Spaces of Tilings, Finite Telescopic Approximations and Gap-Labeling

Jean Bellissard[1,2], **Riccardo Benedetti**[3], **Jean-Marc Gambaudo**[4]

[1] Georgia Institute of Technology, School of Mathematics, Atlanta, GA 30332-0160, USA
[2] Institut Universitaire de France
[3] Dipartimento di Matematica, via F. Buonarroti 2, 56127 Pisa, Italy
[4] Centro de Modelamiento Matematico, UMI CNRS 2807, Universidad de Chile, Blanco Encalada 2120, Santiago, Chile

Abstract: The continuous Hull of a repetitive tiling T in $\mathbb{R}^d$ with the Finite Pattern Condition (FPC) inherits a minimal $\mathbb{R}^d$-lamination structure with flat leaves and a transversal Γ_T which is a Cantor set. This class of tiling includes the Penrose & the Amman Benkker ones in $2D$, as well as the icosahedral tilings in $3D$. We show that the continuous Hull, with its canonical $\mathbb{R}^d$-action, can be seen as the projective limit of a suitable sequence of branched, oriented and flat compact d-manifolds. As a consequence, the longitudinal cohomology and the K-theory of the corresponding C^*-algebra $\mathcal{A}_T$ are obtained as direct limits of cohomology and K-theory of ordinary manifolds. Moreover, the space of invariant finite positive measures can be identified with a cone in the d^{th} homology group canonically associated with the orientation of $\mathbb{R}^d$. At last, the *gap labeling theorem holds*: given an invariant ergodic probability measure μ on the Hull the corresponding Integrated Density of States (IDS) of any selfadjoint operators affiliated to $\mathcal{A}_T$ takes on values on spectral gaps in the $\mathbb{Z}$-module generated by the occurrence probabilities of finite patches in the tiling.

1. Introduction

Let $\mathcal{L}$ be a discrete subset of $\mathbb{R}^d$. Following the ideas developed in [32] for $r > 0$, $\mathcal{L}$ is *r-uniformly discrete* whenever every open ball of radius r meets $\mathcal{L}$ on one point at most. For $R > 0$, $\mathcal{L}$ is *R-relatively dense* whenever every open ball of radius R meets $\mathcal{L}$ on one point at least. $\mathcal{L}$ is a *Delone set* [32] if it is both uniformly discrete and relatively dense. $\mathcal{L}$ is *repetitive* whenever given any finite subset $p \subset \mathcal{L}$, and any $\epsilon > 0$ there is $R > 0$ such that in any ball of radius R there is a subset p' of $\mathcal{L}$ which is a distance at most ϵ of some translated of p. $\mathcal{L}$ has *finite type* whenever $\mathcal{L} - \mathcal{L}$ is discrete.

With each discrete set $\mathcal{L}$ is associated the Radon measure $\nu^{\mathcal{L}} \in \mathfrak{M}(\mathbb{R}^d)$ supported by $\mathcal{L}$ and giving mass one to each point of $\mathcal{L}$ [8]. The weak*-topology on $\mathfrak{M}(\mathbb{R}^d)$ (seen as the dual space of the space $C_c(\mathbb{R}^d)$ of continuous functions with compact support), endows the set of discrete subsets of $\mathbb{R}^d$ with a metrizable topology. For such a topology the subset of an r-uniformly discrete set in $\mathbb{R}^d$ is compact [8]. If $\mathcal{L}$ is r-uniformly discrete let Ω be the closure of the set $\{\mathrm{T}^a\mathcal{L} = \mathcal{L} + a; a \in \mathbb{R}^d\}$ of its translated. Ω is compact. Then $(\Omega, \mathbb{R}^d, \mathrm{T})$ becomes a topological dynamical system called the *Hull of* $\mathcal{L}$. If $\omega \in \Omega$, let $\mathcal{L}_\omega$ denote the uniformly discrete subset of $\mathbb{R}^d$ corresponding to ω. The subset $\Gamma = \{\omega \in \Omega; 0 \in \mathcal{L}_\omega\}$ is called the *canonical transversal*.

The Hull of an r-discrete set $\mathcal{L} \subset \mathbb{R}^d$ is minimal if and only if $\mathcal{L}$ is repetitive [32]. In such a case $\mathcal{L}$ is necessarily Delone. If, in addition, $\mathcal{L}$ has finite type, then its canonical transversal Γ is a Cantor set.

The Hull can also be seen as a *lamination* [20] or a *foliated space* [34], namely a foliation with non smooth transverse structure. On the other hand [41], the construction of the Voronoi cells from the point set of atomic positions, leads to a *tiling* of $\mathbb{R}^d$ by polyhedra touching face to face, from which the point set can be recovered by a dual construction. Hence, the construction of the Hull can equivalently be performed from three equivalent complementary point of view: (i) as a dynamical system, (ii) as a lamination or foliated space, (iii) as a tiling. This latter point of view permits to select constraints using the tiling language more easily than using the language of point sets. The tilings that have mostly attracted attention are those with a finite number of bounded patches modulo translations, the so called *finite pattern condition* (FPC) and satisfying *repetitivity*. Such tilings are equivalent to repetitive finite type Delone sets. The Penrose tiling in $2D$, the Amman Benkker (or octagonal) one or the various icosahedral tilings used to describe quasicrystals [28] belong to this class. However the *pinwheel tiling* [39] is excluded of the present study but is the subject of a future publication [12]. In this work we prove the following results whenever $\mathcal{L}$ is a repetitive, finite type Delone set in $\mathbb{R}^d$:

Theorem 1.1 (Main results). *Let $\mathcal{L}$ be a repetitive Delone set of finite type with Hull Ω.*

1. *There is a projective family* $\cdots \to B_{n+1} \stackrel{f_{n+1}}{\to} B_n \to \cdots$ *of* **compact, branched, oriented, flat manifolds** *(BOF) of dimension d, with the f_n's being BOF-submersions (in particular, $Df_n = id$), such that the Hull of $\mathcal{L}$ is conjugate by an homeomorphism to the inverse limit $\varprojlim(B_n, f_n)$. The $\mathbb{R}^d$-action is induced by the infinitesimal parallel transport by constant vector fields in each of the B_n's.*
2. *Let $H_d(\Omega, \mathbb{R})$ be the d-longitudinal homology group defined as the inverse limit $\varprojlim \left(H_d(B_n, \mathbb{R}), f_n^*\right)$. Then $H_d(\Omega, \mathbb{R})$ has a canonical positive cone induced by the orientation of the B_n's, which is in one-to-one correspondence with the space of $\mathbb{R}^d$-invariant positive finite measures on Ω.*
3. *If the $\| f_n^* \|$'s are uniformly bounded in n, the Hull is uniquely ergodic. If, in addition,* $\dim H_d(B_n, \mathbb{R}) = N < \infty$ *there is no more than N invariant ergodic probability measures on Ω.*
4. *Let $\mathcal{A} = \mathcal{C}(\Omega) \rtimes \mathbb{R}^d$ be the crossed product C^*-algebra associated with the Hull. Then through the Thom-Connes isomorphism, $K_*(\mathcal{A}) = \varinjlim \left(K_{*+d}(B_n), f_n^*\right)$.*
5. *Any $\mathbb{R}^d$-invariant ergodic probability measure μ defines canonically a trace $\mathcal{T}_\mu$ on $\mathcal{A}$ together with an induced measure μ_{tr} on the transversal Γ [16]. Then the image by $\mathcal{T}_\mu$ of the group $K_0(\mathcal{A})$ coincides with $\int_\Sigma d\mu_{tr}\, C(\Sigma, \mathbb{Z})$, namely the $\mathbb{Z}$-module generated by the occurrence numbers (w.r.t. μ) of patches of finite size of $\mathcal{L}$ (gap labeling theorem).*

The main motivation for such a work comes from the description of aperiodic solids [8]. The discrete set $\mathcal{L}$ corresponds to the positions of the atomic nuclei in the limit of zero temperature. The notion of uniform discreteness corresponds to the existence of a minimum distance between atoms due to the Coulomb repulsion of positively charged nuclei. The relative density of $\mathcal{L}$ expresses the absence of empty regions of arbitrary radius. Hence, a solid or a liquid can be represented by a Delone set. The notion of finite type is a restrictive way of expressing the rigidity of the solid. Many aperiodic solids, such as perfect crystals or quasicrystals, can be described by Delone sets of finite type. The repetitivity expresses the existence of a long range order.

It has been argued [4, 5] that the mathematical description of aperiodic solids can be made through the construction of the so-called *Noncommutative Brillouin Zone* (NCBZ) to replace the Bloch theory used for periodic crystals. It has been shown that the C^*-algebra $\mathcal{A} = \mathcal{C}(\Omega) \rtimes \mathbb{R}^d$ is the non-commutative analog, for aperiodic solids, of the space of continuous functions on the Brillouin zone for periodic solids. Then $\mathcal{A}$ inherits a structure of a Noncommutative Riemannian Manifold [4, 5, 10]. The present work gives a lot of details about the topology of such manifolds. In particular, it raises the problem of whether or not this topology is accessible to physical experiments in one way or another.

The present paper also solves a longstanding conjecture, known by physicists since the mid-eighties and given explicitly in [8] concerning the *gap labeling theorem*. It was proved for the first time by Johnson & Moser [29] in 1982 for the Schödinger operators with almost periodic potentials and simultaneously by Bellissard [3] for two dimensional electrons on a square lattice submitted to a magnetic field. The relation with the K-theory of the NCBZ was established immediately [3] (see [5] for other references). It was established in full generality for $d = 1$ in [5] using the Pimsner-Voiculescu exact sequence [36]. The same method was used in [23] to extend the result for $d = 2$. In [7], the result for $d = 2$ was reestablished by using the Kasparov spectral sequence. In the case for which the Hull is given by a $\mathbb{Z}^d$-action on a Cantor set X, Hunton and Forrest [21], using the technique of spectral sequences, have proved that the K-group is isomorphic to the group cohomology of $\mathbb{Z}^d$ with values in the group $\mathcal{C}(X, \mathbb{Z})$. While this latter result does not lead to the computation of the set of gap labels in general, it permits to compute the K-group in many practical situations that occur in physics [22, 24] as well as the set of gap labels for $d = 3$ [9]. The proof of this conjecture for tilings that satisfy repetitivity and FPC in an arbitrary dimension is one of the main results of this paper. It is important to notice that similar results have been obtained independently and almost simultaneously by Benameur & Oyono [11] and by Kaminker & Putnam [30] for the case of a $\mathbb{Z}^d$-action on a Cantor set. However, the Hull of a repetitive Delone set of finite type is in general not conjugate to the suspension of a $\mathbb{Z}^d$-action on a Cantor set. Nevertheless a recent result by Sadun & Williams [40] shows that it is orbit equivalent to such an action. Therefore the result by Hunton & Forrest applies to the present case.

This paper is organized as follows. In Sect. 2, the basic notions on Delone sets, tiling theory and laminations are summarized. Section 3 is devoted to the notion of *branched oriented flat manifolds* (BOF). In particular some details are given on the way to compute their homology, cohomology. It is shown that the homology of highest degree admits a positive cone canonically associated with the orientation. Section 4 is devoted to the notion of *expanding flattening sequences* that permits to built the projective family $\cdots \to B_{n+1} \overset{f_{n+1}}{\to} B_n \to \cdots$. In particular the various limits for the homology and the cohomology groups are analyzed in detail. Moreover it is shown how tiling spaces fit in this framework. Section 5 concerns the study of invariant measures. The last Sect. 6 is devoted to the K-theory of the C^*-algebra canonically associated with the Hull and to the proof of the gap labeling theorem.

2. Delone Sets, Tilings and Laminations

In this section, basic notions on Delone sets, Tilings and Laminations are reviewed.

2.1. Delone sets. Let $\mathcal{L}$ be a discrete subset of $\mathbb{R}^d$. Then [32, 8]:

Definition 2.1. 1. *$\mathcal{L}$ is* **uniformly discrete** *if there is $r > 0$ such that any open ball $B(x;r)$ of radius r meets $\mathcal{L}$ on at most one point. Then $\mathcal{L}$ will be called r-discrete.*
2. *$\mathcal{L}$ is* **relatively dense** *if there is $R > 0$ such that any closed ball $\overline{B(x;R)}$ of radius R meets $\mathcal{L}$ on at least one point. Then $\mathcal{L}$ will be called R-dense.*
3. *$\mathcal{L}$ is called a* **Delone** *(or* **Delauney***) set if it is both uniformly discrete and relatively dense. If $\mathcal{L}$ is r-discrete and R-dense, it will be called (r,R)-Delone.*
4. *$\mathcal{L}$ has* **finite type** *if the subset $\mathcal{L} - \mathcal{L} = \{y - y' \in \mathbb{R}^d\,;\ y, y' \in \mathcal{L}\}$ of $\mathbb{R}^d$ is discrete.*
5. *$\mathcal{L}$ is* **repetitive** *if given any finite subset $p \subset \mathcal{L}$ any given $\epsilon > 0$, there is $R > 0$ such that in any ball $B(x;R)$ there is a subset $p' \subset \mathcal{L} \cap B(x;R)$ that is a translated of p within the distance ϵ: namely there is $a \in \mathbb{R}^d$ such that the Hausdorff distance between p' and $p + a$ is less than ϵ.*
6. *$\mathcal{L}$ is aperiodic if there is no $a \in \mathbb{R}^d$ such that $\mathcal{L} + a = \mathcal{L}$.*

Note that repetitivity implies relative denseness. As in [8], with each discrete set $\mathcal{L}$ is associated the Radon measure

$$\nu^{\mathcal{L}} = \sum_{y \in \mathcal{L}} \delta_y.$$

In the following, any discrete set will be identified with its associated measure. The set of Radon measures $\mathcal{M}(\mathbb{R}^d)$ is endowed with the weak-$*$ topology relative to the space $\mathcal{C}_c(\mathbb{R}^d)$ of complex valued continuous functions on $\mathbb{R}^d$ with compact support. It is proved in [8] that the closure of the space of discrete subsets of $\mathbb{R}^d$ is the space $QD(\mathbb{R}^d)$ of *point sets with multiplicity*, namely the set of pairs $(\mathcal{L}, \underline{n})$, where $\mathcal{L}$ is discrete and $\underline{n} : x \in \mathcal{L} \mapsto n_x \in \mathbb{N}_*$. The identification with Radon measures is given by

$$\nu^{\mathcal{L},\underline{n}} = \sum_{y \in \mathcal{L}} n_y\, \delta_y.$$

Such a set can be interpreted as a family of *atoms* sitting on points of $\mathcal{L}$, where there are exactly n_x such atoms sitting at x. Note that this weak topology induces the following topology on the set of discrete sets: a sequence $(\mathcal{L}_n)_{n\in\mathbb{N}}$ converges to $\mathcal{L}$ if and only if for any bounded open set Λ, the sequence $\mathcal{L}_n \cap \Lambda$ converges to $\mathcal{L} \cap \Lambda$ for the Hausdorff metric.

Lemma 2.2 ([8]). *The set $UD_r(\mathbb{R}^d)$ of Radon measures on $\mathbb{R}^d$ of the form $\nu^{\mathcal{L}}$ where $\mathcal{L}$ is r-discrete, is compact and metrizable in $\mathcal{M}(\mathbb{R}^d)$.*

The translation group $\mathbb{R}^d$ acts on $\mathcal{M}(\mathbb{R}^d)$ through $\mathrm{T}^a\mu(f) = \mu(f(.+a))$ if $f \in \mathcal{C}_c(\mathbb{R}^d)$.

Definition 2.3. *Let $\mathcal{L}$ be an r-discrete subset of $\mathbb{R}^d$. Its Hull Ω is the closure, in $\mathcal{M}(\mathbb{R}^d)$ of the family $\{\nu^{\mathcal{L}+a}\ ;\ a \in \mathbb{R}^d\}$ of its translated.*

It follows immediately from Lemma 2.2 that the Hull of a uniformly discrete set is compact and metrizable. Moreover, $\mathbb{R}^d$ acts on Ω through homeomorphisms. Hence $(\Omega, \mathbb{R}^d)$ is a topological dynamical system [26]. The following result can be found in [8]

Proposition 2.4. *Let $\mathcal{L}$ be an r-discrete subset of $\mathbb{R}^d$ with Hull Ω. Then any element $\omega \in \Omega$ is a Radon measure supported by an r-discrete set $\mathcal{L}_\omega$ such that $\omega = \nu^{\mathcal{L}_\omega}$. Moreover, if $\mathcal{L}$ is (r, R)-Delone (resp. Delone and repetitive, resp. Delone of finite type), then so are the $\mathcal{L}_\omega$'s.*

An important object is the *canonical transversal* defined by

Definition 2.5. *Let $\mathcal{L}$ be an r-discrete subset of $\mathbb{R}^d$ with Hull Ω. Its canonical transversal is the subset $\Gamma \subset \Omega$ defined by those ω's for which $\mathcal{L}_\omega$ contains the origin $0 \in \mathbb{R}^d$.*

It is easy to prove that Γ is closed in Ω. It is also easy to show that if $0 \in \mathcal{M}(\mathbb{R}^d)$ is not an element of the Hull (in which case $\mathcal{L}_0 = \emptyset$), then every orbit of Ω meet Γ. Moreover, the intersection of the orbit of ω with Γ is precisely $\mathcal{L}_\omega$ if the orbit is identified with $\mathbb{R}^d$: namely $\mathcal{L}_\omega = \{x \in \mathbb{R}^d\ ;\ \mathrm{T}^{-x}\omega \in \Gamma\}$. The following results can be found in [32, 31]

Theorem 2.6. *(i) Let $\mathcal{L}$ be an r-discrete subset of $\mathbb{R}^d$ with Hull Ω. Then the dynamical system $(\Omega, \mathbb{R}^d)$ is minimal if and only if $\mathcal{L}$ is repetitive (therefore $\mathcal{L}$ is Delone).*
(ii) Let $\mathcal{L}$ be an r-discrete set of finite type, then its canonical transversal is completely disconnected.
(iii) If $\mathcal{L}$ is an aperiodic repetitive Delone set, then for any element ω in its Hull, $\mathcal{L}_\omega$ is aperiodic.

Note that the transversal may have isolated points. For example, if $\mathcal{L}$ is periodic its transversal is a finite set. Proposition 2.24 [31] below shows however that if $\mathcal{L}$ is an aperiodic finite type and repetitive Delone set the transversal has no isolated point.

2.2. Voronoi tiling. Let $\mathcal{L}$ be an (r, R)-Delone set. If $x \in \mathcal{L}$ the *Voronoi cell* V_x is the open set

$$V_x = \{y \in \mathbb{R}^d\ ;\ |x - y| < |x' - y|,\ \forall x' \in \mathcal{L} \setminus \{x\}\}.$$

Concretely, V_x is the intersection of the opened half spaces containing x and bounded by an hyperplane located halfway between x and another point $x' \in \mathcal{L}$. Therefore V_x is convex. Moreover, since $\mathcal{L}$ is R-dense, V_x is a polyhedron enclosed in the open ball centered at x with radius R. Since $\mathcal{L}$ is r-discrete it contains the open ball centered at x with radius r. It is easy to see that two different Voronoi cells have empty intersection while the closure of all Voronoi cells cover $\mathbb{R}^d$. Thus the Voronoi cells gives a *tiling* (see Sect. 2.3) of $\mathbb{R}^d$ with at most a countable number of prototiles.

Definition 2.7. *Let $\mathcal{L}$ be a Delone set in $\mathbb{R}^d$. Then its Voronoi tiling is the tiling defined by the Voronoi cells.*

The following result is elementary.

