Wellposedness for the Navier–Stokes flow in the exterior of a rotating obstacle

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SUMMARY

In this paper we study the Navier–Stokes boundary-initial value problem in the exterior of a rotating obstacle, in two and three spatial dimensions. We prove the local in time existence and uniqueness of strong solutions. Moreover, we show that the solutions are global in time, in two spatial dimensions.

KEY WORDS: Navier–Stokes equation; incompressible fluid; rigid bodies

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

We consider a rigid obstacle represented by a closed and bounded set $\mathcal{O}(t)\subset\mathbb{R}^n$, $t\geqslant 0$, $n\in\{2,3\}$, which is rotating in a viscous incompressible fluid occupying the exterior domain $\Omega(t)=\mathbb{R}^n\backslash\mathcal{O}(t)$. Moreover, we assume that the fluid is homogeneous and of density one.

We shall assume that the motion of the fluid is described by the classical Navier–Stokes equations, whereas the obstacle is rotating about the x_3-axis, centred at the origin, with angular velocity $\omega=1$, for $n=2$ and $\omega=(0,0,1)^{\mathrm{t}}$ for $n=3$, respectively. Here and hereafter, the superscript t denotes the transposed of a matrix and all vectors are column ones; $x=(x_1,x_2,x_3)^{\mathrm{t}}$, $\nabla_x=(\partial/\partial x_1,\partial/\partial x_2,\partial/\partial x_3)^{\mathrm{t}}$ and so on. The fact that the obstacle is rotating with constant angular velocity ω implies that the domain occupied by the fluid, respectively, by the solid, at instant t are given by

$$\Omega(t)=\{Q(t)y,\ y\in\Omega(0)\},\quad \mathcal{O}(t)=\{Q(t)y,\ y\in\mathcal{O}(0)\}$$

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where

$$Q(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

for $n = 2$, and where

$$Q(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $n = 3$.

Hence, we can write the full system of equations modelling the motion of the fluid around the rotating obstacle as

$$\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla) v + \nabla q = f, \quad x \in \Omega(t), \; t \in (0, T) \tag{1}$$

$$\operatorname{div} v = 0, \quad x \in \Omega(t), \; t \in (0, T) \tag{2}$$

$$v(x, t) = \omega \times x, \quad x \in \partial \mathcal{O}(t), \; t \in (0, T) \tag{3}$$

$$v(x, 0) = v_0(x), \quad x \in \Omega(0) \tag{4}$$

For $n = 2$, ω is a scalar quantity and the notation $\omega \times x$ is understood as $\omega x^{\perp}$, where $x^{\perp} = \binom{-x_2}{x_1}$ for each $x = \binom{x_1}{x_2} \in \mathbb{R}^2$.

In the above system the unknowns are $v(x, t)$ (the Eulerian velocity field of the fluid), $q(x, t)$ (the pressure of the fluid).

Moreover, $f(x, t)$ is the force acting on the fluid. For x, $y \in \mathbb{R}^n$, the notation $x \cdot y$ stands for the inner product of x and y and $|x|$ stands for the corresponding norm. Moreover, we have denoted by $\partial \mathcal{O}(t)$ the boundary of the rigid body at instant t. The positive constant ν is the viscosity of the fluid.

Only a few results on well posedness of problem (1)–(4) are available. As far as we know, a local in time existence and uniqueness result of 'mild' solutions (in three spatial dimensions) has been proved by Hishida [1] for initial data possessing the same regularity as in the paper of Fujita and Kato [2]. The above mentioned paper uses a non-trivial generalization of the semigroup method of Fujita and Kato [2]. Moreover, very recently, a local in time existence result of strong solutions (in three spatial dimensions) has been proved by Galdi and Silvestre [3]. In this paper, the authors show that if the initial velocity v_0, in a suitable norm, and the magnitude of ω do not exceed a certain constant depending only on the viscosity and on the regularity of $\Omega(0)$, then the solution is global in time. However, the authors do not make any reference to uniqueness properties of the solution. Both works above mentioned deal with the problem by writing the equations of the system in a frame attached to the obstacle.

The aim of this paper is to prove a global in time existence and uniqueness result in the two-dimensional case, and a local in time result in the three-dimensional case, of strong solutions of problem (1)–(4). The method that we use, in order to show the local *existence and* *uniqueness* of strong solutions, is inspired by the approach in Reference [4], where the body undergoes a translation and a rotation which are to be determined from equilibrium conditions on the boundary. The key step of this approach is to use a new change of variables instead of the simple rotation used in most of the previous literature (see, for instance, References [1,3] and the references cited therein). We then prove that the local solution to our problem is actually global in time in two spatial dimensions.

In the sequel, we denote $\Omega=\Omega(0)$ and $\mathcal{O}=\mathcal{O}(0)$. As usual, for $m\in\mathbb{N}$ and for $p\in[1,\infty]$, we denote by $W^{m,p}(\Omega)$ the Sobolev spaces formed by the functions in $L^p(\Omega)$ which have distributional derivatives, up to the order m, in $L^p(\Omega)$, and we set $H^m(\Omega)=W^{m,2}(\Omega)$.

Denote

$$\mathscr{H}^m(t)=[H^m(\Omega(t))]^n, \quad \mathscr{H}^m=\mathscr{H}^m(0)$$

$$\mathscr{L}^p(t)=[L^p(\Omega(t))]^n, \quad \mathscr{L}^p=\mathscr{L}^p(0)$$

Let us define the function spaces

$$L^2(0,T;H^2(\Omega(t))), \quad H^1(0,T;L^2(\Omega(t))), \quad C([0,T],H^1(\Omega(t))) \quad \text{and} \quad L^2(0,T;H^1(\Omega(t)))$$

which will be extensively used in the sequel. To do this, let $\psi:\mathbb{R}^n\times[0,\infty[\to\mathbb{R}^n$ be such that for each $t\geqslant 0$, $\psi|_\Omega(\cdot,t)$ is a C^∞-diffeomorphism from Ω into $\Omega(t)$. Moreover, suppose that the mappings

$$(y,t)\mapsto D_t D_y^\alpha \psi(y,t), \quad \alpha\in\mathbb{N}^n$$

exist, are continuous and of bounded support in Ω. In Section 3, we shall construct a change of variables from Ω into $\Omega(t)$, which satisfies the properties stated above.

Let $v(\cdot,t)$, $t\geqslant 0$ be a family of functions with $v(\cdot,t):\Omega(t)\to\mathbb{R}^n$. Denote $u(y,t)=v(\psi(y,t),t)$, for all $t\geqslant 0$ and for all $y\in\Omega$. Then the functions spaces introduced above are defined by

$$L^2(0,T;H^2(\Omega(t)))=\{v:u\in L^2(0,T;H^2(\Omega))\}$$

$$H^1(0,T;L^2(\Omega(t)))=\{v:u\in H^1(0,T;L^2(\Omega))\}$$

$$C([0,T],H^1(\Omega(t)))=\{v:u\in C([0,T],H^1(\Omega))\}$$

$$L^2(0,T;H^1(\Omega(t)))=\{v:u\in L^2(0,T;H^1(\Omega))\}$$

Moreover, let us denote by $\mathscr{U}(0,T;\Omega(t))$ the space of strong solutions for the velocity, defined by

$$\mathscr{U}(0,T;\Omega(t))=L^2(0,T;\mathscr{H}^2(t))\cap C([0,T],\mathscr{H}^1(t))\cap H^1(0,T;\mathscr{L}^2(t)) \tag{5}$$

Roughly speaking the functions in the above spaces are time-dependent vector fields defined, at each instant t, on the rotating domain $\Omega(t)$ and which lie in classical Sobolev spaces (with respect to the space variable).

Finally, denote by $\hat{H}^1(\Omega)$ the homogeneous Sobolev space

$$\hat{H}^1(\Omega)=\{q\in L^2_{\mathrm{loc}}(\bar{\Omega})\mid \nabla q\in[L^2(\Omega)]^n\}$$

where $q \in L^2_{\text{loc}}(\bar{\Omega})$ means that $q \in L^2(\Omega \cap B_0)$ for all open balls $B_0 \subset \mathbb{R}^n$ with $B_0 \cap \Omega \neq \emptyset$. We identify two functions of $\hat{H}^1(\Omega)$ if they differ by a constant.

The main results of the paper are:

Theorem 1.1
Suppose that $\partial\mathcal{O}$ is a $C^{2+\mu}$-boundary, with $\mu \in (0,1)$. Let $f \in L^2_{\text{loc}}(0,\infty;[W^{1,\infty}(\mathbb{R}^n)]^n)$ and $v_0 \in \mathscr{H}^1$ be such that

$$\begin{cases} \operatorname{div} v_0 = 0 & \text{in } \Omega \\ v_0(x) = \omega \times x, & x \in \partial\mathcal{O} \end{cases}$$

Consider C_0 such that $\|v_0\|_{\mathscr{H}^1} \leqslant C_0$. Then, there exists a time T_0 depending only on C_0 such that Equations (1)–(4) admit a unique strong solution

$$(v,q) \in \mathscr{U}(0,T;\Omega(t)) \times L^2(0,T;\hat{H}^1(\Omega(t)))$$

for any $T \in (0,T_0)$.

Moreover, we can choose T_0 such that one of the following alternatives holds true:

(i) $T_0 = \infty$.
(ii) The function $t \mapsto \|v(t)\|_{\mathscr{H}^1(t)}$ is not bounded in $[0,T_0)$.

Theorem 1.2
Assume that the hypothesis in Theorem 1.1 hold true and suppose that $n=2$. Then, the strong solution given in Theorem 1.1 is global in time, i.e. alternative (i) in Theorem 1.1 holds true.

Remark 1.3
The existence of solutions for problems (1)–(4), with initial data satisfying the same assumptions as in Reference [2], has been investigated in Reference [1], in three space dimensions. In that work, the author proved the local in time existence and uniqueness of 'mild' solutions, i.e. solutions which are more regular than weak ones and less regular than strong ones. Moreover, the main result in Reference [3] yields the existence of a strong solution, without any reference to uniqueness properties. The novelty of our results consists in the fact that we obtain a solution which is more regular than in Reference [1] and that we show its uniqueness.

The plan of this paper is as follows: in Section 2, we sketch the main steps of the proof of our main result. In Section 3 we introduce the change of variables, which plays a central role in Section 4. In Section 4.1, we study the linearized problem associated to (1)–(4). In Section 4.2, we give the estimates needed in order to carry out the fixed point procedure. In Section 4.3 we implement our fixed point procedure to conclude the proof of Theorem 1.1. Finally, in Section 5 we prove that the solution is global in two spatial dimensions.

2. OUTLINE OF THE MAIN PROOFS

For the sake of simplicity, we shall prove Theorem 1.1 in the case where $f \equiv 0$.

