Higher-order spectral analysis and weak asymptotic stability of convex processes

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Abstract

This paper deals with the asymptotic stability analysis of a discrete dynamical inclusion whose right-hand side is a convex process. We provide necessary and sufficient conditions for weak asymptotic stability, and obtain sharp estimates for the asymptotic null-controllability set. These estimates involve not only standard, but also higher-order spectral information on the convex process and its adjoint.

Keywords: Convex process; Discrete dynamical inclusion; Eigenvalue analysis; Asymptotic stability

1. Introduction

This paper deals with the asymptotic stability analysis of a discrete dynamical system of the form

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$$x(k+1) \in F\big(x(k)\big), \quad \forall k = 0, 1, \ldots. \tag{1}$$

As state space, consider a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. The multivalued operator $F : H \rightrightarrows H$ is assumed to be a convex process in the sense that

$$\operatorname{gr} F = \big\{(s, v) \in H \times H \colon\ v \in F(s)\big\}$$

is a convex cone containing the origin. This geometric property imposed on the graph of F amounts to saying that

$$0 \in F(0),$$
$$F(\alpha s) = \alpha F(s), \quad \forall \alpha > 0,\ \forall s \in H,$$
$$F(s_1) + F(s_2) \subset F(s_1 + s_2), \quad \forall s_1, s_2 \in H.$$

A trajectory of F refers to a sequence $x : \mathbb{N} \to H$ satisfying the evolution law (1). Thus,

$$S_F(\xi) = \big\{x : \mathbb{N} \to H \colon\ x \text{ solves (1) and } x(0) = \xi\big\}$$

corresponds to the set of all trajectories of F emanating from the initial state $\xi \in H$. Observe that the multivalued operator $S_F : H \rightrightarrows H^{\mathbb{N}}$ enjoys the same properties as F, namely, normalization, positive homogeneity, and super-additivity.

Definition 1.1. F is said to be weakly asymptotically stable if

$$\forall \xi \in H, \quad \exists x \in S_F(\xi) \quad \text{such that} \quad \lim_{k \to \infty} x(k) = 0,$$

that is to say, from every initial state emanates a trajectory of F that, in the long run, becomes arbitrarily close to the origin.

Weak asymptotic stability is a concept that speaks by itself and does not need any further introduction. Definition 1.1 has been considered by authors like Phat [10,11] and Smirnov [12], among others. The purpose of this note is not only providing necessary and sufficient conditions for weak asymptotic stability, but also deriving sharp estimates for the set

$$\mathcal{K}_\infty(F) = \left\{\xi \in H \colon\ \lim_{k \to \infty} x(k) = 0 \text{ for some } x \in S_F(\xi)\right\}.$$

We say that $\mathcal{K}_\infty(F)$ is the *asymptotic null-controllability set* of F. We are borrowing the terminology of control theory because (1) can be seen as a generalization of the control model

$$x(k+1) = Ax(k) + Bu(k), \quad u(k) \in P,$$

where P is a closed convex cone in a given Hilbert space, and A and B are continuous linear operators.

Two remarks are useful for putting our study in the right perspective: firstly, $\mathcal{K}_\infty(F)$ is a convex cone containing the origin; and, secondly,

$$\mathcal{K}_\infty(F) \subset \operatorname{dom} S_F \subset \operatorname{dom} F,$$

with $\operatorname{dom} F = \{\xi \in H \colon\ F(\xi) \neq \emptyset\}$ and $\operatorname{dom} S_F = \{\xi \in H \colon\ S_F(\xi) \neq \emptyset\}$ being the domains of F and S_F, respectively. Needless to say, the convex process F cannot be weakly asymptotically stable unless it is nonempty-valued everywhere.

2. Upper and lower estimates for $\mathcal{K}_\infty(F)$

In the classic framework of a linear evolutionary system

$$x(k+1) = Ax(k), \quad \forall k = 0, 1, \dots,$$

weak asymptotic stability simply means that

$$\lim_{n\to\infty} A^n \xi = 0, \quad \forall \xi \in H. \tag{2}$$

A convergence condition like (2) can be formulated also in the context of the difference inclusion (1). To do this, one has to introduce first the set

$$F^n(\xi) = [F \circ F \circ \cdots \circ F](\xi), \tag{3}$$

with F appearing n times on the right-hand side of (3), and $\circ$ denoting composition. So,

$$v \in F^n(\xi) \quad \Leftrightarrow \quad \begin{cases} \text{there is a chain } \{v_0, \dots, v_n\} \text{ with } v_0 = \xi,\ v_n = v, \\ \text{and } v_{r+1} \in F(v_r) \text{ for } r = 0, 1, \dots, n-1. \end{cases}$$

The interpretation of $F^n(\xi)$ is clear: it corresponds to the set all states that can be reached by the multivalued system (1) after n steps starting from ξ. Next, one has to check whether the successive reachable sets

$$F^1(\xi),\ F^2(\xi),\ F^3(\xi),\ \dots$$

get closer or not to the origin. More precisely, one has to see what happens with the distance

$$\operatorname{dist}\bigl[0, F^n(\xi)\bigr] = \inf_{v\in F^n(\xi)} \|v\|$$

as n goes to ∞. This way of proceeding leads to the upper estimate:

Proposition 2.1. *For any convex process F, one has*

$$\mathcal{K}_\infty(F) \subset \Bigl\{\xi \in H\colon \lim_{n\to\infty} \operatorname{dist}\bigl[0, F^n(\xi)\bigr] = 0\Bigr\}. \tag{4}$$

Proof. It is enough to observe that every $x \in S_F(\xi)$ satisfies the composite evolution law

$$x(n) \in F^n(\xi), \quad \forall n = 0, 1, \dots, \tag{5}$$

where the standard convention $F^0 = I$ (identity operator) is in order. □

Remark. A sequence $x : \mathbb{N} \to H$ satisfying (5) can be seen as a sort of generalized trajectory of F emanating from ξ. In contrast with a usual trajectory, the state $x(n+1)$ is not necessarily obtained from $x(n)$ by performing one extra iteration.

In what follows, the symbol F^{-n} denotes the inverse of F^n. Thus, $F^{-n}(0) = \{\xi \in H\colon 0 \in F^n(\xi)\}$ corresponds to the set of all states that can be brought to the origin in n steps. In view of this interpretation,

$$\mathcal{K}(F) = \bigcup_{n\geqslant 1} F^{-n}(0)$$

is called the *finite-time null-controllability set* of F. We use the term *stationary* to refer to a sequence $x : \mathbb{N} \to H$ such that $x(k) = 0$ for all k above a certain threshold. Observe that

$$\xi \in \mathcal{K}(F) \quad \Leftrightarrow \quad \text{there is an } x \in S_F(\xi) \text{ which is stationary.} \tag{6}$$

The following counterpart of Proposition 2.1 can be proven in a straightforward manner. As usual, the notation "cl" stands for topological closure.

