Parameter uniform numerical method for singularly perturbed turning point problems exhibiting boundary layers

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Abstract

This article presents a numerical method to solve singularly perturbed turning point problems exhibiting two exponential boundary layers. Classical finite-difference schemes do not yield parameter uniform convergent results on a uniform mesh, in general (Robust Computational Techniques for Boundary Layers, Chapman & Hall, London, CRC Press, Boca Raton, FL, 2000). In order to overcome this difficulty, we propose an appropriate piecewise uniform (Shishkin) mesh and apply the classical finite-difference schemes on this mesh. Error estimates are derived by decomposing the solution into smooth and singular components. The present method is layer resolving as well as parameter uniform convergent. Numerical examples are presented to show the applicability and efficiency of the method.

Keywords: Singularly perturbed turning point problems; Boundary layer; Asymptotic approximation; Finite-difference schemes; Piecewise uniform (Shishkin) mesh

1. Introduction

Singular perturbation problems (SPPs) model convection–diffusion process in applied mathematics that arise in diverse areas, including linearized Navier–Stokes equation at high Reynolds number,

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heat transport problems with large Peclet numbers, magneto-hydrodynamic duct problems at Hartman numbers and the drift–diffusion equation of semiconductor device modeling. Boundary and interior layers are usually present in the solutions of SPPs. These layers are thin regions in the domain where the gradient of the solution steepens as the singular perturbation parameter ε approaches zero.

In general, classical numerical methods may give rise to difficulties for small values of the singular perturbation parameter ε. More precisely, finite-difference schemes based on centered or upwind differences on uniform meshes yield error bounds, in the maximum norm, which depend on an inverse power of ε. To resolve these problems, either additional information about the solution may be used to produce accurate efficient methods, which may involve a priori modification of the mesh or operator, or an attempt may be made to produce a posteriori adaptive methods. For more details about the numerical methods, the readers may refer to the books of Miller et al. [8], Roos et al. [14] and Farrell et al. [5].

In this paper, we treat the following singularly perturbed two-point boundary value problem with a turning point at $x = 0$:

$$Lu(x) \equiv \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad x \in D = (-1, 1), \tag{1.1}$$

$$u(-1) = A, \quad u(1) = B, \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter, a, b and f are sufficiently smooth functions such that

$$\begin{aligned} &a(0) = 0, \quad a'(0) \leqslant 0, \\ &|a(x)| \geqslant a_0 > 0 \quad \text{for } 0 < |x| \leqslant 1, \\ &b(x) \geqslant b_0 > 0, \quad \forall x \in \bar{D} = [-1, 1], \\ &|a'(x)| \geqslant \frac{|a'(0)|}{2}, \quad \forall x \in \bar{D}. \end{aligned} \tag{1.3}$$

With the above assumptions, the turning point problem (TPP) (1.1)–(1.2) possesses a unique solution exhibiting two boundary layers of exponential type at both end points $x = -1, 1$ [2].

In [6], Jayakumar and Ramanujam proposed a numerical method for a singularly perturbed DE without turning points. They have used the classical and exponentially fitted difference (EFD) schemes (see, for example, [3]) to obtain the numerical solution, respectively, in the outer and inner regions. Recently, Natesan et al. [12] presented a numerical technique to solve SPP without turning points. Vigo-Aguiar and Natesan [15] introduced a domain decomposition method for a class of singular perturbation problems and implemented it in a parallel machine.

In general, the numerical treatment of TPP is more difficult than the SPPs without turning points, because the coefficient of the convection term vanishes inside the domain of interest. Natesan and Ramanujam suggested a computational method for the TPP (1.1)–(1.2) using classical and EFD schemes in [10]. All these methods need the knowledge of an asymptotic approximation of the exact solution to determine the so-called transition boundary condition. Another technique known as initial-value technique was suggested in [11] for the singularly perturbed TPP (1.1)–(1.2) in which the numerical solution is obtained by solving suitable initial and terminal value problems. In [9], the authors analyzed the piecewise uniform meshes for the TPP (1.1)–(1.2).

Miller et al. [8] used the classical schemes on piecewise uniform meshes (known as Shishkin meshes) to solve singularly perturbed BVPs of convection–diffusion and reaction–diffusion problems

subject to Dirichlet boundary conditions without turning points. The principal aim of this paper is to provide layer resolving parameter uniform convergent numerical method for the TPP (1.1)–(1.2). For this, we suggest an appropriate piecewise-uniform mesh and apply the classical finite-difference schemes on this mesh. Then ε-uniform error estimates are derived and some numerical examples are included to support the theoretical estimates.

Before concluding the introduction section, we present some of the earlier works for singularly perturbed TPPs. Abrahamsson [1] derived a priori estimates for the solutions of SPPs with a turning point. The qualitative aspects of these problems, like existence, uniqueness and asymptotic behavior of the solution was studied by O'Malley [13] and Wasow [16]. A set of general sufficient conditions for a uniformly convergent scheme is obtained by Farrell [4]. Berger et al. [2] modified the El-Mistikway-Werle scheme for TPPs.

