On regions of existence and nonexistence of solutions for a system of p–q-Laplacians [1]

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Abstract. We give a new region of existence of solutions to the superhomogeneous Dirichlet problem

$$\begin{aligned} &-\Delta_p u = v^\delta, \quad v > 0 \text{ in } B,\\ &-\Delta_q v = u^\mu, \quad u > 0 \text{ in } B,\\ &u = v = 0 \quad \text{on } \partial B, \end{aligned} \tag{S_R}$$

where B is the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$. Here $\delta, \mu > 0$ and $\Delta_m u = \text{div}(|\nabla u|^{m-2}\nabla u)$ is the m-Laplacian operator for $m > 1$.

Keywords: m-Laplacian, energy identities

1. Introduction and main results

Consider the quasilinear elliptic system

$$\begin{aligned} &-\Delta_p u = v^\delta, \quad v > 0 \text{ in } B,\\ &-\Delta_q v = u^\mu, \quad u > 0 \text{ in } B,\\ &u = v = 0 \quad \text{on } \partial B, \end{aligned} \tag{S_R}$$

where B is the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$. Here $\delta, \mu > 0$ and

$$\Delta_m u = \text{div}\big(|\nabla u|^{m-2}\nabla u\big)$$

is the m-Laplacian operator for $m > 1$.

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In view of the invariance of problem (S_R) under rotations, it is natural to look for radially symmetric solutions. If we still denote by u, v the solutions as functions of $r = |x|$, we obtain the system of ODE's

$$\begin{aligned} -\big(r^{N-1}\big|u'(r)\big|^{p-2}u'(r)\big)' &= r^{N-1}\big|v(r)\big|^{\delta}, \quad 0 < r < R, \\ -\big(r^{N-1}\big|v'(r)\big|^{q-2}v'(r)\big)' &= r^{N-1}\big|u(r)\big|^{\mu}, \quad 0 < r < R, \end{aligned} \tag{1.1}$$

with appropriate boundary conditions. We are primarily interested in the existence of (regular) solutions of (1.1), i.e., $(u, v) \in (C^1[0, R] \cap C^2(0, R])^2$ satisfying (1.1) and $u'(0) = v'(0) = 0$, $u(R) = v(R) = 0$.

Clearly, either both u and v are identically 0, or both u and v are strictly positive and decreasing on $[0, R)$.

Observe that system (S_R) is homogeneous in the sense that if (u, v) is a solution, then $(\lambda u, \nu v)$ is also a solution provided that $\lambda,\ \nu > 0$ and $\lambda^{1-p} = \nu^{\delta}$ and $\nu^{1-q} = \lambda^{\mu}$. So it is natural to call the system *superhomogeneous* when

$$d := \delta\mu - (p-1)(q-1) > 0, \quad \delta > 0,\ \mu > 0. \tag{H_1}$$

In case that $p = q = 2$, condition (H_1) is usually called *superlinear condition* and it is equivalent to the condition

$$\frac{1}{\delta+1} + \frac{1}{\mu+1} < 1.$$

It has been shown in [2,6,11], and [12] that under (H_1), when $N > 2$, a necessary and sufficient condition for the existence of radial solutions to (S_R) is

$$\frac{1}{\delta+1} + \frac{1}{\mu+1} > \frac{N-2}{N}.$$

In case that $m = p = q \neq 2$ and $\delta = \mu$ (see Remark A.1 in the Appendix) we have that if (u, v) is a solution, then $u = v$ and hence the system reduces to an equation. It follows then from results of [10] that a solution exists in that case (for $m < N$) if and only if

$$\frac{1}{\delta+1} > \frac{N-m}{Nm}. \tag{1.2}$$

Apart from these cases no necessary and sufficient condition for the existence of solutions is known. Sufficient conditions have been obtained in [3] where a-priori estimates are established by means of a blow up method in the sense of Gidas and Spruck, see [5], and a degree argument. In [1], the problem has been studied in a bounded convex domain with C^2 boundary.

The main goal of this paper is to exhibit a new region of existence of solutions to (S_R). This is done in Theorem 1.5.

To our knowledge, when $p \neq q$ or $p = q \neq 2$ and $\delta \neq \mu$, there are no nonexistence results (of Pohozaev type) in the literature. In Theorem 1.7 we provide such a region of nonexistence.

An important ingredient in the proof of our main result Theorem 1.5 is the observation that under condition (H_1), the absence of positive "ground states" implies existence of solutions for (S_R). The result is contained implicitly in [3,4], but for the sake of completeness we state it in Proposition 1.1 below and we outline its proof in the Appendix.

Proposition 1.1. *Let $p,\ q > 1,\ \delta, \mu > 0$ be such that (H_1) holds. If the system*

$$\begin{aligned} -\Delta_p u &= |v|^\delta, \quad v > 0 \text{ in } \mathbb{R}^N, \\ -\Delta_q v &= |u|^\mu, \quad u > 0 \text{ in } \mathbb{R}^N \end{aligned} \tag{S_∞}$$

has no radially symmetric solution (u, v) in $(C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\}))^2$, then system (S_R) possesses a nontrivial solution for any $R > 0$.

Remark 1.2. We do not know if the converse of this proposition is true as it is in the case of a single equation or the case of the system with $p = q = 2$, see [2,6,11,12].

Remark 1.3. If $p \geqslant N$, then $-\Delta_p u \geqslant 0$ and $u \geqslant 0$ in $\mathbb{R}^N$ imply $u = \text{Const.}$, see [7–9]. Hence from $-\Delta_p u = 0$ it follows that $v = 0$, and from the second equation it follows that $u = 0$. Therefore, if $p \geqslant N$ or/and $q \geqslant N$ it follows from Proposition 1.1 that (S_R) possesses at least one solution (u, v). Hence in Theorem 1.7 we may assume without loss of generality that $\max\{p, q\} < N$.

Remark 1.4. In [3] is has been shown that if

$$\max\left\{\alpha - \frac{N-p}{p-1}, \beta - \frac{N-q}{q-1}\right\} \geqslant 0, \tag{1.3}$$

where

$$\alpha = \frac{1}{d}\left[p(q-1) + \delta q\right], \qquad \beta = \frac{1}{d}\left[q(p-1) + \mu p\right] \tag{1.4}$$

and (H_1), then the assumptions of Proposition 1.1 are satisfied. Hence in this case, that is, when (1.3) is satisfied, the existence of solutions to (S_R) follows. Observe that in case that $p = q = m$ and $\delta = \mu$, the condition (1.3) is equivalent to

$$\frac{m-1}{\delta + m - 1} > \frac{N-m}{m}, \quad m > 1,$$

which is more restrictive than (1.2). Hence condition (1.3) is not optimal.