The first step in the proof of Theorem 1.1 is to reduce system (1)–(4) to a problem in the cylindrical domain $\Omega \times (0,T)$. To this end, we use a change of variables, which coincides with the simple rotation used in the previous literature (see, for instance, Reference [1]) in a neighbourhood of the rotating body, but it equals to the identity far from the rotating body. We then obtain a system equivalent to (1)–(4), which has the form

$$\frac{\partial u}{\partial t} - \nu(\mathbf{L}u) + (\mathbf{M}u) + (\mathbf{N}u) + (\mathbf{G}p) = 0 \quad \text{in } \Omega \times (0,T)$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega \times (0,T)$$

$$u(y,t) = \omega \times y \quad \text{on } \partial\mathcal{O} \times (0,T)$$

$$u(y,0) = v_0(y), \quad y \in \Omega$$

The unknowns of this system are $u(y,t)$ and $p(y,t)$. $(\mathbf{L}u)$ is the transformed of Δv, $(\mathbf{M}u)$ is a linear term in u and in ∇u, whereas $(\mathbf{N}u)$ is a non-linear term corresponding to $(v \cdot \nabla)v$. All the coefficients depend smoothly on the time t, so $(\mathbf{L}u)$ is close to Δu and $(\mathbf{G}p)$ is close to ∇p for small t.

We next construct a smooth and compactly supported function $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$\operatorname{div} \Lambda = 0 \qquad \text{in } \mathbb{R}^n$$

$$\Lambda(y) = \omega \times y \quad \text{in a ball containing } \mathcal{O}$$

Thus, if we define $U(y,t) = u(y,t) - \Lambda(y)$ and $P(y,t) = p(y,t)$, with (u,p) satisfying the above conditions, then it is easy to verify that (U,P) is the solution of the equivalent problem:

$$(\mathscr{P}) \begin{cases} \dfrac{\partial U}{\partial t} - \nu(\mathbf{L}U) + (\mathbf{M}U) + (\mathbf{N}U) + (\mathbf{B}U) + (\mathbf{G}P) = F & \text{in } \Omega \times (0,T) \\ \operatorname{div} U = 0 & \text{in } \Omega \times (0,T) \\ U(y,t) = 0 & \text{on } \partial\mathcal{O} \times (0,T) \\ U(y,0) = U_0(y), & y \in \Omega \end{cases}$$

where $(\mathbf{B}U)$ is a linear term in U and in ∇U. This term also contains Λ and $\nabla\Lambda$. Moreover, F and U_0 are given by

$$F = \nu(\mathbf{L}\Lambda) - (\mathbf{M}\Lambda) - (\mathbf{N}\Lambda)$$

$$U_0 = v_0 - \Lambda$$

The solutions (U,P) of $(\mathscr{P})$ can be seen as a fixed point of the mapping

$$\mathscr{N} : (W,Q) \mapsto (U,P)$$

where (U,P) is the solution of the Stokes problem:

$$\frac{\partial U}{\partial t} - \nu\Delta U + \nabla P = F \quad \text{in } \Omega \times (0,T)$$

$$\operatorname{div} U = 0 \quad \text{in } \Omega \times (0,T)$$

$$U(y,t)=0 \quad \text{on} \;\; \partial\mathcal{O}\times(0,T)$$
$$U(y,0)=U_0(y), \quad y\in\Omega$$

with

$$F=\nu(\mathbf{L}\Lambda)-(\mathbf{M}\Lambda)-(\mathbf{N}\Lambda)+\nu((\mathbf{L}-\Delta)W)-(\mathbf{M}W)-(\mathbf{N}W)-(\mathbf{B}W)+((\nabla-\mathbf{G})Q)$$

For T_0 small enough, we show that there exists a closed ball $\mathcal{K}$ (in an appropriate Banach space) such that $\mathcal{N}$ maps $\mathcal{K}$ into $\mathcal{K}$ and such that the restriction of $\mathcal{N}$ to this ball is a contraction for all $T\in(0,T_0)$. This will prove the local in time existence and uniqueness of the strong solution. The last step is to prove that, in two spatial dimensions, our solution is global in time, by means of suitable *a priori* estimates.

3. THE TRANSFORMED EQUATIONS

In this section, we describe the change of variables and we state the transformed equations in the cylindrical domain $\Omega\times(0,T)$.

3.1. The change of variables

Denote by $d(\mathcal{O})$ the diameter of the set $\mathcal{O}$ and let $r>d(\mathcal{O})$. Moreover, denote by B_r the open ball centred at the origin and of radius r. Since $\mathcal{O}(t)$ is the image of $\mathcal{O}$ by a rotation centred at the origin, it is clear that

$$\mathcal{O}(t)\subset B_r \quad \forall t\geqslant 0$$

Let $\theta\in C^\infty(\mathbb{R}^n,\mathbb{R})$ be a cut-off function, whose support is contained in B_{2r}, and with $\theta\equiv 1$ in $\overline{B_r}$. In the sequel, we shall use the function $\Lambda:\mathbb{R}^n\to\mathbb{R}^n$ defined by

$$\Lambda(x)=-\frac{1}{2}\,\mathrm{curl}(\theta|x|^2\omega)=\begin{cases}\begin{pmatrix}-\dfrac{1}{2}\dfrac{\partial\theta}{\partial x_2}|x|^2-\theta x_2\\[2ex] \dfrac{1}{2}\dfrac{\partial\theta}{\partial x_1}|x|^2+\theta x_1\end{pmatrix} & \text{for } n=2\\[6ex] \begin{pmatrix}-\dfrac{1}{2}\dfrac{\partial\theta}{\partial x_2}|x|^2-\theta x_2\\[2ex] \dfrac{1}{2}\dfrac{\partial\theta}{\partial x_1}|x|^2+\theta x_1\\[2ex] 0\end{pmatrix} & \text{for } n=3\end{cases} \tag{6}$$

Λ is clearly a C^∞ function. Moreover, we have the following result, whose proof can be easily obtained from (6).

Lemma 3.1
The mapping Λ defined by (6) has the following properties:

1. $\Lambda = 0$ outside B_{2r}.
2. $\operatorname{div} \Lambda = 0$ in $\mathbb{R}^n$.
3. $\Lambda(x) = \omega \times x$, if $x \in \mathcal{O}(t)$ and $t \geqslant 0$.

Next, consider the time-dependent vector field $X(\cdot, t)$ satisfying

$$\begin{aligned} \frac{\partial X}{\partial t}(y,t) &= \Lambda(X(y,t)), \quad t>0 \\ X(y,0) &= y \in \mathbb{R}^n \end{aligned} \tag{7}$$

with Λ given by (6).

Lemma 3.2
For all $y \in \mathbb{R}^n$, the initial-value problem (7) admits a unique solution $X(y,\cdot): [0,\infty) \to \mathbb{R}^n$, which is a C^∞ function in $\mathbb{R}^n \times [0,\infty)$. Moreover, for all $t \geqslant 0$, we have that the mapping $y \mapsto X(y,t)$ is a C^∞-diffeomorphism from $\mathbb{R}^n$ onto itself and from Ω onto $\Omega(t)$. Furthermore, its inverse $Y(x,t)$ is also a C^∞ function in $\mathbb{R}^n \times [0,\infty[$.

Proof
Since Λ is a C^∞ function, from the classical Cauchy–Lipschitz–Picard Theorem, it follows that (7) admits a unique maximal solution $X(y,\cdot)$, defined, say, on $[0,T_1[$, which is a C^∞ function in $[0,T_1)$.

Moreover, since $\Lambda = 0$ outside B_{2r} it clearly follows that the solution X of (7) does not blow up in finite time.

The global existence and uniqueness of the solutions of (7) imply that, for all $t \geqslant 0$, $X(\cdot,t)$ is one to one and onto in $\mathbb{R}^n$.

Moreover, since Λ is a C^∞ mapping, from a classical result (see, for instance, Reference [5, Corollary 4.1, p. 101]), we obtain that X is a C^∞ mapping from $\mathbb{R}^n \times [0,\infty)$ onto $\mathbb{R}^n$.

Furthermore, since the inverse $Y(\cdot,t)$ satisfies the similar initial-value problem,

$$\begin{aligned} \frac{\partial Y}{\partial t}(x,t) &= -\Lambda(Y(x,t)), \quad t>0 \\ Y(x,0) &= x \in \mathbb{R}^n \end{aligned} \tag{8}$$

it follows that $X(\cdot,t)$ is a C^∞-diffeomorphism from $\mathbb{R}^n$ onto itself.

On the other hand, we can check that for all $y \in \mathcal{O}$, the function

$$\tilde{X}(y,t) = Q(t)y$$

satisfies (7). In fact, since $\tilde{X}(y,t) \in \mathcal{O}(t)$ (by the definition of $\mathcal{O}(t)$), and by using assertion 3 in Lemma 3.1, we have that $\Lambda(\tilde{X}(y,t)) = \omega \times \tilde{X}(y,t)$. Thus, by means of simple calculations, we see that if $y \in \mathcal{O}$ and $t \geqslant 0$ then

$$\frac{\partial \tilde{X}}{\partial t}(y,t) = Q(\omega \times y) = (Q\omega) \times (Qy) = \omega \times \tilde{X}(y,t) = \Lambda(\tilde{X}(y,t))$$

Therefore, by using again the uniqueness of the solution of (7), we get that for all $t \geqslant 0$, $X(\cdot,t)(\mathcal{O}) \subset \mathcal{O}(t)$ and similarly that $Y(\cdot,t)(\mathcal{O}(t)) \subset \mathcal{O}$. Hence, for all $t \geqslant 0$, $X(\cdot,t)(\mathcal{O}) = \mathcal{O}(t)$ and thus we get that $X(\cdot,t): \Omega \to \Omega(t)$ is a C^∞-diffeomorphism. □

Our change of variables satisfies the following useful property, which follows from the condition $\operatorname{div} \Lambda = 0$, via a classical result due to Liouville (see, for instance, Reference [6, p. 249]).

Lemma 3.3
Let X be as in Lemma 3.2 and denote, for each $t \geqslant 0$:

$$J_X = \left(\frac{\partial X_i}{\partial y_j} \right)_{i,j}, \quad i,j \in \{1, \ldots, n\}$$

the Jacobian matrix of the transformation $y \mapsto X(y,t)$. Then

$$\det J_X(y,t) = 1 \quad \forall y \in \mathbb{R}^n \quad \forall t \geqslant 0$$

3.2. The equations in the cylindrical domain

Define

$$u(y,t) = J_Y(X(y,t),t) v(X(y,t),t) \quad \forall y \in \Omega \quad \forall t \geqslant 0 \tag{9}$$

i.e. for each $i \in \{1, \ldots, n\}$

$$u_i(y,t) = \sum_{j=1}^{n} \frac{\partial Y_i}{\partial x_j}(X(y,t),t) v_j(X(y,t),t)$$

and

$$p(y,t) = q(X(y,t),t) \quad \forall y \in \Omega \quad \forall t \geqslant 0 \tag{10}$$

where X and Y are as in Lemma 3.2.

By using the fact that

$$J_X(y,t) J_Y(X(y,t),t) = \mathrm{Id} \quad \forall (y,t) \in \Omega \times [0,\infty[$$

where Id is the identity matrix, we obtain the following result:

Lemma 3.4
Suppose that X and u are defined as before. Then,

$$\operatorname{div} u(y,t) = \operatorname{div} v(X(y,t),t) \quad \forall (y,t) \in \Omega \times [0,\infty[$$

For a proof see, for instance, Reference [7, Proposition 2.4].

