Collapsing steady states of the Keller–Segel system

Manuel del Pino[1] **and Juncheng Wei**[2]

[1] Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile
[2] Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail: delpino@dim.uchile.cl and wei@math.cuhk.edu.hk

Abstract
We consider the boundary value problem:

$$\Delta u - au + \varepsilon^2 \mathrm{e}^u = 0, \qquad u > 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

which is equivalent to the stationary Keller–Segel system from chemotaxis. Here $\Omega \subset \mathbb{R}^2$ is a smooth and bounded domain. We show that given any two non-negative integers k, l with $k + l \geqslant 1$, for ε sufficiently small, there exists a solution u_ε for which $\varepsilon^2 \mathrm{e}^{u_\varepsilon}$ develops asymptotically k interior Dirac deltas with weight 8π and l boundary deltas with weight 4π. Location of blow-up points is characterized explicitly in terms of Green's function of the Neumann problem.

1. Introduction

Chemotaxis is one of the simplest mechanisms for aggregation of biological species. The term refers to a situation where organisms, for instance bacteria, move towards high concentrations of a chemical which they secrete. A basic model in chemotaxis was introduced by Keller and Segel [27]. They considered an advection–diffusion system consisting of two coupled parabolic equations for the concentration of the considered species and that of the chemical released, represented, respectively, by positive quantities $v(x,t)$ and $u(x,t)$ defined on a bounded, smooth domain Ω in $\mathbb{R}^N$ under no-flux boundary conditions. The system reads as follows:

$$\begin{aligned} &v_t = \Delta v - \nabla(v \nabla u), \\ &\tau u_t = \Delta u - u + v, \\ &u, v > 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

Here τ is a positive constant. After the seminal works by Nanjudiah [32] and Childress and Percus [8], many contributions have been made to the understanding of different analytical aspects of this system and its variations. We refer the reader for instance to [3, 4, 7, 9, 15, 17, 22–26, 30–44]. It is well known that if space dimension is $N = 2$ then classical solutions may blow up in finite time. The structure of this phenomenon has been widely treated in the literature for the two-dimensional case. It is known that the blow-up for the quantity v (whose mass is clearly preserved in time) takes place as a finite sum of Dirac measures at points with masses greater than or equal to 8π or 4π, respectively, depending on whether they are located inside the domain or at the boundary. This phenomenon, commonly referred to as 'chemotactic collapse', has become fairly well understood. Asymptotic local profiles, forms of stability of blow-up and dynamics post blow-up, have also been analysed. Relatively less is known about steady states of the problem, namely solutions of the elliptic system:

$$\begin{aligned} &\Delta v - \nabla(v\nabla u) = 0 \text{ in } \Omega,\\ &\Delta u - u + v = 0 \text{ in } \Omega,\\ &u, v > 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{2}$$

Steady states are of basic importance for the understanding of the global dynamics of the system since, as pointed out in [33], a Lyapunov functional for (1) is present, (see [21, 39]). This problem was first studied by Schaaf [36] in the one-dimensional case. Biler [3] established the existence of non-trivial solutions of (2) in higher dimensions in the radially symmetric case. In the general two-dimensional domain case, Wang and Wei [44], independently of Senba and Suzuki [37], proved the following result: given any positive number λ with $\lambda \in (0, (1/|\Omega|) + \lambda_1)\backslash\{4\pi m\}_{m=1,\ldots}$ (where λ_1 is the first positive eigenvalue of $-\Delta$ with Neumann boundary condition), there exists a non-constant solution to (2) with $\int_\Omega v = \lambda|\Omega|$.

The purpose of this paper is to construct non-trivial solutions to (2) with masses in the v coordinate close to $4\pi m$ for each given $m \geqslant 1$. More precisely, if $2k + l = m$, we are able to find solutions which exhibit in the limit l Dirac measures on the boundary and k inside the domain, with respective weights 4π and 8π. Our main result reads as follows.

Theorem 1.1. *Given non-negative integers k, l with $k + l \geqslant 1$, there exists a family of non-constant solutions $(u_\varepsilon, v_\varepsilon)$, $\varepsilon > 0$ to problem (2) such that*

$$\lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon = 4\pi(2k + l).$$

More precisely, up to subsequences, there exist k points $\xi_1, \ldots, \xi_k \in \Omega$ and l points $\xi_{k+1}, \ldots, \xi_{k+l} \in \partial\Omega$ such that as $\varepsilon \to 0$,

$$v_\varepsilon \rightharpoonup \sum_{i=1}^{k} 8\pi\delta_{\xi_i} + \sum_{i=1}^{l} 4\pi\delta_{\xi_{k+i}}.$$

This limiting phenomenon for steady states is a form of chemotactic collapse which would be interesting to relate with that present in finite time blow-up. Stability of these solutions opens as a basic question for a better understanding of global dynamics of (1). Hints toward a global phase portrait in a simplified radially symmetric model have been found in [4]; in particular connections between blow-up and steady state solutions are conjectured.

As we will state in theorem 1.2 below, much more accurate information on these solutions is available; in particular location of points ξ_i is explicitly described in terms of Green's function of the domain.

A basic feature of problem (2) is that it can be reduced to a scalar equation as follows. It is easy to check that solutions of (2) satisfy the relation

$$\int_\Omega v|\nabla(\log v - u)|^2 = 0,$$

so that $v = \varepsilon^2 e^u$ for some positive constant, ε. Thus system (2) is equivalent to the boundary value problem

$$\Delta u - u + \varepsilon^2 e^u = 0, \qquad u > 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{3}$$

In what follows, we assume that $N = 2$ and look for non-trivial solutions of this problem when $\varepsilon > 0$ is a small number.

In [37, 40], Senba and Suzuki characterized the asymptotic behaviour of families of solutions to (3) with uniformly bounded mass as $\varepsilon \to 0$. For $y \in \bar{\Omega}$ we denote by $G(x, y)$ Green's function of the problem

$$\Delta_x G - G + \delta_y = 0 \text{ in } \Omega, \qquad \frac{\partial G}{\partial \nu_x} = 0 \text{ on } \partial\Omega. \tag{4}$$

The *regular part* of $G(x, y)$ is defined depending on whether y lies in the domain or on its boundary as

$$H(x, y) = \begin{cases} G(x, y) + \dfrac{1}{2\pi} \log|x - y|, & \text{if } y \in \Omega, \\ G(x, y) + \dfrac{1}{\pi} \log|x - y|, & \text{if } y \in \partial\Omega. \end{cases} \tag{5}$$

In this way, $H(\cdot, y)$ is of class $C^{1,\alpha}$ in $\bar{\Omega}$. In [37] and [40], the following fact was established: if u_ε is a family of solutions to problem (3) such that

$$\lim_{\varepsilon^2 \to 0} \varepsilon^2 \int_\Omega e^{u_\varepsilon} = \lambda_0 > 0$$

then there exist non-negative integers k, l $m \geqslant 1$ for which $\lambda_0 = 4\pi(2k + l)$. Moreover, up to subsequences, there exist points ξ_i, $i = 1, \dots, m$ with $\xi_i \in \Omega$ for $i \leqslant k$ and $\xi_i \in \partial\Omega$ for $k < i \leqslant m$ for which

$$u_\varepsilon(x) \to \sum_{i=1}^{k} 8\pi G(x, \xi_j) + \sum_{i=k+1}^{m} 4\pi G(x, \xi_i),$$

uniformly on compact subsets of $\bar{\Omega} \setminus \{\xi_1, \dots, \xi_m\}$. Moreover, the m-tuple $(\xi_1, \dots, \xi_m)$ is a critical point of the functional,

$$\varphi_m(x_1, \dots, x_m) = \sum_{i=1}^{m} c_i^2 H(x_i, x_i) + \sum_{i \neq j} c_i c_j G(x_i, x_j), \tag{6}$$

where $c_i = 8\pi$ for $i = 1, \dots, k$ where $c_i = 4\pi$ for $i = k + 1, \dots, m$, defined on $\Omega^k \times (\partial\Omega)^l$ where with no ambiguity we set $\varphi_m(x_1, \dots, x_m) = +\infty$ if $x_i = x_j$ for some $i \neq j$.

Our main result then establishes the reciprocal of this property: let k, l be any non-negative integers such that $k + l \geqslant 1$. Then for any ε sufficiently small, problem (3) admits a solution u_ε satisfying the above properties. In order to make this statement more precise, let us denote

$$U_{\mu,a} = \log \frac{8\mu^2}{(\mu^2 + |x - a|^2)^2}, \qquad \mu > 0, \qquad a \in \mathbb{R}^2. \tag{7}$$

It is well known that these functions correspond to all solutions to the problem

$$\Delta u + e^u = \text{ in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^u < +\infty. \tag{8}$$

Our main result can be specified as follows.

Theorem 1.2. *Let k, l be non-negative integers with $k + l \geqslant 1$. Then for all sufficiently small ε there is a solution, u_ε, to problem (4) with the following properties:*

(1) u_ε has exactly $k+l$ local maximum points ξ_i^ε, $i = 1, \ldots, k+l$ such that $\xi_i^\varepsilon \in \Omega$, for $i \leqslant k$ and $\xi_i^\varepsilon \in \partial\Omega$ for $k+1 \leqslant i \leqslant k+l$. Furthermore

$$\lim_{\varepsilon\to 0} \varphi_m(\xi_1^\varepsilon, \ldots, \xi_m^\varepsilon) = \min_{\Omega^k \times (\partial\Omega)^l} \varphi_m.$$

(2) There are constants $\mu_j > 0$ such that

$$u_\varepsilon(x) = \sum_{j=1}^m U_{\varepsilon\mu_j, \xi_j^\varepsilon}(x) + O(1).$$

(3) $\varepsilon^2 \int_\Omega \mathrm{e}^{u_\varepsilon} \to 4\pi(2k + l)$ as $\varepsilon \to 0$.

The existence of a global minimum for the function φ_m in $\Omega^k \times (\partial\Omega)^l$ follows from properties of the Green's function; see the proof of lemma 7.1. In reality, associated with each *topologically non-trivial* for φ_m, a bubbling solution at a corresponding critical point exists, (see section 8).