We are now in a position to state our main results.

Theorem 1.5. *Suppose $N \geqslant 2$ and that $\delta,\ \mu > 0$ satisfy (H_1).*

(1) *Let*

$$\frac{2N}{N+1} < p \leqslant 2 \quad \text{and} \quad \frac{2N}{N+1} < q \leqslant 2. \tag{1.5}$$

Then problem (S_R) possesses a solution (u, v) provided that

$$\frac{1}{\delta + 1} + \frac{1}{\mu + 1} > \frac{N - \underline{m}}{N(\underline{m} - 1)}, \tag{1.6}$$

where $\underline{m} = \min\{p, q\}$.

(2) *Let*

$$2 \leqslant p < N \quad \text{and} \quad 2 \leqslant q < N. \tag{1.7}$$

Then problem (S_R) *possesses a solution* (u, v) *provided that*

$$\frac{1}{\delta+1} + \frac{1}{\mu+1} > \frac{N(\overline{m}-1)-\overline{m}}{N(\overline{m}-1)}, \tag{1.8}$$

where $\overline{m} = \max\{p, q\}$.

Remark 1.6. Observe that when $p = q = 2$, conditions (1.6) and (1.8) are the same and they are optimal, see [2,6,11,12]. When $m = p = q \neq 2$, $m < 2$, and $\delta = \mu$, condition (1.6) reads

$$\frac{2}{\delta+1} > \frac{N-m}{N(m-1)}.$$

Since (1.2), which is optimal, can be rewritten as

$$\frac{2}{\delta+1} > \frac{N-m}{N(m-1)} \frac{2(m-1)}{m}, \quad \text{with } \frac{2(m-1)}{m} < 1,$$

it follows that condition (1.6) is not optimal.

When $m < 2$, we note that (1.6) gives a new region of existence provided that

$$\frac{N(m-1)}{N-m} < \frac{(2N+1)m - 3N}{N-m},$$

which holds if $m > \frac{2N}{N+1}$. Since $2N/(N+1) < 2$, there is always room for some $m < 2$, as is shown in Fig. 1.

When $m > 2$, we note that (1.8) gives a new region of existence provided that

$$\frac{N(m-1)}{N-m} < \frac{N(m-1)+m}{N(m-1)-m},$$

which holds if

$$m < \frac{(3N+1+\sqrt{(N-1)(N+7)})N}{2N^2+2}.$$

Since the right-hand side of this inequality is greater than 2 for $N > 2$, there is always room for some $m > 2$, as is shown in Fig. 2.

Finally we have

Theorem 1.7. *Suppose that* $N > 2$ *and* $\delta, \mu > 0$ *satisfy* (H_1).

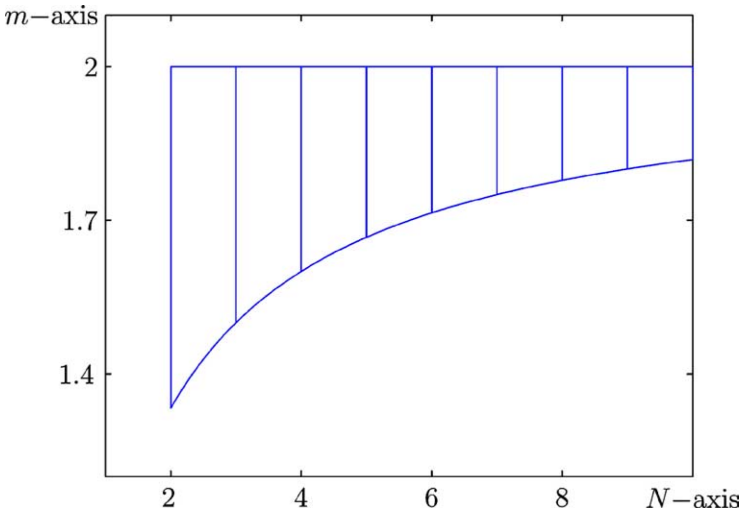

Fig. 1. The values of $m < 2$ for which we obtain a new region of existence.

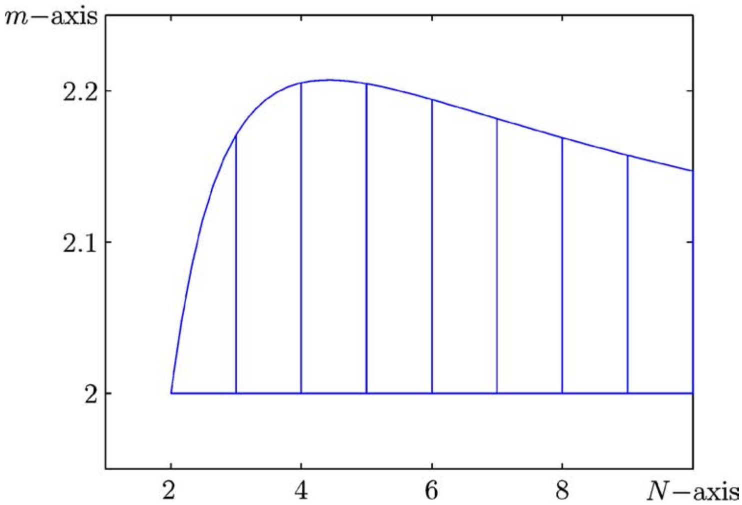

Fig. 2. The values of $m > 2$ for which we obtain a new region of existence.

(1) *If* $2 \leqslant p,\ q < N$, *and*

$$\frac{1}{\delta+1}+\frac{1}{\mu+1}\leqslant\frac{N-\overline{m}}{N(\overline{m}-1)}, \tag{1.9}$$

where $\overline{m}=\max\{p,q\}$, *then system* (S_R) *has no solutions* (*regular or not*).

(2) *If* $N/(N-1) < p,\ q \leqslant 2$ *and*

$$\frac{1}{\delta+1}+\frac{1}{\mu+1}\leqslant\frac{N(\underline{m}-1)-\underline{m}}{N(\underline{m}-1)}, \tag{1.10}$$

where $\underline{m}=\min\{p,q\}$, *then system* (S_R) *has no solutions* (*regular or not*).

