# Network Games with Atomic Players 

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#### Abstract

We study network and congestion games with atomic players that can split their flow. This type of games readily applies to competition among freight companies, telecommunication network service providers, intelligent transportation systems and manufacturing with flexible machines. We analyze the worst-case inefficiency of Nash equilibria in those games and conclude that although self-interested agents will not in general achieve a fully efficient solution, the loss is not too large. We show how to compute several bounds for the worst-case inefficiency, which depend on the characteristics of cost functions and the market structure in the game. In addition, we show examples in which market aggregation can adversely impact the aggregated competitors, even though their market power increases. When the market structure is simple enough, this counter-intuitive phenomenon does not arise.


## 1 Introduction

In this paper, we study network games with atomic players that can split flow among multiple routes. This type of games readily applies to competition among freight companies, telecommunication network service providers, intelligent transportation systems and, by considering the generalization to congestion games, it also applies to manufacturing with flexible machines. This class of network games was first discussed by [17. More recently, [25, 24, 8] considered a similar model from the perspective of the price of anarchy, which is the framework of this paper.

Consider a directed network $G=(V, A)$ and players that wish to route flow between origin-destination (OD) pairs. We denote the set of all players by $[K]=$ $\{1, \ldots, K\}$. Each player $k \in[K]$ has to choose a flow $x^{k} \in \mathbb{R}_{+}^{A}$ that routes $d_{k}$ units of flow from $s_{k}$ to $t_{k}$. Note that players can divide their flows among many paths. We refer collectively to the flows for all players by $\vec{x}:=\left(x^{1}, \ldots, x^{K}\right)$. In addition, to simplify notation we henceforth let $x:=\sum_{k \in[K]} x^{k}$ be the aggregate flow induced by all $K$ players.

As arcs are subject to adverse congestion effects, we associate a cost function $c_{a}(\cdot): \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$to every arc. These functions map the total flow on the arc $x_{a}$ to its per-unit cost $c_{a}\left(x_{a}\right)$. Cost functions are assumed nondecreasing, differentiable and convex although for some of our results the convexity assumption can
be slightly relaxed. In addition, we only consider separable cost functions, i.e., the cost in one arc only depends on the flow in the same arc. The goal of each competitor is to send its demand minimizing its total $\operatorname{cost} C^{k}(\vec{x}):=\sum_{a \in A} x_{a}^{k} c_{a}\left(x_{a}\right)$. To understand the dependence between the inefficiency of equilibria and the cost functions, we let $\mathcal{C}$ be an arbitrary but fixed set that contains the allowable cost functions. Typical choices include polynomials of degree at most $r$ with $r$ fixed, and the delay functions of $\mathrm{M} / \mathrm{M} / 1$ queues.

A strategy distribution $\vec{x}$ is a Nash equilibrium when no player has an incentive to unilaterally change her strategy. In other words, the best reply strategy for player $k$ is the flow $x^{k}$ that solves the following optimization problem in which flows $x^{i}$ are fixed for $i \neq k$. For ease of notation, we introduce a reverse arc with zero cost between $t^{k}$ and $s^{k}$.

$$
\begin{aligned}
\left(\mathrm{NE}^{k}\right) \quad \min C^{k}(\vec{x}) & \\
\sum_{(u, v) \in A} x_{(u, v)}^{k}-\sum_{(v, w) \in A} x_{(v, w)}^{k} & =0 \quad \text { for all } v \in V \\
x_{\left(t^{k}, s^{k}\right)}^{k} & =d_{k} \\
x_{a}^{k} & \geq 0 \quad \text { for all } a \in A .
\end{aligned}
$$

Note that our assumptions guarantee that these optimization problems are convex, which implies that an equilibrium always exists 22. The uniqueness of equilibria is a longstanding open question, except for some particular cases [17, 1]. Using the convexity of $C^{k}(\vec{x})$ and the first order optimality conditions of $\left(\mathrm{NE}^{k}\right)$, we can characterize equilibria with a variational inequality. Indeed, $\vec{x}$ is at equilibrium if and only if, for all $k \in[K], x^{k}$ solves

$$
\begin{equation*}
\sum_{a \in A} c_{a}^{k}\left(\vec{x}_{a}\right)\left(y_{a}^{k}-x_{a}^{k}\right) \geq 0 \text { for any feasible flow } y^{k} \text { for player } k \tag{1}
\end{equation*}
$$

Here, the modified cost function $c_{a}^{k}\left(\vec{x}_{a}\right):=c_{a}\left(x_{a}\right)+x_{a}^{k} c_{a}^{\prime}\left(x_{a}\right)$ is the derivative with respect to $x_{a}^{k}$ of the term $x_{a}^{k} c_{a}\left(x_{a}\right)$ in $C^{k}(\vec{x})$. Intuitively, the second term accounts for player $k$ 's ability to set prices in the arc.

