A dynamical system approach to the construction of singular solutions of some degenerate elliptic equations

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Abstract

We study the existence of singular separable solutions to the 2-dimensional quasilinear equation $-\nabla \cdot (|\nabla u|^{p-2}\nabla u) + |u|^{q-1}u = 0$ under the form $u(r,\theta) = r^{-\beta}\omega(\theta)$. We obtain the full description of the set of such solutions by combining a 2-dimensional shooting method with a phase plane analysis approach.

Keywords: p-Laplacian; Singularities; Phase plane analysis; Poincaré map

1. Introduction

Let $1 < p < q+1$ be real numbers, and $(r,\theta) \in [0,\infty) \times S^1$ the polar coordinates in $\mathbb{R}^2$. The aim of this article is to give a complete description of the set of separable solutions of the degenerate elliptic equation with absorption

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u) + |u|^{q-1}u = 0 \tag{1.1}$$

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in $\mathbb{R}^2\backslash\{0\}$ in the form

$$u(r,\theta) = r^{-\beta}\omega(\theta), \quad (r,\theta)\in(0,\infty)\times S^1, \tag{1.2}$$

for some $\beta\in\mathbb{R}$. For homogeneity reasons

$$\beta = \beta_q = p/(q+1-p), \tag{1.3}$$

while ω is a 2π-periodic solution of

$$-\frac{d}{d\theta}[(\beta^2\omega^2+\omega'^2)^{\frac{p-2}{2}}\omega'] - a_{p,q}[\beta^2\omega^2+\omega'^2]^{\frac{p-2}{2}}\omega + \omega|\omega|^{q-1} = 0, \tag{1.4}$$

where $a_{p,q} = \beta_q((\beta_q+1)(p-1)-1)$.

If $p=2$, then $\beta_q = 2/(q-1)$ and (1.4) reduces to

$$-\omega'' - \left(\frac{2}{q-1}\right)^2\omega + \omega|\omega|^{q-1} = 0, \tag{1.5}$$

which is the Euler–Lagrange equation of the functional E defined by

$$E(\omega) = \frac{1}{2}\int_{S^1}\left(\omega'^2 - \left(\frac{2}{q-1}\right)^2\omega^2 + \frac{2}{q+1}|\omega|^{q+1}\right)d\theta. \tag{1.6}$$

Moreover, there exists an obvious first integral (usually called the Painlevé first integral), obtained here by a simple multiplication by ω',

$$\frac{d}{d\theta}\left(\omega'^2 + \left(\frac{2}{q-1}\right)^2\omega^2 - \frac{2}{q+1}|\omega|^{q+1}\right) = 0. \tag{1.7}$$

Those two observations make (1.5) easy to integrate. If we put $\lambda = 4/(q-1)^2$, we replace (1.5) by the equation

$$-\omega'' - \lambda\omega + \omega|\omega|^{q-1} = 0, \tag{1.8}$$

and denote by $\mathscr{E}_\lambda$ the set of its solutions. If $k(\lambda)$ is the largest integer smaller than $\sqrt{|\lambda|}$, the following result is proved in [3].

Theorem A. *If* $\lambda\leqslant 0$, $\mathscr{E}_\lambda = \{0\}$. *If* $0<\lambda\leqslant 1$, $\mathscr{E}_\lambda = \{0, \lambda^{1/(q-1)}, -\lambda^{1/(q-1)}\}$. *If* $\lambda>1$, $\mathscr{E}_\lambda$ *has* $3+k(\lambda)$ *connected components*: $\mathscr{E}^0_\lambda$, $\mathscr{E}^+_\lambda$, $\mathscr{E}^-_\lambda$, *and* $\mathscr{E}^k_\lambda$ $(1\leqslant k\leqslant k(\lambda))$, *where*

(i) $\mathscr{E}^0_\lambda = \{0\}$, $\mathscr{E}^+_\lambda = \{\lambda^{1/(q-1)}\}$, *and* $\mathscr{E}^-_\lambda = \{-\lambda^{1/(q-1)}\}$;

(ii) *for each* $1\leqslant k\leqslant k(\lambda)$, $\mathscr{E}^k_\lambda$ *is the set of all solutions to* 1.8 *with least anti-period* π/k, *and* $\mathscr{E}^k_\lambda = \{\omega(.+\alpha)\colon \alpha\in S^1\}$.

The above result can be interpreted via the bifurcation approach since when $\lambda = \lambda_k = k^2$ the linearized problem

$$-\psi'' - \lambda\psi = 0, \tag{1.9}$$

is singular and any couple $(\lambda_k, 0)$ is a bifurcation point from which a branch of solutions (λ, ω) is issued. Moreover, any of these branches of solutions can be continued for $\lambda > \lambda_k$, and there exists no other solution.

When $p \neq 2$, (1.4) is not the Euler equation of any functional, and this makes the problem much more difficult to study. It is natural to introduce the set of singular separable p-harmonic functions, i.e. the set of solutions of

$$-\nabla \cdot (|\nabla v|^{p-2}\nabla v) = 0, \tag{1.10}$$

in $\mathbb{R}^2 \backslash \{0\}$ which are written in the form

$$v(r,\theta) = r^{-\gamma}\phi(\theta), \quad (r,\theta) \in (0,\infty) \times S^1.$$

Then ϕ is a 2π-periodic solution of

$$-\frac{d}{d\theta}[(\gamma^2\phi^2 + \phi'^2)^{\frac{p-2}{2}}\phi'] - b_{p,\gamma}[\gamma^2\phi^2 + \phi'^2]^{\frac{p-2}{2}}\phi = 0, \tag{1.11}$$

where $b_{p,\gamma} = \gamma((\gamma+1)(p-1)-1)$. The set of solutions of (1.11) has been characterized by Kichenassamy and Véron [8] (and Kroll and Mazja [9] in the regular case). They proved

Theorem B. *Assume $p > 1$, then for each positive integer k there exists a $\gamma_k \in \mathbb{R}$ and $\phi_k : \mathbb{R} \to \mathbb{R}$ with least anti-period π/k, of class C^∞ such that*

$$v(x) = v(r,\theta) = r^{-\gamma_k}\phi_k(\theta), \tag{1.12}$$

is p-harmonic in $\mathbb{R}^2 \backslash \{0\}$; γ_k is the positive root of

$$(\gamma+1)^2 = (1+1/k)^2(\gamma^2 + \gamma(p-2)/(p-1)). \tag{1.13}$$

The couple (γ_k, ϕ_k) is unique, up to translation and homothety over ϕ_k.

Let $\mathscr{E}_{p,q}$ be the set of 2π-periodic solutions of (1.4). We define

$$\ell_{p,q} = \left[\left(\frac{p}{q+1-p}\right)^{p-1}\left(\frac{pq}{q+1-p} - 2\right)\right]^{1/(q+1-p)}, \tag{1.14}$$

which exists only if $(p-2)q > 2(1-p)$, or equivalently $(\beta_q + 1)(p-1) > 1$. The main result of this article is the following.