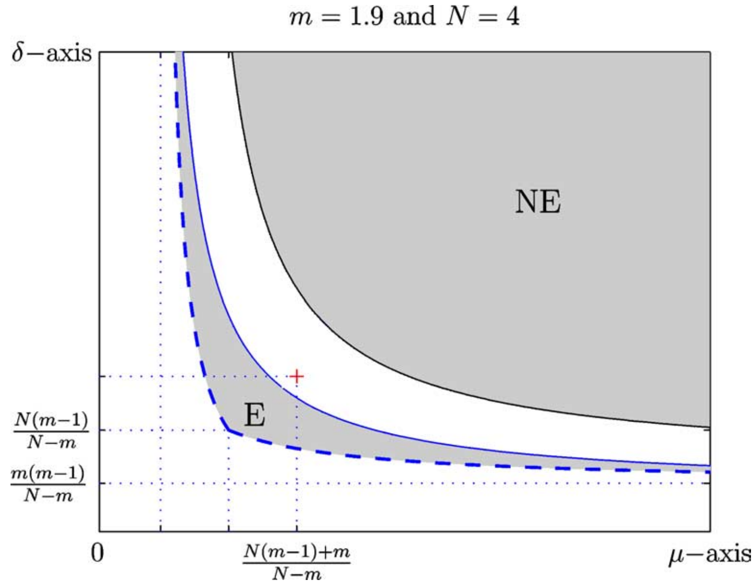

Fig. 3. The nonexistence region in grey is given by (1.10), and the new existence region is the one bounded by equality in (1.3) and by equality in (1.6).

Remark 1.8. In case that $p = q = 2$, (1.9)–(1.10) are optimal, but when $m = p = q \neq 2$ they are not. Indeed, if we set $\mu = \delta$ in (1.9), we obtain

$$\mu \geqslant \frac{(2N+1)m - 3N}{N-m} > \frac{N(m-1)+m}{N-m} \quad \text{for all } m > 2,$$

and if we set $\mu = \delta$ in (1.10), we obtain

$$\mu \geqslant \frac{N(m-1)+m}{N(m-1)-m} > \frac{N(m-1)+m}{N-m} \quad \text{for all } m < 2.$$

Since $\mu \geqslant \frac{N(m-1)+m}{N-m}$ is the optimal range for the case of one equation, our claim follows.

For $N = 4$, in Fig. 3 we show the new region of existence and the nonexistence region for the case $m = p = q = 1.9$, and in Fig. 4 we show the new region of existence and the nonexistence region for the case $m = p = q = 2.1$.

Our article is organized as follows. In Section 2 we give a Pohozaev type identity which is the key to prove our main results and we prove them. Finally in the Appendix we prove Proposition 1.1.

2. A Pohozaev type identity and proof of our main results

Our main theorems are based on the two following lemmas, which give appropriate generalizations of the Pohozaev identity used to deal with the case $p = q = 2$, see [6].

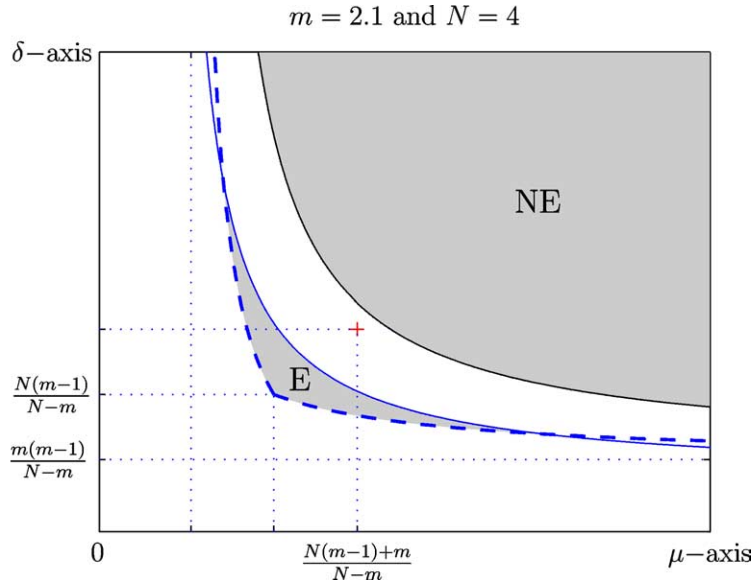

Fig. 4. The nonexistence region in grey is given by (1.9). The new existence region is bounded by equality in (1.8) and the dashed curve which is given by equality (1.3).

Lemma 2.1. *Let* $(u, v) \in (C^1[0,\infty) \cap C^2(0,\infty))^2$ *be a solution of the system*

$$\begin{aligned} &-(r^{N-1}|u'|^{p-2}u')' = r^{N-1}v^{\delta},\\ &-(r^{N-1}|v'|^{q-2}v')' = r^{N-1}u^{\mu}, \qquad\qquad (S^r_\infty)\\ &u(r) > 0, \qquad v(r) > 0, \quad r \in [0,\infty), \end{aligned}$$

with δ, $\mu > 0$, *and assume that either*

$$2N/(N+1) \leqslant p \leqslant q \leqslant 2 \quad \text{or} \quad 2 \leqslant p \leqslant q < N.$$

Let us define

$$\begin{aligned} E_1(r) = {}& r^{N+k_1-2}|u'|^{p-1}|v'|^{q-1}\\ &- \frac{N}{\delta+1} r^{N-1}|u'|^{p-1}\int_r^\infty s^{k_1-2}|v'|^{q-1}\,\mathrm{d}s - \frac{N}{\mu+1} r^{N-1}|v'|^{q-1}\int_r^\infty s^{k_1-2}|u'|^{p-1}\,\mathrm{d}s\\ &+ r^N \int_r^\infty s^{k_1-2}|v'|^{q-1}v^\delta\,\mathrm{d}s + r^N\int_r^\infty s^{k_1-2}|u'|^{p-1}u^\mu\,\mathrm{d}s, \quad r \in (0,\infty), \end{aligned} \tag{2.1}$$

where

$$k_1 = \begin{cases} p + \dfrac{N-p}{p-1}(p-2) & \text{if } 2N/(N+1) \leqslant p \leqslant q \leqslant 2,\\ \dfrac{q}{q-1} & \text{if } 2 \leqslant p \leqslant q. \end{cases}$$