The rest of the paper is organized as follows. Section 2 presents some analytical results giving bounds for the derivatives of the solution of the TPP (1.1)–(1.2). Uniform convergence on Shishkin meshes is proved in Section 3. Section 4 provides numerical examples and the paper concludes with a discussion.

For any given function $g(x) \in \mathscr{C}^k(\bar{D})$ (k a nonnegative integer), let us denote

$$\|g\|_k = \sum_{i=0}^{k} \max_{x \in \bar{D}} |g^{(i)}(x)|.$$

2. The continuous problem

Bounds for the solution of the TPP (1.1)–(1.2) and its derivatives are derived in this section. Further, we analyze the asymptotic behavior of the solution and obtain bounds for the smooth and singular components of the analytic solution separately. Hereinafter, we shall denote the subdomains of $\bar{D}$, as $D_1 = [-1, -\delta]$, $D_2 = [-\delta, \delta]$ and $D_3 = [\delta, 1]$, where $0 < \delta \leqslant \frac{1}{2}$.

In the following, we first prove that the operator L as defined in (1.1) satisfies a minimum principle. Then we state a stability estimate for the solution of the TPP (1.1)–(1.2).

Lemma 2.1 ((Minimum principle) (Berger et al. [2]). *Let y be a smooth function satisfying $y(-1) \geqslant 0$, $y(1) \geqslant 0$ and $Ly(x) \leqslant 0$, $\forall x \in D$. Then $y(x) \geqslant 0$, $\forall x \in \bar{D}$.*

Proof. The proof is by contradiction. Assume that there exist a point $p \in \bar{D}$ such that $y(p) < 0$. It follows from the given boundary values that $p \notin \{-1, 1\}$. Define the function $w(x) = y(x) \exp(a_0(1+x)/2\varepsilon)$ and note that $w(p) < 0$.

Choose a point $q \in D$ and that $w(q) = \min_D w(x) < 0$. Therefore, from the definition of $q, w'(q) = 0$ and $w''(q) \geqslant 0$. But then

$$Ly(q) = \exp(-a_0(1+q)/2\varepsilon) \left[\varepsilon w''(q) + (a(q) - a_0)w'(q) - \frac{a_0}{2\varepsilon}\left(a(q) + b(q) - \frac{a_0}{2}\right) w(q)\right] > 0,$$

which is a contradiction. Thus we obtain $y(x) \geqslant 0$, $\forall x \in \bar{D}$. □

An immediate consequence of the minimum principle is the following uniform stability estimate.

Lemma 2.2 (Berger et al. [2]). *Consider the TPP* (1.1)–(1.2). *If* $u(x)$ *is the solution of this TPP, then for some positive constant* C, *we have*

$$\|u(x)\| \leqslant C\left[\max\{|A|, |B|\} + \frac{1}{b_0}\|f\|\right], \quad \forall x \in \bar{D}.$$

Proof. Let us define the comparison functions

$$\Psi^{\pm}(x) = \left[\max\{|A|, |B|\} + \frac{1}{b_0}\|f\|\right] \pm u(x).$$

One can obtain the required estimate by applying the minimum principle (Lemma 2.1) to the comparison function $\Psi^{\pm}(x)$. □

The following theorem gives estimates for u and its derivatives in the interval D_1 and D_3 which exclude the turning point $x = 0$.

Theorem 2.3 (Berger et al. [2]). *If* $u(x)$ *is the solution of* (1.1)–(1.2) *and* a, b *and* $f \in \mathscr{C}^j(\bar{D})$, $j > 0$, *then there exist positive constants* η *and* C *depending only on* $S_1(j)$ *such that*

$$|u^{(k)}(x)| \leqslant \begin{cases} C[1 + \varepsilon^{-k}\exp(-2\eta(1+x)/\varepsilon)], & k = 1(1)j+1, \ x \in D_1, \\ C[1 + \varepsilon^{-k}\exp(-2\eta(1-x)/\varepsilon)], & k = 1(1)j+1, \ x \in D_3, \end{cases}$$

where $S_1(j) = \{\|a\|_j, \|b\|_j, \|f\|_j, a_0, (1-\delta), u(-1), u(1), u(-\delta), u(\delta), j\}$, $a(x) > 0$, *for* $x \in D_1$ *and* $a(x) < 0$, *for* $x \in D_3$.

Let us denote $\beta = b(0)/a'(0)$, and β_l, β_s be fixed positive constants such that $\beta_l < 1 < \beta_s$ and $\beta_l \leqslant |\beta| \leqslant \beta_s$. Define $S_2(j) = \{\|a\|_j, \|b\|_j, \|f\|_j, \beta_s, b_0, |A|, |B|, j\}$. Now, we state a theorem from [2] which bounds the solution of (1.1)–(1.2) and its derivatives in the interval D_2 which contains the turning point $x = 0$.

