Algebraic Topology for Minimal Cantor Sets

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Abstract. It will be shown that every minimal Cantor set can be obtained as a projective limit of directed graphs. This allows to study minimal Cantor sets by algebraic topological means. In particular, homology, homotopy and cohomology are related to the dynamics of minimal Cantor sets. These techniques allow to explicitly illustrate the variety of dynamical behavior possible in minimal Cantor sets.

1 Introduction

A minimal Cantor set is a dynamical system defined by a continuous map on the Cantor set whose orbits are dense. These dynamical systems have been widely studied, mainly by symbolic dynamical means. See for example [BSY], [BH], [D], [DKL], [Du], [DHS], [G], [GPS] and [W]. In this paper we study in a self contained manner minimal Cantor sets by algebraic topological means without any use of symbolic dynamics. In particular, we show that classical concepts such as homology, homotopy and cohomology are related to the dynamics of minimal Cantor sets.

The first result is a structure Theorem which says that every minimal Cantor set can be obtained as a projective limit of directed graphs.

It is then possible to define a homology group for the minimal Cantor as a projective limit of the homology groups of the directed graphs, the Cech-homology of the suspension of the minimal Cantor set. This allows to identify the set of invariant measures with a cone in this homology group.

Examples of uniquely ergodic minimal Cantor sets, minimal Cantor sets with finitely many ergodic measures were already known, see [D] and [W]. The projective limit structure allows us to construct such examples, using elementary linear algebra, in a very explicit way. Using this homological approach an example of a minimal Cantor set whose set of ergodic (probability) measures is homeomorphic to a n-dimensional sphere, is also presented.

As invariant measures are related to homology it will be shown that entropy is linked to homotopy of the directed graphs. Using a homotopical argument minimal Cantor sets with positive and even infinite topological entropy are constructed. Combining homological and homotopical arguments uniquely ergodic examples of minimal Cantor sets with infinite topological entropy are explicitly given. See also [G].

It is also possible to define a cohomology group over S^1 for the minimal Cantor as a direct limit of the cohomology groups of the directed graphs. Using this cohomology group, the minimal Cantor sets which admit a semi-conjugation to an irrational rotation of the circle are identified. In particular, the group of rotation numbers which allow such semi-conjugations is defined. Explicit examples of minimal Cantor sets are constructed which do not allow semi-conjugations to rotations, do have non-trivial semi-conjugations to circle rotations and to minimal torus shift of arbitrary dimension.

2 Minimal Cantor sets

A Cantor set is a perfect 0-dimensional compact metric space. It can be covered by a partition of clopen sets[1] with arbitrary small diameters. It follows that a Cantor set can be seen in many different ways as the projective limit of finite sets labeling the elements of a successive sequence of partitions (see for instance [Mi1], [Mi2] for an interesting use of this idea). A dynamical system given by a continuous map on a Cantor set is called a *minimal Cantor set* if all orbits are dense. In this section we will give a combinatorial description of minimal Cantor sets. The idea is to make clopen covers of the Cantor set reflecting the action of the map. The same idea was used for studying minimal Cantor sets appearing in unimodal dynamics ([M]) but turned out to be strong enough to describe abstract minimal Cantor sets.

Let $f\colon C \to C$ be a minimal Cantor set. We are going to construct arbitrarily small covers consisting of clopen sets which represent the dynamics of f. Unless otherwise stated all considered subsets of C will be clopen.

The construction of such a cover $\mathcal{X}$ starts with the choice of a partition $\mathcal{P}$ of C. The partition $\mathcal{P}$ is used for getting control on the size of the sets in $\mathcal{X}$. Choose $U_0 \subset P \in \mathcal{P}$. The cover $\mathcal{X}$ of C will consist of clopen sets whose points pass trough the same sets of the partition $\mathcal{P}$ before they return to U_0. The definition can be given inductively. Let $\mathcal{X} = \bigcup_{n\geq 0} \mathcal{X}(n)$ where $\mathcal{X}(n)$ is defined as follows. Let

$$\begin{aligned} \mathcal{X}(0) &= \{U_0\} \\ \mathcal{X}(n+1) &= f_{\mathcal{P}}^{-1}(\mathcal{X}(n)). \end{aligned}$$

The pullback $f_{\mathcal{P}}^{-1}(\mathcal{X}(n))$ consists of the sets

$$(f^{-1}(V) \cap Q) - U_0$$

where $V \in \mathcal{X}(n)$ and $Q \in \mathcal{P}$. Observe that the definition depends only on $U_0 \subset C$ and the partition $\mathcal{P}$. The collection $\mathcal{X} = \cup \mathcal{X}(n)$ is a pairwise disjoint clopen cover of C. In particular it is finite. The cover $\mathcal{X}$ reflects the dynamics of f: the image of every set $V \in \mathcal{X}(n+1)$ is a subset of some set in $\mathcal{X}(n)$, $n \geq 0$.

The *path* of a set $V_n \in \mathcal{X}(n)$ is

$$\lambda(V_n) = \{V_n, V_{n-1}, \ldots, V_1, V_0 = U_0\}$$

[1] i.e. closed open sets

where $V_j \in \mathcal{X}(j)$ and $f(V_{j+1}) \subset V_j$. The following Lemma summarizes how this cover $\mathcal{X}$ reflects the action of f.

Lemma 2.1

a) *$\mathcal{X}$ is a pairwise disjoint clopen cover of C;*

b) *let $U_1, U_2, \ldots, U_d \in \mathcal{X}$ be all the sets such that $f(U_0) \cap U_j \neq \emptyset$ for $j = 1, \ldots, d$. Then*

$$C = \cup_{j=1}^{d} \lambda(U_j);$$

c) *the diameter of every $V \in \mathcal{X}$ is smaller than* $\mathrm{mesh}(\mathcal{P})$.

Although the above properties follow directly from the definition they form the fundamental tool for describing minimal Cantor sets. In particular the above defined cover can be considered to be an approximation of the map f. It can be naturally represented by a directed graph. The elements of $\mathcal{X}$ serve as vertices and the action of f defines the edges.

Definition 2.2 *A directed topological graph X is called a combinatorial cover iff*

a) *X is finite and the set of vertices carries the discrete topology;*

b) *X is irreducible (every two vertices can be connected by a directed path);*

c) *except for one vertex $0_X \in X$, every vertex of X has exactly one out going edge. This vertex 0_X is called the splitting vertex and can have more out going edges.*

A vertex $y \in X$ is called an image of a vertex $x \in X$ if there is an edge going from x to y. The shortest directed path $\lambda(x)$ from a vertex $x \in X$ to 0_X is called the path of x.

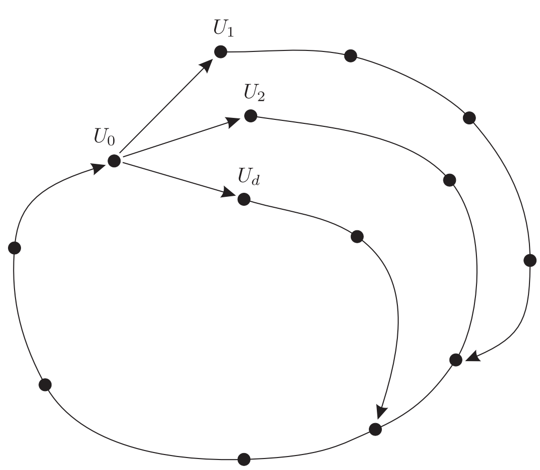

Figure 1. A combinatorial cover.

The above constructed cover $\mathcal{X}$ can be considered as a combinatorial cover. The combinatorial covers as described in Lemma 2.1 consist of a finite number of loops, corresponding to the path of the sets $U_1, U_2, \dots, U_d$, starting and ending in the splitting vertex.