It is important to remark about the analogy existing between our results and those known for the Liouville-type equation:

$$\begin{aligned} &\Delta u + \varepsilon^2 \mathrm{e}^u = 0 \qquad \text{in } \Omega, \\ &u = 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{9}$$

Asymptotic behaviour of solutions of (9) for which $\varepsilon^2 \int_\Omega \mathrm{e}^u$ remains uniformly bounded is well understood after the works [5, 28, 29]: $\varepsilon^2 \mathrm{e}^u$ approaches a superposition of Dirac deltas in the interior of Ω. Construction of solutions with this behaviour has been achieved in [1, 11, 16]. Related constructions for problems involving nonlinear, exponential boundary conditions, for which boundary concentration appears, have been performed in [12] and [13]. A special feature of the problem treated in this paper is the presence of **mixed** boundary–interior bubbling solutions. A similar phenomenon had only been observed in [19], for a different Neumann singularly perturbed problem. To capture such solutions, we use the so-called 'localized energy method'—a combination of the Lyapunov–Schmidt reduction method and variational techniques. Namely, we first use the Lyapunov–Schmidt reduction method to convert the problem into a finite dimensional one, for a suitable asymptotic *reduced energy*, related with φ_m. Such a scheme has been used in many works; see for instance [2, 10, 11, 14, 16, 18–20, 35] and references therein. Our approach shares elements with those in [11]; however, a different, more delicate functional setting has to be introduced. In what remains of this paper we shall prove theorem 1.2.

2. Ansatz for the solution

Given $\xi_j \in \bar{\Omega}$, $\mu_j > 0$ we define

$$u_j(x) = \log \frac{8\mu_j^2}{(\varepsilon^2\mu_j^2 + |x - \xi_j|^2)^2}.$$

The choice of ξ_j and μ_j will be made later on.

The ansatz is

$$U(x) = \sum_{j=1}^m (u_j(x) + H_j^\varepsilon(x)), \tag{10}$$

where H_j^ε is a correction term defined as the solution of

$$\begin{aligned} &-\Delta H_j^\varepsilon + H_j^\varepsilon = -u_j \qquad \text{in } \Omega, \\ &\frac{\partial H_j^\varepsilon}{\partial \nu} = -\frac{\partial u_j}{\partial \nu} \qquad \text{on } \partial\Omega. \end{aligned} \tag{11}$$

Lemma 2.1. *For any* $0 < \alpha < 1$,

$$H_j^\varepsilon(x) = c_j H(x, \xi_j) - \log 8\mu_j^2 + O(\varepsilon^\alpha) \tag{12}$$

uniformly in $\bar{\Omega}$, *where* H *is the regular part of Green's function defined in (5).*

We will give the proof of this lemma at the end of the section.

It will be convenient to work with the scaling of u given by

$$v(y) = u(\varepsilon y) + 4\log\varepsilon.$$

If u is a solution of (3) then v satisfies

$$\begin{aligned} &-\Delta v + \varepsilon^2(v - 4\log\varepsilon) = \mathrm{e}^v \qquad \text{in } \Omega_\varepsilon, \\ &\frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega_\varepsilon, \end{aligned} \tag{13}$$

where $\Omega_\varepsilon = \Omega/\varepsilon$. With this scaling u_j becomes

$$v_j(y) = \log \frac{8\mu_j}{(\mu_j^2 + |y - \xi_j'|^2)^2},$$

where $\xi_j' = \xi_j/\varepsilon$ and where we will write ν for the exterior normal unit vector to $\partial\Omega$ and $\partial\Omega_\varepsilon$.

Note that $u_j + H_j^\varepsilon$ satisfies

$$\begin{aligned} &-\Delta(u_j + H_j) + \varepsilon^2(u_j + H_j) = \mathrm{e}^{v_j} \qquad \text{in } \Omega_\varepsilon, \\ &\frac{\partial(u_j + H_j)}{\partial \nu} = 0 \qquad \text{on } \partial\Omega_\varepsilon. \end{aligned} \tag{14}$$

We will seek a solution v of (13) of the form

$$v = V + \phi,$$

where

$$V(y) = U(\varepsilon y) + 4\log\varepsilon \tag{15}$$

and U is defined by (10). Problem (13) can be stated so as to find ϕ a solution to

$$\begin{aligned} &-\Delta\phi + \varepsilon^2\phi = \mathrm{e}^V\phi + N(\phi) + R \qquad \text{in } \Omega_\varepsilon, \\ &\frac{\partial \phi}{\partial \nu} = 0 \qquad \text{on } \partial\Omega_\varepsilon, \end{aligned} \tag{16}$$

where the 'nonlinear term' is

$$N(\phi) = \mathrm{e}^V(\mathrm{e}^\phi - 1 - \phi) \tag{17}$$

and the 'error term' is given by

$$R = \Delta V - \varepsilon^2(V - 4\log\varepsilon) + \mathrm{e}^V. \tag{18}$$

At this point it is convenient to make a choice of the parameters μ_j, the objective being to make the error term small. We claim that if

$$\log 8\mu_j^2 = c_j H(\xi_j, \xi_j) + \sum_{i \neq j} c_i G(\xi_i, \xi_j) \tag{19}$$

then we achieve the following behaviour for R: for any $0 < \alpha < 1$ there exists C independent of ε such that

$$|R(y)| \leqslant C\varepsilon^{\alpha} \sum_{j=1}^{m} \frac{1}{1 + |y - \xi_j'|^3} \qquad \forall y \in \Omega_\varepsilon \tag{20}$$

and for $W = \mathrm{e}^V$

$$W(y) = \sum_{j=1}^{m} \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j|^2)^2}(1 + \theta_\varepsilon(y)), \qquad \forall y \in \Omega_\varepsilon \tag{21}$$

with θ_ε satisfying the following estimate:

$$|\theta_\varepsilon(y)| \leqslant C\varepsilon^{\alpha} + C\varepsilon \sum_{j=1}^{m} |y - \xi_j'| \qquad \forall y \in \Omega_\varepsilon.$$

Proof of (21).

$$\begin{aligned} W(y) &= \varepsilon^4 \exp\left(\sum_{i=1}^{m} u_j(\varepsilon y) + H_i^{\varepsilon}(\varepsilon y)\right) \\ &= \exp\left(\sum_{i=1}^{m}\left(\log \frac{8\mu_i^2}{(\mu_i^2 + |y - \xi_i'|^2)^2} + H_i^{\varepsilon}(\varepsilon y)\right)\right). \end{aligned}$$

Let us fix a small constant $\delta > 0$ and consider this expression for $|y - \xi_j'| < \delta/\varepsilon$:

$$W(y) = \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2} \exp\left(H_j^{\varepsilon}(\varepsilon y) + \sum_{i \neq j}^{m}\left[\log \frac{8\mu_i^2}{(\varepsilon^2\mu_i^2 + \varepsilon^2|y - \xi_i'|^2)^2} + H_i^{\varepsilon}(\varepsilon y)\right]\right).$$

Using (12) and the fact that H is $C^1(\partial\Omega^2)$ we have

$$\begin{aligned} H_i^{\varepsilon}(\varepsilon y) &= c_i H(\varepsilon y, \xi_i) - \log(8\mu_i^2) + O(\varepsilon^{\alpha}) \qquad \forall y \in \Omega_\varepsilon \\ &= c_i H(\xi_j, \xi_i) - \log(8\mu_i^2) + O(\varepsilon^{\alpha}) + O(\varepsilon|y - \xi_j'|) \qquad \forall y \in \Omega_\varepsilon. \end{aligned}$$

Hence for $|y - \xi_j'| < \delta/\varepsilon$,

$$\begin{aligned} &H_j^{\varepsilon}(\varepsilon y) + \sum_{i \neq k}^{m}\left(\log \frac{8\mu_i^2}{(\varepsilon^2\mu_i^2 + \varepsilon^2|y - \xi_i'|^2)^2} + H_i^{\varepsilon}(\varepsilon y)\right) \\ &\quad = c_j H(\xi_j, \xi_j) - \log(8\mu_j^2) + \sum_{i \neq j}^{m}\left(\log \frac{8\mu_i^2}{|\xi_j - \xi_i|^4} + c_i H(\xi_j, \xi_i) - \log(8\mu_i^2)\right) \\ &\qquad + O(\varepsilon^{\alpha}) + O(\varepsilon|y - \xi_j'|). \\ &\quad = c_j H(\xi_j, \xi_j) - \log(8\mu_j^2) + \sum_{i \neq j}^{m} c_i G(\xi_j, \xi_i) + O(\varepsilon^{\alpha}) + O(\varepsilon|y - \xi_j'|) \\ &\quad = O(\varepsilon^{\alpha}) + O(\varepsilon|y - \xi_j'|) \end{aligned}$$

by the choice of μ_j (cf (19)). Therefore

$$W(y) = \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2}(1 + O(\varepsilon^{\alpha}) + O(\varepsilon|y - \xi_j'|)) \qquad \forall |y - \xi_j'| < \frac{\delta}{\varepsilon}. \tag{22}$$

If $|y - \xi_j'| > \delta/\varepsilon$ for all $j = 1, \ldots, m$ we have $W = O(\varepsilon^4)$, and this together with (22) implies (21). □

Proof of (20). We defined $R = \Delta V - \varepsilon^2(V - 4\log\varepsilon) + \mathrm{e}^V$ with V given by (15).

By our definition and (14),

$$R = \varepsilon^4 \mathrm{e}^{\sum_{j=1}^m (u_j + H_j)} - \sum_{j=1}^m \mathrm{e}^{v_j}.$$

For $|y - \xi_j'| < \delta/\varepsilon$, we have according to (22)

$$\begin{aligned} R &= \mathrm{e}^{v_j}(1 + O(\varepsilon^\alpha) + O(\varepsilon|y - \xi_j'|)) - \mathrm{e}^{v_j} + O(\varepsilon^4) \\ &= O(\mathrm{e}^{v_j}(\varepsilon^\alpha + \varepsilon|y - \xi_j'|)) + O(\varepsilon^4), \end{aligned}$$

which proves (20). □

Proof of lemma 2.1. The boundary condition satisfied by H_j^ε is

$$\frac{\partial H_j^\varepsilon}{\partial \nu} = -\frac{\partial u_j}{\partial \nu} = 4\frac{(x - \xi_j)\cdot\nu(x)}{\varepsilon^2\mu_j^2 + |x - \xi_j|^2}. \tag{23}$$

Thus for $\xi_j \in \Omega$, $j = 1, \ldots, k$, we have

$$\frac{\partial H_j^\varepsilon}{\partial \nu} = 4\frac{(x - \xi_j)\cdot\nu(x)}{|x - \xi_j|^2} + O(\varepsilon^2) \qquad \text{on } \partial\Omega. \tag{24}$$

For $\xi_j \in \partial\Omega$, $j = k+1, \ldots, k+l$, we have

$$\lim_{\varepsilon\to 0} \frac{\partial H_j^\varepsilon}{\partial \nu}(x) = 4\frac{(x - \xi_j)\cdot\nu(x)}{|x - \xi_j|^2} \qquad \forall x \neq \xi_j. \tag{25}$$