Lemma 2.8. *Let $\mathcal{L}$ be an (r, R)-Delone set. If $x, x' \in \mathcal{L}$, the closure of their Voronoi cells intersect along a common face.*

Two points $x \neq x'$ of $\mathcal{L}$ will be called *nearest neighbors* whenever the closure of their Voronoi cell intersect along a face of codimension one. They are *neighbors* if the closure of their Voronoi cell intersect. The *Voronoi graph* is the graph with $\mathcal{L}$ as a vertex set and pairs of nearest neighbor sites in $\mathcal{L}$ as a bond set. The *Voronoi distance* of $x, x' \in \mathcal{L}$ is given by the smallest integer n such that there is a sequence $(x_0 = x, x_i, \cdots, x_n = x')$ of vertices such that x_k and x_{k+1} are nearest neighbors.

A protocell of the Voronoi tiling of $\mathcal{L}$ is an open set V such that there is $x \in \mathcal{L}$ with $V_x - x = V$ (see Sect. 2.3). A protocell always contains the origin. The following result is elementary:

Lemma 2.9. *Let $\mathcal{L}$ be an (r, R)-Delone set. Then $\mathcal{L}$ has finite type if and only if it has a finite number of protocells. In this case, the set of protocells of $\mathcal{L}_\omega$ coincides with the one of $\mathcal{L}$ for any ω in the Hull.*

Consequently, the Voronoi tiling of a finite type Delone set satisfies the *finite pattern condition* (see Sect. 2.3, Definition 2.14 and Remark 2.16). This is because the Voronoi tiles touch face-to-face. The notion of *repetitivity* is similar for tilings and Delone sets (see Definition 2.21).

Let now $\mathcal{L}$ be repetitive with finite type. As before, Ω will denote the Hull of the uniformly discrete set $\mathcal{L}$, whereas Γ will denote its canonical transversal. A *box* in Ω is an open subset homeomorphic to $\Sigma \times \mathcal{U}$, where Σ is an open subset of Γ and $\mathcal{U}$ is an open subset of $\mathbb{R}^d$.

Lemma 2.10. *Let $\mathcal{L}$ be an r_0-uniformly discrete subset of $\mathbb{R}^d$ with Hull Ω and transversal Γ. Then the map $\phi : (\omega, x) \in \Gamma \times B(0; r_0) \mapsto \mathrm{T}^x\omega \in \Omega$ defines a homeomorphism onto its image.*

Proof. Let $\mathcal{O}$ be the image of this map, namely

$$\mathcal{O} = \{\omega \in \Omega\,;\ \exists x \in \mathbb{R}^d,\ \|x\| < r_0,\ \mathrm{T}^{-x}\omega \in \Gamma\}.$$

The map ϕ is obviously continuous and onto. Moreover it is one-to-one because, if $\mathrm{T}^{-x}\omega = \mathrm{T}^{-y}\omega'$ with $(\omega, x), (\omega', y) \in \Gamma \times B(0; r_0)$ then $\mathcal{L}_{\omega'} = \mathcal{L}_\omega + y - x$. Since the origin 0 belongs to both $\mathcal{L}_{\omega'}$ and $\mathcal{L}_\omega$, it follows that $x - y \in \mathcal{L}_\omega$. Since $\mathcal{L}_\omega$ is r_0-discrete, then $x = y$ for otherwise they must satisfy $2r_0 > \|x - y\| \geq 2r_0$. In particular $\omega = \omega'$. On the other hand, let $\omega \in \mathcal{O}$. Then $B(0; r_0) \cap \mathcal{L}_\omega \neq \emptyset$ by definition. But since $\mathcal{L}_\omega$ is r_0-discrete this intersection is made of a unique point x_ω which depends continuously on ω. Then $\nu_\omega = \mathrm{T}^{-x_\omega}\omega \in \Gamma$. The map $\omega \in \mathcal{O} \mapsto (\nu_\omega, x_\omega) \in \Gamma \times B(0; r_0)$ is continuous and defines the inverse of ϕ. □

Since similar results will be proved later, the following proposition will not be proved here. It can be seen as a straightforward consequence of the theory of finite type repetitive Delone sets (see Theorem 2.6).

Proposition 2.11. *Let $\mathcal{L}$ be a repetitive (r, R)-Delone set in $\mathbb{R}^d$ of finite type, with Hull Ω and transversal Γ. Let V be a protocell of $\mathcal{L}$.*

(i) *Let Γ_V be the subset of Γ such that the Voronoi cell of the origin in $\mathcal{L}_\omega$ is V. Then Γ_V is a clopen set.*

(ii) *Let B_V be the set*

$$B_V = \{\omega \in \Omega\,;\ \exists x \in V\,,\, \mathrm{T}^{-x}\omega \in \Gamma_V\}.$$

Then the map $\phi : (\omega, x) \in \Gamma_V \times V \mapsto \mathrm{T}^x\omega \in B_V$ is a homeomorphism. In particular B_V is a box.

(iii) *Two such boxes have empty intersection, whereas their closure cover Ω. Moreover, the closures of two such boxes intersect on a set homeomorphic to $\Sigma \times F$, where F is a common face of the corresponding protocells and Σ is a clopen subset of Γ.*

2.3. Tilings. This subsection is devoted to a few important properties of tilings in the usual d-dimensional Euclidean space $\mathbb{R}^d$. Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^d$. The vector space $\mathbb{R}^d$ is *oriented* by stipulating that the standard basis is positive. In this section, a *cell* will be a bounded connected open subset of $\mathbb{R}^d$. A *punctured cell* is a pair (C, x), where C is a cell in $\mathbb{R}^d$ and $x \in C$. A *tile* (*resp. punctured tile*) will be the closure of a cell (*resp.* punctured cell). In particular, a tile is a compact set which is the closure of its interior. If t is a punctured tile, then x_t will denote the distinguished point in its interior. Two tiles t_1, t_2 are i-equivalent whenever there is an isometry of $\mathbb{R}^d$ transforming t_1 into t_2. Two punctured tiles $(\overline{C_1}, x_1)$ and $(\overline{C_2}, x_2)$ are i-equivalent if there is an isometry g such that $g(\overline{C_1}) = \overline{C_2}$ and $g(x_1) = x_2$. They are t-equivalent whenever there is $a \in \mathbb{R}^d$ such that $t_2 = t_1 + a$ (with a similar definition for punctured tiles). A prototile of type i (*resp.* t) is an i-equivalent class (*resp.* t-equivalent class) of tiles (and similarly for punctured tiles). Let X be a *countable* set of prototiles.

Definition 2.12. *For $s \in \{i, t\}$ a tiling with tile s-types in X is a countable family T of tiles of $\mathbb{R}^d$ such that:*

(i) *The s-equivalent class of each $t \in T$ belongs to X.*
(ii) *T covers $\mathbb{R}^d$.*
(iii) *Any two distinct tiles of T have disjoint interiors.*

If X is punctured, then the set of points $\mathcal{L}_T = \{x_t; t \in T\}$ is the set of **vertices** *of the tiling. A tiling T is polyhedral if all its tiles are closed polyhedron homeomorphic to a closed ball in $\mathbb{R}^d$.*

Note that the tiles of a polyhedral tiling need not be convex. Given a countable set X of prototiles, it is not known in general whether the set of tilings with type tiles in X is empty or not. However, many examples of finite X's are known to give a rich set of tilings. For example, tiles of the Penrose tiling or the Voronoi tiling of a quasi crystal [41], belong to finitely many t-types, while tiles of the pinwheel tiling [39] belong only to two tile i-types but to an infinite number of tile t-types. The following obvious result establishes an equivalence with Sect. 2.1

Proposition 2.13. *Let X be a countable set of punctured prototiles. If the maximum diameter of a prototile is finite the set of vertices of any tiling with tile types in X is relatively dense. If the distance of the distinguished point of a prototile to its complement is bounded below uniformly in X then the set of vertices is uniformly discrete.*

Given T a tiling with tile s-types in X (with $s = i, t$), a *patch* is the interior of a union a finitely many tiles. A *pattern* of T is the s-equivalence class of a patch of T. A pattern defines a finite collection of elements of X but two distinct patterns may define the same such collection. The diameter of a pattern is the diameter of any patch it contains: since two patches define the same pattern if and only if they are isometric, they have same diameter. In this paper, only tiling with a finite number of tile t-types are considered.

Hypothesis 2.1. *1. The set of prototiles X is finite;*
2. If $T(X)$ denotes the set of tilings for which the tile t-type of each tile belongs to X, then $T(X) \neq \emptyset$. $T(X)$ is called the **tiling space***.*

It follows from Proposition 2.13 that the vertices of such a tiling form a Delone set. The group $(\mathbb{R}^d, +)$ acts on $T(X)$ by translations, namely if $a \in \mathbb{R}^d$ and if T is a tiling, then $\mathrm{T}^a(T)$ denotes the tiling $T = \{t + a; t \in T\}$. The tiling space $T(X)$ is endowed with a distance (hence with the induced topology) defined as follows: let A denote the set of $\epsilon \in (0, 1)$ such that there exist x and x' in $B(0; \epsilon)$ for which $(T + x) \cap B(0; 1/\epsilon) = (T' + x') \cap B(0; 1/\epsilon)$, then:

$$\begin{aligned} \delta(T, T') &= \inf A\,, &\quad \text{if} \quad A \neq \emptyset\,,\\ \delta(T, T') &= 1\,, &\quad \text{if} \quad A = \emptyset\,. \end{aligned}$$

Hence the diameter of $T(X)$ is bounded by 1 and the $\mathbb{R}^d$-action on $T(X)$ is continuous. If X is punctured, this topology is in general strictly finer than the weak*-topology defined on the Delone sets of vertices (see Sect. 2.1). For each $T \in T(X)$, $o(T) = \mathrm{T}^{\mathbb{R}^d}(T)$ denotes its *orbit*. δ restricts to any orbit $o(T)$ and the induced topology is finer that the one induced by $\mathbb{R}^d$. The *continuous Hull* of T is the closure Ω_T of $o(T)$ in $T(X)$. It is invariant for the $\mathbb{R}^d$-action on $T(X)$. Thus $(\Omega_T, \mathbb{R}^d, \mathrm{T})$ is a topological dynamical system. The following sub-class of tilings is remarkable:

Definition 2.14. *A tiling T satisfies the* **finite pattern condition***, and then T will be called FPC, if for any $R > 0$, there are, up to translation, only finitely many patterns with diameter smaller than R.*

Remark 2.15. If X is punctured, the set of vertices of any FPC tiling with tile type in X is a finite type Delone set.

Remark 2.16. Polyhedral tilings in $\mathbb{R}^d$ in which tiles are touching face-to-face are FPC if and only if they have a finite number of prototiles. This applies to the Voronoi tiling of a Delone set.

Remark 2.17. The set $T = \{[-1, 1]^{\times 2} + (2m, 2n + \xi_m)\,;\ (m, n) \in \mathbb{Z}^2\}$ is a tiling in $\mathbb{R}^2$ with one prototile. If $(\xi_m)_{m\in\mathbb{Z}}$ is a family of independent random variables with continuous distribution, it is almost surely not FPC.

An equivalence between finite type Delone sets and FPC punctured tilings is given by

Theorem 2.18 ([31]). *If a tiling $T \in T(X)$ satisfies the FPC, then Ω_T is compact. Moreover, if T is punctured, the weak*-topology defined by the set of vertices $\mathcal{L}_T$ coincides with the δ-metric topology. In particular the continuous Hull Ω_T coincides with the Hull of $\mathcal{L}_T$.*

As for Delone sets, if T is FPC, then each tiling in Ω_T is FPC.

Definition 2.19. *Let X be punctured and satisfy Hypothesis 2.1 and let $T \in T(X)$. The closed set Γ_T of tilings $T' \in \Omega_T$ such that one vertex coincides with $0 \in \mathbb{R}^d$ is the* **canonical transversal**.

The following result [31] is equivalent to Theorem 2.6 (ii)

Proposition 2.20. *If a punctured tiling $T \in T(X)$ is FPC, then its canonical transversal Γ_T is compact and completely disconnected.*

Definition 2.21. *A tiling $T \in T(X)$ is* **repetitive** *if for any patch in T there is $R > 0$ such that, for every x in $\mathbb{R}^d$, there exists a translated of this patch belonging to T and contained in the ball $B(x; R)$.*

As for Delone set (see Theorem 2.6),

Proposition 2.22. *If a tiling $T \in T(X)$ is both FPC and repetitive, then Ω_T is minimal, namely each orbit is dense in Ω_T.*

Definition 2.23. *A tiling $T \in T(X)$ is* **aperiodic** *if there exists no $x \neq 0$ in $\mathbb{R}^d$ such that $\mathrm{T}^x(T) = T$; it is* **strongly aperiodic** *if all tilings in Ω_T are aperiodic.*

Consequently [31]:

Proposition 2.24. *If an aperiodic tiling $T \in T(X)$ is both FPC and repetitive then it is strongly aperiodic. In this case, any canonical transversal Γ_T is a Cantor set.*

Definition 2.25. *Tilings that are aperiodic, FPC and repetitive are called* **perfect**.

As the main object is the dynamical system $(\Omega_T, \mathbb{R}^d, \mathrm{T})$, this suggests the following equivalence relation on tilings of $\mathbb{R}^d$.

Definition 2.26. *Two tilings T and T' of $\mathbb{R}^d$ are* Ω-**equivalent** *if there exists a homeomorphism $\phi : \Omega_T \to \Omega_{T'}$ which conjugates the two $\mathbb{R}^d$-actions.*

2.4. Tilings versus Laminations. Let M be a compact metric space and assume there exist a cover of M by open sets U_i and homeomorphisms called *charts* $h_i : U_i \to V_i \times T_i$, where V_i is an open set in $\mathbb{R}^d$ and T_i is some topological space. These open sets and homeomorphisms define an atlas of a (d)-*lamination* structure with d-dimensional leaves on M, if the *transition maps* $h_{i,j} = h_j \circ h_i^{-1}$ read on their domains of definitions:

$$h_{i,j}(x,t) = (f_{i,j}(x,t),\ \gamma_{i,j}(t)),$$

where $f_{i,j}$ and $\gamma_{i,j}$ are continuous in the t variable and $f_{i,j}$ is smooth in the x variable. Two atlases are *equivalent* if their union is again an atlas. A (d)-*lamination* is the data of a metric compact space M together with an equivalence class of atlas $\mathfrak{L}$.

We call a *slice* of a lamination a subset of the form $h_i^{-1}(V_i \times \{t\})$. Notice that from the very definition of a lamination, a slice associated with some chart h_i intersects at most one slice associated with another chart h_j. The *leaves* of the lamination are the smallest connected subsets that contain all the slices they intersect. Leaves of a lamination inherit a d-manifold structure, thus at any point in the lamination, it is possible to define the tangent space to the leaf passing through this point. A lamination is *orientable* if there exists in $\mathfrak{L}$ an atlas made of charts $h_i : U_i \to V_i \times T_i$ and orientations associated with each V_i preserved by the restrictions $f_{i,j}$ of the transition maps to the leaves. $\mathfrak{L}$ is *oriented* if one fixes one global orientation.

Definition 2.27. *Given a lamination* $(M, \mathfrak{L})$*, a* **transversal** *of* $\mathfrak{L}$ *is a compact subset* Γ *of* M *such that, for any leaf* L *of* $\mathfrak{L}$*,* $\Gamma \cap L$ *is non empty and is a discrete subset with respect to the* d*-manifold topology of the leaf* L*.*

The laminations related to tilings have special properties:

Definition 2.28. *An oriented* d*-lamination*$(M, \mathfrak{L})$ *is flat if:*

(i) *there exists in* $\mathfrak{L}$ *a maximal atlas made of charts* $h_i : U_i \to V_i \times T_i$ *with transition maps given, on their domain of definition, by*

$$h_{i,j}(x,t) = (x + a_{i,j}, \gamma_{i,j}(t)), \quad a_{i,j} \in \mathbb{R}^d.$$

(ii) *Every leaf of* $\mathfrak{L}$ *is an oriented flat* d*-manifold isometric to* $\mathbb{R}^d$*.*

Definition 2.29. *Let* $(M, \mathfrak{L})$ *be an oriented, flat* d*-lamination. A* **box** *is the domain of a chart of the maximal atlas of* $\mathfrak{L}$*. For any point* x *in a box* B *with coordinates* (p_x, c_x) *in the chart* h*, the slice* $h^{-1}(V \times \{c_x\})$ *is called the* **horizontal** *and the Cantor set* $h^{-1}(\{p_x\} \times C)$ *is called the* **vertical** *of* x *in* B*.*

Since a transition map transforms horizontals into horizontals and verticals into verticals, these definitions make sense.

Definition 2.30. *A lamination* $(M, \mathfrak{L})$ *is* **tilable** *if it is flat and admits a transversal* Γ *which is a Cantor set.*

On a tilable lamination $(M, \mathfrak{L})$ it is possible to define a parallel transport along each leaf of the lamination. We denote by $Par(M, \mathfrak{L})$ the set of parallel vector-fields on $(M, \mathfrak{L})$. This set is parameterized by $\mathbb{R}^d$. This gives a meaning to the notion of translation in the lamination by a vector u in $\mathbb{R}^d$, defining this way a dynamical system $(M, \mathrm{T}_{\mathfrak{L}})$. The lamination is *minimal* if such a dynamical system is minimal.

Let $T \in T(X)$ be a perfect tiling constructed from a finite collection of protocells $X = \{p_1, \ldots, p_k\}$. For each protocell p_i in X, let $y_i \in p_i$. The family $Y = (y_1, \ldots, y_n)$ determines a punctured version X_Y of X. Let $\Gamma_{T,Y}$ be the canonical transversal associated with Y. From Proposition 2.24, $\Gamma_{T,Y}$ is a Cantor set. The following result is straightforward

Proposition 2.31. *Let* $F(\Omega_T)$ *be the set of compact subsets of* Ω_T *endowed with the Hausdorff distance. Then the map* $Y \in p_1 \times \cdots \times p_k \mapsto \Gamma_{T,Y} \in F(\Omega_T)$ *is continuous.*

Let P be a patch containing a cell containing $0 \in \mathbb{R}^d$. Then let $C_{T,P}$ be the set of tilings in $\Gamma_{T,Y}$ the restriction of which to P coincides with P: this is a clopen set. The topology of $\Gamma_{T,Y}$ is generated by the countable family of such clopen sets. Then $C_{T,P}$ is called the *acceptance domain* of P.

