a constant $c > 0$ such that

$$G(x,y) \geqslant c$$

for all $y \in \partial\Omega$ and all $x \in \overline{\Omega}$. If $h \in L^1(\partial\Omega)$, $h \geqslant 0$, from the representation formula (108) we see that the conclusion holds. $\square$

Let u_ε be family of solutions to (1). If $\varepsilon e^{u_\varepsilon}$ is unbounded in $L^1(\partial\Omega)$ then by Lemma 9.2 we see that for a subsequence $u_\varepsilon \nearrow \infty$ uniformly in $\overline{\Omega}$.

Proof of Theorem 1.1. The first part of this proof is an adaptation of the argument used in [2]. Since we assume that $\varepsilon e^{u_\varepsilon}$ is bounded in $L^1(\partial\Omega)$ we can select a sequence $\varepsilon_j \to 0$ and a Radon measure $\gamma \geqslant 0$ in $\partial\Omega$ such that $\varepsilon_j e^{u_j} \rightharpoonup \gamma$ weakly in the sense of Radon measures in $\partial\Omega$ where

$$u_j = u_{\varepsilon_j}.$$

We keep this notation throughout the rest of this section.

Claim. *There is a constant $\beta_0 > 0$, such that if for some $x \in \partial\Omega$ we have*

$$\gamma(\{x\}) \leqslant \beta_0$$

then there exists $R > 0$ so that

$$\limsup_j \|u_j\|_{L^\infty(\Omega\cap B_R(x))} < \infty. \tag{109}$$

Indeed, fix some $p > 1$ and choose $\beta_0 = \frac{\pi}{4p}$. Let $B_R(x)$ denote the open ball with center at x and radius $R > 0$. Note that $\gamma(B_R(x)) \to \gamma(\{x\})$ as $R \to 0^+$ so we can select $R > 0$ so that

$$\gamma(B_{2R}(x)) \leqslant 2\beta_0$$

and from now on we fix this $R > 0$ depending only on x.

By standard properties of the weak convergence of Radon measures

$$\limsup_j \varepsilon_j \int_{\partial\Omega\cap B_R(x)} e^{u_j}\, ds \leqslant \gamma(\overline{B}_R(x)) \leqslant 2\beta_0. \tag{110}$$

Let $a_j = \varepsilon_j e^{u_j} \chi_{B_R(x)}$ and v_j be the solution of

$$\begin{cases} -\Delta v_j + v_j = 0 & \text{in } \Omega, \\ \dfrac{\partial v_j}{\partial \nu} = a_j & \text{on } \partial\Omega. \end{cases}$$

Let also $b_j = \varepsilon_j e^{u_j} - a_j$ and $w_j = u_j - v_j$. Note that $b_j \equiv 0$ in $B_R(x)$ therefore by elliptic estimates

$$\|w_j\|_{L^\infty(B_{R/2}(x))} \leqslant C\|w_j\|_{L^1(B_R(x))} \leqslant C.$$

Therefore

$$\begin{aligned}\int_{\partial\Omega\cap B_{R/2}(x)} \left(\varepsilon_j e^{u_j}\right)^p &= \varepsilon_j^p \int_{\partial\Omega\cap B_{R/2}(x)} e^{pw_j} e^{pv_j} \\ &\leqslant C\,\varepsilon_j^p \int_{\partial\Omega\cap B_{R/2}(x)} e^{pv_j} \\ &\leqslant C\,\varepsilon_j^p \int_{\partial\Omega} \exp\left(k_j \frac{v_j}{\|a_j\|_{L^1(\partial\Omega)}}\right), \qquad (111)\end{aligned}$$

where $k_j = p\|a_j\|_{L^1(\partial\Omega)}$. But observe that by (110) and the definition of a_j we have

$$\limsup_j k_j = \limsup_j \, p\|a_j\|_{L^1(\partial\Omega)} \leqslant 2p\beta_0 < \pi.$$

Hence from (111) and Lemma 9.1 we find

$$\int_{\partial\Omega\cap B_{R/2}(x)} \left(\varepsilon_j e^{u_j}\right)^p \, ds \leqslant C\varepsilon_j^p \to 0.$$

This inequality and elliptic estimates imply that

$$\limsup_j \|u_j\|_{L^\infty(\Omega\cap B_{R/4}(x))} < \infty$$

which is the desired conclusion.