The regular part of Green's function, $H(x, \xi_j)$, satisfies

$$\begin{aligned} -\Delta_x H(x,\xi_j) + H(x,\xi_j) &= -\frac{4}{c_j}\log\frac{1}{|x - \xi_j|} \qquad x \in \Omega, \\ \frac{\partial H}{\partial \nu_x}(x,\xi_j) &= \frac{4}{c_j}\frac{(x - \xi_j)\cdot\nu(x)}{|x - \xi_j|^2} \qquad x \in \partial\Omega. \end{aligned}$$

For the difference $z_\varepsilon(x) = H_j^\varepsilon(x) + \log 8\mu_j^2 - c_j H(x, \xi_j)$ we have

$$\begin{aligned} -\Delta z_\varepsilon + z_\varepsilon &= -\log\frac{1}{(\varepsilon^2\mu_j^2 + |x - \xi_j|^2)^2} + \log\frac{1}{|x - \xi_j|^4} \qquad \text{in } \Omega, \\ \frac{\partial z_\varepsilon}{\partial \nu} &= \frac{\partial H_j^\varepsilon}{\partial \nu} - 4\frac{(x - y)\cdot\nu(x)}{|x - y|^2} \qquad \text{on } \partial\Omega. \end{aligned}$$

We claim that for any $p > 1$ there exists $C > 0$ such that

$$\left\| \frac{\partial H_j^\varepsilon}{\partial \nu} - 4\frac{(x - \xi_j)\cdot\nu(x)}{|x - \xi_j|^2} \right\|_{L^p(\partial\Omega)} \leqslant C\varepsilon^{1/p}. \tag{26}$$

For this it will be convenient to observe first that for $\xi_j \in \partial\Omega$

$$|(x - \xi_j)\cdot\nu(x)| \leqslant C|x - \xi_j|^2 \qquad \forall x \in \partial\Omega, \tag{27}$$

which can be proved, for example, assuming that $\xi_j = 0$ and that near the origin $\partial\Omega$ is the graph of a function $G : (-a, a) \to \mathbb{R}$ with $G(0) = G'(0) = 0$. Now

$$\frac{\partial H_j^\varepsilon}{\partial \nu} - 4\frac{(x - \xi_j)\cdot\nu(x)}{|x - \xi_j|^2} = \varepsilon^2\mu_j^2\frac{(x - \xi_j)\cdot\nu(x)}{|x - \xi_j|^2(\varepsilon^2\mu_j^2 + |x - \xi_j|^2)}. \tag{28}$$

By (27)

$$\left| \frac{\partial H_j^{\varepsilon}}{\partial \nu} - 4 \frac{(x-\xi_j)\cdot \nu(x)}{|x-\xi_j|^2} \right| \leqslant C \frac{\varepsilon^2}{\varepsilon^2 \mu_j^2 + |x-\xi_j|^2}. \tag{29}$$

Fix $\rho > 0$ small. Then

$$\left| \frac{\partial H_j^{\varepsilon}}{\partial \nu} - 4 \frac{(x-\xi_j)\cdot \nu(x)}{|x-\xi_j|^2} \right| \leqslant C\varepsilon^2 \qquad \forall |x-\xi_j| \geqslant \rho, \qquad x \in \partial\Omega. \tag{30}$$

Now let $p > 1$. Changing variables $x - \xi_j = \varepsilon y$ we have

$$\int_{B_\rho(\xi_j)\cap\partial\Omega} \left| \frac{\varepsilon^2}{(\varepsilon^2\mu_j^2 + |x-\xi_j|^2)} \right|^p \mathrm{d}x = C\varepsilon \int_{B_{\rho/\varepsilon}(0)\cap\Omega_\varepsilon} \left| \frac{1}{\mu_j^2 + |y|^2} \right|^p \mathrm{d}y$$
$$\leqslant C\varepsilon \int_0^{\rho/\varepsilon} \frac{1}{(1+s^2)^p} \mathrm{d}s$$
$$\leqslant C\varepsilon.$$

Combining this with (29) and (30) we conclude that (26) holds.

For $p > 1$ let us now estimate

$$\left\| \log \frac{1}{|x-\xi_j|^2} - \log \frac{1}{\varepsilon^2\mu_j^2 + |x-\xi_j|^2} \right\|_{L^p(\Omega)}^p = \int_{B_{10\varepsilon\mu_j}(\xi_j)\cap\Omega} \ldots + \int_{\Omega\setminus B_{10\varepsilon\mu_j}(\xi_j)} \ldots = I_1 + I_2.$$

For I_1 observe that

$$\int_{B_{10\varepsilon\mu_j}(\xi_j)\cap\Omega} \left| \log \frac{1}{|x-\xi_j|^2} \right|^p \mathrm{d}x \leqslant C \int_0^{C\varepsilon} |\log r|^p r \,\mathrm{d}r \leqslant C\varepsilon^2 (\log 1/\varepsilon)^p.$$

The same bound is true for the integral of $|\log(1/\varepsilon^2\mu_j^2 + |x-\xi_j|^2)|^p$ in $B_{10\varepsilon\mu_j}(\xi_j)\cap\Omega$. Hence

$$|I_1| \leqslant C\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^p.$$

Let us estimate I_2 as follows:

$$\left| \log \frac{1}{|x-\xi_j|^2} - \log \frac{1}{\varepsilon^2\mu_j^2 + |x-\xi_j|^2} \right| \leqslant \frac{C\varepsilon}{|x-\xi_j|}.$$

Take $1 < p < 2$ and integrate

$$|I_2| \leqslant C\varepsilon^p \int_{10\mu\varepsilon}^{D} r^{1-p} \,\mathrm{d}r \leqslant C\varepsilon^p,$$

where D is the diameter of Ω. In conclusion, for any $1 < p < 2$, we have

$$\left\| \log \frac{1}{|x-\xi_j|^2} - \log \frac{1}{\varepsilon^2\mu_j^2 + |x-\xi_j|^2} \right\|_{L^p(\Omega)} \leqslant C\varepsilon.$$

By L^p theory

$$\|z_\varepsilon\|_{W^{1+s,p}(\Omega)} \leqslant C \left(\left\| \frac{\partial z_\varepsilon}{\partial \nu} \right\|_{L^p(\partial\Omega)} + \|\Delta z_\varepsilon\|_{L^p(\Omega)} \right) \leqslant C\varepsilon^{1/p}$$

for any $0 < s < 1/p$. By Morrey embedding we obtain

$$\|z_\varepsilon\|_{C^\gamma(\bar{\Omega})} \leqslant C\varepsilon^{1/p}$$

for any $0 < \gamma < (1/2) + (1/p)$. This proves the result (with $\alpha = 1/p$). □

Remark. The convergence (25) is not uniform in general because $(\partial H_j^\varepsilon/\partial\nu)(\xi_j) = 0$ while the function $x \mapsto 2((x-\xi_j)\cdot\nu(x)/|x-\xi_j|^2)$ can be extended continuously to ξ_j with a value equal to the curvature of $\partial\Omega$ at ξ_j.

3. Solvability of a linear equation

The main result of this section is the solvability of the following linear problem: given h find $\phi, c_{11}, \ldots, c_{mJ_m}$ such that

$$
\begin{aligned}
&-\Delta\phi + \varepsilon^2\phi = W\phi + h + \sum_{j=1}^{m}\sum_{i=1,J_j} c_{ij}\chi_j Z_{ij} \qquad \text{in } \Omega_\varepsilon,\\
&\frac{\partial\phi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega_\varepsilon,\\
&\int_{\Omega_\varepsilon} \chi_j Z_{ij}\phi = 0 \qquad \forall j = 1,\ldots,m,\ i = 1, J_j,
\end{aligned}
\tag{31}
$$

where $m = k + l$, W is a function that satisfies (21), $h \in L^\infty(\Omega_\varepsilon)$ and Z_{ij}, χ_j are defined as follows: $J_j = 2$ if $j = 1,\ldots,k$ and $J_j = 1$ if $j = k+1,\ldots,k+l$.

Let z_{ij} be

$$z_{0j} = \frac{1}{\mu_j} - 2\frac{\mu_j}{\mu_j^2 + y^2}, \qquad z_{ij} = \frac{y_i}{\mu_j^2 + y^2}.$$

It is well known that any solution to

$$\Delta\phi + \mathrm{e}^{v_j}\phi = 0,\ |\phi| \leqslant C(1+|y|)^\sigma \tag{32}$$

is a linear combination of z_{ij}, $i = 0, 1, 2$. (See lemma 2.1 of [6].)

Next we choose a large but fixed number R_0 and non-negative smooth function $\chi : \mathbb{R} \to \mathbb{R}$ so that $\chi(r) = 1$ for $r \leqslant R_0$ and $\chi(r) = 0$ for $r \geqslant R_0 + 1$, $0 \leqslant \chi \leqslant 1$.

For $j = 1,\ldots,k$ (corresponding to **interior** bubble case), we define

$$\chi_j(y) = \chi(|y - \xi_j'|), \qquad Z_{ij}(y) = z_{ij}(y), \quad i = 0, 1, 2, \quad j = 1,\ldots,k. \tag{33}$$

For $j = k+1,\ldots,k+l$ (corresponding to **boundary** bubble case), we have to strengthen the boundary first. More precisely, at the boundary point $\xi_j \in \partial\Omega$, we assume that $\xi_j = 0$ and the unit outward normal at ξ_j is $-\mathbf{e}_2 = (0,-1)$. Let $G(x_1)$ be the defining function for the boundary $\partial\Omega$ in a neighbourhood $B_\rho(\xi_j)$ of ξ_j, that is, $\Omega \cap B_\rho(\xi_j) = \{(x_1,x_2) | x_2 > G(x_1), (x_1,x_2) \in B_\rho(\xi_j)\}$. Then, let $F_j : B_\rho(\xi_j) \cap N \to \mathbb{R}^2$ be defined by

$$F_j = (F_{j,1}, F_{j,2}), \qquad \text{where } F_{j,1} = x_1 + \frac{x_2 - G(x_1)}{1 + |G'(x_1)|^2} G'(x_1), \qquad F_{j,2} = x_2 - G(x_1). \tag{34}$$

Then we set

$$F_j^\varepsilon(y) = \frac{1}{\varepsilon} F_j(\varepsilon y). \tag{35}$$

Note that F_j preserves the Neumann boundary condition. Define

$$\chi_j(y) = \chi(|F_j^\varepsilon(y)|),\ Z_{ij}(y) = z_{ij}(F_j^\varepsilon(y)) \qquad i = 0, 1 \qquad j = k+1,\ldots,k+l. \tag{36}$$

It is important to note that

$$\Delta Z_{0j} + \mathrm{e}^{v_j} Z_{0j} = O(\varepsilon(1 + |y - \xi_j'|)^{-3}) \tag{37}$$

since

$$\nabla z_{0j} = O\left(\frac{1}{(1+|y-\xi_j^{'}|)^3}\right).$$

All the above functions depend on ε but we omit this dependence in the notation. Furthermore, now all Z_{ij} satisfy the Neumann boundary condition (since F_j preserves the Neumann boundary condition).