The projection $h_{\mathcal{X}}\colon C \to \mathcal{X}$ is given by $h_{\mathcal{X}}(x) = V$ iff $x \in V \in \mathcal{X}$. This projection preserves the action of f and the graph structure, that is if $x \in V$ and $f(x) \in W$ then $h_{\mathcal{X}}(W)$ is an image of $h_{\mathcal{X}}(V)$.

The (combinatorial) cover $\mathcal{X}$ is considered to be an approximation of f. The next step is to make a consistent sequence of finer and finer (combinatorial) covers.

Assume that the (combinatorial) cover $\mathcal{X}_n$ is defined using the partition $\mathcal{P}_n$, with $\text{mesh}(\mathcal{P}_n) \leq \frac{1}{n}$, and $U_0 \in \mathcal{X}_n$ as splitting vertex. Choose a clopen partition $\mathcal{P}_{n+1}$ which refines $\mathcal{X}_n$. This can be done by taking $\text{mesh}(\mathcal{P}_{n+1})$ small enough. Assume it smaller than $\frac{1}{n+1}$. Choose a clopen set $U' \subset U_0$ such that for some $U_1 \in \mathcal{X}_n$ $f(U') \subset U_1$. Choose $P \in \mathcal{P}_{n+1}$ with $P \cap U' \neq \emptyset$. Finally choose a clopen $U \subset P \cap U'$ and construct $\mathcal{X}_{n+1}$ by using U and $\mathcal{P}_{n+1}$. By construction, the cover $\mathcal{X}_{n+1}$ refines the cover $\mathcal{X}_n$. Let $\pi_n\colon \mathcal{X}_{n+1} \to \mathcal{X}_n$ be the projection between the combinatorial covers induced by the inclusion map.

Definition 2.3 *A map $\pi\colon Y \to X$ between two combinatorial covers is called a combinatorial refinement iff*

a) *π preserves the graph structure;*
b) $\pi(0_Y) = 0_X$*;*
c) *There is an image $1_X \in X$ of $0_X \in X$ such that*

$$1_X = \pi(\{V \,|\, V \text{ image of } 0_Y \in Y\}).$$

The inclusion of $\mathcal{X}_{n+1}$ into $\mathcal{X}_n$ is denoted by $\pi_n\colon \mathcal{X}_{n+1} \to \mathcal{X}_n$. The inductive construction of $\mathcal{X}_n$ was done such that

Lemma 2.4 *The projection $\pi_n\colon \mathcal{X}_{n+1} \to \mathcal{X}_n$ is a combinatorial refinement.*

Consider the sequence of refinements $\pi_n\colon \mathcal{X}_{n+1} \to \mathcal{X}_n$, $n \geq 1$. The projective limit

$$\mathcal{X} = \lim \mathcal{X}_n$$

will be a topological graph. In particular, the edges can be described by a continuous function. This follows from the following observation. In general, the image of a vertex $U \in \mathcal{X}_{n+1}$ is not well defined. However, by construction it follows directly that

$$\pi_n(\{V \in \mathcal{X}_{n+1} \,|\, V \text{ is image of } U\}) = U' \in \mathcal{X}_n$$

is uniquely defined. The graph structure on $\mathcal{X}$ can be described by the map $g\colon \mathcal{X} \to \mathcal{X}$ defined by

$$g(\{U_n\}) = \{U'_n\}.$$

This map is continuous.

Consider the projections $h_n = h_{\mathcal{X}_n} \colon C \to \mathcal{X}_n$. Because the maps h_n and π_n are inclusions they commute, $h_n = \pi_n \circ h_{n+1}$. Hence, there is a limit map $h\colon C \to \mathcal{X}$ which is continuous and onto. In fact h is a homeomorphism. To show this it is sufficient to proof that h is injective. Take two points $x, y \in C$. Because of Lemma 2.1(c) we know that $\mathrm{mesh}(\mathcal{X}_n) \to 0$, the covers $\mathcal{X}_n$ are going to separate points. So there exists an $n \geq 1$ and $U, V \in \mathcal{X}_n$ with $x \in U$ and $y \in V$ and $U \neq V$. Hence $h_n(x) \neq h_n(y)$ which means $h(x) \neq h(y)$. Moreover, by construction we have that h conjugates f to g.

Consider a sequence of combinatorial refinements $\pi_n \colon X_{n+1} \to X_n$ and take the corresponding projective limit

$$X = \lim X_n.$$

The graph structure of this topological graph can, as we saw above, be described by a continuous map $f\colon X \to X$. If this system is a minimal Cantor set it is called a *combinatorially obtained* minimal Cantor set. We proved

Theorem 2.5 *Every minimal Cantor set can be conjugated to a combinatorially obtained minimal Cantor set.*

In general the dynamical systems obtained by taking projective limits of directed graphs will not be minimal Cantor sets. Let us finish this section describing the, very weak, restriction needed to be made on the refinements to obtain minimal Cantor sets.

Let $f\colon X \to X$ be obtained by taking the projective limit corresponding to the combinatorial refinements $\pi_n \colon X_{n+1} \to X_n$. The vertices in the graphs X_n form a finite clopen partition of X. The space X has arbitrarily fine finite clopen covers. Hence it is zero-dimensional and compact.

Corresponding to the chosen representation of f there is a special point $0 = \{0_{X_n}\} \in X$. If there is a directed path of length t from a vertex $U \in X_n$ to a vertex $V \in X_n$ which doesn't pass through 0_{X_n} then $f^t(U) \subset V$. This implies that every orbit in X accumulates at 0. So $\omega(x) \supset \omega(0)$ for $x \in X$ and X is a minimal set iff $\omega(0) = X$. For studying $\omega(0)$ we need to know the intersection properties between loops of X_{n+1} with loops of X_n.

Let L_n consist of the images of 0_{X_n}, that is $U_1, \ldots, U_{d_n}$. The set L_n labels the loops of X_n. In particular denote the image which contains $f(0)$ by $U_1 \in L_n$. For $U_i \in L_n$ and $V_j \in L_{n+1}$ define $w_{ij} = \#\{T \in \lambda(V_j) \,|\, T \subset U_i\}$. The loop $\lambda(U_j)$ of X_{n+1} passes w_{ij} times through the loop $\lambda(U_i)$ of X_n. The matrix W_n with entries w_{ij} is called the *winding matrix* corresponding to $\pi_n \colon X_{n+1} \to X_n$.

The intersection properties can be summarized by the graph L whose vertices are $\cup L_n$ together with edges from $V_j \in L_{n+1}$ to $U_i \in L_n$ with weight w_{ij}.

The matrix W_{mn} describes in the same way the projection from X_{m+1} to X_n. Clearly, $W_{mn} = \Pi_{j=m-1}^{n} W_j$. The graph L is said to be 2-*connected* if for every $n \geq 1$ there exists an $m \geq 1$ such that all entries of the first column of W_{mn} are at least 2. If the first column of W_{mn} are positive then the path $\lambda(V_1)$, $V_1 \in L_{m+1}$

passes through all loops of X_n. If this holds for all n the orbit of 0 will be dense. We need the 2 in the definition to be sure that X is a Cantor set and not just a periodic orbit, the loop of V_1, $V_1 \in L_{m+1}$ passes through the loops of X_n in at least 2 ways.

