A NOTE ON TILINGS AND TRANSLATION SURFACES

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Abstract. Consider a tiling $\mathcal{T}$ of the 2-dimensional Euclidean space made with copies up to translation of a finite number of polygons meeting each other full edge to full edge. In this paper, we prove that, associated with $\mathcal{T}$, there exists a tiling of a (compact) translation surface, made with copies up to translation of some of the polygons used to construct $\mathcal{T}$.
Furthermore, when $\mathcal{T}$ is repetitive, there exists a tiling of a translation surface, made with copies up to translation of arbitrarily large polygons chosen in a finite collection of patches of $\mathcal{T}$; each of these patches containing copies of all the polygons used to construct $\mathcal{T}$.

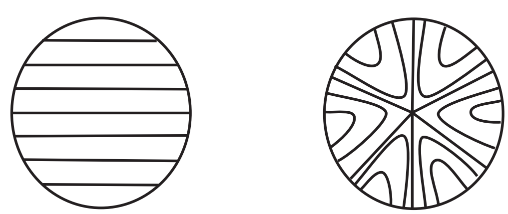
1. Introduction

Consider the Euclidean 2-space $\mathbb{R}^2$. A *polygonal tiling* of $\mathbb{R}^2$ is a countable collection of polygons called *tiles* such that:

- the union of the tiles covers the whole plane;
- whenever two tiles intersect, they do it along their boundaries and full edge to full edge.

We say that a finite collection of polygons $\mathcal{P}$ *tiles* $\mathbb{R}^2$ if there exists a polygonal tiling of $\mathbb{R}^2$ such that each of its tiles is a translated copy of one element in $\mathcal{P}$.

Consider now a topological compact orientable surface with no boundary component $\mathcal{S}$ and, on this surface, a finite set of points $p_1, \ldots, p_n$. Let $\mathcal{S}'$ be the punctured surface $\mathcal{S} \setminus \{p_1, \ldots, p_n\}$. A *translation structure* on $\mathcal{S}$ is the data of a maximal atlas on $\mathcal{S}'$ whose transition maps are translations and $\mathcal{S}$ equipped with such a structure is called a *translation surface*. This translation structure allows to associate with any direction θ in $\mathbb{R}^2$, an orientable line field $\mathcal{L}_\theta$ on $\mathcal{S}'$ and thus a non negative integer with each singularity p_i, which is the index of the line field at p_i. This number is clearly independent of the direction θ. The translation structure also defines on $\mathcal{S}'$ a flat metric and thus a distance $d_{\mathcal{S}'}$. Furthermore, the completion of the metric space $(\mathcal{S}', d_{\mathcal{S}'})$ is the closed surface $\mathcal{S}$. Any compact orientable surface can be equiped with a translation structure and the 2-torus is the only translation surface with no singularities. Translation structures are well-known objects; on the one hand they allow to give a combinatorial description of quadratic differentials on Riemann surfaces (see [HM]); on the other hand, they turn out to be a key tool in the study of the dynamics of rational billiards (see [KZ]) .



1. Local models of a translation surface

We say that a finite collection of polygons $\mathcal{P}$ in $\mathbb{R}^2$ *tiles* a translation surface $\mathcal{S}$ if there exists a finite collection of geodesic polygons in $\mathcal{S}$ such that :

- the collection of geodesic polygons covers $\mathcal{S}$;
- whenever two geodesic polygons intersect they do it along their boundaries and full edge to full edge;
- the set of singularities of the the translation surface is contained in the set of vertices of the geodesic polygons;

- for any of these geodesic polygons, there exists a chart of the maximal atlas associated with the translation structure, which maps the interior of this geodesic polygon to the interior of one of the polygons in $\mathcal{P}$.