Let S denote the set

$$S = \{\, x \in \partial\Omega \mid \gamma(\{x\}) > \beta_0 \,\}.$$

Then S is finite and for every $x \in \partial\Omega\backslash S$ we have that u_j is bounded in a neighborhood of x. Therefore u_j is bounded in compact subsets of $\partial\Omega\backslash S$ and so $\varepsilon_j e^{u_j} \to 0$ uniformly on compact subsets of $\partial\Omega\backslash S$. This shows that the support of m is contained in S and therefore we can write

$$\gamma = \sum_{j=1}^m a_j \delta_{\xi_j},$$

where $a_j > 0$ and $\xi_j \in \partial\Omega$. From the preceding remarks we see that $u_j \to u^*$ and u^* satisfies

$$\begin{cases} -\Delta u^* + u^* = 0 & \text{in } \Omega, \\ \dfrac{\partial u^*}{\partial \nu} = \displaystyle\sum_{j=1}^{m} a_j \delta_{\xi_j} & \text{on } \partial\Omega. \end{cases}$$

From this it follows that

$$u^*(x) = \frac{1}{2\pi} \sum_{j=1}^{m} a_j G(x, \xi_j). \tag{112}$$

We shall now prove (7) through Pohozaev-type identities in balls around the singularities. Let us concentrate on ξ_1 and assume that it is located in the origin. For the computations we will make a change of variables to flatten the boundary of Ω around 0. Pick some radius R_0 small enough and consider a map $\Psi : H \cap B_{R_0} \to \Omega \cap B_r$, where $H = \{ (y_1, y_2) \mid y_2 > 0\}$. We can choose Ψ to be a conformal diffeomorphism, C^3 up $\partial H \cap B_{R_0}$, and such that $\Psi(0) = 0$ and $D\Psi(0) = I$ (after rotation of Ω). Define

$$\tilde{u}_j(y) = u_j(\Psi(y)), \quad y \in H \cap B_{R_0}.$$

Then $\tilde{u}_j$ satisfies

$$\begin{cases} -\Delta \tilde{u}_j + b(y)\tilde{u}_j = 0 & \text{in } H \cap B_{R_0}, \\ \dfrac{\partial \tilde{u}_j}{\partial \nu} = \varepsilon_j h(y) e^{\tilde{u}_j} & \text{on } \partial H \cap B_{R_0}, \end{cases} \tag{113}$$

where b and h are smooth functions, given by

$$b(y) = |\det D\Psi(y)|,$$
$$h(y) = |D\Psi(y)e_1|$$

and $e_1 = (1, 0)$. Note, since we assume $D\Psi(0) = I$ we can drop the absolute values in b and h.

For simplicity we will drop the index j in $\tilde{u}_j$ and we write the partial derivative $\frac{\partial}{\partial y_k}$ with a subscript $(\cdot)_{y_k}$, e.g. $\frac{\partial \tilde{u}_i}{\partial y_1} = \tilde{u}_{y_1}$. We use the convention of summation over repeated indices, and denote by ν the exterior normal vector to $\partial(H \cap B_R)$. ν_1 and ν_2 are the components of ν and we write a partial derivative with respect to ν as $\frac{\partial \tilde{u}}{\partial \nu} = \tilde{u}_\nu$.