In order to write the equations satisfied by $u(y,t)$ and $p(y,t)$ we define for each $i \in \{1, \ldots, n\}$

$$(\mathbf{L}u)_i = \sum_{j,k=1}^{n} \frac{\partial}{\partial y_j} \left(g^{jk} \frac{\partial u_i}{\partial y_k} \right) + 2 \sum_{j,k,l=1}^{n} g^{kl} \Gamma^i_{jk} \frac{\partial u_j}{\partial y_l}$$
$$+ \sum_{j,k,l=1}^{n} \left\{ \frac{\partial}{\partial y_k} (g^{kl} \Gamma^i_{jl}) + \sum_{m=1}^{n} g^{kl} \Gamma^m_{jl} \Gamma^i_{km} \right\} u_j \tag{11}$$

$$(\mathbf{N}u)_i = \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial y_j} + \sum_{j,k=1}^{n} \Gamma^i_{jk} u_j u_k \tag{12}$$

$$(\mathbf{M}u)_i = \sum_{j=1}^{n} \frac{\partial Y_j}{\partial t} \frac{\partial u_i}{\partial y_j} + \sum_{j,k=1}^{n} \left\{ \Gamma^i_{jk} \frac{\partial Y_k}{\partial t} + \frac{\partial Y_i}{\partial x_k} \frac{\partial^2 X_k}{\partial t \partial y_j} \right\} u_j \tag{13}$$

$$(\mathbf{G}p)_i = \sum_{j=1}^{n} g^{ij} \frac{\partial p}{\partial y_j} \tag{14}$$

where, for each $i, j, k \in \{1, \ldots, n\}$, we have denoted (see, for instance, Reference [8])

$$g^{ij} = \sum_{k=1}^{n} \frac{\partial Y_i}{\partial x_k}(X(y,t),t) \frac{\partial Y_j}{\partial x_k}(X(y,t),t) \quad \text{(metric contravariant tensor)} \tag{15}$$

$$g_{ij} = \sum_{k=1}^{n} \frac{\partial X_i}{\partial y_k}(y,t) \frac{\partial X_j}{\partial y_k}(y,t) \quad \text{(metric covariant tensor)} \tag{16}$$

and

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left\{ \frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_l} \right\} \quad \text{(Christoffel's symbol)} \tag{17}$$

With this notation, we have

Proposition 3.5
The pair (v, q) satisfies

$$(v,q) \in \mathcal{U}(0,T;\Omega(t)) \times L^2(0,T;\hat{H}^1(\Omega(t)))$$

together with (1)–(4) if and only if the pair (u, p) defined by (9)–(10) satisfies the condition

$$(u,p) \in \mathcal{U}(0,T;\Omega) \times L^2(0,T;\hat{H}^1(\Omega))$$

together with

$$\frac{\partial u}{\partial t} - v(\mathbf{L}u) + (\mathbf{M}u) + (\mathbf{N}u) + (\mathbf{G}p) = 0 \quad \text{in } \Omega \times (0,T) \tag{18}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega \times (0,T) \tag{19}$$

$$u(y,t) = \omega \times y \quad \text{on } \partial\mathcal{O} \times (0,T) \tag{20}$$

$$u(y,0) = v_0(y), \quad y \in \Omega \tag{21}$$

Proof
The equivalence between (1) and (18) has been established in Theorem 2.5 from Inoue–Wakimoto [7]. The equivalence between (2) and (19) follows from Lemma 3.4. The facts that (4) is equivalent to (21) follows directly from the change of variables. We still have to show that (3) is equivalent to (20).

To this end, consider $y \in \mathcal{O}$. Then, the corresponding solution to problem (7) is $X(y,t) = Q(t)y$. Thus, for any $y \in \mathcal{O}$, we have that

$$J_X(y,t) = Q(t), \quad J_Y(X(y,t),t) = Q^t(t)$$

In particular, for all $(y,t) \in \partial\mathcal{O} \times (0,T)$, we have that

$$u(y,t) = J_Y(X(y,t),t)v(X(y,t),t) = Q^t(t)[\omega \times X(y,t)]$$
$$= \begin{cases} \omega Q^t(t)[Q(t)y]^{\perp} = \omega y^{\perp} & \text{if } n=2 \\ [Q^t(t)\omega] \times [Q^t(t)Q(t)y] = \omega \times y & \text{if } n=3 \end{cases}$$

This concludes the proof of the proposition. □

4. PROOF OF THEOREM 1.1

4.1. *The linearized problem*

The aim of this section is to prove our local in time existence and uniqueness result for (1)–(4), i.e. Theorem 1.1. We recall that we have assumed that $f \equiv 0$. From Proposition 3.5 the local in time existence for system (1)–(4) is equivalent to the local existence of a solution for (18)–(21).

In the remaining part of this work, we denote by $\mathcal{U}$ the Banach space

$$\mathcal{U} = L^2(0,T;\mathcal{H}^2) \cap C([0,T],\mathcal{H}^1) \cap H^1(0,T;\mathcal{L}^2) \tag{22}$$

endowed with the norm

$$\|U\|_{\mathcal{U}} = \|U\|_{L^2(0,T;\mathcal{H}^2)} + \|U\|_{L^\infty(0,T;\mathcal{H}^1)} + \|U\|_{H^1(0,T;\mathcal{L}^2)}$$

Let (u,p) be a solution of (18)–(21) and we put $U(y,t) = u(y,t) - \Lambda(y)$, $P(y,t) = p(y,t)$, where Λ is given by (6). It is easy to check that (U,P) satisfies:

$$\frac{\partial U}{\partial t} - \nu(\mathbf{L}U) + (\mathbf{M}U) + (\mathbf{N}U) + (\mathbf{B}U) + (\mathbf{G}P) = F \quad \text{in } \Omega \times (0,T) \tag{23}$$

$$\operatorname{div} U = 0 \quad \text{in } \Omega \times (0,T) \tag{24}$$

$$U(y,t) = 0 \quad \text{on } \partial\mathcal{O} \times (0,T) \tag{25}$$

$$U(y,0) = U_0(y), \quad y \in \Omega \tag{26}$$

where, for each $i \in \{1,\ldots,n\}$, we have denoted

$$(\mathbf{B}U)_i = \sum_{j=1}^{n} \left(U_j \frac{\partial \Lambda_i}{\partial y_j} + \Lambda_j \frac{\partial U_i}{\partial y_j} \right) + 2 \sum_{j,k=1}^{n} \Gamma^i_{j,k} U_j \Lambda_k \tag{27}$$

and

$$\begin{aligned} F &= \nu(\mathbf{L}\Lambda) - (\mathbf{M}\Lambda) - (\mathbf{N}\Lambda) \\ U_0 &= v_0 - \Lambda \end{aligned} \tag{28}$$

Thus, problem (18)–(21) is equivalent to problem (23)–(28), which has homogeneous boundary conditions.

The solution of the above problem can be seen as a fixed point of the mapping $\mathcal{N}:(W,Q)\mapsto(U,P)$, defined from

$$\mathscr{U}\times L^2(0,T;\hat{H}^1(\Omega))$$

onto itself, where (U,P) satisfies the classical Stokes system

$$\frac{\partial U}{\partial t} - \nu\Delta U + \nabla P = F \quad \text{in } \Omega\times(0,T) \tag{29}$$

$$\operatorname{div} U = 0 \quad \text{in } \Omega\times(0,T) \tag{30}$$

$$U(y,t) = 0 \quad \text{on } \partial\mathcal{O}\times(0,T) \tag{31}$$

$$U(y,0) = U_0(y), \quad y\in\Omega \tag{32}$$

with

$$F = \nu(\mathbf{L}\Lambda) - (\mathbf{M}\Lambda) - (\mathbf{N}\Lambda) + \nu((\mathbf{L}-\Delta)W) - (\mathbf{M}W) - (\mathbf{N}W) - (\mathbf{B}W) + ((\nabla-\mathbf{G})Q) \tag{33}$$

and with the operators **L**, **M**, **N**, **G** defined by (11)–(17) and **B** defined by (27).

The main result of this subsection yields the existence, the uniqueness and an estimate of the solutions for linear problem (29)–(32):

Proposition 4.1
Let $T>0$, $F\in L^2(0,T;\mathscr{L}^2)$ and $U_0\in\mathscr{H}^1$ be such that

$$\operatorname{div} U_0 = 0 \quad \text{in } \Omega$$

$$U_0 = 0 \quad \text{on } \mathcal{O}$$

Then, the linear problem (29)–(32) admits a unique strong solution

$$U\in\mathscr{U}, \quad P\in L^2(0,T;\hat{H}^1(\Omega))$$

Moreover, (U,P) satisfies

$$\begin{aligned} &\|U\|_{L^2(0,T;\mathscr{H}^2)} + \|U\|_{L^\infty(0,T;\mathscr{H}^1)} + \|U\|_{H^1(0,T;\mathscr{L}^2)} + \|\nabla P\|_{L^2(0,T;\mathscr{L}^2)} \\ &\quad\leqslant C\exp(T)(\|U_0\|_{\mathscr{H}^1} + \|F\|_{L^2(0,T;\mathscr{L}^2)}) \end{aligned} \tag{34}$$

for some constant $C>0$ depending only on Ω and on ν.

In order to prove this result, we shall use a semigroup approach.

We have to introduce some function spaces:

$$\mathscr{V}(\Omega) = \{\phi \in C_0^\infty(\Omega, \mathbb{R}^n) \mid \operatorname{div} \phi = 0\}$$

$$\mathbb{H} = \text{the closure of } \mathscr{V}(\Omega) \text{ in } \mathscr{L}^2$$

$$\mathbb{V} = \text{the closure of } \mathscr{V}(\Omega) \text{ in } \mathscr{H}^1$$

According to classical results (see, for instance, Reference [9]), we have that

$$\mathbb{H} = \{v \in \mathscr{L}^2 \mid \operatorname{div} v = 0 \text{ in } \mathscr{D}'(\Omega),\ v \cdot n = 0 \text{ in } H^{-1/2}(\partial\Omega)\}$$

$$\mathbb{V} = \{v \in \mathscr{H}_0^1 \mid \operatorname{div} v = 0 \text{ in } L^2(\Omega)\}$$

Let $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ be the linear operator defined by

$$D(A) = \mathscr{H}^2 \cap \mathbb{V} \tag{35}$$

$$AU = \mathbb{P}(-v\Delta U) \quad \forall U \in D(A) \tag{36}$$

where $\mathbb{P} : \mathscr{L}^2 \to \mathbb{H}$ denotes the orthogonal projector on the Hilbert space $\mathbb{H}$. We have that

Proposition 4.2
The operator A defined by (35) and (36) is selfadjoint and positive. Moreover, for any $U \in D(A)$, we have that

$$\|U\|_{\mathscr{H}^2} \leqslant C\|(I + A)U\|_{\mathscr{L}^2} \tag{37}$$

for some constant $C > 0$ depending only on Ω and on v.

The proof of this proposition can be easily obtained by means of standard arguments and by using Theorem 2.1 from Reference [10], so that we skip it.

Proof of Proposition 4.1
First, we remark that $D(A^{1/2}) = \mathbb{V}$. In fact, the graph norm of $D(A^{1/2})$ is

$$\|U\|^2_{D(A^{1/2})} = (U, U)_{\mathscr{L}^2} + (AU, U)_{\mathscr{L}^2} = \|U\|^2_{\mathscr{L}^2} + v\|\nabla U\|^2_{[L^2(\Omega)]^{n^2}}$$

for all $U \in D(A)$. Thus, this norm is equivalent to $\mathscr{H}^1$ norm.

On the other hand, since $\mathscr{V}(\Omega) \subset D(A) \subset \mathbb{V}$, we conclude that $D(A^{1/2}) = \mathbb{V}$.