Proposition 2.2. *Let F be a convex process. Then,*

(a) $\mathcal{K}(F)$ *is a convex cone contained in* $\mathcal{K}_\infty(F)$;
(b) $\{\xi \in H\colon \lim_{n\to\infty} \operatorname{dist}[\xi, F^{-n}(0)] = 0\} \subset \operatorname{cl}[\mathcal{K}_\infty(F)]$.

Proof. Since $\{F^{-n}(0)\}_{n\geqslant 1}$ is a collection of convex cones arranged in a nondecreasing order

$$F^{-1}(0) \subset F^{-2}(0) \subset F^{-3}(0) \subset \cdots,$$

it follows that $\mathcal{K}(F)$ is a convex cone, and

$$\operatorname{dist}\bigl[\xi, \mathcal{K}(F)\bigr] = \inf_{n\geqslant 1} \operatorname{dist}\bigl[\xi, F^{-n}(0)\bigr] = \lim_{n\to\infty} \operatorname{dist}\bigl[\xi, F^{-n}(0)\bigr], \quad \forall \xi \in H.$$

Hence

$$\operatorname{cl}\bigl[\mathcal{K}(F)\bigr] = \Bigl\{\xi \in H\colon \lim_{n\to\infty} \operatorname{dist}\bigl[\xi, F^{-n}(0)\bigr] = 0\Bigr\}.$$

The observation (6) yields the remaining part of the proposition. □

It is natural to ask how large is the gap between the set on the left-hand side of Proposition 2.2(b), and the set on the right-hand side of (4). In general, one should not expect to have

$$\Bigl\{\xi \in H\colon \lim_{n\to\infty} \operatorname{dist}\bigl[\xi, F^{-n}(0)\bigr] = 0\Bigr\} = \operatorname{cl}\Bigl\{\xi \in H\colon \lim_{n\to\infty} \operatorname{dist}\bigl[0, F^{n}(\xi)\bigr] = 0\Bigr\}, \tag{7}$$

but, if this equality occurs, then the cones $\mathcal{K}(F)$ and $\mathcal{K}_\infty(F)$ have necessarily the same closure. This observation is behind the formulation of next result. Recall that a convex process $F : H \rightrightarrows H$ is said to be

$$\begin{cases} \textit{strict} & \text{if } \operatorname{dom} F = H; \\ \textit{closed} & \text{if } \operatorname{gr} F \text{ is a closed set;} \\ \textit{coercive} & \text{if } \operatorname{dist}[0, F(s)] \to \infty \text{ as } \|s\| \to \infty; \\ \textit{nonexpansive} & \text{if } \operatorname{dist}[0, F(s)] \leqslant \|s\|, \ \forall s \in H. \end{cases}$$

Theorem 2.1. *Let F be a closed convex process such that F^{-1} is coercive and nonexpansive. Then, $\mathcal{K}_\infty(F)$ is contained in the closure of $\mathcal{K}(F)$.*

Proof. Coercivity of F^{-1} has been added just to make sure that all the iterates F^{-n} are closed. Now, under the condition

$$\forall n \geqslant 1, \quad F^{-n} \text{ is a closed convex process,} \tag{8}$$

the nonexpansive behavior of F^{-1} guarantees equality (7). To see this point, a number of elements must be brought to the discussion. First of all, nonexpansiveness of F^{-1} implies that each F^{-n} is strict and

$$\|F^{-n}\| \leqslant \|F^{-1}\|^n \leqslant 1. \tag{9}$$

Here the expression

$$\|G\| = \sup_{\|u\| \leqslant 1} \operatorname{dist}\big[0, G(u)\big]$$

is used to measure the "magnitude" of a strict convex process $G : H \rightrightarrows H$. Another concept to be recalled is that of lower-limit of a collection $\{C_n\}_{n \geqslant 1}$ of sets lying in a metric space Z. By definition, one has

$$\liminf_{n\to\infty} C_n = \Big\{z \in Z\colon \lim_{n\to\infty} \operatorname{dist}[z, C_n] = 0\Big\}.$$

The lower-limit of a collection $\{G_n\}_{n \geqslant 1}$ of convex processes $G_n : H \rightrightarrows H$ is a new convex process whose graph is given by

$$\operatorname{gr}\Big[\liminf_{n\to\infty} G_n\Big] = \liminf_{n\to\infty}[\operatorname{gr} G_n].$$

With this notation at hand, it becomes clear that

$$\Big\{\xi \in H\colon \lim_{n\to\infty} \operatorname{dist}\big[0, F^n(\xi)\big] = 0\Big\} = \Big\{\xi \in H\colon\ 0 \in \liminf_{n\to\infty}\big[F^n(\xi)\big]\Big\}$$

is contained in the closed convex cone

$$\Big\{\xi \in H\colon\ 0 \in \Big[\liminf_{n\to\infty} F^n\Big](\xi)\Big\} = \Big[\liminf_{n\to\infty} F^n\Big]^{-1}(0) = \Big[\liminf_{n\to\infty} F^{-n}\Big](0).$$

On the other hand,

$$\operatorname{cl}\big[\mathcal{K}(F)\big] = \liminf_{n\to\infty}\big[F^{-n}(0)\big].$$

So, everything boils down to checking whether

$$\Big[\liminf_{n\to\infty} F^{-n}\Big](0) = \liminf_{n\to\infty}\big[F^{-n}(0)\big]. \tag{10}$$

According to the general theory of lower-limits [3], equality (10) follows from (8) and

$$\sup_{n \geqslant 1} \|F^{-n}\| < \infty.$$

The above uniform boundedness condition is ensured, of course, by (9). □

Remark. The magnitude of a convex process can be used as preliminary test for checking weak asymptotic stability. Indeed, a strict convex process F is weakly asymptotically stable if $\|F\| < 1$. In view of this result, special attention must be devoted to the case $\|F\| \geqslant 1$.

Some refinements are possible in Theorem 2.1. For instance, the inclusion $\mathcal{K}_\infty(F) \subset \operatorname{cl}[\mathcal{K}(F)]$ still holds if F is a convex process such that

$$\text{for some } p \geqslant 1, \quad F^{-p} \text{ is closed, coercive, and nonexpansive.}$$

To see this, apply Theorem 2.1 to the composite operator F^p, and observe that

$$\mathcal{K}_\infty(F) \subset \mathcal{K}_\infty\big(F^p\big) \quad \text{and} \quad \mathcal{K}(F) = \mathcal{K}\big(F^p\big).$$

We shall not insist too much on Theorem 2.1 because its describes a situation that is rather abnormal: the finite-time null-controllability set is rich enough to capture all the information contained in $\mathcal{K}_\infty(F)$.

While trying to evaluate the asymptotic null-controllability set, the inclusion $\mathcal{K}(F) \subset \mathcal{K}_\infty(F)$ is the first thing that comes to mind. However, simple examples show that this lower estimate can be very rough. To get a better estimate, spectral information on the operator F must be brought into the picture.