Theorem 2.32. *Let* T *be a perfect tiling with prototcells* $X = \{p_1, \ldots, p_f\}$*. Then there is a minimal tilable lamination* $(\Omega, \mathfrak{L})$ *such that:*

(i) $(\Omega, \mathrm{T}_{\mathfrak{L}})$ *and* (Ω_T, T) *are conjugate dynamical systems.*
(ii) *For any* $Y = (y_1, \ldots, y_k) \in p_1 \times \cdots \times p_k$*, the set* $\Gamma_{T,Y}$ *is a transversal of the lamination.*

Proof. Fix one $\Gamma_{T,Y}$ and choose a tiling T' in $\Gamma_{T,Y}$ and a patch P' of T' that contains a cell containing 0. For each such pair (T', P') let $C_{(T',P')}$ be the acceptance domain of P'. Then a box $U_{(T',P')} = \phi_{T',P'}(P' \times C_{(T',P')})$ is defined through the map $\phi_{(T',P')}$ defined by

$$\begin{aligned} P' \times C_{(T',P')} &\mapsto U_{(T',P')} \\ (v, T") &\mapsto \mathrm{T}^v(T"). \end{aligned}$$

Since T is perfect, the orbit of T' is dense so that as P' varies, the $U_{(T',P')}$ form an open cover of Ω_T. By compactness, a finite subcover $U_1, \ldots, U_n$ can be extracted. Let $P_1, \ldots, P_n$ be the corresponding patches, let C_i denote the clopen set $C_{(T',P_i)}$ and let ϕ_i be the map $\phi_{(T',P_i)}$. Each clopen set C_i can be decomposed into a partition $C_{i,j}$ of clopen sets, where $1 \le j \le k_i$, of arbitrary small diameter. Since any tiling in Ω_T is perfect, this decomposition can be chosen such that

1. For each $i = 1, \ldots, n$, and each $j = 1, \ldots, k(i)$ the restriction of ϕ_i to $P_i \times C_{i,j}$ is an homeomorphism onto its image denoted by $U_{i,j}$.
2. For each pair (i_1, i_2) in $\{1, \ldots, n\}^{\times 2}$ and for each j_1 in $\{1, \ldots, k(i_1)\}$ there exists at most one j_2 in $\{1, \ldots, k(i_2)\}$ such that $U_{i_1,j_1} \cap U_{i_2,j_2} \neq \emptyset$.

If $T" \in U_{i_1,j_1} \cap U_{i_2,j_2}$ there are $(v_s, T_s) \in P_{i_s} \times C_{i_s}$ $(s = 1, 2)$ such that $\phi_{i_1}(v_1, T_1) = T" = \phi_{i_2}(v_2, T_2)$. Moreover, the vector $a = v_2 - v_1$ is independent of the choice of $T"$. For indeed the transition map $\phi_{i_2}^{-1} \circ \phi_{i_1}$ is defined on $P_{i_1} \cap (P_{i_2} - a) \times C_{i_1,j_1} \cap (\mathrm{T}^{-a} C_{i_2,j_2})$ and is given by $\phi_{i_2}^{-1} \circ \phi_{i_1}(v_1, T_1) = (v_2, T_2) = (v_1 + a, \mathrm{T}^a(T_1))$. It follows that Ω_T can be endowed with a structure of tilable lamination $\mathfrak{L}$, where the leaf containing a tiling T' in Ω_T is its $\mathbb{R}^d$-orbit. By construction, the set $\Gamma_{T,Y}$ is a transversal of the lamination and it is plain to check that any other $\Gamma_{T,Y'}$ is also a transversal. A translation in the lamination $(\Omega_T, \mathfrak{L})$ defined using $Par(\Omega_T, \mathfrak{L})$ coincides by construction with a usual translation acting on Ω_T, so the last statement is immediate and minimality is a direct consequence of the minimality of (Ω_T, T). □

Remark 2.33. Recently L. Sadun and R.F. Williams [40] have proved that the continuous Hull of a perfect tiling of $\mathbb{R}^d$ is homeomorphic to a bundle over the d-torus whose fiber is a Cantor set, a much nicer object than a lamination. Unfortunately, this homeomorphism is not a conjugacy: it does not commute with translations. However this shows that every tiling is orbit equivalent to a $\mathbb{Z}^d$-action.

2.5. Box decompositions. Tilings may look like very rigid objects. Through their connections with laminations and dynamical systems, however, tiling spaces are easier to handle. It leads to new results on tilings. For this purpose and by analogy with Markov partitions in Dynamical Systems, let us introduce the notion of *box decomposition* of a tilable lamination.

Definition 2.34. *A* **well oriented** d**-cube** *in* $\mathbb{R}^d$ *is an open set of the form* $(a_1, b_1) \times \cdots \times (a_d, b_d)$ *with* $a_i < b_i$ *real numbers. A* **block** *in* $\mathbb{R}^d$ *is a connected set which is the interior of a finite union of closures of well oriented* d*-cubes.*

Definition 2.35. *Let* $(M, \mathfrak{L})$ *be a tilable lamination. Let* $B \subset M$ *be a box of the form* $h^{-1}(P \times C)$ *in a chart* h *in* $\mathfrak{L}$ *defined in a neighborhood of the closure* $\overline{B}$ *where* C *is a clopen subset of a Cantor set.*

(i) *When P is a well oriented d-cube, B is called a box* **of cubic type**;
(ii) *When P is a block, B is called a box* **of block type**.

These definitions are independent of the choice of the chart in $\mathfrak{L}$.

Definition 2.36. *For a box B (of block or cubic type) of the form $h^{-1}(P \times C)$ in a defining chart h, the set $h^{-1}(\partial P \times C)$ is its* **vertical boundary**.

Lemma 2.37. *Let $B_1, \dots, B_m$ be a finite collection of boxes of block type in a tilable lamination. Then, there exists a finite collection of boxes of cubic type in $B'_1 \dots B'_p$ such that:*

(i) *the B'_l's are pairwise disjoint,*
(ii) *the closure of $\cup_{l=1}^{l=p} B'_l$ coincides with the closure of $\cup_{j=1}^{j=m} B_j$,*
(iii) *if a B'_l intersects a B_j then it is contained in this B_j.*

Proof. The case of two boxes B_1, B_2 will be considered first. For $i = 1, 2$, let h_i be the chart defining B_i so that $h_i(B_i) = P_i \times C_i$. On its domain of definition, the transition map $h_{1,2}$ reads:

$$h_{1,2}(x,t) = h_1 \circ h_2^{-1}(x,t) = (x - a, \gamma(t)), \quad a \in \mathbb{R}^d .$$

Since, for $i = 1, 2$, C_i is completely disconnected, there is a partition in clopen sets $C_{i,j}, \ \ j = 1, \dots, k(i)$ with arbitrary small diameters. This partition can be chosen so that $P_1 \times C_{1,j_1} \cap h_{1,2}(P_2 \times C_{2,j_2}) = P_1 \cap (P_2 - a) \times C'$ (or may be empty) where C' is a clopen subset of C_{1,j_1}. Similarly $P_2 \times C_{2,j_2} \cap h_{2,1}(P_1 \times C_{1,j_1}) = P_2 \cap (P_1 + a) \times C"$, where $C"$ is a clopen subset of C_{2,j_2}. It follows that $h_1^{-1}(P_1 \times C_{1,j_1}) \cup h_2^{-1}(P_2 \times C_{2,j_2})$ is the reunion of five disjoint boxes of block type $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$, where

$$A_1 = h_1^{-1}(P_1 \times (C_{1,j_1} \setminus C')), \quad A_5 = h_2^{-1}(P_2 \times (C_{2,j_2} \setminus C")),$$

$$A_2 = h_1^{-1}(P_1 \setminus (P_2 - a) \times C'), \quad A_4 = h_2^{-1}(P_2 \setminus (P_1 + a) \times C"),$$

$$A_3 = h_1^{-1}(P_1 \cap (P_2 - a) \times C') = h_2^{-1}(P_2 \cap (P_1 + a) \times C").$$

Since P_1, P_2, $P_1 \cap (P_2 - a)$, $P_1 \setminus (P_2 - a)$ and $P_2 \setminus (P_1 + a)$ are blocks, they are also finite unions of well oriented d-cubes $P'_1, \dots, P'_m$. This decomposition in d-cubes induces a finite collection of disjoint (open) cubic boxes with closure given by the closure of $h_1^{-1}(P_1 \times C_{1,j_1}) \cup h_2^{-1}(P_2 \times C_{1,j_2})$. Applying the same procedure for all pairs (i_1, j_1) and (i_2, j_2) leads to the desired finite collection of cubic boxes. The result for a collection of n boxes in general is obtained by induction. □

Definition 2.38. *A tilable lamination $(M, \mathfrak{L})$ admits a* **box decomposition of cubic type** *(resp. of* **block type***) if there exists a finite collection of boxes of cubic type (resp. of block type) $\mathcal{B} = \{B_1, \cdots, B_n\}$ such that:*

(i) *the B_i's are pairwise disjoint,*
(ii) *the union of the closures of the B_i's covers M.*

Proposition 2.39. *Any tilable lamination $(M, \mathfrak{L})$ admits a box decomposition of cubic type.*

Proof. For each $x \in M$, let U contain x and be the domain of a chart $h : U \to V \times T$. Let P be a well oriented d-cube in V and let C be a small clopen set in T such that $x \in U' = h_x^{-1}(P \times C)$. This gives a cover of M by boxes of cubic type. Since M is compact, there is a finite subcover by boxes of cubic types. The result is then a consequence of Lemma 2.37. □

In terms of tilings, this last result implies that the tilings induced on each leaf of the lamination by the intersection of the leaf with the vertical boundaries of the boxes $B_1, \ldots, B_n$ defined in Proposition 2.39 are made with at most n-types of well oriented d-cubes. In general, it is not true that we recover the whole lamination by considering the hull of these induced tilings. As it will be shown in [12], expansivity is required for a tilable lamination to be a tiling space (two distinct leaves cannot stay arbitrarily close one to the other).

However, when the tilable lamination is already given from a tiling space, Theorem 2.32 and Proposition 2.39 yield with no difficulty:

Corollary 2.40. *For any perfect tiling T of $\mathbb{R}^d$ there is a perfect tiling T' made with a finite number of well oriented cubic prototiles which is Ω-equivalent to T.*

2.6. Zooming. The last result shows that it is enough to restrict the study to perfect tilings having a finite number prototiles made of well oriented d-cubes (namely *cubic type tilings*). However, several constructions made in this paper require using *block type tilings*, namely tiling defined by a finite number of prototiles each being a block. For these reasons, unless otherwise stated, all box decompositions will be of block type. It is worth remarking that the constructions that we perform in the following work for tilable laminations and not only for tiling spaces. Finally, let us notice that the construction leading to Proposition 2.39 leaves a lot of room. This freedom allows us to introduce the following definition.

Definition 2.41. *Let $\mathcal{B}$ and $\mathcal{B}'$ be two box decompositions of block type of a same tilable lamination $(M, \mathfrak{L})$. $\mathcal{B}'$ is* **zoomed out** *of $\mathcal{B}$ if*

(i) *for each point x in a box B' in $\mathcal{B}'$ and in a box B in $\mathcal{B}$, the vertical of x in B' is contained in the vertical of x in B,*
(ii) *the vertical boundaries of the boxes of $\mathcal{B}'$ are contained in the vertical boundaries of the boxes of $\mathcal{B}$,*
(iii) *for each box B' in $\mathcal{B}'$, there exists a box B in $\mathcal{B}$ such that $B \cap B' \neq \emptyset$ and the vertical boundary of B does not intersect the vertical boundary of B'.*

Proposition 2.42. *Consider a minimal tilable lamination $(M, \mathfrak{L})$. Then, for any box decomposition of block type $\mathcal{B}$, there exists another box decomposition of block type zoomed out of $\mathcal{B}$.*

Proof.

Step 0. Building a finite repetitive Delone set. Let $\mathcal{B} = \{B_1, \ldots B_n\}$. Each box B_i of the box decomposition reads $P_i \times C_i$ in a chart. We consider the image $\hat{C}_i$ under the parametrization of $\{p_i\} \times C_i$ where p_i is the barycenter of P_i. Let $\hat{C} = \cup_{i=1}^n \hat{C}_i$. Choose a leaf L of the lamination and a point x in $L \cap \hat{C}$. Identifying (L, x) with $(\mathbb{R}^d, 0)$, it is clear that the set $L \cap \hat{C}$ is a repetitive Delone set in $\mathbb{R}^d$ which has finite type. Each point y in this Delone set is the barycenter of a block P_y which is a translated copy of one of the P_i's.

Step 1. Voronoi decomposition. Let $\delta > 0$ be small enough and consider a clopen (closed open) set $\hat{C}'_1$ in the vertical $\hat{C}_1$ with diameter smaller than δ and containing the point x. Using again the identification $(L, x) = (\mathbb{R}^d, 0)$, we easily check that the set $L \cap \hat{C}'_1$ is again a repetitive Delone set in $\mathbb{R}^d$ which has finite type. The clopen set $\hat{C}'_1$ can be decomposed in a finite number of disjoint clopen sets $\hat{C}'_{1,1}, \dots \hat{C}'_{1,m}$ which are characterized by the property that for each pair of points y and y' in some $\hat{C}'_{1,i}$, the Voronoi cells associated with y' is the image the Voronoi cell associated with y under the translation of the vector $y' - y$. We denote the translation class of these Voronoi cells $V_{1,j}$. This partition of $\hat{C}'_1$ yields a box decomposition $\mathcal{B}_V = \{B_{1,1}, \dots, B_{1,m}\}$ of $(M, \mathfrak{L})$, called the *Voronoi box decomposition* associated with C_1, whose boxes read in the charts:

$$B_{1,j} = V_{1,j} \times C'_{1,j},$$

where $\{p_1\} \times C'_{1,i}$ is the image in the chart of $\hat{C}'_{1,j}$.

Step 2. Zooming.. We perform the following surgery on the Voronoi cells of the Delone set $L \cap \hat{C}'_1$. For each point y in $L \cap \hat{C}'_1$ with Voronoi cell V_y we associate the block:

$$P'_y = \cup_{y' \in V_y \cap \hat{C}} P_{y'}.$$

It may happen that there exists y' in $\hat{C}$ which belongs to two Voronoi cells V_z and $V_{z'}$. In order to get a partition of $\mathbb{R}^d$ we have to avoid this ambiguity. For this purpose, we make a choice in a coherent way, that is to say so that, whenever z and z' both belong to some $L \cap \hat{C}_{1,j}$, then two blocks P'_z and $P'_{z'}$ are tranlated copies one of the other. We denote by $P'_{1,j}$ the translation class of these polyhedra. This gives us a new box decomposition $\mathcal{B}' = \{B'_1, \dots, B'_m\}$, deduced from the Voronoi box decomposition $\mathcal{B}_V$ and whose boxes read in the charts:

$$B'_j = P'_{1,j} \times C'_{1,j}.$$

It is straightforward to check that, when δ is small enough, the box decomposition $\mathcal{B}'$ is zoomed out of the box decomposition $\mathcal{B}$. □

Let now $\mathcal{B} = \{B_1, \dots, B_n\}$ and $\mathcal{B}' = \{B'_1, \dots, B'_m\}$ be two box decompositions of the same tilable lamination $(M, \mathfrak{L})$ such that $\mathcal{B}'$ is zoomed out of $\mathcal{B}$ and let T and T' be the two tilings induced by both decompositions on the same leaf L of the lamination. The connected components of the intersection of L with the B'_j's define the tiles of T'. Each tile t' of T' is tiled with tiles of T and surrounded by other tiles of T. The union of the tiles of T whose closures intersect the closure of t' and whose interiors do not intersect the interior of the tile t' is called the *first corona* of the tile t'.

Definition 2.43. *We say that $\mathcal{B}'$* **forces its border** [1] *if for each tile t' of the tiling induced on each leaf of $(M, \mathfrak{L})$ by the box decomposition $\mathcal{B}'$, the first corona of t' depends only (up to translation) on the box B'_j which contains t'.*

Proposition 2.42 can be improved as follows

[1] This concept was first introduced in the context of tilings (see [31]). The present definition agrees with the standard one.

Theorem 2.44. *Consider a minimal tilable lamination $(M, \mathfrak{L})$. Then, for any box decomposition of block type $\mathcal{B}$, there exists another box decomposition of block type zoomed out of $\mathcal{B}$ that forces its border.*

Proof. Consider a box decomposition of block type $\mathcal{B}' = \{B'_1, \ldots, B'_m\}$ zoomed out of $\mathcal{B}$. Since the tiling T and T' respectively induced by $\mathcal{B}$ and $\mathcal{B}'$ have finite type, it follows that there exist only finitely many possible first coronas for any tile of T'. In particular, fix j in $\{1, \ldots, m\}$ and consider the box B'_j which reads in a chart $P'_{1,j} \times C'_{1,j}$. There exists a finite partition of $C'_{1,j}$ in clopen sets $C'_{1,j,1} \ldots C'_{1,j,k}$ of $C'_{1,j}$ such that, for all $l = 1, \ldots, k$, the tiles of T' which correspond (in the charts) to the connected components of $L \cap P'_{1,j} \times C'_{1,j,l}$ have the same first corona up to translation. The new boxes which read in the charts $P'_{1,j} \times C'_{1,j,l}$ define a box decompostion zoomed out of $\mathcal{B}$ that forces its border. □

Corollary 2.45. *Let $(M, \mathfrak{L})$ be a minimal tilable lamination. Then, for any box decomposition of block type $\mathcal{B}$, there exists a sequence $(\mathcal{B}^{(n)})_{n \geq 0}$ of box decompositions of block type such that*

(i) $\mathcal{B}^{(0)} = \mathcal{B}$,
(ii) *for each $n \geq 0$, $\mathcal{B}^{(n+1)}$ is zoomed out of $\mathcal{B}^{(n)}$ and forces its border.*

In terms of tilings, this last result can be interpreted as follows. Let L be a leaf of the lamination and let $n \in \mathbb{N}$. Then let $T^{(n)}$ be the tiling on this leaf induced by its intersection with the vertical boundaries of the boxes of the box decomposition $\mathcal{B}^{(n)}$ defined in Theorem 2.44. This sequence of tilings is *nested*, namely

Definition 2.46. *A sequence $T^{(n)}_{n \geq 0}$ of tilings is called* **nested** *if the following properties hold*

(i) *the t-tiles types of $T^{(n)}$ are defined by a finite number of block prototiles,*
(ii) *each tile $t^{(n+1)}$ of $T^{(n+1)}$ is a connected patch of $P^{(n)}$ of $T^{(n)}$ and contains at least one tile of $T^{(n)}$ its interior,*
(iii) *the patch $Q^{(n)}$ of $T^{(n)}$ made with all the tiles of $T^{(n)}$ that touch $P^{(n)}$ is uniquely determined up to translation* [2]*,*
(iv) *the tiling $T^{(n)}$ is Ω-conjugate to $T^{(0)}$.*

Let $(M, \mathfrak{L})$ be a minimal tilable lamination with a box decomposition $\mathcal{B}$. Then, two points in M are equivalent if they are in the same box of $\mathcal{B}$ and on the same vertical in this box. The quotient space is a *branched d-manifold* that inherits from its very construction some extra structures. Section 3 is devoted to an axiomatic approach of these objects. Performing this quotient operation for a sequence of box decompositions zoomed out one of the others as in Corollary 2.45 gives rise to a sequence of such branched manifolds whose study is reported in Sect. 4.

3. BOF-d-Manifolds

This section is devoted to the definition and the study of *branched oriented flat* (briefly: BOF) d-manifolds. There are several ways of defining these objects. For a polyhedral

[2] This is a generalization of the classical definition of "forcing its border" for substitution tilings (see [31]).

FPC tiling, the number of prototiles is finite and their faces are parallel to a finite number of hyperplanes. Hence the local models for branching points will be built out of polyhedral sectors with boundaries parallel to one of these hyperplanes. This is the way chosen in this work.

In the rest of this paper then, $\mathcal{E}$ will denote a family of $n \geq d$ non zero vectors in $\mathbb{R}^d$ generating $\mathbb{R}^d$. Then given $x \in \mathbb{R}^d$ let $F_e(x) = \{y \in \mathbb{R}^d\,;\ \langle e|y-x\rangle = 0\}$ be the affine hyperplane through x and perpendicular to $e \in \mathcal{E}$. The BOF manifolds that will be considered here are defined through this $\mathcal{E}$-geometry. Cubic type tiling is obtained by choosing $n = d$ and the elements of $\mathcal{E}$ are proportional to the vectors of the canonical basis of $\mathbb{R}^d$. The case of an octagonal tiling in $\mathbb{R}^2$ corresponds to $n = 4$ with $\{e_1, e_2\}$ the canonical basis and $e_3 = (e_1 + e_2)/\sqrt{2}$, while $e_4 = (-e_1 + e_2)/\sqrt{2}$. For the Penrose tiling in $\mathbb{R}^2$, $n = 5$ and the e_i's are five unit vectors with angle $2\pi/5$ between them.