Proposition 2.6 *The map $f\colon X \to X$ is a minimal Cantor set if and only if L is 2-connected.*

3 Invariant measures and Homology

In this section we are going to discuss the space $\mathcal{M}(X)$ consisting of signed invariant measures of a minimal Cantor set $f\colon X \to X$. A signed invariant measure is the difference of two finite measures. On $\mathcal{M}(X)$ we use the following norm

$$|\mu| = \sup_{\phi \in B^0(X)} |\int \phi d\mu|,$$

where $B^0(X)$ stands for the unit ball on the space of continuous functions on X equipped with the sup norm. All measure spaces under consideration will be equipped with similar norms.

Fix a projective limit representation for the minimal Cantor set $f\colon X \to X$, say $X = \lim X_n$ with $\pi_n\colon X_{n+1} \to X_n$ the corresponding projections. The number of loops in X_n is d_n. Consider the space of signed measures on X_n, the σ-algebra is generated by the elements of X_n. Each loop of X_n carries an "invariant measure". More precisely, let $\lambda(U_1), \dots, \lambda(U_{d_n})$ be the loops of X_n. The measure ν_j^n on X_n has $\lambda(U_j)$ as support and

$$\nu_j^n(A) = 1 \text{ iff } A \in \lambda(U_j).$$

The first homology group $H_1(X_n)$ is the vector space generated by these measures ν_j^n. Formally, $H_1(X_n)$ is a measure space. However, the generators correspond to the loops of the graph X_n and we can also think about $H_1(X_n)$ as the first homology group of the graph X_n.

The inclusion $p_n\colon X \to X_n$ induces a map

$$(p_n)_*\colon \mathcal{M}(X) \to H_1(X_n),$$

the $(p_n)_*$-image of a measure in $\mathcal{M}(X)$ is the measure obtained when the σ-algebra is restricted to the one generated by the sets of the cover X_n.

Lemma 3.1 *The map $\pi_n\colon X_{n+1} \to X_n$ induces a linear map*

$$(\pi_n)_*\colon H_1(X_{n+1}) \to H_1(X_n)$$

which represented using the bases above equals the winding matrix of π_n,

$$(\pi_n)_* = W_n.$$

Proof. Observe that every measure in $H_1(X_n)$ is determined by its values on the U_j's,

$$(p_n)_*(\mu) = \Sigma_{j=1}^{d_n} \mu(U_j)\nu_j^n.$$

Furthermore a computation shows

$$(\pi_n)_*(\mu) = \Sigma_{i=1}^{d_n}\{\Sigma_{j=1}^{d_{n+1}} w_{ij}\mu(U_j)\}\nu_i^n$$

which proves $(\pi_n)_* = W_n$. □

The set $\mathcal{I}(X) \subset \mathcal{M}(X)$ consists of the invariant measures of $f\colon X \to X$. Define

$$H_1^+(X_n) = \{\Sigma_{j=1}^{d_n}\alpha_j\nu_j^n \mid \alpha_j \geq 0\}.$$

Again denote the composition $W_nW_{n+1}\ldots W_m$ by W_{mn} and let

$$I(X_n) = \bigcap_{j=n+1}^{\infty} W_{jn}(H_1^+(X_j)).$$

The sets $I(X_n)$ are cones in $H_1(X_n)$. Clearly $W_n(I(X_{n+1})) = I(X_n)$. Hence the projective limit $\lim_{W_n} I(X_n)$ is well defined. Finally, let $\mathcal{P}(X) \subset \mathcal{I}(X)$ and $P(X_n) \subset I(X_n)$ consist of corresponding probability measures.

Because all maps under consideration are inclusion maps the induced maps $(p_n)_*\colon \mathcal{M}(X) \to H_1(X_n)$ satisfy $(p_{n+1})_* \circ W_n = (p_n)_*$. Furthermore they are closed continuous maps. This enables us to extend the maps $(p_n)_*$ to a bounded map

$$p_*\colon \mathcal{M}(X) \to \lim_{W_n} H_1(X_n).$$

Proposition 3.2 *The map*

$$p_*\colon \mathcal{I}(X) \to \lim_{W_n} I(X_n)$$

is an isomorphism. In particular the map

$$p_*\colon \mathcal{P}(X) \to \lim_{W_n} P(X_n)$$

is as such.

Proof. Observe that $(p_m)_*(\mathcal{I}(X)) \subset H_1^+(X_m)$ for all $m \geq 1$ and $(p_n)_* = W_nW_{n+1} \ldots W_{m-1} \circ (p_m)_*$. Finally because W_n is a non-negative matrix we get $W_m(H_1^+ (X_{m+1})) \subset H_1^+(X_m)$. Hence, $(p_n)_*(\mathcal{I}(X)) \subset I(X_n)$ for all n, This implies $p_*(\mathcal{I}(X)) \subset \lim_{W_n} I(X_n)$.

Every point in $\lim_{W_n} I(X_n)$ gives rise to a positive additive set function on the clopen sets of X. It gives rise to an invariant measure, the map $p_*\colon \mathcal{I}(X) \to \lim_{W_n} I(X_n)$ is onto. □

A minimal Cantor set is said to have *bounded combinatorics* if it can be obtained combinatorially such that the winding matrices W_n are positive and the size and entries of these matrices are uniformly bounded.

Proposition 3.3 *Let $f\colon X \to X$ be a minimal Cantor set with representation $X = \lim X_n$.*

a) *If the number of loops in X_n is uniformly bounded by d then f has at most d ergodic invariant probability measures.*
b) *If f has bounded combinatorics then it is uniquely ergodic.*

Proof. To prove the first statement we may assume that $d_n = d$ for all $n \geq 1$. Normalize the basis measures from Lemma 3.1 to probability measures, $\mu_j^m = \frac{1}{t_j^m}\nu_j^m$ where $\{\nu_j^m \mid j = 1, \dots, d\}$ is the basis and t_j^m the period of the corresponding loop.

Let $P_m \subset H_1^+(X_m)$ be the set of probability measures and $P_n^m = W_{nm}(P_m)$. Because P_m is the convex hull of the $\{\mu_j^m\}$, P_n^m is the convex hull of the measures $\mu_j^{nm} = W_{nm}(\mu_j^m)$. By taking a subsequence we may assume that the measures μ_j^{nm} converge to measures $\mu_j \in P_n$ for $j = 1, \dots, d$. Because $P(X_n) = \bigcap P_n^m$ we get that $P(X_n)$ equals the convex hull of the measures $\{\mu_j \mid j = 1, \dots, d\}$. Hence it is the convex hull of at most d points. Suppose that X had more than d ergodic measures. Then, for n large enough, the projection of these ergodic measures would be distinct extremal points of $P(X_n)$ which has at most d extremal points.

To prove the second statement we have to show that $\mathcal{I}(X)$ is one-dimensional. The hyperbolic distance between two points $x, y \in H_1^+(X_n)$ is

$$\mathrm{hyp}(x, y) = -\ln \frac{(m + l) \cdot (m + r)}{l \cdot r},$$

where m is the length of the line segment $[x, y]$ and l, r are the length of the connected components of $T \setminus [x, y]$. The line segment T is the largest line segment in $H_1^+(X_n)$ containing $[x, y]$. Positive matrices contract the hyperbolic distances on the positive cones. The winding matrices W_n have uniformly bounded size and entries. This implies that the contraction is uniform.