3. A sharper lower estimate for $\mathcal{K}_\infty(F)$

In this section we obtain a sharper lower estimate for $\mathcal{K}_\infty(F)$ by using tools of spectral analysis. To start with, we introduce:

Definition 3.1. The resolvent of F at $\lambda \in \mathbb{R}$ is the operator $R_\lambda F : H \rightrightarrows H$ given by

$$(R_\lambda F)(v) = (F - \lambda I)^{-1}(v) = \big\{\xi \in H \colon\ v \in (F - \lambda I)(\xi)\big\}.$$

The *n-order resolvent* of F at $\lambda \in \mathbb{R}$ is the iterated composition

$$R_\lambda^n F = (R_\lambda F) \circ (R_\lambda F) \circ \cdots \circ (R_\lambda F) = (F - \lambda I)^{-n}.$$

As usual, an eigenvalue of F is understood as a number $\lambda \in \mathbb{R}$ satisfying $\lambda a \in F(a)$ for some $a \neq 0$. Such $a \in H$ is called an eigenvector of F associated to the eigenvalue λ. The set

$$\Lambda(F) = \{\lambda \in \mathbb{R} \colon\ \lambda \text{ is an eigenvalue of } F\}$$

is referred to as the point spectrum of F. For convex processes, the concept of eigenvalue has been extensively discussed in the last decade. It has known interesting applications not only in control theory [2,4,6,11,14], but also in other areas.

One could introduce a sort of n-order point spectrum by simply writing

$$\Lambda_n(F) = \big\{\lambda \in \mathbb{R} \colon\ \big(R_\lambda^n F\big)(0) \neq \{0\}\big\},$$

but such idea is of no use. Indeed, next lemma shows that $\Lambda_n(F) = \Lambda(F)$ for every $n \geqslant 1$.

Lemma 3.1. *Let F be a convex process. For $\lambda \in \mathbb{R}$, the following three conditions are equivalent*:

(a) $(F - \lambda I)^{-1}(0) \neq \{0\}$;
(b) $(R_\lambda^n F)(0) \neq \{0\}$ *for every* $n \geqslant 1$;
(c) $(R_\lambda^n F)(0) \neq \{0\}$ *for some* $n \geqslant 1$.

Proof. Since $\{(R_\lambda^n F)(0)\}_{n \geqslant 1}$ is a collection of convex cones arranged in a nondecreasing order, the only nontrivial implication is (c) $\Rightarrow$ (a). Let $n \geqslant 1$ be an integer such that

$(R^n_\lambda F)(0)$ contains a nonzero vector, say $\xi \in H$. Then, there is a chain $\{v_0, \dots, v_n\}$ satisfying the end-point conditions $v_0 = \xi$, $v_n = 0$, and such that

$$v_{r+1} + \lambda v_r \in F(v_r) \quad \text{for } r = 0, 1, \dots, n-1. \tag{11}$$

For $r = n-1$, one gets $\lambda v_{n-1} \in F(v_{n-1})$. If $v_{n-1} \neq 0$, then we are done. Otherwise, we write (11) for $r = n-2$, obtaining in this way $\lambda v_{n-2} \in F(v_{n-2})$. We apply the same argument as before, and continue proceeding backward until the desired conclusion is attained. $\square$

Although the idea of dealing with higher-order eigenvalues is fruitless, introducing higher-order eigenvectors does make sense:

Definition 3.2. Let $n \geqslant 1$. An *n-order eigenvector* of F associated to $\lambda \in \mathbb{R}$ is any nonzero vector belonging to $(R^n_\lambda F)(0)$. A nonzero vector in

$$\Phi_F(\lambda) = \bigcup_{n \geqslant 1} \big(R^n_\lambda F\big)(0)$$

is called a finite-order eigenvector of F associated to λ.

For linear operators, the notion of finite-order eigenvector is certainly known. For convex processes, such a notion appears in the work by Smirnov [12]. Observe that $\Phi_F(\lambda)$ is precisely the finite-time null-controllability set of the shifted process $F - \lambda I$. In particular,

$$\Phi_F(0) = \mathcal{K}(F).$$

So, next result can be seen as complement to Proposition 2.2(a).

Proposition 3.1. *Let F be a convex process. Then,*

$$\big(R^n_\lambda F\big)(0) \subset \mathcal{K}_\infty(F), \quad \forall n \geqslant 1,\ \forall \lambda \in]0, 1[. \tag{12}$$

Proof. Fix $\lambda \in]0, 1[$ and $n \geqslant 1$. If $\xi \in (R^n_\lambda F)(0)$, then there is a chain $\{v_0, \dots, v_n\}$ satisfying the end-point conditions $v_0 = \xi$, $v_n = 0$, and the recursive relation (11). Extend this chain by setting $v_r = 0$ for all $r > n$. Consider now the sequence $x : \mathbb{N} \to H$ given by

$$x(k) = \sum_{r=0}^{k} C^k_r \lambda^{k-r} v_r, \quad \forall k = 0, 1, \dots, \tag{13}$$

with

$$C^k_r = \frac{k!}{r!(k-r)!}. \tag{14}$$

We claim that $x \in S_F(\xi)$. To start with, observe that $x(0) = v_0 = \xi$. To prove (1), we proceed by induction. One clearly has

$$x(1) = \lambda v_0 + v_1 \in F\big(x(0)\big).$$

The induction hypothesis is that (1) holds for $k = N - 1$. We need to prove that (1) holds for $k = N$. From the very definition of x, it follows that

$$x(N+1) = \lambda x(N) + \sum_{r=1}^{N} C_{r-1}^{N-1} \lambda^{N-r} (v_{r+1} + \lambda v_r) \tag{15}$$

and

$$x(N) = \lambda x(N-1) + \sum_{r=1}^{N} C_{r-1}^{N-1} \lambda^{N-r} v_r. \tag{16}$$

Both formulas have been proved and exploited in a different context by Alvarez, Correa and Gajardo [1]. Their proofs are quite technical and do not need to be reproduced here. It is essentially a matter of playing with the general properties of the combinatorial numbers defined by (14). Recall now that

$$v_{r+1} + \lambda v_r \in F(v_r), \quad \forall r \geqslant 0.$$

Since F is a convex process, it follows that

$$\sum_{r=1}^{N} C_{r-1}^{N-1} \lambda^{N-r} (v_{r+1} + \lambda v_r) \in F\left(\sum_{r=1}^{N} C_{r-1}^{N-1} \lambda^{N-r} v_r \right). \tag{17}$$

