Homogenization of a class of nonlinear eigenvalue problems

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In this article we study the asymptotic behaviour of the eigenvalues of a family of nonlinear monotone elliptic operators of the form $\mathcal{A}_\varepsilon = -\operatorname{div}(a_\varepsilon(x, \nabla u))$, which are sub-differentials of even, positively homogeneous convex functionals, under the assumption that the operators G-converge to an operator $\mathcal{A}_{\mathrm{hom}} = -\operatorname{div}(a_{\mathrm{hom}}(x, \nabla u))$. We show that any limit point λ of a sequence of eigenvalues λ_ε is an eigenvalue of the limit operator $\mathcal{A}_{\mathrm{hom}}$, where λ_ε is an eigenvalue corresponding to the operator $\mathcal{A}_\varepsilon$. We also show the convergence of the sequence of first eigenvalues λ_ε^1 to the corresponding first eigenvalue of the homogenized operator.

1. Introduction

Nonlinear monotone elliptic operators like the p-Laplacian operator,

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

are important in describing problems of nonlinear elasticity, reaction–diffusion, non-Newtonian fluid mechanics, geophysics, etc., which, very often, need to be studied over media having a highly non-homogeneous character. In practice, these problems can be analysed numerically only by using the methods of the theory of homogenization. In brief, the method consists of associating a limit problem, asymptotically by letting the parameter which models the scale of material homogeneities tend to zero. The solution of the homogenized limit problem serves as an approximation to the solution of the the original physical problem. For an introduction and applications of this theory to a variety of physical problems, we refer the reader to the texts [2, 9].

In this article, we study the eigenvalues of nonlinear monotone elliptic operators in heterogeneous media by using the theory of homogenization. As is well known, a knowledge of these eigenvalues can be useful in solving questions on bifurcation and stability (see [17]). In view of the general philosophy that numerical calculations are difficult to perform on models of heterogeneous media, the best way to obtain some knowledge of these eigenvalues is by using the eigenvalues of the homogenized nonlinear operator as approximations.

There is a vast literature treating the question of homogenization of nonlinear stationary elliptic equations: for variational problems involving a single equation or systems in varying domains or non-varying domains, etc., and also for fully nonlinear equations in the set-up of viscosity solutions. Similarly, nonlinear eigenvalue problems have been extensively studied since the 1950s because of their many applications. It is beyond the scope of this article to provide extensive references on any of these topics, but we direct the reader to some relevant ones, by Pankov [14] and Chiado Piat *et al*. [4] for the homogenization of nonlinear monotone elliptic operators, and Browder [3], Rabinowitz [15], Coffman [5] and Lindqvist [12] for nonlinear eigenvalue problems. Surprisingly, there seems to be very little literature available on the homogenization of nonlinear eigenvalue problems (see [8]) in spite of its apparent importance, although the homogenization of eigenvalue problems for linear operators has been extensively studied, starting with Kesavan [10, 11].

Let us now initiate the study of the problem.

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain having a Lipschitz boundary. Let c, α, β and p be positive constants satisfying $0 < \alpha < \beta$ and $1 < p < \infty$. Let q be such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We consider nonlinear monotone operators $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ of the form $-\operatorname{div}(a(x, \nabla u))$ whose coefficients $a : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ belong to the class of functions satisfying the following hypotheses.

(H0) $a(\cdot,\cdot)$ is of Carathéodory type. Namely, $a(x,\cdot)$ is a continuous function for almost every x in Ω and $a(\cdot,\xi)$ is a measurable function for every ξ in $\mathbb{R}^m$.

(H1) (monotonicity) $0 \leqslant (a(x,\xi_1) - a(x,\xi_2)) \cdot (\xi_1 - \xi_2)$ $\forall \xi_1, \xi_2 \in \mathbb{R}^m$, almost everywhere (a.e.) $x \in \Omega$.

(H2) (uniform ellipticity) $\alpha|\xi|^p \leqslant a(x,\xi) \cdot \xi$ $\forall \xi \in \mathbb{R}^m$, a.e. $x \in \Omega$.

(H3) (growth) $|a(x,\xi)| \leqslant \beta|\xi|^{p-1}$ $\forall \xi \in \mathbb{R}^m$, a.e. $x \in \Omega$.

Let us define $\Phi(x,\xi_1,\xi_2) := a(x,\xi_1)\cdot\xi_1 + a(x,\xi_2)\cdot\xi_2$ for all ξ_1, ξ_2 in $\mathbb{R}^m$ and for all x in Ω. Let $\delta = \min(\frac{1}{2}p, (p-1))$.

(H4) (equi-continuity)

$$|a(x,\xi_1) - a(x,\xi_2)| \leqslant c\Phi(x,\xi_1,\xi_2)^{(p-1-\delta)/p}(a(x,\xi_1) - a(x,\xi_2)) \cdot (\xi_1 - \xi_2)^{\delta/p}$$

for all $\xi_1, \xi_2 \in \mathbb{R}^m$, a.e. $x \in \Omega$.

(H5) (positive homogeneity) $a(x,t\xi) = t^{p-1}a(x,\xi)$ $\forall \xi \in \mathbb{R}^m$, $\forall t > 0$, a.e. $x \in \Omega$.

(H6) (oddness) $a(x,-\xi) = -a(x,\xi)$ $\forall \xi \in \mathbb{R}^m$, a.e. $x \in \Omega$.

(H7) (cyclical monotonicity) The function a is said to be cyclically monotone if

$$\sum_{i=1}^{n} \langle a(x,\xi_i), \xi_{i+1} - \xi_i \rangle \leqslant 0$$

$\forall \xi_1, \dots, \xi_{n+1} \in \mathbb{R}^m$, $\xi_{n+1} = \xi_1$, $\forall n \in \mathbb{N}$, a.e. $x \in \Omega$.

(H8) (strong monotonicity) Let $\gamma = \max(p, 2)$ and let Φ be as above. We say that a is strongly monotone if it satisfies

$$\alpha|\xi_1 - \xi_2|^\gamma \Phi(x, \xi_1, \xi_2)^{1-(\gamma/p)} \leqslant \langle a(x, \xi_1) - a(x, \xi_2), \xi_1 - \xi_2 \rangle$$

$\forall \xi_1, \xi_2 \in \mathbb{R}^m$, a.e. $x \in \Omega$.