Then for $r \in (0,\infty)$ we have

$$
\begin{aligned}
E_1'(r) = {} & \Big(k_1 - N + \frac{N}{\delta+1} + \frac{N}{\mu+1}\Big) r^{N+k_1-3} |u'|^{p-1} |v'|^{q-1} \\
& - \frac{N}{\delta+1} r^{N-1} v^\delta \int_r^\infty s^{k_1-2} |v'|^{q-1}\, \mathrm{d}s + N r^{N-1} \int_r^\infty s^{k_1-2} |v'|^{q-1} v^\delta \, \mathrm{d}s \\
& - \frac{N}{\mu+1} r^{N-1} u^\mu \int_r^\infty s^{k_1-2} |u'|^{p-1}\, \mathrm{d}s + N r^{N-1} \int_r^\infty s^{k_1-2} |u'|^{p-1} u^\mu \, \mathrm{d}s. \qquad (2.2)
\end{aligned}
$$

Proof. We prove first that E_1 is well defined. Since u and v are decreasing functions we have that for any $T > r$ it holds that

$$
\int_r^T s^{k_1-2} |v'|^{q-1} v^\delta \, \mathrm{d}s \leqslant v^\delta(r) \int_r^T s^{k_1-2} |v'|^{q-1}\, \mathrm{d}s,
$$

and

$$
\int_r^T s^{k_1-2} |u'|^{p-1} u^\mu \, \mathrm{d}s \leqslant u^\mu(r) \int_r^T s^{k_1-2} |u'|^{p-1}\, \mathrm{d}s,
$$

thus it is sufficient to prove that the first two integrals in (2.1) are well defined.

In order to do so, we recall that from [3, Lemma 2.1] or [4, Proposition V.1], we have

$$
\big|u'(r)\big| \leqslant C r^{-\alpha-1}, \qquad \big|v'(r)\big| \leqslant C r^{-\beta-1}
$$

for some $C > 0$ and r large.

We deal first with the case $2N/(N+1) \leqslant p \leqslant q \leqslant 2$. We will see first that $1-k_1+(\alpha+1)(p-1) > 0$, which is the condition to have the second integral in (2.1) well defined. Indeed,

$$
1 - k_1 + (\alpha+1)(p-1) = p - k_1 + \alpha(p-1) = \frac{N-p}{p-1}(2-p) + \alpha(p-1) > 0.
$$

For the first integral we have

$$
1 - k_1 + (\beta+1)(q-1) = q - k_1 + \beta(q-1) \geqslant p - k_1 + \beta(q-1) > 0,
$$

hence the first integral appearing in (2.1) is also well defined.

For the case $2 \leqslant p \leqslant q$, it can be easily verified that $\alpha(p-1) > k_1 - 1 - (p-1)$ and $\beta(q-1) > k_1 - 1 - (q-1)$. We only verify the first inequality: as $p \geqslant 2$, we have that

$$
k_1 - 1 - (p-1) = \frac{1 - (q-1)(p-1)}{q-1} \leqslant 0,
$$

and $\alpha(p-1) > 0$, thus the first two integrals in (2.1) are well defined.

Now (2.2) follows by direct differentiation using that (u,v) is a solution to (S_∞^r). $\square$

Our second lemma is essentially the same Pohozaev identity as in Lemma 2.1, but in $(0, R]$.

Lemma 2.2. *Let* $(u, v) \in (C^1[0, R] \cap C^2(0, R])^2$ *be a solution of the system* (S_R) *with* δ, $\mu > 0$, *and assume that either*

$$N/(N-1) < p \leqslant q \leqslant 2 \quad \text{or} \quad 2 \leqslant p \leqslant q < N.$$

Let us define

$$\begin{aligned} E_2(r) = {} & r^{N+k_2-2}|u'|^{p-1}|v'|^{q-1} \\ & - \frac{N}{\delta+1} r^{N-1}|u'|^{p-1} \int_r^R s^{k_2-2}|v'|^{q-1}\,\mathrm{d}s - \frac{N}{\mu+1} r^{N-1}|v'|^{q-1} \int_r^R s^{k_2-2}|u'|^{p-1}\,\mathrm{d}s \\ & + r^N \int_r^R s^{k_2-2}|v'|^{q-1}v^\delta\,\mathrm{d}s + r^N \int_r^R s^{k_2-2}|u'|^{p-1}u^\mu\,\mathrm{d}s, \quad r \in (0, R], \end{aligned} \tag{2.3}$$

where

$$k_2 = \begin{cases} q + \dfrac{N-q}{q-1}(q-2) & \text{if } 2 \leqslant p \leqslant q < N, \\ \dfrac{p}{p-1} & \text{if } N/(N-1) < p \leqslant q \leqslant 2. \end{cases}$$

Then for $r \in (0, R)$ *we have*

$$\begin{aligned} E_2'(r) = {} & \left(k_2 - N + \frac{N}{\delta+1} + \frac{N}{\mu+1}\right) r^{N+k_2-3}|u'|^{p-1}|v'|^{q-1} \\ & - \frac{N}{\delta+1} r^{N-1} v^\delta \int_r^R s^{k_2-2}|v'|^{q-1}\,\mathrm{d}s + N r^{N-1} \int_r^R s^{k_2-2}|v'|^{q-1}v^\delta\,\mathrm{d}s \\ & - \frac{N}{\mu+1} r^{N-1} u^\mu \int_r^R s^{k_2-2}|u'|^{p-1}\,\mathrm{d}s + N r^{N-1} \int_r^R s^{k_2-2}|u'|^{p-1}u^\mu\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

Now we can prove our main results.

Proof of Theorem 1.5. In view of Proposition 1.1, in order to prove our theorem we only need to prove that under assumption (1.6) or (1.8) system (S_∞) does not possess any radial solution. We will argue by contradiction by assuming that there exists a radially symmetric solution (u, v) to (S_∞). The idea is to have E_1 strictly increasing with $\lim_{r\to 0^+} E_1(r) = 0$ and $\lim_{r\to\infty} E_1(r) = 0$ which will give a contradiction.

We prove first (1), and start by proving that $\lim_{r\to 0} E_1(r) = 0$. Since u and v are regular, a simple application of L'Hôpital's rule gives

$$\lim_{r\to 0}|u'(r)|^{p-1}/r = v(0)^\delta/N \quad \text{and} \quad \lim_{r\to 0}|v'(r)|^{q-1}/r = u(0)^\mu/N.$$

Therefore we need $N + k_1 > 0$, or equivalently, $p > 3N/(2N+1)$. But $p > 2N/(N+1) > 3N/(2N+1)$ if $N > 1$, hence $\lim_{r\to 0} E_1(r) = 0$.