Theorem 2.4 (Berger et al. [2]). *Assume that* $\beta < 0$. *If* $u(x)$ *is the solution of* (1.1)–(1.2) *and* a, b *and* $f \in \mathscr{C}^j(\bar{D})$, $j > 0$, *then there exists a positive constant* C *depending only on* $S_2(j)$ *such that*

$$|u^{(k)}(x)| \leqslant C, \quad \forall x \in D_2, \ k = 0(1)j.$$

Remark 2.5. The choice $\delta = 1/2$ can be found in [2].

2.1. Bounds for the smooth and singular components

Hereinafter, we denote the generic positive constant independent of the mesh size, mesh points and the perturbation parameter ε by C.

We decompose the solution u of (1.1)–(1.2) into smooth and singular components as

$$u = v_0 + \varepsilon y_1 + w_0. \tag{2.1}$$

Here, v_0 satisfies the following reduced problem:

$$a(x)v_0'(x) - b(x)v_0(x) = f(x), \quad x \in D. \tag{2.2}$$

Now, applying the differential operator L and the boundary conditions as given in (1.1)–(1.2) to the asymptotic approximation (2.1), we obtain

$$Lu(x) \equiv Lv_0(x) + \varepsilon L y_1(x) + Lw_0(x) = f(x), \tag{2.3}$$

$$u(-1) = v_0(-1) + \varepsilon y_1(-1) + w_0(-1) = A, \tag{2.4}$$

$$u(1) = v_0(1) + \varepsilon y_1(1) + w_0(1) = B, \tag{2.5}$$

where y_1 and w_0 satisfy the following problems, respectively:

$$L y_1(x) = -v_0''(x), \quad x \in D, \tag{2.6}$$

$$y_1(-1) = 0, \quad y_1(1) = 0, \tag{2.7}$$

$$Lw_0(x) = 0, \quad x \in D, \tag{2.8}$$

$$w_0(-1) = A - v_0(-1), \quad w_0(1) = B - v_0(1). \tag{2.9}$$

Now, we will bound the smooth and singular components and their respective derivatives separately. In this section, the variable k appears in the derivatives will take values in $0 \leqslant k \leqslant 3$, but one can obtain a similar results for any finite value of k. Eq. (2.2) is independent of ε, and having smooth coefficients a, b and f. From these assumptions, one can have

$$|v_0^{(k)}(x)| \leqslant C, \quad \forall x \in D_1 \cup D_3.$$

Further, the BVP (2.6)–(2.7) which defines y_1 is similar to the BVP (1.1)–(1.2), then from Theorem 2.3, we have the following bound:

$$|y_1^{(k)}(x)| \leqslant \begin{cases} C[1 + \varepsilon^{-k} e_1(x, a_0)], & \forall x \in D_1, \\ C[1 + \varepsilon^{-k} e_2(x, a_0)], & \forall x \in D_3, \end{cases}$$

where $e_1(x, a_0) = \exp(-a_0(1 + x)/\varepsilon)$ and $e_2(x, a_0) = \exp(-a_0(1 - x)/\varepsilon)$.

Following the approach as found in [8], the bounds for the singular component w_0 and its derivatives are obtained in D_1. In a similar fashion, one can prove an analogous result in D_3. Let us define the two functions

$$\Psi^{\pm}(x) = |w_0(-1)| e_1(x, a_0) \pm w_0(x).$$

It can be easily verified that $\Psi^{\pm}(-1) \geqslant 0$, $\Psi^{\pm}(-\delta) \geqslant 0$, and $L\Psi^{\pm}(x) \leqslant 0$. Then from the minimum principle (Lemma 2.1), we have $\Psi^{\pm}(x) \geqslant 0$, and hence

$$|w_0(x)| \leqslant C e_1(x, a_0), \quad \forall x \in D_1.$$

Also $w_0(x)$ can be written as $w_0(x) = w_0(-\delta)\phi(x) + w_0(-1)(1 - \phi(x))$, where

$$\phi(x) = \frac{\int_{-1}^{x} \exp(-A(t)/\varepsilon)\, \mathrm{d}t}{\int_{-1}^{0} \exp(-A(t)/\varepsilon)\, \mathrm{d}t} \quad \text{and} \quad A(x) = \int_{-1}^{x} a(s)\, \mathrm{d}s.$$

Now $w_0'(x) = [w_0(-\delta) - w_0(-1)]\phi'(x)$, and so

$$|w_0'(x)| \leqslant C|\phi'(x)| \leqslant C\varepsilon^{-1}e_1(x, a_0), \quad \forall x \in D_1.$$