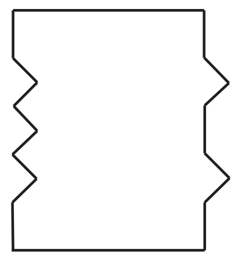
The main result of this paper reads as follows:

Theorem 1.1. *Assume that a finite set $\mathcal{P}$ of polygonal disks in $\mathbb{R}^2$ tiles $\mathbb{R}^2$. Then, there exists a translation surface $\mathcal{S}$ such that $\mathcal{P}$ tiles $\mathcal{S}$.*

Remark 1:
A polygonal tiling $\mathcal{T}$ of $\mathbb{R}^2$ is *periodic* if there exists in the vector space $\mathbb{R}^2$, a pair of independent directions called *periods*, u_1 and u_2, such that $\mathcal{T}$, $\mathcal{T}+u_1$ and $\mathcal{T}+u_2$ coincide. For a finite collection of polygons $\mathcal{P}$ that tiles $\mathbb{R}^2$ periodically, Theorem 1.1 is obviously true. Indeed, let $\mathcal{T}$ be a periodic tiling constructed with $\mathcal{P}$ and let u_1 and u_2 be 2 independent periods of $\mathcal{T}$. The quotient space $\mathbb{R}^2/\mathbb{Z}u_1 + \mathbb{Z}u_2$ is a 2-torus equipped with a translation structure (inherited from the translation structure of $\mathbb{R}^2$) with no singularity which is also tiled by $\mathcal{P}$.
Remark 2:
In 1966, R. Berger proved that the problem of knowing whether a finite collection of polygons tiles $\mathbb{R}^2$ is undecidable [Ber] (see also R. M. Robinson [Rob]). This undecidability is a consequence of the fact that there exists a finite collection of polygons $\mathcal{P}$ which tiles $\mathbb{R}^2$ but cannot tile $\mathbb{R}^2$ periodically. R. Berger gave the first example of such a collection of polygons. In our language, Berger proved that there exists a finite collection of polygons $\mathcal{P}$ in $\mathbb{R}^2$ which tiles $\mathbb{R}^2$ but does not tile the 2-torus seen as a translation surface. Theorem 1.1 asserts that nevertheless there exists a translation surface (with singularity) that can be tiled by $\mathcal{P}$.
Remark 3:
The problem of knowing whether a finite collection of polygons $\mathcal{P}$ can or cannot tile a translation surface is is two-sided.
On the one hand, two types of local rules must be satisfied.

- *The edges rule :* For any edge $\mathcal{E}$ of a polygon in $\mathcal{P}$ there exists an edge of a polygon in $\mathcal{P}$ which is parallel to $\mathcal{E}$ but with inverse orientation (where the orientation of the edges is the one induced by the orientation of $\mathbb{R}^2$).
- *The vertices rule :* For any vertex $\mathcal{V}_1$ of a polygon in $\mathcal{P}$ there exists a finite sequence of vertices of polygons in $\mathcal{P}$, $\mathcal{V}_2, \ldots, \mathcal{V}_p$ such that the angles associated with each vertex $\mathcal{V}_l$, for l going from 1 to p, add up to a multiple of 2π.

On the other hand, these two rules are not enough to insure that $\mathcal{P}$ can tile a translation surface. When $\mathcal{P}$ is reduced to the single polygon given in Figure 2, both local rules are clearly satisfied, however $\mathcal{P}$ cannot tile any translation surface.



2. A polygon that does not tile

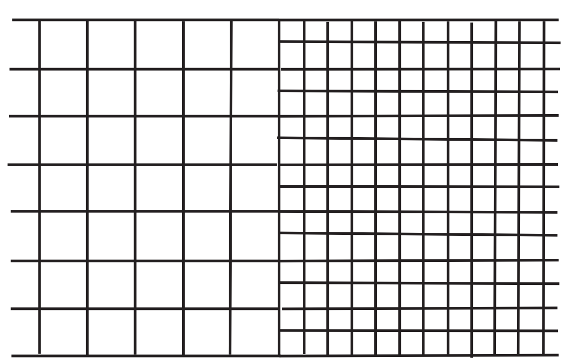
Remark 4:
It is important to notice that Theorem 1.1 does not say that all the polygons in $\mathcal{P}$ are used to tile the translation surface. Consider for instance the set of polygons $\mathcal{P} = \{P_1, P_2, P_3\}$ where:

- P_1 is the unit square in $\mathbb{R}^2$ with edges parallel to the canonical basis of $\mathbb{R}^2$;
- P_2 is the same square dilated by a factor 2;
- and P_3 is the same square has P_2 with three extra vertices respectively located at the center of the left vertical side, at a distance 1/3 of the upper left corner on the upper horizontal side, at a distance 1/3 of the lower left corner on the lower horizontal side.

