On the structure of positive radial solutions to an equation containing a p-Laplacian with weight

Marta García-Huidobro[a,*], Raúl Manásevich[b], Cecilia S. Yarur[c]

[a]Departamento de Matemática, Universidad Católica de Chile, Casilla 306, Correo 22, Santiago, Chile
[b]Departamento de Ingeniería Matemática, FCFM, Universidad de Chile, Casilla 170, Correo 3, Santiago, Chile
[c]Departamento de Matemática y C.C., Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile

Abstract

Let $A, B : (0,\infty) \mapsto (0,\infty)$ be two given weight functions and consider the equation

$$(P) \qquad -\operatorname{div}\Big(A(|x|)|\nabla u|^{p-2}\nabla u\Big) = B(|x|)|u|^{q-2}u, \quad x \in \mathbb{R}^n,$$

where $q > p > 1$. By considering positive radial solutions to this equation that are bounded, we are led to study the initial value problem

$$\begin{cases} -\Big(a(r)|u'|^{p-2}u'\Big)' = b(r)(u^+)^{q-1}, & r \in (0,\infty), \\ u(0) = \alpha > 0, \quad \lim\limits_{r\to 0} a(r)|u'(r)|^{p-1} = 0, \end{cases}$$

where $a(r) = r^{(N-1)}A(r)$ and $b(r) = r^{(N-1)}B(r)$. By means of two key functions m and B_q defined below, we obtain several new results that allow us to classify solutions to this initial

*Corresponding author. Fax: +56 2 5525916

E-mail addresses: mgarcia@mat.puc.cl (M. García-Huidobro), manasevi@dim.uchile.cl (R. Manásevich), cyarur@usach.cl (C.S. Yarur).

value problem as being respectively crossing, slowly decaying, or rapidly decaying. We also generalize several results in Clément et al. (Asymptotic Anal. 17 (1998) 13–29), Kawano et al. (Funkcial. Ekvac 36 (1993) 121–145), Yanagida and Yotsutani (Arch. Rational Mech. Anal. 124 (1993) 239–259), Yanagida and Yotsutani (J. Differential Equations 115 (1995) 477–502), Yanagida and Yotsutani (Arch. Rational Mech. Anal. 134 (1996) 199–226).

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1. Introduction

In this paper we will study the structure of positive radial solutions to the equation

$$(P)\qquad -\text{div}\left(A(|x|)|\nabla u|^{p-2}\nabla u\right) = B(|x|)|u|^{q-2}u, \quad x \in \mathbb{R}^N,$$

where $q > p > 1$ and $A, B : (0,\infty) \mapsto (0,\infty)$ are two functions that satisfy some regularity and growth conditions that we will state later in this section.

The case $p = 2$ and $A(|x|) = 1$ was considered by Kawano et al. [9], and by Yanagida and Yotsutani in [19–21]. In [9, Theorem 1], a very general condition was given so that the nature of the solution to the problem

$$\begin{aligned} -\left(r^{N-1}u'\right)' &= r^{N-1}B(r)(u^+)^{q-1} \\ u(0) = \alpha > 0,& \quad u'(0) = 0, \end{aligned} \tag{1.1}$$

could be determined. In fact the following result was proved.

Theorem KYY. *Let the weight B in* (1.1) *satisfy $B \in C^1(0,\infty)$, $B(r)\geqslant 0$ and $B(r) \not\equiv 0$ on* $(0,\infty)$, *$rB \in L^1(0,1)$. Let*

$$\tilde{G}(r) = \frac{1}{q}r^N B(r) - \frac{N-2}{2}\int_0^r t^{N-1}B(t)\,dt,$$

and assume that $\tilde{G}$ satisfies $\tilde{G}(r) \not\equiv 0$ on $(0,\infty)$, *and there exists $R_1 \in [0,\infty)$ such that $\tilde{G}(r)\geqslant 0$ for $r \in [0, R_1]$ and $\tilde{G}'(r)\leqslant 0$ for $r \in (R_1,\infty)$. Then the solutions to* (1.1) *can be classified into one of the following types. Either*