3.1. Local models. For $r > 0$, and $x = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$, let $B(x;r)$ be the Euclidean open ball centered at x with radius $r > 0$. Instead of using the Euclidean norm, it may be convenient to use the $\mathcal{E}$-norm defined by

$$\|x\|_{\mathcal{E}} = \max_{e\in\mathcal{E}} |\langle e|x\rangle|.$$

Since $\mathcal{E}$ is generating, this is a norm on $\mathbb{R}^d$ equivalent to the Euclidean one. Correspondingly, the open $\mathcal{E}$-ball centered at x of radius r will be defined as

$$\mathcal{B}_{\mathcal{E}}(x;r) = \{y \in \mathbb{R}^d\,;\ \|y-x\|_{\mathcal{E}} < r\}\,.$$

Such an $\mathcal{E}$-ball is a convex polyhedron and there are $\eta_{\pm} > 0$ depending only on $\mathcal{E}$ such that $B(x;\eta_- r) \subset \mathcal{B}_{\mathcal{E}}(x;r) \subset B(x;\eta_+ r)$.

Remark 3.1. If $n = d$ and if the elements of $\mathcal{E}$ are proportional to the vectors of the canonical basis of $\mathbb{R}^d$, an $\mathcal{E}$-ball is a *well oriented d-cube* (see Def. 2.34). Since the elements of $\mathcal{E}$ may not have the same length, a well oriented d-cube is not necessarily an hypercube, namely its sides may have different lengths.

To describe the various sectors separated by the hyperplanes $F_e(x)$'s in such balls, it is convenient to use a combinatorial description, through binary code theory. Here $\mathcal{C}_n$ will denote the set $\{+1,-1\}^{\mathcal{E}}$. An element of $\mathcal{C}_n$ is then a family $\epsilon = (\epsilon_e)_{e\in\mathcal{E}}$ with $\epsilon_e = \pm 1$. $\mathcal{C}_n$ is endowed with the Hamming distance $d_{\mathcal{H}}(\epsilon,\epsilon') = \#\{e \in \mathcal{E}\,;\ \epsilon_e \neq \epsilon'_e\}$. A subset $A \subset \mathcal{C}_n$ is *connected*, whenever given any two points ϵ and ϵ' in A, there is a *path* γ joining them, namely $\gamma = \{\epsilon^{(0)} = \epsilon, \epsilon^{(1)}, \ldots, \epsilon^{(r-1)}, \epsilon^{(r)} = \epsilon'\}$ with $d_{\mathcal{H}}(\epsilon^{(s-1)}, \epsilon^{(s)}) = 1$ for $s = 1, \ldots, r$. Such a path is known under the name of *Gray code* in information theory. For $e \in \mathcal{E}$ let $\eta_e : \mathcal{C}_n \mapsto \mathcal{C}_n$ be one step in the direction e, namely $(\eta_e(\epsilon))_{e'} = (-1)^{\delta_{ee'}}\epsilon_{e'}$. It is clear that $\eta_e^2 = id$ whereas η_e and $\eta_{e'}$ commute. Then $d_{\mathcal{H}}(\epsilon,\epsilon') = 1$ if and only if there is $e \in \mathcal{E}$ such that $\epsilon' = \eta_e(\epsilon)$. Given $A \subset \mathcal{C}_n$ connected, a *boundary code* of A is an $\epsilon \in A$ such that there is $e \in \mathcal{E}$ with $\eta_e(\epsilon) \notin A$. The set of such points is denoted by ∂A. If $\epsilon \in \partial A$ then $I(\epsilon)$ will denote the set of $e \in \mathcal{E}$ such that $\eta_e(\epsilon) \notin A$. A code $\epsilon \notin A$ is *adjacent* to A if there is $e \in \mathcal{E}$ such that $\eta_e(\epsilon) \in A$. The set of adjacent points is denoted by δA. Two connected disjoint sets A, B are adjacents, if $\partial A \cap \delta B \neq \emptyset$. This is equivalent to $\delta A \cap \partial B \neq \emptyset$ and the adjacency relation between A and B will be denoted $\langle A, B\rangle$.

Given $\epsilon \in \mathcal{C}_n$ the set

$$O_\epsilon(x;r) = \{y \in B(x;r)\,;\ \epsilon_e\langle e|y-x\rangle \geq 0\,,\ \forall e \in \mathcal{E}\}\,,$$

is closed in $B(x;r)$ but, since $B(x;r)$ is open, it is not closed in $\mathbb{R}^d$. It is a convex cone with boundaries defined by the $F_e(x)$'s. It may have an empty interior though, as can be seen on the example of the octagonal tiling. It will be called an *$\mathcal{E}$-orthant* if it has a non empty interior. It is clear that the $\mathcal{E}$-orthants of a ball cover it. Let then $\mathcal{C}_\mathcal{E}$ be the subset of $\mathcal{C}_n$ of codes $\epsilon \in \mathcal{C}_n$ for which $O_\epsilon(x;r)$ has a non empty interior. $\mathcal{C}_\mathcal{E}$ is not empty since the union of all $\mathcal{E}$-sectors is the ball $B(x;r)$. Moreover if $O_\epsilon(x;r)$ is an $\mathcal{E}$-orthant, so is $O_{-\epsilon}(x;r) = -O_\epsilon(x;r)$. Hence $\epsilon \in \mathcal{C}_\mathcal{E}$ if and only if $-\epsilon \in \mathcal{C}_\mathcal{E}$, whenever $-\epsilon$ denotes the code $(-\epsilon_e)_{e\in\mathcal{E}}$.

Lemma 3.2. *Two $\mathcal{E}$-orthants of $B(x;r)$ have a union with connected interior if and only if the Hamming distance of their code is one.*

Proof. Let ϵ and ϵ' be the codes of the two $\mathcal{E}$-orthants. Since they are distinct, their Hamming distance in non zero. If their Hamming distance is one, there is a unique $e \in \mathcal{E}$ such that $\epsilon' = \eta_e(\epsilon)$. Then $O_\epsilon(x;r) \cup O_{\epsilon'}(x;r)$ is the set of $y \in B(x;r)$ such that $\epsilon_{e'}\langle e'|y-x\rangle \geq 0$ for all $e' \neq e$. This is obviously a convex set so it is connected. Since it has a non empty interior, by hypothesis, its interior is also connected.

If the Hamming distance is greater than one, there are at least two indices $e \neq e'$ for which the coordinates of the two codes disagree and therefore the union of the two orthants is contained in the set $C_+ \cup C_-$ of $y \in B(x;r)$ where $C_\pm = \{y\,;\,,\ \pm\epsilon_f\langle f|y-x\rangle \geq 0\ f = e, e'\}$. The intersections of $C_\pm$ with the 2-plane generated by e, e' are convex cones $\hat{C}_\pm$ that are symmetric around the point of intersection $\hat{x}$ of the traces of the hyperplanes $F_e(x)$ and $F_{e'}(x)$. Thus these cones intersect on $\hat{x}$ only and the interior of their union does not contain $\hat{x}$ so that this interior cannot be connected. □

In particular two such $\mathcal{E}$-orthants intersect along a face of codimension 1. As a consequence, since the ball is connected, the set $\mathcal{C}_\mathcal{E}$ is connected.

An $\mathcal{E}$-orthant O centered at x is stratified as follows. A point in ∂O has degree l whenever it belongs to an affine subspace of dimension l obtained as an intersection of $F_e(x)$'s and is not contained in any other such intersection of lower dimension. Then $\partial_l O$ will denote this set of points, with the convention that $\partial_d O$ is the interior of O. Then $\partial_0 O = \{x\}$. It is simple to check that $\partial_l O \cap \partial_{l-1} O = \emptyset$ and that $\overline{\partial_l O} = \partial_l O \cup \overline{\partial_{l-1} O}$ for $l = 1, \ldots, d$.

An *$\mathcal{E}$-sector* S is an open connected subset of $B(x;r)$ which is the interior of the union of $\mathcal{E}$-orthants. Since S is connected, thanks to Lemma 3.2 the set $A \subset \mathcal{C}_\mathcal{E}$ of codes of the orthants building S is automatically connected. S will then be denoted by $S(x;r,A)$,

$$S(x;r,A) = \mathrm{Int}\left(\bigcup_{\epsilon\in A} O_\epsilon(x;r)\right).$$

The boundary $\partial S(x;r,A)$ of this sector is the set of points of its closure that are in $B(x;r)$ and not in $S(x;r,A)$. This boundary is a finite union of $\mathcal{E}$-orthants contained in some of the hyperplanes $F_e(x)$ of the form $F_e(x;r,\epsilon) = F_e(x) \cap O_\epsilon(x;r) \cap B(x;r)$. Then $F_e(x;r,\epsilon)$ does not depend on ϵ_e so that ϵ can be chosen in $\mathcal{C}_n(e) = \{+1,-1\}^{\times\mathcal{E}\setminus\{e\}}$. The following result is straightforward

Lemma 3.3. *Given a connected subset $A \subset \mathcal{C}_{\mathcal{E}}$, $\partial S(x; r, A)$ is the union of the $\mathcal{E}$-faces $F_e(x; r, \epsilon)$ where ϵ runs through the elements of ∂A and $e \in I(\epsilon)$,*

$$\partial S(x; r, A) \;=\; \bigcup_{\epsilon \in \partial A} \bigcup_{e \in I(\epsilon)} F_e(x; r, \epsilon)\,.$$

Similarly, if A and B are two disjoint adjacent connected subsets of $\mathcal{C}_{\mathcal{E}}$,

$$\overline{S(x; r, A)} \cap \overline{S(x; r, B)} \cap B(x; r) \;=\; \bigcup_{\epsilon \in \partial A \cap \delta B} \bigcup_{e \in I(\epsilon)} F_e(x; r, \epsilon)\,.$$

Clearly each $F_e(x; r, \epsilon)$ defines an orthant in an affine subspace of dimension $(d - 1)$ corresponding to the set $\mathcal{E}_e$ which is made of the projections on F_e of the vectors of $\mathcal{E} \setminus \{e\}$. Therefore $\partial S(x; r, A)$ is the closure of a union of $\mathcal{E}_e$-sectors. Thanks to Lemma 3.3 such sectors are attached to the connected components of ∂A.

As for $\mathcal{E}$-orthants, an $\mathcal{E}$-sector S is stratified in a similar way. $\partial_d S = S$ while $\partial_l S$ is the union of points in $\partial_l O \cap \partial S$ for any $\mathcal{E}$-orthant O building S. It is important to note that $\partial_l S$ may be empty for l small enough. An l-face of S is the closure of a connected component of $\partial_l S$.

In what follows, $\mathcal{P}_{\mathcal{E}}$ will denote the set of connected subsets of $\mathcal{C}_{\mathcal{E}}$ and $N : \mathcal{P}_{\mathcal{E}} \mapsto \mathbb{N}$ denotes a map, namely for each $A \in \mathcal{P}_{\mathcal{E}}$, $N(A) \in \mathbb{N}$ is a given integer. It will be assumed that $N(\emptyset) = 0$. A subset $\pi \subset \mathcal{P}_{\mathcal{E}}$ will be called *admissible* if $\emptyset \in \pi$ and $\mathcal{C}_{\mathcal{E}} \notin \pi$. The *subgraph* of N is the set

$$G_\pi(N) \;=\; \{(A, n) \in \pi \times \mathbb{N};\; 0 \le n \le N(A)\}.$$

Since $\emptyset \in \pi$ it follows that $(\emptyset, 0) \in G_\pi(N)$. The (π, N)-*preball* centered at x with radius R is the set $\hat{B}(x; r, \pi, N) = B(x; r) \times G_\pi(N)$. Given $(A, n) \in G_\pi(N))$ the subset $B(x; r) \times \{(A, n)\}$ is called a *sheet*. On such a preball, let $\sim$ be the equivalence relation defined by $(y, A, n) \sim (y', A', n')$ if and only if (i) either $(y, A, n) = (y', A', n')$, (ii) or $A \neq A'$ and then $y = y' \in B(x; r) \setminus S(x; r, A) \cup S(x; r, A')$. It is straightforward to check that this is an equivalence relation.

Definition 3.4. *An open BOF-$\mathcal{E}$-ball centered at x and of size r is a quotient set of the form $B(x; r, \pi, N) = \hat{B}(x; r, \pi, N)/\sim$ for some admissible family $\pi \subset \mathcal{P}_{\mathcal{E}}$ and some map $N : \mathcal{P}_{\mathcal{E}} \mapsto \mathbb{N}$ such that $N(\emptyset) = 0$. A closed BOF-$\mathcal{E}$-ball is obtained in the same way by using the closed ball $\overline{B(x; r)}$ instead.*

Hence a BOF-$\mathcal{E}$-ball is a local model describing branched topological spaces. Actually, a BOF-$\mathcal{E}$-ball $\mathcal{D}$, with underlying (Euclidean) ball $\mathcal{B}$, is also endowed with a canonical structure of *flat branched* Riemannian manifold (with boundary if closed). For indeed $\mathcal{D}$ comes from a preball $\hat{D}$ which is the disjoint finite union of copies of $\mathcal{B}$ (its sheets). Each such sheet can be seen as a flat Riemannian manifold if endowed with the Euclidean structure of $\mathbb{R}^d$. Hence, the tangent space at $x \in \mathcal{D}$ is simply $T_x\mathcal{D} = \mathbb{R}^d$, and the tangent bundle is just the trivial one $\mathcal{D} \times \mathbb{R}^d$. Similarly, $\mathcal{D}$ is given a canonical orientation induced by the one of its underlying $\mathcal{E}$-ball. The canonical map $p : [(y, A, n)] \in \mathcal{D} \mapsto y \in \mathcal{B}$ is differentiable. The boundary of $\mathcal{D}$ is the inverse image of the boundary of $\mathcal{B}$ by p. The canonical Euclidean metric on $\mathbb{R}^d$ (namely $ds^2 = dx_1^2 + \cdots + dx_d^2$) induces a *branched flat metric* on $\hat{\mathcal{D}}$ and then on $\mathcal{D}$ by projection. Any ball isometric to $\mathcal{B}$ embedded in $\mathcal{D}$ with the same center is called a *smooth sheet* of $\mathcal{D}$.

A BOF-$\mathcal{E}$-ball $\mathcal{D}$ is stratified as follows. Let $\partial_d\mathcal{D}$ be the set of points of $\mathcal{D}$ coming from the open set $S(x; r, A) \times \{(A, n)\}$ in various sheets of $\hat{\mathcal{D}}$. Then the complement of $\partial_d\mathcal{D}$ projects in $\mathcal{B}$ into the set of boundaries of $\mathcal{E}$-orthants. Let then $\partial_l\mathcal{D}$ be the set of points z coming from $\partial_l S$ for S a sheet of $\hat{\mathcal{D}}$. Then

(i) $\overline{\mathcal{D}} = \overline{\partial_d\mathcal{D}}$,
(ii) $\overline{\partial_l\mathcal{D}} = \partial_l\mathcal{D} \cup \overline{\partial_{l-1}\mathcal{D}}$ for $1 \le l \le d$,
(iii) $\partial_l\mathcal{D} \cap \partial_{l-1}\mathcal{D} = \emptyset$.

It follows that the family $\{\partial_l\mathcal{D};\ 0 \le l \le d\}$ is a partition of $\mathcal{D}$.

Definition 3.5. *Let $\mathcal{D}$ be a BOF-$\mathcal{E}$-ball, then*

(i) *the set $\partial_l\mathcal{D}$ is called its l-**stratum**,*
(ii) *$\mathcal{D}$ has type p whenever $\partial_{d-p}\mathcal{D} \neq \emptyset$ while $\partial_{d-p-1}\mathcal{D} = \emptyset$.*

It follows that a type 0 BOF-$\mathcal{E}$-ball is a Euclidean ball. A type 1 BOF-$\mathcal{E}$-ball centered at x is branching along an hyperplane of the form $\langle e|y - x\rangle = 0$ for some $e \in \mathcal{E}$ in its underlying d-ball. More generally a BOF-$\mathcal{E}$-ball of type p is branching along the intersection of p distinct such hyperplanes in generic position. Moreover, if $\mathcal{D}$ is a BOF-$\mathcal{E}$-ball, any point in $\partial_l\mathcal{D}$ is the center of a BOF-$\mathcal{E}$-ball of type $d - l$.

In order to define a global geometry the next step requires the definition of the relevant class of coordinate transformations. Since the previous geometry is flat and differs from the usual Euclidean one only because of branching, the following class will be sufficient

Definition 3.6. *A continuous map $f : \mathcal{D} \to \mathcal{D}'$ between two BOF-$\mathcal{E}$-balls of the same size $r > 0$ is a* **local BOF-submersion** *(onto its image) if:*

1) *f is C^1 with respect to the branched C^1-structures.*
2) *For every open smooth sheet D of $\mathcal{D}$, $f(D)$ is a smooth sheet of $\mathcal{D}'$.*
3) *There is a translation T on $\mathbb{R}^d$ such that the restriction of f to each smooth sheet of D coincides with the restriction of T.*

Once a basis of $\mathbb{R}^d$ is fixed, giving corresponding trivializations of the two tangent bundles, then the differential df_x of f at any point x of $\mathcal{D}$ induces the identity on $\mathbb{R}^d$.

Definition 3.7. *A local BOF-submersion f is a* **local BOF-isometry** *if and only if it is bijective. In such a case $f(\mathcal{D})$ is a BOF-$\mathcal{E}$-ball, f^{-1} is a local BOF-isometry and the stratification associated with $\mathcal{D}$ is mapped on the stratification associated with $f(\mathcal{D})$ stratum by stratum.*

3.2. BOF-d-manifolds. In this section, the local models defined in the previous section will be used to define global ones. Here again the family $\mathcal{E}$ ($n = |\mathcal{E}| \ge d$) is given and is supposed to generate $\mathbb{R}^d$.

Definition 3.8. *A* **BOF-manifold** *of dimension d, B is a compact, connected metrizable topological space endowed with a maximal atlas $\{U_j, \phi_j\}$ such that:*

1) *There is a generating family $\mathcal{E}$ of $n \ge d$ vectors in $\mathbb{R}^d$ such that every $\phi_j : U_j \mapsto W_j$ is a homeomorphism onto an open set W_j of some open BOF-$\mathcal{E}$-ball.*

2) *For any BOF-$\mathcal{E}$-ball $\mathcal{D}$ embedded into $\phi_j(U_i \cap U_j)$, the restriction of $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ to $\mathcal{D}$ is a local BOF-isometry onto a BOF-$\mathcal{E}$-ball embedded into $\phi_i(U_i \cap U_j)$.*

Hence for every point x of a BOF-manifold B, there exists a neighborhood U of x and a chart $\phi : U \to \phi(U)$ such that $\phi(U)$ is a BOF-$\mathcal{E}$-ball and $\phi(x) = 0$. The type of the BOF-$\mathcal{E}$-ball $\phi(U)$ is uniquely determined by x. Such a neighborhood U is called a *normal neighborhood* of x.