On the other hand, the induction hypothesis $x(N) \in F(x(N-1))$ yields

$$\lambda x(N) \in F\big(\lambda x(N-1)\big). \tag{18}$$

We now sum up (17) and (18), and use the super-additivity of F. In view of formulas (15), (16), what we are getting is precisely the relation (1) for $k = N$. In short, we have shown that $x \in S_F(\xi)$. To complete the proof, it remains to check that $x(k) \to 0$ as $k \to \infty$. For k large enough (in fact, for $k \geqslant n$), expression (13) becomes

$$x(k) = \sum_{r=0}^{n-1} C_r^k \lambda^{k-r} v_r.$$

Thus,

$$\|x(k)\| \leqslant M \sum_{r=0}^{n-1} C_r^k \lambda^{k-r} \quad \text{with } M = \max\big\{\|v_0\|, \ldots, \|v_{n-1}\|\big\}.$$

A matter of computation yields

$$\begin{aligned}
\|x(k)\| &\leqslant M \sum_{r=0}^{n-1} \lambda^{k-r} \frac{(n-1)!}{r!(n-1-r)!} \frac{k!}{(k-r)!} \frac{(n-1-r)!}{(n-1)!} \\
&\leqslant M \lambda^{k-(n-1)} \sum_{r=0}^{n-1} C_r^{n-1} k(k-1) \cdots (k-r+1) \\
&\leqslant M \lambda^{k-n+1} \sum_{r=0}^{n-1} C_r^{n-1} k^r.
\end{aligned}$$

The last sum being equal to $(1+k)^{n-1}$, one gets finally the estimate

$$\|x(k)\| \leqslant M\lambda^{k-n+1}(1+k)^{n-1},$$

with a right-hand side term going to 0 as $k \to \infty$. $\square$

Inclusion (12) does not hold when $\lambda \in]-1, 0[$. To handle the case of a negative λ, we are led to introduce:

Definition 3.3. The bilateral resolvent of F at $\lambda \in \mathbb{R}$ is the operator $B_\lambda F : H \rightrightarrows H$ given by

$$(B_\lambda F)(v) = (F - \lambda I)^{-1}(v) \cap -(F - \lambda I)^{-1}(-v).$$

The n-order bilateral resolvent of F at λ is the composition $B_\lambda^n F = (B_\lambda F) \circ (B_\lambda F) \circ \cdots \circ (B_\lambda F)$.

The definition of $B_\lambda F$ may look strange at first sight, but it is motivated by a simple geometric consideration. In fact,

$$\mathrm{gr}(B_\lambda F) = \mathrm{gr}(R_\lambda F) \cap -\mathrm{gr}(R_\lambda F)$$

is the largest linear space contained in the graph of $R_\lambda F$. The "bilateral" counterpart of Definition 3.2 reads as follows:

Definition 3.4. Let $n \geqslant 1$. An n-order bilateral eigenvector of F associated to $\lambda \in \mathbb{R}$ is any nonzero vector belonging to $(B_\lambda^n F)(0)$. A nonzero vector in

$$\Psi_F(\lambda) = \bigcup_{n \geqslant 1} \big(B_\lambda^n F\big)(0)$$

is called a finite-order bilateral eigenvector of F associated to λ.

Bilateral eigenvectors emerge as natural mathematical objects while dealing with nonlinear convex processes. The set $(F - \lambda I)^{-1}(0)$ may contain a nonzero vector ξ, but not its opposite $-\xi$. If this happens, the eigenvector ξ is not of the bilateral type. Without further ado, we state:

Proposition 3.2. *Let F be a convex process. Then,*

$$\big(B_\lambda^n F\big)(0) \subset \mathcal{K}_\infty(F), \quad \forall n \geqslant 1,\ \forall \lambda \in]-1, 0[. \tag{19}$$

Proof. Fix $\lambda \in]-1, 0[$ and $n \geqslant 1$. If $\xi \in (B_\lambda^n F)(0)$, then we can construct a chain $\{v_0, \ldots, v_n\}$ such that $v_0 = \xi$, $v_n = 0$, and

$$v_{r+1} \in (B_\lambda F)^{-1}(v_r) \quad \text{for } r = 0, 1, \ldots, n-1.$$

The above recurrence relation breaks down into

$$\left.\begin{array}{l} v_{r+1} + \lambda v_r \in F(v_r) \\ -v_{r+1} + \lambda(-v_r) \in F(-v_r) \end{array}\right\} \quad \text{for } r = 0, 1, \ldots, n-1. \tag{20}$$

As in the proof of Proposition 3.1, we extend this chain by setting $v_r = 0$ for all $r > n$, and then we consider the sequence $x : \mathbb{N} \to H$ given by expression (13). It has already been shown that $x(k) \to 0$ as $k \to \infty$, so the crucial question is the following one: does x belong to $S_F(\xi)$? The answer is yes, and for proving this fact one can proceed as in Proposition 3.1. This time, however, one must play simultaneously with both recursive relations stated in (20). The bilateral aspect of (20) is not to be neglected. □

The results stated in this section can be presented in a more compact manner by introducing a suitable notation. The distinction between the usual case and the bilateral one is implicit in the definition of the mapping

$$\lambda \in \mathbb{R} \rightrightarrows \Theta_F(\lambda) = \begin{cases} \Psi_F(\lambda) & \text{if } \lambda \in]-1, 0[, \\ \mathcal{K}(F) & \text{if } \lambda = 0, \\ \Phi_F(\lambda) & \text{if } \lambda \in]0, 1[\,. \end{cases}$$

Recall that the Minkowski sum of a finite number of convex cones coincides with the convex hull of these cones. Inspired by this fact, we use the integration symbol

$$\int_{]-1,1[} \Theta_F(\lambda)\, d\lambda = \mathrm{co}\Bigg[\bigcup_{\lambda \in]-1,1[} \Theta_F(\lambda) \Bigg] \tag{21}$$

to denote the convex hull of the family $\{\Theta_F(\lambda)\}_{\lambda \in]-1,1[}$.

Theorem 3.1. *Let F be a convex process. Then,*

$$\int_{]-1,1[} \Theta_F(\lambda)\, d\lambda \subset \mathcal{K}_\infty(F). \tag{22}$$

Proof. By Propositions 2.2(a), 3.1, and 3.2, we know that $\Theta_F(\lambda) \subset \mathcal{K}_\infty(F)$ whenever $\lambda \in]-1, 1[$. The desired conclusion is obtained by passing to the convex hull. □

While computing the integral (21), only the eigenvalues of F need to be taken into account. By way of example,

$$\int_{]-1,1[} \Theta_F(\lambda)\, d\lambda = \Theta_F(\lambda_1) + \cdots + \Theta_F(\lambda_p) \quad \text{if } \Lambda(F) \cap]-1, 1[= \{\lambda_1, \ldots, \lambda_p\}.$$

Since the point spectrum of a convex process is not necessarily finite, carrying out the computation of (21) is not always as easy as above.

The different values of λ do not play an identical role, so it may be useful to split the integral (21) in the form

$$\int_{]-1,1[} \Theta_F(\lambda)\, d\lambda = \int_{]-1,0[} \Psi_F(\lambda)\, d\lambda + \mathcal{K}(F) + \int_{]0,1[} \Phi_F(\lambda)\, d\lambda. \tag{23}$$

The right-hand side of (23) consists of three components, each one having its own interpretation. The last two components are convex cones, while the first one is a linear subspace.