Theorem 1. *Assume $q+1>p>1$. If $\beta_q \leqslant (2-p)/(p-1)$, $\mathscr{E}_{p,q}=\{0\}$. If $(2-p)/(p-1)<\beta_q \leqslant \gamma_1$, $\mathscr{E}_{p,q}=\{0,\ell_{p,q},-\ell_{p,q}\}$. If $\beta_q>\gamma_1$, let $k(q)$ be the largest integer such that $\beta_q>\gamma_{k(q)}$. Then $\mathscr{E}_{p,q}$ has $3+k(q)$ connected components; $\mathscr{E}^0_{p,q}$, $\mathscr{E}^+_{p,q}$, $\mathscr{E}^-_{p,q}$, and $\mathscr{E}^k_{p,q}$ $(1\leqslant k\leqslant k(q))$, where*

(i) $\mathscr{E}^0_{p,q}=\{0\}$, $\mathscr{E}^+_{p,q}=\{\ell_{p,q}\}$, *and* $\mathscr{E}^-_{p,q}=\{-\ell_{p,q}\}$;

(ii) *for each* $1\leqslant k\leqslant k(q)$, $\mathscr{E}^k_{p,q}$ *is the set of solutions to* (1.4), *with least anti-period* π/k, *and* $\mathscr{E}^k_{p,q}=\{\phi(.+\alpha):\ \alpha\in S^1\}$.

The proof of this result is difficult and based upon two completely different points of view: a 2-dimensional shooting method, and a phase plane analysis. The shooting method consists in proving the existence of a positive solution of (1.1) in an angular sector $\{(r,\theta):\ r>1,\ 0<\theta<\theta_k\}$ of the 2-dimensional plane, and subject to Dirichlet conditions on the lateral boundary of the sector. Here $\theta_k=\pi/k$ for some positive integer k. Thanks to the assumption on β, it will be proved that this solution is bounded from below and from above by two terms with the same decay order, $C_i(\theta)r^{-\beta}$ for some functions C_i $(i=1,2)$. From this two-side estimate, these follows a precise asymptotic behavior (as $r\to\infty$) which shows the existence of at least one positive solution of (1.4) on $(0,\theta_k)$ vanishing at the end points. The non-existence is proved by the strong maximum principle. Surprisingly (and contrary to the semilinear case $p=2$), uniqueness cannot be obtained directly, neither from (1.1) nor from (1.4). Thus, we immerge this equation into a more general class of 2-dimensional differential systems and prove, by a phase plane analysis that the period of periodic solutions of such systems is a strictly monotone function of some shooting parameter. The dynamical systems approach for constructing solutions of non-linear equations is usually settled upon the invariant manifold theory: either its utilization is implicit as in [2] for constructing the very singular solution of the semilinear heat equation with absorption, or the theory is used in full as in [1,5,6] when studying ground states of a wide class of semilinear elliptic equations. In [7], this approach is combined with the use of the Mel'nikov function on invariant manifold in order to prove sharp asymptotics. The theory of internal isolated singularities is developed in [12–14].

2. The shooting method

We start this section with two key observations:

(A) By multiplying (1.4) by ω and integrating over $(0,2\pi)$ we get

$$\int_0^{\pi}[\beta^2\omega^2+\omega'^2]^{\frac{p-2}{2}}\omega'^2\,d\theta-a_{p,q}\int_0^{\pi}[\beta^2\omega^2+\omega'^2]^{\frac{p-2}{2}}\omega^2 d\theta+\int_0^{\pi}|\omega|^{q+1}\,d\theta=0.$$

Thus, there is no non-trivial solution if $a_{p,q}\leqslant 0$ or equivalently $\beta_q\leqslant(2-p)/(p-1)$. On the contrary, if $a_{p,q}>0$ there exists always two non-trivial constant solutions, $\ell_{p,q}$, defined by (1.14), and $-\ell_{p,q}$. Moreover, it is worth noticing that if $p \geqslant 2$ it never occurs that $\beta_q \leqslant (2-p)/(p-1)$, therefore $\mathscr{E}_{p,q}$ is never reduced to the zero function. In any case, we shall always assume $a_{p,q} > 0$.

(B) At the point θ_0 where $\omega(\theta_0)$ vanishes, $\omega'(\theta_0)$ is not zero. However, this is far from obvious except in the case $p = 2$, since the Cauchy–Lipschitz theorem does not hold at points where ω and ω' vanish. This fact will be the consequence of the following structure result.

Proposition 1. *Let $q+1 > p > 1$ such that $a_{p,q} > 0$. If ω is a non-constant 2π-periodic solution of* (1.4), *there exists a positive integer k such that ω has least period $2\pi/k$. Moreover there exists $\alpha \in S^1$ such that $\omega_\alpha(.) = \omega(. - \alpha)$ vanishes at* 0 *and π/k, is positive on $(0, \pi/k)$ and satisfies $\omega_\alpha'(0) = -\omega_\alpha'(\pi/k) > 0$.*

Proof. If ω is a non-constant solution of (1.4) it is bounded from above (resp. from below) by $\ell_{p,q}$ (resp. $-\ell_{p,q}$). This follows from the maximum principle, since at a point of positive maximum (resp. negative minimum), ω'' exists (the equation is not degenerate) and the solution is locally C^2 (see [10]). It is also a consequence of the fact that the function

$$U_{\mathrm{M}} : x \mapsto \ell_{p,q}|x|^{-\beta_q}, \quad \forall x \neq 0, \tag{2.1}$$

is the maximal solution of (1.1) in $\in \mathbb{R}^2 \setminus \{0\}$, as it follows from the constructions in [4] (see also Vazquez' universal a priori estimate [11]). Let us assume that ω achieves positive values, and let θ_1 be such that $\omega(\theta_1) = \max\{\omega(\theta)\colon\ \theta \in S^1\}$. If ω would keep a constant sign, it would have a positive minimum, $\omega(\theta_2) < \ell_{p,q}$. Since $\omega''(\theta_2)$ exists at such a point (at this point, the equation is not degenerate since $\omega(\theta_2) > 0$) it would be non-negative and (1.4) would imply

$$\omega^q(\theta_2) \geqslant a_{p,q}\beta^{p-2}\omega^{p-1}(\theta_2) \Rightarrow \omega(\theta_2) \geqslant \ell_{p,q},$$

a contradiction. Therefore, ω is not always positive on S^1.