Equation (31) will be solved for $h \in L^\infty(\Omega_\varepsilon)$ but we will be able to estimate the size of the solution in terms of the following norm:

$$\|h\|_\infty = \sup_{y\in\Omega_\epsilon} |h(y)|, \qquad \|h\|_* = \sup_{y\in\Omega_\varepsilon} \frac{|h(y)|}{\varepsilon^2 + \sum_{j=1}^m (1+|y-\xi_j'|)^{-2-\sigma}}, \tag{38}$$

where we fix $0 < \sigma < 1$ although the precise choice will be made later on.

Proposition 3.1. *Let $d > 0$ and m a positive integer. Then there exist $\varepsilon_0 > 0$, C such that for any $0 < \varepsilon < \varepsilon_0$, any family of points $(\xi_1, \ldots, \xi_m) \in \mathcal{M}_\delta$ and any $h \in L^\infty(\Omega_\varepsilon)$ there is a unique solution $\phi \in L^\infty(\Omega_\varepsilon)$, $c_{ij} \in \mathbb{R}$ to (31). Moreover*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C \log\frac{1}{\varepsilon}\|h\|_*.$$

We begin by stating an *a priori* estimate for solutions of (31) satisfying orthogonality conditions with respect to Z_{ij}, $i = 0, 1, J_j$, $j = 1, \ldots, m$.

Lemma 3.2. *There are $R_0 > 0$ and $\varepsilon_0 > 0$ so that for $0 < \varepsilon < \varepsilon_0$ and any solution ϕ of (31) with orthogonality conditions*

$$\int_{\Omega_\varepsilon} Z_{ij}\chi_j\phi = 0 \qquad \forall i = 0, \ldots, J_j \qquad \forall j = 1, \ldots, m \tag{39}$$

we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\|h\|_*,$$

where C is independent of ε.

For the proof of this lemma we need to construct a suitable barrier.

Lemma 3.3. *For $\varepsilon > 0$ small enough there exist $R_1 > 0$ and*

$$\psi : \Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi_j') \to \mathbb{R}$$

smooth and positive so that

$$-\Delta\psi + \varepsilon^2\psi - W\psi \geqslant \sum_{j=1}^m \frac{1}{|y-\xi_j'|^{2+\sigma}} + \varepsilon^2 \qquad \text{in } \Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi_j'),$$

$$\frac{\partial\psi}{\partial\nu} \geqslant 0 \qquad \text{on } \partial\Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi_j'),$$

$$\psi > 0 \qquad \text{in } \Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi_j'),$$

$$\psi \geqslant 1 \qquad \text{on } \Omega_\varepsilon \cap \left(\bigcup_{j=1}^m \partial B_{R_1}(\xi_j')\right).$$

The constants $R_1 > 0$, $c > 0$ can be chosen independently of ε and ψ is bounded uniformly:

$$0 < \psi \leqslant C \qquad \text{in } \Omega_\varepsilon \Big\backslash \bigcup_{j=1}^{m} B_{R_1}(\xi_j').$$

Proof of lemma 3.2. We take $R_0 = 2R_1$, R_1 being the constant of lemma 3.3. Thanks to the barrier ψ of that lemma we deduce that the following maximum principle holds in $\Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi_j')$: if $\phi \in C^2(\Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi_j'))$ satisfies

$$-\Delta\phi + \varepsilon^2\phi \geqslant W\phi \qquad \text{in } \Omega_\varepsilon \Big\backslash \bigcup_{j=1}^{m} B_{R_1}(\xi_j'),$$

$$\frac{\partial\phi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega_\varepsilon \Big\backslash \bigcup_{j=1}^{m} B_{R_1}(\xi_j'),$$

$$\phi \geqslant 0 \qquad \text{on } \Omega_\varepsilon \cap \left(\bigcup_{j=1}^{m} \partial B_{R_1}(\xi_j') \right),$$

then $\phi \geqslant 0$ in $\Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi_j')$.

Let h be bounded and ϕ a solution to (31) satisfying (39). Following [11] we first claim that $\|\phi\|_{L^\infty(\Omega_\varepsilon)}$ can be controlled in terms of $\|h\|_*$ and the following inner norm of ϕ:

$$\|\phi\|_i = \sup_{\Omega_\varepsilon \cap (\bigcup_{j=1}^m B_{R_1}(\xi_j'))} |\phi|.$$

Indeed, set

$$\tilde{\phi} = C_1\psi(\|\phi\|_i + \|f\|_*),$$

with C_1 a constant independent of ε. By the above maximum principle we have $\phi \leqslant \tilde{\phi}$ and $-\phi \leqslant \tilde{\phi}$ in $\Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{R_1}(\xi_j')$. Since ψ is uniformly bounded we deduce

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C(\|\phi\|_i + \|f\|_*), \tag{40}$$

for some constant C independent of ϕ and ε.

We prove the lemma by contradiction. Assume that there exists a sequence $\varepsilon_n \to 0$, points $(\xi_1^n, \ldots, \xi_m^n) \in \mathcal{M}_\delta$ and functions ϕ_n, f_n and h_n with $\|\phi_n\|_{L^\infty(\Omega_{\varepsilon_n})} = 1$ and $\|h_n\|_* \to 0$ so that for each n ϕ_n solves (31) and satisfies (39). By (40) we see that $\|\phi_n\|_i$ stays away from zero. For one of the indices, say j, we can assume that $\sup_{B_{R_1}(\xi_j')} |\phi_n| \geqslant c > 0$ for all n. Consider $\hat{\phi}_n(z) = \phi_n(z - \xi_j')$ and let us translate and rotate Ω_{ε_n} so that Ω_{ε_n} approaches the upper half-plane $\mathbb{R}^2_+$ and $\xi_j' = 0$. Then by elliptic estimates $\hat{\phi}_n$ converges uniformly on compact sets to a non-trivial solution of

$$\Delta\phi + \mathrm{e}^{v_j}\phi = 0, \; |\phi| \leqslant C.$$

Thus $\hat{\phi}$ is a linear combination of z_{ij}, $i = 0, \ldots, J_j$. On the other hand we can take the limit in the orthogonality relations (39), observing that limits of the functions Z_{ij} are just rotations and translations of z_{ij}, and we find $\int_{\mathbb{R}^2_+} \chi\hat{\phi}\, z_{ij} = 0$ for $i = 0, J_j$. This contradicts the fact that $\hat{\phi} \not\equiv 0$. $\square$

Proof of lemma 3.3.

We take

$$\psi_{1j}(r) = 1 - \frac{1}{r^\sigma} \qquad \text{where } r = |y - \xi_j^{'}|. \tag{41}$$

Then

$$-\Delta\psi_{1j} = \sigma^2 \frac{1}{r^{2+\sigma}}.$$

If $\xi_j^{'} \in \Omega_\varepsilon$ then we have

$$\frac{\partial\psi_{1j}}{\partial\nu_\varepsilon} = O(\varepsilon^{1+\sigma}) \qquad \text{on } \partial\Omega_\varepsilon.$$

If $\xi_j^{'} \in \Omega_\varepsilon$ and $|y-\xi_j^{'}| > R$, we have

$$\frac{\partial\psi_{1j}}{\partial\nu_\varepsilon} = \sigma \frac{\langle y-\xi_j^{'}, \nu_\varepsilon\rangle}{r^{2+\sigma}}.$$

As before, we write $\partial\Omega_\varepsilon$ near ξ_j' as the graph $\{(y_1, y_2) : y_2 = (1/\varepsilon)G(\varepsilon y_1)\}$ with $G(0) = 0$ and $G'(0) = 0$. Then

$$\begin{aligned}\frac{\partial\psi_{1j}}{\partial\nu} &= \frac{\sigma}{r^{2+\sigma}} \frac{1}{\sqrt{G'(\varepsilon y_1)^2+1}} \left(-y_1 G'(\varepsilon y_1), \frac{1}{\varepsilon}G(\varepsilon(y_1)\right) \\ &= \frac{\sigma}{r^{2+\sigma}} \frac{1}{\sqrt{O(\delta^2)+1}} O(\varepsilon r^2) \qquad \forall R_1 < r < \delta/\varepsilon \\ &= O\left(\frac{\varepsilon}{r^\sigma}\right) \qquad \forall R_1 < r < \delta/\varepsilon.\end{aligned}$$

Combining together, we see that

$$\frac{\partial\psi_{1j}}{\partial\nu_\varepsilon} = o(\varepsilon) \qquad \text{on } \partial\Omega_\varepsilon. \tag{42}$$

Now let ψ_0 be the unique solution of

$$\Delta\psi_0 - \varepsilon^2\psi_0 + \varepsilon^2 = 0 \quad \text{in } \Omega_\varepsilon, \qquad \frac{\partial\psi_0}{\partial\nu_\varepsilon} = \varepsilon \quad \text{on } \partial\Omega_\varepsilon.$$

Set

$$\psi = \sum_{j=1}^m \psi_{1j} + C\psi_0. \tag{43}$$

Then for $|y-\xi_j^{'}| > R$, $j = 1, \ldots, k+l$ where R is large

$$-\Delta\psi + \varepsilon^2\psi - W\psi \geqslant C\varepsilon^2 + \sigma^2 \sum_{j=1}^m \frac{1}{|y-\xi_j^{'}|^{2+\sigma}} - CW \geqslant \frac{\sigma^2}{2} \sum_{j=1}^m \frac{1}{|y-\xi_j^{'}|^{2+\sigma}} + \varepsilon^2 \tag{44}$$

since

$$W \leqslant \sum_{j=1}^m \frac{1}{1+|y-\xi_j^{'}|^4}.$$

On $\partial\Omega_\varepsilon$,

$$\frac{\partial\psi}{\partial\nu_\varepsilon} \geqslant \frac{\varepsilon}{2}.$$

It is easy to see that $(4/\sigma^2)\psi$ satisfies all the properties of the lemma. □

We will establish next an *a priori* estimate for solutions to problem (31) that satisfy orthogonality conditions with respect to Z_{ij}, $i = 1, J_j$ only.

Lemma 3.4. *For ε sufficiently small, if ϕ solves*

$$-\Delta\phi+\varepsilon^2\phi+W\phi=h \qquad \text{in } \Omega_\varepsilon, \qquad \frac{\partial\phi}{\partial\nu}=0 \qquad \text{on } \partial\Omega_\varepsilon \tag{45}$$

and satisfies

$$\int_{\Omega_\varepsilon} Z_{ij}\chi_j\phi=0 \qquad \forall j=1,\dots,m, \qquad i=1,J_j \tag{46}$$

then

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)}\leqslant C\log\frac{1}{\varepsilon}\|h\|_*, \tag{47}$$

where C is independent of ε.