Following [12], we also consider situations in which some OD pairs are controlled by individual players while others are controlled by infinitely many of them, each in charge of an arbitrary small portion of demand. These games can be viewed as limits of games in which the number of players tends to infinity but some of them retain market power to set prices while others are relegated to be price-takers, and their equilibria can be characterized by a variational inequality similar to (1). The extreme case in which all users are price-takers was first considered by Wardrop [28]. In this case, the game becomes nonatomic and the corresponding solution concept is usually called a Wardrop equilibrium. In other words Wardrop equilibria are those for which all flow-carrying paths are of minimum cost (among all paths serving the same OD pair). Except were otherwise stated, all results in this paper are valid for the three classes of equilibria that we just mentioned because we work with arbitrary market powers.

To measure the quality of equilibria, we need to introduce a social cost function. The most common measure for network models is the sum of the costs among all players. This is easily computed as $C(\vec{x})=C(x):=\sum_{k \in[K]} C^{k}(\vec{x})=$ $\sum_{a \in A} x_{a} c_{a}\left(x_{a}\right)$. (Notice that the total cost does not depend on $\vec{x}$ directly, but rather on the total flow $x$.) A socially optimal flow is a solution $\vec{x}^{\text {opr }}$ that minimizes $C(x)$ among all feasible solutions $\vec{x}$. It is well known that a system optimum can have a strictly lower social cost than an equilibrium 20, 10. Moreover, system optima may even be better for all users compared to an equilibrium [4]; the problem is that users may still have an incentive to deviate from it so it may not be stable.

The main objective of this article is to study how much efficiency is lost when competition arises. To that extent, [16] proposes to use the worst-case inefficiency (in terms of ratio) of the social cost of equilibria with respect to that of social optima as a way to quantify the impact of not being able to coordinate players in a game. This ratio became known as the price of anarchy [18. Roughgarden and Tardos initiated the study of the price of anarchy in network games by proving that the inefficiency loss because of selfish behavior in nonatomic network games with affine cost functions is at most $33 \%$ [25]. Following their work, a series of papers generalized these results to nonatomic network and congestion games, under less and less restrictive assumptions [23, 27, 7, 5, 19]. Finally, [26, 8] generalize the results to arbitrary nonatomic congestion games by noticing that for the characterization of equilibria with variational inequalities, the network structure is unnecessary.

In addition to nonatomic games, [25, 11, 2, 6, consider the atomic case with unsplittable flows, meaning that competitors have to choose a single path to route all the demand from their (single) origins to their (single) destinations.

Our Results. Section 2 shows an upper bound on the price of anarchy for arbitrary networks when costs belong to a given set of cost functions. In addition, we provide a lower bound that arises from a particular instance. Both the lower and upper bounds are strictly higher than the price of anarchy in the nonatomic case implying that price-setting behavior can hurt the system. These are the first bounds of this type for atomic games. Section 3 concentrates in games with a single OD pair. First, we provide bounds on the price of anarchy that depend on the variability of the market power of the different players. To the best of our knowledge, this is the first bound that depends on the market concentration. It measures the price to pay - in the worst case - when going from monopolies to oligopolies to markets with numerous similar players. Then, in Section 3.2, we study the case in which players are symmetric, i.e., they all share the same OD pair and they all control the same amount of flow. In this setting we are able to give a potential function that characterizes equilibria and use this to show that equilibria with atomic players are at least as efficient as the Wardrop equilibrium. All results in this paper generalize to the more general atomic congestion games with divisible demands.

Due to space limitations, several proofs in this extended abstract are omitted or only sketched. More details and further results can be found in the full version of this article available at the authors' website.

## 2 Atomic Games with General Players

In this section we study the price of anarchy for atomic games with arbitrary networks and arbitrary configuration of OD pairs. Players with arbitrarily large market power may coexist with price-taking players. As a warm-up exercise and before considering arbitrary sets $\mathcal{C}$, we derive a bound on the price of anarchy for the case in which cost functions are affine functions. To this end, we define an optimization problem whose first order optimality conditions correspond to the equilibrium conditions. In particular, this optimization problem implies that the equilibrium is essentially unique (although, as 3] points out, this is implied by [22]).

