On the stable hole solutions in the complex Ginzburg–Landau equation

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Abstract

We show numerically that the one-dimensional quintic complex Ginzburg–Landau equation admits four different types of stable hole solutions. We present a simple analytic method which permits to calculate the region of existence and approximate shape of stable hole solutions in this equation. The analytic results are in good agreement with numerical simulations.

Keywords: Ginzburg–Landau equation; Stable hole solutions

1. Introduction

The study of amplitude equations, which can be derived in the vicinity of symmetry-breaking instabilities, has been useful in order to gain an insight into non-equilibrium phenomena in spatially extended systems [1]. The cubic complex Ginzburg–Landau (G–L) equation, which describes the large-scale modulations of the bifurcated solutions near oscillatory instabilities, is well known and shows a rich spatio-temporal behavior including chaos [2]. However, this equation does not admit stable localized structures, like pulses or holes, because the trivial solution zero is not an attractor of the system. The hole solutions of Nozaki and Bekki are structurally unstable [3]. To obtain stable localized solutions, in particular, holes, we focus on the

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quintic complex G–L equation:

$$\partial_t A = \mu A + \beta|A|^2 A + \gamma|A|^4 A + D\partial_{xx} A \ . \tag{1}$$

The subscript x denotes a partial derivative with respect to x, $A(x,t) = R\exp \mathrm{i}\phi$ is a complex field, and the parameters β, γ, and D are in general complex. Nevertheless, Eq. (1) admits stable stationary hole solutions with most parameters being real [4]. In this article we shall consider μ, γ and D as real; $\beta = \beta_r + \mathrm{i}\beta_i$. The signs of the parameters $\beta_r > 0$ and $\gamma < 0$ are chosen in order to guarantee that the bifurcation is subcritical and saturates to quintic order. Due to the existence of the parameter β_i the system is non-variational and admits stable localized structures. Recent numerical simulations of Eq. (1) with periodic boundary conditions for $\mu = -0.06$, $\beta_r = 1.125$, $\gamma = -0.859375$ and $D = 1$ lead to four different types of holes as a function of β_i and the initial conditions [5]: stationary 2π-holes, whose modulus does not touch zero (Fig. 1(a)), stationary π-holes of the first kind, whose modulus touches zero (Fig. 1(b)), stationary π-holes of the second kind, whose modulus touches zero (Fig. 1(c)), and left- (Fig.1(d)) or right-moving π-holes. Stationary π-holes of the first and second kind have qualitatively different slopes around $R = 0$.

In this article we present a simple analytic method which enables us to calculate the region of existence and approximate shape of stable hole solutions. This analytic approach has been successfully used in previous studies