The set

$$I(X_n) = \bigcap_{j=n+1}^{\infty} W_{jn}(H_1^+(X_n))$$

is one-dimensional because of the uniform contraction of each W_n. In particular, by using Proposition 3.2, we have that $\mathcal{I}(X)$ is one-dimensional. The map f has only one invariant probability measure. $\square$

There exist minimal Cantor sets which can be combinatorially obtained by covers X_n which all have $d \geq 1$ loops and have $d - 1$ ergodic invariant probability measures. To describe such an example arrange the projections π_n such that the corresponding winding matrices W_n have all entries equal to 1 except for the diagonal entries of the second and following columns which equal a large number w_n. In particular, the basis measure μ_j^{n+1} which is concentrated on the j^{th} loop of X_{n+1} will be projected to a measure very close to μ_j^n, the measure concentrated on the j^{th} loop of X_n. By taking a sequence w_n growing fast enough we can assure that $\mathcal{I}(X_n)$ is a cone spanned by $d-1$ different ergodic measures. The corresponding minimal Cantor set has $d-1$ ergodic invariant (probability) measures.

Theorem 3.4 *For every n there exists a minimal Cantor whose set of ergodic invariant (probability) measures is homeomorphic to the sphere S^n.*

Proof. For all dimensions the idea for the construction is the same. We will give the proof for $n = 1$.

Let X_1 be a combinatorial cover with three tubes. Hence the probability measures $P_1 \subset H_1(X_1)$ form a 2-dimensional simplex. The convex hull of a finite set $E \subset P_1$ is denoted by $\text{hull}(E)$. A set E is called the set of extremal points of $\text{hull}(E)$ if $\text{hull}(E') \neq \text{hull}(E)$ for every strict subset $E' \subset E$.

We are going to define the combinatorial covers X_{n+1} and the projections $\pi_n\colon X_{n+1} \to X_n$ inductively. Suppose $X_1, X_2, \ldots, X_n$ and the corresponding projections are defined. Using the notation of the proof of Proposition 3.3 we get that P_1^n, the projection of the probability measures P_n into P_1, form a convex set spanned by $E_n = \{\mu_j^{1n} \,|\, j = 1, \ldots, d_n\}$, where d_n is the number of loops in X_n. The induction hypothesis assumes that E_n is the set of extremal points of P_1^n. Assume that the measures in E_n are ordered in such a way that $\text{hull}(\{\mu_j^{1n}, \mu_{j+1}^{1n}\})$, $j = 1, \ldots, d_n - 1$ and $\text{hull}(\{\mu_{d_n}^{1n}, \mu_1^{1n}\})$ are the sides of P_1^n.

The cover X_{n+1} is going to have $d_{n+1} = 2d_n$ loops. For every loop in X_n there is a loop in X_{n+1} which passes a_n times through this given loop and exactly once through all other loops in X_n. This gives a group of d_n loops in X_{n+1}.

For every pair $\{\mu_j^n, \mu_{j+1}^n\}$, $j = 1, \ldots, d_n - 1$ and the pair $\{\mu_{d_n}^n, \mu_1^n\}$ there will be a loop in X_{n+1} which passes b_n times through both corresponding supporting loops in X_n and exactly once through all other loops in X_n. This gives another group of d_n loops in X_{n+1}. All loops of X_{n+1} are going to pass at least once through all loops of X_n. This is to assure that $X = \lim X_n$ becomes a minimal Cantor set.

Observe that by choosing the number a_n very big, the measures on the loops of the first group are going to have their masses concentrated mainly on the loop through which it passes a_n times. Their projections into P_n are going to converge to the corresponding measure μ_j^n. By choosing the number b_n very big, the measures on the loops of the second group are going to be equally concentrated over the two loops through which it passes b_n times. Their projections are going to converge to the mean of the corresponding measures: $\frac{1}{2}(\mu_j^n + \mu_{j+1}^n)$.

Let $E_{n+1}^1(a)$ consist of the projections in P_1 of the measures concentrated on the loops of the first group in X_{n+1} when constructed with a and $E_{n+1}^2(b)$ consist of the projections of the measures in the second group when constructed with b.

By the discussion above we know that $E^1_{n+1}(a)$ converges to E_n when $a\to\infty$ and $E^2_{n+1}(b)$ converges to the middle points of the sides of $\mathrm{hull}(E_n)$ when $b\to\infty$.

We are going to define the values a_n and b_n inductively. Assume that E_j, $j=1,\dots,n$ is defined inductively together with finite sets E'_j, $1,\dots,n$, satisfying

1) $\mathrm{hull}(E'_j)$ is strictly contained in $\mathrm{hull}(E_j)$;

2) The annulus $A_j=\mathrm{hull}(E_j)\setminus\mathrm{hull}(E'_j)$ satisfies the metrical property
$$\epsilon(A_j)\le 2l_{j-1};$$
where $\epsilon(A_j)$ is the length of the longest straight line in A_n and l_n the longest side of $\mathrm{hull}(E_n)$;

3) $A_{j-1}\supset A_j$ and $l_j\le 0.6 l_{j-1}$.

Let us define E_{n+1} and E'_{n+1} extending the above property. By taking a'_{n+1} sufficiently big, $E^1_{n+1}(a'_{n+1})$ converges to E_n, and we can manage so that the annulus $A=\mathrm{hull}(E_n)-\mathrm{hull}(E^1_{n+1}(a'_{n+1}))$ is part of A_n and $\epsilon(A)\le 2l_n$. Let $E'_{n+1}=E^1_{n+1}(a'_{n+1})$. By the same reason as above there is an a_{n+1} such that $\mathrm{hull}(E'_{n+1})$ lies strictly in $\mathrm{hull}(E^1_{n+1}(a_{n+1}))$. We may assume that all points in $E^1_{n+1}(a_{n+1})$ have distance to E_n less than $0.01l_n$. Because all loops in X_{n+1} pass through all loops of X_n the $\mathrm{hull}(E^1_{n+1}(a_{n+1}))$ lies strictly inside $\mathrm{hull}(E_n)$. Now take b_{n+1} such that $E^2_{n+1}(b_{n+1})\cap\mathrm{hull}(E^1_{n+1}(a_{n+1}))=\emptyset$. By taking b_{n+1} big enough we may assume that $l_{n+1}\le 0.6l_n$.

Let $E_{n+1}=E^1_{n+1}(a_{n+1})\cup E^2_{n+1}(b_{n+1})$ and $E'_{n+1}=E^1_{n+1}(a'_{n+1})$. This finishes the inductive definition.

Claim 3.5 *The set $P(X_1)=\bigcap P^n_1=\bigcap\mathrm{hull}(E_n)\subset P_1$ is a strictly convex disk, every line connecting two points on the boundary intersects the boundary only in the begin and end point.*

Observe that the topological boundary of $P(X_1)$ equals $\bigcap A_n$. Hence if the boundary of $P(X_1)$ contains a straight line L then $L\subset A_n$ for every n. So $|L|\le\epsilon(A_n)\to 0$. Contradiction.

First we will show that $\mathcal{P}(X)=\lim_{A_n}P(X_n)$ contains a set homeomorphic to a circle. Take a refining sequence of equal distributed partitions of the circle S^1 with d_n pieces. Let $\phi_{1n}\colon S^1\to\partial\,\mathrm{hull}(E_n)$ be the homeomorphism which maps the pieces of the n^{th} partition linearly to the sides of $\mathrm{hull}(E_n)$. These homeomorphisms can be factorized as $\phi_{1n}=W_{1n}\circ\phi_n$, where $\phi_n\colon S^1\to\partial P_n$ maps the pieces of the n^{th} partition linearly onto the corresponding sides of P_n. Let $\phi_{nm}=W_{nm}\circ\phi_m$.

If the ϕ_{1n} are chosen coherently then this sequence converges to an embedding $h_1\colon S^1\to P_1$. The construction implies that all $\phi_{nm}\colon S^1\to P_n$ converge to a continuous map $h_n\colon S^1\to P_n$ satisfying $h_n=W_n\circ h_{n+1}$. Furthermore the factorization shows that all $h_n\colon S^1\to\gamma_n=h_n(S^1)\subset P_n$ are embeddings of the circle and $W_n\colon\gamma_{n+1}\to\gamma_n$ is a homeomorphism. Let $\gamma=\lim_{W_n}\gamma_n\subset\mathcal{P}(X)$. This γ is homeomorphic to S^1.