Consider an affine cost function of the form $c(x)=q x+r$. Let us define a modified cost function $\hat{c}: \mathbb{R}_{+}^{K} \rightarrow \mathbb{R}_{+}$by $\hat{c}(\vec{x}):=\frac{q}{2}\left(\sum_{k \in[K]} x^{k}\right)^{2}+\frac{q}{2} \sum_{k \in[K]}\left(x^{k}\right)^{2}+$ $r \sum_{k \in[K]} x^{k}$. It is easy to see that $\hat{c}(\vec{x})$ is convex, or strictly convex when $q>0$. We define Problem (NLP-NE) as the minimization of the potential function $\hat{C}(\vec{x}):=\sum_{a \in A} \hat{c}_{a}\left(\vec{x}_{a}\right)$ among all feasible flows $\vec{x}$. Strict convexity implies that there is a single solution to the previous problem. As its first-order optimality conditions coincide with the conditions that characterize a Nash equilibrium, the latter has to be unique. In addition, (NLP-NE) can be used to approximate a Nash equilibrium up to a fixed additive term in polynomial time [21]. One cannot expect to do better than an additive approximation because an equilibrium may require irrational numbers. We remark that this approach can be easily extended to the setting of games with a mix of atomic and nonatomic players introduced by 12 .

As in other settings [25, 15, this potential function can be used to derive bounds on the price of anarchy but those bounds turn out to be loose. Using the variational inequality displayed in (1), we can prove a stronger upper bound on the price of anarchy for atomic congestion games. The upper bound we provide below originates in [24] using ideas from [7]. Let us define, for $c \in \mathcal{C}$,

$$
\begin{equation*}
\alpha^{K}(c):=\sup _{\vec{x}, \vec{y} \in \mathbb{R}_{+}^{K}} \frac{x c(x)}{y c(y)+\sum_{k \in[K]}\left(x^{k}-y^{k}\right) c^{k}(\vec{x})} . \tag{2}
\end{equation*}
$$

We remind the reader that we have defined $\vec{x}:=\left(x^{1}, \ldots, x^{K}\right)$ and $x:=\sum_{k \in[K]} x^{k}$, and similarly for $\vec{y}$. For this definition and the ones below to work, we shall assume that $0 / 0=0$. Roughgarden proved that $\alpha^{K}(\mathcal{C}):=\sup _{c \in \mathcal{C}} \alpha^{K}(c)$ is an upper bound on the price of anarchy of atomic games [24]. To slightly simplify the calculations, we define

$$
\beta^{K}(c):=\sup _{\vec{x}, \vec{y} \in \mathbb{R}_{+}^{K}} \frac{\sum_{k \in[K]}\left\{\left(c^{k}(\vec{x})-c(y)\right) y^{k}+\left(c(x)-c^{k}(\vec{x})\right) x^{k}\right\}}{x c(x)},
$$



Fig. 1. Example with price of anarchy larger than $4 / 3$
and $\beta^{K}(\mathcal{C}):=\sup _{c \in \mathcal{C}} \beta^{K}(c)$. It is straightforward to see that $\beta^{K}(\mathcal{C}) \geq 0$ and that $\alpha^{K}(\mathcal{C})=\left(1-\beta^{K}(\mathcal{C})\right)^{-1}$ when $\beta^{K}(\mathcal{C})<1$. To simplify notation we will not explicitly distinguish the case of $\beta^{K}(\mathcal{C}) \geq 1$ and assume that $\left(1-\beta^{K}(\mathcal{C})\right)^{-1}=$ $+\infty$ in such a case. We now give a bound on the price of anarchy that depends on $\beta^{K}(\mathcal{C})$. Note that this bound on the price of anarchy is also valid for the mixed atomic and nonatomic games.

Proposition 2.1 ( $[\mathbf{2 4}]$ ). Consider an atomic congestion game with $K$ players and with separable cost functions drawn from $\mathcal{C}$. Let $\vec{x}^{\mathrm{NE}}$ be a Nash equilibrium and $\vec{x}^{\mathrm{OPT}}$ be a social optimum. Then, $C\left(x^{\mathrm{NE}}\right) \leq\left(1-\beta^{K}(\mathcal{C})\right)^{-1} C\left(x^{\text {opr }}\right)$.