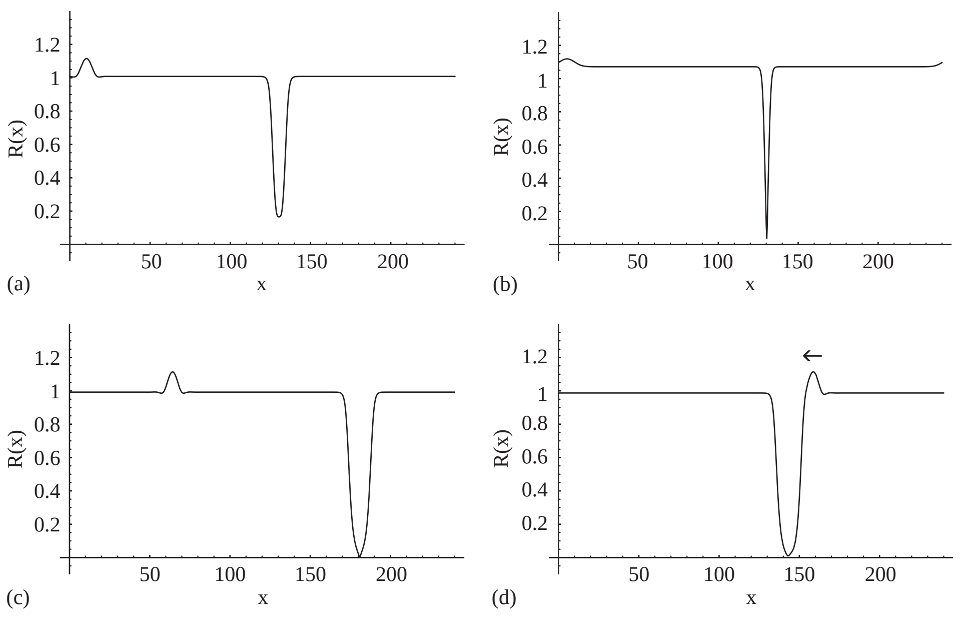

Fig. 1. The modulus R is shown for different stable holes. (a) Stationary 2π-hole for $\beta_i = 0.480$. (b) Stationary π-hole of the first kind for $\beta_i = 0.480$. (c) Stationary π-hole of the second kind for $\beta_i = 0.495$. (d) Left-moving π-hole for $\beta_i = 0.502$.

dealing with the appearance mechanism of stationary, oscillating and moving pulses in the quintic G–L equation [6–9] and breathing localized solutions in reaction–diffusion systems [10,11].

2. Analytical approach

Carrying out the following Ansatz: $r = R(x)$; $\phi = \Omega t + \theta(x)$, Eq. (1) reduces to

$$0 = \mu R + \beta_r R^3 + \gamma R^5 + D(R_{xx} - R\theta_x^2) , \tag{2}$$

$$R\Omega = \beta_i R^3 + D(R\theta_{xx} + 2R_x\theta_x) . \tag{3}$$

The strategy to approximately calculate $R(x)$, $\theta(x)$, and Ω consists in considering that $\theta_x(x)$ (the wave vector) is constant ($-p$ for the left side, $+p$ for the right side) in almost all the domains (*outside the core*) except in a narrow domain around the center of the hole (*core*), where $\theta_x(x)$ is considered to be a cubic function (see Fig. 2).

Outside the core ($|x| > x_*$), Eqs. (2) and (3) lead to

$$0 = (\mu - Dp^2)R + \beta_r R^3 + \gamma R^5 + DR_{xx} , \tag{4}$$

$$R\Omega = \beta_i R^3 \pm 2DpR_x . \tag{5}$$

Asymptotically, for $x \to \infty$, from the above equations we obtain R_0, the asymptotic value of the modulus of the stationary hole, and its frequency Ω: $R_0 = \sqrt{\frac{-\beta_r - \sqrt{\beta_r^2 - 4\gamma(\mu - Dp^2)}}{2\gamma}}$, $\quad \Omega = -\frac{\beta_i}{2\gamma}(\beta_r + \sqrt{\beta_r^2 - 4\gamma(\mu - Dp^2)})$.

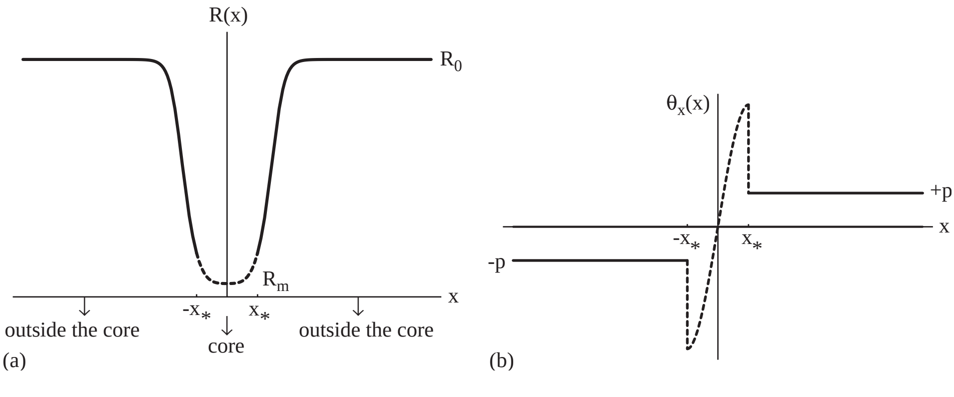

Fig. 2. Analytical approximation for the stationary hole. The space is divided into two regions: *outside the core*, where the wave vector is constant, and *core*, where the wave vector is a cubic function. The value $x = x_*$ corresponds to the local maximum of the cubic function. (a) Modulus of the pulse. (b) Wave vector.