EXAMPLE 1.1. The prototype for such functions is $a(x, \xi) := |A(x)\xi \cdot \xi|^{(p-2)/2} A(x)\xi$, where $A(\cdot)$ is a measurable function with values in the set of $m \times m$ symmetric matrices which satisfies

$$\alpha'|\xi|^2 \leqslant A(x)\xi \cdot \xi, \qquad |A(x)\xi| \leqslant \beta'|\xi| \quad \forall \xi \in \mathbb{R}^m, \text{ a.e. } x \in \Omega \tag{1.1}$$

for some positive constants α' and β'. It can then be checked that the function a verifies all the above hypotheses for some positive constants α and β which depend on α' and β'.

For a satisfying the above hypotheses, we consider the eigenvalue problem

$$\left.\begin{aligned} -\operatorname{div}(a(x, \nabla u)) = \lambda|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.2}$$

A real number λ such that the equation (1.2) admits a non-trivial solution u is called an eigenvalue of the nonlinear, positively homogeneous monotone operator $-\operatorname{div}(a(x, \nabla u))$ and u is called a corresponding eigenfunction. In this case, by taking u as a test function in the eigenvalue problem, we note that

$$\lambda = \frac{\int_\Omega a(x, \nabla u) \cdot \nabla u \, \mathrm{d}x}{\int_\Omega |u|^p \, \mathrm{d}x}. \tag{1.3}$$

Thus, any eigenvalue of the problem is non-negative, owing to the monotonicity of the function a. The infimum of these quotients will be called the first eigenvalue λ^1 for the problem (1.2):

$$\lambda^1 = \inf_{\substack{u \in W_0^{1,p}(\Omega), \\ u \neq 0}} \frac{\int_\Omega a(x, \nabla u) \cdot \nabla u \, \mathrm{d}x}{\int_\Omega |u|^p \, \mathrm{d}x}. \tag{1.4}$$

This terminology is justified later, in § 3. In the case of the p-Laplacian, the first eigenvalue is thus the reciprocal of the best constant $C(p)$ in the Poincaré inequality

$$\int_\Omega |u|^p \, \mathrm{d}x \leqslant C(p) \int_\Omega |\nabla u|^p \, \mathrm{d}x \quad \forall u \in W_0^{1,p}(\Omega),$$

and this underlines the importance of studying the first eigenvalue problem in general.

Let $\varepsilon > 0$ be a small parameter which represents the scale of heterogeneity. We consider a family of functions a_ε satisfying hypotheses (H0)–(H8). For example, we could describe a periodic heterogeneous medium (scale ε) using coefficients $a_\varepsilon(x, \xi) = a(x/\varepsilon, \xi)$ where a is a function satisfying (H0)–(H8) on $\mathbb{R}^m \times \mathbb{R}^m$, which is in addition periodic (scale 1) in the variable x. Let λ_ε be an eigenvalue of the corresponding eigenvalue problem (1.5):

$$\left.\begin{aligned} -\operatorname{div}(a_\varepsilon(x,\nabla u_\varepsilon)) = \lambda_\varepsilon |u_\varepsilon|^{p-2}u_\varepsilon \quad \text{in } \Omega,\\ u_\varepsilon = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.5}$$

We will study the asymptotic behaviour of such sequences of eigenvalues λ_ε as the parameter $\varepsilon \to 0$.

This article is organized as follows. In § 2 we state the main results of our paper. In § 3 we introduce some notation, definitions and general preliminary results that will be used for proving these results, while the proofs themselves are given in § 4.

2. Main results

The problem of convergence of the eigenvalues of the nonlinear operators

$$-\operatorname{div}(a_\varepsilon(x,\nabla u))$$

is related to the convergence of their respective inverse operators. We say, following [4], that a family of maximal monotone operators $\mathcal{A}_\varepsilon := -\operatorname{div}(a_\varepsilon(x,\nabla u))$ G-converges to an operator $\mathcal{A}_{\text{hom}} := -\operatorname{div}(a_{\text{hom}}(x,\nabla u))$ (where a_{hom} is possibly multi-valued) if the following property holds.

Proposition 2.1. *For every sequence $h_\varepsilon \in W^{-1,q}(\Omega)$ converging strongly to some $h \in W^{-1,q}(\Omega)$, and whenever v_ε is a weak solution in $W_0^{1,p}(\Omega)$ of*

$$\left.\begin{aligned} -\operatorname{div}(a_\varepsilon(x,\nabla v_\varepsilon)) = h_\varepsilon \quad \text{in } \Omega,\\ v_\varepsilon = 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.1}$$

we have $v_\varepsilon \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$ and $a_\varepsilon(x,\nabla v_\varepsilon) \rightharpoonup a_{\text{hom}}(x,\nabla v)$ weakly in $L^q(\Omega)$. This implies, of course, that v solves the equation (in the weak sense)

$$\left.\begin{aligned} -\operatorname{div}(a_{\text{hom}}(x,\nabla v)) = h \quad \text{in } \Omega,\\ v = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.2}$$

We write $\mathcal{A}_\varepsilon \xrightarrow{\text{G}} \mathcal{A}_{\text{hom}}$.

The main results of the paper can be now be stated as follows.

Theorem 2.2. *Let us assume that the a_ε satisfy the hypotheses (H0)–(H7) and that there exists a function a_{hom} such that the operators $\mathcal{A}_\varepsilon \xrightarrow{\text{G}} \mathcal{A}_{\text{hom}}$. Let λ_ε be an eigenvalue of the operator $\mathcal{A}_\varepsilon$ and let u_ε be a corresponding eigenfunction. Then any limit point λ of the sequence λ_ε is an eigenvalue of the limit operator $\mathcal{A}_{\text{hom}}$ and there exists a subsequence of u_ε which converges weakly to an eigenfunction corresponding to λ for the limit operator.*

Theorem 2.3. *The totality of the properties (H0)–(H8) is stable under G-convergence, that is, if the a_ε satisfy (H0)–(H8) and a_{hom} is such that $\mathcal{A}_\varepsilon \xrightarrow{\text{G}} \mathcal{A}_{\text{hom}}$, then a_{hom} also satisfies (H0)–(H8).*

THEOREM 2.4. *Assume that the a_ε satisfy the hypotheses (H0)–(H8) and that there exists a function $a_{\rm hom}$ such that $\mathcal{A}_\varepsilon \xrightarrow{\rm G} \mathcal{A}_{\rm hom}$. Let λ^1_ε and $\lambda^1_{\rm hom}$ be the first eigenvalues of the operators $\mathcal{A}_\varepsilon$ and $\mathcal{A}_{\rm hom}$, defined using (1.4) with a_ε and $a_{\rm hom}$, respectively, in place of a. The sequence λ^1_ε then converges to $\lambda^1_{\rm hom}$ as $\varepsilon \to 0$.*

REMARK 2.5. In the statement of the above theorems we have assumed the G-convergence of the operators. This is not restrictive, since if the a_ε are measurable coefficients which satisfy (H1)–(H3), then the operators $\mathcal{A}_\varepsilon$ G-converge (up to a subsequence) to a maximal monotone operator $\mathcal{A}_{\rm hom}$ whose coefficient $a_{\rm hom}$ is measurable and satisfies (H1)–(H3). We refer to [4, theorem 4.1] for this well-known compactness result. On the other hand, the hypotheses (H4)–(H7) are all important in the context of a well-posed eigenvalue problem. We assume (H8) for technical reasons.