Proof. Using (1) and the definition of $\beta^{K}(\mathcal{C})$ in order, we get that

$$
\begin{aligned}
C\left(x^{\mathrm{NE}}\right) & =\sum_{a \in A} \sum_{k \in[K]}\left\{\left(c_{a}\left(x_{a}^{\mathrm{NE}}\right)-c_{a}^{k}\left(\vec{x}_{a}^{\mathrm{NE}}\right)\right) x_{a}^{\mathrm{NE}, k}+c_{a}^{k}\left(\vec{x}_{a}^{\mathrm{NE}}\right) x_{a}^{\mathrm{NE}, k}\right\} \\
& \leq \sum_{a \in A} \sum_{k \in[K]}\left\{\left(c_{a}\left(x_{a}^{\mathrm{NE}}\right)-c_{a}^{k}\left(\vec{x}_{a}^{\mathrm{NE}}\right)\right) x_{a}^{\mathrm{NE}, k}+c_{a}^{k}\left(\vec{x}_{a}^{\mathrm{NE}}\right) x_{a}^{\mathrm{opT}, k}\right\} \leq \beta^{K}(\mathcal{C}) C\left(x^{\mathrm{NE}}\right)+C\left(\vec{x}^{\mathrm{opT}}\right) .
\end{aligned}
$$

Although [24, 8] independently claimed (by providing different proofs) that the price of anarchy in the atomic case cannot exceed that of the nonatomic case, in Fig. 1 we present an instance with affine cost functions that has a price of anarchy larger than $\alpha$ (affine functions) $=4 / 3$. The top OD pair is controlled by a single player while the bottom one is nonatomic. At Nash equilibrium, the common arc has 0.9 and 1 units of demand coming from the atomic and nonatomic OD pairs, respectively, and the total cost is 3.89 . Under the social optimum, the common arc has 1 and 0 units of demand and the total cost is 2.9. Dividing, we get a price of anarchy of approximately 1.341 . Moreover, optimizing over the parameters, we can get an instance with a price of anarchy of approximately 1.343 . Notice that the nonatomic OD pair is not necessary to be worse than in nonatomic games. We could construct a similar example with a finite number of players. That would require replacing the nonatomic OD pair by $K-1$ atomic players, each controlling $1 /(K-1)$ units of demand. If $K$ is large, both equilibria are similar by continuity (e.g., 13] proves that equilibria in atomic games converge to those in nonatomic games when players lose market power).

We now provide a concrete expression for the price of anarchy under specific sets of cost functions. The key is to first obtain a simpler expression for $\beta^{K}(c)$.

Theorem 2.2. Assume that $x c(x)$ is a convex function. Defining $\beta^{\infty}(c):=$ $\sup _{0 \leq y \leq x} \frac{y\left(c(x)-c(y)+c^{\prime}(x) y / 4\right)}{x c(x)}$, we have that $\beta^{K}(c) \leq \beta^{\infty}(c)$.

Proof. Starting from the definition of $\beta^{K}(c)$, we get

$$
\begin{align*}
\beta^{K}(c) & =\sup _{\vec{x}, \vec{y} \in \mathbb{R}_{+}^{K}} \frac{x c(x)-y c(y)+\sum_{k \in[K]} c^{k}(\vec{x})\left(y^{k}-x^{k}\right)}{x c(x)} \\
& =\sup _{\vec{x}, \vec{y} \in \mathbb{R}_{+}^{K}} \frac{y c(x)-y c(y)+c^{\prime}(x)\left(\sum_{k \in[K]} y^{k} x^{k}-\sum_{k \in[K]}\left(x^{k}\right)^{2}\right)}{x c(x)} \tag{3}
\end{align*}
$$

As $c$ is nondecreasing, $c^{\prime}(x) \geq 0$. Thus, assuming w.l.o.g. that $x^{1} \geq x^{k}$ for all $k \in[K]$, to make (3) as big as possible we have to set $\left(y^{1}, \ldots, y^{K}\right)$ to $(y, 0, \ldots, 0)$. It follows that

$$
\begin{equation*}
\beta^{K}(c)=\sup _{\vec{x} \in \mathbb{R}_{+}^{K} ; x^{1}=\max (\vec{x}) ; y \in \mathbb{R}_{+}} \frac{y(c(x)-c(y))+c^{\prime}(x)\left(x^{1} y-\sum_{k \in[K]}\left(x^{k}\right)^{2}\right)}{x c(x)} \tag{4}
\end{equation*}
$$

To find the best choice of $\vec{x}$, it is enough to solve $\max \left\{x^{1} y-\sum_{k \in[K]}\left(x^{k}\right)^{2}: \vec{x} \in\right.$ $\left.\mathbb{R}_{+}^{K}, x^{1}=\max (\vec{x})\right\}$. By symmetry, an optimal solution to this problem satisfies $x^{2}=\cdots=x^{K}$. Therefore, we replace $x_{1}$ by $u$ and the rest of the $x_{k}$ by $v$, and solve

$$
\begin{equation*}
\max _{u \geq v \geq 0 ; u+(K-1) v=x} u y-u^{2}-(K-1) v^{2} . \tag{5}
\end{equation*}
$$