Claim 3.6 *Let* $\mathrm{Erg}(X)$ *be the set of ergodic (probability) measures of* f. *Then*

$$\mathrm{Erg}(X) = \gamma.$$

First we will show that $\gamma \subset \mathrm{Erg}(X)$. Let $\mu \in \gamma$. Suppose that μ is not ergodic then $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ where the μ_i's are two invariant probability measures. Consider $(p_1)_*(\mu) = \alpha(p_1)_*(\mu_1) + (1-\alpha)(p_1)_*(\mu_2)$. Because $(p_1)_*(\mu) \in \gamma_1$ and $P(X_1)$ is strictly convex this is only possible if $(p_1)_*(\mu_1) = (p_1)_*(\mu_2) = (p_1)_*(\mu)$. But $W_n\colon \gamma_{n+1} \to \gamma_n$ is bijective which implies $\mu = \mu_1 = \mu_2$. Contradiction, $\gamma \subset \mathrm{Erg}(X)$.

To finish the proof of Theorem 3.5 we have to show that every invariant measure can be disintegrated over the supports of the ergodic measures in γ. For $m > n$ let $\mu_j^m, j \leq d_m$ be the probability measure concentrated on the j^{th} loop and μ_j^{nm} the projection of μ_j^m into P_n.

Let $\mu \in \mathcal{P}(X)$. Then for $m \geq n$, $(p_m)_*(\mu) = \Sigma_{j=1}^{d_m}\alpha_j^m\mu_j^m$ with $\alpha_j^m \geq 0$ and $\Sigma_{j=1}^{d_m}\alpha_j^m = 1$. So $(p_n)_*(\mu) = \Sigma_{j=1}^{d_m}\alpha_j^m\mu_j^{nm}$ which induces a discrete measure ϵ_{nm} on P_n by $\epsilon_{nm}(\mu_j^{nm}) = \alpha_j^m$. By taking subsequences we may assume that for all $n \geq 1$ the sequences ϵ_{nm}, $m \geq n$ will converge weakly to a measure ϵ_n. Clearly the support of ϵ_n is part of γ_n and we may assume $(W_n)_*(\epsilon_{n+1}) = \epsilon_n$. Hence there is also an induced measure ϵ on γ.

Let $\phi\colon X \to \mathbf{R}$ be a function which is constant on the vertices of X_n. Now for every $m \geq n$

$$\begin{aligned}\int \phi d\mu &= \Sigma_{j=1}^{d_m}\alpha_j^m \int \phi d\mu_j^{nm} \\ &= \Sigma_{j=1}^{d_m}\epsilon_{nm}(\mu_j^{nm}) \int \phi d\mu_j^{nm} \\ &= \int [\int \phi d\nu] d\epsilon_{nm}.\end{aligned}$$

The function on P_n defined by $\nu \to \int \phi d\nu$ is continuous and because $\epsilon_{nm} \to \epsilon_n$ weakly $\int \phi d\mu = \int [\int \phi d\nu] d\epsilon_n$. Using the fact that ϕ is piecewise constant it follows easily that the same formula holds on γ with the measure ϵ concentrated on γ and by using standard arguments the formula can be shown to hold for measurable functions ϕ,

$$\int \phi d\mu = \int [\int \phi d\nu] d\epsilon.$$

This desintegration shows that a measure whose corresponding measure ϵ is not concentrated in a single point of γ, is not ergodic. Hence $\gamma = \mathrm{Erg}(X)$. □

A variation of the above construction could be to approximate every point in E_n by two close points in E_{n+1}. In this way the set of ergodic measures will be homeomorphic to a Cantor set. Countable sets of ergodic measures can be obtained by approximating every point in E_n by one in E_{n+1} except for one special chosen point which is approximated by two points in E_{n+1}, one of which is the special one for E_{n+1}. Even we can make examples having $\mathrm{Erg}(X)$ to be a union of manifolds, Cantor sets and discrete parts.

4 Entropy and homotopy

In Section 3 it was shown that invariant measures are homological objects. The prove of the theorem below shows that entropy reflects homotopical properties of the system. Other examples of the type described in this theorem were already constructed in [G] using symbolic dynamical methods.

Theorem 4.1 ([G]) *There exist uniquely ergodic minimal Cantor sets with infinite topological entropy.*

Let $C \subset \mathbf{R}^d$ be an open cone. The *hyperbolic* distance on C is defined as follows. For $x, y \in C$ let $T \subset C$ be the maximal line segment containing x, y. Then

$$\mathrm{hyp}_C(x,y) = -\ln \frac{(m+l)\cdot(m+r)}{l \cdot r},$$

where m is the length of the line segment $[x,y]$ between x and y and l, r are the lengths of the connected components of $T \setminus [x,y]$. In the case when l (or r) is infinite the hyperbolic distance is defined to be $\mathrm{hyp}_C(x,y) = -\ln \frac{m+r}{r}$. The hyperbolic distance on the positive cone in $\mathbf{R}^d$ is denoted by hyp_d. Let

$$C_s^d = \{x \in \mathbf{R}^d \,|\, \mathrm{hyp}_d(x, \mathbf{1}) < s\}$$

The proof of the following lemma is a continuity argument.

Lemma 4.2 *For every $d \geq 1, \epsilon > 0$ there exists $s = s(d,\epsilon)$ such that*

$$\mathrm{hyp}_d(x,y) \leq \epsilon \cdot \mathrm{hyp}_{C_s^d}(x,y),$$

for $x, y \in C_s^d$.

The elements in the set $W_a^d = \{0, 1, \ldots, d-1\}^a$ are called words of length a. For every word $w = (w_i)_{i=1,2,\ldots,a}$ the frequency vector $\nu_w \in \mathbf{R}^d$ is defined as follows

$$\nu_w(j) = \frac{1}{a} \cdot \#\{i \leq a \,|\, w_i = j\}.$$

Let

$$V_a^{d,s} = \{w \in W_a^d \,|\, w_1 = 0, \nu_w \in C_s^d\}.$$

Lemma 4.3

$$\lim_{a \to \infty} \frac{\# V_a^{d,s}}{d^a} = 1.$$

Proof. The statement is a reformulation of the Birkhoff Ergodic Theorem applied to the full shift over d symbols. $\square$

The construction of an example with infinite entropy is a generalization of the construction of an example with positive entropy. For expository reasons we first present the finite entropy example and the main part of the construction.

Fix a small $\delta > 0$. Let X be a combinatorial cover with d loops and $s = s(d, \frac{1}{2})$ given by Lemma 4.2. Lemma 4.3 assures that we can choose $a \gg 1$ such that

$$\# V_a^{d,s} \geq (1-\delta) \cdot d^a.$$

The combinatorial refinement $\pi \colon X' \to X$ is defined as follows. Each word in $V_a^{d,s}$ can be interpreted as a path through X which starts to follow the 0-loop of X. Let X' be a combinatorial cover whose loops are in 1 to 1 correspondence with the words in $V_a^{d,s}$. The projection π is intrinsically defined. Choose one of the loops of X' to be the 0-loop of X'.