We will return to this remark later, in § 3. Let us now make some remarks on the literature and possible open problems.

In the linear case $a_\varepsilon(x,\xi) = A_\varepsilon(x)\xi$, when the A_ε are symmetric matrices which satisfy the coercivity and boundedness condition (1.1), the spectrum of the operator $\mathcal{A}_\varepsilon$ is known to be a discrete sequence tending to infinity and the kth eigenvalue is given by the Rayleigh–Ritz variational characterization

$$\lambda^k_\varepsilon := \min_{V\in F_k} \max_{u\in V\setminus\{0\}} \frac{\int_\Omega A_\varepsilon(x)\nabla u\cdot\nabla u\,\mathrm{d}x}{\int_\Omega |u|^2\,\mathrm{d}x},$$

where F_k denotes the set of all k-dimensional subspaces of $H^1_0(\Omega)$. It was shown, in this case, by Kesavan [10,11] that the G-convergence of the sequence of operators $\mathcal{A}_\varepsilon$ implies the convergence of their spectrum and that the sequence of kth eigenvalues λ^k_ε converges to the kth eigenvalue of the limit operator. In the nonlinear case, precise knowledge of the set of eigenvalues of the nonlinear operator $\mathcal{A}_\varepsilon$ does not exist, and, besides, it is hard to extend the notion of spectrum. However, nonlinear odd homogeneous maximal cyclically monotone operators like the p-Laplacian do admit an infinite unbounded sequence of eigenvalues given as min–max critical values of the corresponding Rayleigh quotient (see [4,12] for the precise definition). An interesting open question is whether these min–max values converge to the min–max values of the limit operator.

3. Preliminaries

3.1. Notation

We shall use the usual notations for L^p spaces, Sobolev spaces, smooth functions with compact support, and the space of bounded continuous functions on a domain Ω. The symbol $|\cdot|_p$ will be used for the L^p norm and $\langle\cdot,\cdot\rangle$ will be used for the inner product in any inner product space or for any duality functional, for example, between $W^{-1,q}(\Omega)$ and $W^{1,p}_0(\Omega)$. The inner product in $\mathbb{R}^m$ shall also be denoted by '$\cdot$' for convenience. We shall denote the extended real line by $\bar{\mathbb{R}}$. The symbol χ_A will denote the characteristic function of a set A. Namely, the function which takes the value 1 on the set A and the value 0 on the complement of A. We shall write $\omega \subset\subset \Omega$ to mean that ω is a relatively compact subset of Ω. We shall denote the Lebesgue measure of a measurable set A by $|A|$.

3.2. On functions

Let X be a real reflexive Banach space (possibly $\mathbb{R}^m$). Given a proper function $F : X \to \bar{\mathbb{R}}$, its *sub-differential* is a multi-valued function $\partial(F) : X \rightrightarrows X^*$ whose values at a point $x \in X$ is the set:

$$\{x^* \in X^* : F(y) \geqslant F(x) + \langle x^*, y - x\rangle \text{ for all } y \in X\}.$$

Clearly, $\partial(F)(x) = \emptyset$ if $x \notin \mathrm{dom}(F)$. The graph of a multi-function $\mathcal{A} : X \rightrightarrows X^*$, denoted by $\mathrm{Gr}(\mathcal{A})$, is the set

$$\{(u, h) \in X \times X^* : u \in \mathrm{dom}(\mathcal{A}),\ h \in \mathcal{A}(u)\},$$

where $\mathrm{dom}(\mathcal{A}) := \{u \in X : \mathcal{A}(u) \neq \emptyset\}$. Recall that a multi-function $\mathcal{A} : X \rightrightarrows X^*$ is *monotone* if and only if it satisfies the inequality:

$$\langle \eta_1 - \eta_2, \xi_1 - \xi_2\rangle \geqslant 0 \quad \forall \xi_1, \xi_2 \in X \text{ and } \forall \eta_i \in \mathcal{A}(\xi_i),\ i = 1, 2.$$

It is said to be *strictly monotone* if the inequality is strict for all $\xi_1 \neq \xi_2$. It is *cyclically monotone* if and only if it satisfies the condition

$$\sum_{i=1}^{n} \langle \eta_i, \xi_{i+1} - \xi_i\rangle \leqslant 0, \quad \forall \xi_1, \ldots, \xi_{n+1} \in X,\ \xi_{n+1} = \xi_1,\ \forall \eta_i \in \mathcal{A}(\xi_i),\ \forall n \in \mathbb{N}.$$

It is easy to see that if $\mathcal{A}$ is cyclically monotone, then it is also monotone. A multi-function $\mathcal{A}$ is said to be maximal (cyclically) monotone if it is (cyclically) monotone and its graph is not strictly contained in the graph of another (cyclically) monotone multi-function. We recall the following well-known characterization of sub-differentials of convex functions for which we refer to [16, theorem 12.25] or [1, proposition 3.64].

LEMMA 3.1. *A multi-valued function $\mathcal{A} : X \rightrightarrows X^*$ is the sub-differential of a lower semi-continuous proper convex function f if and only if it is maximal cyclically monotone. The function f is uniquely determined up to an additive constant.*

We now state some properties, which we need, for functions defined on $\mathbb{R}^m$, though some of these hold equally for infinite-dimensional reflexive Banach spaces. It will be useful to remember that a continuous (cyclically) monotone function is necessarily maximal (cyclically) monotone (cf. [16, example 12.7]). Another well-known result on convex functions says that if f is a convex function, then its sub-differential a is single valued at a point if and only if it is differentiable there (see [15, theorem 25.1]). Also, recall that a function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be *even* if $f(-\xi) = f(\xi)$ for all $\xi \in \mathbb{R}^m$. We now prove a result useful in the remainder of the paper.