The optimal solution satisfies that $u=\min \{x / K+y(K-1) / 2 K, x\}$. Plugging in $x^{1}=\min \{x / K+y(K-1) / 2 K, x\}$ and $x^{k}=\max \{x / K-y / 2 K, 0\}$ for $k=$ $2, \ldots, K$ in (4), we have that

$$
\beta^{K}(c) \leq \sup _{x, y \in \mathbb{R}_{+}^{2}} \frac{y c(x)-y c(y)+c^{\prime}(x)\left(\frac{y^{2}}{4}-\frac{(x-y / 2)^{2}}{K}\right)}{x c(x)}
$$

Under the convexity assumption, a calculation shows that the optimal solution for the RHS is achieved at $y \leq x$, from where the result follows.

The definition $\beta^{\infty}(\mathcal{C})$ is very similar to that of $\beta(\mathcal{C}):=\sup _{0 \leq y \leq x} \frac{y(c(x)-c(y))}{x c(x)}$, which provides a bound on the price of anarchy for nonatomic games [7]. The only difference between the two expressions is the last term in the numerator of $\beta^{\infty}(\mathcal{C})$, which penalizes equilibria in the case of atomic players. Extending the arguments of [7] to $\beta^{\infty}(\mathcal{C})$, we can prove the following result. In particular, that allows us to conclude that the price of anarchy is at most $3 / 2,2.464$ and 7.826 , for affine, quadratic and cubic cost functions, respectively.

Proposition 2.3. If $\mathcal{C}$ only contains polynomials of degree at most $r$, the price of anarchy is at most $\left(1-\max _{0 \leq u \leq 1} u\left(1-u^{r}+r u / 4\right)\right)^{-1}$.


Fig. 2. Example with pseudo-approximation guarantee larger than 2

### 2.1 Pseudo-approximations

We now concentrate on pseudo-approximation results (also known as bicreteria results) which compare the Nash equilibrium to a social optimum in an instance with expanded demands. Roughgarden and Tardos proved that the social cost of a Wardrop equilibrium is bounded by that of a social optimum of a game with demands doubled [25]. They extended the pseudo-approximation bound to atomic games, which was based on a characterization of equilibria of atomic congestion games. Unfortunately, this characterization is not correct. Figure 2 presents an example for which the Nash equilibrium is more costly than the system optimum with demands doubled. The top OD pair is atomic and the bottom one is nonatomic. Consider $M:=(1-\varepsilon)^{n}+n(1 / 4-\varepsilon)(1-\varepsilon)^{n-1}$, where $\varepsilon$ is such that $(1-\varepsilon)^{n}<1 / n$. The parameters $M$ and $\varepsilon$ are chosen so that the Nash equilibrium is the flow in which the nonatomic demand routes all its $3 / 4$ units of flow in the middle arc and the atomic player splits its flow in $1 / 4-\varepsilon$ along the middle arc and the rest in the other. The social cost of the equilibrium equals $(1-\varepsilon)^{n+1}+(1 / 4+\varepsilon) M$. Consider the flow routing twice the demand in which $\varepsilon$ units of flow take the top arc, $1-\varepsilon$ units take the middle arc, and $3 / 2$ units take the bottom arc. Therefore, the social cost of the system optimum is at most $\varepsilon M+(1-\varepsilon)^{n+1}+3 /(2 n)$. Comparing the two costs, we conclude that in order to find a counterexample we need to find $n$ and $\varepsilon$ such that $n(1-\varepsilon)^{n}<1$ and $M n / 6>1$. This is achieved by taking $\varepsilon=0.1$ and $n=34$. Modifying the example slightly, we can obtain a counterexample with polynomials of degree 26 . On the other hand, if we allow polynomials of arbitrary degree, it can be seen that the cost of the Nash equilibrium can be made arbitrarily higher than that of the system optimum with demands doubled.

In addition, one cannot expect to prove a theorem of this type with a constant expansion factor if arbitrary cost functions are allowed. To see this, consider the same example as in Fig. 2 and a parameter $0<\delta<1$. The nonatomic demand is $1-\delta$, the demand of the atomic player is $2 \delta$, and the cost functions, from top to bottom, are 2 , a step function that is 0 for $x \leq 1$ and 1 otherwise, and 0 . It can be seen that there is one equilibrium with total cost equal to $2 \delta$ while the system optimum when the demand is amplified by $1 /(2 \delta)$ has zero cost. The example can be worked out for polynomial cost functions (of arbitrary high degree). The previous discussion leads us to the following result.