Therefore, $U_0 \in D(A^{1/2})$ and by applying Lemma 3.3 and Theorem 3.1 from Reference [11] together with Proposition 4.2, we obtain that the initial-value problem

$$U' + AU = \mathbb{P}F, \quad U(0) = U_0 \tag{38}$$

admits a unique solution $U \in L^2(0, T; D(A)) \cap C([0, T], D(A^{1/2})) \cap H^1(0, T; \mathbb{H})$. Moreover, there exists $K > 0$, depending on Ω, on v and on T, which is non-decreasing with respect to T, such that

$$\|U\|_{L^2(0,T;D(A))} + \|U\|_{C([0,T],D(A^{1/2}))} + \|U\|_{H^1(0,T;\mathbb{H})} \leqslant K(\|U_0\|_{D(A^{1/2})} + \|\mathbb{P}F\|_{L^2(0,T;\mathbb{H})}) \tag{39}$$

In our particular case, by multiplying the first equation of (38) by U and by AU, successively, it is easy to prove that $K = C\exp(T)$, for some constant $C > 0$ depending only on Ω and on ν.

On the other hand, if we multiply the first equation of (38) by $\phi \in \mathscr{V}(\Omega)$, we obtain that

$$\int_\Omega [U'(t) - \nu\Delta U(t) - F(t)] \cdot \phi \, \mathrm{d}y = 0, \quad \text{a.e. in } (0,T)$$

Thus, by applying Propositions 1.1 and 1.2 from Reference [12], we obtain the existence of $P \in L^2(0,T;\hat{H}^1(\Omega))$ such that

$$U'(t) - \nu\Delta U(t) + \nabla P(t) = F(t) \quad \text{in } \Omega \times (0,T) \tag{40}$$

Moreover, since $U(t) \in \mathbb{V}$, we have that

$$\operatorname{div} U = 0 \quad \text{in } \Omega \times (0,T)$$

$$U = 0 \quad \text{on } \mathcal{O} \times (0,T)$$

The above relations and (40) imply that (U,P) is a strong solution to problem (29)–(32). In order to prove the uniqueness of the solution, it suffices to remark that all solution of (29)–(32) is also solution of (38), whose solutions are unique. Finally, estimate (34) follows directly from (39) and (37). □

4.2. *Estimates on the coefficients*

The aim of this subsection is to provide some estimates on the operators $\mathbf{L}, \mathbf{M}, \mathbf{N}, \mathbf{G}$ defined by (11)–(17) and $\mathbf{B}$ defined by (27). These estimates are essential in our fixed point procedure.

The basic ingredient in the definition of the operators $\mathbf{L}, \mathbf{M}, \mathbf{N}, \mathbf{G}$ and $\mathbf{B}$ is the vector field Λ. From definition (6) of Λ, it clearly follows that there exists a constant $K = K(r) > 0$ verifying

$$\|D^\alpha \Lambda\|_{[L^\infty(\mathbb{R}^n)]^n} \leqslant K, \quad \forall \alpha \in \mathbb{N}^n, \quad |\alpha| \leqslant 3 \tag{41}$$

The result below yields estimates of the change of variables mappings X and Y.

Lemma 4.3

Let $T > 0$. Then, there exists a constant $K = K(r) > 0$ such that the function X defined by (7) satisfies:

$$\left\| \frac{\partial X_i}{\partial y_j} \right\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leqslant \exp(KT)$$

$$\left\| \frac{\partial^2 X_i}{\partial y_j \partial y_k} \right\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leqslant KT \exp(2KT)$$

$$\left\| \frac{\partial^3 X_i}{\partial y_j \partial y_k \partial y_l} \right\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leqslant KT \exp(2KT)[1 + 2KT\exp(KT)]$$

for all $i, j, k, l \in \{1, \ldots, n\}$.

Moreover, the same estimates above are still valid if we replace $\partial X_i/\partial y_j$ by $\partial Y_i/\partial x_j$, $\partial^2 X_i/\partial y_j \partial y_k$ by $\partial^2 Y_i/\partial x_j \partial x_k$ and $\partial^3 X_i/\partial y_j \partial y_k \partial y_l$ by $\partial^3 Y_i/\partial x_j \partial x_k \partial x_l$, respectively.

Proof
For each $j \in \{1, \ldots, n\}$ define $z_j(y,t) = (\partial X/\partial y_j)(y,t)$. By differentiating both sides of Equation (7) with respect to y_j, we have that z_j satisfies

$$\begin{aligned} \frac{\partial z_j}{\partial t} &= J(y,t) z_j(y,t), \quad t > 0 \\ z_j(y,0) &= e_j \end{aligned} \tag{42}$$

where $J(y,t) = ((\partial \Lambda_i/\partial x_j)(X(y,t)))_{ij}$ and $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.
Thus, from the Gronwall lemma together with (41), we deduce from (42) that

$$\|z_j\|_{[L^\infty(\mathbb{R}^n \times (0,T))]^n} \leqslant \exp(KT) \tag{43}$$

which proves the estimates for the first derivatives of X.
Next, by differentiating Equation (42) with respect to y_k, we obtain again from the Gronwall lemma combined to (41) and (43), the estimates for the second derivatives of X. The proof of the estimates for the third derivatives is similar.
Finally, the proof of the estimates for the derivatives of Y is similar, so we skip it. □

The lemma above allows us to get new estimates:

Corollary 4.4
There exists a constant $K = K(r) > 0$ such that for all $m, l \in \{1, \ldots, n\}$

$$\left\| \frac{\partial X_m}{\partial y_l} - \delta_{ml} \right\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leqslant nKT \exp(KT) \tag{44}$$

$$\|g_{ml} - \delta_{ml}\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leqslant nKT \exp(KT)[1 + n \exp(KT)] \tag{45}$$

where δ_{ml} denotes the Kronecker's symbol.
Moreover, the same estimates above are still valid if we replace $\partial X_m/\partial y_l$ by $\partial Y_m \partial x_l$ and g_{ml} by g^{ml}, respectively.

Proof
From the definition of the function $X(y,t)$, we have that for each $m, l \in \{1, \ldots, n\}$

$$\frac{\partial X_m}{\partial y_l}(y,0) = \delta_{ml}$$

Therefore, from the mean value theorem, and since X is a C^2 function, we get that for any $t \in (0,T)$, there exists $\xi \in (0,t)$ such that

$$\begin{aligned} \frac{\partial X_m}{\partial y_l}(y,t) - \delta_{ml} &= \frac{\partial^2 X_m}{\partial t \partial y_l}(y,\xi)(t-0) = \frac{\partial \Lambda_m}{\partial y_l}(X(y,\xi))t \\ &= t \sum_{k=1}^{n} \frac{\partial \Lambda_m}{\partial x_k}(X(y,\xi)) \frac{\partial X_k}{\partial y_l}(y,\xi) \end{aligned}$$

Thus, from Lemma 4.3 together with (41), the above equality yields estimates (44).

Moreover, from the definition of g_{ml}, we have that

$$|g_{ml}-\delta_{ml}|=\left|\sum_{k=1}^{n}\left(\frac{\partial X_m}{\partial y_k}-\delta_{mk}\right)\frac{\partial X_l}{\partial y_k}+\sum_{k=1}^{n}\delta_{mk}\left(\frac{\partial X_l}{\partial y_k}-\delta_{kl}\right)\right|$$

Therefore, from Lemma 4.3 together with estimate (44), the above equation yields estimate (45). Similarly, we obtain the estimates for the first derivatives of Y and for g^{ml}, respectively. □

4.3. Proof of the local existence and uniqueness result

From Proposition 4.1, we have that the definition of the mapping $\mathcal{N}$ makes sense. For $T,R>0$ define

$$\mathcal{K}=\{(W,Q)\in\mathcal{U}\times L^2(0,T;\hat{H}^1(\Omega))\,|\,\|W\|_{\mathcal{U}}+\|\nabla Q\|_{L^2(0,T;\mathcal{L}^2)}\leqslant R\} \tag{46}$$

In the rest of this subsection, we shall denote by K a constant satisfying the following:

(K) K is positive and depends only on r (with $r>d(\mathcal{O})$) and on n (the spatial dimension).

The aim of this subsection is to prove that for T_0 small enough, and for R large enough, we have that $\mathcal{N}(\mathcal{K})\subset\mathcal{K}$ and that $\mathcal{N}:\mathcal{K}\to\mathcal{K}$ is a contraction for any $T\in(0,T_0)$.

Lemma 4.5

Suppose that $(W,Q)\in\mathcal{K}$, and that $\mathbf{L},\mathbf{M},\mathbf{G}$ and $\mathbf{B}$ are given by (11), (13), (14) and (27), respectively. Then, there exist constants K,K_1 satisfying (K) such that

1. $\|\nu((\mathbf{L}-\Delta)W)\|_{L^2(0,T;\mathcal{L}^2)}\leqslant KTR\exp(K_1T)$.
2. $\|(\mathbf{M}W)\|_{L^2(0,T;\mathcal{L}^2)}\leqslant KT^{1/2}R\exp(K_1T)$.
3. $\|(\mathbf{B}W)\|_{L^2(0,T;\mathcal{L}^2)}\leqslant KT^{1/2}R\exp(K_1T)$.
4. $\|(\nabla-\mathbf{G})Q\|_{L^2(0,T;\mathcal{L}^2)}\leqslant KTR\exp(K_1T)$.

Proof

From (11), it follows that for each $i\in\{1,\ldots,n\}$:

$$(\mathbf{L}W)_i=\sum_{j,k=1}^{n}g^{jk}\frac{\partial^2W_i}{\partial y_j\partial y_k}+\sum_{j,k=1}^{n}\frac{\partial g^{jk}}{\partial y_j}\frac{\partial W_i}{\partial y_k}+2\sum_{j,k,l=1}^{n}g^{kl}\Gamma^i{}_{jk}\frac{\partial W_j}{\partial y_l}$$
$$+\sum_{j,k,l=1}^{n}\left\{\frac{\partial g^{kl}}{\partial y_k}\Gamma^i_{jl}+g^{kl}\frac{\partial\Gamma^i_{jl}}{\partial y_k}+\sum_{m=1}^{n}g^{kl}\Gamma^m_{jl}\Gamma^i_{km}\right\}W_j \tag{47}$$

Next, we estimate all coefficients which appear in expression of $(\mathbf{L}W)_i$.