PROPOSITION 3.2. *Let f be a differentiable function on $\mathbb{R}^m$ whose gradient is a. We have the following properties:*

(i) *if a is strictly monotone, then f is strictly convex;*

(ii) *if f is positively homogeneous of degree p, then $pf(\xi) = a(\xi)\cdot\xi$ for all $\xi \in \mathbb{R}^m$;*

(iii) *letting* $f(0) = 0$, *a is positively homogeneous of degree* $p-1$ *if and only if* f *is positively homogeneous of degree* p;

(iv) *a is odd if and only if* f *is even;*

(v) *if a satisfies the conditions (H2), (H3) and (H5), then* $\alpha|\xi|^p \leqslant pf(\xi) \leqslant \beta|\xi|^p$ *for all* $\xi \in \mathbb{R}^m$;

(vi) *if* f *is convex and a satisfies (H3), then*

$$|f(s) - f(s')| \leqslant \beta(|s|^{p-1} + |s'|^{p-1})|s - s'| \quad \forall s, s' \in \mathbb{R}^m. \tag{3.1}$$

Proof. (i) This property is well known and can be deduced very easily, for example, from [7, theorem 5.3.1, ch. I].

(ii) We obtain the identity by differentiating the relation $f(t\xi) = t^p f(\xi)$ with respect to t and then setting $t = 1$.

(iii) We have

$$f(\xi) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}(f(s\xi))\,\mathrm{d}s = \int_0^1 a(s\xi) \cdot \xi\,\mathrm{d}s.$$

From this it follows that if a is positively $p-1$ homogeneous, then f is positively p homogeneous. The converse is straightforward to prove by differentiating $f(t\xi) = t^p f(\xi)$ with respect to ξ.

(iv) The arguments for the proof of this property are similar to the arguments for (iii).

(v) This, again, follows from (ii) and the hypotheses on a.

(vi) The property being symmetric in s and s', it is sufficient to estimate $f(s)-f(s')$. Since f is convex, its tangent always lies below its graph. Using the tangent at s we have

$$f(s) - f(s') \leqslant -a(s) \cdot (s' - s).$$

Therefore, using the hypothesis on a, we obtain

$$f(s) - f(s') \leqslant |a(s)||s' - s| \leqslant \beta|s|^{p-1}|s' - s| \leqslant \beta(|s|^{p-1} + |s'|^{p-1})|s' - s|.$$

This completes the proof. □

We obtain the following lemma as an immediate corollary to lemma 3.1 and proposition 3.2 using the fact a in the lemma is single valued.

Lemma 3.3. *Given any a satisfying (H0)–(H8) there exists a Carathéodory function* f *which is even, positively* p*-homogeneous strictly convex and differentiable in the variable* ξ *satisfying* $\alpha|\xi|^p \leqslant pf(x,\xi) \leqslant \beta|\xi|^p$ *for all* $\xi \in \mathbb{R}^m$ *a.e.* x *in* Ω *such that* $\nabla_\xi f(x,\xi) = a(x,\xi)$. *It can be normalized such that it satisfies* $f(x,0) = 0$.

We now calculate the sub-differential of some functionals for later use.

Lemma 3.4. *Let $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a measurable function which is differentiable and convex function with respect to ξ satisfying a p-growth condition. Suppose also that its derivative a (with respect to ξ) satisfies (H3). Let $F : L^p(\Omega) \to \bar{\mathbb{R}}$ be defined by*

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(x, \nabla u)\,\mathrm{d}x & \text{if } u \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise}, \end{cases} \tag{3.2}$$

and let G be its restriction to $W_0^{1,p}(\Omega)$. Then, the following are true.

(i) *G is weakly lower semi-continuous on $W_0^{1,p}(\Omega)$. It has a Gateaux derivative $\mathcal{A}$ everywhere on $W_0^{1,p}(\Omega)$ given by the operator $-\operatorname{div}(a(x, \nabla u))$.*

(ii) *F is proper, convex and lower semi-continuous for the strong topology on $L^p(\Omega)$. The sub-differential $\partial(F)$ has the domain*

$$\operatorname{dom}(\partial(F)) = \{u \in W_0^{1,p}(\Omega) : -\operatorname{div}(a(x, \nabla u)) \in L^q(\Omega)\}. \tag{3.3}$$

Furthermore, one has $\partial(F)(u) = -\operatorname{div}(a(x, \nabla u))$ for all $u \in \operatorname{dom}(\partial(F))$.

Proof. (i) The weak lower semi-continuity is a standard fact (see [6, example 1.24]). Let us calculate the Gateaux derivative of the functional G in the direction $\varphi \in W_0^{1,p}(\Omega)$ at a given $u \in W_0^{1,p}(\Omega)$. For any $t > 0$, using proposition 3.2(vi), we get

$$g(t, x) := \frac{f(x, \nabla(u + t\varphi)) - f(x, \nabla u)}{t} \leqslant \beta(|\nabla u|^{p-1} + t^{p-1}|\nabla\varphi|^{p-1})|\nabla\varphi| \quad \text{a.e. } x.$$

Since, f is differentiable $g(t, x) \to a(x, \nabla u) \cdot \nabla\varphi$ a.e. x as $t \to 0$. On the other hand, $|g(t, x)|$ is bounded above by the integrable function $\beta(|\nabla u|^{p-1} + t^{p-1}|\nabla\varphi|^{p-1})|\nabla\varphi|$ a.e. in Ω. So, by the Lebesgue dominated convergence theorem, we have

$$\lim_{t\to 0} \frac{G(u + t\varphi) - G(u)}{t} = \lim_{t\to 0} g(t, x)\,\mathrm{d}x = \int_\Omega \nabla_\xi f(x, \nabla u) \cdot \nabla\varphi\,\mathrm{d}x.$$

The result follows by integration by parts in the last expression.

(ii) The convexity follows from the convexity of f and it is proper because of the p-growth assumption. We refer, once again, to [6, example 1.24 and proposition 2.10] for the lower semi-continuity of F.

Now, we calculate the sub-differential of this convex function. Let us denote the set on the right-hand side of (3.3) by $\mathcal{V}$.

Claim (1). *$\mathcal{V} \subset \operatorname{dom}(\partial(F))$ and $-\operatorname{div}(a(x, \nabla u)) \in \partial(F)(u)$ for any $u \in \mathcal{V}$.*

To prove this claim, for any $v \in W_0^{1,p}(\Omega)$, the (sub-)differentiability of the convex function f gives

$$\begin{aligned} F(v) - F(u) &= \int_\Omega (f(x, \nabla v) - f(x, \nabla u))\,\mathrm{d}x \\ &\geqslant \int_\Omega \nabla_\xi f(x, \nabla u) \cdot (\nabla v - \nabla u)\,\mathrm{d}x. \end{aligned}$$

Since v and u belong to $W_0^{1,p}(\Omega)$, after integrating by parts in the last inequality we get

$$F(v) \geqslant F(u) - \int_\Omega \operatorname{div}(a(x, \nabla u))(v - u)\, \mathrm{d}x.$$

On the other hand, the above inequality holds trivially for $v \in L^p(\Omega)$, so the claim is proved.