Proposition 2.4. Let $\vec{x}^{\mathrm{NE}}$ be a Nash equilibrium and, for an arbitrary $\alpha>1$, let $\vec{x}^{\text {opr }}$ be a social optimum of the game when demands are multiplied by $\alpha$. Then,
there exists an instance of the atomic network game with convex and increasing cost functions such that $C\left(x^{\mathrm{NE}}\right)>C\left(x^{\mathrm{opT}}\right)$.

In view of the previous negative results, we now prove a pseudo-approximation result for atomic games that hinges on ideas of [8]. The following proposition provides a bound that depends on the allowable cost functions $\mathcal{C}$. For example, in the case of affine cost functions, the expansion factor for which the social cost of equilibria is bounded by that of the expanded system optimum is $4 / 3$.

Proposition 2.5. Let $\vec{x}^{\mathrm{Ne}}$ be a Nash equilibrium of an atomic congestion game with $K$ players and with separable cost functions drawn from $\mathcal{C}$. If $\vec{x}^{\text {opt }}$ denotes a social optimum of the game with demands multiplied by $1+\beta^{K}(\mathcal{C})$, then $C\left(x^{\mathrm{NE}}\right) \leq$ $C\left(x^{\mathrm{opt}}\right)$.

Proof. Consider the flow $\vec{y}=\vec{x}^{\text {ort }}$ that optimally routes $\left(1+\beta^{K}(\mathcal{C})\right) d_{k}$ units of demand from $s_{k}$ to $t_{k}$ for $k \in[K]$. Then,

$$
\begin{aligned}
& C\left(x^{\mathrm{NE}}\right)=\left(1+\beta^{K}(\mathcal{C})\right) \sum_{a \in A} \sum_{k \in[K]}\left\{\left(c_{a}\left(x_{a}^{\mathrm{NE}}\right)-c_{a}^{k}\left(\vec{x}_{a}^{\mathrm{NE}}\right)\right) x_{a}^{\mathrm{NE}}, k\right. \\
&+c_{a}^{k}\left(\vec{x}_{a}^{\mathrm{NE}}\right) x_{a}^{\mathrm{NE}}, k \\
& \leq \beta^{K}(\mathcal{C}) C\left(x^{\mathrm{NE}}\right) \\
& \leq\left(1+\beta^{K}(\mathcal{C})\right) \sum_{a \in A} \sum_{k \in[K]}\left\{\left(c_{a}\left(x_{a}^{\mathrm{NE}}\right)-c_{a}^{k}\left(\vec{x}_{a}^{\mathrm{NE}}\right)\right) x_{a}^{\mathrm{NE}, k}+\frac{c_{a}^{k}\left(\vec{x}_{a}^{\mathrm{NE}}\right) y_{a}^{k}}{1+\beta^{K}(\mathcal{C})}\right\}-\beta^{K}(\mathcal{C}) C\left(x^{\mathrm{NE}}\right),
\end{aligned}
$$

where the inequality follows using (11) with $y_{a}^{k} /\left(1+\beta^{K}(\mathcal{C})\right)$. As $c_{a}\left(x_{a}^{\mathrm{NE}}\right)-c_{a}^{k}\left(\vec{x}_{a}^{\mathrm{NE}}\right) \leq 0$,

$$
\begin{aligned}
C\left(x^{\mathrm{NE}}\right) & \leq \sum_{a \in A} \sum_{k \in[K]}\left\{\left(c_{a}\left(x_{a}^{\mathrm{NE}}\right)-c_{a}^{k}\left(\vec{x}_{a}^{\mathrm{NE}}\right)\right) x_{a}^{\mathrm{NE}}, k\right. \\
& \leq c_{a}^{k}\left(\vec{x}_{a}^{\mathrm{NE}}(\mathcal{C}) C\left(x^{k}\right)+C(y)-\beta^{K}(\mathcal{C}) C\left(x^{\mathrm{NE}}\right)\right. \\
& (\mathcal{C}) C\left(x^{\mathrm{NE}}\right)=C(y)
\end{aligned}
$$

Remark 2.6. Note that the example above shows that for general cost functions (continuous and convex), $\beta^{\infty}(\mathcal{C})$ is unbounded.