Proposition 4.4 *There exist uniquely ergodic minimal Cantor sets with arbitrary high entropy.*

Proof. Choose d_0, T_0 and $\delta > 0$ such that

$$\frac{\ln d_0}{T_0} + 2\ln(1-\delta) \gg 1$$

and let X_0 be a combinatorial cover which has d_0 loops all of length T_0. Now define inductively the combinatorial refinements

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

by $X_{N+1} = (X_N)'$ and let X be the inverse limit. We will use the following notation. The number of loops in X_N is denoted by d_N and s_N, a_N are the numbers defining the refinement X_{N+1}. In particular,

$$d_{N+1} = \# V_{a_N}^{d_N, s_N} \geq (1-\delta) \cdot d_N^{a_N}.$$

Observe that the periods of the loops in each X_N are the same, say T_N. By construction we get

$$T_{N+1} = a_N \cdot T_N.$$

In particular,

$$T_N = T_0 \cdot \Pi_{i=0}^{N-1} a_i.$$

Let $N > N_1 \geq N_0$ and $n = \Pi_{i=N_1}^{N-1} a_i$. Then to each loop λ of X_N, of period T_N, we can assign a word $w_\lambda \in W_n^{d_{N_1}}$ describing the order in which the loop of X_N passes through the loops of X_{N_1}. By construction we have

Claim 4.5 *If $w_{\lambda_1} = w_{\lambda_2}$ then $\lambda_1 = \lambda_2$.*

Claim 4.6 *X is a uniquely ergodic Cantor set.*

Proof. Observe that every loop of X_{N+1} passes at least twice through every loop of X_N. In fact every loop passes many times trough any loop of X_N. This implies that X is a minimal Cantor set.

By construction, the positive cone in $H_1(X_{N+1})$ is mapped by the winding matrix W_N into the $C^{d_N}_{s_N} \subset H_1(X_N)$. In particular,

$$\text{hyp}_{d_N}(W_N x, W_N y) \leq \frac{1}{2} \cdot \text{hyp}_{d_{N+1}}(x, y).$$

The hyperbolic distances are contracted uniformly, X is uniquely ergodic. □

Claim 4.7 *The entropy of X is larger than $\frac{\ln d_0}{T_0} + 2\ln(1-\delta) \gg 1$.*

Proof. Let $h_N = \frac{\ln d_N}{T_N}$. The construction was done such that $d_{N+1} \geq (1-\delta) \cdot d_N^{a_N}$ and $T_{N+1} = a_N \cdot T_N$. This implies

$$h_{N+1} \geq h_N + \frac{\ln(1-\delta)}{T_{N+1}}.$$

By using $a_N \geq 2$ we get

$$\limsup_{N\to\infty} h_N \geq \frac{\ln d_0}{T_0} + 2\ln(1-\delta).$$

Let $S(T, \epsilon)$ be the number of points in the largest set consisting of points which can be pairwise separated ϵ apart within T steps. Then, see [B], the entropy of X is

$$h = \lim_{\epsilon\to 0} \limsup_{T\geq 1} \frac{\ln S(T,\epsilon)}{T}.$$

Let $\epsilon > 0$ be given and let X_{N_1} and be such that all the vertices of X_{N_1} are at least ϵ apart. Let $N \geq N_1$ and $E_N \subset X_N$ be the set of initial points of the loops of X_N. Claim 4.5 implies that E_N consists of points which can be separated ϵ apart within T_N steps. Hence,

$$\begin{aligned}\limsup_{N\to\infty} \frac{\ln S(T_N,\epsilon)}{T_N} &\geq \limsup_{N\to\infty} \frac{\ln \#E_N}{T_N} \\ &= \limsup_{N\to\infty} \frac{\ln d_N}{T_N} \\ &\geq \frac{\ln d_0}{T_0} + 2\ln(1-\delta).\end{aligned}$$

This implies that the entropy h of this example satisfies

$$h \geq \frac{\ln d_0}{T_0} + 2\ln(1-\delta).$$

□

This finishes the construction of minimal Cantor sets with arbitrary high entropy. □

The example with infinite entropy is a limit of minimal Cantor sets with increasing entropy. The limiting process will be described by combinatorial covers X_n^k, $k \leq K$, $k \leq n \leq K$ which will be defined inductively in K such that

- X_n^k is a refinement of X_{n-1}^k;
- There are projections $X_n^{k+1} \to X_n^k$ which commute with the refinements $X_n^k \to X_{n-1}^k$ and map 0-loops to 0-loops.
- The induced projections $H_1(X_n^{k+1}) \to H_1(X_n^k)$ map positive cones onto positive cones.
- The induced projections $H_1(X_n^{k_2}) \to H_1(X_{n-1}^{k_1})$, $K \geq k_2 \geq k_1$, $n \leq N$ contract uniformly the hyperbolic distance of the positive cones.
- The number of loops of X_n^k is denoted by d_n^k. Each loop has the same period, denoted by T_n^k, and
$$T_{n+1}^k = a_n T_n^k, a_n \geq 2$$
$$d_{n+1}^{n+1} \geq (1-\delta) \cdot (d_n^n)^{2a_n}.$$

Assume X_n^k, $k \leq n \leq K$ are defined.

Claim 4.8 *There exists $C_s^{d_K^K} \subset H_1(X_K^K)$ such that $C' = \pi(C_s^{d_N^N})$, where $\pi\colon H_1(X_K^K) \to H_1(X_K^k)$ is the induced projection, satisfies*

$$\mathrm{hyp}_{d_K^k}(x,y) \leq \frac{1}{2} \cdot \mathrm{hyp}_{C'}(x,y),$$

for $x, y \in C'$.

The cone $C_s^{d_K^K} \subset H_1(X_K^K)$ is used to define, as before,

$$X_{K+1}^K = (X_K^K)'.$$

Let a_K be the corresponding number used to define X_{K+1}^K. To each loop λ in X_{K+1}^K and each $k \leq K$ we can assign a word

$$\lambda \mapsto w_\lambda \in \pi_1(X_K^k).$$

Let X_{K+1}^k be a combinatorial refinement of X_K^k such that each word which arises is represented exactly once by a loop. This construction induces projections

$$X_{K+1}^k \to X_{K+1}^{k-1}$$

which commute with the refinements. The choice of the 0 loop in X^K_{K+1} determines the 0-loop in each X^k_{K+1}.

The m^{th} multiple $X^{(m)}$ of a combinatorial cover X is a combinatorial cover which is obtained by re placing each loop of X by m copies. The 0 loop of $X^{(m)}$ is chosen to be one of the copies of the 0 loop of X. To finish the inductive definition we define

$$X^{K+1}_{K+1} = (X^K_{K+1})^{(2)}.$$

It is easily seen that the definition of X^k_{K+1}, $k \leq K+1$, satisfy the previous conditions.

Let X^k be the projected limit of

$$X^k_k \leftarrow X^k_{k+1} \leftarrow X^k_{k+2} \leftarrow \cdots .$$

Observe that the induced maps

$$X^{K+1}_{K+1} \to X^K_K$$

are combinatorial refinements. Let X be the projected limit of

$$X^1_1 \leftarrow X^2_2 \leftarrow X^3_3 \leftarrow \cdots$$

$$\begin{array}{ccccccccc}
\ddots & \vdots & & \vdots & & \vdots & & & \vdots \\
\nwarrow & \uparrow & & \uparrow & & \uparrow & & & \uparrow \\
 & X^{K-1}_{K-1} & \leftarrow & X^{K-1}_{K} & \leftarrow & X^{K}_{K+1} & \leftarrow & \cdots & X^{K-1} \\
 & & \nwarrow & \uparrow & & \uparrow & & & \uparrow \\
 & & & X^{K}_{K} & \leftarrow & X^{K}_{K+1} & \leftarrow & \cdots & X^{K} \\
 & & & & \nwarrow & \uparrow & & & \uparrow \\
 & & & & & X^{K+1}_{K+1} & \leftarrow & \cdots & X^{K+1} \\
 & & & & & & \nwarrow & & \uparrow \\
 & & & & & & & \ddots & \vdots \\
 & & & & & & & & \\
 & & & & & & & & X
\end{array}$$

The following Proposition reformulates the Theorem 4.1.