Since the tiles have to meet full edge to full edge, P_1 can be glued with P_3 only if P_1 is on the right side of P_3 and P_3 can be glued with P_2 only if P_3 is on the right side of P_2 . Thus, there are 3 types of tilings constructed with $\mathcal{P}$:

- *Type 1:* the periodic tiling made only with copies of P_1;
- *Type 2:* the periodic tiling made only with copies of P_2;
- *Type 3:* the tiling for which there exists a vertical stripe made with copies of P_3, the left side of this stripe being tiled only with copies of P_2 and the right side only with copies of P_1.

However, it is easy to show that, except the 2-tori tiled with P_1 and the 2-tori tiled with P_2, there is no translation surface tiled with copies of both P_1 and P_2.



3. A non repetitive tiling

A *patch* P of a polygonal tiling is a polygon in $\mathbb{R}^2$ which is the union of tiles of $\mathcal{T}$. A tiling $\mathcal{T}$ is *repetitive* if for each patch P, there exists $R > 0$ such that each ball with radius R in $\mathbb{R}^2$ contains a translated copy of of P. Clearly the tilings of type 3 described above are not repetitive.
In the case of repetitive tilings, Theorem 1.1 can be improved as follows:

Theorem 1.2. *Assume that a finite set $\mathcal{P}$ of polygonal disks in $\mathbb{R}^2$ tiles $\mathbb{R}^2$ giving rise to a repetitive tiling $\mathcal{T}$. Then, for any $r > 0$, there exists a finite collection of polygons $\mathcal{Q}$ such that:*

- *the polygons in $\mathcal{Q}$ are patches of $\mathcal{T}$;*
- *each patch in $\mathcal{Q}$ contains a ball with radius r;*
- *each patch in $\mathcal{Q}$ contains a translated copy of each polygon in $\mathcal{P}$;*
- *there exists a translation surface $\mathcal{S}$ such that $\mathcal{Q}$ tiles $\mathcal{S}$.*

The link between tilings of $\mathbb{R}^2$ and tilings of a translation surface goes through the notion of branched translation surface that is developed in Section 2. In Section 3 we recall some background on tiling spaces seen as dynamical systems. Finally Sections 4 and 5 are respectively devoted to the proofs of Theorems 1.1 and 1.2.

2. Branched Translation surfaces

A branched translation surface can be described using its local models.

- Choose an integer $n > 0$ and a real number $\epsilon > 0$.
- Consider n disks $D_1, \dots, D_n$ with radius ϵ, each of them being chosen in a different translation surface. We can manage so that when ϵ is small enough, if a singularity of a translation surface is in one of the disks, it is at its center.
- For each $l = 1$ to n, consider the disk D_l partitioned in $k(l)$ geodesic sectors $S_1, \dots, S_{k(l)}$ issued from the origin of the disk.
- Whenever there exists a map from the interior of one sector to another sector (possibly in another disk) which is an isometry whose derivative read in the corresponding charts, is the identity, we can identify both sectors using this map or not[1].
- A *local model*, denoted by $\mathcal{LM}$, is a connected component of the disjoint union of the disks $D_1, \dots, D_n$ quotiented by the above identifications for ϵ small enough.

The *singular locus*, denoted by *Sing*, of the model type consists in those points for which any neighborhood is not homeomorphic to a disk. This set is included in the set defined by the projection to the quotient of the edges of each sector. Notice that the identification process allows to define a tangent space at each point of the local model, including on the singular locus. Notice also that the translation structure on the disks induces an atlas of $\mathcal{LM} \setminus Sing$ whose transition maps are translations.
There are 3 types of local models:

- *The face type* corresponds to the case when $n = 1$. The singular locus is then empty.
- *The edge type* corresponds to the case when:
  - $n > 1$

[1] This means that if this identification is possible we are allowed to make it or not.