Definition 3.9. *The* **injectivity radius of** $x \in B$*, denoted by $inj_B(x)$, is the sup of the size of any normal neighborhood of x in B. The* **injectivity radius of** B*, denoted $inj(B)$ is defined by:*

$$\mathrm{inj}(B) = \inf_{x \in B} \mathrm{inj}_B(x) .$$

Definition 3.10. *For $0 \le l \le d$, the l-***face** *of a BOF-manifold B is the set of points x in B for which there exits a chart (U, ϕ) in the atlas such that x is in U and $\phi(x)$ belongs to the l-stratum of the BOF-$\mathcal{E}$-ball $\phi(U)$. This property is independent on the choice of the chart (U, ϕ) as long as x is in U. An l-***region** *is a connected component of the l-face of B. The finite partition of B into l-regions, for $0 \le l \le d$, is called the* **natural stratification** *of B. The union of all the l-regions for $0 \le l \le d-1$, forms the* **singular locus** *$Sing(B)$ of B.*

Note that, in particular, any oriented flat d-torus is a BOF-d-manifold with one d-region. In general, any d-region of a BOF-manifold B is a connected (in general non compact) d-manifold naturally endowed with a $(X, G) = (\mathbb{R}^d, \mathbb{R}^d)$-structure [13], where the group $G = \mathbb{R}^d$ acts on $X = \mathbb{R}^d$ by translation. This means that any region admits a d-manifold atlas such that all the transition maps on connected domains are restriction of translations.

All the objects that have been associated with any open BOF-$\mathcal{E}$-ball, such as the tangent bundle, its trivializations, the orientation, the branched C^1-structure, the branched flat metric and so on, globalize to any BOF-d-manifold. In particular, there is a natural notion of parallel transport on B with respect to the flat metric, whence the notion of *parallel vector field* on B. Let $Par(B)$ denote the set of parallel vector fields on B. Fixing the canonical trivialization of the tangent bundle TB leads to a natural isomorphism

$$\rho_B : \mathbb{R}^d \mapsto Par(B) ,$$

defined by identifying $\mathbb{R}^d$ with any tangent space $T_{\tilde{x}}B$, and by associating to every vector $v \in \mathbb{R}^d$ the parallel vector field $\rho_B(v)$ obtained by parallel transport of v, starting from $\tilde{x}$. This does not depend on the choice of the base point $\tilde{x}$. The following definition is based on Lemma 3.1

Definition 3.11. *A BOF-manifold B of dimension d has* **cubic faces** *(resp.* **polyhedral faces***) if, for each d-region R of B, there exists a C^1 injective map $f : R \mapsto \mathbb{R}^d$ such that $f(R)$ is the interior of a well oriented d-cube (resp. of a polyhedron homotopic to a sphere) in $\mathbb{R}^d$ and the differential of f, read in the charts of B, satisfies $df_x = \mathbf{1}$ at each point x in R.*

In particular, the d-regions of BOF-manifolds with polyhedral faces do not carry any topology (namely there are contractile).

3.3. BOF-submersion. From now on any BOF-manifold will have dimension d.

Definition 3.12. *A continuous map $f : B \to B'$ between BOF-manifolds is a* **BOF-submersion** *if:*

1) *f is C^1 and surjective.*
2) *For every $x \in B$ and for every normal neighborhood $\mathcal{D}'$ of $f(x)$ in B', there exists a normal neighborhood $\mathcal{D}$ of x in B such that $f(\mathcal{D}) \subset \mathcal{D}'$ and, read in the corresponding charts, $f|_{\mathcal{D}} : \mathcal{D} \mapsto f(\mathcal{D})$ is a local BOF-submersion.*
3) *For each region R of B, there exists a region R' of B' such that f is a diffeomorphism from R' to R. In particular, the singular set of B is mapped into the singular set of B',*

$$Sing(B) \subset f^{-1}(Sing(B'))\,.$$

Remark 3.13. Notice that the pre-image of a normal neighborhood $U' \subset B'$ (with radius $r > 0$) is a finite union of disjoint normal neighborhoods $U_1, \ldots, U_n$ in B (with radius $r > 0$) and that $f : U_i \to U'$ is a C^1 bijection whose differential is the identity when read in the charts. This implies in particular that

$$inj(B') \;\leq\; inj(B)\,.$$

Remark 3.14. In much the same way, a BOF-submersion f send an l-region into an l-region as well. This can be seen locally, since then $d_x f = \mathbf{1}$. Globally, this can be proved by using a covering by normal neighborhoods. Similarly, the inverse image of an l-region is a finite union of l-regions.

3.4. Cycles and positive weight systems. Let B be a BOF-manifold of dimension d. Let F be a $d-1$-region. If $x \in F$, then x admits a neighbourhood $\mathcal{D}$ which is homeomorphic to an open $\mathcal{E}$-ball centered at x of type 1. Namely the singular locus of $\mathcal{D}$ is simply $\mathcal{D} \cap F$ and can be seen in a chart as the piece of one of the hyperplanes $F_e(x)$ contained in $\mathcal{D}$. Such a $\mathcal{D}$ is divided into two half balls, defined respectively by the sectors $S(x; r, A_{e,\pm})$, where r is the radius of $\mathcal{D}$ and $A_{e,\pm}$ is the set of codes in $\mathfrak{C}_{\mathcal{E}}$ such that $\epsilon_e = \pm 1$. Then denote by $\mathcal{D}_\pm$ these two sectors. If $\mathcal{D}'$ is another such ball centered at $x' \in F$ with non empty intersection with $\mathcal{D}$, then, since the change of charts are isometries, the definition of $\mathcal{D}'_\pm$ agrees with the one for $\mathcal{D}_\pm$, namely $\mathcal{D}_+$ does not intersect $\mathcal{D}'_-$, neither do $\mathcal{D}_-$ and $\mathcal{D}'_+$, while $\mathcal{D}_+$ intersects $\mathcal{D}'_+$ as well as $\mathcal{D}_-$ intersects $\mathcal{D}'_-$. Therefore, the union U of such neighborhoods is an open set containing F and is divided in a unique way into two disjoint open subsets of $U \setminus F$, U_+ and U_-. These two sets define the two *sides* of F. Each side intersects various d-regions branching along F denoted by $R_1^\pm, \ldots, R_{m_\pm}^\pm$. A family $w(R_s^\pm)$ of complex numbers obeys *Kirchhoff's law* at F if

$$\sum_{s=1}^{m_+} w(R_s^+) \;=\; \sum_{s'=1}^{m_-} w(R_{s'}^-)\,. \qquad \textbf{Kirchhoff's law}$$

Definition 3.15. (i) *A* **non negative** *(resp.* **positive***)* **weight** *w on a BOF-manifold B of dimension d is a function assigning to each d-region R a non negative real number $w(R) \geq 0$ (resp. $w(R) > 0$) obeying Kirchhoff's law at each $(d-1)$ region.*

(ii) *Then $W(B)$, $W^*(B)$ will denote the sets of non negative and positive weights of B respectively. The total mass of a weight $w \in W(B)$ is the sum of the $w(R)$'s over all d-regions. Then, if $m \geq 0$, $W^m(B)$ will denote the set of non negative weight of mass m.*
(iii) *A pair (B, w) where B is a BOF manifold and $w \in W^*(B)$ will be called a measured BOF-manifold.*

Remark 3.16. Note that the definition of non negative weights does not involve any region orientation. It makes sense even for non-orientable branched manifolds [14].

If the orientation of a BOF-manifold B is taken into account, then it is possible to define the homology of B as follows. Each stratum $\partial_l B$ can be decomposed into a finite number $a_l \in \mathbb{N}$ of l-regions. Let an orientation of each of them be chosen so that d-regions be orientated with the orientation induced by B. Let $\mathbb{A}$ be a commutative ring, which will be $\mathbb{Z}$ or $\mathbb{R}$ in practice. Let then $C_l(B, \mathbb{A})$ be the free $\mathbb{A}$-module generated by the l-regions. By convention, if R is such a region, $-R$ will denote the same region with opposite orientation. Moreover, an element of $C_l(B, \mathbb{A})$ can be seen as a map $R \mapsto f(R) \in \mathbb{A}$ defined on the set of l-regions, such that $f(-R) = -f(R)$. An element of $C_l(B, \mathbb{A})$ will be called an *l-chain* of B. Hence $C_l(B, \mathbb{A})$ is isomorphic to the free module $\mathbb{A}^{a_l}$.

Remark 3.17. With the exception of the d-regions orientation, there are no canonical choices for the above orientations in general. On the other hand, it is clear that two different choices implement a linear automorphism of $\mathbb{A}^{a_l}$. However, the rest of this work is independent of such a choice.

Given an oriented d-region R and an oriented $(d-1)$-region F contained in the closure of R, the two orientations *match* if F is positively oriented whenever its normal points outside R. In much the same way, if R is an oriented l-region, R can be seen as an open set in some l-dimensional real affine space with same orientation. Therefore if F is an oriented $(l-1)$-region contained in its closure, the same definition of orientation matching applies. If the orientation does not match then $-F$ has a matching orientation. In such a case, let bR be the set of $(l-1)$ regions contained in the closure of R with matching orientation. Let then the *boundary operator* be the $\mathbb{A}$-linear map defined by

$$b_{i+1} : C_{i+1}(B, \mathbb{A}) \mapsto C_i(B, \mathbb{A}), \quad (b_{i+1} f)(F) = \sum_{R; F \in bR} f(R). \qquad (1)$$

Conventionally, $C_{d+1}(B, \mathbb{A}) = 0$ so that $b_{d+1} = 0$. It is a well-known fact that $b_i \circ b_{i+1} = 0$. Therefore, the $\mathbb{A}$-module $\mathcal{Z}_i(B, \mathbb{A}) = \text{Ker } b_i$, called the space of *$i$-cycles* of B, contains the $\mathbb{A}$-module, $\mathcal{B}_i(B, \mathbb{A}) = \text{Im } b_{i+1}$, called the space of *$i$-boundaries* of B. Then, the quotient $H_i(B, \mathbb{A}) = \mathcal{Z}_i(B, \mathbb{A})/\mathcal{B}_i(B, \mathbb{A})$ is called the i^{th} *homology group* of B. A standard result of algebraic topology insures that (up to an $\mathbb{A}$-module isomorphism) $H_i(B, \mathbb{A})$ is a topological invariant of B that coincides with the i^{th}-singular homology of B (see for example [43]). As can be seen from Eq. (1), a positive weight w can be defined as a d-cycle positive on positively oriented d regions. The Kirchhoff law is simply expressed by $b_d w = 0$. Since $b_{d+1} = 0$ such d-cycles are the elements of $H_d(B, \mathbb{R})$. This allows to introduce the positive cones $H_d^{>0}(B, \mathbb{R})$ and $H_d^{\geq 0}(B, \mathbb{R})$ for the positive and non negative weights so that

$$W(B) = H_d^{\geq 0}(B, \mathbb{R}), \quad W^*(B) = H_d^{>0}(B, \mathbb{R}).$$

The dual module $C^i(B, \mathbb{A})$ of $C_i(B, \mathbb{A})$ is called the module of *i-co-chains* of B. It can be identified with $\mathbb{A}^{a_i}$ using the dual basis of the above distinguished basis of $C_i(B, \mathbb{A})$. Then the *differential* are the linear operators transposed to the boundaries, namely

$$d_i : C^i(B, \mathbb{A}) \to C^{i+1}(B, \mathbb{A}),$$

is defined by $\langle d_i c | f \rangle = \langle c | b_i f \rangle$ if $c \in C^i(B, \mathbb{A})$ and $f \in C_{i+1}(B, \mathbb{A})$. This gives another complex with $\mathcal{Z}^i(B, \mathbb{A}) = \mathrm{Ker} d_i$ is the space of *i-cocycles*, whereas $\mathcal{B}^i(B, \mathbb{A}) = \mathrm{Im} d_{i-1}$ is the set of *i-coboundaries*. The i^{th} *cohomology group* with cœfficients in $\mathbb{A}$ is $H^i(B, \mathbb{A}) = \mathcal{Z}^i(B, \mathbb{A})/\mathcal{B}^i(B, \mathbb{A})$. In particular $\mathcal{Z}^d(B, \mathbb{A}) = C^d(B, \mathbb{A})$ is the free module spanned by the characteristic functions of the d-regions. Hence if $w = \sum_R w(R)\ e_R$ is a d-chain (where e_R is the basis in $\mathcal{Z}_d(B, \mathbb{A})$ indexed by the d-regions) and if $\alpha \in H^d(B, \mathbb{A})$ is represented by the d-co-cycle $\alpha = \sum_R \alpha(R)\ e^*_R$ (where e^*_R is the dual basis), then

$$\alpha(w) = \langle w | \alpha \rangle = \sum_R \alpha(R)\ w(R).$$

Remark 3.18. It follows immediately that *any d-cohomology class can be contracted against a positive measure* on B. This would not be possible if B were not orientable. Nevertheless, $< w|c >$ can be defined through the previous formula, without knowing that w is a d-cycle. Thus the previous formula does make sense if B is not orientable. If B is orientable, however, $< w|c > = < w|[c] >$, where $[c]$ is the co-homology class represented by c. It follows that the natural pairing $\langle\cdot|\cdot\rangle$ induces a pairing

$$\langle\cdot|\cdot\rangle : W(B) \times H^d(B, \mathbb{A}) \to \mathbb{R}, \quad \langle w | \alpha \rangle = \alpha(w).$$

Thanks to Definition 3.12, the following result is immediate

Proposition 3.19. *If $f : B \to B'$ is a BOF-submersion then:*

1) *there exists a natural linear map (well defined up to the mild ambiguity indicated in Remark 3.17) $f_* : \mathcal{Z}_d(B, \mathbb{A}) \to \mathcal{Z}_d(B', \mathbb{A})$ such that $f_*(W^m(B)) \subset W^m(B')$, $f_*(W^{*,m}(B)) \subset W^{*,m}(B')$, for every $m \in \mathbb{R}^+$.*
2) *There exists a natural module map $f^* : C^d(B', \mathbb{A}) \to C^d(B, \mathbb{A})$ such that $[\alpha](f_*(w)) = [f^*(\alpha)](w)$, for every $\alpha \in C^d(B', \mathbb{A})$, every $w \in \mathcal{Z}_d(B, \mathbb{A})$, where $[\alpha] \in H^d(B, \mathbb{A})$ is the co-homology class represented by the co-chain α.*
3) *There exists a module map $f^* : H^d(B', \mathbb{A}) \to H^d(d, \mathbb{A})$ defined by $f^*([\alpha]) = [f^*(\alpha)]$.*

In particular, as a consequence of this definition, the pairing $\langle w | \alpha \rangle$ is invariant by an BOF-submersion.

Lemma 3.20. *Let B be a BOF-d-manifold and $\mu \in W(B)$. Let $c \in \mathcal{Z}^d(B, \mathbb{R})$ represent a class of $H^d(B, \mathbb{Z})$. Then there exists a family of integers (m_R), where R varies in the family of d-regions, such that*

$$\langle \mu | c \rangle = \sum_R m_R \mu(R).$$

Moreover, all such linear combinations with integer coefficients arise in this way.

Proof. As μ is a d-cycle the value of $\langle \mu | c \rangle$ does not change if a coboundary is added to c. As c represents a class in $H^d(B, \mathbb{Z})$, c differs from a suitable $c' \in \mathcal{Z}^d(B, \mathbb{Z})$ by a coboundary. The first statement of the lemma follows. Since every d-cochain in $C^d(B, \mathbb{Z})$ is a d-cocycle, the other statement follows. □

4. Tilings & Expanding Flattening Sequences

4.1. Expanding flattening sequences.

Definition 4.1. *A BOF-submersion $f : B \to B'$ satisfies the* **flattening condition***, if for every $x \in B$ and for every normal neighborhood $\mathcal{D}'$ of $f(x)$, there exists a small enough normal neighborhood $\mathcal{D}$ of x in B such that $f(\mathcal{D}) \subset \mathcal{D}'$ and, read in corresponding charts, $f| : \mathcal{D} \to f(\mathcal{D})$ is a local BOF-submersion that maps $\mathcal{D}$ on one single sheet of $f(\mathcal{D})$.*

Definition 4.2. *An* **expanding flattening sequence** *EFS is a sequence $\mathcal{F} = \{f_i : B_{i+1} \to B_i\}_{i\in\mathbb{N}}$ of submersions such that:*

(i) *the sequence of injectivity radius of the B_i's is a strictly increasing sequence that goes to $+\infty$ with i;*
(ii) *for each $i \in \mathbb{N}$ the map f_i satisfies the flattening condition.*

With an EFS $\mathcal{F}$ are associated the following "inverse" or "direct" sequences ($\mathbb{A} = \mathbb{Z}, \mathbb{R}$):

i) Inverse sequences in homology,

$$Z_d(\mathcal{F}, \mathbb{A}) = \{(f_i)_* : Z_d(B_{i+1}, \mathbb{A}) \to Z_d(B_i, \mathbb{A})\}_{i\in\mathbb{N}},$$
$$W(\mathcal{F}, \mathbb{A}) = \{(f_i)_* : W(B_{i+1}, \mathbb{A}) \to W(B_i, \mathbb{A})\}_{i\in\mathbb{N}},$$
$$W^*(\mathcal{F}, \mathbb{A}) = \{(f_i)_* : W^*(B_{i+1}, \mathbb{A}) \to W^*(B_i, \mathbb{A})\}_{i\in\mathbb{N}}\ .$$

ii) Direct sequences of cohomology,

$$C^d(\mathcal{F}, \mathbb{A}) = \{(f_i)^* : C^d(B_i, \mathbb{A}) \to C^d(B_{i+1}, \mathbb{A})\}_{i\in\mathbb{N}},$$
$$H^d(\mathcal{F}, \mathbb{A}) = \{(f_i)^* : H^d(B_i, \mathbb{A}) \to H^d(B_{i+1}, \mathbb{A})\}_{i\in\mathbb{N}}\ .$$

iii) Inverse sequence of parallel transport,

$$\mathrm{Par}\ (\mathcal{F}) = \{df_i : \mathrm{Par}\ (B_{i+1}) \to \mathrm{Par}\ (B_i)\}_{i\in\mathbb{N}}\ .$$

Associated with these "inverse" (*resp.* "direct") sequences, are their *inverse limits* (*resp.* the direct limits) which will be relevant in the next sections. Given an inverse sequence of maps $\mathcal{X} = \{X_i \xleftarrow{\tau_i} X_{i+1}\}_{i\in\mathbb{N}}$, its *projective limit* is defined by

$$\lim_{\leftarrow} \mathcal{X} \ = \ \{(x_0, x_1, \dots, x_n, \dots) \in \prod_{i\geq 0} X_i\ ;\ \tau_i(x_{i+1}) = x_i\ ,\ \forall i \in \mathbb{N}\}.$$

Then for every $j \geq 0$ there exists a natural map $p_j : \lim\limits_{\leftarrow} \mathcal{X} \to X_j$ given by the j^{th} projection. If the X_j's are topological spaces and the maps are continuous, the set $\lim\limits_{\leftarrow} \mathcal{X}$ is a topological space if endowed with the finest topology making the p_j's continuous. In particular, this topology coincides with the product topology, so that, if all the X_i's are compact, $\lim\limits_{\leftarrow} \mathcal{X}$ is compact. Similarly, given any direct sequence of maps $\mathcal{Y} = \{Y_i \stackrel{\tau_i}{\to} Y_{i+1}\}$, its *inductive limit* is defined as

$$\lim_{\rightarrow} \mathcal{Y} \;=\; \bigcup_{j\in\mathbb{N}} \{(x_j, x_{j+1}, \dots, x_n, \dots) \in \prod_{n\geq j} Y_n \;;\; \tau_n(x_n) = x_{n+1}\,,\ \forall n \geq j\}.$$

Let $i_j : Y_j \to \lim\limits_{\rightarrow} \mathcal{Y}$ be defined by $i_j(y) = (y_n)_{n\geq j}$, where $y_j = y$ and $y_{n+1} = \tau_n(y_n)$ whenever $n \geq j$. Then if the Y_j's are topological spaces and the maps continuous maps, the direct limit set is a topological space with the coarsest topology making the i_j's continuous. In general however, the inductive limit of compact spaces need not be compact. On the other hand, if the factors X_j or Y_j are all $\mathbb{A}$-modules and if the maps τ_n are module homomorphisms, then so are the projective and inductive limits.