Let us denote by $I_1\ (\theta_0, \tilde{\theta}_0)$, the connected component containing θ_1 of the $\theta \in S^1$, where $\omega(\theta) > 0$. Thus, ω is positive on $(\theta_0, \tilde{\theta}_0)$ (identifying S^1 with $[0, 2\pi)$ we can assume $\theta_0 < \tilde{\theta}_0$). Put

$$C_{\theta_0,\tilde{\theta}_0} = \{x = (r,\theta) \in \mathbb{R}^2\colon\ r > 0, \theta \in (\theta_0, \tilde{\theta}_0)\}.$$

Then $u(x) = u(r,\theta) = r^{-\beta}\omega(\theta)$ is a positive solution of (1.1) in the angular domain $C_{\theta_0,\tilde{\theta}_0} \setminus \{0\}$. In order to prove that $\omega'(\theta_0) > 0$, we consider $a \in \mathbb{R}^2$ such that the open disk $D_1(a)$, of radius 1 and center a is included into $C_{\theta_0,\tilde{\theta}_0}$ and tangent, at the point b and $\tilde{b}$ to the two half-lines $L_0 = \{(r,\theta)\colon\ r > 0,\ \theta = \theta_0\}$ and $\tilde{L}_0 = \{(r,\theta)\colon\ r > 0,\ \theta = \tilde{\theta}_0\}$. Although it plays no role in the sequel, the polar coordinates of a are $[\sin((\theta_0 + \tilde{\theta}_0)/2)]^{-1}$ and $(\theta_0 + \tilde{\theta}_0)/2$. Let $D_{(1+p)/2p}(a)$ be the disk of radius $(1+p)/2p$ and center a. Since u is positive in $D_1(a)$ its minimum on $\partial D_{(1+p)/2p}(a)$ is positive. We denote it by η. Set

$$w_{A,B}(x) = A(1-|x-a|) + B(1-|x-a|)^2,$$

where A and B are positive parameters to be chosen such that

(i) $w_{A,B} \leqslant \eta$ on $\partial D_{(1+p)/2p}(a) \Leftrightarrow A(p-1)/2p + B(p-1)^2/4p^2 \leqslant \eta$,

(ii) $-\nabla \cdot (|\nabla w_{A,B}|^{p-2} \nabla w_{A,B}) + w_{A,B}^q \leqslant 0$ in $D_1(a) \backslash D_{(1+p)/2p}(a)$. If we set $\rho = x - a$, this last inequality is equivalent to

$$\begin{aligned} &-|A+2B(1-\rho)|^{p-2}[2(p-1)B - \rho^{-1}(A+2B(1-\rho))] \\ &\quad + (1-\rho)^q(A+B(1-\rho))^q \leqslant 0, \quad \forall \rho \in ((1+p)/2p, 1). \end{aligned}$$

Since

$$2(p-1)B - \rho^{-1}(A+2B(1-\rho)) \geqslant \frac{2p(B-A)}{p+1}, \quad \forall \rho \in ((1+p)/2p, 1),$$

requirements (i) and (ii) are fulfilled as soon as we take $0 < A < B$, small enough. By the maximum principle, $w_{A,B} \leqslant u$ in $D_1(a) \backslash D_{(1+p)/2p}(a)$. Both u and $w_{A,B}$ vanish at the point b, and on L_0 and $\partial D_1(a)$, respectively. Thus, if v denotes the outward normal derivative to $D_1(a)$ at b,

$$|b|^{-\beta} \omega'(\theta_0) = \frac{\partial u}{\partial v}(b) \leqslant \frac{\partial w_{A,B}}{\partial v}(b) = -A.$$

Here, we have used the fact that u is at least $C^{1,\alpha}$ (for some $0 < \alpha < 1$) by the regularity theory of degenerate elliptic equations, and so is ω. Thus, the right and left derivatives of ω at θ_0 coincide. Consequently, $\beta^2 \omega(\theta_0^+)^2 + \omega'(\theta_0^+)^2 \neq 0$ and the Cauchy–Lipschitz existence and local uniqueness theorem applies at the point θ_0. Since the equation is odd, the function $\theta \mapsto -\omega(\theta_0 - \theta)$ is a solution of (1.4) in some interval $(\theta_0 - \delta, \theta_0)$ on the left of θ_0. Therefore,

$$\omega(\theta_0 - \theta) = -\omega(\theta), \quad \forall \theta \in (\theta_0 - \delta, \theta_0 + \delta) \subset (2\theta_0 - \tilde{\theta}_0, \tilde{\theta}_0). \tag{2.2}$$

Since $\omega > 0$ on $(\theta_0, \tilde{\theta}_0)$, the equation is not degenerate on $(\theta_0 - \delta, \theta_0]$. This implies that the symmetry relation (2.2) holds on the interval $[2\theta_0 - \tilde{\theta}_0, \tilde{\theta}_0]$. Again, because the equation is not degenerate on this interval and $\omega'(\theta_1) = 0$, there holds

$$\omega(\theta) = \omega(2\theta_1 - \theta), \quad \forall \theta \in (0, \theta_1).$$

Consequently, ω is $2(\tilde{\theta}_0 - \theta_0)$-periodic, with anti-period $\tilde{\theta}_0 - \theta_0$. The necessary and sufficient condition insuring ω to be 2π-periodic is that there exists a positive integer k such that $\tilde{\theta}_0 - \theta_0 = \pi/k$. Another consequence of the non-degeneracy of the equation is that two solutions with the same period only differs by a phase coefficient α. This implies the last statement. $\square$

Theorem 2. *Let* $q+1>p>1$, $k\in\mathbb{N}_*$, $\theta_k=\pi/k$, *then*

(i) *If* $\beta>\gamma_k$ *there exists a positive solution* ω *of* (1.4) *in* $(0,\theta_k)$ *which vanishes at* $\theta=0$ *and* $\theta=\theta_k$.

(ii) *If* $\beta\leqslant\gamma_k$, *the only solution of* (1.4) *in* $(0,\theta_k)$ *which vanishes at* $\theta=0$ *and* $\theta=\theta_k$ *is the zero function.*

Proof. For $R\geqslant 0$, we set

$$C^R_{0,\theta_k}=\{x=(r,\theta):\ r>R,0<\theta<\theta_k\},$$

and $C_{0,\theta_k}=C^0_{0,\theta_k}$. If a solution ω of (1.4) in $(0,\theta_k)$ vanishing at the two end points exists, the function

$$(r,\theta)\mapsto u_k(r,\theta)=r^{-\beta}\omega(\theta),$$

is a solution of (1.1) in C^R_{0,θ_k} for any $r>R$ vanishing on the lateral boundary

$$\partial_{\mathrm{L}}C^R_{0,\theta_k}=\{x=(r,0):\ r>R\}\cup\{x=(r,\theta_k):\ r>R\}.$$

Step 1. Construction of an approximate solution: We define the function $h\in C(\mathbb{R}^2)$ by

$$h(x)=\begin{cases}2-|x| & \text{if } |x|\leqslant 2,\\ 0 & \text{if } |x|\geqslant 2.\end{cases}$$

For $n>R\geqslant 1$ we set $C^{R,n}_{0,\theta_k}=C^R_{0,\theta_k}\backslash\overline{C^n_{0,\theta_k}}$. Let $\varepsilon>0$ to be chosen, and v_n the solution (obtained by minimization) of

$$\begin{aligned}&-\nabla\cdot(|\nabla v_n|^{p-2}\nabla v_n)+|v_n|^{q-1}v_n=0 \quad \text{in } C^{1,n}_{0,\theta_k},\\ &v_n=\varepsilon h \quad \text{on } \partial C^{1,n}_{0,\theta_k}.\end{aligned}\tag{2.3}$$

The function v_n is non-negative. We choose ε such that

$$\varepsilon h(x)\leqslant U_{\mathrm{M}}(x),\quad \forall x\in\mathbb{R}^2.$$

(remember that U_{M}, defined by (2.1) is the maximal solution of (1.1) in $\mathbb{R}^2\backslash\{0\}$). By monotonicity and the maximum principle, we have

$$1<n_1<n_2\Rightarrow v_{n_1}\leqslant v_{n_2}\leqslant U_{\mathrm{M}} \quad \text{in } C^{1,n_1}_{0,\theta_k}.$$

When n tends to infinity, u_n increases and converges to some u which is a positive solution of (1.1) in C^1_{0,θ_k}, with boundary value εh on $\partial C^1_{0,\theta_k}$. Moreover,

$$u(x) \leqslant U_{\mathrm{M}}(x) = \ell_{p,q}|x|^{-\beta} \quad \text{in } C^1_{0,\theta_k}.$$

Notice that the decay at infinity and the monotone operators theory are enough to ensure the uniqueness of u.