We now verify that $\lim_{r\to\infty} E_1(r) = 0$. We have the bounds near infinity given by

$$u(r) \leqslant Cr^{-\alpha}, \qquad |u'(r)| \leqslant Cr^{-\alpha-1}, \qquad v(r) \leqslant Cr^{-\beta}, \qquad |v'(r)| \leqslant Cr^{-\beta-1}$$

for some $C > 0$ and r large, see [3, Lemma 2.1] or [4, Proposition V.1]. Next, by observing that by the definition of α, β we have

$$1 - \delta\beta = -(\alpha+1)(p-1) \quad \text{and} \quad 1 - \mu\alpha = -(\beta+1)(q-1),$$

in order that $\lim_{r\to\infty} E(r) = 0$ it is sufficient to show that

$$N + k_1 - 2 - (\alpha+1)(p-1) - (\beta+1)(q-1) < 0.$$

This last inequality is equivalent to

$$N + k_1 - p - q < \frac{(p-1)[p(q-1)+\delta q] + (q-1)[q(p-1)+\mu p]}{\delta\mu - (p-1)(q-1)}.$$

Calling $A = \delta + q - 1$, $B = \mu + p - 1$ and $L = N + k_1 - p - q$, this reads

$$L < \frac{q(p-1)A + p(q-1)B}{AB - (p-1)A - (q-1)B},$$

and since the denominator is positive, we have then to prove that

$$L < \frac{(q-1)(L+p)}{\delta + q - 1} + \frac{(p-1)(L+q)}{\mu + p - 1}. \tag{2.5}$$

Since $p \leqslant q \leqslant 2$, we have that $\delta + q - 1 \leqslant \delta + 1$ and $\mu + p - 1 \leqslant \mu + 1$, and thus, using assumption (1.6) (with $\underline{m} = p$), we have (using also that $0 < L + p < L + q$)

$$\frac{(L+p)(N-p)}{N} \leqslant \frac{(q-1)(L+p)}{\delta + q - 1} + \frac{(p-1)(L+q)}{\mu + p - 1}.$$

Therefore we have to prove that $L < (L+p)(N-p)/N$, which is equivalent to $k_1 - q < 0$. Using now that $p < 2$, we have

$$k_1 = p + \frac{N-p}{p-1}(p-2) < q,$$

proving (2.5) and thus $E_1(\infty) = 0$.

We prove next that under the assumptions of the theorem we have $E_1'(r) > 0$ for all $r > 0$. Since by the choice of k_1

$$k_1 - N + \frac{N}{\delta+1} + \frac{N}{\mu+1} = -\frac{(N-p)}{p-1} + \frac{N}{\delta+1} + \frac{N}{\mu+1},$$

we have by assumption (1.6) that the first term in (2.2) is indeed positive. Let us set now

$$G(p,\mu,u)(r) = N\int_r^\infty s^{k_1-2}|u'|^{p-1}u^\mu\,\mathrm{d}s - \frac{N}{\mu+1}u^\mu\int_r^\infty s^{k_1-2}|u'|^{p-1}\,\mathrm{d}s, \tag{2.6}$$

where $k_1 = p + \frac{N-p}{p-1}(p-2)$. With this notation, for $r \in (0,\infty)$, we have that $E_1'(r)$ can be written as

$$\begin{aligned} E_1'(r) = {} & \Big(k_1 - N + \frac{N}{\delta+1} + \frac{N}{\mu+1}\Big) r^{N+k_1-3}|u'|^{p-1}|v'|^{q-1} \\ & + r^{N-1}G(p,\mu,u)(r) + r^{N-1}G(q,\delta,v)(r). \end{aligned} \tag{2.7}$$

By differentiating both sides in (2.6) with respect to r we obtain

$$G'(p,\mu,u)(r) = \Big(-N + \frac{N}{\mu+1}\Big) r^{k_1-2}|u'|^{p-1}u^\mu + \frac{N\mu}{\mu+1}u^{\mu-1}|u'|\int_r^\infty s^{k_1-2}|u'|^{p-1}\,\mathrm{d}s. \tag{2.8}$$

Using now that $(r^{N-1}|u'|^{p-1})' \geqslant 0$, we have $s^{N-1}|u'|^{p-1}(s) \geqslant r^{N-1}|u'|^{p-1}(r)$ for $s \geqslant r$, and consequently, using that $p \leqslant 2$, we find that $(s^{(N-1)/(p-1)}|u'|(s))^{p-2} \leqslant (r^{(N-1)/(p-1)}|u'|(r))^{p-2}$ for $s \geqslant r$. Therefore,

$$\int_r^\infty s^{k_1-2}|u'|^{p-1}\,\mathrm{d}s = \int_r^\infty \big(s^{(N-1)/(p-1)}|u'|(s)\big)^{p-2}\big|u'(s)\big|\,\mathrm{d}s \leqslant r^{k_1-2}u(r)|u'|^{p-2}. \tag{2.9}$$

Replacing (2.9) into (2.8), we obtain

$$G'(p,\mu,u)(r) \leqslant \Big(-N + \frac{N}{\mu+1}(\mu+1)\Big) r^{k-2}|u'|^{p-1}u^\mu = 0,$$

hence $G'(p,\mu,u)(r) \leqslant 0$ for all $r > 0$, and since $G(p,\mu,u)(\infty) = 0$, we have $G(p,\mu,u)(r) \geqslant 0$ for all $r > 0$. Thus the term in the third line in (2.2) is also positive.

Finally we show that $G(q,\delta,v)(r) \geqslant 0$ for all $r > 0$, proving that the term in the second line of (2.2) is also positive. Indeed, we define $\bar{k} = q + \frac{N-q}{q-1}(q-2)$ and note that $k_1 \leqslant \bar{k}$ when $q \geqslant p$. We proceed as above using the following inequality

$$\begin{aligned} \int_r^\infty s^{k_1-2}|v'|^{q-1}\,\mathrm{d}s &= \int_r^\infty s^{k_1-\bar{k}}\big(s^{(N-1)/(q-1)}|v'|(s)\big)^{q-2}\big|v'(s)\big|\,\mathrm{d}s \\ &\leqslant r^{k_1-\bar{k}}\int_r^\infty \big(s^{(N-1)/(q-1)}|v'|(s)\big)^{q-2}\big|v'(s)\big|\,\mathrm{d}s \leqslant r^{k_1-2}v(r)|v'|^{q-2}. \end{aligned}$$

Therefore $E_1'(r) > 0$ for all $r > 0$ in contradiction with $E_1(0^+) = E_1(\infty) = 0$. Thus under the assumptions of the theorem there cannot exist radially symmetric solutions to (S_∞) and we can use Proposition 1.1 to obtain the existence of at least one solution to (S_R) for any positive R.