(C) *the solution $u(\cdot,\alpha)$ of* (1.1) *has a first positive zero in* $(0,\infty)$ *for every $\alpha > 0$, or*
(S) *the solution $u(\cdot,\alpha)$ of* (1.1) *is positive in* $(0,\infty)$ *and $\lim_{r\to\infty} r^{N-2}u(r,\alpha) = \infty$ for every $\alpha > 0$, or*
(M) *there exists a unique $\alpha^* > 0$ such that the solution $u(\cdot,\alpha)$ of* (1.1) *satisfies*
- *$u(r,\alpha) > 0$ for all $r > 0$ with $\lim_{r\to\infty} r^{N-2}u(r,\alpha) = \infty$ whenever $\alpha \in (0,\alpha^*)$.*

- *$u(r,\alpha^*)>0$ for all $r>0$ with $\lim_{r\to\infty} r^{N-2}u(r,\alpha^*)=\ell\in(0,\infty)$.*
- *$u(\cdot,\alpha)$ has a first zero for any $\alpha\in(\alpha^*,\infty)$.*

Yanagida and Yotsutani [21] under the additional condition that B is such that

$$\frac{rB'(r)}{B(r)} \text{ is decreasing and nonconstant in } (0,\infty), \tag{1.2}$$

were able to discriminate the nature of the solutions using q as a parameter. They first set

$$\sigma := \lim_{r\to 0}\frac{rB'(r)}{B(r)}, \quad \ell := \lim_{r\to\infty}\frac{rB'(r)}{B(r)}, \tag{1.3}$$

with $-\infty\leqslant\ell<\sigma\leqslant\infty$, where we notice that by the assumptions on B it must be that $\sigma>-2$, and $\sigma>\ell$. Then, defining the two critical numbers

$$q_\sigma := \frac{2(N+\sigma)}{N-2}, \quad q_\ell := \max\left\{2, \frac{2(N+\ell)}{N-2}\right\}, \tag{1.4}$$

they proved the following result in [21].

Theorem YY. Let $N>2$, let $q>1$ and let $B\in C^1(0,\infty)$ be a positive function satisfying $rB\in L^1(0,1)$ and (1.2).

(i) If $2<q\leqslant q_\ell$, then for any $\alpha>0$, the solution $u(\cdot,\alpha)$ of (1.1) has a first positive zero in $(0,\infty)$.

(ii) If $q\geqslant q_\sigma$, then for any $\alpha>0$, the solution $u(\cdot,\alpha)$ of (1.1) is positive in $(0,\infty)$ and $\lim_{r\to\infty} r^{N-2}u(r,\alpha)=\infty$.

(iii) If $q_\ell<q<q_\sigma$, then there exists a unique $\alpha^*>0$ such that the solution $u(\cdot,\alpha)$ of (1.1) satisfies
- $u(r,\alpha)>0$ for all $r>0$ with $\lim_{r\to\infty} r^{N-2}u(r,\alpha)=\infty$ whenever $\alpha\in(0,\alpha^*)$.
- $u(r,\alpha^*)>0$ for all $r>0$ with $\lim_{r\to\infty} r^{N-2}u(r,\alpha^*)=\ell\in(0,\infty)$.
- $u(\cdot,\alpha)$ has a first zero for any $\alpha\in(\alpha^*,\infty)$.

In this form Yanagida and Yotsutani extended previous results dealing with the particular form of $B(r)=\frac{1}{1+r^\gamma}$, for which the equation in problem (1.1) is known as Matukuma equation, see [9,18]. Related results to those of Yanagida and Yotsutani for the Matukuma equation, or more generally Matukuma-type equations, can be found in [11–13,17] and the references therein.

In this paper we consider the more general case than (1.1) given by

$$(IVP) \qquad \begin{cases} -(a(r)|u'|^{p-2}u')' = b(r)(u^+)^{q-1}, & r\in(0,\infty),\\ u(0)=\alpha>0, \quad \lim_{r\to 0} a(r)|u'(r)|^{p-1}=0, \end{cases}$$

where $q > p > 1$ and $a,\ b \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ with $\mathbb{R}^+ = (0, \infty)$. The case of radial positive solutions to problem (P) corresponds to $A(r) = r^{-(N-1)}a(r)$ and $B(r) = r^{-(N-1)}b(r)$.