Step 2. Estimate from below: Let ϕ_k be a solution of (1.11) with corresponding exponent $\gamma = \gamma_k$, normalized by

$$0 \leqslant \phi_k \leqslant \max_{0,\theta_k} \phi_k = 1.$$

We set $\sigma = \beta/\gamma_k$. Then $\sigma > 1$. We claim that, for $\eta > 0$ small enough,

$$(r,\theta) \mapsto V_\eta(r,\theta) = \eta r^{-\beta} \phi_k^\sigma(\theta),$$

is a non-negative subsolution of (1.1) in C^1_{0,θ_k} which vanishes on the lateral boundary $\partial_{\mathrm{L}} C^1_{0,\theta_k}$. If we denote

$$\mathscr{P}(V) = -\nabla \cdot (|\nabla V|^{p-2} \nabla V) + |V|^{q-1} V,$$

and

$$\mathscr{D}(\psi) = -\frac{d}{d\theta}[(\beta^2\psi^2 + \psi'^2)^{\frac{p-2}{2}}\psi'] - a_{p,q}[\beta^2\psi^2 + \psi'^2]^{\frac{p-2}{2}}\psi + |\psi|^{q-1}\psi,$$

then

$$\mathscr{P}(V_\eta) = r^{-q\beta}\mathscr{D}(\eta\phi_k^\sigma).$$

Put $\psi = \eta\phi_k^\sigma$. By a straightforward computation one obtains

$$(\beta^2\psi^2 + \psi'^2)^{\frac{p-2}{2}} = \eta^{p-2}\sigma^{p-2}\phi_k^{(\sigma-1)(p-2)}(\gamma_k^2\phi_k^2 + {\phi_k}'^2)^{\frac{p-2}{2}},$$

and

$$\begin{aligned}\frac{d}{d\theta}[(\beta^2\psi^2 + \psi'^2)^{\frac{p-2}{2}}\psi'] &= \eta^{p-1}\sigma^{p-1}\frac{d}{d\theta}[\phi_k^{(\sigma-1)(p-1)}(\gamma_k^2\phi_k^2 + {\phi_k}'^2)^{\frac{p-2}{2}}{\phi_k}'] \\ &= \eta^{p-1}\sigma^{p-1}\phi_k^{(\sigma-1)(p-1)}\frac{d}{d\theta}[(\gamma_k^2\phi_k^2 + {\phi_k}'^2)^{\frac{p-2}{2}}{\phi_k}'] \\ &\quad + (\sigma-1)(p-1)\eta^{p-1}\sigma^{p-1}\phi_k^{(\sigma-1)(p-1)-1}(\gamma_k^2\phi_k^2 + {\phi_k}'^2)^{\frac{p-2}{2}}{\phi_k}'^2.\end{aligned}$$

Since

$$-\frac{d}{d\theta}[(\gamma_k^2\phi_k^2 + {\phi_k}'^2)^{\frac{p-2}{2}}{\phi_k}'] = b_{p,\gamma_k}(\gamma_k^2\phi_k^2 + {\phi_k}'^2)^{\frac{p-2}{2}}\phi_k,$$

with $b_{p,\gamma_k} = \gamma_k((\gamma_k+1)(p-1)-1)$, it follows that

$$\eta^{1-p}\mathscr{D}(\psi) = \eta^{q+1-p}\phi_k^{q\sigma} + \sigma^{p-2}\phi_k^{(\sigma-1)(p-1)-1}(\gamma_k^2\phi_k^2 + {\phi_k'}^2)^{\frac{p-2}{2}}[(\sigma b_{p,\gamma_k} - b_{p,q})\phi_k^2 - \sigma(\sigma-1)(p-1){\phi_k'}^2].$$

But $\sigma b_{p,\gamma_k} - b_{p,q} = \beta(\gamma_k - \beta)(p-1) = -\gamma_k^2\sigma(\sigma-1)(p-1)$. Therefore,

$$\begin{aligned}\eta^{1-p}\mathscr{D}(\psi) &= \eta^{q+1-p}\phi_k^{q\sigma} - (p-1)(\sigma-1)\sigma^{p-1}\phi_k^{(\sigma-1)(p-1)-1}(\gamma_k^2\phi_k^2 + {\phi_k'}^2)^{\frac{p}{2}} \\ &\leqslant \eta^{q+1-p}\phi_k^{q\sigma} - (p-1)(\sigma-1)\sigma^{p-1}\phi_k^{\sigma(p-1)}. \end{aligned} \tag{2.4}$$

Since $\sigma > 1$, the right-hand side of (2.4) is non-positive for η small enough. We can also impose $\eta\phi_k \leqslant \varepsilon$ in order to have $V_\eta \leqslant u(x)$ if $|x| = 1$. By the maximum principle, V_η is dominated by u in C^1_{0,θ_k}. This implies

$$\eta\phi_k^\sigma(x/|x|) \leqslant |x|^\beta u(x) \leqslant \ell_{p,q} \quad \text{in } C^1_{0,\theta_k}. \tag{2.5}$$

Step 3. Asymptotic behavior: For $R > 0$, we define u_R by $u_R = R^\beta u(Rx)$. The function u_R satisfies (1.1) in $C^{1/R}_{0,\theta_k}$. By the degenerate elliptic equation regularity theory, the set of functions $\{u_R\}$ remains bounded in the $C^{1,\alpha}_{\text{loc}}$-topology of $\overline{C_{0,\theta_k}}\setminus\{0\}$. Since

$$\frac{d}{dR}(R^\beta(2 - R|x|)_+^\beta) = \beta R(2 - R|x|)_+^{\beta-1}(2 - 2R|x|) \leqslant 0 \quad \text{for } |x| \geqslant 1/R,$$

there holds

$$R'^\beta(2 - R'|x|)_+^\beta \leqslant R^\beta(2 - R|x|)_+^\beta \quad \text{for } |x| \geqslant 1/R,$$

for $0 < R < R'$. Because $h(x) = (2 - |x|)_+$, it follows by the maximum principle

$$R' \geqslant R \Rightarrow u_{R'} \leqslant u_R \in C_{0,\theta_k}. \tag{2.6}$$

Thus, there exists a function u^* such that u_R decreases and converges to u^* as $R \to \infty$ in $C^1_{\text{loc}}(\overline{C_{0,\theta_k}}\setminus\{0\})$. The function u^* is a solution of (1.1) in C_{0,θ_k} which vanishes on $\partial_{\text{L}} C_{0,\theta_k}$. Because of (2.5), u^* satisfies

$$\eta\phi_k^\sigma(x/|x|) \leqslant |x|^\beta u^*(x) \leqslant \ell_{p,q} \quad \text{in } C_{0,\theta_k}. \tag{2.7}$$

Finally,

$$\lim_{R\to\infty} R^\beta u(Rr,\theta) = u^*(r,\theta) = r^{-\beta}\lim_{R\to\infty}(Rr)^\beta u(Rr,\theta) = r^{-\beta}u^*(1,\theta).$$

If we define ω by $\omega(\theta) = u^*(1, \theta)$, then $u^*(r, \theta) = r^{-\beta}\omega(\theta)$, and

$$\eta\phi_k^\sigma(\theta) \leqslant \omega(\theta) \leqslant \ell_{p,q}, \quad \forall\theta \in [0, \theta_k]. \tag{2.8}$$

This implies that ω is a positive solution of (1.4) on $(0, \theta_k)$ which vanishes at the two end points.