4. Dualization

The purpose of this section is deriving upper estimates for the convex cone $\mathcal{K}_\infty(F)$. The set

$$\mathcal{M}(F) = \Big\{\xi \in H\colon\ \lim_{n\to\infty} \mathrm{dist}\big[0, F^n(\xi)\big] = 0\Big\} \tag{24}$$

is a fairly sharp upper bound for $\mathcal{K}_\infty(F)$, but (24) is not always easy to be evaluated. To obtain easily computable bounds, we look at the spectral information contained in the adjoint process of F.

Recall that the *adjoint* (or transpose) of the convex process $F : H \rightrightarrows H$ is the convex process $F^* : H \rightrightarrows H$ defined by

$$(u, z) \in \mathrm{gr}\, F^* \quad \Leftrightarrow \quad \langle z, s\rangle \leqslant \langle u, v\rangle, \quad \forall (s, v) \in \mathrm{gr}\, F.$$

We assume that the reader is familiar with this transposition mechanism. Being consistent with the notation introduced in the previous section, we write $(R_\lambda F^*)(0) = (F^* - \lambda I)^{-1}(0)$. The symbol

$$P^- = \big\{\xi \in H\colon\ \langle \xi, w\rangle \leqslant 0,\ \forall w \in P\big\}$$

refers to the negative polar cone of $P \subset H$.

Proposition 4.1. *Let F be a convex process. Then,*

$$\mathcal{K}_\infty(F) \subset \big[\big(R_\lambda F^*\big)(0)\big]^-, \quad \forall \lambda \geqslant 1. \tag{25}$$

Proof. Let $\xi \in \mathcal{K}_\infty(F)$ and $\lambda \geqslant 1$. Suppose that $\xi \notin [(R_\lambda F^*)(0)]^-$, that is to say, $\langle w, \xi\rangle > 0$ for some vector $w \in (R_\lambda F^*)(0)$. By definition of $(R_\lambda F^*)(0)$, such w satisfies

$$\langle w, v\rangle \geqslant \langle \lambda w, s\rangle, \quad \forall (s, v) \in \mathrm{gr}\, F. \tag{26}$$

On the other hand, there is a trajectory $x \in S_F(\xi)$ such that $\lim_{k\to\infty} x(k) = 0$. Plugging into (26), one gets

$$\big\langle w, x(k+1)\big\rangle \geqslant \big\langle \lambda w, x(k)\big\rangle, \quad \forall k = 0, 1, \ldots.$$

Hence, for each $k \in \{0, 1, \ldots\}$, one can write $\langle w, x(k)\rangle \geqslant \lambda^k \langle w, \xi\rangle$, and therefore

$$\big\|x(k)\big\| \geqslant \frac{\langle w, \xi\rangle}{\|w\|} > 0.$$

That x remains away from the origin is, of course, a contradiction. Therefore, the state ξ must be in $[(R_\lambda F^*)(0)]^-$. $\square$

As we shall see next, it is possible to sharpen the estimate (25) by using higher-order spectral information on F^*. However, this is not just a matter of writing

$$\mathcal{K}_\infty(F) \subset \big[\big(R_\lambda^n F^*\big)(0)\big]^-, \quad \forall n \geqslant 1,\ \forall \lambda \geqslant 1. \tag{27}$$

Inclusion (27) is correct for $n = 1$, but an extra term is missing when $n \geqslant 2$. This point will be clarified after stating two auxiliary lemmas. The first lemma has to do with a certain kernel function $\Gamma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined recursively by

$$\Gamma(r+1, k) = \begin{cases} \sum_{p=0}^{k-1} \Gamma(r, p) & \text{if } k \geqslant 1, \\ 0 & \text{if } k = 0, \end{cases}$$

with $\Gamma(0, k) = 1$ for every $k \geqslant 0$.

Lemma 4.1. *For $r \geqslant 1$ and $k \in \{0, 1, \dots, r-1\}$, one has $\Gamma(r, k) = 0$.*

Proof. The proof is accomplished by induction on r. $\square$

The purpose of the second lemma is establishing a link between the trajectories of F and the stationary trajectories of $F^* - \lambda I$. The function $\Gamma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ plays here a relevant role.

Lemma 4.2. *Let F be a convex process. Take $\lambda \geqslant 0$ and $x \in S_F(\xi)$. Then, for any stationary trajectory $y \in S_{F^*-\lambda I}(w)$, one has*

$$\langle x(k), w \rangle \geqslant \sum_{r=0}^{k} \Gamma(r, k) \lambda^{k-r} \langle y(r), \xi \rangle, \quad \forall k \geqslant 0. \tag{28}$$

Proof. The length of a stationary sequence $y : \mathbb{N} \to H$ is the smallest integer $N \geqslant 1$ such that $y(k) = 0$ for every $k \geqslant N$. For $N = 1$, the lemma amounts to saying that

$$0 \in (F^* - \lambda I)(w) \quad \Rightarrow \quad \langle x(k), w \rangle \geqslant \lambda^k \langle w, \xi \rangle, \quad \forall k \geqslant 0.$$

This particular situation has been taken care of in the proof of Proposition 4.1. Suppose (28) holds for any trajectory $y \in S_{F^*-\lambda I}(w)$ of length N. We shall prove that (28) remains true also for trajectories of length $N + 1$. So, pick up any

$$y \in S_{F^*-\lambda I}(w) \quad \text{such that} \quad y(k) = 0, \ \forall k \geqslant N + 1.$$

The shifted trajectory $k \mapsto \tilde{y}(k) = y(k+1)$ satisfies $\tilde{y} \in S_{F^*-\lambda I}(y(1))$ and $\tilde{y}(k) = 0$ for any $k \geqslant N$, so the induction hypothesis yields

$$\langle x(k), y(1) \rangle \geqslant \sum_{r=0}^{k} \Gamma(r, k) \lambda^{k-r} \langle y(r+1), \xi \rangle, \quad \forall k \geqslant 0. \tag{29}$$

Denote by a_k the sum on the right-hand side of (29). Since $y(1) + \lambda w \in F^*(w)$, one has

$$\langle x(k), w \rangle \geqslant \langle x(k-1), y(1) \rangle + \lambda \langle x(k-1), w \rangle.$$

Hence,

$$\begin{aligned} \langle x(k), w \rangle &\geqslant a_{k-1} + \lambda \langle x(k-1), w \rangle \geqslant a_{k-1} + \lambda \big[a_{k-2} + \lambda \langle x(k-2), w \rangle \big] \\ &\geqslant \cdots \geqslant \lambda^k \langle \xi, w \rangle + \sum_{j=1}^{k} a_{k-j} \lambda^{j-1}. \end{aligned}$$