By integrating Eq. (5) it is possible to obtain an explicit expression for $R(x)$:

$$R(x) = \frac{R_0}{\sqrt{1 + \exp\left\{ -\dfrac{\Omega}{Dp}(|x| - x_0) \right\}}} \,, \tag{6}$$

where x_0 is a constant to be determined, which is related to the translational symmetry of Eq. (5).

Inside the core ($|x| < x_*$) we assume that $R(x) = R_m + \varepsilon x^2 + \delta x^4$ and $\theta_x = \alpha x - \beta x^3$, where R_m is the height of the hole at $x = 0$. Then $x_* = \sqrt{\alpha/3\beta}$.

From Eqs. (2) and (3) we can calculate the values of ε, δ, β and α:

$$\varepsilon = \frac{-\mu R_m - \beta_r R_m^3 - \gamma R_m^5}{2D}, \quad \delta = \frac{D R_m \alpha^2 - \mu\varepsilon - 3\beta_r R_m^2 \varepsilon - 5\gamma R_m^4 \varepsilon}{12D} \,,$$

$$\beta = \frac{\varepsilon(3\beta_i R_m^2 + 5D\alpha - \Omega)}{3DR_m}, \quad \alpha = \frac{\Omega - \beta_i R_m^2}{D} \,. \tag{7}$$

Imposing continuity of the amplitude $R(x)$ at $x = x_*$ we obtain

$$x_0 = x_* + \frac{Dp}{\Omega} \ln\left| \frac{R_0^2}{r_c^2} - 1 \right| ,$$

where $r_c = R(x_*)$. The continuity of the first derivative of the amplitude $R(x)$ leads to a first relation between R_m and p:

$$f(p, R_m) \equiv \frac{r_c \Omega (R_0^2 - r_c^2)}{2DpR_0^2} - 2\varepsilon x_* - 4\delta x_*^3 = 0 \,. \tag{8}$$

In order to obtain a second relation between R_m and p we use a *consistency relation* by multiplying Eq. (3) by $R(x)$ and integrating from 0 to ∞.

$$\Omega \int_0^\infty R^2 \, \mathrm{d}x = \beta_i \int_0^\infty R^4 \, \mathrm{d}x + DpR_0^2 \,. \tag{9}$$

Taking into account that expression (6) represents an exact solution of Eq. (3) the above consistency relation reduces to

$$g(p, R_m) \equiv \Omega \int_0^{x_*} R^2 \, \mathrm{d}x - \beta_i \int_0^{x_*} R^4 \, \mathrm{d}x - Dpr_c^2 = 0 \,. \tag{10}$$

3. Appearance of stationary holes

Thus we have constructed approximate expressions for $R(x)$ and $\theta_x(x)$ in all the domains in terms of two unknown parameters, namely, R_m and p. The existence of stationary holes is related to the intersection between the curves $f(p, R_m) = 0$ and $g(p, R_m) = 0$.

For Eq. (1), and for fixed parameters μ, β_r, γ and D we found the following scenario: there exists a critical value β_{ic1} so that for $\beta_i < \beta_{ic1}$ the curves $f(p, R_m) = 0$

and $g(p, R_m) = 0$ do not intersect at any point, suggesting that there are no holes (see Fig. 3(a)). For $\beta_i > \beta_{ic1}$ the curves intersect in two points leading to a stable and an unstable hole via a saddle-node bifurcation (see Fig. 3(b)). By further increasing β_i we find another critical value β_{ic2} so that for $\beta_i > \beta_{ic2}$ there still exists an intersection between the curves $f = g = 0$ predicting an unstable hole, but the stable hole disappears (see Fig. 3(c)).

To verify this analytical prediction in a concrete case we fix the parameters of Eq. (1): $\mu = -0.06$; $\beta_r = 1.125$; $\gamma_r = -0.859375$; and $D = 1$. A numerical simulation of Eq. (1) with periodic (or Neumann) boundary conditions leads to stable stationary holes in the range $\beta_i \in [0.456, 0.503]$, which is in good agreement with our theoretical prediction. In Fig. 3(c) we can see that the holes disappear when the height of the center of the hole disappears. This fact has been numerically confirmed. In Fig. 4(a) we show the real part of the field A (thin line) and the modulus $R(x)$ of the hole for $\beta_i = 0.48$ obtained by a direct numerical simulation. In Fig. 4(b) we draw the corresponding hole predicted by us (see Fig. 3(c)).