Step 4. Non-existence: Let us assume $\beta \leqslant \gamma_k$ and there exists a positive solution ω of (1.4) in $(0, \theta_k)$ vanishing at the two end points. In this case, $\sigma = \beta/\gamma_k \leqslant 1$. We still define V_η by

$$V_\eta(r, \theta) = \eta r^{-\beta}\phi_k^\sigma(\theta),$$

where $\eta > 0$ and obtain, with $\psi = \eta\phi_k^\sigma$,

$$\eta^{1-p}\mathscr{D}(\psi) = \eta^{q+1-p}\phi_k^{q\sigma} - (p-1)(\sigma-1)\sigma^{p-1}\phi_k^{(\sigma-1)(p-1)-1}(\gamma_k^2\phi_k^2 + {\phi_k}'^2)^{\frac{p}{2}} > 0.$$

We choose $\eta = \eta_0 > 0$ as the smallest parameter such that $\eta\phi_k^\sigma \geqslant \omega$. Notice that this is possible since both ω and ϕ_k^σ are C^1 and positive in the interval $(0, \theta_k)$ on the end points of which ${\phi_k}'$ does not vanish.

Case 1: There exists $\theta_0 \in (0, \theta_k)$ such that

$$\eta\phi_k^\sigma(\theta) \geqslant \omega(\theta), \quad \forall\theta \in [0, \theta_k] \quad \text{and} \quad \eta\phi_k^\sigma(\theta_0) = \omega(\theta_0).$$

Notice that the above configuration always holds if $\sigma < 1$. By the mean value theorem,

$$(\beta^2\psi^2 + \psi'^2)^{(p-2)/2}\psi' - (\beta^2\omega^2 + \omega'^2)^{(p-2)/2}\omega' = a(\psi' - \omega') + b(\psi - \omega),$$

where

$$\begin{aligned} b = {} & (p-2)(\beta^2(\omega + t(\psi - \omega))^2 + (\omega' + t(\psi' - \omega'))^2)^{(p-4)/2} \\ & \times (\omega + t(\psi - \omega))(\omega' + t(\psi' - \omega')) \end{aligned}$$

and

$$\begin{aligned} a = {} & (p-2)(\beta^2(\omega + t(\psi - \omega))^2 + (\omega' + t(\psi' - \omega'))^2)^{(p-4)/2} \\ & \times (\omega + t(\psi - \omega))(\omega' + t(\psi' - \omega'))^2 \\ & + (\beta^2(\omega + t(\psi - \omega))^2 + (\omega' + t(\psi' - \omega'))^2)^{(p-2)/2}, \end{aligned}$$

for some $t \in (0, 1)$. Moreover,

$$\psi(\theta_0) = \omega(\theta_0) = \Theta > 0 \quad \text{and} \quad \psi'(\theta_0) = \omega'(\theta_0) = \Lambda.$$

Therefore,

$$b(\theta_0) = (p-2)(\beta^2\Theta^2 + \Lambda^2)^{(p-4)/2}\Theta\Lambda$$

and

$$a(\theta_0) = (\beta^2\Theta^2 + \Lambda^2)^{(p-4)/2}(\beta^2\Theta^2 + (p-1)\Lambda^2).$$

Thus, $a(\theta_0) > 0$, and this property holds in a neighborhood of θ_0. Since the two equations are not degenerate, the functions a and b are C^1 in a neighborhood of θ. Similarly,

$$(\beta^2\psi^2 + \psi'^2)^{(p-2)/2}\psi - (\beta^2\omega^2 + \omega'^2)^{(p-2)/2}\omega = c(\psi' - \omega') + d(\psi - \omega),$$

and

$$\psi^q - \omega^q = e((\psi - \omega)),$$

for some bounded C^1 functions c, d and e. From this we have

$$\mathscr{D}(\psi) - \mathscr{D}(\omega) = -\frac{d}{d\theta}(a(\psi' - \omega') + b(\psi - \omega)) - a_{p,q}c(\psi' - \omega') + (d + e)(\psi - \omega).$$

Since $\mathscr{D}(\psi) - \mathscr{D}(\omega) \geqslant 0$,

$$-\frac{d}{d\theta}(a(\psi' - \omega')) - (a_{p,q}c + b)(\psi' - \omega') + (d + e - b')(\psi - \omega)_+ \geqslant 0, \qquad (2.9)$$

holds, with $\psi - \omega \geqslant 0$ and $(\psi - \omega)(\theta_0) = 0$. By the strong maximum principle $\psi - \omega \equiv 0$ in a neighborhood of θ_0. Therefore, the set of points where ψ and ω coincide is open in $(0, \theta_0)$. Since it is closed by continuity, it implies $\psi - \omega \equiv 0$ in the whole interval $(0, \theta_0)$. Contradiction.

Case 2: $\sigma = 1$ and there exists $\eta > 0$ such that

$$\eta\phi_k(\theta) > \omega(\theta), \quad \forall \theta \in (0, \theta_k) \qquad \text{and} \qquad \eta{\phi_k}'(0) = \omega'(0) \qquad \text{or}$$
$$\eta{\phi_k}'(\theta_k) = \omega'(\theta_k).$$

We proceed as above, using the fact that ${\phi_k}'$ does not vanish at the end points. Therefore, $\psi - \omega \geqslant 0$ in a neighborhood of 0 (or θ_k similarly). The inequality (2.9) holds with a strongly elliptic operator. Since $(\psi - \omega)(\theta_0) = 0$, the Hopf boundary lemma implies $(\psi' - \omega')(\theta_0) > 0$, which contradicts the tangency of the two graphs at $\theta = 0$. □

The proof of uniqueness of ω will be obtained by the phase plane analysis developed in the next section.

3. The phase plane analysis

3.1. Dynamical system and critical points

In this section, we will consider the following more general ordinary differential equation (ODE) in the real variable t:

$$-\frac{d}{dt}[(\beta^2\omega^2+\omega'^2)^{\frac{p-2}{2}}\omega']-\alpha[\beta^2\omega^2+\omega'^2]^{\frac{p-2}{2}}\omega+g(\omega)=0, \tag{3.1}$$

where $\alpha>0$ and g is a differentiable and odd function that satisfies the following hypotheses:

(H1) $\alpha\beta^{p-2}\in \mathrm{Im}(F)$, where $F(x):=\frac{g(x)}{x|x|^{p-2}}$,
(H2) F is a strictly increasing function.