## 3 Atomic Games with a Single OD Pair

In this section we concentrate on games played on networks with arbitrary topology in which all $K$ players share the same source $s$ and sink $t$. Single-source single-sink instances are easier to analyze because the total flow can be decomposed by player in an arc-by-arc fashion. This decomposition will allow us to provide improved results compared to the general case. The presentation is divided into two sections: In the first we consider the case in which different players control different amounts of demand, resulting in different market shares. In the second part, we consider the case of symmetric players in which all players have the same demand to route through the network. These two alternatives have previously been considered by [17], although it is assumed that the network only consists of parallel links.

### 3.1 Variable Market Power

We consider the case in which different players have different market power as they control different amounts of demand. To that extent, we define the Herfindahl index by $H:=\sum_{k \in[K]}\left(d_{k} / D\right)^{2}$, where $D:=\sum_{k \in[K]} d_{k}$ is the total demand. This index is a number between $1 / K$ and 1. A higher index means that the market is less competitive, and the case of $H=1$ corresponds to a monopoly. The case in which $H=1 / K$ corresponds to symmetric players (see next section). The following proposition reinterprets the definition of $\beta^{K}(c)$ to improve the bound given by Theorem [2.2. The proof can be found in the full paper.

Proposition 3.1. If we only consider instances with a single OD pair, the constant $\beta^{K}(c)$ is at most $\sup _{0 \leq y \leq x} \frac{y\left(c(x)-c(y)+c^{\prime}(x) y H / 4\right)}{x c(x)}$.

The difference compared to the expression provided by Theorem 2.2 is the factor $H$ in the last term of the numerator. Observe that as $H \leq 1$, this result can only reduce the price of anarchy. Moreover, if each player controls at most a fraction $\phi(K)$ of the demand such that $\phi(K) \rightarrow 0$ when $K \rightarrow \infty$, the price of anarchy is asymptotically equal to that in the nonatomic game. Indeed, the worst case for the market power variability is that there are $1 / \phi(K)$ players, each controlling a fraction $\phi(K)$ of the demand, while the rest of the players control an infinitesimal. In that case $H \leq(1 / \phi(K)) \phi(K)^{2}=\phi(K) \rightarrow 0$. For example, in an oligopoly with $K$ players that control a total demand equal to $K$, but in which $K / \ln K$ players control $\ln K$ units of demand each and the rest of the players do not have market power, the analysis above shows that this oligopoly approaches the nonatomic game when $K$ grows.

For the case of affine cost functions, the price of anarchy can be bounded by $(4-H) /(3-H)$. This generalizes that the price of anarchy is equal to $4 / 3$ for nonatomic games $(H=0)$ and at most $3 / 2$ in general (arbitrary $H$ ). Nevertheless, we know that when $H=1$, the price of anarchy equals 1 . By perturbing the monopolistic case we can show that the price of anarchy for the case of a single OD pair is strictly less than $3 / 2$. However, this analysis is quite technical and it is unlikely to provide a bound that is tight.

Dafermos and Sparrow [9] proved that nonatomic games with general networks and cost functions of the form $c_{a}\left(x_{a}\right)=q_{a} x_{a}^{p}$ for non-negative constants $q_{a}$ and $p$, have fully efficient Nash equilibria. The situation for atomic games is totally different: a carefully constructed instance with linear cost functions (constant times flow) implies that the price of anarchy under linear costs is at least 1.17. For games with a single OD pair, [1] proves that the flow that divides a system optimum proportionally to the market power of different players is a Nash equilibrium, implying that the price of anarchy is 1 .

Providing bounds that depend on the market concentration for multiple OD pairs is an interesting question that our work leaves open. Our techniques do not easily extend to multiple OD pairs because it is not clear how to create a feasible flow arc by arc. Nevertheless, as (11) holds when competitors have to
route from multiple origins to multiple destinations, these results can presented with more general assumptions.

### 3.2 Symmetric Players

When all players have the same demand $d$ to route through the network, 17 shows that there is a unique Nash equilibrium. Our first contribution in this section is to provide a convex optimization problem whose optimum is the unique equilibrium. This implies that the game with symmetric players is a potential game. To facilitate notation, we add a reverse arc between $t$ and $s$ with zero cost.

$$
\begin{align*}
& \min \quad \sum_{a \in A} x_{a} c_{a}\left(x_{a}\right)+(K-1) \sum_{a \in A} \int_{0}^{x_{a}} c_{a}(\tau) d \tau  \tag{SNE}\\
& \sum_{(u, v) \in A} x_{(u, v)}-\sum_{(v, w) \in A} x_{(v, w)}=0 \quad \text { for all } v \in V \\
& x_{(t, s)}=d K \\
& x_{a} \geq 0 \quad \text { for all } a \in A .
\end{align*}
$$

Interestingly, (SNE) consists in finding a feasible flow that minimizes a convex combination between the objective functions of the problems used to compute a system optimum and a Nash equilibrium of a nonatomic game. When there is a single player the second part vanishes leaving the social cost only. Instead, when there are many players the second part is dominant and the social cost becomes negligible. It turns out that a solution is optimal for (SNE) if and only if it is a Nash equilibrium. Therefore, if the cost functions are strictly increasing, there is exactly one Nash equilibrium. By comparing the KKT conditions of (SNE) and ( $\mathrm{NE}^{k}$ ), we get the following result.