Proof. Let ϕ satisfy (45) and (46). We will modify ϕ to satisfy all orthogonality relations in (39) and for this purpose we consider modifications with compact support of the functions Z_{0j}. Let $R>R_0+1$ be large and fixed.

Let

$$a_{0j}=\frac{1}{\mu_j((4/c_j)\log(1/\varepsilon R)+H(\xi_j,\xi_j))}. \tag{48}$$

Set

$$\hat Z_{0j}(y)=Z_{0j}(y)-\frac{1}{\mu_j}+a_{0j}G(\xi_j,\varepsilon y). \tag{49}$$

Note that by our definition, $\hat Z_{0,j}$ satisfies the Neumann boundary condition.

Let η be radial smooth cut-off function on $\mathbb{R}^2$ so that

$$0\leqslant\eta\leqslant1, \qquad |\nabla\eta|\leqslant C \text{ in } \mathbb{R}^2, \qquad \eta\equiv1 \text{ in } B_R(0) \text{ and } \eta\equiv0 \text{ in } \mathbb{R}^2\setminus B_{R+1}(0).$$

Set

$$\eta_j(y)=\eta(|y-\xi_j'|) \quad \text{for } j=1,\dots,k, \qquad \eta_j(y)=\eta(F_j^\varepsilon(y)) \quad \text{for } j=k+1,\dots,k+l. \tag{50}$$

Now define

$$\tilde Z_{0j}=\eta_j Z_{0j}+(1-\eta_j)\hat Z_{0j}. \tag{51}$$

Given ϕ satisfying (45) and (46) let

$$\tilde\phi=\phi+\sum_{j=1}^m d_j\tilde Z_{0j}, \qquad \text{where } d_j=-\frac{\int_{\Omega_\varepsilon} Z_{0j}\chi_j\phi}{\int_{\Omega_\varepsilon} Z_{0j}^2\chi_j}.$$

Estimate (47) is a direct consequence of the following claim.

Claim.

$$|d_j|\leqslant C\log\frac{1}{\varepsilon}\|h\|_* \qquad \forall j=1,\dots,m. \tag{52}$$

We start proving this by observing, using the notation $L=-\Delta+\varepsilon^2-W$, that

$$L(\tilde\phi)=h+\sum_{j=1}^m d_jL(\tilde Z_{0j}) \qquad \text{in } \Omega_\varepsilon, \tag{53}$$

and

$$\frac{\partial\tilde\phi}{\partial\nu}=0 \qquad \text{on } \partial\Omega_\varepsilon. \tag{54}$$

Thus by lemma 3.2 we have

$$\|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} \leqslant C \sum_{j=1}^{m} |d_j| \|L(\tilde{Z}_{0j})\|_* + C\|h\|_*. \tag{55}$$

Multiplying equation (53) by $\tilde{Z}_{0k}$, integrating by parts and using (54) we find

$$\sum_{j=1}^{m} d_j \int_{\Omega_\varepsilon} L(\tilde{Z}_{0j})\tilde{Z}_{0k} \leqslant C\|h\|_* \left[1 + \sum_{j=1}^{m} \|L(\tilde{Z}_{0j})\|_*\right] + C \sum_{j=1}^{m} |d_j| \|L(\tilde{Z}_{0j})\|_*^2. \tag{56}$$

We now measure the size of $\|L(\tilde{Z}_{0j})\|_*$. To this end, we have for $|y - \xi_j'| > R$, according to (37),

$$L(\hat{Z}_{0j}) = -\mathrm{e}^{v_j} Z_{0j} + W\hat{Z}_{0j} + O(\varepsilon(1 + |y - \xi_j'|)^{-3}) = \mathrm{e}^{v_j}\left(a_{0j}G(\xi_j, \varepsilon y) - \frac{1}{\mu_j}\right)$$
$$+O(\varepsilon(1 + |y - \xi_j'|)^{-3}). \tag{57}$$

Thus

$$\|(1 - \eta_j)L(\hat{Z}_{0j})\|_* \leqslant \frac{C}{\log(1/\varepsilon)}, \tag{58}$$

where the number C depends in principle on the chosen large constant R.

So

$$L(\tilde{Z}_{0j}) = \eta_j L(Z_{0j}) + (1 - \eta_j)L(\hat{Z}_{0j}) + 2\nabla\eta_j\nabla(Z_{0j} - \hat{Z}_{0j}) + \Delta\eta_j(Z_{0j} - \hat{Z}_{0j})$$
$$= O(\varepsilon^{2+\alpha}) + (1 - \eta_j)\mathrm{e}^{v_j}\left(a_{0j}G(\xi_j, \varepsilon y) - \frac{1}{\mu_j}\right) + 2\nabla\eta_j\nabla(Z_{0j} - \hat{Z}_{0j})$$
$$+\Delta\eta_j(Z_{0j} - \hat{Z}_{0j}). \tag{59}$$

Note that for $r = |y - \xi_j'| \in (R, R + 1)$, we have

$$\hat{Z}_{0j} - Z_{0j} = a_{0j}G(\xi_j, \varepsilon y) - \frac{1}{\mu_j}$$
$$= a_{0j}\left(\frac{4}{c_j}\log\frac{1}{\varepsilon|\xi_j' - y|} + H(\xi_j, \varepsilon y)\right) - \frac{1}{\mu_j}.$$

Hence we derive that for $r \in (R, R + 1)$

$$\hat{Z}_{0j} - Z_{0j} = \frac{C}{\log(1/\varepsilon)}\log\frac{1}{r} + O\left(\frac{\varepsilon^\alpha}{\log(1/\varepsilon)}\right), \nabla(\hat{Z}_{0j} - Z_{0j}) = -\frac{C}{\log(1/\varepsilon)}\frac{1}{r}$$
$$+O\left(\frac{\varepsilon^\alpha}{\log(1/\varepsilon)}\right). \tag{60}$$

From (59) and (60), we conclude that

$$\|L(\tilde{Z}_{0j})\|_* \leqslant \frac{C}{\log(1/\varepsilon)}. \tag{61}$$

Now we estimate the left-hand side integral of (56). From (59), we see that for $j \neq k$,

$$\int_{\Omega_\varepsilon} L(\tilde{Z}_{0j})\tilde{Z}_{0k} = O(\varepsilon^\alpha) + \int_{\Omega_\varepsilon} O\left(\frac{1}{\log(1/\varepsilon)}(|\eta_j'| + |\Delta\eta_j|)\right)\tilde{Z}_{0,k} = O\left(\left(\frac{1}{\log(1/\varepsilon)}\right)^2\right).$$

For $j = k$, we decompose

$$\int_{\Omega_\varepsilon} L(\tilde{Z}_{0k})\tilde{Z}_{0k} = I + II + O(\varepsilon),$$

where

$$II = \int_{\Omega_\varepsilon} O(\varepsilon^{2+\alpha}) + (1-\eta_k)\mathrm{e}^{v_j}\left(a_{0k}G(\xi_k,\varepsilon y) - \frac{1}{\mu_k}\right)\tilde{Z}_{0k}$$
$$= O(\varepsilon^{\alpha}) + O\left(\frac{1}{R}\frac{1}{\log(1/\varepsilon)}\right)$$

and

$$I = \int_{\Omega_\varepsilon} (2\nabla\eta_k\nabla(Z_{0k} - \hat{Z}_{0k}) + \Delta\eta_k(Z_{0k} - \hat{Z}_{0k}))\tilde{Z}_{0k}.$$

Thus integrating by parts we find

$$I = \int \nabla\eta\nabla(Z_{0k} - \hat{Z}_{0k})\hat{Z}_{0k} - \int \nabla\eta_j(Z_{0k} - \hat{Z}_{0k})\nabla\hat{Z}_{0k} + O(\varepsilon).$$

Now, we observe that in the considered region, $r \in (R, R+1)$ with $r = |y-\xi_k^{'}|$, $|\hat{Z}_{0k} - Z_{0k}| \leqslant C/\log(1/\varepsilon)$ while $|\nabla Z_{0k}^{'}| \leqslant 1/R^3 + 1/(R\log(1/\varepsilon))$. Thus

$$\left|\int \nabla\eta_j(Z_{0k} - \hat{Z}_{0k})\nabla\hat{Z}_{0k}\right| \leqslant \frac{D}{R^3}\frac{1}{\log(1/\varepsilon)},$$

where D may be chosen independent of R. Now

$$\int \nabla\eta_k\nabla(Z_{0k} - \hat{Z}_{0k})\hat{Z}_{0k} = \int_R^{R+1} \eta^{'}\left(a_{0k}\frac{1}{r} + O(\varepsilon)\right)\hat{Z}_{0k}r\,\mathrm{d}r$$
$$= a_{0k}\int_R^{R+1}\eta^{'}\left(1 + O(\varepsilon) + O\left(\frac{1}{R}\right)\right)$$
$$= -\frac{E}{\log(1/\varepsilon)}\left[1 + O\left(\frac{1}{R}\right)\right],$$

where E is a positive constant independent of ε. Thus we conclude, choosing R large enough, that $I \sim -\frac{E}{\log(1/\varepsilon)}$. Combining this and the estimate for II we find

$$\int_{\Omega_\varepsilon} L(\tilde{Z}_{0k})\tilde{Z}_{0k} = -\frac{E}{\log(1/\varepsilon)}\left[1 + O\left(\frac{1}{R}\right)\right], \qquad \int_{\Omega_\varepsilon} L(\tilde{Z}_{0j})\tilde{Z}_{0k} = O\left(\frac{1}{R}\frac{1}{\log(1/\varepsilon)}\right)$$
$$\text{for } j \neq k. \tag{62}$$

This, combined with (56), proves the lemma. □

Proof of proposition 3.1.

First we prove that for any $\phi, d_1, \ldots, d_m$ solution to (31) the bound

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\log\frac{1}{\varepsilon}\|h\|_* \tag{63}$$

holds.

The previous lemma yields

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\log\frac{1}{\varepsilon}\left(\|h\|_* + \sum_{j=1}^{m}\sum_{i=1}^{J_j}|c_{ij}|\right). \tag{64}$$

So it suffices to estimate the values of the constants c_{ij}.

To this end, we multiple (31) by Z_{ij} and integrate to find

$$\int_{\Omega_\varepsilon} L(\phi)(Z_{ij}) = \int_{\Omega_\varepsilon} hZ_{ij} + c_{ij}\int_{\Omega_\varepsilon}\chi_j|Z_{ij}|^2. \tag{65}$$

Note that for $i \neq 0$

$$Z_{ij} = O\left(\frac{1}{1+|y-\xi_j|}\right).$$

So

$$\int_{\Omega_\varepsilon} h Z_{ij} = O(\|h\|_*) \tag{66}$$

and

$$\int_{\Omega_\varepsilon} L(\phi) Z_{ij} = \int_{\Omega_\varepsilon} L(Z_{ij})\phi = O(\varepsilon\|\phi\|_\infty). \tag{67}$$

Substituting (66) and (67) into (65), we obtain (64).