  - each disk is divided in two sectors (with angle π) along a diameter Δ which has the same direction for each disk;
  - identifications are performed so that, for at least one of these disks, the local model is not homeomorphic to a face type model.
- *The vertex type* corresponds to the case when :
  - $n > 1$;
  - there exists at least a disk for which the number of sectors is greater than or equal to 2;
  - identifications are performed so that, for at least one of these disks, the local model is not homeomorphic to a face type model nor to an edge type model.

A *local translation* is a map from an open set in a local model to an open set in another local model such that:

- it is a homeomorphism onto its image;
- it maps the singular locus on the singular locus;
- outside the singular locus, it is a translation when read in the charts.

A branched translation surface $\mathcal{S}_b$ is a compact metric space such that, there exists a cover of $\mathcal{S}_b$ with open sets U_α such that each U_α is homeomorphic to a local model and such that the transition maps are local translations where they are defined. We denote by $Sing(\mathcal{S}_b)$ the singular locus of a branched translation surface $\mathcal{S}_b$. It induces a natural stratification of the branched translation surface in faces, edges and vertices. A branched translation surface is *regular* if none of the local models used to build the branched translation surface contains a singularity of a flat surface (at its center).
It is clear that a translation surface is a branched translation surface with no singular locus.

3. Background on Tilings

3.1. **Tilings versus dynamics.** Let $\mathcal{P}$ be a collection of polygons in $\mathbb{R}^2$ and let $\Omega(\mathcal{P})$ be the set of all tilings of $\mathbb{R}^2$ equipped with an origin 0, that can be constructed using the polygons in $\mathcal{P}$. We will assume in the remainder of this paper that $\Omega(\mathcal{P})$ is not empty. The group $\mathbb{R}^2$ acts naturally on $\Omega(\mathcal{P})$:

$$(\mathcal{T}, u) \mapsto \mathcal{T} + u.$$

The set $\Omega(\mathcal{P})$ is also equipped with a natural metrizable topology. A metric δ defining this topology can be chosen as follows: Consider in $\Omega(\mathcal{P})$ two tilings T and T'. Let $B_\epsilon(0)$ stand for the open ball with radius ϵ centered at 0 in $\mathbb{R}^2$ and let A denote the set of ϵ in $(0,1)$ such that there exists u in $\mathbb{R}^2$, ϵ-close to Identity, such that $T + u \cap B_{1/\epsilon}(0) = T' \cap B_{1/\epsilon}(0)$. Then :

$$\delta(T, T') = \begin{cases} \inf A & \text{if} A \neq \emptyset \\ 1 & \text{if} A = \emptyset \end{cases}$$

The set $\Omega(\mathcal{P})$ equipped with this topology is clearly compact and the $\mathbb{R}^2$-action is continuous.

3.2. **Tilings versus solenoids.** Let us now consider the subset $\Omega_0(\mathcal{P})$ of $\Omega(\mathcal{P})$ which consists in those tilings in $\Omega(\mathcal{P})$ such that the origin 0 coincides with a vertex of one of its tiles.This subset is compact and totally disconnected. It follows that $\Omega(\mathcal{P})$ has a laminated structure: it is locally homeomorphic to the product of an open set in $\mathbb{R}^2$ by a totally disconnected set. This local structure can be described more precisely (see [BG] for more details). Consider an open set U in $\mathbb{R}^2$ and a clopen (closed open) subset C in $\Omega_0(\mathcal{P})$. For U and C with small enough diameter, the map:

$$U \times C \to \Omega(\mathcal{P})$$
$$(u, c) \mapsto \phi(u, c) = c + u,$$

is a homeomorphism onto its image. There exists a a finite set of such parametrizations : $\phi_i : U_i \times C_i \to \Omega(\mathcal{P})$, whose ranges cover $\Omega(\mathcal{P})$ and such that whenever the range of two parametrizations ϕ_i and ϕ_j intersect, the transition map $\phi_j^{-1} \circ \phi_i$ reads where it is defined:

$$(*) \quad \phi_j^{-1} \circ \phi_i(u, c) \; = \; (u + u_{i,j}, c - u_{i,j}),$$

where $u_{i,j}$ is a vector in $\mathbb{R}^2$ which depends only on the two parametrizations. We say that $\Omega(\mathcal{P})$ equipped with such an atlas is a $\mathbb{R}^2$-*solenoid*. In the following it will be more convenient to consider $\Omega(\mathcal{P})$ equipped with a maximal atlas, that is to say, a maximal collection of parametrizations whose associated transition maps have the above prescribed form $(*)$.