But

$$a_{k-j}\lambda^{j-1} = \lambda^{j-1}\sum_{r=0}^{k-j}\Gamma(r,k-j)\lambda^{k-j-r}\big\langle y(r+1),\xi\big\rangle,$$

so one gets

$$\big\langle x(k),w\big\rangle \geqslant \lambda^k\langle\xi,w\rangle + \sum_{j=1}^{k}\sum_{r=0}^{k-j}\Gamma(r,k-j)\lambda^{k-r-1}\big\langle y(r+1),\xi\big\rangle.$$

A careful permutation of the summation order produces

$$\big\langle x(k),w\big\rangle \geqslant \lambda^k\langle\xi,w\rangle + \sum_{r=0}^{k-1}\sum_{p=r}^{k-1}\Gamma(r,p)\lambda^{k-r-1}\big\langle y(r+1),\xi\big\rangle.$$

Lemma 4.2 yields then the inequality

$$\big\langle x(k),w\big\rangle \geqslant \lambda^k\langle\xi,w\rangle + \sum_{r=0}^{k-1}\sum_{p=0}^{k-1}\Gamma(r,p)\lambda^{k-r-1}\big\langle y(r+1),\xi\big\rangle.$$

Due to the definition of Γ, one obtains

$$\big\langle x(k),w\big\rangle \geqslant \lambda^k\langle\xi,w\rangle + \sum_{r=0}^{k-1}\Gamma(r+1,k)\lambda^{k-r-1}\big\langle y(r+1),\xi\big\rangle,$$

that is to say,

$$\big\langle x(k),w\big\rangle \geqslant \sum_{r=0}^{k}\Gamma(r,k)\lambda^{k-r}\big\langle y(r),\xi\big\rangle.$$

This proves (28) for $k \geqslant 1$. The case $k = 0$ can be checked directly. □

We now are ready to incorporate the term that is missing in (27). In the theorem stated below, we use the notation

$$P^{\triangle} = H\backslash P^{+} = \big\{\xi \in H\colon\ \langle\xi,w\rangle < 0 \text{ for some } w \in P\big\}$$

to indicate the complement of $P^{+} = -P^{-}$.

Theorem 4.1. *Let F be a convex process. Then,*

$$\mathcal{K}_\infty(F) \subset \big[\big(R_\lambda^n F^*\big)(0)\big]^{-} \cup \big[\big(R_\lambda^{n-1} F^*\big)(0)\big]^{\triangle}, \quad \forall n \geqslant 2,\ \forall\lambda \geqslant 1. \tag{30}$$

Proof. Fix $n \geqslant 2$ and $\lambda \geqslant 1$. Let ξ be in $\mathcal{K}_\infty(F)$, that is to say, $\lim_{k\to\infty} x(k) = 0$ for some $x \in S_F(\xi)$. Suppose, to the contrary, that ξ does not belong to the right-hand side of (30). In such a case,

$$\langle\xi,w\rangle > 0 \quad \text{for some } w \in (R_\lambda^n F^*)(0),$$

and $\xi \in [(R_\lambda^{n-1} F^*)(0)]^+$. By a monotonicity argument, one has in fact

$$\xi \in \big[\big(R_\lambda^{n-r} F^*\big)(0)\big]^+, \quad \forall r = 1, \ldots, n-1.$$

Since $w \in (R_\lambda^n F^*)(0)$, one can construct a stationary trajectory $y \in S_{F^*-\lambda I}(w)$ of length n. By Lemma 4.2, the relation (28) necessarily holds. Observe that

$$y(r) \in \big(R_\lambda^{n-r} F^*\big)(0), \quad \forall r = 1, \ldots, n-1,$$

and therefore $\langle y(r), \xi\rangle \geqslant 0$ for $r = 1, \ldots, n-1$. Keeping in mind that $\lambda \geqslant 1$, one arrives at

$$\big\langle x(k), w\big\rangle \geqslant \lambda^k \langle w, \xi\rangle \geqslant \langle w, \xi\rangle > 0,$$

contradicting in this way the convergence of $x(k)$ toward the origin. □

Next example shows that formula (27) is not correct for $n = 2$. So, the extra term in (30) should not be neglected.

Example. Consider the closed convex process $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ given by

$$F(s) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} s + \mathbb{R}^2_+.$$

One can check that

$$\xi = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \in \mathcal{K}_\infty(F).$$

On the other hand,

$$F^*(u) = \begin{cases} \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] u & \text{if } u \in \mathbb{R}^2_+, \\ \emptyset & \text{if } u \notin \mathbb{R}^2_+, \end{cases}$$

and, therefore, $\Lambda(F^*) = \{1\}$. So, choose $\lambda = 1$. A simple matter of computation shows that

$$w = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \big(R_\lambda^2 F^*\big)(0).$$

Since $\langle w, \xi\rangle > 0$, it follows that $\xi \notin [(R_\lambda^2 F^*)(0)]^-$. This proves that $\mathcal{K}_\infty(F)$ is not contained in $[(R_\lambda^2 F^*)(0)]^-$. What is happening here is that ξ belongs to the set $[(R_\lambda^{2-1} F^*)(0)]^\triangle$.

Inclusions (25) and (30) do not hold when $\lambda \leqslant -1$. As we have learned already, dealing with negative eigenvalues requires using a bilateral approach. Next result involves the orthogonal space of

$$\big(B_\lambda F^*\big)(0) = \big(F^* - \lambda I\big)^{-1}(0) \cap -\big(F^* - \lambda I\big)^{-1}(0).$$

Proposition 4.2. *Let F be a convex process. Then,*

$$\mathcal{K}_\infty(F) \subset \big[\big(B_\lambda F^*\big)(0)\big]^\perp, \quad \forall \lambda \leqslant -1. \tag{31}$$

Proof. Let $\xi \in \mathcal{K}_\infty(F)$ and $\lambda \leqslant -1$. Suppose that $\xi \notin [(B_\lambda F^*)(0)]^\perp$, that is to say,

$$\langle w, \xi \rangle \neq 0 \quad \text{for some } w \in \big(B_\lambda F^*\big)(0).$$

Such w is a bilateral eigenvector of F^* associated to λ. The double inclusion

$$\lambda w \in F^*(w), \quad \lambda(-w) \in F^*(-w)$$

leads to the equality

$$\langle w, v \rangle = \langle \lambda w, s \rangle, \quad \forall (s, v) \in \operatorname{gr} F. \tag{32}$$

As in the proof of Proposition 4.1, there is a trajectory $x \in S_F(\xi)$ such that $\lim_{k\to\infty} x(k) = 0$. Plugging into (32), one gets

$$\big\langle w, x(k+1)\big\rangle = \big\langle \lambda w, x(k)\big\rangle, \quad \forall k = 0, 1, \ldots.$$

Hence, for each $k \in \{0, 1, \ldots\}$, one has $\langle w, x(k)\rangle = \lambda^k \langle w, \xi \rangle$, and therefore

$$\big\|x(k)\big\| \geqslant \lambda^k \frac{\langle w, \xi \rangle}{\|w\|}.$$

The inequality $\langle w, \xi \rangle > 0$ allows us to write

$$\big\|x(2k)\big\| \geqslant \frac{\langle w, \xi \rangle}{\|w\|} > 0,$$

whereas the inequality $\langle w, \xi \rangle < 0$ yields

$$\big\|x(2k+1)\big\| \geqslant -\frac{\langle w, \xi \rangle}{\|w\|} > 0.$$

In both cases, x admits a subsequence that remains away from the origin. This contradicts the fact that $x(k) \to 0$ as $k \to \infty$. In conclusion, ξ must be in $[(B_\lambda F^*)(0)]^\perp$. $\square$

The bilateral counterpart of Theorem 4.1 reads as follows. The symbol $L^\diamond = H \setminus L^\perp$ indicates the complement of the orthogonal space $L^\perp$.