The second and third derivatives of w_0 can be estimated immediately by using earlier results in Eq. (2.8). Thus, we have

$$|w_0^{(k)}(x)| \leqslant C\varepsilon^{-k}e_1(x, a_0), \quad \forall x \in D_1.$$

Since, $u^{(k)} = v_0^{(k)} + \varepsilon y_1^{(k)} + w_0^{(k)}$, the earlier estimates yield,

$$|v_0^{(k)} + \varepsilon y_1^{(k)}| \leqslant C[1 + \varepsilon^{(1-k)}e_1(x, a_0)], \quad \forall x \in D_1,$$

$$|w_0^{(k)}| \leqslant C\varepsilon^{-k}e_1(x, a_0), \quad \forall x \in D_1.$$

In particular, this shows that the smooth component $v_0 + \varepsilon y_1$ and its first derivative are bounded for all values of ε. However, y_1 can now be decomposed in the same manner as was u, leading immediately to $y_1 = v_1 + \varepsilon v_2 + w_1$, where one has

$$|v_1^{(k)}(x)| \leqslant C, \quad |v_2^{(k)}(x)| \leqslant C[1 + \varepsilon^{-k}e_1(x, a_0)], \quad \forall x \in D_1,$$

$$|w_1^{(k)}(x)| \leqslant C\varepsilon^{-k}e_1(x, a_0), \quad \forall x \in D_1.$$

Combining these two decompositions, we get $u = v + w$, where $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2$ and $w = w_0 + \varepsilon w_1$. Since, $u^{(k)} = v^{(k)} + w^{(k)}$, and the above estimates yield

$$|v^{(k)}(x)| \leqslant C[1 + \varepsilon^{(2-k)}e_1(x, a_0)], \quad \forall x \in D_1,$$

$$|w^{(k)}(x)| \leqslant C\varepsilon^{-k}e_1(x, a_0), \quad \forall x \in D_1.$$

The following theorem provides bounds for the smooth and singular components as given above.

Theorem 2.6. *One can decompose the solution u of the TPP (1.1)–(1.2) as*

$$u = v + w,$$

where, for $0 \leqslant k \leqslant 3$, the smooth component v satisfies

$$|v^{(k)}(x)| \leqslant C[1 + \varepsilon^{(2-k)}e(x, a)], \quad \forall x \in \bar{D}$$

and the singular component w satisfies

$$|w^{(k)}(x)| \leqslant C\varepsilon^{-k}e(x, a), \quad \forall x \in \bar{D},$$

where $e(x, a) = e_1(x, a) + e_2(x, a)$.

Proof. Theorem 2.4 guarantees that the solution of the TPP (1.1)–(1.2) and its derivatives are smooth in the interval D_2. Hence, the proof is an immediate consequence of the above estimates on $v^{(k)}(x)$ and $w^{(k)}(x)$. □

In Farrell et al. [5], it has been proved that the classical finite-difference schemes on uniform meshes are not globally parameter uniform convergence for singularly perturbed two-point boundary

value problems. This motivates us to devise the piecewise-uniform mesh for the TPP (1.1)–(1.2). The details are given in the following section.

3. Difference scheme on a piecewise uniform mesh

In this section, we show that one can obtain ε-uniform convergence for the classical scheme, if it is applied on piecewise uniform meshes, known as Shishkin meshes. Consider the classical upwind scheme on a piecewise uniform mesh D_ε^N, $N \geqslant 4$ which is constructed by dividing the domain $\bar{D}$ into three subintervals $D_L = [-1, -1+\tau]$, $D_C = [-1+\tau, 1-\tau]$ and $D_R = [1-\tau, 1]$ such that $\bar{D} = D_L \cup D_C \cup D_R$.

The transition parameter τ is chosen to be

$$\tau = \min\left\{\frac{1}{4}, K\varepsilon \ln N\right\}, \quad K \geqslant \frac{1}{\min\{a_0, b_0\}} \quad \text{in } D. \tag{3.1}$$

Then D_ε^N is obtained by putting a uniform mesh with $N/4$ mesh elements in both D_L and D_R, and a uniform mesh with $N/2$ elements in D_C. Let us denote $D_\varepsilon^N = \{x_i\}_1^{N-1}$. The resulting fitted finite-difference scheme for the TPP (1.1)–(1.2) is given below:

$$L^N U(x_i) \equiv \varepsilon\delta^2 U(x_i) + a(x_i) D^* U(x_i) - b(x_i) U(x_i) = f(x_i), \quad x_i \in D_\varepsilon^N, \tag{3.2}$$

$$U(0) = A, \quad U(1) = B, \tag{3.3}$$

where

$$D^+ Z_i = \frac{Z_{i+1} - Z_i}{x_{i+1} - x_i}, \quad D^- Z_i = \frac{Z_i - Z_{i-1}}{x_i - x_{i-1}}, \quad \delta^2 Z_i = \frac{2(D^+ Z_i - D^- Z_i)}{x_{i+1} - x_{i-1}},$$

$$D^* Z_i = \begin{cases} D^+ Z_i & \text{if } a(x_i) > 0, \\ D^- Z_i & \text{if } a(x_i) < 0. \end{cases}$$

In this section, we follow the approach of [8] for the error analysis of the above numerical scheme. First, we shall prove the following discrete minimum principle and then a uniform stability result, similar to the continuous one as given in Lemmas 2.1 and 2.2.