Proposition 4.9
X^k is a uniquely ergodic minimal Cantor set and

$$h_{X^k} \to \infty.$$

X is a uniquely ergodic minimal Cantor set.
There are factor maps $X \to X^k$. In particular, the entropy of X is infinite.

Proof. Every loop of X^k_n passes at least twice through every loop of X^k_{n-1}. This implies that every X^k is a minimal Cantor set. Every loop of X^{K+1}_{K+1} passes at least twice through every loop of X^K_K: X is a minimal Cantor set.

The induced maps $H_1(X^{k_2}_n) \to H_1(X^{k_1}_{n-1})$, $k_2 \geq k_1$, contract the hyperbolic distance on the corresponding positive cones. This implies that X and all X_k are uniquely ergodic.

Each X^k is an example as in Proposition 4.4. In particular,

$$h_{X^k} \geq \frac{\ln d^k_k}{T^k_k} + 2\ln(1-\delta) = h_k.$$

Observe that

$$d^{k+1}_{k+1} \geq \{(1-\delta)\cdot (d^k_k)^{a_k}\}^2$$

and

$$T^{k+1}_{k+1} = a_k \cdot T^k_k.$$

Hence,

$$h_{k+1} \geq 2\cdot h_k + 2\ln(1-\delta) + \frac{2\ln(1-\delta)}{T^{k+1}_{k+1}}.$$

By taking δ small enough we can assure that $h_k \to \infty$. □

5 Semi-conjugations to circle rotations and cohomology

In this section we are going to study semi-conjugations between minimal Cantor sets and rotation on the circle. In particular, we will construct for every minimal Cantor set a topological invariant $P_X \subset S^1$. The invariant P_X is a countable subgroup of S^1 and it is defined by first defining the cohomology group of the minimal Cantor set X.

Let X be a minimal Cantor set and suppose it can be combinatorially obtained by the refinements

$$\pi_n \colon X_{n+1} \to X_n,$$

where X_n has d_n loops. The corresponding winding matrix is W_n. Let

$$H^1(X_n, S^1) = T^{d_n}$$

be the first cohomology group of the graph X_n, the group of functionals on $H_1(X_n)$. This group is isomorphic to $S^1 \times \cdots \times S^1 = T^{d_n}$, the d_n dimensional torus. Let $\mu_j \in H_1(X_n), j = 1, \dots, d_n$ correspond to the j^{th} loop of X_n. The value of an element $\underline{\theta} = (\theta_1, \dots, \theta_{d_n}) \in S^1 \times \cdots \times S^1 = T^{d_n} = H^1(X_n, S^1)$ on the cycle $\mu = \sum_{j=1}^{d_n} a_j \mu_j \in H_1(X_n)$ is given by

$$\underline{\theta}(\mu) = \sum_{j=1}^{d_n} a_j \theta_j \in S^1.$$

The projection $\pi_n \colon X_{n+1} \to X_n$ induces a linear map $(\pi_n)^* \colon H^1(X_n, S^1) \to H^1(X_{n+1}, S^1)$ given by

$$(\pi_n)^*(\underline{\theta})(\mu) = \underline{\theta}((\pi_n)_*(\mu)).$$

It is easily seen that by using the basis generated by the loops in X_n we get

Lemma 5.1 *Let W_n^T be the transpose of the winding matrix W_n. Then*

$$(\pi_n)^* = W_n^T.$$

In the sequel we will be working on these bases. We define the first cohomology group of X as the direct limit of the sequence

$$H^1(X_0, S^1) \to H^1(X_1, S^1) \to H^1(X_2, S^1) \to \cdots H^1(X_n, S^1) \to \cdots H^1(X, S^1)$$

where the maps are the induced maps $(\pi_n)^*$.

Consider the situation when the minimal Cantor set $f \colon X \to X$ admits a semi-conjugation to the rotation of the circle over ρ, $R_\rho \colon S^1 \to S^1$. That means, there is a continuous map $h \colon X \to S^1$ with

$$h \circ f = R_\rho \circ h.$$

Let $U \subset X$ correspond to a vertex of X_n which is the first vertex of a loop and $V = f(U)$ corresponds to the image vertex of U. Then $h(V) = h(U) + \rho$. The same holds for every vertex and its image in any loop on X_n. Let $x \in U$ and t_j^n be the length of the loop starting at U. Then

$$h(f^{t_j^n}(x)) = h(x) + \rho \cdot t_j^n.$$

So, passing through the j^{th} loop of X_n will cause a jump over $\rho \cdot t_j^n$ in the circle. The cohomology group $H^1(X_n, S^1)$ allows us to keep track of the total jump made in the circle when passing through the loops of X_n. In particular, consider the following map $\gamma_n \colon S^1 \to H^1(X_n, S^1)$ defined by

$$\rho \mapsto (\rho \cdot t_j^n),$$

where t_n^i are the periods of the loops of X_n. The map γ_n commutes with the maps W_n^T. Hence, γ_n extends to a map

$$\gamma\colon S^1 \to H^1(X, S^1).$$

The stable set $W^s(X) \subset H^1(X, S^1)$ is defined as

$$W^s(X) = \{\underline{\theta} = (\theta_n)_{n \geq n_0} \in H^1(X, S^1) \mid \theta_n \to 0\}.$$

Definition 5.2 *The set of rotation numbers for X is*

$$P_X = \gamma^{-1}(W^s(X)).$$

Lemma 5.3 *The set P_X is a topological invariant of the minimal Cantor set X. Moreover, it is a subgroup of S^1.*

Proof. The construction of the set of rotation numbers implies immediately that it is a topological invariant. The group structure of P_X follows from the fact that the map γ is a morphism and the maps $(\pi_n)^*$ are morphisms. □

Lemma 5.4 *Let X be a minimal Cantor set. If there exists a continuous $h\colon X \to S^1$ which semi-conjugates X with a rotation of the circle over $\rho \in S^1$ then*

$$\rho \in P_X.$$

Proof. Consider the j^{th} loop of X_n and take a point $x \in 0_{X_n}$ which will follow this loop. 0_{X_n} corresponds to a small set in X. In particular, the diameter $|h(0_{X_n})|$ of $h(0_{X_n})$ can be taken arbitrary small by taking n large enough. This is because of the continuity of h. Observe,

$$|\gamma_n(\rho)_j - 0| = |\rho \cdot t_j^n - 0| = |h(f^{t_j^n}(x)) - h(x)| \leq |h(0_{X_n})| \to 0,$$

when $n \to \infty$. So $\gamma(\rho) \in W^s(X)$. □

Proposition 5.5 *Let $W_n = (w_{ij}^n)$ be the winding matrices of a representation for the minimal Cantor set X. Assume there is a $K \geq 0$ such that for all $n \geq 0$*

$$\sum_j w_{ij}^n \leq K,$$

If for some $\delta < 1$ the rotation number $0 \neq \rho \in P_X$ has the property

$$|\gamma_n(\rho) - 0| \leq C\delta^n,$$

for all $n \geq 0$ then the minimal Cantor set X is semi-conjugated with the rotation R_ρ.