From (15) and from Lemma 4.3, it clearly follows that for all $j,k\in\{1,\ldots,n\}$:

$$\|g^{jk}\|_{L^\infty(\mathbb{R}^n\times(0,T))}\leqslant n\exp(2KT) \tag{48}$$

Moreover, from (15), and by means of simple calculations, we have that

$$\frac{\partial g^{jk}}{\partial y_l}=\sum_{m=1}^{n}\sum_{p=1}^{n}\frac{\partial X_p}{\partial y_l}\left(\frac{\partial^2Y_j}{\partial x_p\partial x_m}\frac{\partial Y_k}{\partial x_m}+\frac{\partial^2Y_k}{\partial x_p\partial x_m}\frac{\partial Y_j}{\partial x_m}\right)$$

Thus, from Lemma 4.3, we deduce from above equation that for all $j, k, l \in \{1, \ldots, n\}$:

$$\left\| \frac{\partial g^{jk}}{\partial y_l} \right\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leqslant 2n^2 KT \exp(4KT) \tag{49}$$

Similarly, from (16) and from Lemma 4.3, we get that for all $i, j, l \in \{1, \ldots, n\}$:

$$\left\| \frac{\partial g_{ij}}{\partial y_l} \right\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leqslant 2nKT \exp(3KT) \tag{50}$$

Furthermore, by using (48) combined to (50), we obtain from (17) that for all $i, j, k \in \{1, \ldots, n\}$:

$$\|\Gamma^k_{ij}\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leqslant 3n^3 KT \exp(5KT) \tag{51}$$

Moreover, by means of simple calculations, by using (48) combined to (49) and to (50), we obtain from (17) that for all $i, j, k, l \in \{1, \ldots, n\}$:

$$\left\| \frac{\partial \Gamma^k_{ij}}{\partial y_l} \right\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leqslant 3n^3 KT \exp(5KT)\{1 + KT \exp(KT)[3 + 2n \exp(KT)]\} \tag{52}$$

On the other hand, according to a classical Sobolev embedding, we have that

$$\|W\|_{L^2(0,T;\mathscr{H}^1)} \leqslant T^{1/2} \|W\|_{L^\infty(0,T;\mathscr{H}^1)} \leqslant T^{1/2} R \tag{53}$$

Thus, by using (48)–(52), we obtain from (47) that for each $i \in \{1, \ldots, n\}$:

$$\begin{aligned} \|((\mathbf{L} - \Delta)W)_i\|_{L^2(0,T;\mathscr{L}^2)} \leqslant & \left\| \sum_{j,k=1}^n (g^{jk} - \delta_{jk}) \frac{\partial^2 W_i}{\partial y_j \partial y_k} \right\|_{L^2(0,T;\mathscr{L}^2)} \\ & + \{2n^3 KT \exp(4KT) + 6n^5 KT \exp(7KT) \\ & + 3n^6 KT \exp(7KT)(1 + 2nKT \exp(2KT) + 3n^4 KT \exp(5KT) \\ & + KT \exp(KT)[3 + 2n \exp(KT)])\} T^{1/2} R \end{aligned}$$

Therefore, from (45) in Corollary 4.4, we obtain assertion 1. Similarly, we prove assertion 2.

Moreover, from (14) we have that for each $i \in \{1, \ldots, n\}$:

$$((\nabla - \mathbf{G})Q)_i = \sum_{j=1}^n (\delta_{ij} - g^{ij}) \frac{\partial Q}{\partial y_j}$$

Thus, again from (45) in Corollary 4.4, we deduce from above equation assertion 4.

Finally, from (27), we have that assertion 3 is a direct consequence of relations (41), (51) and (52). □

The following useful lemma has been proved in Reference [13].

Lemma 4.6

Let $\mathscr{U}$ be the space defined by (22). Then, for any $V, W \in \mathscr{U}$, we have that $(W \cdot \nabla)V \in L^{5/2}(0, T; \mathscr{L}^2)$, and for all $i, j \in \{1, \ldots, n\}$, we have that $W_i V_j \in L^\infty(0, T; L^2(\Omega))$. Moreover, there exists a constant $C>0$, depending only on Ω, such that

$$\|(W\cdot\nabla)V\|_{L^{5/2}(0,T;\mathscr{L}^2)} \leqslant C\|W\|_{L^\infty(0,T;\mathscr{H}^1)}\|V\|^{1/5}_{L^\infty(0,T;\mathscr{H}^1)}\|V\|^{4/5}_{L^2(0,T;\mathscr{H}^2)} \tag{54}$$

$$\|W_iV_j\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\|W\|_{L^\infty(0,T;\mathscr{H}^1)}\|V\|_{L^\infty(0,T;\mathscr{H}^1)} \tag{55}$$

The previous lemma allows us to get estimates on the non-linear term $(\mathbf{N}W)$.

Corollary 4.7
There exist constants K_1 satisfying (K) and $C=C(n,\Omega)>0$ such that for all $(W,Q)\in\mathscr{K}$:

$$\|(\mathbf{N}W)\|_{L^2(0,T;\mathscr{L}^2)}\leqslant CT^{1/10}R^2\exp(K_1T)$$

Proof
By using relation (54) in Lemma 4.6, we have that

$$\|(W\cdot\nabla)W\|_{L^{5/2}(0,T;\mathscr{L}^2)}\leqslant C\|W\|^{6/5}_{L^\infty(0,T;\mathscr{H}^1)}\|W\|^{4/5}_{L^2(0,T;\mathscr{H}^2)}\leqslant CR^2$$

Thus, by the Hölder inequalities, we deduce that

$$\|(W\cdot\nabla)W\|_{L^2(0,T;\mathscr{L}^2)}\leqslant T^{1/10}\|(W\cdot\nabla)W\|_{L^{5/2}(0,T;\mathscr{L}^2)}\leqslant CR^2T^{1/10}$$

To conclude, we use the above relation together with (51) and (55) in Lemma 4.6. □

Corollary 4.8
For R large enough and T_0 small enough, we have that $\mathscr{N}(\mathscr{K})\subset\mathscr{K}$, for any $T\in(0,T_0)$.

Proof
Let $(W,Q)\in\mathscr{K}$ and set $\mathscr{N}(W,Q)=(U,P)$, i.e. from definition of the mapping $\mathscr{N}$ (see Section 4.1), we have that (U,P) is the solution of (29)–(32), with F given by (33). Therefore, according to (34) in Proposition 4.1, combined to Lemma 4.5 and to Corollary 4.7, we obtain that

$$\begin{aligned}\|U\|_{\mathscr{U}}+\|\nabla P\|_{L^2(0,T;\mathscr{L}^2)} \leqslant{}& C\exp(T)\{\|U_0\|_{\mathscr{H}^1}+C(\Lambda)+2KTR\exp(K_1T)\\ &+2KT^{1/2}R\exp(K_1T)+C_1T^{1/10}R^2\exp(K_1T)\}\end{aligned} \tag{56}$$

where $C(\Lambda)>0$ is a constant depending only on Λ. Let $T_0\leqslant 1$, $R\geqslant 1$ and set

$$\tilde{C}=C\exp(1),\quad \tilde{C}_2=2K\exp(K_1),\quad \tilde{C}_1=C_1\exp(K_1),\quad \tilde{C}_3=\tilde{C}_1+2\tilde{C}_2 \tag{57}$$

Therefore, from (56) we clearly have that for all $T\in(0,T_0)$:

$$\begin{aligned}\|U\|_{\mathscr{U}}+\|\nabla P\|_{L^2(0,T;\mathscr{L}^2)} &\leqslant \tilde{C}\{\|U_0\|_{\mathscr{H}^1}+C(\Lambda)+\tilde{C}_2T_0R+\tilde{C}_2T_0^{1/2}R+\tilde{C}_1T_0^{1/10}R^2\}\\ &\leqslant \tilde{C}\{\|U_0\|_{\mathscr{H}^1}+C(\Lambda)+\tilde{C}_3T_0^{1/10}R^2\}\end{aligned} \tag{58}$$

On the other hand, let $R>0$ and $T_0>0$ be such that

$$R \geqslant \max\left(1, \frac{1}{2\tilde{C}\tilde{C}_3}, 2\tilde{C}\{\|U_0\|_{\mathscr{H}^1} + C(\Lambda)\}\right) \tag{59}$$

$$T_0 \leqslant \left(\frac{1}{2\tilde{C}\tilde{C}_3 R}\right)^{10} \tag{60}$$

Thus, according to the above equations and from (58), we clearly have that for any $T \in (0, T_0)$

$$\|U\|_{\mathscr{U}} + \|\nabla P\|_{L^2(0,T;\mathscr{L}^2)} \leqslant R$$

which means that $(U,P) \in \mathscr{K}$. This achieves the proof of the corollary. □

In order to apply the Banach fixed point theorem, we have to show that for R and T_0 as in the previous corollary, the mapping $\mathscr{N} : \mathscr{K} \to \mathscr{K}$ is a contraction for any $T \in (0, T_0)$. Let us consider $(W^1, Q^1), (W^2, Q^2) \in \mathscr{K}$. In the sequel, we denote by F^i the function defined by (33) corresponding to (W^i, Q^i), $i \in \{1,2\}$, and by $\mathscr{N}(W^i, Q^i) = (U^i, P^i)$, $i \in \{1,2\}$. Moreover, we set

$$W = W^1 - W^2, \quad Q = Q^1 - Q^2; \quad U = U^1 - U^2, \quad P = P^1 - P^2; \quad F = F^1 - F^2$$

Then, (U,P) is the solution of the Stokes problem

$$\begin{aligned} \frac{\partial U}{\partial t} - \nu \Delta U + \nabla P = F \quad & \text{in } \Omega \times (0,T) \\ \operatorname{div} U = 0 \quad & \text{in } \Omega \times (0,T) \\ U(y,t) = 0 \quad & \text{on } \partial\mathcal{O} \times (0,T) \\ U(y,0) = 0, \quad & y \in \Omega \end{aligned}$$

where

$$F = \nu((\mathbf{L} - \Delta)W) - (\mathbf{M}W) - (\mathbf{N}W^1) + (\mathbf{N}W^2) - (\mathbf{B}W) + ((\nabla - \mathbf{G})Q) \tag{61}$$

As before, we first estimate the terms which appear in the expression of F. For the linear terms we have

Lemma 4.9
There exist constants K, K_1 satisfying (K) such that for all $(W,Q) \in \mathscr{U} \times L^2(0,T;\hat{H}^1(\Omega))$

1. $\|\nu((\mathbf{L} - \Delta)W)\|_{L^2(0,T;\mathscr{L}^2)} \leqslant KT \, \|W\|_{\mathscr{U}} \exp(K_1 T)$.
2. $\|(\mathbf{M}W)\|_{L^2(0,T;\mathscr{L}^2)} \leqslant KT^{1/2} \, \|W\|_{\mathscr{U}} \exp(K_1 T)$.
3. $\|(\mathbf{B}W)\|_{L^2(0,T;\mathscr{L}^2)} \leqslant KT^{1/2} \, \|W\|_{\mathscr{U}} \exp(K_1 T)$.
4. $\|((\nabla - \mathbf{G})Q)\|_{L^2(0,T;\mathscr{L}^2)} \leqslant KT \, \|\nabla Q\|_{L^2(0,T;\mathscr{L}^2)} \exp(K_1 T)$.

The proof of this lemma is the same as that from Lemma 4.5, so that we skip it.