CLAIM (2). $\operatorname{dom}(\partial(F)) \subset \mathcal{V}$ *and, for any* $u \in \operatorname{dom}(\partial(F))$*, one has* $\partial(F)(u) = -\operatorname{div}(a(x, \nabla u))$.

Let us prove this claim. Let $u \in \operatorname{dom}(\partial(F))$ and $s \in \partial(F)(u)$. Therefore, by definition,

$$F(v) - F(u) \geqslant \langle s, v - u\rangle_{L^q(\Omega), L^p(\Omega)} \quad \forall v \in L^p(\Omega).$$

Clearly, $u \in W_0^{1,p}(\Omega)$. We fix any $\varphi \in W_0^{1,p}(\Omega)$ and any $t > 0$, and choose $v = u + t\varphi$ in the above inequality, to obtain

$$\int_\Omega \frac{f(x, \nabla(u + t\varphi)) - f(x, \nabla u)}{t}\, \mathrm{d}x \geqslant \langle s, \varphi\rangle_{L^q(\Omega), L^p(\Omega)}.$$

We can pass to the limit, as $t \to 0$, on the right-hand side, by part (i) of lemma 3.4, and we obtain

$$\int_\Omega \nabla_\xi f(x, \nabla u) \cdot \nabla\varphi\, \mathrm{d}x \geqslant \langle s, \varphi\rangle_{L^q(\Omega), L^p(\Omega)}.$$

By changing φ to $-\varphi$ in the above inequality we obtain, in fact, an equality for every $\varphi \in W_0^{1,p}(\Omega)$. Therefore, we have $-\operatorname{div}(a(x, \nabla u)) = s$ in the distributional sense. But $s \in L^q(\Omega)$ to begin with. This proves the second claim.

The proof is complete owing to claims 1 and 2. □

3.3. On the eigenvalue problem (1.2)

Our purpose, here, is to give a useful characterization of eigenvalues of (1.2) for any a that satisfies (H0)–(H8). Besides, we show the existence of eigenvalues by showing that λ^1 defined by (1.4) is the least eigenvalue for (1.2).

Using lemma 3.3 and proposition 3.2(ii), we first rewrite (1.4) in a convenient form:

$$\lambda^1 = \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} \mathcal{R}(u), \quad \text{where } \mathcal{R}(u) := \frac{p \int_\Omega f(x, \nabla u)\, \mathrm{d}x}{\int_\Omega |u|^p\, \mathrm{d}x}. \tag{3.4}$$

We make the following important observation.

PROPOSITION 3.5. *There is a one-to-one correspondence between the eigenvalue problem (1.2) and the Euler–Lagrange equation at critical points of the functional* $\mathcal{R}$*, defined above in (3.4), on* $W_0^{1,p}(\Omega) \setminus \{0\}$*. The eigenvalues are the critical values. The infimum* λ^1 *of* $\mathcal{R}$ *on* $W_0^{1,p}(\Omega) \setminus \{0\}$*, if it is attained, is the least eigenvalue.*

Proof. It follows from lemma 3.4(i) that $\mathcal{R}$ has a Gateaux derivative on $W_0^{1,p}(\Omega) \setminus \{0\}$. It can be seen that

$$\nabla_G \mathcal{R}(u) = 0 \iff -\operatorname{div}(a(x, \nabla u)) = \mathcal{R}(u)|u|^{p-2}u.$$

Therefore, clearly if u is a critical point then $\mathcal{R}(u)$ is an eigenvalue. Conversely, if (λ, u) satisfy (1.2) for some non-zero $u \in W_0^{1,p}(\Omega)$, then, by (1.3), $\lambda = \mathcal{R}(u)$. Therefore, by the above equivalence we have $\nabla_G \mathcal{R}(u) = 0$. This proves the first part.

In particular, if $\lambda^1 = \min_{W_0^{1,p}(\Omega)\setminus\{0\}} \mathcal{R}$, then it is not only an eigenvalue but the least eigenvalue, since any eigenvalue is in the range of $\mathcal{R}$. □

The following lemma, together with the proposition above, implies that λ^1 of (1.4) is the least eigenvalue for the problem (1.2).

LEMMA 3.6. *The infimum in the definition (1.4) is attained.*

Proof. We need to show that the infimum is attained in (3.4). Take a minimizing sequence u_n in $W_0^{1,p}(\Omega)$. We may assume, without loss of generality, that $|u_n|_p = 1$ thanks to the fact that both the numerator and denominator in (3.4) are positively p-homogeneous. The quotients

$$\frac{p\int_\Omega f(x, \nabla u_n)\,\mathrm{d}x}{\int_\Omega |u_n|^p\,\mathrm{d}x}$$

form a bounded sequence of real numbers. So, using the ellipticity property (H2), we obtain

$$\alpha|\nabla u_n|^p \leqslant C.$$

Thus, in addition, u_n is bounded in $W_0^{1,p}(\Omega)$. Assume, without loss of generality, that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. Furthermore, by the compact inclusion of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$, it follows that $u_n \to u$ strongly in $L^p(\Omega)$ and hence, $\lim_{n\to\infty}|u_n|_p = |u|_p = 1$. Thus, by the lower semi-continuity property (see lemma 3.4(i)) we have

$$\frac{p\int_\Omega f(x, \nabla u)\,\mathrm{d}x}{\int_\Omega |u|^p\,\mathrm{d}x} \leqslant \liminf_{n\to\infty} \frac{p\int_\Omega f(x, \nabla u_n)\,\mathrm{d}x}{\int_\Omega |u_n|^p\,\mathrm{d}x},$$

whereas, u_n being a minimizing sequence, the right-hand side in the above inequality is exactly λ^1. This shows that the minimum in (3.4) is attained for the above u. □

3.4. Comments on hypotheses (H0)–(H8)

We have seen in remark 2.5 that the hypotheses (H0)–(H3) ensure the G-convergence of the operators. In principle, the G-limit could be a multi-valued operator whose coefficient a_{hom} is a multi-valued function. We avoid this by assuming (H4), which is stable with respect to G-convergence (see [4, § 7]), and which implies that the coefficients are locally Hölder continuous with exponent $\delta/(p-\delta)$. On the other hand, the positive homogeneity and cyclical monotonicity conditions, (H5) and (H7), respectively, are used in the previous subsection to put the eigenvalue problem in a nice framework and to show existence. The structure of the eigenvalue problem has a richer structure if f is even and strictly convex, in which case the quotient functional admits an unbounded infinite sequence of min–max type critical values (see [4]). We can guarantee this, owing to proposition 3.2(i), by assuming (H6) and (H8). The advantage is that (H8) is stable under G-limits and can be used in proving the stability of the properties (H5)–(H7). Another use of (H8) lies in the fact that this implies the uniqueness of the solution to the stationary problem (2.1).