Theorem 4.2. *Let F be a convex process. Then,*

$$\mathcal{K}_\infty(F) \subset \big[\big(B_\lambda^n F^*\big)(0)\big]^\perp \cup \big[\big(B_\lambda^{n-1} F^*\big)(0)\big]^\diamond, \quad \forall n \geqslant 2,\ \forall \lambda \leqslant -1. \tag{33}$$

Proof. It it essentially a matter of adjusting Lemma 4.2 to the present situation. One can show the relation

$$\big\langle x(k), w\big\rangle = \sum_{r=0}^{k} \Gamma(r,k) \lambda^{k-r} \big\langle y(r), \xi \big\rangle, \quad \forall k \geqslant 0, \tag{34}$$

for any stationary $y : \mathbb{N} \to H$ such that

$$y(0) = w \quad \text{and} \quad \big(y(r), y(r+1)\big) \in \operatorname{gr}\big(F^* - \lambda I\big) \cap -\operatorname{gr}\big(F^* - \lambda I\big), \quad \forall r \geqslant 0. \tag{35}$$

The implication (35) $\Rightarrow$ (34) is true, regardless of the choice of $\lambda \in \mathbb{R}$. Having clarified this point, fix now $n \geqslant 2$ and $\lambda \leqslant -1$. Let ξ be an initial state that can be brought asymptotically to the origin by means of a trajectory x of F. If ξ were not in the right-hand side of (33), then $\langle \xi, w\rangle \neq 0$ for some $w \in (B_\lambda^n F^*)(0)$, and

$$\xi \in \left[\left(B_\lambda^{n-r} F^*\right)(0)\right]^\perp, \quad \forall r = 1, \ldots, n-1.$$

Since $w \in (B_\lambda^n F^*)(0)$, one can construct a trajectory y as in (35). The relation (34) leads then to a contradiction. Indeed, (34) is inconsistent with the convergence of $x(k)$ toward the origin. □

5. Weak asymptotic stability results

Most of the heavy work has been accomplished in Sections 3 and 4. It is time now to state some general asymptotic stability results based on our estimates for the set $\mathcal{K}_\infty(F)$. We begin by writing down a sufficient condition for weak asymptotic stability. The notation

$$\operatorname{cone}\{a_1, \ldots, a_p\} = \{\mu_1 a_1 + \cdots + \mu_p a_p \colon \mu_1 \geqslant 0, \ \ldots, \ \mu_p \geqslant 0\}$$

refers to the convex cone generated by $\{a_1, \ldots, a_p\}$. As usual, $\operatorname{span}\{b_1, \ldots, b_q\}$ denotes the linear space spanned by $\{b_1, \ldots, b_q\}$.

Theorem 5.1. *Let $F : H \rightrightarrows H$ be a convex process. Suppose that the domain of F is representable in the form*

$$\operatorname{dom} F = \operatorname{span}\{b_1, \ldots, b_q\} + \operatorname{cone}\{c_1, \ldots, c_r\} + \operatorname{cone}\{a_1, \ldots, a_p\}, \tag{36}$$

where

$$\begin{cases} c_1, \ldots, c_r & \textit{are vectors in } \mathcal{K}(F); \\ a_1, \ldots, a_p & \textit{are finite-order eigenvectors of } F \\ & \textit{associated to eigenvalues lying in }]0,1[\,; \\ b_1, \ldots, b_q & \textit{are finite-order bilateral eigenvectors of } F \\ & \textit{associated to eigenvalues lying in }]-1,0[\,. \end{cases}$$

Then, F is weakly asymptotically stable relative to its domain, that is to say, $\mathcal{K}_\infty(F) =$ $\operatorname{dom} F$.

Proof. The set on the right-hand side of (36) is contained in the integral (23), which in turns is contained in $\mathcal{K}_\infty(F)$. □

Remark. One or two components in (36) could be missing, that is to say, there is no need to have all three of them at the same time. For instance, if F does not possess finite-order bilateral eigenvectors, then we simply drop the first term in (36).

Weak asymptotic stability of F can be secured if the space H is representable in the form

$$H = \operatorname{span}\{b_1, \ldots, b_q\} + \operatorname{cone}\{c_1, \ldots, c_r\} + \operatorname{cone}\{a_1, \ldots, a_p\},$$

but this would require H to be finite dimensional. In an infinite dimensional setting, it is more reasonable to write

$$H = \overline{\mathrm{span}}\{b_j\colon\ j \in J_1\} + \overline{\mathrm{cone}}\{c_j\colon\ j \in J_2\} + \overline{\mathrm{cone}}\{a_j\colon\ j \in J_3\}, \tag{37}$$

where J_1, J_2, J_3 are countable index sets. The symbols $\overline{\mathrm{cone}}(Q)$ and $\overline{\mathrm{span}}(Q)$ refer, respectively, to the closed convex conic hull, and the closed linear hull of $Q \subset H$.

Theorem 5.2. *Let $F : H \rightrightarrows H$ be a convex process. Assume that the space H is representable in the form* (37)*, where*

$$\begin{cases} \text{each } c_j & \text{belongs to } \mathcal{K}(F); \\ \text{each } a_j & \text{is a finite-order eigenvector of } F \\ & \text{associated to an eigenvalue lying in }]0,1[\,; \\ \text{each } b_j & \text{is a finite-order bilateral eigenvector of } F \\ & \text{associated to an eigenvalue lying in }]-1,0[\,. \end{cases}$$

Then, $\mathcal{K}_\infty(F)$ is dense in H.

Proof. The set on the right-hand side of (37) is contained in the closure of (23), which in turns is contained in the closure of $\mathcal{K}_\infty(F)$. The conclusion is that $\mathrm{cl}[\mathcal{K}_\infty(F)] = H$. □

Finally, we write down a necessary condition for weak asymptotic stability.

Theorem 5.3. *A convex process $F : H \rightrightarrows H$ fails to be weakly asymptotically stable if any of the following conditions occurs*:

(a) *F^* admits a finite-order eigenvector with associated eigenvalue in* $[1,\infty[$;
(b) *F^* admits a finite-order bilateral eigenvector with associated eigenvalue in* $]-\infty,-1]$.