Lemma 3.1. *Assume that the mesh function Y_i satisfies $Y_0 \geqslant 0$, $Y_N \geqslant 0$. Then $L^N Y_i \leqslant 0$, for $1 \leqslant i \leqslant N-1$ implies that $Y_i \geqslant 0$, $\forall\, 0 \leqslant i \leqslant N$.*

Proof. Let us choose k in such a way that $Y_k = \min_i Y_i$. If $Y_k \geqslant 0$, then there is nothing to prove. Suppose that $Y_k < 0$, then the proof is completed by showing that this leads to a contradiction. From the boundary values, it is clear that $k \notin \{0, N\}$, $Y_{k+1} - Y_k \geqslant 0$ and $Y_k - Y_{k-1} \leqslant 0$. Hence,

$$L^N Y_k = \begin{cases} \dfrac{2\varepsilon}{x_k - x_{k-1}}\left(\dfrac{Y_{k+1} - Y_k}{x_{k+1} - x_k} - \dfrac{Y_k - Y_{k-1}}{x_k - x_{k-1}}\right) + a_k\left(\dfrac{Y_{k+1} - Y_k}{x_{k+1} - x_k}\right) - b_k Y_k \geqslant 0 & \text{if } a_k > 0, \\ \dfrac{2\varepsilon}{x_k - x_{k-1}}\left(\dfrac{Y_{k+1} - Y_k}{x_{k+1} - x_k} - \dfrac{Y_k - Y_{k-1}}{x_k - x_{k-1}}\right) + a_k\left(\dfrac{Y_k - Y_{k-1}}{x_k - x_{k-1}}\right) - b_k Y_k \geqslant 0 & \text{if } a_k < 0 \end{cases}$$

with a strict inequality if $Y_k - Y_{k-1} < 0$ and $Y_{k+1} - Y_k > 0$. But this contradicts the assumption that $L^N Y_i \leqslant 0$ for $1 \leqslant i \leqslant N-1$. Hence, $Y_{k+1} = Y_k = Y_{k-1}$. Repeating the same argument by replacing $k-1$ by $k-2$, and so on, we have $Y_0 = Y_1 = Y_2 = \cdots = Y_k = Y_{k+1} < 0$, which is the required contradiction. Hence, it follows that $Y_k \geqslant 0$, and we have $Y_i \geqslant 0,\ \forall\, 0 \leqslant i \leqslant N$. □

Lemma 3.2. *If Z_i is any mesh function such that $Z_0 = Z_N = 0$. Then*

$$|Z_i| \leqslant \frac{1}{a^*} \max_{1 \leqslant j \leqslant N-1} |L^N Z_j|, \quad \forall 0 \leqslant i \leqslant N,$$

where

$$a^* = \begin{cases} -a_0 & \text{if } 0 \leqslant i \leqslant N/2, \\ a_0 & \text{if } (N/2)+1 \leqslant i \leqslant N. \end{cases}$$

Proof. Let us define

$$M^{\mp} = \frac{1}{a^*} \max_{1 \leqslant j \leqslant N-1} |L^N Z_j|.$$

Introduce the two mesh functions $Y_i^{\pm}$ defined by $Y_i^{\pm} = M^{\mp} x_i \pm Z_i$. Clearly $Y_0^{\pm} \geqslant 0,\ Y_N^{\pm} \geqslant 0$ and

$$L^N Y_i^{\pm} = M^{\mp}(a_i - b_i x_i) \pm L^N Z_i \leqslant 0,$$

since $a_i \geqslant a_0 > 0,\ \forall x_i < 0,\ 1 \leqslant i \leqslant N/2$, and $a_i \leqslant -a_0 < 0,\ \forall x_i > 0,\ (N/2)+1 \leqslant i \leqslant N-1$. The discrete minimum principle (Lemma 3.1) then implies that $Y_i \geqslant 0$, for $0 \leqslant i \leqslant N$. □

With the above continuous and discrete results, we are in a position to provide the ε-uniform convergence result in the following.