For the non-linear term we have

Lemma 4.10
There exist constants K_1 satisfying (K) and $C = C(n,\Omega) > 0$ such that for all (W^1, Q^1), $(W^2, Q^2) \in \mathscr{K}$:

$$\|(\mathbf{N}W^1) - (\mathbf{N}W^2)\|_{L^2(0,T;\mathscr{L}^2)} \leqslant CRT^{1/10}\,\|W\|_{\mathscr{U}}\;\exp(K_1 T)$$

Proof
We clearly have that

$$(W^1 \cdot \nabla)W^1 - (W^2 \cdot \nabla)W^2 = (W^1 \cdot \nabla)W + (W \cdot \nabla)W^2$$

Thus, from relation (54) in Lemma 4.6 combined to the above relation, we obtain that

$$\|(W^1 \cdot \nabla)W^1 - (W^2 \cdot \nabla)W^2\|_{L^{5/2}(0,T;\mathscr{L}^2)} \leqslant C(\|W^1\|_{\mathscr{U}}\|W\|_{\mathscr{U}} + \|W\|_{\mathscr{U}}\|W^2\|_{\mathscr{U}}) \leqslant CR\|W\|_{\mathscr{U}}$$

Therefore, from the Hölder inequalities, the above relation implies that

$$\|(W^1 \cdot \nabla)W^1 - (W^2 \cdot \nabla)W^2\|_{L^2(0,T;\mathscr{L}^2)} \leqslant CRT^{1/10}\,\|W\|_{\mathscr{U}} \tag{62}$$

On the other hand,

$$\sum_{j,k=1}^{n} \Gamma^i_{jk} W^1_j W^1_k - \sum_{j,k=1}^{n} \Gamma^i_{jk} W^2_j W^2_k = \sum_{j,k=1}^{n} \Gamma^i_{jk} W^2_j W_k + \sum_{j,k=1}^{n} \Gamma^i_{jk} W_j W^1_k$$

Thus, from relations (51) and (55) in Lemma 4.6, combined to the above relation, we obtain that

$$\left\| \sum_{j,k=1}^{n} \Gamma^i_{jk} W^1_j W^1_k - \sum_{j,k=1}^{n} \Gamma^i_{jk} W^2_j W^2_k \right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant 3n^5 KT\exp(5KT)CR\|W\|_{\mathscr{U}}$$

Finally, from above relation together with (62), we get the result. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1
From Lemmas 4.9 and 4.10 together with relation (61), combined to (34), we obtain that

$$\begin{aligned}\|U\|_{\mathscr{U}} + \|\nabla P\|_{L^2(0,T;\mathscr{L}^2)} \leqslant\; & C\exp(T)\{KT\|W\|_{\mathscr{U}}\exp(K_1T) + 2KT^{1/2}\|W\|_{\mathscr{U}}\exp(K_1T) \\ & + KT\|\nabla Q\|_{L^2(0,T;\mathscr{L}^2)}\exp(K_1T) \\ & + C_1RT^{1/10}\|W\|_{\mathscr{U}}\;\exp(K_1T)\}\end{aligned} \tag{63}$$

Therefore, for $R > 0$ and $T_0 > 0$ as in (59) and (60), respectively, we get from (63) that

$$\begin{aligned}\|U\|_{\mathscr{U}} + \|\nabla P\|_{L^2(0,T;\mathscr{L}^2)} &\leqslant \tilde{C}\{(\tilde{C}_2/2)T_0\,\|W\|_{\mathscr{U}} + \tilde{C}_2T_0^{1/2}\,\|W\|_{\mathscr{U}} + (\tilde{C}_2/2)T_0\,\|\nabla Q\|_{L^2(0,T;\mathscr{L}^2)} \\ &\quad + \tilde{C}_1RT_0^{1/10}\,\|W\|_{\mathscr{U}}\} \\ &\leqslant \tilde{C}\tilde{C}_3T_0^{1/10}R\{\|W\|_{\mathscr{U}} + \|\nabla Q\|_{L^2(0,T;\mathscr{L}^2)}\}\end{aligned}$$

where $\tilde{C}$, $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ are defined by (57). Then, for R and T_0 as in (59) and (60) respectively, we get from above inequality that $\mathcal{N} : \mathcal{K} \to \mathcal{K}$ is a contraction for all $T \in (0, T_0)$. This proves that $\mathcal{N}$ admits a unique fixed point, and consequently problem (23)–(28) admits a unique strong solution

$$(U, P) \in \mathcal{U} \times L^2(0, T; \hat{H}^1(\Omega))$$

for any $T \in (0, T_0)$.

Finally, we have to prove that we can choose the time T_0 such that one of the alternatives (i) or (ii) holds true. The result follows in a standard manner from the fact that the local existence time T_0 obtained in Corollary 4.8 is uniform with respect to v_0, provided that $\|v_0\|_{\mathcal{H}^1} \leqslant C_0$. We have thus proved our local existence and uniqueness result. □

5. PROOF OF THE GLOBAL EXISTENCE AND UNIQUENESS IN 2-D

According to Theorem 1.1, in order to get global existence, it suffices to show that the H^1 norm of v does not blow up in finite time. Therefore, the aim of this section is to prove that the mapping

$$t \mapsto \|v(t)\|_{\mathcal{H}^1(t)}$$

is bounded on $[0, T)$ for all $T \in (0, T_0)$, where T_0 is the maximal time existence of the solution v.

We introduce a new change of variables. Let

$$w(y, t) = v(y, t) - \Lambda(y) \tag{64}$$

where (v, q) is the solution to problem (1)–(4) given by Theorem 1.1, and Λ is defined by (6). Clearly (w, q) satisfies

$$\frac{\partial w}{\partial t} - \nu \Delta w + (w \cdot \nabla) w + \nabla q = F - (w \cdot \nabla)\Lambda - (\Lambda \cdot \nabla) w \quad \text{in } \Omega(t) \tag{65}$$

$$\operatorname{div} w = 0 \quad \text{in } \Omega(t) \tag{66}$$

$$w(x, t) = 0 \quad \text{on } \partial\mathcal{O}(t) \tag{67}$$

$$w(x, 0) = w_0(x) \quad \text{in } \Omega \tag{68}$$

where

$$F = \nu \Delta \Lambda - (\Lambda \cdot \nabla)\Lambda \tag{69}$$

$$w_0 = v_0 - \Lambda \tag{70}$$

Next, we shall give some results which are valid in two or in three space dimensions.

Lemma 5.1
Let (v, q) be the strong solution to problem (1)–(4) given by Theorem 1.1. Then, v is bounded in the energy space

$$L^\infty(0,T;\mathscr{L}^2(t)) \cap L^2(0,T;\mathscr{H}^1(t))$$

i.e. for any $T \in (0, T_0)$ there exists a constant $C_T > 0$ such that

$$\|v\|_{L^\infty(0,T;\mathscr{L}^2(t))} + \|v\|_{L^2(0,T;\mathscr{H}^1(t))} \leqslant C_T$$

Proof
Clearly, it suffices to prove the estimate in the lemma for the function w defined by (64). If we take the inner product in $\mathscr{L}^2(t)$ of (65) with w and we integrate by parts the second term on the left-hand side, we obtain that

$$\int_{\Omega(t)} \frac{\partial w}{\partial t} \cdot w \,\mathrm{d}x + v \int_{\Omega(t)} |\nabla w(t)|^2 = (F, w)_{\mathscr{L}^2(t)} - ((w \cdot \nabla)\Lambda, w)_{\mathscr{L}^2(t)} \quad \text{a.e. in } (0,T) \qquad (71)$$

On the other hand, since Λ is the velocity of the domain $\Omega(t)$, by applying the Reynolds transport theorem in the first term on the left-hand side of (71), combined to the fact that $\operatorname{div} \Lambda = 0$ in $\mathbb{R}^n$, we obtain that a.e. in $(0, T)$

$$\begin{aligned} \int_{\Omega(t)} \frac{\partial w}{\partial t} \cdot w \,\mathrm{d}x &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} |w|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\Omega(t)} \Lambda \cdot \nabla[|w|^2] \,\mathrm{d}x \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} |w|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\Omega(t)} [(\Lambda \cdot \nabla)w] \cdot w \,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} |w|^2 \,\mathrm{d}x \end{aligned} \qquad (72)$$

Moreover, by using (41) and assertion 1 in Lemma 3.1, we get from (69) that

$$\|F\|_{\mathscr{L}^2(t)} \leqslant C \quad \forall t \in [0, T] \qquad (73)$$

where $C = \sqrt{\mu(B_{2r})}K[v + K]$, and $\mu(B_{2r})$ denotes the Lebesgue measure of the ball B_{2r}.

Therefore, by applying the Hölder inequality in (71) and by using (72), (73) combined to (41), we obtain that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{\mathscr{L}^2(t)} + v\|\nabla w(t)\|^2_{[L^2(\Omega(t))]^{n^2}} &\leqslant C\|w\|_{\mathscr{L}^2(t)} + K\|w\|^2_{\mathscr{L}^2(t)} \\ &\leqslant M\|w\|_{\mathscr{L}^2(t)}(1 + \|w\|_{\mathscr{L}^2(t)}) \\ &\leqslant \frac{M^2}{2}\|w\|^2_{\mathscr{L}^2(t)} + \frac{1}{2}(1 + \|w\|_{\mathscr{L}^2(t)})^2 \\ &\leqslant 1 + \left(1 + \frac{M^2}{2}\right)\|w\|^2_{\mathscr{L}^2(t)} \quad \text{a.e. in } (0,T) \end{aligned}$$

In the above inequality K is the constant in (41) and $M=\max(C,K)$. Thus, by integrating the above estimate with respect to t, we obtain that for all $\tau\in[0,T]$:

$$\frac{1}{2}\|w(\tau)\|^2_{\mathscr{L}^2(\tau)}+\nu\int_0^\tau\|\nabla w(t)\|^2_{[L^2(\Omega(t))]^{n^2}}\,\mathrm{d}t$$

$$\leqslant\frac{1}{2}\|w_0\|^2_{\mathscr{L}^2}+T+\left(1+\frac{M^2}{2}\right)\int_0^\tau\|w(t)\|^2_{\mathscr{L}^2(t)}\,\mathrm{d}t$$

$$\leqslant\frac{1}{2}\|w_0\|^2_{\mathscr{L}^2}+T+(2+M^2)\int_0^\tau\left\{\frac{1}{2}\|w(t)\|^2_{\mathscr{L}^2(t)}+\nu\int_0^t\|\nabla w(s)\|^2_{[L^2(\Omega(s))]^{n^2}}\,\mathrm{d}s\right\}\mathrm{d}t$$

Finally, by applying the Gronwall lemma in the above estimate, we obtain that

$$\frac{1}{2}\|w(\tau)\|^2_{\mathscr{L}^2(\tau)}+\nu\int_0^\tau\|\nabla w(t)\|^2_{[L^2(\Omega(t))]^{n^2}}\,\mathrm{d}t\leqslant\left(\frac{1}{2}\|w_0\|^2_{\mathscr{L}^2}+T\right)\exp[(2+M^2)T]$$

for all $\tau\in[0,T]$. We have thus proved the result in the lemma. □

Next, we prove a technical result, which will be used in the proof of Theorem 1.2.