Take now $0 < R < R_0$ and multiply the equation in (113) by $\tilde{u}_{y_1}$ and integrate on $H \cap B_R$ to find

$$\int_{H\cap B_R} -\tilde{u}_{y_i y_i}\tilde{u}_{y_1} + b(y)\tilde{u}\tilde{u}_{y_1} = 0.$$

Integrating by parts, and using the boundary condition in (113) we get

$$\begin{aligned} &\varepsilon \int_{\partial H\cap B_R} he^{\tilde{u}}\tilde{u}_{y_1} + \int_{\partial B_R\cap H} \tilde{u}_\nu \tilde{u}_{y_1} \\ &= \int_{H\cap B_R} \tilde{u}_{y_i}\tilde{u}_{y_i y_1} + b\tilde{u}\tilde{u}_{y_1} = \int_{H\cap B_R} \frac{1}{2}(\tilde{u}_{y_i}^2)_{y_1} + \frac{1}{2} b\,(\tilde{u}^2)_{y_1} \\ &= \frac{1}{2}\int_{\partial B_R\cap H} (|\nabla \tilde{u}|^2 + b\tilde{u}^2)\nu_1 - \frac{1}{2}\int_{H\cap B_R} b_{y_1}\tilde{u}^2. \end{aligned} \tag{114}$$

Integrating by parts the first term in (114) we find

$$\int_{\partial H\cap B_R} he^{\tilde{u}}\tilde{u}_{y_1} = he^{\tilde{u}}\Big|_{-R}^{R} - \int_{\partial H\cap B_R} h' e^{\tilde{u}}$$

and substituting in (114) we obtain

$$\begin{aligned} \varepsilon he^{\tilde{u}}\Big|_{-R}^{R} - \varepsilon \int_{\partial H\cap B_R} h' e^{\tilde{u}} + \int_{\partial B_R\cap H} \tilde{u}_\nu \tilde{u}_{y_1} = \frac{1}{2}\int_{\partial B_R\cap H} (|\nabla \tilde{u}|^2 + b\tilde{u}^2)\nu_1 \\ - \frac{1}{2}\int_{H\cap B_R} b_{y_1}\tilde{u}^2. \end{aligned} \tag{115}$$

Before we take the limit as $j \to \infty$ we recall that $\Psi : H \cap B_{R_0} \to \Omega$ is a conformal map, smooth up to $\partial H \cap B_{R_0}$, and that we assumed $\Psi(0) = 0$ and $D\Psi(0) = I$. Using complex variables $z = y_1 + iy_2$, and expanding Ψ in its Taylor series, we have

$$\Psi(z) = z + \frac{c}{2}z^2 + O(z^3), \tag{116}$$

where $O(z^3)$ denotes a quantity which is bounded by $|z|^3$ for z in a fixed neighborhood of the origin. Let α, $\beta \in \mathbb{R}$ denote the real and imaginary parts of $c = \Psi''(0)$ that is $c = \alpha + i\beta$. Then β is the curvature of the $\partial\Omega$ at 0 and α is curvature at 0 of $t \mapsto \Psi(0, t)$ which is a curve transverse to $\partial\Omega$. We can modify Ψ to prescribe this number. Indeed, consider a change of variables

$$z = \frac{1}{\lambda}(e^{\lambda w} - 1),$$

where w is in a neighborhood of the origin. For $\lambda \in \mathbb{R}$ this map restricted to a neighborhood of the origin sends the upper half-plane in itself, the real line into the real line, and the lower half-plane into itself again. A computation shows that the expansion of Ψ in the variable w is

$$\Psi(w) = w + \tfrac{1}{2}(c+\lambda)w^2 + O(w^3).$$

Let $\tilde{u}^*(y) = u^*(\Psi(y))$ denote the limit function in the y coordinates, and observe that

$$\tilde{u}_j \to \tilde{u}^* \quad \text{in } C^1_{\mathrm{loc}}(\overline{H} \cap B_R - \{0\}).$$