Theorem 3.3. *Let u and U be, respectively, the solutions of* (1.1)–(1.2) *and* (3.2)–(3.3). *Then, for sufficiently large N, we have the following estimate*:

$$\sup_{0 < \varepsilon \leqslant 1} \|u - U\| \leqslant C N^{-1} (\ln N)^2.$$

Proof. The solution U of the discrete problem is decomposed in an analogous manner as that of the continuous solution u. Thus $U = V + W$, where V is the solution of the inhomogeneous problem given by

$$L^N V = f, \quad V(-1) = v(-1), \quad V(1) = v(1)$$

and W is the solution of the homogeneous problem

$$L^N W = 0, \quad W(-1) = w(-1), \quad W(1) = w(1).$$

The error can be written in the form

$$U - u = (V - v) + (W - w)$$

and so the errors in the smooth and singular components of the solution can be estimated separately.

The estimate of the smooth component is obtained using the following stability and consistency argument. We consider the local truncation error

$$L^N(V-v) = (L - L^N)v = \varepsilon \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \delta^2 \right) v + a \left(\frac{\mathrm{d}}{\mathrm{d}x} - D^* \right) v.$$

Then, by local truncation error estimates, we obtain

$$|L^N(V-v)(x_i)| \leqslant \begin{cases} \dfrac{\varepsilon}{3}(x_{i+1} - x_{i-1})|v^{(3)}| + \dfrac{a(x_i)}{2}(x_{i+1} - x_i)|v^{(2)}| & \text{if } a(x_i) > 0, \\ \dfrac{\varepsilon}{3}(x_{i+1} - x_{i-1})|v^{(3)}| + \dfrac{a(x_i)}{2}(x_i - x_{i-1})|v^{(2)}| & \text{if } a(x_i) < 0 \end{cases}$$

and Theorem 2.6 yields,

$$|L^N(V-v)(x_i)| \leqslant CN^{-1}.$$

Now, applying Lemma 3.2 to the mesh function $(V-v)(x_i)$, we can easily obtain

$$|(V-v)(x_i)| \leqslant CN^{-1}. \tag{3.4}$$

To estimate the local truncation error of the singular component $L^N(W-w)$, the argument depends on whether $\tau = 1/4$ or $\tau = K\varepsilon \ln N$.

The mesh is uniform in the first case and also $K\varepsilon \ln N \geqslant 1/4$. Therefore, the local truncation error is bounded in the standard way as done above. More precisely,

$$|L^N(W-w)(x_i)| \leqslant \begin{cases} \dfrac{\varepsilon}{3}(x_{i+1} - x_{i-1})|w^{(3)}| + \dfrac{a(x_i)}{2}(x_{i+1} - x_i)|w^{(2)}| & \text{if } a(x_i) > 0, \\ \dfrac{\varepsilon}{3}(x_{i+1} - x_{i-1})|w^{(3)}| + \dfrac{a(x_i)}{2}(x_i - x_{i-1})|w^{(2)}| & \text{if } a(x_i) < 0. \end{cases}$$

Application of Theorem 2.6 to the above inequalities gives

$$|L^N(W-w)(x_i)| \leqslant C\varepsilon^{-2}N^{-1}.$$

But in the present case, $\varepsilon^{-1} \leqslant 4K \ln N$ and so

$$|L^N(W-w)(x_i)| \leqslant CN^{-1}(\ln N)^2.$$

Now, applying Lemma 3.2 to the mesh function $(W-w)(x_i)$, we then have

$$|(W-w)(x_i)| \leqslant CN^{-1}(\ln N)^2. \tag{3.5}$$

In the second case the mesh is piecewise uniform with the mesh spacing $4\tau/N$ in the subintervals D_L, D_R and $2\tau/N$ in the subinterval D_C. A different argument is used to bound $|W-w|$ in each of these subintervals.

In the subinterval with no boundary layer $D_C = [-1+\tau, 1-\tau]$, both W and w are small, and because $|W-w| \leqslant |W| + |w|$, it suffices to bound W and w separately. Actually $D_C = [-1+\tau, 0] \cup [0, 1-\tau]$, but here, we consider only the subinterval $[0, 1-\tau]$ for our discussion since one can obtain a similar

estimate in the same way for the subinterval $[-1+\tau, 0]$. Note first that in the subinterval $[0, 1-\tau]$

$$\frac{w_0'(x)}{w_0(1)} = -[1-\exp(-a_0/\varepsilon)]\phi'(x) > 0 \quad \text{and} \quad \frac{w_0(0)}{w_0(1)} = \exp(-a_0/\varepsilon).$$

Thus $w_0(0)/w_0(1)$ is positive and increasing in the interval $(0,1)$. It follows that for all $x \in [0, 1-\tau]$

$$0 \leqslant \frac{w_0(0)}{w_0(1)} \leqslant \frac{w_0(1-\tau)}{w_0(1)}$$

and so

$$|w_0(x)| \leqslant |w_0(1-\tau)|.$$

The same is true for w_1 and since $w = w_0 + \varepsilon w_1$, it follows that

$$|w(x)| \leqslant |w(1-\tau)|, \quad \forall x \in [0, 1-\tau].$$

Using the estimate of $|w|$ and the relation $\tau = K\varepsilon \ln N$ it follows that

$$|w(x)| \leqslant C\mathrm{e}^{-a_0\tau/\varepsilon} = CN^{-1}, \quad \forall x \in [0, 1-\tau]. \tag{3.6}$$