Lemma 5.2

Let $w\in L^2(0,T;\mathscr{H}^2(t))\cap C([0,T],\mathscr{H}^1(t))\cap H^1(0,T;\mathscr{L}^2(t))$ be such that $w|_{\partial\mathcal{O}(t)}=0$ for all $t\in[0,T]$. Define

$$\frac{\mathrm{d}^\Lambda w}{\mathrm{d}t}=\frac{\partial w}{\partial t}+(\Lambda\cdot\nabla)w$$

Then, we have that for a.e. $t\in(0,T)$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}|\nabla w|^2\,\mathrm{d}x=-\int_{\Omega(t)}\Delta w\cdot\frac{\mathrm{d}^\Lambda w}{\mathrm{d}t}\,\mathrm{d}x-\int_{\Omega(t)}\nabla w:[\nabla w\cdot\nabla\Lambda]\,\mathrm{d}x \tag{74}$$

Proof

By differentiating with respect to the time the equation

$$w(X(y,t),t)=0,\quad y\in\partial\mathcal{O},\ t\in(0,T)$$

we obtain that

$$\frac{\partial w}{\partial t}+\left(\frac{\partial X}{\partial t}\cdot\nabla\right)w=0,\quad y\in\partial\mathcal{O},\ t\in(0,T)$$

Thus, since $\partial X/\partial t=\Lambda(X)$ (see (7)), by putting $x=X(y,t)$, we obtain that

$$\frac{\partial w}{\partial t}+(\Lambda\cdot\nabla)w=0,\quad x\in\partial\mathcal{O}(t),\ t\in(0,T)$$

On the other hand, with the notation introduced in the lemma, we clearly have that the above equation reads as

$$\frac{\mathrm{d}^\Lambda w}{\mathrm{d}t}=0\quad\text{on }\partial\mathcal{O}(t),\ t\in(0,T) \tag{75}$$

Moreover, simple calculations show that

$$\frac{\mathrm{d}^{\Lambda}}{\mathrm{d}t}(\nabla w) = \nabla\left(\frac{\mathrm{d}^{\Lambda} w}{\mathrm{d}t}\right) - \nabla w \cdot \nabla\Lambda \tag{76}$$

Next, we follow the proof in two steps. In the first step, we assume that

$$\frac{\mathrm{d}^{\Lambda} w}{\mathrm{d}t} \in L^2(0,T;\mathscr{H}^1(t)) \tag{77}$$

From the Reynolds transport theorem, combined to the fact that $\operatorname{div}\Lambda = 0$ in $\mathbb{R}^n$, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} |\nabla w|^2 \mathrm{d}x = \frac{1}{2}\int_{\Omega(t)} \frac{\mathrm{d}^{\Lambda}}{\mathrm{d}t}\left(|\nabla w|^2\right)\mathrm{d}x = \int_{\Omega(t)} \nabla w : \frac{\mathrm{d}^{\Lambda}}{\mathrm{d}t}(\nabla w)\,\mathrm{d}x \tag{78}$$

On the other hand, integrating by parts, and by using (75)–(77), we obtain that

$$\begin{aligned}\int_{\Omega(t)} \nabla w : \frac{\mathrm{d}^{\Lambda}}{\mathrm{d}t}(\nabla w)\,\mathrm{d}x &= \int_{\Omega(t)} \nabla w : \left[\nabla\left(\frac{\mathrm{d}^{\Lambda} w}{\mathrm{d}t}\right) - \nabla w \cdot \nabla\Lambda\right]\mathrm{d}x \\ &= -\int_{\Omega(t)} \Delta w \cdot \frac{\mathrm{d}^{\Lambda} w}{\mathrm{d}t}\,\mathrm{d}x - \int_{\Omega(t)} \nabla w : [\nabla w \cdot \nabla\Lambda]\,\mathrm{d}x \end{aligned}\tag{79}$$

Hence, from (78) and (79) we get the result in the lemma, if w satisfies (77).

The second step is to prove formula (74) in the general case. To this end, let $z(y,t) = w(X(y,t),t)$, where $X(y,t)$ is defined in (7). Since X is smooth and its derivative with respect to the time is a compactly supported function, it clearly follows from regularity of w that

$$z \in L^2(0,T;\mathscr{H}^2) \cap C([0,T],\mathscr{H}^1_0) \cap H^1(0,T;\mathscr{L}^2) \tag{80}$$

Moreover, from Lemma 3.3, and making the change of variables $x = X(y,t)$ in (74), we formally obtain, after some calculations, that for a.e. $t \in (0,T)$:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} |\nabla z \cdot J_Y|^2\,\mathrm{d}y &= -\int_{\Omega}\left(g^{kl}\frac{\partial^2 z_i}{\partial y_k \partial y_l} + (\nabla z)_{ik}\Delta Y_k\right)\frac{\partial z_i}{\partial t}\,\mathrm{d}y \\ &\quad - \int_{\Omega} [\nabla z \cdot J_Y] : [\nabla z \cdot J_Y \cdot (\nabla\tilde{\Lambda} \cdot J_Y)]\,\mathrm{d}y \end{aligned}\tag{81}$$

where Y is the inverse of X, $J_Y = J_Y(X(y,t),t)$ is the jacobian matrix of Y composed with X, $\tilde{\Lambda}(y,t) = \Lambda(X(y,t))$ and g^{kl} is defined by (15).

On the other hand, let $V = \mathscr{H}^2 \cap \mathscr{H}^1_0$ be endowed with the norm $\|\phi\|_V = \|(I - \Delta)\phi\|_{\mathscr{L}^2}$. According to classical results about of elliptic regularity for Laplacian (see, for instance, Reference [14, p. 181]), it clearly follows that this norm is equivalent to the usual H^2 norm. Moreover, let $H = \mathscr{H}^1_0$. With these definitions, it clearly follows that V', the dual space of V with respect to the pivot space H, is $V' = \mathscr{L}^2$. Thus, from (80) it follows that

$$z \in L^2(0,T;V), \quad \frac{\partial z}{\partial t} \in L^2(0,T;V')$$

Therefore, according to a classical result (see, for instance, Reference [12, pp. 261–262]), we have that there exists a sequence $z^n \in C^\infty([0,T];V)$ $(n\in\mathbb{N})$ such that

$$\begin{aligned} z^n &\to z \quad \text{strongly in } L^2(0,T;V) \\ \frac{\partial z^n}{\partial t} &\to \frac{\partial z}{\partial t} \quad \text{strongly in } L^2(0,T;V') \end{aligned} \tag{82}$$

Let $E=\{t\in(0,T): z^n(t)\to z(t) \text{ strongly in } V\}$. After passage to a subsequence, it follows that E is a set of complement negligible in $(0,T)$. Let $s,t\in E$. Thus, by integrating (81) (which is valid for z^n taking into account its regularity) from s to t, we obtain that

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla z^n(t)\cdot J_Y(t)|^2\,\mathrm{d}y - \frac{1}{2}\int_\Omega |\nabla z^n(s)\cdot J_Y(s)|^2\,\mathrm{d}y \\ &\quad = -\int_s^t\int_\Omega \left(g^{kl}(\tau)\frac{\partial^2 z_i^n}{\partial y_k \partial y_l}(\tau) + (\nabla z^n)_{ik}(\tau)\Delta Y_k(\tau)\right)\frac{\partial z_i^n}{\partial t}(\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &\qquad -\int_s^t\int_\Omega [\nabla z^n(\tau)\cdot J_Y(\tau)] : [\nabla z^n(\tau)\cdot J_Y(\tau)\cdot(\nabla\tilde{\Lambda}(\tau)\cdot J_Y(\tau))]\,\mathrm{d}y\,\mathrm{d}\tau \end{aligned}$$

From (82), and since Y is a smooth function, we can pass to the limit as $n\to\infty$ in the above equation and we obtain the same for z, which implies (81). This concludes the proof of the lemma. □

We are now in a position to prove that the H^1 norm of $v(t)$ does not blow up in finite time.

Proposition 5.3
Let (v,q) be the strong solution to problem (1)–(4) as in Theorem 1.1 and suppose that $n=2$. Then, the mapping $t\mapsto \|v(t)\|_{\mathscr{H}^1(t)}$ is bounded in $[0,T)$ for all $T\in(0,T_0)$, i.e. there exists a constant $C_T>0$ such that

$$\|v(t)\|_{\mathscr{H}^1(t)} \leqslant C_T \quad \forall t\in[0,T)$$

Proof
Again, it suffices to prove that $\|w(t)\|_{\mathscr{H}^1(t)}\leqslant C_T$ for all $t\in[0,T)$, where w is defined by (64). In order to show this result, consider the function φ defined by $\varphi=(\mathrm{d}^\Lambda w/\mathrm{d}t)-(w\cdot\nabla)\Lambda$, where we have used the notation introduced in the previous lemma. This function has the following properties:

1. $\varphi\in L^2(0,T;\mathscr{L}^2(t))$ for any $T\in(0,T_0)$.
2. $\operatorname{div}\varphi=0$ in $\mathscr{D}'(\Omega(t))$, a.e. in $(0,T)$.
3. $\varphi\cdot n=0$ in $H^{-1/2}(\partial\mathcal{O}(t))$, a.e. in $(0,T)$.

The two first properties above are direct from the regularity of v. Property 3 is a direct consequence of (75) combined to the fact that $w|_{\partial\mathcal{O}(t)}=0$.

On the other hand, notice that, according to the definition of φ, Equation (65) can be rewritten as

$$\varphi - \nu\Delta w + (w\cdot\nabla)w + \nabla q = F - 2(w\cdot\nabla)\Lambda \quad \text{in } \Omega(t),\ t\in(0,T)$$

By taking the inner product in $\mathscr{L}^2(t)$ with φ, the above relation implies that a.e. in $(0,T)$:

$$\|\varphi\|^2_{\mathscr{L}^2(t)} - v(\Delta w, \varphi)_{\mathscr{L}^2(t)} + ((w\cdot\nabla)w,\varphi)_{\mathscr{L}^2(t)} = (F,\varphi)_{\mathscr{L}^2(t)} - 2((w\cdot\nabla)\Lambda),\varphi)_{\mathscr{L}^2(t)} \qquad (83)$$

Next, we need to integrate by parts the second term in the left-hand side of Equation (83). To this end, we apply Lemma 5.2. Hence, we have that

$$\begin{aligned} -v\int_{\Omega(t)} \Delta w\cdot\varphi\,\mathrm{d}x &= -v\int_{\Omega(t)} \Delta w\cdot\frac{\mathrm{d}^\Lambda w}{\mathrm{d}t}\,\mathrm{d}x + v\int_{\Omega(t)}\Delta w\cdot[(w\cdot\nabla)\Lambda]\,\mathrm{d}x \\ &= \frac{v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}|\nabla w|^2\,\mathrm{d}x + v\int_{\Omega(t)}\nabla w : [\nabla w\cdot\nabla\Lambda]\,\mathrm{d}x \\ &\quad - v\int_{\Omega(t)}\nabla w:\nabla[(w\cdot\nabla)\Lambda]\,\mathrm{d}x \end{aligned} \qquad (84)$$

Thus, by replacing (84) in (83) and rearranging the terms, we get that

$$\begin{aligned} &\|\varphi\|^2_{\mathscr{L}^2(t)} + \frac{v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w(t)\|^2_{[L^2(\Omega(t))]^4} \\ &\quad = v\int_{\Omega(t)}\nabla w:\nabla[(w\cdot\nabla)\Lambda]\,\mathrm{d}x - v\int_{\Omega(t)}\nabla w:[\nabla w\cdot\nabla\Lambda]\,\mathrm{d}x - ((w\cdot\nabla)w,\varphi)_{\mathscr{L}^2(t)} \\ &\qquad + (F,\varphi)_{\mathscr{L}^2(t)} - 2((w\cdot\nabla)\Lambda,\varphi)_{\mathscr{L}^2(t)} \quad \text{a.e. in } (0,T) \end{aligned}$$

On the other hand, by applying the Hölder inequality in the above estimate, and by using (41) and (73), we deduce that

$$\begin{aligned} &\|\varphi\|^2_{\mathscr{L}^2(t)} + \frac{v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w(t)\|^2_{[L^2(\Omega(t))]^4} \\ &\quad \leqslant 2vK\|\nabla w\|^2_{[L^2(\Omega(t))]^4} + vK\|\nabla w\|_{[L^2(\Omega(t))]^4}\|w\|_{\mathscr{L}^2(t)} + \|(w\cdot\nabla)w\|_{\mathscr{L}^2(t)}\|\varphi\|_{\mathscr{L}^2(t)} \\ &\qquad + C\|\varphi\|_{\mathscr{L}^2(t)} + 2K\|w\|_{\mathscr{L}^2(t)}\|\varphi\|_{\mathscr{L}^2(t)} \quad \text{a.e. in } (0,T) \end{aligned} \qquad (85)$$