Let $\mathcal{F}$ be an EFS. Then its projective limit is a compact space $\Omega(\mathcal{F})$. In much the same way the corresponding inverse sequences of weight spaces give rise to projective limits $\mathcal{M}^m(\mathcal{F}) = \lim\limits_{\leftarrow} W^m(\mathcal{F}, \mathbb{R})$, $\mathcal{M}^{*m}(\mathcal{F}) = \lim\limits_{\leftarrow} W^{*m}(\mathcal{F}, \mathbb{R})$. They also give rise to

$$\mathcal{M}^*(\mathcal{F}) \;=\; \bigcup_{m\in\mathbb{R}^+} \mathcal{M}^{*m}(\mathcal{F})\,, \quad \mathcal{M}(\mathcal{F}) \;=\; \bigcup_{m\in\mathbb{R}^+} \mathcal{M}^m(\mathcal{F})\,.$$

Standard results in cohomology theory lead to

Proposition 4.3. $\lim\limits_{\rightarrow} H^d(\mathcal{F}, \mathbb{A}) = H^d(\Omega(\mathcal{F}), \mathbb{A})$.

Since $\mathcal{M}(\mathcal{F}) = \lim\limits_{\leftarrow} Z_d^{\geq 0}(\mathcal{F}, \mathbb{R})$ and thanks to Proposition 3.19, there is a natural pairing

$$\mathcal{M}(\mathcal{F}) \times H^d(\Omega(\mathcal{F}), \mathbb{R}) \mapsto \mathbb{R} \qquad < \mu | h >= h(\mu)$$

defined as follows : for $h = (h_j, \dots, h_s, \dots)$ with $h_{s+1} = f_s^*(h_s)$, and $\mu = (\mu_0, \dots, \mu_s, \dots)$, then $< \mu | h >=< \mu_j | h_j >$.

The following result is a direct consequence of Lemma 3.20.

Corollary 4.4. *Let* $\mu = (\mu_1, \dots, \mu_n, \dots)$ *be in* $\mathcal{M}(\mathcal{F})$*,* $c = (c_0, \dots, c_n, \dots)$ *be a class in* $H^d(\Omega(\mathcal{F}), \mathbb{Z})$ *and let* s *be big enough so that* $< \mu, c >=< \mu_s, c_s >$*, then :*

$$< \mu, c >= m_1\mu_{s,1} + \dots + m_{p(s)}\mu_{s,p(s)},$$

where B_s *has* $p(s)$ d*-regions,* $\mu_{s,i}$ *is the weights of the* i^{th} d*-region of* B_s *and the* m_i*'s are integers. Moreover, all such linear combinations with integer coefficients arise in this way.*

In the next section it will be shown that $\Omega(\mathcal{F})$ is actually a tilable lamination and that any tilable lamination can be obtained in this way. In much the same way, the inductive limit $\omega_{\mathcal{F}}$ of Par $(\mathcal{F})$ will be shown to act on $\Omega(\mathcal{F})$ to make it a dynamical system supporting an action of $\mathbb{R}^d$ which is a semi-conjugacy with the usual action of $\mathbb{R}^d$ on tiling spaces. In Sect. 5, $\mathcal{M}(\mathcal{F})$ will be identified with the set of invariant measures on $\Omega(\mathcal{F})$. In Sect. 6 the direct limit of the K-theory of the B_i will be investigated. The application to the gap-labeling will follow.

4.2. From EFS to tilings. From now on, to simplify, let all the BOF-manifolds in the EFS $\mathcal{F}$ have regions given by blocks only. And let $\omega_{\mathcal{F}}$ be the inverse limit $\omega_{\mathcal{F}} = \varprojlim \text{Par}\,(\mathcal{F})$ defined previously.

Proposition 4.5. *The set $\omega_{\mathcal{F}}$ is naturally isomorphic to $\mathbb{R}^d$ and acts on $\Omega(\mathcal{F})$ and $\Omega(\mathcal{F})$ is a tilable lamination.*

Proof. For any BOF-d-manifold B_i, there exists a natural isomorphism $\rho_{B_i} : \mathbb{R}^d \to \text{Par}(B_i)$. For any BOF-submersion $f_i : B_{i+1} \to B_i$, the induced map $df : \text{Par}(B_i) \to \text{Par}(B_{i+1})$ satisfies $(\rho_{B_{i+1}})^{-1} \circ df \circ \rho_{B_i} = id$. Consequently $\varprojlim \text{Par}(\mathcal{F})$ is isomorphic to the inverse limit $\varprojlim \mathbb{R}^d$ associated with the inverse sequence $id : \mathbb{R}^d \to \mathbb{R}^d$. Thus $\omega_{\mathcal{F}}$ is isomorphic to $\mathbb{R}^d$.

Then $\omega_{\mathcal{F}}$ acts on $\Omega(\mathcal{F})$ as follows. From Remark 3.13, there is $r > 0$ such that for each point $x = (x_0, \ldots, x_i, \ldots)$ in $\Omega(\mathcal{F})$, there is a sequence of normal neighborhoods U_i with radius r around each point x_i in B_i such that for each $i > 0$, $f(U_i)$ is one single sheet D_{i-1} of U_{i-1} and thus $f(U_i) = D_{i-1}$. Hence if $u \in \text{Par}(B_i)$ satisfies $\|u\| < r$, the point $x_i + u \in U_i$ is well defined and, since f_i is a BOF-submersion, $f_i(x_i + u) = f_i(x_i) + u \in D_{i-1}$. This gives a meaning to the notion of "small" translation of the point $(x_0, \ldots, x_i, \ldots)$. Now, any vector v in $\omega_{\mathcal{F}}$ can be decomposed into a sum $v = u_1 + \cdots + u_m$, where $\|u_l\| < r$ for $l = 1, \ldots, m$. For a point x in $\Omega(\mathcal{F})$, $x + v$ can be defined as $x + v = (\ldots(x + u_1) + u_2) + \ldots) + u_m)$. It is plain to check that this definition is independent of the decomposition of v.

For $i \geq 0$, let $\pi_i : \Omega(\mathcal{F}) \to B_i$. It is plain to check that small neighbors of the preimages of the d-regions of the B_i's are the domains of charts of an atlas which give to $\Omega(\mathcal{F})$ a structure of tilable lamination. □

Each region of the first BOF-d-manifold B_0 is a well oriented d-cube. Let X denote the set of prototiles made with all well oriented d-cubes. With any point $x = (x_0, \ldots, x_n, \ldots)$ in $\Omega(\mathcal{F})$ we associate a tiling $h(x)$ in $T(X)$ made with prototiles in X as follows:

- Since the injectivity radius of the B_i's goes to infinity with i, there exists, for $i \geq 0$ a sequence of normal neighborhoods U_i of x_i with radius r_i, where r_i is an increasing sequence going to infinity with i.
- From this sequence of normal neighborhoods we extract a sequence of (premiages by chart maps of) sheets $D_i \subset U_i$ such that $D_i \subset f_i(D_{i+1})$. These sheets are well oriented d-cubes centered at x_i.
- Consider the translated copies $D'_i = D_i - x_i$. The d-cubes D'_i are centered at 0 in $\mathbb{R}^d$, they have an increasing radius going to infinity with i. Furthermore all these d-cubes are tiled with the protoliles in X and, for each $i > 0$, the tiling of D'_i coincides with the tiling of D'_{i-1} in D'_{i-1}.
- the limit of this process defines a single tiling $h(x)$ in $T(X)$.

The proofs of the following properties are plain.

Proposition 4.6. *(i) The map $h : \Omega(\mathcal{F}) \to T(X)$ is continuous, injective and realizes a semi-conjugacy between the dynamical systems $(\Omega(\mathcal{F}), \omega_{\mathcal{F}})$ and $(\Omega_{h(x)}), \omega)$ where ω stands for the restriction of the standard $\mathbb{R}^d$-action on $T(X)$;*
(ii) all tilings in $h(\Omega(\mathcal{F}))$ satisfy the finite pattern condition.

So far, the minimality of the dynamical system $(\Omega(\mathcal{F}), \omega_{\mathcal{F}})$ has not been considered. A simple criterion for minimality is the following. An EFS $\mathcal{F}$ verifies the *repetitivity condition* if for each $i \geq 0$ and each flat sheet D_i in B_i, namely the pre-image by a chart map of a sheet, there exists $p > 0$ such that each d-region of B_{i+p} covers D_i under the composition $f_i \circ f_{i+1} \circ \cdots \circ f_{i+p-1}$.

Proposition 4.7. *Let $\mathcal{F}$ be an EFS that satisfies the repetitivity condition, then the dynamical system $(\Omega(\mathcal{F}), \omega_{\mathcal{F}})$ is minimal.*

Proof. The proof is exactly the same as the one for tilings (Proposition 1.4). □

4.3. From tilings to EFS. So far, given a repetitive EFS, a compact space of perfect tilings made with cubic prototiles has been constructed together with its corresponding continuous Hull. On the other hand, a correspondence between perfect tilings and minimal tilable laminations has been established in Sect. 2. What is left to show in this section is to associate with any perfect tiling, made of a finite number of well oriented cubic prototiles, a repetitive EFS.

Let T be a perfect tiling with the set of prototiles $X = \{p_1, \ldots, p_n\}$. Here the p_i's are well oriented d-cubes. To get the first BOF-d-manifold B_0 let $\tilde{B}_0$ be the disjoint union of the prototiles. Then two points $x_1 \in p_1$ and $x_2 \in p_2$ will be identified if there exist a tile t_1 with t-type p_1 and a tile t_2 with t-type p_2 such that the translated copy of x_1 in t_1 coincides with the translated copy of x_2 in t_2. This gives B_0. Then a map $\pi_0 : \Omega_T \to B_0$ is defined as follows: if $T' \in \Omega_T$ is a tiling, let p be the prototile corresponding to the tile of T' containing the origin or $\mathbb{R}^d$; then $\tilde{x}$ is the point of $p \subset \tilde{B}_0$ associated with this origin and $x = \pi_0(T')$ is the representative of $\tilde{x}$ in B_0. Then π_0 defines a box decomposition $\mathcal{B}^{(0)}$ (see Definition 2.35) of the Hull equipped with its tilable lamination structure $(\Omega_T, \mathcal{L})$: the sets $\pi_0^{-1}(int(p_i))$, for $i = 1, \ldots, n$ are the n boxes of the box decomposition. In addition, the tiling $T_0 = T$ can be seen as the trace of this box decomposition on some leaf L of this lamination. B_0 can be seen as an approximation of the Hull. But it is still a poor one.

To improve upon the description of the Hull, a sequence $\mathcal{B}^{(i)}$, $i \geq 0$ of box decompositions such that, for each $i \geq 0$, $\mathcal{B}^{(i+1)}$ is zoomed out of $\mathcal{B}^{(i)}$ and forces its border (see Corollary 2.45). Through the leaf L, this gives a nested sequence of tilings $(T^{(i)})_{i \geq 0}$, where $T^{(0)} = T$ (see Definition 2.46). The BOF-d-manifolds B_i's are constructed as follows:

- First the BOF-d-manifold B'_i is built from $T^{(i)}$ in the same way as B_0 from T_0. This gives a map $\pi'_i : \Omega_T \to B'_i$.
- Since the tiles of $T^{(i)}$ are not well oriented d-cubes, but blocks, the BOF-d-manifold B'_i is not necessarily cellular.
- Each region of the BOF-d-manifold B'_i is tiled with the prototiles in X. The BOF-d-manifold B_i coincides with B'_i but is tiled with the prototiles in X. Hence the d-regions of B_i are prototiles in X.

For $i \geq 0$, B_i is a cellular BOF-d-manifold. Its d-regions are well oriented d-cubes. The map $\pi'_i : \Omega_T \to B'_i$ induces canonically a map $\pi_i : \Omega_T \to B_i$ and there exists a canonical map $f_i : B_{i+1} \to B_i$ defined by

$$f_i(x) = \pi_i(\pi_{i+1}^{-1}(x)).$$

This definition makes sense since the vertical of a point of $(\Omega_T, \mathcal{L})$ in a box of $\mathcal{B}^{(i+1)}$ is included in its vertical in a box of $\mathcal{B}^{(i)}$ (see Definition 2.42). Moreover, for each $i \geq 0$, the map $f_i : B_{i+1} \to B_i$ is a BOF-submersion and satisfies

$$f_i \circ \pi_{i+1} = \pi_i .$$

In addition, this sequence of BOF manifold is an EFS because each box decomposition $\mathcal{B}^i$, for $i \geq 1$ forces its border (see for instance [31]) implying the flattening condition. Let F denote this EFS, so that $\Omega(\mathcal{F}) = \lim\limits_{\leftarrow} \mathcal{F}$ and let $\pi : \Omega_T \to \Omega(\mathcal{F})$ be the map defined by $\pi(x) = (\pi_0(x), \pi_1(x), \ldots, \pi_n(x), \ldots)$.

Proposition 4.8. *The map* $\pi : \Omega_T \to \Omega(\mathcal{F})$ *is a conjugacy between* (Ω_T, ω) *and* $(\Omega(\mathcal{F}), \omega_{\mathcal{F}})$.

Proof. The proof is straightforward. □

5. Invariant Measures

So far, the same minimal dynamical system has been described in four different ways:

- as the continuous Hull of a Delone set;
- as the continuous Hull of a perfect tiling;
- or as a minimal tilable lamination;
- or as a minimal expanding flattening sequence.

The interplay between these points of view will provide a combinatorial description of finite invariant measures of this dynamical system. A description of transverse measures of a measurable groupoid can be found in [16]. It applies in particular to foliations. In the context of laminations see [20] for a more complete description.

Let $(M, \mathcal{L})$ be a lamination with d-dimensional leaves. Let an atlas in $\mathcal{L}$ be fixed once and for all, with charts $h_i : U_i \to V_i \times T_i$, where V_i is an open set in $\mathbb{R}^d$ and T_i is some topological space. As already explained in Sect. 2.4, on their domains of definitions, the transition maps $h_{i,j} = h_j \circ h_i^{-1}$ satisfy:

$$h_{i,j}(x, t) = (f_{i,j}(x, t),\ \gamma_{i,j}(t)),$$

where $f_{i,j}$ and $\gamma_{i,j}$ are continuous in the t variable and $f_{i,j}$ is smooth in the x variable.

Definition 5.1. *Let* $(M, \mathcal{L})$ *be a lamination. A* **finite transverse invariant measure** *on* $(M, \mathcal{L})$ *(or, for short, a* **transverse measure***) is a family* $\mu^t = (\mu_i)_i$ *where for each i,* μ_i *is a finite positive measure on* T_i*, such that if B is a Borelian subset of* T_i*, contained in the domain of definition of the transition map* γ_{ij} *then*

$$\mu_i(B) = \mu_j(\gamma_{ij}(B)).$$

This definition shows that a transverse measure does not depend upon the choice of an atlas in its equivalence class. In a similar way, a (longitudinal) k-differential form on $(M, \mathcal{L})$ is the data of k-differential forms on the open sets V_i that are mapped onto one another by the differential of the transition maps f_{ij}. Let $A^k(M, \mathcal{L})$ denote the set of longitudinal k-differential forms on $(M, \mathcal{L})$.

Definition 5.2. *A foliated cycle is a linear form from $A^d(M, \mathcal{L})$ to $\mathbb{R}$ which is positive on positive forms and vanishes on exact forms.*

With any transverse measure μ^t, a foliated cycle of degree d can be canonically associated as follows. If $\omega \in A^d(M, \mathcal{L})$ has its support in one of the domains U_i of a chart, ω can be seen as a d-form on $V_i \times T_i$. Then ω can be integrated on the slice $V_i \times \{t\}$ for each $t \in T_i$, giving a real valued function on T_i that can be integrated with respect to μ_i, defining $C_{\mu^t}(\omega)$. In general, a partition of unity $\{\phi_i\}_i$ associated with the cover of M by chart domains, will allow to decompose any longitudinal d-form into a finite sum of forms with support in the domain of a chart. This gives a continuous linear form

$$\mathcal{C}_{\mu^t}(\omega) = \sum_i \mathcal{C}_{\mu^t}(\phi_i \omega)$$

from $A^d(M, \mathcal{L})$ to $\mathbb{R}$ that, thanks to the compatibility condition satisfied by the μ_i's, does not depend upon the choice of the partition of the unity. Hence $\mathcal{C}_{\mu^t}$ is a current of degree d. $\mathcal{C}_{\mu^t}$ is positive for positive forms (namely $\mathcal{C}_{\mu^t}$ is *positive*). Moreover, the invariance of μ^t shows that $\mathcal{C}_{\mu^t}$ vanishes on exact forms. The foliated cycle $\mathcal{C}_{\mu^t}$ is called *the Ruelle-Sullivan current* associated with the transverse invariant measure μ^t. As shown in [44], any such positive closed current defines in a unique way a transverse measure. Thus both points of view, transverse invariant measure and foliated cycle, are equivalent.

Let now $(M, \mathfrak{L})$ be a tilable lamination. Parallel transport can be defined along leaves leading to a meaningful definition of translation by $u \in \mathbb{R}^d$, if u is seen as a constant vector field. Hence, $(M, \omega_{\mathfrak{L}})$ becomes a dynamical system. Every finite positive $\mathbb{R}^D$-invariant measure μ on M, defines a transverse measure on the lamination as follows. For any Borelian subset of a transverse set T_i,

$$\mu_i(B) = \lim_{r \to 0^+} \frac{1}{\lambda_d(B(r))} \mu(h_i^{-1}(B(r) \times B)),$$

where λ_d stands for the Lebesgue measure in $\mathbb{R}^d$ and $B(r) \subset V_i$ is a ball of radius r. Conversely, let μ^t be a transverse measure on $(M, \mathfrak{L})$. A finite positive $\mathbb{R}^d$-invariant measure on M can be defined as follows, Let $f : M \to \mathbb{R}$ be a continuous function with support in the chart domain U_i's. Then the map $f \circ h_i^{-1}$ is defined on $V_i \times T_i$. By integrating $f \circ h_i^{-1}$ on the sheets $V_i \times \{t\}$ against the Lebesgue measure λ_d of $\mathbb{R}^d$, a real valued map on T_i is defined that can be integrated against the transverse measure μ_i to get a real number denoted by $\int f d\mu$. When the support of f is not in one of the U_i's, a partition of the unity $\{\phi_i\}_i$ associated with the cover of M by the U_i's will allow to extend the definition of the integral as

$$\int f d\mu = \sum_i \int f \phi_i d\mu.$$

This defines a positive finite measure on M which does not depend on the choice of the atlas in $\mathcal{L}$ nor upon the chosen partition of the unity. By construction, it is invariant under the $\mathbb{R}^d$-action. It is also plain that the existence of a finite measure on M invariant for the $\mathbb{R}^d$-action is in correspondence with a finite measure on a transversal Γ invariant under the action of the holonomy groupoid (see [16]). Thus, for a tilable lamination $(M, \mathfrak{L})$ the following four points of view are equivalent:

- A finite transverse invariant measure;
- a foliated cycle;
- a finite measure on M invariant for the $\mathbb{R}^d$-action;
- a finite measure on a transversal Γ invariant for the holonomy groupoid action.