To obtain a similar bound on W an auxiliary mesh function $\bar{W}$ is defined analogous to W except that the coefficient $a(x)$ in the difference operator L^N is replaced by its lower bound a_0. Then, from Lemma 7.5 of [8],

$$|W(x_i)| \leqslant |\bar{W}(x_i)|, \quad \forall 0 \leqslant i \leqslant N.$$

Furthermore, the same lemma leads immediately to

$$|W(x_i)| \leqslant CN^{-1}, \quad \forall N/2 \leqslant i \leqslant 3N/4. \tag{3.7}$$

From the estimates obtained in (3.6) and (3.7), we have in the subinterval $[0, 1-\tau]$

$$|(W-w)(x_i)| \leqslant CN^{-1}, \quad \forall N/2 \leqslant i \leqslant 3N/4. \tag{3.8}$$

On the other hand in the subinterval D_R the classical argument once again leads to the following estimate of the local truncation error for $(3N/4)+1 \leqslant i \leqslant N-1$:

$$|L^N(W-w)(x_i)| \leqslant C\varepsilon^{-2}|x_{i+1} - x_{i-1}| = 2C\varepsilon^{-2}\tau N^{-1}.$$

Also, $|W(1) - w(1)| = 0$ and $|W(x_{3N/4}) - w(x_{3N/4})| \leqslant |W(x_{3N/4})| + |w(x_{3N/4})| \leqslant CN^{-1}$ from (3.8). Introducing the barrier function

$$\Phi_i = (x_i - (1-\tau))C_1\varepsilon^{-2}\tau N^{-1} + C_2N^{-1}$$

it follows that for a suitable choice of C_1 and C_2 the mesh functions

$$\Psi_i^{\pm} = \Phi_i \pm (W-w)(x_i)$$

satisfy the inequalities $\Psi_{3N/4}^{\pm} \geqslant 0$, $\Psi_N^{\pm} = 0$ and

$$L^N\Psi_i^{\pm} \leqslant 0, \quad (3N/4)+1 \leqslant i \leqslant N-1.$$

Application of Lemma 3.1 to the function $\Psi_i^{\pm}$ yields

$$\Psi_i^{\pm} \geqslant 0, \quad (3N/4)+1 \leqslant i \leqslant N$$

and it follows that

$$|(W-w)(x_i)| \leqslant \Phi_i \leqslant C_1\varepsilon^{-2}\tau^2 N^{-1} + C_2 N^{-1}.$$

Since $\tau = K\varepsilon \ln N$, we have

$$|(W-w)(x_i)| \leqslant CN^{-1}(\ln N)^2. \tag{3.9}$$

Combining the estimates as given in (3.8) and (3.9), we obtain

$$|(W-w)(x_i)| \leqslant CN^{-1}(\ln N)^2, \quad \forall N/2 \leqslant i \leqslant N. \tag{3.10}$$

A similar estimate as that of (3.10) can be obtained for the subinterval $[-1, 0]$, that is, for $0 \leqslant i \leqslant N/2$. Since

$$|U - u| \leqslant |V-v| + |W-w|,$$

inequalities (3.4) and (3.10) then gives the required result. □

4. Numerical examples

This section presents two numerical examples to show the applicability and efficiency of the method. The numerical results are given in the form of tables. The maximum nodal errors and order of convergence are estimated by using the exact solution (when it is available) and the double mesh principle (in the absence of exact solution). Both of the following examples have a turning point at $x = 1/2$.

Example 4.1. Consider the following singularly perturbed turning point problem [7]:

$$\varepsilon u''(x) - 2(2x-1)u'(x) - 4u(x) = 0, \quad x \in (0,1),$$

$$u(0) = 1, \quad u(1) = 1.$$

The exact solution is given by

$$u(x) = \mathrm{e}^{-2x(1-x)/\varepsilon}.$$

The exact solution is used to calculate the maximum nodal error, more precisely, we determine the maximum error as

$$E_\varepsilon^N = \max_{x_i \in \bar{D}_\varepsilon^N} |u(x_i) - U^N(x_i)| \quad \text{and} \quad E^N = \max_\varepsilon E_\varepsilon^N,$$

where u denotes the exact solution, and U^N stands for the numerical solution obtained by using N mesh intervals in the domain $\bar{D}_\varepsilon^N$. In addition, the rate of convergence is calculated by

$$p = \log_2 \left(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}} \right).$$

The estimated maximum pointwise error and the rate of convergence are presented in Tables 1 and 2.