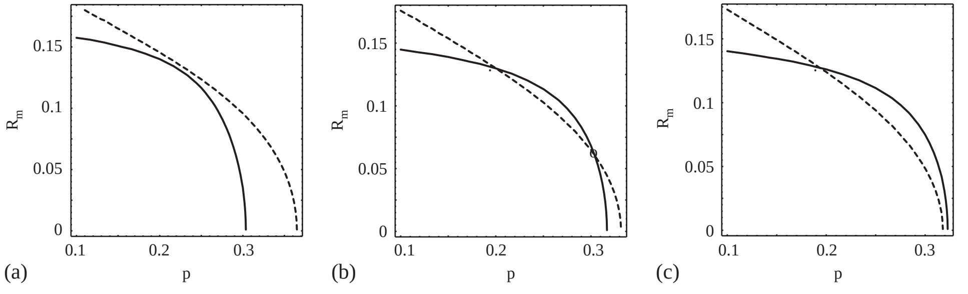

Fig. 3. Values of the parameters are: $\mu = -0.06$; $\beta_r = 1.125$; $\gamma_r = -0.859375$; and $D = 1$. $\beta_{ic1} = 0.43$ and $\beta_{ic2} = 0.50$. (a) $\beta_i = 0.4 < \beta_{ic1}$. (b) $\beta_i = 0.48 > \beta_{ic1}$. The intersection between the curves $f(p, R_m) = 0$ (continuous line) and $g(p, R_m) = 0$ (dashed line) predicts one unstable (solid circle) and one stable hole (open circle). (c) $\beta_i = 0.51 > \beta_{ic2}$.

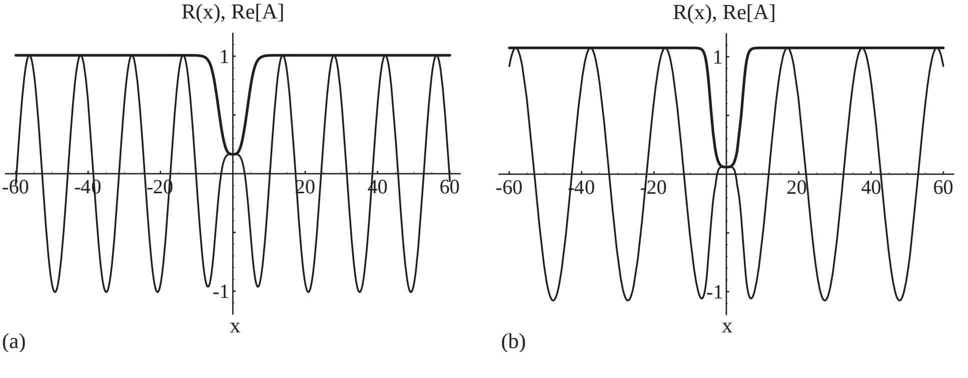

Fig. 4. Values of the parameters are: $\mu = -0.06$; $\beta_r = 1.125$; $\beta_i = 0.48$; $\gamma_r = -0.859375$; $D = 1$. The thin continuous line represents $\mathrm{Re}\, A$ and the thick line represents the modulus of the hole $R(x)$. (a) Numerical simulation. (b) Analytical prediction.

4. Conclusions

In this article we show that the quintic complex G–L equation admits four different types of stable hole solutions with most parameters being real. In addition, we present a simple analytic method which enables us to calculate the region of existence and approximate shape of stationary 2π-holes. We conjecture that the method is also useful for the study of the other kind of holes.

Acknowledgements

O.D. wishes to express his gratitudes for the support of FAI (Project ICIV-001-04, Universidad de los Andes) and FONDECYT (Project 1050660). O.D. also thanks Helmut Brand for stimulating discussions and hospitality during his stay at the University of Bayreuth, Germany. E.T. wishes to thank FONDECYT (Project 1020374) and FONDAP (Project 11980002).

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