We add that the conditions (H0)–(H8) are not completely independent of each other. It can be seen easily that (H8) implies (H1)–(H2) and that (H4) implies (H3) in addition to the continuity of the coefficient.

3.5. On the convergence of operators and functionals

We refer to [1] for a discussion of the following notions. Let X be a real reflexive Banach space and let X^* denote its topological dual.

A family of maximal monotone operators $\mathcal{A}_\varepsilon : X \rightrightarrows X^*$ is said to *graph converge* to an operator $\mathcal{A} : X \rightrightarrows X^*$ if and only if the following property holds.

For every $(u, h) \in \mathrm{Gr}(\mathcal{A})$ there exists a sequence $(u_\varepsilon, h_\varepsilon) \in \mathrm{Gr}(\mathcal{A}_\varepsilon)$ such that $(u_\varepsilon, h_\varepsilon)$ converges to (u, h) strongly in the product topology.

We shall denote the graph convergence of the operators by $\mathcal{A}_\varepsilon \xrightarrow{\text{graph}} \mathcal{A}$.

Let F_ε be a family of functionals defined on a real reflexive Banach space X into $\bar{\mathbb{R}}$. It is said that F_ε *Mosco converges* to a functional $F : X \to \bar{\mathbb{R}}$ if and only if the following two conditions are fulfilled:

(i) $u_\varepsilon \rightharpoonup u$ *weakly* in X implies that $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geqslant F(u)$,

(ii) for every $u \in X$ there exists a sequence $u_\varepsilon \to u$ *strongly* in X such that $\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leqslant F(u)$.

We write $F_\varepsilon \xrightarrow{\text{Mosco}} F$.

We end this section by recalling the following well-known and powerful theorem (see [1, theorem 3.66]) connecting Mosco convergence of convex functionals and the graph convergence of their differentials.

Proposition 3.7. *Let X be a reflexive Banach space. For any sequence of lower semi-continuous, proper, convex functions F_ε (normalized so that $F_\varepsilon(0) = 0$) the following are equivalent:*

(i) $F_\varepsilon \xrightarrow{\text{Mosco}} F$,

(ii) $\partial F_\varepsilon \xrightarrow{\text{graph}} \partial F$.

4. Proof of the main results

We start by proving theorem 2.2.

Proof of theorem 2.2. Let λ_ε be a sequence of eigenvalues where, for each $\varepsilon > 0$, λ_ε is an eigenvalue of the nonlinear operator $\mathcal{A}_\varepsilon$, and let u_ε be a corresponding eigenfunction. We may assume without loss of generality that u_ε has been normalized, so that $|u_\varepsilon|_p = 1$. Let us suppose that λ is a limit point of the sequence λ_ε. We shall show that λ is an eigenfunction of the limit operator $\mathcal{A}_\varepsilon$.

Since $|u_\varepsilon|_p = 1$ we conclude that $h_\varepsilon := \lambda_\varepsilon |u_\varepsilon|^{p-2} u_\varepsilon$ is a bounded sequence in $L^q(\Omega)$. By the compact inclusion $L^q(\Omega) \to W^{-1,q}(\Omega)$ we conclude that $h_\varepsilon \to h$ strongly in $W^{-1,q}(\Omega)$ for a subsequence which we continue to index by ε. Therefore, using the assumption that $\mathcal{A}_\varepsilon \xrightarrow{\mathrm{G}} \mathcal{A}_{\mathrm{hom}}$, we conclude that u_ε converges weakly in $W_0^{1,p}(\Omega)$ to the solution u of the equation:

$$\begin{aligned} -\operatorname{div}(a_{\mathrm{hom}}(x,\nabla u)) = h \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega. \end{aligned}$$

To complete the proof, we show that $h = \lambda|u|^{p-2}u$ and $u \neq 0$. First, note that the compact inclusion of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ together with the weak convergence $u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ implies that $u_\varepsilon \to u$ strongly in $L^p(\Omega)$. By passing to a subsequence, if necessary, we may assume that $u_\varepsilon \to u$ a.e. in Ω. Let $\delta > 0$ be any positive number. Then, using Egoroff's theorem, we can find a measurable subset Ω_δ satisfying $|\Omega \setminus \Omega_\delta| < \delta$ such that $u_\varepsilon \to u$ uniformly on Ω_δ. Let χ_{Ω_δ} be the characteristic function of the set Ω_δ and let φ be any bounded continuous function on Ω. Therefore, by passing to the limit in

$$\int_\Omega \chi_{\Omega_\delta} h_\varepsilon \varphi \,\mathrm{d}x = \int_\Omega \chi_{\Omega_\delta} \lambda_\varepsilon |u_\varepsilon|^{p-2} u_\varepsilon \varphi \,\mathrm{d}x,$$

we obtain

$$\int_\Omega \chi_{\Omega_\delta} h \varphi \,\mathrm{d}x = \int_\Omega \chi_{\Omega_\delta} \lambda |u|^{p-2} u \varphi \,\mathrm{d}x.$$

We can now pass to the limit as $\delta \to 0$, using the fact that $|\Omega \setminus \Omega_\delta| \to 0$ when $\delta \to 0$, to conclude that

$$\int_\Omega h\varphi \,\mathrm{d}x = \int_\Omega \lambda |u|^{p-2} u\varphi \,\mathrm{d}x$$

for all $\varphi \in C_b(\Omega)$. This shows that $h = \lambda|u|^{p-2}u\varphi$ a.e. in Ω. It is also clear that $|u|_p = 1$ owing to the strong convergence of u_ε in $L^p(\Omega)$ and the fact that $|u_\varepsilon|_p = 1$ for all $\varepsilon > 0$. This concludes the proof. □

We will need the following lemma for the proof of theorem 2.3. It is an example of an application of the compensated compactness principle (see [13, 18]) in its simplest form.

LEMMA 4.1. *Suppose that* $\mathcal{A}_\varepsilon \xrightarrow{\mathrm{G}} \mathcal{A}_{\mathrm{hom}}$. *Let* $h \in W^{-1,q}(\Omega)$ *be given and* v_ε *be the solution of (2.1) (with* $h_\varepsilon = h$*). Then, given any sequence* $w_\varepsilon \rightharpoonup w$ *weakly in* $W_0^{1,p}(\Omega)$ *and any function* $\varphi \in \mathcal{D}(\Omega)$*, we have*

$$\int_\Omega a_\varepsilon(x,\nabla v_\varepsilon)\cdot\nabla w_\varepsilon \varphi \,\mathrm{d}x \to \int_\Omega a_{\mathrm{hom}}(x,\nabla v)\cdot\nabla w\varphi \,\mathrm{d}x, \tag{4.1}$$

where v *is the solution of the homogenized equation (2.2) with right-hand side* h.