Table 1
Maximum pointwise errors E_ε^N, and ε uniform errors E^N for Example 4.1

ε	Number of mesh points N						
	16	32	64	128	256	512	1024
1.0e-00	0.0079	0.0038	0.0019	0.0009	0.0005	0.0002	0.0001
1.0e-01	0.1354	0.0785	0.0432	0.0229	0.0118	0.0060	0.0030
1.0e-02	0.1753	0.1156	0.0786	0.0487	0.0293	0.0170	0.0096
1.0e-03	0.1792	0.1176	0.0798	0.0494	0.0298	0.0172	0.0097
1.0e-04	0.1796	0.1177	0.0800	0.0495	0.0298	0.0172	0.0097
1.0e-05	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
1.0e-06	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
1.0e-07	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
1.0e-08	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
1.0e-09	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
E^N	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097

Table 2
Rate of convergence for Example 4.1

ε	Number of mesh points N					
	16	32	64	128	256	512
1.0e-00	1.0559	1.0000	1.0780	0.8480	1.3219	1.0000
1.0e-01	0.7865	0.8617	0.9157	0.9566	0.9758	1.0000
1.0e-02	0.6007	0.5565	0.6906	0.7330	0.7854	0.8244
1.0e-03	0.6077	0.5594	0.6919	0.7292	0.7929	0.8264
1.0e-04	0.6097	0.5570	0.6926	0.7321	0.7929	0.8264
1.0e-05	0.6084	0.5583	0.6926	0.7321	0.7929	0.8264
1.0e-06	0.6084	0.5583	0.6926	0.7321	0.7929	0.8264

Example 4.2. Consider the nonhomogeneous TPP [10]:

$$\varepsilon u''(x) - 2(2x-1)u'(x) - 4u(x) = 4(4x-1), \quad x \in (0,1),$$
$$u(0) = 1, \quad u(1) = 1.$$

The exact solution of this problem is not available, in order to calculate the maximum pointwise error and rate of convergence, we use the double mesh principle. Define the double mesh differences to be

$$G_\varepsilon^N = \max_{x_i \in \bar{D}_\varepsilon^N} |U^N(x_j) - U^{2N}(x_j)| \quad \text{and} \quad G^N = \max_\varepsilon G_\varepsilon^N,$$

Table 3
Maximum pointwise errors G_ε^N, and ε uniform errors G^N for Example 4.2

ε	Number of mesh points N						
	16	32	64	128	256	512	1024
1.0e-00	0.0098	0.0048	0.0023	0.0012	0.0006	0.0003	0.0001
1.0e-01	0.1693	0.1065	0.0609	0.0332	0.0174	0.0089	0.0045
1.0e-02	0.1334	0.1094	0.0832	0.0591	0.0360	0.0221	0.0127
1.0e-03	0.1390	0.1126	0.0848	0.0600	0.0365	0.0224	0.0129
1.0e-04	0.1396	0.1129	0.0849	0.0601	0.0366	0.0225	0.0129
1.0e-05	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
1.0e-06	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
1.0e-07	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
1.0e-08	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
1.0e-09	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
G^N	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129

Table 4
Rate of convergence for Example 4.2

ε	Number of mesh points N					
	16	32	64	128	256	512
1.0e-00	1.0297	1.0614	0.9386	1.0000	1.0000	1.5850
1.0e-01	0.6687	0.8063	0.8753	0.9321	0.9672	0.9839
1.0e-02	0.2861	0.3950	0.4934	0.7152	0.7040	0.7992
1.0e-03	0.3039	0.4091	0.4991	0.7171	0.7044	0.7961
1.0e-04	0.3063	0.4112	0.4984	0.7155	0.7019	0.8026
1.0e-05	0.3063	0.4095	0.5001	0.7155	0.7019	0.8026
1.0e-06	0.3063	0.4095	0.5001	0.7155	0.7019	0.8026

where $U^N(x_j)$ and $U^{2N}(x_j)$, respectively, denote the numerical solutions obtained using N and $2N$ mesh intervals. Further, we calculate the parameter-robust orders of convergence as

$$q = \log_2 \left(\frac{G_\varepsilon^N}{G_\varepsilon^{2N}} \right).$$

The numerical results for the present example are presented in Tables 3 and 4.

5. Discussion

The proposed numerical method uses the classical upwind difference scheme on a piecewise-uniform mesh (Shishkin mesh). In general, the numerical treatment of TPPs is much more complicate than singular perturbation problems without turning points. This is mainly because the convection coefficient $a(x)$ vanishes inside the domain of interest. To preserve the stability of the difference scheme

we use both the forward and backward difference schemes depending on the sign of $a(x)$. The present method dose not require any information about the asymptotic approximation, and easy to implement. Finally, one can notice the efficiency and accuracy of the present method from the maximum pointwise error, and the rate of convergence as provided in the previous section, which reflect the theoretical error estimates derived in this article.

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