Now consider the Hilbert space

$$H = \left\{\phi \in H^1(\Omega_\varepsilon) : \int_{\partial\Omega_\varepsilon} \chi_j Z_{ij}\phi = 0 \qquad \forall j = 1,\ldots,m, \qquad i = 1, J_j\right\}$$

with the norm $\|\phi\|^2_{H^1} = \int_{\Omega_\varepsilon} |\nabla\phi|^2 + \varepsilon^2\phi^2$. Equation (31) is equivalent to finding $\phi \in H$ such that

$$\int_{\Omega_\varepsilon} (\nabla\phi\nabla\psi + \varepsilon^2\phi\psi) - \int_{\Omega_\varepsilon} W\phi\psi = \int_{\partial\Omega_\varepsilon} h\psi \qquad \forall\psi \in H.$$

By Fredholm's alternative this is equivalent to the uniqueness of solutions to this problem, which is guaranteed by (63). □

The result of proposition 3.1 implies that the unique solution $\phi = T(h)$ of (31) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of all functions h in L^∞ for which $\|h\|_* < \infty$, into L^∞.

It is important for later purposes to understand the differentiability of the operator T with respect to the variables, $\xi'_{k,l}$. Fix $h \in \mathcal{C}_*$ and let $\phi = T(h)$. We want to compute derivatives of ϕ with respect to, say, $\xi'_{k,l}$. Similarly to that of [11], we obtain the following estimate

$$\|\partial_{\xi'_{k,l}} T(h)\|_\infty \leqslant C(\log(1/\varepsilon))^2\|h\|_*, \qquad \text{for all } k = 1,\ldots,m, \quad l = 1, J_k. \tag{68}$$

4. The nonlinear problem

Consider the nonlinear equation

$$\begin{aligned}
& -\Delta\phi + \varepsilon^2\phi - W\phi = R + N(\phi) + \sum_{ij} c_{ij}\chi_j Z_{ij} \qquad \text{in } \Omega_\varepsilon,\\
& \frac{\partial\phi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega_\varepsilon,\\
& \int_{\Omega_\varepsilon} \chi_j Z_{ij}\phi = 0 \qquad \forall j = 1,\ldots,m, \quad i = 1, J_j,
\end{aligned} \tag{69}$$

where W is as in (21) and N, R are defined in (17) and (18), respectively.

Lemma 4.1. *Let $m > 0$, $d > 0$. Then there exist $\varepsilon_0 > 0$, $C > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and any $(\xi_1,\ldots,\xi_m) \in \mathcal{M}_\delta$ the problem (69) admits a unique solution ϕ, $c_1,\ldots,c_m$ such that*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\varepsilon^\alpha|\log\varepsilon|, \tag{70}$$

where α is any number in the interval $(0, 1)$. Furthermore, the function $\xi' \to \phi(\xi') \in C(\bar{\Omega}_\varepsilon)$ is C^1 and

$$\|D_{\xi'}\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\,\varepsilon^\alpha|\log\varepsilon|^2. \tag{71}$$

Proof. The proof of this lemma can be done along the lines of those of lemma 4.1 of [11]. We omit the details. □

5. Variational reduction

In view of lemma 4.1, given $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{M}_\delta$, we define $\phi(\xi)$ and $c_{ij}(\xi)$ to be the unique solution to (69) satisfying the bound (70).

Given $\xi = (\xi_1, \ldots, \xi_m) \in \Omega^k \times (\partial\Omega)^l$ we write

$$U(\xi) = \sum_{j=1}^{m}(u_j(x) + H_j^\varepsilon(x)),$$

where the ansatz is defined in (10). Set

$$F_\varepsilon(\xi) = J_\varepsilon(U(\xi) + \tilde{\phi}(\xi)), \tag{72}$$

where J_ε is the functional defined by

$$J_\varepsilon(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 + v^2) - \varepsilon^2 \int_\Omega \mathrm{e}^v \tag{73}$$

and

$$\tilde{\phi}(\xi)(x) = \phi(\xi)\left(\frac{x}{\varepsilon}\right), \qquad x \in \Omega. \tag{74}$$

Lemma 5.1. *If $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{M}_\delta$ is a critical point of F_ε then $u = U(\xi) + \tilde{\phi}(\xi)$ is a critical point of J_ε, that is, a solution to (4).*

Proof. Let

$$I_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2 + \varepsilon^2 v^2 - \varepsilon^4 \int_{\Omega_\varepsilon} \mathrm{e}^v.$$

Then $F_\varepsilon(\xi) = J_\varepsilon(U(\xi) + \tilde{\phi}(\xi)) = I_\varepsilon(V(\xi') + \phi(\xi'))$ where $\xi' = \xi/\varepsilon$. Therefore

$$\frac{\partial F_\varepsilon}{\partial \xi_{k,l}} = \frac{1}{\varepsilon}\frac{\partial I_\varepsilon(V(\xi') + \phi(\xi'))}{\partial \xi'_{k,l}} = \frac{1}{\varepsilon} D I_\varepsilon(V(\xi') + \phi(\xi'))\left[\frac{\partial V(\xi')}{\partial \xi'_{k,l}} + \frac{\partial \phi(\xi')}{\partial \xi'_{k,l}}\right].$$

Since $v = V(\xi') + \phi(\xi')$ solves (69)

$$\frac{\partial F_\varepsilon}{\partial \xi_{k,l}} = \frac{1}{\varepsilon} \sum_{i=1,J_j,j=1,\ldots,m} c_{ij} \int_{\partial\Omega_\varepsilon} \chi_j Z_{ij} \left[\frac{\partial V(\xi')}{\partial \xi'_{k,l}} + \frac{\partial \phi(\xi')}{\partial \xi'_{k,l}}\right].$$

Let us assume that $DF(\xi) = 0$. From the previous equation we conclude that

$$\sum_{j=1,\ldots,m,i=1,J_j} c_{ij} \int_{\Omega_\varepsilon} \chi_j Z_{ij} \left[\frac{\partial V(\xi')}{\partial \xi'_{k,l}} + \frac{\partial \phi(\xi')}{\partial \xi'_{k,l}}\right] = 0 \qquad \forall k = 1, \ldots, m, \qquad l = 1, J_k.$$

Since $\|\partial\phi(\xi')/\partial\xi'_{k,l}\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\varepsilon^\alpha |\log\varepsilon|^2$ and $(\partial V(\xi')/\partial\xi'_{k,l}) = \pm Z_{kl} + o(1)$ where $o(1)$ is in the L^∞ norm, it follows that

$$\sum_{j=1,\ldots,m,J=1,J_j} c_{ij} \int_{\partial\Omega_\varepsilon} \chi_j Z_{ij}(\pm Z_{kl} + o(1)) \qquad \forall k = 1, \ldots, m,$$

which is a strictly diagonal dominant system. This implies that $c_{ij} = 0$ $\forall i = 1, \ldots, m$, $j = 1, J_j$. □

In order to solve for critical points of the function F, a key step is its expected closeness to the function $J_\varepsilon(U)$, which we will analyse in the next section.

Lemma 5.2. *The following expansion holds*

$$F_\varepsilon(\xi) = J_\varepsilon(U) + \theta_\varepsilon(\xi),$$

where

$$|\theta_\varepsilon| + |\nabla\theta_\varepsilon| \to 0,$$

uniformly on $\mathcal{M}_\delta$.

Proof. Let $\tilde{\theta}(\xi') = I_\varepsilon(V+\phi) - I_\varepsilon(V)$. In order to get the proof of this lemma, we need to show that

$$|\tilde{\theta}| + \varepsilon^{-1}|\nabla_{\xi'}\tilde{\theta}_\varepsilon| = o(1).$$

Taking into account $DI_\varepsilon(V+\phi)[\phi] = 0$, a Taylor expansion and an integration by parts give

$$\begin{aligned} I_\varepsilon(V+\phi) - I_\varepsilon(V) &= \int_0^1 D^2 I_\varepsilon(V+t\phi)[\phi]^2\,(1-t)\,\mathrm{d}t \\ &= \int_0^1 \left(\int_{\Omega_\varepsilon} [N(\phi)+R]\phi + \int_{\Omega_\varepsilon} \mathrm{e}^V[1-\mathrm{e}^{t\phi}]\phi^2\right)(1-t)\,\mathrm{d}t, \end{aligned} \tag{75}$$

so we get

$$I_\varepsilon(V+\phi) - I_\varepsilon(V) = \tilde{\theta}_\varepsilon = O(\varepsilon^{2\alpha}|\log\varepsilon|^3),$$

taking into account that $\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\varepsilon^\alpha|\log\varepsilon|$. Let us differentiate with respect to $\xi'_{k,l}$

$$\partial_{\xi'_{k,l}}[I_\varepsilon(V+\phi) - I_\varepsilon(V)] = \int_0^1 \left(\int_{\partial\Omega_\varepsilon} \partial_{\xi'_{k,l}}[(N(\phi)+R)\phi] + \int_{\Omega_\varepsilon} \partial_{\xi'_{k,l}}[\mathrm{e}^V[1-\mathrm{e}^{t\phi}]\phi^2]\right)(1-t)\,\mathrm{d}t.$$

Using the fact that $\|\partial_{\xi'}\phi\|_* \leqslant C\varepsilon^\alpha|\log\varepsilon|^2$ and the estimates of the previous sections we get

$$\partial_{\xi'_{k,l}}[I_\varepsilon(V+\phi) - I_\varepsilon(V)] = \partial_{\xi'_{k,l}}\tilde{\theta}_\varepsilon = O(\varepsilon^{2\alpha}|\log\varepsilon|^4).$$

The continuity in ξ of all these expressions is inherited from that of ϕ and its derivatives in ξ in the L^∞ norm. The proof is complete. $\square$

6. Expansion of the energy

Lemma 6.1. *Let* μ_j *be given by (19). Then for any* $0 < \alpha < 1$,

$$\begin{aligned} J_\varepsilon(U) &= (8\pi k + 4\pi l)(\beta - 1 + \log 8) + 2(8\pi k + 4\pi l)\log\frac{1}{\varepsilon} \\ &\quad -\frac{1}{2}\sum_{j=1}^m c_j\left[c_j H(\xi_j,\xi_j) + \sum_{i,i\neq j} c_i G(\xi_i,\xi_j)\right] + O(\varepsilon^\alpha), \end{aligned}$$

where

$$\beta = \int_0^\infty \frac{1}{(1+x^2)^2}\log\frac{1}{(1+x^2)^2}x\,\mathrm{d}x.$$