Taking the limit in (115) we get

$$-\alpha a_1 + \int_{\partial B_R \cap H} \tilde{u}^*_\nu \tilde{u}^*_{y_1} = \frac{1}{2}\int_{\partial B_R \cap H} |\nabla \tilde{u}^*|^2 \nu_1 + \frac{1}{2}\int_{\partial B_R \cap H} b\, \tilde{u}^{*2} \nu_1 - \frac{1}{2}\int_{H \cap B_R} b_{y_1} \tilde{u}^{*2}. \tag{117}$$

We rewrite now (112) into a singular and a regular part near the origin

$$u^* = s + w,$$

where

$$s(x) = \frac{a_1}{\pi} \log \frac{1}{|x|} \quad \text{and } w(x) = \frac{a_1}{2\pi} H(x,0) + \frac{1}{2\pi}\sum_{j=2}^{m} a_j G(x,\xi_j).$$

We define then the corresponding functions in the new coordinates

$$\tilde{s}(y) = s(\Psi(y)), \qquad \tilde{w}(y) = w(\Psi(y)), \quad y \in H \cap B_{R_0}.$$

Using this decomposition (117) takes the form

$$\begin{aligned} -\alpha a_1 &+ \int_{\partial B_R \cap H} \tilde{s}_\nu \tilde{s}_{y_1} + \tilde{s}_\nu \tilde{w}_{y_1} + \tilde{s}_{y_1}\tilde{w}_\nu + \tilde{w}_\nu \tilde{w}_{y_1} \\ &= \int_{\partial B_R \cap H} \left(\frac{1}{2}|\nabla \tilde{s}|^2 + \nabla \tilde{s} \nabla \tilde{w} + \frac{1}{2}|\nabla \tilde{w}|^2 \right) \nu_1 \\ &\quad + \int_{\partial B_R \cap H} b \left(\frac{1}{2}\tilde{s}^2 + \tilde{s}\tilde{w} + \frac{1}{2}\tilde{w}^2 \right) \nu_1 \\ &\quad - \int_{H \cap B_R} b_{y_1} \left(\frac{1}{2}\tilde{s}^2 + \tilde{s}\tilde{w} + \frac{1}{2}\tilde{w}^2 \right). \end{aligned} \tag{118}$$

Since w satisfies $-\Delta w + w = -\frac{a_1}{\pi}\log\frac{1}{|x|}$ in Ω we find for $\tilde w$

$$-\Delta\tilde w + b(y)\tilde w = -\frac{a_1}{\pi}m(y) \quad \text{in } H\cap B_{R_0},$$

where

$$m(y) = b(y)\log\frac{1}{|\Psi(y)|}.$$

Multiplying this equation by $\frac{\partial\tilde w}{\partial y_1} = \tilde w_{y_1}$ and integrating on $H\cap B_R$ (similarly as for $\tilde u$)

$$\int_{\partial B_R\cap H}\tilde w_\nu\tilde w_{y_1} - \int_{\partial H\cap B_R}\tilde w_{y_2}\tilde w_{y_1} = \frac12\int_{\partial B_R\cap H}(|\nabla\tilde w|^2 + b\tilde w^2)\nu_1 - \frac12\int_{H\cap B_R} b_{y_1}\tilde w^2 \\ - \frac{a_1}{\pi}\int_{B_R\cap H} m(y)\tilde w_{y_1}.$$

Solving for $\int_{\partial B_R\cap H}\tilde w_\nu\tilde w_{y_1}$ in this equation and replacing in (118) we obtain

$$\begin{aligned} &-\alpha a_1 + \int_{\partial B_R\cap H}\tilde s_\nu\tilde s_{y_1} + \tilde s_\nu\tilde w_{y_1} + \tilde s_{y_1}\tilde w_\nu \\ &\quad= \int_{\partial B_R\cap H}\left(\frac12|\nabla\tilde s|^2 + \nabla\tilde s\nabla\tilde w\right)\nu_1 + \int_{\partial B_R\cap H} b\left(\frac12\tilde s^2 + \tilde s\tilde w\right)\nu_1 \\ &\qquad - \int_{\partial H\cap B_R} b_{y_1}\left(\frac12\tilde s^2 + \tilde s\tilde w\right) - \int_{\partial H\cap B_R}\tilde w_{y_2}\tilde w_{y_1} \\ &\qquad + \frac{a_1}{\pi}\int_{B_R\cap H} m(y)\tilde w_{y_1}. \end{aligned} \tag{119}$$

Now we take the limit in this relation as $R\to 0$.