In order to transform the ODE into a dynamical system put $x=\omega$ and $y=\omega'$, then we get

$$(\mathscr{S})\begin{cases} x'=P(x,y):=y,\\ y'=Q(x,y):=\dfrac{g(x)[\beta^2x^2+y^2]^{\frac{4-p}{2}}-\alpha\beta^2x^3-(\alpha+(p-2)\beta^2)xy^2}{(p-1)y^2+\beta^2x^2}. \end{cases}$$

The only singular point of the system is the origin $(0,0)$. Therefore, at any point in $\mathbb{R}^2\backslash\{0,0\}$ the Cauchy–Lipschitz local existence and uniqueness theorem applies. A direct calculation shows that the critical points of $(\mathscr{S})$, are given by

$$y=0 \quad \text{and} \quad F(x)=\alpha\beta^{p-2}>0. \tag{3.2}$$

Since F is strictly increasing and even, we have then two critical points, $(-c,0)$ and $(c,0)$, where $c=F^{-1}(\alpha\beta^{p-2})$. The linearized system at a point $(x,0)$, is given by the matrix

$$A(x):=\begin{bmatrix} 0 & 1\\ a(x) & 0\end{bmatrix},$$

where

$$a(x):=\beta^{2-p}[-\alpha\beta^{p-2}+g'(x)|x|^{2-p}-(p-2)g(x)|x|^{-p}x].$$

Note that the condition (H2) on F, for $x>0$, is equivalent to $xg'(x)>(p-1)g(x)$ and $a(x)>\beta^{2-p}[x^{1-p}g(x)-\alpha\beta^{p-2}]$. Thus, $a(c)>0$.

Remark 1. Define $H(x) \coloneqq Q(x,0) = x\beta^{2-p}[F(x) - F(c)]$, we have

$$H'(x) = x\beta^{2-p}[F(x) - F(c) + xF'(x)],$$

and then by the condition (H2) on F, we see that $H(x)$ is strictly negative for $0<x<c$, and strictly positive and increasing for $x>c$.

The eigenvalues λ_1 and λ_2 of $A(c)$ are given by the following algebraic system:

$$\begin{cases} \lambda_1 + \lambda_2 = 0, \\ \lambda_1\lambda_2 = -a(c), \end{cases}$$

and the associated eigenvectors are

$$\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

Note that in this case, both $(c,0)$ and $(-c,0)$, are saddle points.

3.2. *Qualitative study of the dynamical system*

Notation 1. We will use the following notation: $\mathcal{O}(x_0, y_0)$ is the orbit $\{(x(t), y(t))\}$, that passes through the point (x_0, y_0) at $t = 0$.

Lemma 1. *Concerning the dynamical system* $(\mathcal{S})$, *we have*

(a) *The x- and y-axis are axes of symmetry, and so the origin is a center of symmetry.*

(b) *Orbit of solutions of any autonomous system is invariant by time shift.*

Proof. The second point is obvious and to see the first point (a), consider the following applications:

$$\Phi_x : (t,x,y) \mapsto (-t,-x,y), \quad \Phi_y : (t,x,y) \mapsto (-t,x,-y).$$

It is clear that, $(\mathcal{S})$ is invariant under Φ_x and Φ_y. Since the Cauchy–Lipschitz theorem applies, any orbit which intersects the x-axis (always perpendicularly) is symmetric with respect to the y-axis, and similarly by exchanging the role of the two axes. □

This will give directly

Lemma 2. *Any orbit which intersects both x- and y-axis is a closed orbit.*

Proof. Without loss of generality (see 1(b)), we can assume that the orbit starts at $(0, y_0)$, at time $t = 0$, Applying in this order the applications Φ_y, Φ_x, and Φ_y, we get the result. $\square$

Now, we define the following subsets:

$$I_1 = \{y_0 > 0:\ \mathcal{O}(0, y_0) \cap Ox \neq \phi\ \text{(finite time)}\},$$

$$I_2 = \{y_0 > 0:\ \mathcal{O}(0, y_0)\ \text{satisfies}\ (\mathcal{P})\},$$

where the property $(\mathcal{P})$ means
there exists $\epsilon_0 > 0$ such that $y(t) \geqslant \epsilon_0,\ \forall t \in \mathbb{R}$.

Lemma 3. *The set I_1 is a non-empty interval.*

Proof. *Step* 1. The set I_1 is not empty. Let $x_0 \in (0, c)$, and consider the orbit $\mathcal{O}(x_0, 0)$. We will show that the trajectory intersects the y-axis at finite time. Using $(\mathcal{S})$ we have

$$x'(0) = 0 \quad \text{and} \quad y'(0) = g(x_0)(\beta x_0)^{2-p} - \alpha x_0 < 0,$$

and then, x and y are decreasing from x_0 and 0, respectively. Note that the orbit cannot cross again the x-axis or the origin, since $y' < 0$.

Thus, if the orbit does not intersect the y-axis the unique possibility is to have a vertical asymptote. In this case, we must have

$$x'(t) \to 0 \quad \text{and} \quad y(t) \to -\infty \ \text{as}\ t \to +\infty.$$

From the equation, $x'(t) = y(t)$, we get $y(t) \to 0$ as $t \to +\infty$, which is not possible.

Therefore, there exists $t_0 > 0$, such that $x(t_0) = 0$, and so $-y(t_0) \in I_1$.

Step 2. The set I_1 is an interval. Let $y_0 \in I_1$, and let us prove that $]0, y_0] \subseteq I_1$. Indeed, if $0 < y_1 < y_0$, then the orbit $\mathcal{O}(0, y_1)$ cannot intersect the orbit $\mathcal{O}(0, y_0)$, and it cannot tend to the origin or any equilibrium, because $x' > 0$. So necessarily it intersects the x-axis. $\square$

Remark 2. By Lemma 2, if $y_0 \in I_1$, then the point $(0, y_0)$ belongs to a periodic orbit.

Lemma 4. *Every periodic orbit intersects the x-axis in the interval $(-c, c)$.*

Proof. By the Poincaré–Bendixon theorem, the bounded open domain of $\mathbb{R}^2$ bordered by a closed orbit contains at least a stationary point, $(-c, 0)$ or $(c, 0)$, or the singular point $(0, 0)$. Thus, any periodic trajectory intersects the x-axis. Assume that $x_1 > c$ is the intersection of a closed orbit with the x-axis. For $t > 0$ small enough, the solution $t \mapsto (x(t), y(t))$ with initial data $(x_1, 0)$ satisfies $x'(t) > 0$ and $y'(t) > 0$ (by Remark 1). Therefore, the x-coordinates are increasing as long as the trajectory belongs to the upper half-plane $y>0$. This implies that the abscissa x_2 of the second intersection point $(x_2, 0)$ with the x-axis satisfies $x_2>x_1$. Since the x-axis is an axis of symmetry, the whole trajectory $O(x_1, 0)$ is obtained by reflection through this axis, and the bounded open domain of $\mathbb{R}^2$ that it encloses contains no stationary or singular point, a contradiction. $\square$

Lemma 5. *The set I_2 is a non-empty open interval.*

Proof. Consider $x_0 = c$ and $y_0>0$, and the associated orbit $\mathcal{O}(x_0, y_0)$, with for some τ, $x(\tau) = c$ and $y(\tau) = y_0$. Since $x'(\tau) = y_0$, $t \mapsto x(t)$ is increasing for $t \geqslant \tau$.