Proof. Define

$$U_j(x) = u_j(x) + H_j^\varepsilon(x),$$

so we may rewrite (10) in an equivalent form $U = \sum_{j=1}^m U_j$. Then

$$\begin{aligned} J_\varepsilon(U) &= \frac{1}{2}\int_\Omega \left|\sum_{j=1}^m \nabla U_j\right|^2 + \frac{1}{2}\int_\Omega \left(\sum_{j=1}^m U_j\right)^2 - \varepsilon^2 \int_\Omega \exp\left(\sum_{j=1}^m U_j\right) \\ &= \sum_{j=1}^m \int_\Omega \frac{1}{2}(|\nabla U_j|^2 + U_j^2) + \frac{1}{2}\sum_{i\neq j}^m \int_\Omega (\nabla U_i \nabla U_j + U_i U_j) \\ &\quad -\varepsilon^2 \int_\Omega \exp\left(\sum_{j=1}^m U_j\right) \\ &= I_A + I_B + I_C. \end{aligned}$$

Let us analyse the behaviour of I_A. Note that U_j satisfies

$$\Delta U_j - U_j + \epsilon^2 \mathrm{e}^{u_j} = 0 \text{ in } \Omega, \qquad \frac{\partial U_j}{\partial \nu} = 0 \text{ on } \partial\Omega$$

which gives

$$\int_\Omega (|\nabla U_j|^2 + U_j^2) = \varepsilon^2 \int_\Omega \mathrm{e}^{u_j}(u_j + H_j^\varepsilon). \tag{76}$$

Let us find the asymptotic behaviour of the expression:

$$\begin{aligned} \int_\Omega (|\nabla U_j|^2 + U_j^2) = \varepsilon^2 \int_\Omega \frac{8\mu_j^2}{(\varepsilon^2\mu_j^2 + |x-\xi_j|^2)^2}\Bigg(&\log \frac{8\mu_j^2}{(\varepsilon^2\mu_j^2 + |x-\xi_j|^2)^2} \\ &+ c_j H(x,\xi_j) + O(\varepsilon^\alpha)\Bigg). \end{aligned}$$

Changing variables $\varepsilon\mu_j y = x - \xi_j$

$$\begin{aligned} \int_\Omega |\nabla U_j|^2 + U_j^2 = \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2}\Bigg(&\log\frac{1}{(1+|y|^2)^2} + c_j H(\xi_j + \varepsilon\mu_j y, \xi_j) \\ &-4\log(\varepsilon\mu_j)\Bigg) + O(\varepsilon^\alpha). \end{aligned}$$

But

$$\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2} = 2c_j + O(\varepsilon), \qquad \int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2}\log\frac{1}{(1+|y|^2)^2} = c_j\beta + O(\varepsilon^\alpha)$$

and for $0 < \alpha < 1$

$$\int_{\Omega_{\varepsilon\mu_j}} \frac{8}{(1+|y|^2)^2}(H(\varepsilon\mu_j y, \xi_j) - H(\xi_j,\xi_j)) = \int_{\Omega_{\varepsilon\mu_j}} \frac{1}{(1+|y|^2)^2} O(\varepsilon^\alpha |y|^\alpha) = O(\varepsilon^\alpha).$$

Therefore

$$\int_\Omega |\nabla U_j|^2 + U_j^2 = 2c_j\beta + c_j^2 H(\xi_j,\xi_j) - 4c_j\log(\varepsilon\mu_j) + O(\varepsilon^\alpha). \tag{77}$$

Thus

$$I_A = \sum_j c_j\beta - 2\sum_{j=1}^m c_j\log(\varepsilon\mu_j) + \sum_{j=1}^m \frac{1}{2}c_j^2 H(\xi_j,\xi_j) + O(\varepsilon^\alpha). \tag{78}$$

We now consider

$$I_B = \frac{1}{2}\sum_{i\neq j}^{m}\int_\Omega (\nabla U_i \nabla U_j + U_i U_j)$$
$$= \frac{\varepsilon^2}{2}\sum_{i\neq j}\int_\Omega \mathrm{e}^{u_i}(u_j + H_j).$$

A similar argument as for I_A shows that

$$I_B = \frac{1}{2}\sum_{i\neq j}^{m} c_i c_j G(\xi_i, \xi_j) + O(\varepsilon^\alpha). \tag{79}$$

Regarding the expression I_C we have

$$I_C = -\varepsilon^2 \int_\Omega \mathrm{e}^{\sum_{k=1}^m U_k} = -\varepsilon^2 \sum_{j=1}^{m}\int_{\Omega\cap B_\delta(\xi_j)} \mathrm{e}^{\sum_{k=1}^m (u_k + H_k^\varepsilon)} + O(\varepsilon^2).$$

Using the definition of u_j and (12) for each term we have

$$\varepsilon^2 \int_{\Omega\cap B_\delta(\xi_j)} \mathrm{e}^{\sum_{k=1}^m (u_k + H_k^\varepsilon)} = \varepsilon^2 \int_{\partial\Omega\cap B_\delta(\xi_j)} \mathrm{e}^{u_j}\mathrm{e}^{c_j H(x,\xi_j) - \log(8\mu_j^2) + O(\varepsilon^\alpha)} E_j(x),$$

where

$$E_j(x) = \exp\left(\sum_{i\neq j}\log\frac{1}{(\varepsilon^2\mu_i^2 + |x-\xi_i|^2)^2} + c_i H(x,\xi_i) + O(\varepsilon^\alpha)\right).$$

Changing variables $\varepsilon\mu_j y = x - \xi_j$ we have

$$\mathrm{e}^{c_j H(\xi_j + \varepsilon\mu_j y, \xi_j) - \log(8\mu_j^2) + O(\varepsilon^\alpha)} = \mathrm{e}^{c_j H(\xi_j,\xi_j) - \log(8\mu_j^2)} + O(\varepsilon^\alpha |y|^\alpha)$$

and

$$E_j(\xi_j + \varepsilon\mu_j y, \xi_j) = \exp\left(\sum_{i, i\neq j} c_i G(\xi_j . \xi_i)\right) + O(\varepsilon^\alpha |y|^\alpha).$$

Therefore, by the definition of μ_j in (19)

$$\varepsilon^2 \int_{\Omega\cap B_\delta(\xi_j)} \mathrm{e}^{\sum_{k=1}^m (u_k + H_k^\varepsilon)} = \varepsilon^2 \int_{\Omega\cap B_\delta(\xi_j)} \mathrm{e}^{u_j + O(\varepsilon^\alpha)} = c_j + O(\varepsilon^\alpha).$$

Thus

$$I_C = -\sum_j c_j + O(\varepsilon^\alpha). \tag{80}$$

Thanks to (78)–(80) we have

$$J_\varepsilon(U) = \sum_{j=1}^{m} c_j(\beta - 1 + \log 8) + 2\sum_{j=1}^{m} c_j \log\frac{1}{\varepsilon}$$
$$+ \sum_{j=1}^{m} c_j \left[-\log(8\mu_j^2) + \frac{1}{2}c_j H(\xi_j,\xi_j) + \frac{1}{2}\sum_{i\neq j} c_i c_j G(\xi_i,\xi_j)\right] + O(\varepsilon^\alpha).$$

Employing (19) again we have

$$J_\varepsilon(U) = \sum_{j=1}^{m} c_j(\beta - 1 + \log 8) + 2\sum_{j=1}^{m} c_j \log\frac{1}{\varepsilon}$$
$$-\frac{1}{2}\sum_{j=1}^{m} c_j\left[\sum_{j=1}^{m} c_j H(\xi_j,\xi_j) + \sum_{i,i\neq j} c_i G(\xi_i,\xi_j)\right] + O(\varepsilon^\alpha). \qquad \square$$

7. Proof of theorem 1.2

Let

$$\varphi_m(\xi) = \sum_{j=1}^{m} c_j\left[c_j H(\xi_j,\xi_j) + \sum_{i,i\neq j} c_i G(\xi_i,\xi_j)\right]. \tag{81}$$

We have the following lemma.

Lemma 7.1. *We have*

$$\min_{\xi\in\partial\mathcal{M}_\delta} \varphi_m(\xi) \to +\infty \text{ as } \delta \to 0. \tag{82}$$

Proof. Let $\xi = (\xi_1,\ldots,\xi_m) \in \partial\mathcal{M}_\delta$. There are two possibilities: either there exists $j_0 \leqslant k$ such that $\mathrm{d}(\xi_{j_0},\partial\Omega) = \delta$ or there exists $i_0 \neq j_0$, $|\xi_{i_0} - \xi_{j_0}| = \delta$.

In the first case, we claim that for all $\xi \in \Omega$

$$H(\xi,\xi) \geqslant C\log\frac{1}{Cd(\xi,\partial\Omega)}. \tag{83}$$

In fact, if ξ is close to the boundary, let ξ_0 be the nearest point of $\partial\Omega$ to ξ. It is easily checked that

$$H(x,\xi) = \frac{1}{2\pi}\log\frac{1}{|x-\xi^*|} + O(1) \qquad \text{as } \mathrm{d}(\xi,\partial\Omega) \to 0 \tag{84}$$

uniformly in Ω, where ξ^* is the reflection of ξ across the boundary, that is, the symmetric point to ξ with respect to ξ_0.

Using the fact that $G(x,y) > 0$, (83) follows from (84)

In the second case, we may assume that there exists a fixed constant C such that $\mathrm{d}(\xi_i,\partial\Omega) \geqslant C$, $i = 1,\ldots,k$, as otherwise it follows into the first case. But then it is easy to see that

$$G(\xi_i,\xi_j) \geqslant C\log\frac{1}{|\xi_i-\xi_j|}. \tag{85}$$

From (83) and (85), the proof of this lemma is complete. $\square$

Proof of theorem 1.2. For $\delta > 0$ sufficiently small, we define a configuration space as

$$\mathcal{M}_\delta := \{\xi = (\xi_1,\ldots,\xi_m) \in (\Omega)^k \times (\partial\Omega)^L \mid \min_{i=1,\ldots,k} \mathrm{d}(\xi_i,\partial\Omega) \geqslant \delta, \min_{i\neq j}|\xi_i-\xi_j| \geqslant \delta\}. \tag{86}$$

According to lemma 6.2, the function $U(\xi) + \tilde{\phi}(\xi)$, where U and $\tilde{\phi}$ are defined by (10) and (74), respectively, is a solution of problem (3) if we adjust ξ so that it is a critical point of $F_\varepsilon(\xi) = J_\varepsilon(U(\xi) + \tilde{\phi}(\xi))$ defined by (72). This is obviously equivalent to finding a critical point of

$$\tilde{F}_\varepsilon(\xi) = -2\left(F_\varepsilon(\xi) - (8\pi k + 4\pi l)(\beta - 1 + \log 8) - 2(8\pi k + 4\pi l)\log\frac{1}{\varepsilon}\right).$$

On the other hand, from lemmas 5.2 and 6.1, we have that for $\xi \in \mathcal{M}_\delta$

$$\tilde{F}_\varepsilon(\xi) = \varphi_m(\xi) + \varepsilon^\alpha \Theta_\varepsilon(\xi), \tag{87}$$

where φ_m is given by (81) and Θ_ε and $\nabla_\xi \Theta_\varepsilon$ are uniformly bounded in the considered region as $\varepsilon \to 0$.