Lemma 9.3. *Recall that $c = \psi''(0) = \alpha + i\beta$. We have*

$$\begin{aligned} \lim_{R\to0}\int_{\partial B_R\cap H}\tilde s_\nu\tilde s_{y_1} &= \frac{3\alpha}{4\pi}a_1^2, \\ \lim_{R\to0}\int_{\partial B_R\cap H}\tilde s_\nu\tilde w_{y_1} &= -a_1\tilde w_{y_1}(0), \\ \lim_{R\to0}\frac12\int_{\partial B_R\cap H}|\nabla\tilde s|^2\nu_1 &= \frac{\alpha}{4\pi}a_1^2, \\ \lim_{R\to0}\int_{\partial B_R\cap H}\nabla\tilde s\nabla\tilde w\nu_1 &= -\frac{a_1}{2}\tilde w_{y_1}(0) \end{aligned} \tag{120}$$

and all other terms in (119) *have limit zero as* $R\to 0$.

We prove this lemma later on.

Proof of (7) completed. Using this lemma together with (119) we obtain

$$-\alpha a_1 + \frac{3\alpha}{4\pi} a_1^2 - a_1 \tilde{w}_{y_1}(0) = \frac{\alpha}{4\pi} a_1^2 - \frac{a_1}{2} \tilde{w}_{y_1}(0)$$

that is

$$\alpha a_1 \left(\frac{a_1}{2\pi} - 1 \right) = \frac{1}{2} a_1 \tilde{w}_{y_1}(0). \tag{121}$$

But $a_1 \neq 0$ and α can be taken to be any real number, so $a_1 = 2\pi$ and $\tilde{w}_{y_1}(0) = 0$, which is equivalent to $\nabla_\tau w(0) = 0$, where τ is tangent to $\partial\Omega$ at 0. $\square$

Proof of Lemma 9.3. We present a proof of (120) only, the others being analogous. Recall that

$$\tilde{s}(y) = s(\Psi(y)), \quad s(x) = \frac{a_1}{\pi} \log \frac{1}{|x|}.$$

Using the expansion (116) for Ψ we have

$$\nabla \tilde{s}(y) = -\frac{a_1}{\pi} \left(\frac{y}{|y|^2} + \frac{1}{2}(\alpha, -\beta) \right) + O(|y|), \tag{122}$$

where we recall that $c = \alpha + i\beta = \Psi''(0)$. One way to see this is to consider the complex valued function $\log \Psi$, and express the expansion of its derivative in terms of $z = y_1 + iy_2$. Using (122) we have

$$\begin{aligned}
\int_{\partial B_R \cap H} \tilde{s}_\nu \tilde{s}_{y_1} &= \frac{a_1^2}{\pi^2} \int_{\partial B_R \cap H} \left(\frac{1}{R} + \frac{1}{2}(\alpha \nu_1 - \beta \nu_2) + O(R) \right) \left(\frac{\nu_1}{R} + \frac{1}{2}\alpha + O(R) \right) ds \\
&= \frac{a_1^2}{\pi^2} \int_{\partial B_R \cap H} \frac{\nu_1}{R^2} + \frac{1}{2R}(\alpha(1 + \nu_1^2) - \beta \nu_1 \nu_2) + O(1)\, ds \\
&= \frac{a_1^2}{\pi^2} \frac{3}{4} \pi \alpha + O(R). \qquad \square
\end{aligned}$$

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