By a solution to (*IVP*) we will understand an absolutely continuous function u defined in the interval $[0, \infty)$ such that $a(r)|u'|^{p-2}u'$ is also absolutely continuous in the open interval $(0, \infty)$ and satisfies the equation in (*IVP*).

We begin Section 2 by showing that if $a^{-1/(p-1)} \notin L^1(1, \infty)$ then the solutions to (*IVP*) must have a zero and hence the qualitative behavior of all solutions to (*IVP*) is known. Also, we will see that $b \in L^1(0, 1)$ is a necessary condition for existence of solutions to (*IVP*). For these reasons we will assume that

$$(H_1) \qquad b \in L^1(0, 1), \qquad a^{-1/(p-1)} \in L^1(1, \infty).$$

Under this assumption, for $r \in (0, \infty)$, we can define the functions

$$h(r) := \int_r^\infty a^{1-p'}(s)\, ds, \quad \beta(r) = \int_0^r b(s)\, ds,$$

and

$$\Gamma(r) = (\beta(r))^{p'-1}(a(r))^{1-p'}, \tag{1.5}$$

where $p' = p/(p-1)$. We will also assume that Γ satisfies the condition

$$(H_2) \qquad \Gamma \in L^1(0, 1),$$

which will be seen to be a necessary condition for the existence of solutions to (*IVP*).

Under these conditions Proposition A.2 in the Appendix tells us that the problem (*IVP*) has a unique solution defined on $[0, \infty)$. Henceforth, we will denote this solution by $u(r, \alpha)$, every time it is necessary to indicate the dependence of the solution on the initial condition. By analogy to the case of $-(r^{N-1}u')'$, we will say that

- $u(r, \alpha)$ is a crossing solution if it has a zero in $(0, \infty)$.
- $u(r, \alpha)$ is a slowly decaying solution if $u(r, \alpha) > 0$ for all $r \in (0, \infty)$ and

$$\lim_{r\to\infty} \frac{u(r)}{h(r)} = \infty.$$

- $u(r, \alpha)$ is a rapidly decaying solution if $u(r, \alpha) > 0$ for all $r \in (0, \infty)$ and

$$\lim_{r\to\infty} \frac{u(r)}{h(r)} = \ell \in (0, \infty).$$

In the case that $u(r, \alpha)$ is a crossing solution, we will denote its (unique) zero by $z(\alpha)$.

In studying the behavior of solutions to (IVP) we will see that the functions

$$m(r) := p' \frac{b(r)h(r)}{\beta(r)|h'(r)|}, \tag{1.6}$$

and

$$B_q(r) := \beta(r)h^{q/p'}(r) \tag{1.7}$$

will play a key role. A function similar to B_q appeared when proving existence of positive solutions to a related Dirichlet problem in a ball, see [4]. In fact we will make strong use of this existence result later.

With the help of the function B_q we now define the following two sets:

$$\mathcal{U} := \left\{ s \geqslant p \;\middle|\; \sup_{0<r<1} B_s(r) < \infty \right\}, \quad \mathcal{W} := \left\{ s \geqslant p \;\middle|\; \sup_{1 \leqslant r < \infty} B_s(r) < \infty \right\},$$

and set

$$\rho_0^* = \sup \mathcal{U} \quad \text{and} \quad \rho_\infty^* = \inf \mathcal{W}, \tag{1.8}$$

with $\rho_\infty^* = \infty$ if $\mathcal{W} = \emptyset$. It is a simple fact, see the proof of Proposition 1.1, that condition (H_2) implies that $p \in \mathcal{U}$ and thus $\mathcal{U} \neq \emptyset$. Observe also that

$$[p, \rho_0^*) \subseteq \mathcal{U} \quad \text{and} \quad (\rho_\infty^*, \infty) \subseteq \mathcal{W} \quad \text{if} \quad \rho_\infty^* < \infty.$$

We will see later that the numbers ρ_0^* and ρ_∞^* generalize the numbers q_σ and q_ℓ in [21]. The following proposition, proved in Section 2, provides formulas to evaluate the numbers ρ_0^* and ρ_∞^* and shows their relationship with the function m.