From the above lemma, the function φ_m is C^1, bounded from below in $\mathcal{M}_\delta$ and such that

$$\varphi_m(\xi_1, \ldots, \xi_m) \to +\infty \text{ as } \delta \to 0.$$

Hence, for δ which is arbitrarily small, φ_m has an absolute minimum M in $\mathcal{M}_\delta$. This implies that $\tilde{F}_\varepsilon(\xi)$ also has an absolute minimum $(\xi_1^\varepsilon, \ldots, \xi_m^\varepsilon) \in \mathcal{M}_\delta$ such that

$$\varphi_m(\xi_1^\varepsilon, \ldots, \xi_m^\varepsilon) \to \min_{\xi \in \mathcal{M}_\delta} \varphi_m(\xi) \text{ as } \varepsilon \to 0.$$

Hence lemma 5.1 guarantees the existence of a solution u_ε for (4). Furthermore, from ansatz (10), we get that, as $\varepsilon \to 0$, u_ε remains uniformly bounded on $\Omega \setminus \cup_{j=1}^m B_\delta(\xi_i^\varepsilon)$ and

$$\sup_{B_\delta(\xi_i^\varepsilon)} u_\varepsilon \to +\infty,$$

for any $\delta > 0$. □

Remark 7.2. Using Ljusternik–Schnirelmann theory, one can get a second, distinct solution satisfying theorem 1.2. The proof is similar to [12].

Remark 7.3. As mentioned in the introduction, one can get a stronger result than theorem 1.2 under the assumption that the function φ_m has, in addition to the ones described in the proof of theorem 1.2, some other critical points in $\hat{\Omega}_m$ with the property of being *topologically non-trivial*, for instance (possibly degenerate) local minima or maxima or saddle points.

Let us define what we mean by *topologically non-trivial* critical point for φ_m. Let Σ be an open set compactly contained in $\mathcal{M}_\delta$ with smooth boundary. We recall that φ_m *links in* Σ *at critical level* $\mathcal{C}$ *relative to* B *and* B_0 if B and B_0 are closed subsets of $\bar{\Sigma}$ with B connected and $B_0 \subset B$ such that the following conditions hold: let us set Γ to be the class of all maps $\Phi \in C(B, \Sigma)$ with the property that there exists a function $\Psi \in C([0, 1] \times B, \Sigma)$ such that

$$\Psi(0, \cdot) = \mathrm{Id}_B, \qquad \Psi(1, \cdot) = \Phi, \qquad \Psi(t, \cdot)|_{B_0} = \mathrm{Id}_{B_0} \quad \text{for all } t \in [0, 1].$$

We assume

$$\sup_{y \in B_0} \varphi_m(y) < \mathcal{C} \equiv \inf_{\Phi \in \Gamma} \sup_{y \in B} \varphi_m(\Phi(y)), \tag{88}$$

and for all $y \in \partial\Sigma$ such that $\varphi_m(y) = \mathcal{C}$, there exists a vector, τ_y, tangent to $\partial\Sigma$ at y such that

$$\nabla\varphi_m(y) \cdot \tau_y \neq 0. \tag{89}$$

Under these conditions a critical point $\bar{y} \in \Sigma$ of φ_m with $\varphi_m(\bar{y}) = \mathcal{C}$ exists. Not only this, any function C^1 close to φ_m inherits such a critical point.

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References

[1] Baraket S and Pacard F 1998 Construction of singular limits for a semilinear elliptic equation in dimension 2 *Calc. Var. Partial Diff. Eqns* **6** 1–38

[2] Bates P, Dancer E N and Shi J 1999 Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability *Adv. Diff. Eqns* **4** 1–69

[3] Biler P 1998 Local and global solvability of some parabolic system modeling chemotaxis *Adv. Math. Sci. Appl.* **8** 715–43

[4] Brenner M P, Constantin P, Kadanoff L P, Schenkel A and Venkataramani S C 1999 Diffusion, attraction and collapse *Nonlinearity* **12** 1071–98

[5] Brezis H and Merle F 1991 Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions *Commun. Partial Diff. Eqns* **16** 1223–53

[6] Chen C-C and Lin C-S 2002 Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces *Commun. Pure Appl. Math.* **55** 728–71

[7] Childress S 1984 *Chemotactic Collapse in Two Dimensions (Lecture Notes in Biomathematics)* vol 55 (Berlin: Springer) pp 61–8

[8] Childress S and Percus J K 1981 Nonlinear aspects of chemotaxis *Math. Biosci.* **56** 217–37

[9] Corrias L, Perthame B and Zaag H 2004 Global solutions of some chemotaxis and angiogenesis systems in high space dimensions *Milan J. Math.* **72** 1–28

[10] Dancer E N and Yan S 1999 Multipeak solutions for a singular perturbed Neumann problem *Pac. J. Math.* **189** 241–62

[11] Del Pino M, Kowalczyk M and Musso M 2005 Singular limits in Liouville-type equations *Calc. Var. Partial Diff. Eqns* **24** 45–82

[12] Dávila J, del Pino M and Musso M 2005 Concentrating solutions in a two-dimensional elliptic problem with exponential Neumann data *J. Funct. Anal.* **27** 430–90

[13] Dávila J, del Pino M, Musso M and Wei J 2006 Singular limits of a two-dimensional boundary value problem arising in corrosion modelling *Arch. Ration. Mech. Anal.* at press

[14] Del Pino M, Felmer P and Musso M 2003 Two-bubble solutions in the super-critical Bahri-Coron's problem *Calc. Var. Partial Diff. Eqns* **16** 113–45

[15] Dolbeault J and Perthame B 2004 Optimal critical mass in the two-dimensional Keller–Segel model in $\mathbb{R}^2$ *C. R. Math. Acad. Sci. Paris* **339** 611–6

[16] Esposito P, Grossi M and Pistoia A 2005 On the existence of blowing-up solutions for a mean field equation *Ann. Inst. H. Poincaré Anal. Non Lineaire* **22** 227–57

[17] Guerra I and Peletier M 2004 Self-similar blow-up for a diffusion-attraction problem *Nonlinearity* **17** 2137–62

[18] Gui C and Wei J 1999 Multiple interior spike solutions for some singular perturbed Neumann problems *J. Diff. Eqns.* **158** 1–27

[19] Gui C and Wei J 2000 On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems *Can. J. Math.* **52** 522–38

[20] Gui C, Wei J and Winter M 2000 Multiple boundary peak solutions for some singularly perturbed Neumann problems *Ann. Inst. H. Poincaré Anal. Non Linéaire* **17** 249–89

[21] Henry D 1981 *Geometric Theory of Semilinear Parabolic Equations* (Berlin: Springer)

[22] Herrero M A and Velazquez J J L 1996 Singularity patterns in a chemotaxis model *Math. Ann.* **306** 583–623

[23] Herrero M A and Velazquez J J L 1996 Chemotactic collapse for the Keller–Segel model *J. Math. Biol.* **35** 177–96

[24] Herrero M A and Velazquez J J L 1997 A blow-up mechanism for a chemotaxis model *Ann. Scuola Norm. Sup. Pisa IV* **35** 633–83

[25] Horstmann D 2002 On the existence of radially symmetric blow-up solutions for the Keller–Segel model *J. Math. Biol.* **44** 463–78

[26] Jager W and Luckhaus S 1992 On explosions of solutions to a system of partial differential equations modelling chemotaxis *Trans. Am. Math. Soc.* **329** 819–24

[27] Keller E F and Segel L A 1970 Initiation of slime mold aggregation viewed as an instability *J. Theor. Biol.* **26** 399–415

[28] Li Y and Shafrir I 1994 Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two *Indiana Univ. Math. J.* **43** 1255–70

[29] Nagasaki K and Suzuki Y 1990 Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities *Asymptotic Anal.* **3** 173–88

[30] Nagai T 1995 Blow-up of radially symmetric solutions to a chemotaxis system *Adv. Math. Sci. Appl.* **5** 581–601

[31] Nagai T 2001 Global existence and blowup of solutions to a chemotaxis system *Proc. Third World Congress of Nonlinear Analysts, Part 2 (Catania, 2000), Nonlinear Anal.* **47** 777–87
[32] Nanjudiah V 1973 Chemotaxis, signal relaying and aggregation morphology *J. Theor. Biol.* **42** 63–105
[33] Nagai T, Senba T and Yoshida K 1997 Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis *Funckcjal. Ekvac.* **40** 411–33
[34] Nagai T, Senba T and Suzuki T 2000 Chemotactic collapse in a parabolic system of mathematical biology *Hiroshima Math. J.* **30** 463–97
[35] Rey O and Wei J 2004 Blow-up solutions for an elliptic Neumann problem with sub-or-supcritical nonlinearity I: $N = 3$ *J. Funct. Anal.* **212** 472–99
[36] Schaaf R 1985 Stationary solutions of chemotaxis systems *Trans. Am. Math. Soc.* **292** 531–56
[37] Senba T and Suzuki T 2000 Some structures of the solution set for a stationary system of chemotaxis *Adv. Math. Sci. Appl.* **10** 191–224
[38] Senba T and Suzuki T 2002 Time global solutions to a parabolic-elliptic system modelling chemotaxis *Asymptotic Anal.* **32** 63–89
[39] Temam R 1988 *Infinite-Dimensional Dynamical Systems in Mechanics and Physics* (New York: Springer)
[40] Senba T and Suzuki T 2002 Weak solutions to a parabolic-elliptic system of chemotaxis *J. Funct. Anal.* **191** 17–51
[41] Velazquez J J L 2002 Stability of some mechanisms of chemotactic aggregation *SIAM J. Appl. Math* **62** 1581–633
[42] Velazquez J J L 2004 Point dynamics in a singular limit of the Keller–Segel model: II. Formation of the concentration regions *SIAM J. Appl. Math.* **64** 1224–48
[43] Velazquez J J L 2004 Well-posedness of a model of point dynamics for a limit of the Keller–Segel system *J. Diff. Eqns* **206** 315–52
[44] Wang G and Wei J 2002 Steady state solutions of a reaction-diffusion system modeling Chemotaxis *Math. Nachr.* **233–234** 221–36