Proof. We can write, using $w\varphi$ as test function in the equation for v_ε, that

$$\int_\Omega a_\varepsilon(x,\nabla v_\varepsilon)\cdot\nabla w_\varepsilon\varphi \,\mathrm{d}x = \langle h, w_\varepsilon\varphi\rangle - \int_\Omega a_\varepsilon(x,\nabla v_\varepsilon)\cdot\nabla\varphi w_\varepsilon \,\mathrm{d}x. \tag{4.2}$$

Then, since $a_\varepsilon(x,\nabla v_\varepsilon) \rightharpoonup a_{\mathrm{hom}}(x,\nabla v)$ weakly in $L^q(\Omega)$ owing to the G-convergence assumption and $w_\varepsilon \to w$ strongly in $L^p(\Omega)$, thanks to Rellich's theorem, we can pass to the limit on the right-hand side to obtain

$$\lim_{\varepsilon\to 0}\int_\Omega a_\varepsilon(x,\nabla v_\varepsilon)\cdot\nabla w_\varepsilon\varphi\,\mathrm{d}x = \langle h, w\varphi\rangle - \int_\Omega a_{\mathrm{hom}}(x,\nabla v)\cdot\nabla\varphi w\,\mathrm{d}x$$
$$= \int_\Omega a_{\mathrm{hom}}(x,\nabla v)\cdot\nabla w\varphi\,\mathrm{d}x,$$

where the last identity follows by taking $w\varphi$ as test function in the equation for v. □

Proof of theorem 2.3. It has already been shown by Chiado Piat *et al.* that properties (H1)–(H4) and (H8) are stable under G-convergence [4, theorems 4.1 and 7.9]. In particular, by (H4), the function a_{hom} is single valued, and continuous in the variable ξ for a.e. $x \in \Omega$, showing that it also satisfies (H0).

Let us now prove that a_{hom} satisfies (H5). Our proof uses a localization method introduced by Tartar [18] and Murat [13] in the linear case for deducing properties of the coefficients of the homogenized operator. Let $\xi \in \mathbb{R}^m$ and $t > 0$ be given. Let ω be any open set with $\omega \subset\subset \Omega$. It is enough to prove that (H5) holds a.e. ω for any such ω. Let u be any $W_0^{1,p}(\Omega)$ function which is identically equal to the function $\xi\cdot x$ on ω. Then, $-\operatorname{div}(a_{\mathrm{hom}}(x,\nabla u))$ belongs to $W^{-1,q}(\Omega)$, since $a_{\mathrm{hom}}(x,\nabla u)\in L^q(\Omega)$, owing to the growth condition (H3). Now, let u_ε be the solution of

$$-\operatorname{div}(a_\varepsilon(x,\nabla u_\varepsilon)) = -\operatorname{div}(a_{\mathrm{hom}}(x,\nabla u)) \quad \text{in } \Omega,$$
$$u_\varepsilon = 0 \quad \text{on } \partial\Omega.$$

Then, by the G-convergence of the operators $\mathcal{A}_\varepsilon$, we find that

$$a_\varepsilon(x,\nabla u_\varepsilon) \rightharpoonup a_{\mathrm{hom}}(x,\nabla u')$$

weakly in $L^q(\Omega)$, where u' is the weak limit in $W_0^{1,p}(\Omega)$ of the sequence u_ε. However, from the equation for u' and the strict monotonicity of a_{hom} (implied by (H8)) we have $\nabla u' = \nabla u$. This implies that $u = u'$, since u and u' are equal to zero on $\partial\Omega$. So, using the positive homogeneity of a_ε, we have

$$a_\varepsilon(x,t\nabla u_\varepsilon) \rightharpoonup t^{p-1}a_{\mathrm{hom}}(x,\nabla u) \text{ weakly in } L^q(\Omega). \tag{4.3}$$

On the other hand, again owing to the homogeneity of the coefficient a_ε, tu_ε is the solution of

$$-\operatorname{div}(a_\varepsilon(x,\nabla(tu_\varepsilon))) = -t^{p-1}\operatorname{div}(a_{\mathrm{hom}}(x,\nabla u)) \quad \text{in } \Omega,$$
$$tu_\varepsilon = 0 \quad \text{on } \partial\Omega.$$

So, using the G-convergence of the operators $\mathcal{A}_\varepsilon$ once again, we have

$$a_\varepsilon(x,\nabla(tu_\varepsilon)) \rightharpoonup a_{\mathrm{hom}}(x,\nabla(tu)) \text{ weakly in } L^q(\Omega). \tag{4.4}$$

From (4.3) and (4.4) we conclude that $a_{\mathrm{hom}}(x,t\nabla u) = t^{p-1}a_{\mathrm{hom}}(x,\nabla u)$ a.e. in Ω. Restricting to ω, we obtain immediately that $a_{\mathrm{hom}}(x,t\xi) = t^{p-1}a_{\mathrm{hom}}(x,\xi)$ a.e. in ω. By the arbitrariness of ω we obtain the same result on Ω. This completes the proof of (H5). Property (H6) is proved in a similar manner and will be left to the reader.

Property (H7) for a_{hom} may be proved similarly, as above. Let

$$\xi_1, \xi_2, \dots, \xi_n, \xi_{n+1} \in \mathbb{R}^m$$

be given, with $\xi_{n+1} = \xi_1$ and ω as above. For $i = 1, \dots, n$, we take $u_i \in W_0^{1,p}(\Omega)$ such that $u_i(x) = \xi_i \cdot x$ a.e. on ω and set $u_{n+1} = u_1$. As above, for any $\varepsilon > 0$, let $u_{i,\varepsilon}$ be a solution of equation (2.1) whose right-hand-side member is

$$-\operatorname{div}(a_{\mathrm{hom}}(x, \nabla u_i)).$$

This choice implies, as before, that $u_{i,\varepsilon} \rightharpoonup u_i$ weakly in $W_0^{1,p}(\Omega)$. Let φ be any non-negative function in $\mathcal{D}(\omega)$. Then, owing to hypothesis (H7) on a_ε, we have

$$\sum_{i=1}^{n} \int_\Omega a_\varepsilon(x, \nabla u_{i,\varepsilon}) \cdot (\nabla u_{i+1,\varepsilon} - \nabla u_{i,\varepsilon}) \varphi \, \mathrm{d}x \leqslant 0.$$