The condition on the winding matrices above means that every loop in X_{n+1} winds at most K times trough the loops of X_n. The proof of this Proposition relies on

Lemma 5.6 *Assume that the winding matrix $W_n = (w_{ij}^n)$ corresponding to $\pi_n\colon X_{n+1} \to X_n$ satisfies*

$$\sum_j w_{ij}^n \leq K.$$

Let $n \geq 1$ and $x \in 0_{X_n}$ be such that

$$\{f(x), f^2(x), \ldots, f^n(x)\} \cap 0_{X_{n+1}} = \emptyset.$$

Then

$$\#\{f(x), f^2(x), \ldots, f^n(x)\} \cap 0_{X_n} \leq K.$$

Proof. The condition on the piece of the orbit of x under consideration implies that this piece has to lie completely within a loop of X_{n+1}. Any loop of X_{n+1} passes at most K times through 0_{X_n}. In particular, this piece of the orbit of x also passes at most K times through 0_{X_n}. □

Let $\rho \in P_X$ be as given in Proposition 5.6 and define the map

$$h\colon \{f^k(0) | k \geq 0\} \to S^1$$

by

$$h(f^k(0)) = k \cdot \rho \in S^1.$$

In order to prove Proposition 5.5 it is enough to check that the map h is uniformly continuous. To do so it is enough to prove the continuity of h in 0. Because of the specific graph structure of X_n and the construction of h the uniform continuity will follow. In particular, if $|h(0_{X_n})| = r$ then for every vertex $U \in X_n$ we have $|h(U)| \leq r$.

Lemma 5.7 *There exists a constant C such that, for any $n \geq 0$ and for any $s \geq 1$ with $f^s(0) \in 0_{X_n}$ we have*

$$|h(f^s(0)) - 0| \leq C \cdot \delta^n.$$

An appropriate decomposition of the orbit of $0 \in X$ is the key of this lemma. Take $s \geq 1$ such that $f^s(0) \in 0_{X_n}$. Let n_1 be the smallest integer so that the orbit $\{f(0), f^2(0), \ldots, f^s(0)\}$ does not visit 0_{X_m} whenever $m \geq n_1$. For $n \leq l < n_1$ define

$$s_l = \max\{0 < k \leq n | f^k(0) \in 0_{X_l}\}.$$

Lemma 5.6 implies that

$$\#\{s_{l+1} < k \leq s_l | f^k(0) \in 0_{X_l}\} \leq K.$$

Observe that

$$\begin{aligned}|h(f^s(0)) - 0| &\le \sum_{l=n}^{n_1-1} |h(f^{s_l}(0)) - h(f^{s_{l+1}}(0))| \\ &\le \sum_{l=n}^{n_1-1} K \cdot |\gamma_l(\rho)| \\ &\le \sum_{l=n}^{n_1-1} K \cdot C \cdot \delta^l \\ &= C_1 \cdot \delta^n.\end{aligned}$$

This finishes the proof of Lemma 5.7 and Proposition 5.5.

5.1 Remarks

- The Fibonacci minimal Cantor set is a minimal Cantor set which can be combinatorially obtained in such a way that the winding matrices are

  $$W_n = W = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

  $n \ge 0$. In this case the first cohomology group $H^1(X, S^1)$ is isomorph with the two dimensional torus T^2. The set $W^s(X) \subset H^1(X, S^1)$ is the stable manifold of the affine torus map W^T. Observe, that W has two eigenvalues, one bigger than 1 and the other smaller than one. The eigenspace of the smaller eigenvalue corresponds to the set $W^s(X)$. The set P_X is generated by one irrational number.

- The minimal Cantor sets obtained when used winding matrices $W_n = (a)$ are called adding machines. The set P_X equals the backward orbit of 0 under the map $x \mapsto ax \mod 1$ on the circle. Observe, that all the rotation numbers are rational and that there are semi-conjugations to the corresponding rational rotations.

- Consider minimal Cantor sets which can be combinatorially obtained with 3×3 winding matrices $W_n = W$, $n \ge 1$ where W has determinant 1 and two eigenvalues with absolute value larger than 1. The absolute value of the third eigenvalue is smaller than 1. In such a case the set W^s corresponds to the eigenspace of the third eigenvalue. By adjusting the first winding matrix we can assure that $P_X = \{0\}$ and hence that the minimal Cantor set does not allow any semi-conjugation to a non-trivial rotation.

- Let X be a minimal Cantor set which has a representation where the winding matrices are all equal, say $W_n = W$, where W is a $d \times d$ matrix. Consider the action of W^T on the d-dimensional torus T^d and let

  $$W^s = \{x \in T^d \,|\, (W^T)^n x \to 0\}.$$

Let
$$Q = \{x \in T^d \,|\, \exists n (W^T)^n x = 0\}.$$

Lemma 5.8 *There exist a subspace $V \subset R^d$ such that*
$$W^s = \{v + q \,|\, v \in V,\, q \in Q\} \subset T^d.$$

Corollary 5.9 *If the generating winding matrix W has determinant 1 then the group P_X is finitely generated. There exists $\rho_1, \rho_2, \ldots, \rho_s \in S^1$ with $s \leq \dim V \leq d-1$ such that*
$$P_X = \{\sum_{i=1}^{s} x_i \rho_i \,|\, x_i \text{ integer}\}.$$

Kroneckers Theorem [HW] implies

Corollary 5.10 *If the generating winding matrix has determinant 1 and P_X has s generators then the minimal Cantor set X can be semi-conjugated to a minimal shift on the s-dimensional torus.*

Proposition 5.11 *If the generating winding matrix has determinant 1 and $\mathrm{codim}(W^s(X)) = \mathrm{codim}(V) = 1$ then P_X has $d-1$ generators and the minimal Cantor set X admits a semi-conjugation to a minimal shift on the $(d-1)$-dimensional torus.*

Proof. Let V be the stable subspace of W^T
$$V = \{x \in \mathbf{R}^d \,|\, (W^T)^n x \to 0\}.$$

The matrix W^T is an isomorphism with integer entries. This implies that V does not contain non zero lattice points.

Let t_j, $j = 1, \ldots, d$ be the periods of the loops of X_1, the first combinatorial cover and $\underline{t}$ the vector whose entries are t_j. Because the codimension of V equals 1 we can find $\rho_1, \ldots, \rho_{d-1} \in P_X$ and integer vectors $\underline{n}_i$ such that
$$V \ni \underline{x}_i = \underline{n}_i + \rho_i \underline{t}, i = 1, \ldots, d-1$$

are independent points in V. We claim that the points $\rho_i \in P_X$ are rationally independent. Assume by contradiction that they are dependent: there are integers $k_1, \ldots, k_{d-1}$ and k such that
$$\sum_{i=1}^{d-1} k_i \rho_i = k.$$

Then
$$V \ni \sum_{i=1}^{d-1} k_i \underline{x}_i = \sum_{i=1}^{d-1} k_i \underline{n}_i + k\underline{t}.$$

This contradicts the fact that V does not contain non zero lattice points and that the $\underline{x}_i$'s are independent. We showed that P_X has $d-1$ generators. In particular, X admits a semi-cojugation to a minimal shift in the $(d-1)$-dimensional torus. $\square$

Don Coppersmith suggested the following elegant set of winding matrices satisfying the condition of the previous Proposition. Let W be a $d \times d$ matrix such that all entries are zero except the entries of the first row and the lower diagonal which all equal 1. The determinant of W equal 1 and codim(W^s) = 1. Also observe that there is some $k > 0$ such that $(W)^k$ is a positive matrixe: indeed a projective limit of combinatorial covers with W as winding matrix defines a minimal Cantor set. This minimal Cantor set is uniquely ergodic and semi-conjugated to a minimal shift on the $(d-1)$-dimensional torus.

An open question is whether a similar cohomological analysis allows to construct a minimal Cantor set which is semi-conjugated to a minimal shift in the infinite dimensional torus.

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