Theorem 3.2. If $x$ solves (SNE), then $\vec{x}=(x / K, \ldots, x / K)$ is a Nash equilibrium of the symmetric game with atomic players.

We make use of the potential function to derive results on the efficiency of equilibria and the monotonicity of the cost when the number of players increase.

Proposition 3.3. Let $\vec{x} \in \mathbb{R}_{+}^{K}$ be a Nash equilibrium in an atomic game with $K$ players who control d units of flow each; and let $\vec{y} \in \mathbb{R}_{+}^{\tilde{K}}$ be a Nash equilibrium in an atomic game with $\tilde{K}<K$ players who control $d K / \tilde{K}$ units of flow each. Then, $C(y) \leq C(x)$.

Proof. Using the optimality of $x$ and $y$ in their respective problems,

$$
\begin{gathered}
\sum_{a \in A} x_{a} c_{a}\left(x_{a}\right)+(K-1) \sum_{a \in A} \int_{0}^{x_{a}} c_{a}(\tau) d \tau \leq \sum_{a \in A} y_{a} c_{a}\left(y_{a}\right)+(K-1) \sum_{a \in A} \int_{0}^{y_{a}} c_{a}(\tau) d \tau \\
\leq \sum_{a \in A} x_{a} c_{a}\left(x_{a}\right)+(\tilde{K}-1) \sum_{a \in A} \int_{0}^{x_{a}} c_{a}(\tau) d \tau+(K-\tilde{K}) \sum_{a \in A} \int_{0}^{y_{a}} c_{a}(\tau) d \tau
\end{gathered}
$$

Thus, $\sum_{a \in A} \int_{0}^{x_{a}} c_{a}(\tau) d \tau \leq \sum_{a \in A} \int_{0}^{y_{a}} c_{a}(\tau) d \tau$, from where

$$
\begin{aligned}
\sum_{a \in A} y_{a} c_{a}\left(y_{a}\right)+(\tilde{K}-1) \sum_{a \in A} \int_{0}^{y_{a}} c_{a}(\tau) d \tau & \leq \sum_{a \in A} x_{a} c_{a}\left(x_{a}\right)+(\tilde{K}-1) \sum_{a \in A} \int_{0}^{x_{a}} c_{a}(\tau) d \tau \\
& <\sum_{a \in A} x_{a} c_{a}\left(x_{a}\right)+(\tilde{K}-1) \sum_{a \in A} \int_{0}^{y_{a}} c_{a}(\tau) d \tau
\end{aligned}
$$

The previous propositions imply that the price of anarchy in symmetric games with $K$ players increases as the number of players increases. Furthermore, it approaches the price of anarchy in the nonatomic case when the number of players goes to infinity. In particular, we can show that, for affine cost functions the price of anarchy is exactly $\left(4 K^{2}\right) /(K+1)(3 K-1)$. The conclusion is that for symmetric games non-atomicity does not degrade the quality of equilibria. This stands in clear contrast to the case of atomic asymmetric games whose price of anarchy is larger than that of nonatomic games. Independently of this work, [14] have studied the effect of collusion in network games. In particular, their results imply that, for an atomic game in a parallel link network and divisible demands, the price of anarchy is at most that of the corresponding nonatomic game. The results we just presented have a similar flavor: we have more restrictive assumptions on the players, but our results are valid for arbitrary networks. We believe that a more general result actually holds. Namely, we conjecture that for atomic networks games with splittable flows and a single OD pair, the price of anarchy is at most that of the corresponding nonatomic game.

The results for symmetric players can be generalized to the asymmetric case (but still users that share a single OD pair) if we assume that all players have a positive flow on all arcs ([17] referred to this assumption by "all-positive flows," and proved that in this case there is a unique Nash equilibrium). For those extensions, we just need to decompose the flow in each arc proportionally to the demand of each player (as we have done in Section 3.11). The assumption guarantees that the decomposition is feasible for all players.

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