We can pass to the limit directly, using lemma 4.1, in the above inequality to conclude that

$$\sum_{i=1}^{n} \int_\Omega a_{\mathrm{hom}}(x, \nabla u_i) \cdot (\nabla u_{i+1} - \nabla u_i) \varphi \, \mathrm{d}x \leqslant 0.$$

As φ vanishes outside ω and $u_i = \xi_i \cdot x$ on ω, we have

$$\sum_{i=1}^{n} \int_\omega a_{\mathrm{hom}}(x, \xi_i) \cdot (\xi_{i+1} - \xi_i) \varphi \, \mathrm{d}x \leqslant 0 \quad \forall \varphi \in \mathcal{D}(\omega).$$

This yields the desired conclusion. Thus, we have proved (H7) and this completes the proof of the theorem. □

We need the following lemma before proving theorem 2.4. Since a_ε and a_{hom} satisfy (H0)–(H8), by lemma 3.3 we can find f_ε and f_{hom} with the properties therein such that $a_\varepsilon(x, \cdot) = \nabla_\xi f_\varepsilon(x, \cdot)$ and $a_{\mathrm{hom}}(x, \cdot) = \nabla_\xi f_{\mathrm{hom}}(x, \cdot)$ a.e. x in Ω.

LEMMA 4.2. *Let the functionals F_ε and F_{hom} be defined as in (3.2) using, respectively, the integrands f_ε and f_{hom}. Then $\partial(F_\varepsilon) \xrightarrow{\text{graph}} \partial(F_{\mathrm{hom}})$ and therefore, F_ε Mosco-converges F_{hom}.*

Proof. Let $(u, h) \in \mathrm{Gr}(F_{\mathrm{hom}})$, which means $h = -\operatorname{div}(a_{\mathrm{hom}}(x, \nabla u))$. Let $u_\varepsilon \in W_0^{1,p}(\Omega)$ be the (unique) solution to the equation $h = -\operatorname{div}(a_\varepsilon(x, \nabla u_\varepsilon))$. Then $u_\varepsilon \in \mathrm{dom}(\partial(F_\varepsilon))$, since $h \in L^q(\Omega)$ and $(u_\varepsilon, h) \in \mathrm{Gr}(F_\varepsilon)$. The G-convergence of the operators $\mathcal{A}_\varepsilon$ to the operator $\mathcal{A}_{\mathrm{hom}}$ and the uniqueness of solutions to the stationary problem (2.1), thanks to (H8), imply that u_ε converges weakly in $W_0^{1,p}(\Omega)$ to the function u. This, of course, implies that u_ε converges strongly to u in $L^p(\Omega)$. Thus, we have verified that $\partial(F_\varepsilon) \xrightarrow{\text{graph}} \partial(F_{\mathrm{hom}})$.

The Mosco convergence follows immediately from proposition 3.7, since we have the graph convergence $\partial(F_\varepsilon) \xrightarrow{\text{graph}} \partial(F_{\mathrm{hom}})$. □

Proof of theorem 2.4. The thesis of the theorem will be established if we show that there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\frac{\int_\Omega p f_{\mathrm{hom}}(x, \nabla u) \, \mathrm{d}x}{\int_\Omega |u|^p \, \mathrm{d}x} \leqslant \liminf_{\varepsilon \to 0} \lambda_\varepsilon^1 \tag{4.5}$$

and

$$\limsup_{\varepsilon\to 0}\lambda_\varepsilon^1 \leqslant \frac{\int_\Omega pf_{\mathrm{hom}}(x,\nabla v)\,\mathrm{d}x}{\int_\Omega |v|^p\,\mathrm{d}x} \quad \forall v\in W_0^{1,p}(\Omega). \tag{4.6}$$

We prove these inequalities by applying the Mosco convergence result $F_\varepsilon \xrightarrow{\text{Mosco}} F_{\mathrm{hom}}$ established in the above lemma. For each $\varepsilon>0$, let u_ε be a function in $W_0^{1,p}(\Omega)$ for which the minimum is attained in the definition (3.4) of λ_ε^1 with f_ε in the place of f. We know that it exists, by proposition 3.6 and it can be chosen such that $|u_\varepsilon|_p=1$. Let λ be any limit point of the family λ_ε^1 and assume further that $\lambda=\lim_{\varepsilon'\to 0}\lambda_{\varepsilon'}^1$ for a subsequence $\varepsilon'\to 0$. The fact that

$$\lambda_\varepsilon^1 = F_\varepsilon(u_\varepsilon) = \int_\Omega pf_\varepsilon(x,\nabla u_\varepsilon)\,\mathrm{d}x \geqslant \alpha|\nabla u_\varepsilon|^p\,\mathrm{d}x$$

and the sequence λ_ε^1 is bounded implies that u_ε is bounded in $W_0^{1,p}(\Omega)$. We choose a subsequence $u_{\varepsilon''}$ such that $u_{\varepsilon''}$ converges weakly in $W_0^{1,p}(\Omega)$ and, therefore, strongly in $L^p(\Omega)$. This implies that $|u|_p=\lim_{\varepsilon''\to 0}|u_{\varepsilon''}|_p=1$. On the other hand, by the defining property (i) of Mosco convergence, we have

$$\lambda = \lim_{\varepsilon''\to 0}\int_\Omega pf_{\varepsilon''}(x,\nabla u_{\varepsilon''})\,\mathrm{d}x \geqslant \int_\Omega pf_{\mathrm{hom}}(x,\nabla u)\,\mathrm{d}x = F_{\mathrm{hom}}(u).$$

Thus, we obtain (4.5). In order to prove the inequality (4.6), given $v\in W_0^{1,p}(\Omega)$, we choose a sequence $v_\varepsilon\to v$ strongly in $L^p(\Omega)$ such that the property (ii) of Mosco convergence holds. Therefore, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0}\lambda_\varepsilon^1 &= \limsup_{\varepsilon\to 0}\int_\Omega pf_\varepsilon(x,\nabla u_\varepsilon)\,\mathrm{d}x \\ &\leqslant \limsup_{\varepsilon\to 0}\frac{\int_\Omega pf_\varepsilon(x,\nabla v_\varepsilon)\,\mathrm{d}x}{\int_\Omega |v_\varepsilon|^p\,\mathrm{d}x} \leqslant \frac{\int_\Omega pf_{\mathrm{hom}}(x,\nabla v)\,\mathrm{d}x}{\int_\Omega |v|^p\,\mathrm{d}x}.\end{aligned}$$

This proves the second inequality. □

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