We next prove (2) and hence we assume $q = \overline{m}$. The proof of $E_1(0^+) = 0$ follows as before. In order to prove that $E_1(\infty) = 0$ we need to prove (2.5) in the case $k_1 = q/(q-1)$. Since $p, q \geqslant 2$, and the function $x \mapsto x/(c+x)$ is strictly increasing in $(0, \infty)$ for any $c > 0$, we have that

$$\frac{p-1}{\mu+p-1} \geqslant \frac{1}{\mu+1} \quad \text{and} \quad \frac{q-1}{\delta+q-1} \geqslant \frac{1}{\delta+1}.$$

Since $q < N$, we have that $L + p = N + k_1 - q > 0$, hence by assumption (1.8), we have

$$\begin{aligned}
\frac{(q-1)(L+p)}{\delta+q-1} + \frac{(p-1)(L+q)}{\mu+p-1} &\geqslant (L+p)\left(\frac{q-1}{\delta+q-1} + \frac{p-1}{\mu+p-1}\right) \\
&\geqslant (L+p)\left(\frac{1}{\delta+1} + \frac{1}{\mu+1}\right) \\
&> (L+p)\left(1 - \frac{k_1}{N}\right),
\end{aligned}$$

and therefore (2.5) will follow if we prove that

$$(L+p)\left(1 - \frac{k_1}{N}\right) \geqslant L. \tag{2.10}$$

Now, (2.10) is equivalent to $(N + k_1 - q)k_1 \leqslant Np$. Since $k_1 \leqslant 2$, $p \geqslant 2$, and $N + k_1 - q \leqslant N$, (2.10) follows and $E_1(\infty) = 0$.

We prove next that $E_1'(r) > 0$ for all $r > 0$. Now the first term in (2.2) is positive by assumption (1.8). We set as before

$$G(p, \mu, u)(r) = N \int_r^\infty s^{k_1-2}|u'|^{p-1}u^\mu \, \mathrm{d}s - \frac{N}{\mu+1}u^\mu \int_r^\infty s^{k_1-2}|u'|^{p-1} \, \mathrm{d}s, \tag{2.11}$$

where now $k_1 = q/(q-1)$, obtaining again that

$$G'(p, \mu, u)(r) = \left(-N + \frac{N}{\mu+1}\right) r^{k_1-2}|u'|^{p-1}u^\mu \tag{2.12}$$

$$+\frac{N\mu}{\mu+1}u^{\mu-1}|u'| \int_r^\infty s^{k_1-2}|u'|^{p-1} \, \mathrm{d}s. \tag{2.13}$$

We claim that $|u'|^{p-1}/r$ is decreasing for all $r > 0$: indeed, since

$$\frac{|u'|^{p-1}}{r} = \frac{1}{r^N} \int_0^r s^{N-1} v^\delta(s) \, \mathrm{d}s,$$

we have that

$$\frac{\mathrm{d}}{\mathrm{d}r} \frac{(|u'|^{p-1})}{r} = \frac{1}{r^N} r^{N-1} v^\delta(r) - N\frac{r^{N-1}}{r^{2N}} \int_0^r s^{N-1} v^\delta(s) \, \mathrm{d}s,$$

and thus, using that v is decreasing in $(0,\infty)$ we find that

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{(|u'|^{p-1})}{r} \leqslant \frac{1}{r^N} r^{N-1} v^\delta(r) - N\frac{r^{N-1}}{r^{2N}}\frac{r^N}{N} v^\delta(r) = 0. \tag{2.14}$$

Since for $\bar{k} = p/(p-1)$, it holds that

$$s^{k_1-2}|u'|^{p-1} = s^{k_1-\bar{k}}\left(\frac{|u'|}{s^{1/(p-1)}}\right)^{p-2}|u'|,$$

and since $p \leqslant q$, also $k_1 \leqslant \bar{k}$. Hence we find that

$$\int_r^\infty s^{k_1-2}|u'|^{p-1}\,\mathrm{d}s \leqslant r^{k_1-2}|u'|^{p-2}\int_r^\infty \bigl|u'(s)\bigr|\,\mathrm{d}s = r^{k_1-2}|u'|^{p-2}u(r).$$

Thus, replacing this estimate into (2.12) we obtain that

$$G'(p,\mu,u)(r) \leqslant \left(-N + \frac{N(\mu+1)}{\mu+1}\right) r^{k_1-2}|u'|^{p-1}u^\mu = 0.$$

Since $G(p,\mu,u)(\infty) = 0$, we have $G(p,\mu,u)(r) \geqslant 0$ for all $r > 0$. Thus the term in the third line of (2.2) is positive.

The same argument, with $\bar{k} = k_1$ can be used to show that $G(q,\delta,v)(r) \geqslant 0$ for all $r > 0$, proving that the term in the second line of (2.2) is also positive and thus $E_1'(r) > 0$ for all $r > 0$. Again we obtain a contradiction and we can use Proposition 1.1 to obtain the existence of at least one solution to (S_R) for any positive R. □

Finally in this section we prove Theorem 1.7.

Proof of Theorem 1.7. We will argue by contradiction assuming that there exists a solution (u,v) to (S_R). Now we will use Lemma 2.2. The idea is to have E_2 decreasing with $E_2(0^+) = 0$ and $E_2(R) > 0$ yielding a contradiction.

We assume $q = \overline{m}$, $p = \underline{m}$, and since the case $p = q = 2$ was proven in [6], we may assume without loss of generality that $q > 2$ for part (1) and $p < 2$ for part (2).

By direct computation we have that

$$E_2(R) = R^{N+k_2-2}\bigl|u'(R)\bigr|^{p-1}\bigl|v'(R)\bigr|^{q-1} > 0.$$

We will show next that $E_2(0^+) = 0$. By [3, Lemma 2.1] or [4, Proposition V.1], we have that any solution (u,v) to (S_R) satisfies

$$u(r) \leqslant Kr^{-\alpha}, \qquad \bigl|u'(r)\bigr| \leqslant Kr^{-\alpha-1}, \qquad v(r) \leqslant Kr^{-\beta}, \qquad \bigl|v'(r)\bigr| \leqslant Kr^{-\beta-1}$$

for some $K > 0$ and $0 < r \ll 1$. Hence in order to show that $E_2(0^+) = 0$ we need

$$N + k_2 - 2 - (\alpha+1)(p-1) - (\beta+1)(q-1) > 0. \tag{2.15}$$

As for (2.5), this last inequality reduces to

$$L > \frac{(q-1)(L+p)}{\delta+q-1} + \frac{(p-1)(L+q)}{\mu+p-1},$$

where $L := N + k_2 - p - q$.