Since for all $x>c$, $Q(x, 0)>0$ and $\lim_{x\to\infty} Q(x, 0) = +\infty$, the orbit could not intersect the x-axis and the limit of y could not be zero as $t \to \infty$. We cannot have a horizontal asymptote: otherwise, $\lim_{t\to\infty} y(t) = r>0$ so this means that $\lim_{x\to\infty} y'(t) = 0$, but in this case $\lim_{x\to\infty} x(t) = \infty$ and $0 = \lim_{x\to\infty} Q(x, r) = \infty$. Thus, necessarily $\lim_{t\to+\infty} y(t) = +\infty$. Therefore, this orbit is bounded below, for $t \geqslant \tau$, by some ϵ_0.

Next we will show that this orbit $\mathcal{O}(x_0, y_0)$ intersects the y-axis. For this, consider the orbit $\tilde{\mathcal{O}}_r(x_0, -y_0)$, with components $(\tilde{x}(t), \tilde{y}(t))$ with $\tilde{x}(0) = x_0$ and $\tilde{y}(0) = -y_0$. By the argument above $\tilde{y}(t) \leqslant -\epsilon_0$ for $t \leqslant 0$, so $\tilde{x}(t)$ decreases from c. Notice again that this orbit $\mathcal{O}(x_0, y_0)$ could not cross the x-axis or the origin. By the same argument of the proof of Lemma 3 this orbit does not have a vertical asymptote, so it must intersect the y-axis in some $y_1<0$ and so $-y_1 \in I_2$.

Now to see that I_2 is an interval, let $y_0 \in I_2$ and $y_2>y_0$. Then immediately $y_2 \in I_2$, since the orbit $\mathcal{O}(0, y_2)$ cannot intersect $\mathcal{O}(0, y_0)$.

Now, we are going to prove that I_2 is an open interval. Indeed, let $y_0 \in I_2$, and consider the orbit $\mathcal{O}(0, y_0)$ with components $(x(t), y(t))$. We know that x is always increasing from 0. There exists then t_1, such that, $x(t_1) = c$ and $y(t_1)>0$. Let $\mathcal{O}_r(\bar{x}, \bar{y})$ be the orbit where, $\bar{x} = c$ and $y(t_1)>\bar{y}>0$. By the same argument, $\mathcal{O}_r(\bar{x}, \bar{y})$ crosses the y-axis at $(0, y_1)$, for some $y_0>y_1$, and therefore $y_1 \in I_2$ and so $y_0 \in (y_1, +\infty) \subset I_2$. $\square$

Remark 3. It is easy to see that I_1 is an interval bounded above by any element of I_2. Therefore, we denote

$$b := \sup I_1 < +\infty.$$

Lemma 6. *For every $x_0 \in (-c, c) \backslash \{0\}$, there exists a unique periodic orbit such that $(x_0, 0)$ belongs to this orbit.*

Proof. By symmetry and same analysis as in the proof of Lemma 3. $\square$

Lemma 7. *I_1 is an open interval.*

Proof. We know that I_1 is an interval. We need to show that $b \notin I_1$. We proceed by contradiction and assume that $b \in I_1$. Then there exists, a first time, $t_0>0$, such that

$y(t_0) = 0$. By Lemma 4, we know that $x(t_0) \in (0, c)$. So for any $x(t_0) < x_1 < c$, and by the last lemma, $(x_1, 0)$ belongs to a periodic orbit $\mathcal{O}(x_1, 0)$. This periodic orbit intersects the y-axis at some $y(t_1) > b$, which is impossible since b is the supremum of I_1. $\square$

Proposition 2. *There exists one and only one heteroclinic orbit which connects the points $(-c, 0)$ and $(c, 0)$ in the upper half-plane, and one heteroclinic orbit that connects $(c, 0)$ and $(-c, 0)$ in the lower half-plane.*

Proof. Consider the orbit $\mathcal{O}(0, b)$, b as above, and let us show that

$$\lim_{t \to +\infty} x(t) = c \quad \text{and} \quad \lim_{t \to +\infty} y(t) = 0.$$

Using the fact that $b \notin I_1$, we obtain, $\lim_{t \to +\infty} x(t) \geqslant c$. Suppose $\lim_{t \to +\infty} x(t) > c$. Then there exists t_0 such that, $x(t_0) = c$ and $y(t_0) > 0$. Proceeding as in the proof of Lemma 5, we get $b \in I_2$, which is impossible since I_2 is open and $I_1 \cup \overline{I_2} = (0, +\infty)$. Thus, $\lim_{t \to \infty} x(t) = c$. From $x' = y$, we obtain

$$\lim_{t \to +\infty} y(t) = 0.$$

The orbit in the lower half-plane is obtained by symmetry. $\square$

Remark 4. The heteroclinic orbit in the upper half-plane is the unstable (resp. stable) trajectory of the point $(-c, 0)$ (resp. $(c, 0)$).

3.3. Variation of the period

In this section, we will consider a general dynamical system

$$(\mathcal{S}') \begin{cases} x' = F(x, y), \\ y' = G(x, y), \end{cases}$$

where F and G are C^1 functions in $\mathbb{R}^2 \backslash \{(0, 0)\}$, in which region the system has no equilibrium. We assume that F (resp. G) is odd with respect to y (resp. x) and even with respect to x (resp. y). As in Lemma 2, these equivariance properties imply that any orbit of $\mathcal{S}'$ which intersects both x- and y-axis is closed. We assume also that there exists an open interval $(0, b)$ $(b > 0)$ of the positive y-axis such that for any $y \in (0, b)$, the positive trajectory of $(\mathcal{S}')$ through $(0, y)$ enters the region $Q_+ = \{(x, y) \in \mathbb{R}^2\colon\ x > 0,\ \ y > 0\}$ and escapes from Q_+ in crossing the positive x-axis in finite time. Let us denote by $T(y)$ the first time such a trajectory intersects the x-axis. The orbit through $(0, y)$ is periodic and symmetric with respect to the coordinates axis, therefore, it is $4T(y)$-periodic. If $(x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\}$, we denote by $M_{(x,y)}(t) = (x(t), y(t))$ the maximal solution of $(\mathcal{S}')$ with initial data (x, y). Since the function T

is continuous, the set

$$\mathscr{R} = \bigcup_{0<y<b} \bigcup_{0<t<T(y)} \{M_{(0,y)}(t)\},$$

is a non-empty open subset of Q_+.

The aim of this section, under some hypothesis on F and G, is to prove a general result on the monotonicity of the period as function of the y coordinates of the initial data.