Furthermore, by applying that

$$(\varepsilon a)\cdot\left(\frac{1}{\varepsilon}b\right) \leqslant \frac{1}{2}\varepsilon^2 a^2 + \frac{1}{2\varepsilon^2}b^2 \quad \forall a,b,\varepsilon > 0 \qquad (86)$$

with ε small enough, and by integrating (85) with respect to t, we get that

$$\begin{aligned} &\frac{1}{4}\int_0^t \|\varphi\|^2_{\mathscr{L}^2(s)}\,\mathrm{d}s + \frac{v}{2}\|\nabla w(t)\|^2_{[L^2(\Omega(t))]^4} \\ &\quad \leqslant \frac{v}{2}\|\nabla w_0\|^2_{[L^2(\Omega)]^4} + 2vK\int_0^t\|\nabla w\|^2_{[L^2(\Omega(s))]^4}\,\mathrm{d}s + vK\int_0^t \|\nabla w\|_{[L^2(\Omega(s))]^4}\|w\|_{\mathscr{L}^2(s)} \\ &\qquad + \int_0^t\|(w\cdot\nabla)w\|^2_{\mathscr{L}^2(s)} + C^2T + 4K^2\int_0^t\|w\|^2_{\mathscr{L}^2(s)}\,\mathrm{d}s \quad \forall t\in[0,T] \end{aligned}$$

Finally, by applying the Cauchy–Schwartz inequality in the above estimate, and by using Lemma 5.1, we get that

$$\frac{1}{4}\int_0^t \|\varphi\|^2_{\mathscr{L}^2(s)}\,\mathrm{d}s + \frac{\nu}{2}\|\nabla w(t)\|^2_{[L^2(\Omega(t))]^4} \leqslant M + \int_0^t \|(w\cdot\nabla)w\|^2_{\mathscr{L}^2(s)}\,\mathrm{d}s \quad \forall t\in[0,T] \tag{87}$$

where $M=(\nu/2)\|\nabla w_0\|^2_{[L^2(\Omega)]^4} + 2\nu K\tilde{C}^2 + \nu K\tilde{C}^2T^{1/2} + C^2T + 4K^2\tilde{C}^2T$ and $\tilde{C}=\tilde{C}(\nu, r, \|v_0\|_{\mathscr{L}^2}, T)>0$ is a constant satisfying

$$\|w\|_{L^\infty(0,T;\mathscr{L}^2(t))} + \|w\|_{L^2(0,T;\mathscr{H}^1(t))} \leqslant \tilde{C} \tag{88}$$

We then have to estimate $(w\cdot\nabla)w$ in terms of the left-hand side of (87). To this end, we use the change of variables $x=Q(t)y$. Consider the functions $z(y,t)=Q^t(t)w(Q(t)y,t)$ and $p(y,t)=q(Q(t)y,t)$, with (w,q) solution of (65)–(70). By means of simple calculations, it is easy to see that

$$(w\cdot\nabla_x)w = Q(z\cdot\nabla_y)z$$

Thus, it follows that

$$\int_{\Omega(t)} |(w\cdot\nabla_x)w|^2\,\mathrm{d}x = \int_\Omega |Q(z\cdot\nabla_y)z|^2\,|\det Q(t)|\,\mathrm{d}y = \int_\Omega |(z\cdot\nabla_y)z|^2\,\mathrm{d}y \tag{89}$$

Therefore, by using a Hölder inequality combined to the continuous Sobolev embedding of $H^{1/2}(\Omega)$ in $L^4(\Omega)$ and to an interpolation inequality (see, for instance, Reference [15, p. 23]), we obtain that

$$\int_\Omega |(z\cdot\nabla_y)z|^2\,\mathrm{d}y \leqslant \|z\|^2_{[L^4(\Omega)]^2}\|\nabla z\|^2_{[L^4(\Omega)]^4} \leqslant C_1\|z\|_{\mathscr{L}^2}\|\nabla z\|_{[L^2(\Omega)]^4}\|\nabla z\|_{[L^2(\Omega)]^4}\|z\|_{\mathscr{H}^2} \tag{90}$$

for some constant $C_1=C_1(\Omega)>0$.

On the other hand, it is easy to see that $z\in D(A)$, where A is defined by (35) and (36). Thus, by using (37) in Proposition 4.2, we get that

$$\|z\|_{\mathscr{H}^2} \leqslant C_2(\|z\|_{\mathscr{L}^2} + \|Az\|_{\mathscr{L}^2}) \tag{91}$$

for some constant $C_2=C_2(\nu,\Omega)>0$.

Furthermore, we can consider (w,q) as the solution of the homogeneous resolvent Stokes problem at some fixed time $t>0$:

$$\begin{aligned} w-\nu\Delta w+\nabla q &= f+w \quad \text{in } \Omega(t)\\ \operatorname{div} w &= 0 \quad \text{in } \Omega(t)\\ w &= 0 \quad \text{on } \partial\Omega(t) \end{aligned}$$

where

$$f = F - 2(w\cdot\nabla)\Lambda - \varphi - (w\cdot\nabla)w \tag{92}$$

By means of simple calculations, it clearly follows that (z, p) satisfies the similar homogeneous resolvent Stokes problem

$$z - \nu\Delta z + \nabla p = \tilde{f} + z \quad \text{in } \Omega$$
$$\operatorname{div} z = 0 \quad \text{in } \Omega$$
$$z = 0 \quad \text{on } \partial\Omega$$

where $\tilde{f}(y,t) = Q^t(t) f(Q(t)y, t)$. Thus, as consequence of Theorem 2.1 in Reference [10], we obtain that

$$\|z\|_{\mathscr{L}^2} + \|Az\|_{\mathscr{L}^2} \leqslant C_3(\|f\|_{\mathscr{L}^2(t)} + \|z\|_{\mathscr{L}^2})$$

for some constant $C_3 = C_3(\nu, \Omega) > 0$.

Finally, from (89)–(91) combined to the above inequality, we deduce that

$$\int_{\Omega(t)} |(w\cdot\nabla_x)w|^2\,\mathrm{d}x \leqslant C_4\|w\|_{\mathscr{L}^2(t)}\|\nabla w\|^2_{[L^2(\Omega(t))]^4}(\|f\|_{\mathscr{L}^2(t)} + \|w\|_{\mathscr{L}^2(t)}) \tag{93}$$

for some constant $C_4 = C_4(\nu, \Omega) > 0$.

On the other hand, from (92) combined to (41), it follows that

$$\|f\|_{\mathscr{L}^2(t)} \leqslant \|F\|_{\mathscr{L}^2(t)} + 2K\|w\|_{\mathscr{L}^2(t)} + \|\varphi\|_{\mathscr{L}^2(t)} + \|(w\cdot\nabla)w\|_{\mathscr{L}^2(t)}$$

Therefore, by replacing the above inequality in (93), we obtain that

$$\begin{aligned}\|(w\cdot\nabla)w\|^2_{\mathscr{L}^2(t)} \leqslant{}& C_4\|w\|^2_{\mathscr{L}^2(t)}\|\nabla w\|^2_{[L^2(\Omega(t))]^4} \\ &+ C_4\|w\|_{\mathscr{L}^2(t)}\|\nabla w\|^2_{[L^2(\Omega(t))]^4}(\|F\|_{\mathscr{L}^2(t)} + 2K\|w\|_{\mathscr{L}^2(t)} \\ &+ \|\varphi\|_{\mathscr{L}^2(t)} + \|(w\cdot\nabla)w\|_{\mathscr{L}^2(t)})\end{aligned} \tag{94}$$

Hence, by integrating the above inequality with respect to t, and by using Lemma 5.1 combined to (73), we obtain that

$$\begin{aligned}\int_0^t \|(w\cdot\nabla)w\|^2_{\mathscr{L}^2(s)}\,\mathrm{d}s \leqslant{}& L + C_4\tilde{C}\int_0^t \|\nabla w\|^2_{[L^2(\Omega(s))]^4}\|\varphi\|_{\mathscr{L}^2(s)}\,\mathrm{d}s \\ &+ C_4\tilde{C}\int_0^t \|\nabla w\|^2_{[L^2(\Omega(s))]^4}\|(w\cdot\nabla)w\|_{\mathscr{L}^2(s)}\,\mathrm{d}s\end{aligned} \tag{95}$$

where $L = C_4\tilde{C}^4 + C_4\tilde{C}^3C + 2K\tilde{C}^4$, and $\tilde{C}, C > 0$ are the constants in (88) and in (73), respectively. Then, by applying (86), we deduce from (95) that for any $\varepsilon > 0$:

$$\begin{aligned}\int_0^t \|(w\cdot\nabla)w\|^2_{\mathscr{L}^2(s)}\,\mathrm{d}s \leqslant{}& L + \frac{C_4^2\tilde{C}^2}{2\varepsilon^2}\int_0^t \|\nabla w\|^4_{[L^2(\Omega(s))]^4}\,\mathrm{d}s \\ &+ \varepsilon^2\int_0^t (\|\varphi\|^2_{\mathscr{L}^2(s)} + \|(w\cdot\nabla)w\|^2_{\mathscr{L}^2(s)})\,\mathrm{d}s\end{aligned}$$

Therefore, by choosing ε small enough in the above inequality, (for instance, $\varepsilon^2 = \frac{1}{5}$), and by replacing in (87), we obtain that

$$\frac{1}{5}\int_0^t \|\varphi\|^2_{\mathscr{L}^2(s)}\,\mathrm{d}s + \frac{2v}{5}\|\nabla w(t)\|^2_{[L^2(\Omega(t))]^4}$$
$$\leqslant \frac{4M}{5} + L + \frac{5C_4^2\tilde{C}^2}{2}\int_0^t \|\nabla w\|^4_{[L^2(\Omega(s))]^4}\,\mathrm{d}s + \frac{1}{5}\int_0^t \|\varphi\|^2_{\mathscr{L}^2(s)}\,\mathrm{d}s \quad \forall t \in [0,T]$$

Hence, from the above inequality, we get that for all $t \in [0,T]$:

$$\|\nabla w\|^2_{[L^2(\Omega(t))]^4} \leqslant \frac{2M}{v} + \frac{5L}{2v} + \frac{25C_4^2\tilde{C}^2}{4v}\int_0^t \|\nabla w\|^4_{[L^2(\Omega(s))]^4}\,\mathrm{d}s$$

Therefore, by applying the Gronwall lemma in the above estimate, we conclude that for any $t \in [0,T]$:

$$\|\nabla w\|^2_{[L^2(\Omega(t))]^4} \leqslant \frac{4M+5L}{2v}\exp\left[\frac{25C_4^2\tilde{C}^2}{4v}\int_0^T \|\nabla w\|^2_{[L^2(\Omega(s))]^4}\,\mathrm{d}s\right]$$

which combined to Lemma 5.1 implies that the mapping

$$t \mapsto \|\nabla w(t)\|_{[L^2(\Omega(t))]^4}$$

is bounded in $[0,T)$. This achieves the proof of our global in time existence and uniqueness result in two spatial dimensions. □

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