We deal first with the case $2 \leqslant p \leqslant q$. Since in this case $\overline{m} = q$, by assumption (1.9) we have

$$\frac{N-q}{(q-1)N} \geqslant \frac{1}{\delta+1} + \frac{1}{\mu+1}.$$

On the other hand, using that $\delta + q - 1 \geqslant \delta + 1$ and $\mu + p - 1 \geqslant \mu + 1$, and $q \geqslant p$ we have

$$(q-1)(L+q)\left(\frac{1}{\delta+1} + \frac{1}{\mu+1}\right) \geqslant \frac{(q-1)(L+p)}{\delta+q-1} + \frac{(p-1)(L+q)}{\mu+p-1},$$

which implies

$$\frac{(L+q)(N-q)}{N} \geqslant \frac{(q-1)(L+p)}{\delta+q-1} + \frac{(p-1)(L+q)}{\mu+p-1}.$$

Hence in order to prove (2.15) it is sufficient that $L > (L+q)(N-q)/N$. But this is equivalent to prove that $q + \frac{N-q}{q-1}(q-2) > p$, which is clearly true by the assumption $q > 2$ and $q \geqslant p$.

Next we deal with the case $N/(N-1) < p \leqslant q \leqslant 2$. Using again the monotonicity of the function $x \mapsto x/(c+x)$, where $c > 0$, using now that $p - 1 \leqslant 1$ and $q - 1 \leqslant 1$, we find that

$$\frac{p-1}{\mu+p-1} \leqslant \frac{1}{\mu+1} \quad \text{and} \quad \frac{q-1}{\delta+q-1} \leqslant \frac{1}{\delta+1}.$$

Hence by assumption (1.10) and using that $L + q > 0$ we find that

$$\begin{aligned}\frac{(q-1)(L+p)}{\delta+q-1} + \frac{(p-1)(L+q)}{\mu+p-1} &\leqslant (L+q)\left(\frac{q-1}{\delta+q-1} + \frac{p-1}{\mu+p-1}\right)\\ &\leqslant (L+q)\left(\frac{1}{\delta+1} + \frac{1}{\mu+1}\right)\\ &\leqslant (L+q)\left(1 - \frac{k_2}{N}\right).\end{aligned}$$

Hence in order to establish (2.15), it is sufficient that

$$(L+q)\left(1 - \frac{k_2}{N}\right) < L.$$

Since this inequality is equivalent to $Nq < (N + k_2 - p)k_2$, and $q < N$, $k_2 \geqslant 2$, a sufficient condition so that it holds is that $N < N + k_2 - p$, which is clearly true since $p < 2$ and $k_2 = p/(p-1)$.

Finally we prove that $E_2'(r) \leqslant 0$ for all $r \in (0, R)$. To this end we define

$$G(q,\delta,v)(r) = N\int_r^R s^{k_2-2}|v'|^{q-1}v^\delta\,\mathrm{d}s - \frac{N}{\delta+1}v^\delta\int_r^R s^{k_2-2}|v'|^{q-1}\,\mathrm{d}s, \tag{2.16}$$

so that

$$E_2'(r) = r^{N-1}\left(k_2 - N + \frac{N}{\delta+1} + \frac{N}{\mu+1}\right)r^{k_2-2}|u'|^{p-1}|v'|^{q-1} + r^{N-1}\bigl(G(q,\delta,v)(r) + G(p,\mu,u)(r)\bigr). \tag{2.17}$$

We claim that $E_2'(r) \leqslant 0$ for $r \in (0, R)$. Indeed, differentiating in (2.16) with respect to r, we obtain

$$G'(q,\delta,v)(r) = \left(-N + \frac{N}{\delta+1}\right)r^{k_2-2}|v'|^{q-1}v^\delta + \delta\frac{N}{\delta+1}v^{\delta-1}|v'|\int_r^R s^{k_2-2}|v'|^{q-1}\,\mathrm{d}s. \tag{2.18}$$

Assume first that $2 \leqslant p \leqslant q$. Using that $(r^{N-1}|v'|^{q-1})' \geqslant 0$ we have

$$s^{N-1}|v'|^{q-1}(s) \geqslant r^{N-1}|v'|^{q-1}(r) \quad \text{for } s \geqslant r,$$

and consequently, $(s^{(N-1)/(q-1)}|v'|(s))^{q-2} \geqslant (r^{(N-1)/(q-1)}|v'|(r))^{q-2}$ for $s \geqslant r$. Hence using that $k_2 - 2 = \frac{N-1}{q-1}(q-1)$ we obtain

$$\int_r^R s^{k_2-2}|v'|^{q-1}\,\mathrm{d}s = \int_r^R \bigl(s^{(N-1)/(q-1)}|v'|(s)\bigr)^{q-2}\bigl|v'(s)\bigr|\,\mathrm{d}s \geqslant r^{k_2-2}v(r)|v'|^{q-2}. \tag{2.19}$$

Thus replacing (2.19) into (2.18), we get

$$G'(q,\delta,v)(r) \geqslant \left(-N + \frac{N}{\delta+1}(\delta+1)\right)r^{k_2-2}|v'|^{q-1}v^\delta = 0,$$

implying $G'(q,\delta,v)(r) \geqslant 0$ for all $r \in (0, R)$.

Similarly we obtain that $G'(p,\mu,u)(r) \geqslant 0$ for all $r \in (0, R)$. Indeed, we define $\bar{k} = p + \frac{N-p}{p-1}(p-2)$ and note that $k_2 \geqslant \bar{k}$ when $q \geqslant p$. We proceed as before, but using the inequality

$$\begin{aligned}\int_r^R s^{k_2-2}|u'|^{p-1}\,\mathrm{d}s &= \int_r^R s^{k_2-\bar{k}}\bigl(s^{(N-1)/(p-1)}|u'|(s)\bigr)^{p-2}\bigl|u'(s)\bigr|\,\mathrm{d}s \\ &\geqslant r^{k_2-\bar{k}}\int_r^R \bigl(s^{(N-1)/(p-1)}|u'|(s)\bigr)^{p-2}\bigl|u'(s)\bigr|\,\mathrm{d}s \geqslant r^{k_2-2}u(r)|u'|^{p-2}.\end{aligned}$$

This implies $G'(p,\mu,u)(r) \geqslant 0$ for all $r \in (0, R)$.