Proof. Consider first the case (a). By applying Lemma 3.1 to the convex process F^*, one sees that (a) is equivalent to

$$F^* \text{ admits an eigenvector with associated eigenvalue in } [1,\infty[\,. \tag{38}$$

Failure of weak asymptotic stability is then a consequence of the estimate (25). A direct way of arriving at the same conclusion is by exploiting the higher-order estimate (30). As far as the case (b) is concerned, one can use the estimate (33), or one can combine (31) with the appropriate bilateral version of Lemma 3.1. □

Remark. The condition (38) was anticipated by Smirnov [12, Theorem 3.1], but only in the context of a closed convex process F defined over a finite dimensional space H. Although Smirnov does consider another condition which is not mentioned in Theorem 5.3, this extra-condition is not always easy to use in practice. Indeed, one is led to compute the maximal subspace $L \subset \mathrm{dom}\, F^* \cap -\mathrm{dom}\, F^*$ enjoying the invariance property $F^*(L) \subset L$, and then one has to check that the eigenvalues of the restriction of F^* to L have absolute values less than 1.

We end this section with an example illustrating the use of condition (b). The example takes place in the infinite dimensional real Hilbert space $H = L^2[-T, T]$ of square integrable functions over the interval $[-T, T]$. The problem under study is a discrete control system of the form

$$\left.\begin{array}{l} x_{k+1}(t) = Ax_k(t) + B\mu_k(t) \\ \mu_k(t) \in P \end{array}\right\} \quad \text{a.e. on } [0, T], \tag{39}$$

where each μ_k is a control variable living in a suitable control space U. Each $x_k = x(k)$ is an element of the state space $L^2[-T, T]$. To fix the ideas, take $U = L^2[-T, T]$ and $B = I$. The convex cone

$$P = \{\pi \in L^2[-T, T]\colon\ \pi = 0 \text{ a.e. on } J_1,\ \pi \geqslant 0 \text{ a.e. on } J_2\}$$

is interpreted as a constraint set for the control variable. Here J_1 and J_2 are two nonzero Lebesgue measure sets in $[-T, T]$ such that $J_1 \cup J_2 = [-T, T]$ and $J_1 \cap J_2 = \emptyset$. The operator A describes the intrinsic behavior of the system when no external control is acting on it. We are considering here the linear continuous self-adjoint operator $A : L^2[-T, T] \to L^2[-T, T]$ given by

$$[As](t) = \beta(t)s(t) \quad \text{a.e. on } [0, T],$$

where $\beta : [-T, T] \to \mathbb{R}$ is a continuous function.

In order to formulate the control problem (39) in the form (1), it is enough to introduce the multivalued operator $F : L^2[-T, T] \rightrightarrows L^2[-T, T]$ given by $F(s) = As + P$. In such a case,

$$F^*(\eta) = \begin{cases} A\eta & \text{if } \eta \in P^+, \\ \emptyset & \text{if } \eta \notin P^+, \end{cases}$$

with

$$P^+ = \{\eta \in L^2[-T, T]\colon\ \eta \geqslant 0 \text{ a.e. on } J_2\}.$$

If there is a nonzero Lebesgue measure set $J \subset J_1$ such that $\beta(t) = \lambda \leqslant -1$ for almost every $t \in J$, then it can be shown that $\lambda \in \Lambda(F^*)$. One sees also that the square integrable function

$$t \in [-T, T] \mapsto \eta(t) = \begin{cases} 1 & \text{if } t \in J, \\ 0 & \text{if } t \notin J, \end{cases}$$

is a bilateral eigenvector of F^* associated to λ. According to Theorem 5.3(b), the existence of such element indicates that the control system (39) is not weakly asymptotically stable.

6. Conclusions

Asymptotic stability analysis of multivalued dynamical systems is a broad theme with many ramifications. The most common framework is that of a differential inclusion

$$\dot{x}(t) \in F(x(t)), \tag{40}$$

where trajectories are sought in a suitable class of functions, and the multivalued operator F has some special structure. It would be hard to draw a complete picture of what has been published in the context of the continuous-time model (40). The Lyapunov function approach is perhaps the most popular one when it comes to deal with asymptotic stability issues. The spectral approach, which is less known, plays a prominent role in the work by Leizarowitz [7] and Smirnov [12,14]. The theory of Lyapunov functions is now fairly well understood, but it is not always easy to construct and exploit such functions. There are good reasons for switching the attention to spectral methods. Once a simple or higher-order eigenvector of F has been computed, one gets immediately a valuable information on the localization of $\mathcal{K}_\infty(F)$. The more we know about the spectral structure of F, the sharper the estimate for $\mathcal{K}_\infty(F)$ we obtain. Another point not to be forgotten is that spectral methods exploits also the duality existing between a convex process and its adjoint.

Passing from a differential inclusion to a difference inclusion is not a mere routine work. Some specific features of the discrete-time model (1) have no counterpart in the context of the continuous-time dynamics (40). Partial results on weak asymptotic stability of discrete-time systems governed by convex processes have been obtained by Phat [10] and Smirnov [13]. It is in relation to [10,13] that our contribution must be evaluated. The reader will notice important changes in the methodology, as well as in the sharpness of the results.

A question that deserve to be studied in the future is the weak asymptotic stability of a positively homogeneous operator F given by

$$\operatorname{gr} F = \bigcup_{j\in J} \operatorname{gr} F_j, \tag{41}$$

with each F_j being a convex process. The index set J may be finite or infinite. For (41), the asymptotic null-controllability set $\mathcal{K}_\infty(F)$ obeys to the rule

$$\bigcup_{j\in J} \mathcal{K}_\infty(F_j) \subset \mathcal{K}_\infty(F).$$

Although some of our results can be extended to this "multiconvex" setting, one should not be over-optimistic. The lack of usual convexity rules out, for instance, the possibility of exploiting standard duality arguments.

A particular case of (41) is that of a linear-selectionable process. The later terminology refers to a positively homogeneous operator F of the form

$$F(x) = \{Ax:\ A \in \mathcal{A}\},$$

with $\mathcal{A}$ being a bundle of linear operators. Molchanov and Pyatnitskiy [8,9] derive asymptotic stability results for difference inclusions governed by a special type of linear-selectionable operators. Their results are not comparable to ours; they are simply different.

As a final remark, we would like to point out that various concepts of weak asymptotic stability can be formulated in the more general context of a multivalued operator which is not necessarily positively homogeneous. There is, in principle, the possibility of bringing such a general operator to the particular framework of a convex process by applying a suitable "differentiation" (or linearization) technique; see, for instance, Frankowska [5] for a comparison between local controllability of a general multivalued operator and global controllability of an associated convex process obtained by differentiation. So, our results could be extended beyond the class of convex processes, keeping in mind, of course, that some information is being lost by the mere fact of differentiating the original operator.

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