From the previous discussion, it is possible to derive the expression of an invariant measure from the point of view of tilings. Let T be a perfect tiling with well oriented d-cubes $X = \{p_1, \dots, p_q\}$ as its set of prototiles. By the previous construction, let $\mathcal{F}$ be the EFS constructed with a sequence of BOF-manifolds B_n, $n \geq 0$ and BOF-manifolds submersions $f_n : B_{n+1} \to B_n$ satisfying the flattening condition. Let $\mathcal{M}(\Omega_T, \omega)$ denote the set of finite measures on Ω_T that are invariant under the $\mathbb{R}^d$-action. Let also $\mathcal{M}^m(\Omega_T, \omega)$ denote the set of finite measures on Ω_T with total mass m.

Let $Y = (y_1, \cdots, y_n)$ be a set of points $y_i \in p_i$, one for each prototile. The transversal $\Omega_{T,Y}$ is a Cantor set on which acts the holonomy groupoid $\mathcal{H}_{T,Y}$. Let $\mathcal{M}(\Omega_{T,Y}, \mathcal{H}_{T,Y})$ denote the set of finite measures on $\Omega_{T,Y}$ that are invariant under the action of the holonomy groupoid $\mathcal{H}_{T,Y}$. With any finite invariant measure μ in $\mathcal{M}(\Omega(T), \omega)$ can be associated a finite *transverse* measure μ^t in $\mathcal{M}(\Omega_{T,Y}, \mathcal{H}_{T,Y})$ and this map is one-to-one.

Since $\Omega_{T,Y}$ is a Cantor set, it can be covered by a partition of clopen sets with arbitrarily small diameters. Such a partition $\mathcal{P}$ is *finer* than another partition $\mathcal{P}'$ if the defining clopen sets of the first one are included in clopen sets of the second one. Let $\mathcal{P}_n$, $n \geq 0$ be a sequence of partitions such that for all $n \geq 0$, $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ and the diameter of the defining clopen sets of $\mathcal{P}_n$ goes to zero as n goes to $+\infty$.

Claim. *A finite measure on $\Omega_{T,Y}$ is given by the countable data of non negative numbers associated with each defining clopen set of each partition $\mathcal{P}_n$ which satisfies the obvious additivity relation.*

The EFS $\mathcal{F}$ provides us such a sequence of partitions $\mathcal{P}_n$ as follows. For each $n \geq 0$, let $F_1, \dots, F_{p(n)}$ be the regions of the BOF-d-manifold B_n. Each F_i is a copy of a prototile in X and consequently there is a marked point y_i in F_i. For $n \geq 0$, $i = 1, \dots, p(n)$, let

$$\mathcal{C}_{n,i} = \pi_n^{-1}(y_i).$$

For n fixed and as i varies from 1 to $n(p)$ the clopen sets $\mathcal{C}_{n,i}$ form a partition $\mathcal{P}_n$ of $\Omega_{T,Y}$. Furthermore, for $n \geq 0$, $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ and the diameter of the clopen sets $\mathcal{C}_{n,i}$ goes to zero as n goes to $+\infty$. It follows that a finite measure on $\Omega_{T,Y}$ is given by the countable data of non-negative weights associated with each defining clopen set $\mathcal{C}_{n,i}$ which satisfies the obvious additivity relation.

The relation between an invariant measure μ in $\mathcal{M}(\Omega_T, \omega)$ and the associated transverse invariant measure μ^t in $\mathcal{M}(\Omega_{T,Y}, \mathcal{H}_{T,Y})$ is given by

Proposition 5.3. *For $n \geq 0$ and i in $\{1, \dots n(p)\}$,*

$$\mu^t(\mathcal{C}_{n,i}) = \frac{1}{\lambda_d(F_i)} \, \mu(\pi_n^{-1}(F_i)).$$

These invariant measures can also be characterized combinatorially. Namely let $\tau_n : \mathcal{M}(\Omega_T, \omega)) \to C_d(B_n, \mathbb{R})$ be the map defined by

$$\tau_n(\mu) = \left(\frac{\mu(\pi_n^{-1}(F_1))}{\lambda_d(F_i)}, \dots, \frac{\mu(\pi_n^{-1}(F_{p(n)}))}{\lambda_d(F_{p(n)})} \right),$$

where the regions $F_1, \dots, F_{p(n)}$ are now ordered and equipped with the natural orientation that allows to identify $C_d(B_n, \mathbb{R})$ with $\mathbb{R}^{p(n)}$.

Remark 5.4. The coordinates of $\tau_n(\mu)$ are the transverse measures associated with μ of clopen sets $\mathcal{C}_{n,i}$ for some $n \geq 0$ and some i in $\{1, \ldots n(p)\}$.

Proposition 5.5. *For any $n \geq 0$, the map τ_n satisfies the following properties:*

(i) $\tau_n(\mathcal{M}(\Omega_T, \omega)) \subset W^\star(B_n)$,
(ii) $f_{n,*} \circ \tau_n = \tau_{n+1}$.

Proof. (i) Let μ be an invariant measure on $\mathcal{M}(\Omega_T, \omega)$. The invariance of μ implies that on each edge of B_n the sum of the transverse measures associated with the regions on one side of the edge is equal to the sum of the transverse measures of regions on the other side of the edge. These are exactly the switching rules (or Kirchoff-like laws) that define $W(B_n)$. The fact that the measure is invariant implies that each region has a strictly positive weight. Thus $\tau_n(\mu)$ is in $W^\star(B_n)$.

(ii) Let $F'_1, \ldots, F'_{p(n+1)}$ be the ordered sequence of regions of B_{n+1} equipped with the natural orientation that allows to identify $C_d(B_{n+1}, \mathbb{R})$ with $\mathbb{R}^{p(n+1)}$. To the linear map $f_{n,*} : C_d(B_{n+1}, \mathbb{R}) \to C_d(B_n, \mathbb{R})$ corresponds a $n(p) \times n(p+1)$ matrix A_n with integer non negative coefficients. The coefficient $a_{i,j,n}$ of the i^{th} line and the j^{th} column is exactly the number of pre images in F'_j of a point in F_i, leading to the relations

$$\frac{\mu(\pi_n^{-1}(F_i))}{\lambda_d(F_i)} = \sum_{j=1}^{j=p(n+1)} a_{i,j,n} \frac{\mu(\pi_{n+1}^{-1}(F'_j))}{\lambda_d(F'_j)},$$

for all $i = 1, \ldots, p(n)$ and all $j = 1, \ldots p(n+1)$. This is exactly the condition $f_{n,*} \circ \tau_n = \tau_{n+1}$. □

If $\mathcal{M}^\star(\mathcal{F}) = \varprojlim W^\star$ the set of invariant measures of (Ω_T, ω) can be characterized as follows

Theorem 5.6.

$$\mathcal{M}(\Omega_T, \omega) \cong \mathcal{M}^\star(\mathcal{F}).$$

Proof. The inclusion

$$\mathcal{M}(\Omega_T, \omega) \subset \mathcal{M}^\star(\mathcal{F})$$

is a direct consequence of Proposition 5.5. Conversely, let $(\beta_0, \ldots, \beta_n, \ldots)$ be an element of $\mathcal{M}^\star(\mathcal{F})$. Thanks to Proposition 5.3, it defines a weight on each clopen set $\mathcal{C}_{n,i}$. The relation $\beta_n = f_{\star n}\beta_{n+1}$ means that this countable sequence of weights satisfies the additivity property and then defines a measure on $\Omega_{T,Y}$. Since the β_n's are cycles, namely they satisfy the switching rules, this measure is a transverse invariant measure, *i.e.* an element in $\mathcal{M}(\Omega_{T,Y}, \mathcal{H}_{T,Y})$. The correspondence between $\mathcal{M}(\Omega_{T,Y}, \mathcal{H}_{T,Y})$ and $\mathcal{M}(\Omega_T, \omega)$ being bijective, the equality is proved. □

Corollary 5.7. • *If the dimension of $H^d(B_n, \mathbb{R})$ is uniformly bounded by N, then for all $m > 0$, $\mathcal{M}^m(\Omega_T, \omega)$ contains at most N ergodic measures;*

• *if furthermore the coefficients of all the matrices $f_{\star n}$ are uniformly bounded then for all $m > 0$, $\mathcal{M}^m(\Omega_T, \omega)$ is reduced to a single point i.e. the dynamical system (Ω_T, ω) is uniquely ergodic.*

Proof. The method used in the proof is standard and can be found in [25] in a similar situation for $d = 1$.

To prove the first statement it is enough to assume that the dimension of the $H^d(B_n, \mathbb{R})$'s is constant and equal to N. The set $\mathcal{M}^m(\Omega_T, \omega)$ is a convex. Its extremal points coincide with the set of ergodic measures. Since $\mathcal{M}^m(\Omega_T, \omega) = \mathcal{M}^{\star m}(\mathcal{F})$, the convex set $\mathcal{M}^m(\Omega_T, \omega)$ is the intersection of the convex nested sets

$$\mathcal{M}(\Omega_T, \omega) = \cap_{n\geq 0} W_n,$$

where

$$W_n = f_{\star 1} \circ \cdots \circ f_{\star n-1} W^\star(B_n).$$

Since each convex cone W_n possesses at most N extremal lines, the limit set $\mathcal{M}^m(\Omega_T, \omega)$ possesses also at most N extremal points and thus at most N ergodic measures.

The second statement is proved if $\mathcal{M}(\Omega_T, \omega)$ is shown to be one-dimensional. Let x and y be two points in the positive cone of $\mathbb{R}^N$ representing one of the W_n's. Let T be the largest line segment containing x and y and contained in the positive cone of $\mathbb{R}^N$. The hyperbolic distance between x and y is given by

$$Hyp(x, y) = -\ln \frac{(m + l)(m + r)}{l.r},$$

where m is the length of the line segment $[x, y]$ and l and r are the length of the connected components of $T \setminus [x, y]$. It is known that positive matrices contract the hyperbolic distance in the positive cone of $\mathbb{R}^N$. Since the matrices corresponding to the maps $f_{\star n}$ are uniformly bounded in sizes and entries, this contraction is uniform. Because of this uniform contraction the set $\mathcal{M}(\Omega_T, \omega)$ is one dimensional. □

Remark 5.8. If it is easy to construct perfect tilings in dimension $d = 1$ which are not uniquely ergodic (see for instance [25]), this question remains unclear in dimension $d \geq 2$.

Since the Ruelle-Sullivan current vanishes on exact d-differential forms, it acts on the de-Rham cohomology group $H^d_{DR}(\Omega(\mathcal{F}))$. The same standard arguments as the one developed in Section 4.1leads to

$$H^d_{DR}(\Omega(\mathcal{F})) = \varinjlim H^d_{DR}(\mathcal{F}).$$

In other words, every co-homology class $[\omega]$ in $H^d_{DR}(\Omega(\mathcal{F}))$ is the direct limit of co-homology classes $[\omega_n]$ in $H^d_{DR}(\Omega(\mathcal{F}))$. It means that for n big enough

$$\mathcal{C}_{\mu^t}([\omega]) = \sum_{i=1}^{p(n)} \frac{\mu(\pi_n^{-1}(F_i))}{\lambda_d(F_i)} \int_{F_i} \omega_n .$$

Let I_n be the standard isomorphism $I_n : H^d_{DR}(B_n) \to H^d(B_n, \mathbb{A})$ (where $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$ depending upon whether the coefficients are in $\mathbb{R}$ or $\mathbb{C}$) defined by

$$< I_n([\omega)], c >= \int_c \omega,$$

for every cycle c in $H_d(B_n, \mathbb{A})$. It will be important in the proof of the gap labeling theorem to consider the integral cohomology classes defined by

$$H_{DR}^{d,int}(B_n) = I_n^{-1} H^d(B_n, \mathbb{Z}).$$

It is the set of classes that take integer values on integers cycles (i.e; cycles in $H_d(B_n, \mathbb{Z})$ leading to consider the inductive limit

$$H_{DR}^{d,int}(\Omega(\mathcal{F})) = \lim_{\rightarrow} H_{DR}^{d,int}(\mathcal{F}).$$

Corollary 4.4 and Remark 5.4 then lead to

Proposition 5.9.

$$\mathcal{C}_{\mu^t}(H_{DR}^{d,int}(\Omega(\mathcal{F}))) = \int_{\Omega_{T,Y}} d\mu^t\, \mathcal{C}(\Omega_{T,Y}, \mathbb{Z}),$$

where $\mathcal{C}(\Gamma_T, \mathbb{Z})$ is the set of integer valued continuous functions on $\Omega_{T,Y}$.

6. C^*-Algebras, K-Theory and Gap-Labeling

In the previous section the continuous Hull Ω_T of a perfect tiling of $\mathbb{R}^d$ has been described in terms of expanding flattening sequences. It has been powerful in describing the ergodic properties of the $\mathbb{R}^d$-action on Ω_T. In this section the topology of this Hull will be analyzed through its K-theory. In doing so, the dynamical system becomes a noncommutative topological space and can be described through its C^*-algebra of continuous functions that turn out to be noncommutative. The gap labeling theorem [5] will be one of the main consequences of this analysis.

6.1. Elements of topological K-theory. This section is a reminder about the classical *topological* K-theory (see for instance [2]).

For any abelian semigroup $(A, +)$ with zero 0, there is a canonical way to associate with A an abelian group $(K(A), +)$ (also called the *Grothendieck* group of A) satisfying a natural universal property. A simple way to construct $K(A)$ is as follows. Consider the product semigroup $A \times A$ and let Δ be its diagonal. The cosets $[(a, b)] = (a, b) + \Delta$ make a partition of $A \times A$. The coset set $K(A) = A \times A/\Delta$ becomes an abelian group under the operation $[(a, b)] + [(c, d)] = [(a + c, b + d)]$. It is plain to check that it is associative and commutative while $[(a, b)] + [(b, a)] = [(0, 0)]$ shows that any element has an opposite. The map

$$\alpha : A \to K(A), \quad \alpha(a) = [a] = [(a, 0)],$$

is a semigroup homomorphism satisfying the following universal property: for any group G and semigroup homomorphism $\gamma : A \to G$ there exists a unique homomorphism $\chi : K(A) \to G$ such that $\gamma = \chi \circ \alpha$. If in addition A satisfies the *cancelation rule* (namely $a + b = c + b \ \Rightarrow\ a = c$), then α is injective. Let $K^+(A)$ denote the image of α. This is a *positive cone* in $K(A)$ with respect to the $\mathbb{Z}$-module structure of $K(A)$ generating the whole $K(A)$. Moreover, every $[(a, b)]$ has the form $[(a, b)] = [a] - [b]$. If for every $a, b \in A$, $a + b = 0 \ \Leftrightarrow\ (a, b) = (0, 0)$, then $K^+(A) \cap -K^+(A) = \{0\}$, hence the relation $[a] - [b] \geq [a'] - [b']$ iff $[a] - [b] - [a'] + [b'] \in K^+(A)$ makes $K(A)$ an ordered group.

Let X be a *compact* topological space. The previous construction can be applied to the semigroup of isomorphism classes of complex vector bundles on X with the operation given by the direct sum $\oplus$. The class of the unique rank 0 vector bundle is the neutral element of this semigroup. The resulting abelian group is denoted by $K^0(X)$. So each element of $K^0(X)$ is of the form $[E]-[F]$, where E and F are (classes of) vector bundles on X; $[E] = [F]$ iff there exists G such that $E \oplus G = F \oplus G$. $K^+(X) = (\{[E] - [0]\}$ and it is the positive cone of an actual order on $K(X)$. The semigroup does not satisfy the cancelation rule though. As X is compact, for every vector bundle F there exist $n \in \mathbb{N}$ and some other vector bundle G such that $[F] \oplus [G] = \epsilon^n$, where ϵ^n denotes the (class of) trivial vector bundle of rank n. Then G will be called a *trivializing complement* of E. Hence $[E] - [F] = [E \oplus G] - [\epsilon^n]$, so that each element β of $K^0(X)$ can be written as $\beta = [H] - [\epsilon^n]$, for some $n \in \mathbb{N}$. If n_0 is the minimum of such n, then $\beta \geq 0$ iff $n_0 = 0$. Moreover if $G \oplus G' = \epsilon^m$, then $E \oplus G = F \oplus G$ implies that $E \oplus \epsilon^m = F \oplus \epsilon^m$. This can be summarized as follows: $[E] = [F]$ if and only if E and F are *stably equivalent*.

Let $\mathcal{C}(X)$ denote the ring of continuous complex valued functions on X. If E is a complex vector bundle over X, let $\Gamma(E)$ be the $\mathcal{C}(X)$-module of continuous sections of E. This defines a functor Γ from the category $\mathcal{B}$ of vector bundles over X to the category $\mathcal{M}$ of $\mathcal{C}(X)$-modules. In particular it induces an equivalence between the category $\mathcal{T}$ of trivial vector bundles to the category $\mathcal{F}$ of free $\mathcal{C}(X)$-modules of finite rank. As X is compact, since for every bundle E there exists a bundle G such that $E \oplus G$ is trivial, it follows that $\Gamma(E)$ can be seen as the image of $\mathcal{C}(X)^n$ under some projection valued continuous map. In particular the category of vector bundles over X coincides with the sub-category $Proj(\mathcal{T})$ (which a priori is smaller) generated by the images of trivial bundles by *projection operators* on trivial bundles. The category $Proj(\mathcal{F})$ of *projective modules* over $\mathcal{C}(X)$ is defined in a similar way. Hence Γ establishes an equivalence between $Proj(\mathcal{T})$ to $Proj(\mathcal{F})$ which is by definition the category of *finitely-generated projective* $\mathcal{C}(X)$-modules (Swann-Serre Theorem [42, 45]). Hence, the construction of $K^0(X)$ can be rephrased in terms of projective modules instead of vector bundles and in such a case it is denoted by $K_0(\mathcal{C}(X))$.

Another important remark is the following. Let $G(n, N)$ be the Grassmannian manifold over the complex field, namely the set of n-dimensional subspaces of $\mathbb{C}^N$. The *tautological* vector bundle over $\mathcal{G}(n, N)$ is the submanifold $\mathcal{EG}(n, N)$ of $\mathcal{G}(n, N) \times \mathbb{C}^N$ of pairs (h, ξ) where h is an n-dimensional subspace of $\mathbb{C}^N$, seen as a point in $\mathcal{G}(n, N)$, and $\xi \in h$. Then h is also the fiber above $h \in G(n, N)$. Given X a topological space and E a vector bundle over X, with dimension n, let $N \geq n$ be such that there is a vector bundle F such that $E \oplus F = \epsilon^N = X \times \mathbb{C}^N$. Then, if x is a point in X the fiber E_x of E above x is a point in $\mathcal{G}(n, N)$ and the map $g_X : x \in X \mapsto E_x \in \mathcal{G}(n, N)$ is continuous. Then E can be seen as the *pull-back* of $\mathcal{EG}(n, N)$ through this map.