For the case $N/(N-1) < p \leqslant q \leqslant 2$, we argue as follows: we set $\bar{k} = q/(q-1) \leqslant k_2$ to obtain

$$\int_r^R s^{k_2-2}|v'|^{q-1}\,\mathrm{d}s = \int_r^R s^{k_2-\bar{k}}\left(\frac{|v'|(s)}{s^{1/(q-1)}}\right)^{q-2}\bigl|v'(s)\bigr|\,\mathrm{d}s \geqslant r^{k_2-2}v(r)|v'|^{q-2}, \tag{2.20}$$

hence in this case we find that

$$G'(q,\delta,v)(r) \geqslant \left(-N + \frac{N}{\delta+1}(\delta+1)\right) r^{k_2-2}|v'|^{q-1}v^{\delta} = 0.$$

Similarly, setting $\bar{k} = k_2$, we find that

$$G'(p,\mu,u)(r) \geqslant \left(-N + \frac{N}{\mu+1}(\mu+1)\right) r^{k_2-2}|v'|^{q-1}v^{\delta} = 0.$$

Since $G(q,\delta,v)(R) = G(p,\mu,u)(R) = 0$, we conclude $G(q,\delta,v)(r) \leqslant 0$ and $G(p,\mu,u)(r) \leqslant 0$ for all $r \in (0,R)$.

Now using that (δ,μ) satisfies (1.9), respectively (1.10), $G(q,\delta,v)(r) \leqslant 0$ and $G(p,\mu,u)(r) \leqslant 0$ for all $r \in (0,R]$, by (2.17) we obtain $E_2'(r) < 0$ for all $r \in (0,R]$, which is a contradiction.

Thus the theorem follows. □

Appendix

We start this section by proving Proposition 1.1.

Proof of Proposition 1.1. In order to prove Proposition 1.1, we will make use of the ideas first used in [3] and later in [4]. For the convenience of the reader, we summarize below the results that we shall use. To this end, we define the operator T associated to system (S_R):

For $(u,v) \in C[0,R] \times C[0,R]$, we set

$$T(u,v)(r) = \left(\int_r^R \left(s^{1-N}\int_0^s t^{N-1}\left|v(t)\right|^{\delta}\,\mathrm{d}t\right)^{1/(p-1)} \mathrm{d}s,\right.$$
$$\left.\int_r^R \left(s^{1-N}\int_0^s t^{N-1}|u(t)|^{\mu}\,\mathrm{d}t\right)^{1/(q-1)} \mathrm{d}s\right),$$

and denote by $B(0,s)$, $s > 0$, the open ball in $C[0,R] \times C[0,R]$ of radius s centered at the origin.

T has the following properties:

(A) (i) T maps $C[0,R] \times C[0,R]$ into $C[0,R] \times C[0,R]$.
 (ii) $(u,v) \in C[0,R] \times C[0,R]$ is a solution to (S_R) if and only if $T(u,v) = (u,v)$.
 (iii) T is completely continuous.
 See [3].

(B) Assume (H_1). If $T(u,v) = (u,v)$, $(u,v) \in C[0,R] \times C[0,R]$, and if there exists $\bar{r} \in [0,R)$ such that $(u,v)(\bar{r}) \neq (0,0)$, then for any $r \in (0,R)$ it must be that $u(r), v(r) > 0$ and $u'(r), v'(r) < 0$ [4, Lemma III.1].

(C) Assume (H_1). Then there exists $\rho_1 > 0$ such that
 (i) For any $\rho \in (0,\rho_1]$, $(0,0)$ is the only fixed point of T in $\overline{B}_{\rho}$.
 (ii) For any $\rho \in (0,\rho_1)$, the Leray–Schauder degree $d_{LS}(I-T, B(0,\rho), 0) = 1$ [4, Proposition III.2].

(D) Assume (H_1). Then we can choose $\eta \in ((p-1)(q-1), \delta\mu)$ and set $\theta = \eta/\delta(q-1)$. Then

$$\frac{p-1}{\delta} < \theta < \frac{\mu}{q-1},$$

and for $(u, v) \in C[0, R] \times C[0, R]$ we can define $T_\mu(u, v) = T(u+\mu, v+\mu^\theta)$. As in [4], $T_0 = T$ and $\{T_\mu\}$ is a family of compact operators satisfying the assumptions of the homotopy theorem on any bounded interval $[0, \bar{\mu}]$. If either

(i) there exists $\bar{\mu} > 0$ such that (u_μ, v_μ) is a solution of $T_\mu(u, v) = (u, v)$ for some $\mu \geqslant 0$, then $\mu \leqslant \bar{\mu}$, or

(ii) there exists $M > 0$ such that (u_μ, v_μ) is a solution of $T_\mu(u, v) = (u, v)$ with $\mu \leqslant \bar{\mu} + 1$, then $\|u_\mu\|_\infty + \|v_\mu\|_\infty \leqslant M$,

is not satisfied, then (S) has a radially symmetric solution $(u, v) \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$ such that u, v are decreasing in $(0, \infty)$ and

$$0 < u(r) \leqslant Cr^{-\alpha}, \qquad 0 < v(r) \leqslant Cr^{-\beta},$$

where $\alpha,\ \beta$ are defined by (1.4) [4, Proposition IV.1].

The proof of Proposition 1.1 is now very simple:

Since (S_∞) does not have any nontrivial solution, by [4, Proposition IV.1], both (D)(i) and (D)(ii) must be satisfied. Hence $d_{LS}(I - T_\mu, B(0, M+1), 0)$ is well defined and has value 0 for all $\mu \leqslant \bar{\mu} + 1$, and by the homotopy theorem

$$\begin{aligned} d_{LS}\big(I - T, B(0, M+1), 0\big) &= d_{LS}\big(I - T_0, B(0, M+1), 0\big) \\ &= d_{LS}\big(I - T_{\bar{\mu}+1}, B(0, M+1), 0\big) = 0. \end{aligned}$$

Hence by (C)(ii) and the excision property of the degree, there exists a nontrivial fixed point of T, which by (A)(ii) is a solution to (S_R). □

Remark A.1. Let $u \geqslant 0$, $v \geqslant 0$ satisfy (S_R) for $p = q = m > 1$ and $\mu = \delta > 0$. Then, since from (S_R), one has that u and v satisfy

$$\begin{aligned} 0 &\geqslant \int_B \big(v^\delta(x) - u^\delta(x)\big)\big(u(x) - v(x)\big)\,\mathrm{d}x \\ &= \int_B \big(|\nabla u|^{m-2}\nabla u - |\nabla v|^{m-2}\nabla v\big) \cdot (\nabla u - \nabla v)\,\mathrm{d}x \geqslant 0, \end{aligned}$$

it follows immediately that $u = v$.

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