Notation 2. For all $(x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\}$ denote by V the vector $(F(x, y), G(x, y))$ and for $\lambda \neq 0$, denote by V_λ the vector $(F(\lambda x, \lambda y), G(\lambda x, \lambda y))$.

The main assumption on V is the following:

(H3) The functions F and G are, respectively, *positively homogeneous of degree* 1 and *superpositively homogeneous of degree* 1 in $\mathscr{R}$, which is

$$F(\lambda x, \lambda y) = \lambda F(x, y) \quad \text{and} \quad G(\lambda x, \lambda y) > \lambda G(x, y) \tag{3.3}$$

for all $(x, y) \in \mathscr{R}$ and all $\lambda > 1$ such that $(\lambda x, \lambda y) \in \mathscr{R}$.

(H4) The function F remains positive in $\mathscr{R}$.

Remark 5. Concerning the assumption on G, we could have equality: $G(\lambda x, \lambda y) = \lambda G(x, y)$ on a discrete subset of each orbit.

In order to show that the function T defined above is strictly increasing, let us consider $\mathscr{O}(0, y_0)$ an orbit with components $M(t) = M_{(0,y_0)}(t) = (x(t), y(t))$, such that $(x(0), y(0)) = (0, y_0) \in \mathscr{R}$, $\lambda > 1$, such that $\lambda\mathscr{O}$, the homothetic of $\mathscr{O}$ with components $(\lambda x(t), \lambda y(t))$ still in $\mathscr{R}$. Notice that $\lambda\mathscr{O}$ and $\mathscr{O}$ have the same period.

Finally, consider the orbit $\mathscr{O}_\lambda$ with components $M_\lambda(t) = (x_\lambda(t), y_\lambda(t)) \in \mathscr{R}$, $\lambda > 1$, such that, $x_\lambda(0) = 0$ and $y_\lambda(0) = \lambda y_0 \in (0, b)$.

Remark 6. The hypothesis on F and G have the following geometrical interpretations:

(i) For all $(x, y) \in \mathscr{R}$, $(\lambda x, \lambda y) \in \mathscr{R}$ the oriented angle (V, V_λ) is positive.

(ii) If for some times t_0 and t_1, $\lambda M(t_0) = M_\lambda(t_1)$ then for $\tau > 0$ small, $M_\lambda(t_1 + t)$ is above the straight line $OM(t_0 + t)$ for all $t \in (0, \tau]$.

To see the second point, remark that the two associated vectors λV and V_λ have the same abscissa $\lambda F(x, y)$, and $G(\lambda x, \lambda y) > \lambda G(x, y)$.

We start by the following lemma:

Lemma 8. *For $\lambda > 1$, the homothetic $\lambda\mathscr{O}$ of the orbit $\mathscr{O}$ is below the orbit $\mathscr{O}_\lambda$ (in $\mathscr{R}$).*

Proof. Let $\delta \in (0, T(y_0))$ and let $\mathcal{O}_{\lambda,\delta}$ denote the orbit through the point $\lambda M(\delta)$. Using the last remark (i) there exists $\varepsilon > 0$ such that $\lambda\mathcal{O}$ is strictly below the curve $\mathcal{O}_{\lambda,\delta}$ for t in the interval $]0, \epsilon]$.

Suppose that there exists a point of intersection between the two curves $\mathcal{O}_{\lambda,\delta}$ and $\lambda\mathcal{O}$ in the set R and let $(\lambda x_1, \lambda y_1) \in \mathcal{O}_\lambda$ be the first point of intersection. Thus, at the point $(\lambda x_1, \lambda y_1)$, the oriented angle between V and V_λ is negative, which contradicts Remark 6(i). Therefore, the curve $\mathcal{O}_{\lambda,\delta}$ is above λO for $t > \delta$. Letting $\delta \to 0$ and using the continuity of the solutions with respect to the initial data in $\mathbb{R}^2 \backslash \{(0,0)\}$ implies that the y-coordinate of any points of $\mathcal{O}_\lambda$ in $\mathcal{R}$ is greater than or equal to the y-coordinate of the point in $\lambda\mathcal{O}$ which has the same x-coordinate. Since equality is impossible, by Remark 6, the proof follows. $\square$

Now we can show the main result of this section:

Theorem 3. *The function $y \mapsto T(y)$ is increasing.*

Proof. By Remark 6(ii) and since $\lambda M(0) = M_\lambda(0)$, there exists $t_0 > 0$ such that for all $t \in]0, t_0]$ the point $M_\lambda(t)$ is above the straight line $OM(t)$.

Let $\tau > 0$ such that, $M_\lambda(t_0 + \tau)$, $M(t_0)$ and the origin are collinear. Then there exists $\lambda' > \lambda$ such that $M_\lambda(t_0 + \tau) = \lambda' M(t_0)$. By the same argument, there exists $t_1 > t_0$, such that, for all $t \in [t_0, t_1]$, $M_\lambda(t + \tau)$ is above the line $OM(t)$. This proves that for all t, $M_\lambda(t)$ is strictly above the straight line $OM(t)$. $\square$

3.4. Proof of Theorem 1

By Proposition 1, we know that all the 2π-periodic (non-constant) solutions of (1.4) are π/k-anti-periodic for some positive integer k. Therefore (and up to a $U(1)$ action), the completion of the proof is reduced to proving the uniqueness holds for positive solutions on the same interval, and that it vanishes at the end points. Let ω and $\tilde{\omega}$ be two solutions of (1.4) on $(0, \theta_k)$ such that

$$\omega(\theta) > 0, \tilde{\omega}(\theta) > 0, \quad \forall \theta \in (0, \theta_k) \quad \text{and} \quad \omega(0) = \tilde{\omega}(0) = \omega(\theta_k) = \tilde{\omega}(\theta_k) = 0. \qquad (3.4)$$

If ω and $\tilde{\omega}$ differ, their derivatives at $\theta = 0$ must be different (since the equation is never degenerate on a trajectory). Therefore, we can assume

$$0 < \omega'(0) < \tilde{\omega}'(0). \qquad (3.5)$$

Moreover, $\tilde{\omega}'(0) < b$ since $\tilde{\omega}$ is a periodic solution. If we put

$$F(x, y) := y$$

and

$$G(x, y) := \frac{|x|^{q-1} x [\beta^2 x^2 + y^2]^{2-p/2} - a_{p,q} \beta^2 x^3 - (a_{p,q} + (p-2)\beta^2) x y^2}{(p-1) y^2 + \beta^2 x^2},$$

then F and G satisfies the regularity and equivariance properties required in (3.3). Moreover, for any λ,

$$F(\lambda x, \lambda y) = \lambda F(x, y)$$

and

$$G(\lambda x, \lambda y) = (\lambda^{q+1-p} - \lambda)\frac{|x|^{q-1}x[\beta^2 x^2 + y^2]^{2-p/2}}{(p-1)y^2 + \beta^2 x^2} + \lambda G(\lambda x, \lambda y).$$

Thus, assumption (3.3) is satisfied in whole Q_+ and the minimal periods of ω and $\tilde{\omega}$ cannot be the same. □

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