Chern classes of a vector bundle E over X are even degree closed integral differential forms $c_i(\beta) \in H^{2i}(X, \mathbb{Z})$ depending only upon the K-theory class $\beta = [E]$ of E. If $f : X \to Y$ is a continuous map between compact spaces, then there is a natural map $f_k^* : K^0(Y) \to K^0(X)$, such that for any $\beta \in K(Y)$, $c_i(f^*(\beta)) = f^*(c_i(\beta))$, where $f^* : H^{2i}(Y, \mathbb{Z}) \to H^{2i}(X, \mathbb{Z})$ is the natural map induced by f on cohomology. In particular, the canonical map $g_X : x \in X \mapsto E_x \in \mathcal{G}(n, N)$, induces maps $g_X^* : K^0(\mathcal{G}(n, N)) \to K^0(X)$ and $g_X^* : H^{2i}(\mathcal{G}(n, N), \mathbb{Z}) \to H^{2i}(X, \mathbb{Z})$ exchanging the Chern classes. Whenever X is a smooth manifold, Chern classes can be constructed as follows: let A be a connection form on E, namely it is a matrix valued one form such that if $s \in \Gamma^\infty(E)$ is a smooth section then $\nabla s = ds + As$ defines a linear map $\nabla : \Gamma^\infty(E) \mapsto \Gamma^\infty(E) \otimes \Lambda^1(X)$ such that

$$\nabla(fs) = f\nabla(s) + df \cdot s\,, \quad f \in \mathcal{C}^\infty(X)\,.$$

Then $\nabla^2 : \Gamma^\infty(E) \mapsto \Gamma^\infty(E) \otimes \Lambda^2(X)$ is the multiplication by a matrix-valued 2-form Ω_A called *curvature* of the connection and it can be shown that

$$c_n = \frac{n!}{(2\pi)^n}\mathrm{Tr}\left(\Omega_A^n\right)\,, \quad 0 \le n \le \dim(X)/2\,,$$

are closed and integral. Chern's theorem asserts that their cohomology class depends only upon the K-theory class of E. A special example is the Grassmannian connection. Through the Swan-Serre theorem E can be described from a projection valued map $P_E : X \mapsto M_N(\mathbb{C})$ (if N is large enough so as to allow E to have a trivializing complement in $X \times \mathbb{C}^N$) as the set of pairs $(x, \xi) \in X \times \mathbb{C}^N$ such that $\xi = P_E(x)\xi$. Then the Grassmannian connection is given by $\nabla s = Pds$ and its curvature is $PdP \wedge dP$, giving

$$c_n = \frac{n!}{(2\pi)^n}\mathrm{Tr}\left((PdP \wedge dP)^n\right)\,, \quad 0 \le n \le \dim(X)/2\,. \tag{2}$$

The group $K^1(X)$ denotes by definition the group $K^0(SX)$, where SX is the *reduced suspension* of X, i.e. the quotient space $X \times \mathbf{S}^1/\{x\} \times \mathbf{S}^1 \cup X \times \{p_1\}$, where x is a marked point in X and p_1 a marked point in the circle $\mathbf{S}^1$ (see [2]). Let $p : X \times \mathbf{S}^1 \to SX$ be the natural projection. If $f : X \to Y$ is as above, $(f \times id) : X \times S^1 \to Y \times \mathbf{S}^1$ induces a continuous map $Sf : SX \to SY$ which can be used to define $f_k^* : K^1(Y) \to K^1(X)$.

In particular, the Bott map gives a projection over the suspension in terms of a unitary valued continuous function over X so that $K^1(X)$ can also be represented by homotopy classes of such maps. If $U : x \in X \mapsto U(x) \in U(N)$ then the odd Chern classes are given by

$$c_{n+1/2} = \frac{(-1)^n}{(2\imath\pi)^{n+1}}\frac{2^n n!^2}{(2n+1)!}\,\mathrm{Tr}\left((U^{-1}dU)^{2n+1}\right)\,, \quad 0 \le n \le \dim(X)/2\,. \tag{3}$$

For a tiling space given by an EFS $\mathcal{F} = \{(B_n, f_n)\}_{n\in\mathbb{N}}$, the K-groups can be obtained as direct limits,

$$K^i(\Omega_T) = \lim_{\to} K^i(\mathcal{F}).$$

The inductive limit implies that any projection valued continuous map on Ω_T can be approximate uniformly by a projection valued continuous map over the B_n's and so is equivalent to a projection in $\mathcal{C}(B_n) \otimes M_N(\mathbb{C})$ for at least one $n \in \mathbb{N}$.

6.2. K-theory and gap labeling theorem. As was remarked in the early eighties, the K-theory of locally compact spaces can be expressed entirely through the C^*-algebra $\mathcal{C}_0(X) \otimes M_N(\mathbb{C})$, which is already non commutative. This allows to extend K-theory to non commutative algebras. In this section only C^*-algebras will be considered, but the theory can be extended to normed algebras invariant by holomorphic functional calculus [15].

In the tiling case, the C^*-algebra of interest is the crossed product $\mathcal{C}(\Omega_T) \rtimes \mathbb{R}^d$. Let $\mathcal{A}_0$ be the dense sub algebra made of continuous functions on $\Omega_T \times \mathbb{R}^d$ with compact support. If μ is an $\mathbb{R}^d$-invariant probability measure on Ω_T, there is a canonical trace $\mathcal{T}_\mu$ defined on $\mathcal{A}_0$ by

$$\mathcal{T}_\mu(A) = \int_{\Omega_T} d\mu(\omega) A(\omega, 0)\,.$$

The main object of the *gap-labeling* question concerns the trace $\mathcal{T}_\mu(P)$ of a projection $P \in \mathcal{A}$.

In order to define K_0, two projections $P,\ Q \in \mathcal{A}$ are equivalent whenever there is an element $U \in \mathcal{A}$ such that $P = UU^*$ and $Q = U^*U$ [15]. If P and Q are orthogonal to each other, the equivalence class of their direct sum $[P \oplus Q]$ depends only upon $[P]$ and $[Q]$, leading to the definition of the addition $[P]+[Q] = [P \oplus Q]$. To make sure that two projections can be always be made mutually orthogonal modulo equivalence, $\mathcal{A}$ must be replaced by $\mathcal{A} \otimes \mathcal{K}$. Here $\mathcal{K}$ is the C^*-algebra of compact operators on a Hilbert space with a countable basis. It can also be defined as the smallest C^*-algebra containing the increasing sequence of finite dimensional matrices $\mathcal{K} = \varinjlim M_n(\mathbb{C})$, where the inclusion of M_n into M_{n+m} $(0 < m)$ is provided by

$$A \in M_n \mapsto i_{n,m}(A) \;=\; \begin{bmatrix} A & 0 \\ 0 & 0_m \end{bmatrix} \in M_{n+m}.$$

The group $K_0(\mathcal{A})$ is the group generated by formal differences $[P] - [Q]$ of equivalent classes of projections in $\mathcal{A} \otimes \mathcal{K}$. Then two equivalent projections have the same trace and since the trace is linear, it defines a group homomorphism $\mathcal{T}_{\mu,*} : K_0(\mathcal{A}) \mapsto \mathbb{R}$ the image of which are called the *gap labels*.

Together with K_0, there is K_1 which is defined as the equivalence classes, under homotopy, of invertible elements in $\varinjlim GL_n(\mathcal{A})$. In the case of $K^1(\mathcal{C}(X))$ this is equivalent to the previous definition in terms of reduced suspensions. Standard results by R. Bott [15] show that $K_1(\mathcal{A})$ is isomorphic to $K_0(\mathcal{A} \otimes \mathcal{C}_0(\mathbb{R}))$ and that $K_1(\mathcal{A} \otimes \mathcal{C}_0(\mathbb{R}))$, which is then nothing but $K_2(\mathcal{A}) = K_0(\mathcal{A} \otimes \mathcal{C}_0(\mathbb{R}^2))$, is actually isomorphic to $K_0(\mathcal{A})$ (Bott's periodicity theorem). Both K_0 and K_1 are discrete abelian groups. They are countable whenever $\mathcal{A}$ is separable. An important property of $K(\mathcal{A}) = K_0(\mathcal{A}) \oplus K_1(\mathcal{A})$ is that it defines a covariant functor which is continuous under taking inductive limits. Namely any *-isomorphism $\alpha : \mathcal{A} \mapsto \mathcal{B}$ between C^*-algebrasinduces a group homomorphism α_* defined by $\alpha_*([P]) = [\alpha(P)]$ for K_0 and similarly for K_1. Moreover, $K(\varinjlim \mathcal{A}_n) = \varinjlim K(\mathcal{A}_n)$.

For the purpose of this paper some more details will be needed in connection with the Bott periodicity. More precisely, a $*$-algebra A is called a *local Banach algebra* (LB) if it is normed and if it is invariant by holomorphic functional calculus. Then $K(A)$ can be defined in a way similar to the case of C^*-algebras. The *suspension* SA is the set of continuous maps $t \in [0, 1] \mapsto a(t) \in A$ such that $a(0) = a(1) = 0$. If A has no unit, let $\tilde{A}$

be the algebra obtained from A by adjoining a unit. The Bott periodicity theorem is based on the properties of the following two maps; (i) the *Bott map* $\beta_A : K_0(A) \mapsto K_1(SA)$, (ii) the index map $\theta_A : K_1(A) \mapsto K_0(SA)$. The Bott map is defined as follows. For $n \in \mathbb{N}$ and for $l \leq n$ let p_l denote the projection

$$p_l = \begin{bmatrix} \mathbf{1}_l & 0 \\ 0 & 0 \end{bmatrix} \in M_n(\mathbb{C}) \subset M_n(\tilde{A}).$$

Let e be an idempotent in $M_n(\tilde{A})$ such that $e - p_l \in M_n(A)$. Then let $f_e : t \in [0, 1] \mapsto f_e(t) = e^{2\imath\pi t} e + (\mathbf{1}_n - e)$. It is clear that $f_e \in M_n(\tilde{SA})$ and is invertible. Moreover, since $f_e f_{p_l}^{-1} - \mathbf{1}_n \mathcal{M}_n(SA)$ it defines unambiguously an element $[f_e f_{p_l}^{-1}]$ of $K_1(SA)$. It can be shown that it depends only upon the class $[e] - [p_n] \in K_0(A)$ and that the map $\beta_A([e] - [p_n]) = [f_e f_{p_l}^{-1}]$ defines a group homomorphism between $K_0(A)$ and $K_1(SA)$ [15].

Similarly the index map θ_A is defined as follows. Let $u \in M_n(\tilde{A})$ be invertible and such that $u - \mathbf{1}_n \in M_n(A)$. Then, for $t \in [0, 1]$,

$$z(t) = R(t) \begin{bmatrix} u^{-1} & 0 \\ 0 & \mathbf{1}_n \end{bmatrix} R(t)^{-1} \begin{bmatrix} u & 0 \\ 0 & \mathbf{1}_n \end{bmatrix}, \quad R(t) = \begin{bmatrix} \cos \frac{\pi t}{2} & \sin \frac{\pi t}{2} \\ -\sin \frac{\pi t}{2} & \cos \frac{\pi t}{2} \end{bmatrix},$$

gives a homotopy between $\mathbf{1}_{2n}$ and the matrix $\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$ by invertible elements such that $z(t) - \mathbf{1}_{2n} \in M_{2n}(A)$. Then $g(t)$ is the loop of idempotents defined by $g(t) = z(t) p_n z(t)^{-1}$ in $M_{2n}(\tilde{A})$. Since it is clear that $g(t) - p_n \in M_{2n}(A)$ for all t's, it defines an element $\theta_A([u]) = [g] - [p_n] \in K_0(SA)$ that depends only upon the class $[u] \in K_1(A)$ of u and θ_A becomes a group homomorphism between $K_1(A)$ and $K_0(SA)$ [15]. Using the Bott periodicity theorem, it can be proved that θ_{SA} is the inverse of β_A [15]. However instead of using the Bott periodicity, it is possible to iterate the procedures as follows. Starting from $g_0 = e \in M_n(A)$ the previous construction leads to a double sequence (i) $g_1, \cdot, g_m, \cdots$ of idempotents with $g_m \in M_{2^m n}(\widetilde{S^{2m} A})$ and $g_m - p_{2^{m-1}n} \in M_{2^m n}(S^{2m} A)$, and (ii) $u_1, \cdots, u_m, \cdots$ of invertible elements such that $u_{m+1} \in M_{2^m n}(\widetilde{S^{2m+1} A})$ and $u_{m+1} - \mathbf{1}_{2^m n} \in M_{2^m n}(S^{2m+1} A)$. Moreover the relation between them is that $[u_{m+1}] = \beta_{S^{2m} A}([g_m] - [p_{2^{m-1}n}])$ whereas $[g_m] - [p_{2^{m-1}n}] = \theta_{S^{2m-1} A}([u_m])$. The following result can be proved by recursion on m

Proposition 6.1. *Let A a local Banach algebra with a tracial state $\mathcal{T}$. Then denoting by $\Delta_m = [0, 1]^{\times m}$, the following relations are satisfied:*

$$c_{2m} = 2(-1)^{m+1} \frac{m!^2}{2m!} c_{2m-1}, \quad c_{2m+1} = 2\imath\pi(-1)^m \frac{(2m+1)!}{m!^2} c_{2m},$$

where

$$c_{2m} = \int_{\Delta_{2m}} \mathcal{T}\left\{ g_m (dg_m)^{\wedge 2m} \right\}, \quad c_{2m+1} = \int_{\Delta_{2m+1}} \mathcal{T}\left\{ (u_{m+1}^{-1} du_{m+1})^{\wedge 2m+1} \right\},$$

As a consequence the following formulæ generalize the one proved by Connes [17]: if $\mathcal{T}_*$ is the map induced by the trace on $K_0(A)$,

$$\mathcal{T}_* \{[e] - [p_l]\} = \frac{1}{(2\imath\pi)^m m!} \int_{\Delta_{2m}} \mathcal{T} \left\{ g_m (dg_m)^{\wedge 2m} \right\} , \tag{4}$$

$$= \frac{(-1)^m}{(2\imath\pi)^{m+1}} \frac{2^m m!^2}{(2m+1)!} \int_{\Delta_{2m+1}} \mathcal{T} \left\{ (u_{m+1}^{-1} du_{m+1})^{\wedge 2m+1} \right\} . \tag{5}$$

The Thom-Connes theorem [17, 15, 19] states that given a C^*-dynamical system $(A, \alpha, \mathbb{R})$ (where $\alpha : s \in \mathbb{R} \mapsto \alpha_s \in \mathrm{Aut}(A)$), there are group isomorphisms $\phi_\alpha^i : K_i(A) \mapsto K_{i+1}(A \rtimes_\alpha \mathbb{R})$. The construction of this isomorphism reduces to the Bott and index maps when the action is trivial, remarking that the convolution C^*-algebra $C^*(\mathbb{R})$ is isomorphic to $\mathcal{C}_0(\mathbb{R})$ through Fourier's transform and that $\mathcal{C}_0(\mathbb{R})$ is isomorphic to the algebra of $\mathcal{C}(0, 1)$ of continuous functions on $(0, 1)$ vanishing at 0 and 1. The Connes construction of ϕ_α^0 relies upon the remark that given an idempotent $e \in A$, it is possible to choose an equivalent idempotent e' (thus giving the same element in $K_0(A)$) and an equivalent $\mathbb{R}$-action α' (thus giving rise to a crossed product isomorphic to $A \rtimes_\alpha \mathbb{R}$) such that $\alpha'(e') = e'$. Then the construction of the Connes map is identical to the Bott map. Using naturality and functoriality, the map ϕ_α^1 can be constructed similarly. The formula (5) for $m = 0$ is then valid provided du is replaced by δu where δ is the generator of one-parameter group of automorphisms α [17].

Applied to the present situation where $A = \mathcal{C}(\Omega_T)$ this gives a group isomorphism ϕ_d between $K_0(\mathcal{C}(\Omega_T) \rtimes \mathbb{R}^d)$ and $K_d(\mathcal{C}(\Omega_T))$ (with $K_i = K_{i+2}$ by Bott's periodicity theorem) and the previous remarks lead to

Theorem 6.2. *[18] Let P be a projection in $\mathcal{A}$ and let $[P]$ be its class in $K_0(\mathcal{A})$.*

(i) If $d = 2m + 1$ is odd, let U be a unitary element of $\mathcal{C}(\Omega_T) \otimes \mathcal{K}$ representing $\phi_d([P])$. Let η be the d-form in $H_{DR}^d(\Omega(\mathcal{F}))$:

$$\eta = \frac{(-1)^m}{(2\imath\pi)^{m+1}} \frac{2^m m!^2}{(2m+1)!} \operatorname{Tr}\left((U^{-1} dU)^{2m+1} \right) .$$

(ii) If $d = 2m$ is even, let $Q_\pm$ be a pair of projections of $\mathcal{C}(\Omega_T) \otimes \mathcal{K}$ with $\phi_d([P]) = [Q_+] - [Q_-]$. Let η be the d-form in $H_{DR}^d(\Omega(\mathcal{F}), \mathbb{C})$,

$$\eta \;=\; \frac{1}{(2\imath\pi)^m m!} \left\{ \operatorname{Tr}\left((Q_+ \, dQ_+ \wedge dQ_+)^m \right) - \operatorname{Tr}\left((Q_- \, dQ_- \wedge dQ_-)^m \right) \right\} .$$

Then, in both cases, $\eta \in H_{DR}^d(\Omega(\mathcal{F}), \mathbb{Z}))$ and

$$\mathcal{T}_\mu(P) \;=\; \mathcal{C}_{\mu^t}([p\eta]).$$

Proof of Theorem 1.1: Combining the previous result with Proposition 5.9, we get:

$$\mathcal{T}_\mu(K_0(\mathcal{A})) \subset \int_{\Omega_{T,Y}} d\mu^t \; C(\Omega_{T,Y}, \mathbb{Z}).$$

The converse inclusion is a standard result: the C^*-algebra $\mathcal{A}$ is *Morita equivalent* to the C^*-algebra $C^*(\Gamma)$ of the groupoid of the transversal, so that they have same K-theory [15]. In this latter algebra the characteristic functions of clopen subsets of Γ are projections with trace given by their integral. Since $\mathcal{C}(\Omega_{T,Y}, \mathbb{Z})$ is generated by such characteristic functions, this proves the other inclusion. Hence, the *gap-labeling theorem* (see Theorem 1.1 stated in the introduction) is proved. □

Remarks about Theorem 6.2: The integral classes defined above are related to Chern classes. The image, say β, of $[P]$ by the Thom-Connes isomorphism can be taken either as an element of $K^0(B_n)$, when d is even, or of $K^1(B_n)$, when d is odd, for n large enough. It turns out (this is contained in the proof of the Thom-Connes theorem) that:

1) when d is even η (which actually "lives" on B_n) represents the Chern class $c_{[d/2]}(\beta) \in H^d(B_n, \mathbb{Z})$; in fact the choice of the normalization constant k_d in Theorem 6.2 ensures that this class is integral. So

$$\mathcal{T}_\mu(P) =< \mu_n | c_{[d/2]}(\beta) > .$$

2) When d is odd

$$\mathcal{T}_\mu(P) =< S\mu_n | c_{[(d+1)/2]}(\beta) > ,$$

where $S\mu_n$ is defined as follows. For $z\mu_n \in Z_k(B_n, \mathbb{A})$, $(\mu_n \times S^1) \in Z_{k+1}(B_n \times S^1$ (it is understood that S^1 has the usual counterclockwise orientation). By using the natural projection $p_n : B_n \times S^1 \to SB_n$, $(\mu_n \times S^1)$ induces a $(k+1)$ (singular) cycle $S\mu_n$ on SB_n, which is called the *suspension* of the cycle μ_n. □

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