3.3. **Canonical box decomposition and branched surface.** A *polygonal box* is the image by a parametrization of a product $U \times C$ where U is the interior of a polygon in $\mathbb{R}^2$ and C is a clopen set in $\Omega_0(\mathcal{P})$. A *polygonal box decomposition* of $\Omega(\mathcal{P})$ is the data of a collection of polygonal boxes such that the closure of the union of these boxes is a cover of $\Omega(\mathcal{P})$. Associated with each polygon $\mathcal{P}_l$ in $\mathcal{P}$, there is a polygonal box $\mathcal{B}_l$ made of all the tilings in $\Omega(\mathcal{P})$ such that 0 belongs to the interior of a translated copy of $\mathcal{P}_l$. The collection $\mathcal{B}(\mathcal{P}) = \{\mathcal{B}_1, \ldots, \mathcal{B}_l\}$ is a box decomposition which is called the *canonical box decomposition* of $\Omega(\mathcal{P})$.
In $\Omega(\mathcal{P})$ we consider the reflexive and symmetric relation $\mathcal{R}$ defined by $\mathcal{T}\,\mathcal{R}\,\mathcal{T}'$ if and only if there exists a box $\mathcal{B}_l$ in $\mathcal{B}(\mathcal{P})$ and a sequence of pairs $(\mathcal{T}_n, \mathcal{T}'_n)_{n\geq 0}$ in $\mathcal{B}_l$ such that:

- $(\mathcal{T}_n, \mathcal{T}'_n)$ tends to $(\mathcal{T}, \mathcal{T}')$ as n goes to $+\infty$;
- for each $n \geq 0$, the origin 0 in $\mathcal{T}_n$ and $\mathcal{T}'_n$ project to a same point in the interior of $\mathcal{P}_l$.

We denote by $\sim$ the equivalence relation generated by the relation $\mathcal{R}$, by $\Omega(\mathcal{P})/\!\sim$ the quotient space and by $\pi : \Omega(\mathcal{P}) \to \Omega(\mathcal{P})/\!\sim$ the associated projection. As a consequence of the rigidity of the transition maps $(*)$, we easily get (see [AP], [BG], [S] and [SW]):

Proposition 3.1. *The quotient space $\Omega(\mathcal{P})/\!\sim$ is a regular branched translation surface whose faces are translated copies of the polygons in $\mathcal{P}$.*

3.4. **Invariant measures and homology.** Consider the set of faces of the branched surface $\mathcal{P} = \{P_1, \ldots P_i, \ldots, P_n\}$ that we equip with an orientation. Consider also the collection $\mathcal{E} = \{E_1, \ldots, E_j, \ldots, E_m\}$ of edges also oriented. Notice that if there is no natural choice for the orientation of the edges, the orientation of $\mathbb{R}^2$ induces a natural orientation on each P_i.
For $i = 1, 2$, the vector space of linear combinations with real coefficients of the oriented edges (resp. faces) is denoted by $C_i(\Omega(\mathcal{P})/\!\sim, \mathbb{R})$, its elements are called *i-chains* and the coefficients are called *coordinates*. By convention, for each 1-chain

or 2-chain c, $-c$ is the chain which corresponds to an inversion of the orientation of the faces and edges.
We define the linear *boundary operator*

$$\partial : C_2(\Omega(\mathcal{P})/\sim, \mathbb{R})) \to C_1(\Omega(\mathcal{P})/\sim, \mathbb{R})$$

which assigns to any face, the sum of the edges at its boundary weighted with a positive sign (resp. negative) if the induced orientation fits (resp. does not fit) with the orientation chosen for these edges. The kernel of the operator ∂ is a vector space of 2-cycles that we denote:

$$H_2(\Omega(\mathcal{P})/\sim, \mathbb{R}) \; = \; \text{Ker } \; \partial.$$

It is well known that (up to an isomorphism), the vector space $H_2(\Omega(\mathcal{P})/\sim, \mathbb{R})$ is a topological invariant of $\Omega(\mathcal{P})/\sim$ that coincides with the 2^{nd} singular homology group of the branched surface $\Omega(\mathcal{P})/\sim$ (see for example [Spa]).
The canonical orientation of the faces allows us to characterize the vector space $H_2(\Omega(\mathcal{P})/\sim, \mathbb{R})$ in $C_2(\Omega(\mathcal{P})/\sim, \mathbb{R})$. For any edge E_j in $\mathcal{E}$ one can split the faces which contain E_j in 2 components: the positive ones for which the orientation on the edge is induced by the natural orientation of the face and the negative ones for which the orientation is different. A 2-chain is a 2-cycle if and only if for each edge E_j the sum of the coordinates of the positive faces is equal to the sum of the coordinates of the negative faces. This gives a set of m linear equations with integer coefficients for n variables (where m is the dimension of $C_1(\Omega(\mathcal{P})/\sim, \mathbb{R})$ and n the dimension of $C_2(\Omega(\mathcal{P})/\sim, \mathbb{R})$). These equations are called the *switching rules*.
Let us call *positive* a 2-cycle with coordinates greater than or equal to zero and denote by $H_2^+(\Omega(\mathcal{P})/\sim, \mathbb{R})$, the closed cone of positive cycles. Finally, let us say that a 2-cycle is *integral* if its coordinates are integers.

Consider the set of finite measures $\mathcal{M}(\Omega(\mathcal{P}))$ on $\Omega(\mathcal{P})$. There exists a natural map:

$$\text{Ev} : \mathcal{M}(\Omega(\mathcal{P})) \to C_2(\Omega(\mathcal{P})/\sim, \mathbb{R})$$

defined by:

$$\text{Ev}(\mu) \; = \; \sum_{i=1}^{i=n} \frac{\mu(\pi^{-1}(P_i))}{\lambda(P_i)}.P_i,$$

where λ stands for the Lebesgue measure in $\mathbb{R}^2$.
The group $\mathbb{R}^2$ acts continuously on the compact metric space $\Omega(\mathcal{P})$ and thus, the cone of finite invariant measures $\mathcal{M}_{\text{inv}}(\Omega(\mathcal{P}))$ is not empty. The smooth structure of the branched surface allows us to show that this cone satisfies the following property:

Proposition 3.2. [BG]

$$\text{Ev}(\mathcal{M}_{\text{inv}}(\Omega(\mathcal{P}))) \; \subset \; H_2^+(\Omega(\mathcal{P})/\sim, \mathbb{R}).$$

4. Proof of Theorem 1.1

Lemma 4.1. *There exist integral 2-cycles in $H_2(\Omega(\mathcal{P})/\sim, \mathbb{R})$ which are positive and different from zero.*

Proof of Lemma 4.1: Let μ be a finite invariant measure in $\mathcal{M}_{\text{inv}}(\Omega(\mathcal{P}))$. Using Proposition 3.2, the 2-chain $\text{Ev}(\mu)$ is a 2-cycle in $H_2^+(\Omega(\mathcal{P})/\sim, \mathbb{R})$ which is not zero. This means that its coordinates $(\mu_1^t, \ldots, \mu_n^t)$ satisfy the switching rules,and are nonnegative and not all of them equal to zero. Let $H_2^{+,+}(\Omega(\mathcal{P})/\sim, \mathbb{R})$ be the intersection of the cone $H_2^+(\Omega(\mathcal{P})/\sim, \mathbb{R})$ with the cone generated by the P_i's with strictly positive coefficients in $\mathrm{Ev}(\mu)$. Since the switching rules consist in a system of linear equations with integer coefficients, the set of rational solutions (rational coordinates) is dense in the set of solutions. It follows that there exist 2-cycles with rational coordinates arbitrarily close to $\mathrm{Ev}(\mu)$ in the interior of $H_2^{+,+}(\Omega(\mathcal{P})/\sim, \mathbb{R})$. By multiplying such a rational cycle by an appropriate integer we conclude the proof of Lemma 4.1. □

Consider now an integral positive 2-cycle different from zero in $H_2(\Omega(\mathcal{P})/\sim, \mathbb{R})$ and with coordinates $(l_1, \ldots, l_n)$, we construct inductively a translation surface tiled with the polygons in $\mathcal{P}$ as follows:

- Let $\mathcal{Q}_0$ be a collection of polygons which consists in n piles of polygons, where, for each $i \in \{1, \ldots, n\}$, the i^{th} pile is made of l_i copies of the polygon P_i .
- Choose a polygon in $\mathcal{Q}_0$, say one copy of P_{i_1} in the i_1^{th} pile. We denote by S_1 the surface which consists in P_{i_1} and call $\mathcal{Q}_1$ the collection $\mathcal{Q}_0$ where one polygon P_{i_1} has been taken out.
- Choose a free edge of S_1 (*i.e.* an edge on the boundary of S_1) and a polygon P_{i_2} in $\mathcal{Q}_1$ such that S_1 and P_{i_2} correspond to faces of different signs sharing this common edge. We denote by S_2 the surface with boundary which consists in gluing S_1 and P_{i_2} along their common edge. We call $\mathcal{Q}_2$ the collection $\mathcal{Q}_1$ where one polygon P_{i_2} has been taken out.
- Choose a free edge in S_2 and a polygon P_{i_3} in $\mathcal{Q}_2$, such that S_2 and P_{i_3} correspond to faces of different signs sharing this common edge. We denote by S_3 the surface with boundary which consists in gluing S_2 and P_{i_3} along their common edge. We call $\mathcal{Q}_3$ the collection $\mathcal{Q}_2$ where one polygon P_{i_3} has been taken out.
- We iterate this process which necessarily has to stop because we started with a finite collection of polygons $\mathcal{Q}_0$.

Assume we reached a step l with a surface S_l which is composed of l'_i polygons P_i for $i = 1, \ldots, n$, and that we cannot go on. This means that for each free edge E of the surface S_l, there is no polygons in Q_l which can share this edge with S_l and with a different sign. This implies that the coordinates l'_i satisfy the switching rules and that the free edge E appears by consecutive pairs (the edges are adjacent) on the boundary of S_l and the sign of S_l is different for both members of the pair. It follows that we can glue both parts of S_l along the edge E and do it for all free edges of S_l. We get this way an oriented closed surface $\mathcal{S}$ which naturally inherits a translation structure and is tiled by polygons in $\mathcal{P}$. This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

The proof of Theorem 1.2 requires more information on tiling spaces dynamics. Let $\mathcal{T}$ be a tiling constructed with a collection of polygons $\mathcal{P}$ and consider the closure $\Omega(\mathcal{T})$ of the $\mathbb{R}^2$-orbit of $\mathcal{T}$ in $\Omega(\mathcal{P})$. It turns out (see for instance [KP]) that the $\mathbb{R}^2$-action is minimal on $\Omega(\mathcal{T})$ if and only if the tiling $\mathcal{T}$ is repetitive.
Assume now that $\mathcal{T}$ is repetitive. A *polygonal box decomposition* of $\Omega(\mathcal{T})$ is the data of a collection of polygonal boxes in $\Omega(\mathcal{P})$ such that the closure of the union

of these boxes is a cover of $\Omega(\mathcal{T})$. In [BG], one can find a proof of the following statement:

Proposition 5.1. [BG] *Let $\mathcal{T}$ be a repetitive polygonal tiling of $\mathbb{R}^2$. For each $r > 0$, there exists a box decomposition $\mathcal{B}_r = \{B_1, \dots, B_{l_r}\}$ of $\Omega(\mathcal{T})$ such that for each $l = 1, \dots, l_r$, the box B_l is the image by a parametrization of a product $U_l \times C_l$ where C_l is a clopen set in $\Omega_0(\mathcal{P})$ and the polygon U_l*

- *is a patch of $\mathcal{T}$;*
- *contains a ball with radius r;*
- *and contains a translated copy of each polygon in $\mathcal{P}$.*

Using this proposition, the proof of Theorem 1.2 follows exactly the same lines as the proof of Theorem 1.1 using the box decomposition $\mathcal{B}_r$ instead of the canonical box decomposition of $\Omega(\mathcal{P})$ and the existence of a finite invariant measure for the $\mathbb{R}^2$-action with support in $\Omega(\mathcal{T})$.

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