Proposition 1.1. *Let assumptions (H_1) and (H_2) hold. Then*

$$\rho_0^* = \max\left\{ p,\, p' \liminf_{r\to 0} \frac{|\log(\beta(r))|}{|\log(h(r))|} \right\}, \quad \rho_\infty^* = \max\left\{ p,\, p' \limsup_{r\to\infty} \frac{|\log(\beta(r))|}{|\log(h(r))|} \right\}. \tag{1.9}$$

In addition, if $m(0) := \lim_{r\to 0} m(r)$ exists, then $\rho_0^* = \max\{p, m(0)\}$. Similarly, if $m(\infty) := \lim_{r\to\infty} m(r)$ exists, then $\rho_\infty^* = \max\{p, m(\infty)\}$.

Sections 3 and 4 are of a technical nature. There, by means of energy functions and comparison lemmas, we obtain some key results that will be used to prove our main theorems in the following section.

Next we will describe our most relevant results, which are given by Theorems 1.1–1.5. The proof of these theorems is the subject of Section 5.

To motivate our first result, let us consider in problem (*IVP*) the particular case given by

$$a(r) = r^{N-1} = b(r), \quad N > p. \tag{1.10}$$

Then $\beta(r) = \frac{r^N}{N}$, $h(r) = \frac{p-1}{N-p} r^{(p-N)/(p-1)}$, $B_q(r) = Cr^{\left(\frac{N-p}{p}\right)\left(\frac{Np}{N-p}-q\right)}$, for some positive constant C, and $m(r) \equiv \frac{Np}{N-p}$. The structure of the positive solutions for this situation is well known and is contained, for example, in [15], Theorem 2.1 in [7], and Theorem 5.1 in [1], see also Theorem 4.1 in [3]. We note that in this case the function m is constantly equal to the Sobolev critical exponent. Motivated by this fact we consider the situation when $m(r) \equiv Const.$, but with a, b not necessarily given by (1.10). We have

Theorem 1.1. *Let the weights a, b satisfy assumptions (H_1) and (H_2) and let $q > p > 1$. Assume that*

$$m(r) = p' \frac{bh}{\beta |h'|}(r) \equiv \rho^*,$$

where ρ^* is a constant. Then the function h is singular at the origin, that is, $h(0) = \infty$, and $\rho^* > p$. Furthermore,

(i) *If $1 < p < q < \rho^*$, then for any $\alpha > 0$ $u(r, \alpha)$ a crossing solution.*
(ii) *If $q = \rho^*$, then u is the rapidly decaying solution given by*

$$u(r, \alpha) = \left(\frac{c}{c\alpha^{1-\frac{\rho^*}{p}} + h^{1-\frac{\rho^*}{p}}} \right)^{p/(\rho^*-p)}, \tag{1.11}$$

where c is a positive constant that depends on α.

(iii) *If $q > \rho^*$, then for any $\alpha > 0$ $u(r, \alpha)$ is a slowly decaying solution.*

The well-known result for a, b given in (1.10) which is quoted above corresponds to $\rho^* = \frac{Np}{N-p}$, which yields

$$1 - \frac{\rho^*}{p} = \frac{-p}{N-p} \quad \text{and} \quad h^{1-\frac{\rho^*}{p}} = c_1 r^{\frac{p}{p-1}},$$

with c_1 a positive constant.

We observe from this result that we can think of the constant ρ^* as a critical number in the sense that solutions to problem (*IVP*) change its behavior depending of the relative position of q with respect to ρ^*. We will see later that something similar will occur when m is not constant, this fact makes this function fundamental in studying the qualitative behavior of the solutions.

Our following result deal with the situation $\inf_{r\in(0,\infty)} m(r) < \sup_{r\in(0,\infty)} m(r)$, and provides an extension of Theorem KYY to our situation. To state this result, let us define the extended real number R_q by

$$R_q := \inf\{r > 0 \mid m(r) < q\}. \tag{1.12}$$

Theorem 1.2. *Let the weights a, b satisfy assumptions (H_1) and (H_2), and let $q > p > 1$ be fixed. Let R_q be defined as in* (1.12). *Assume that $m(r) \not\equiv q$, and assume that the function*

$$(H_3) \qquad r \mapsto (m(r) - q)\beta(r) \text{ is decreasing in } (R_q, \infty).$$

Then the structure of solutions to (IVP) is classified into one of the following types. Either

(C) *the solution $u(\cdot, \alpha)$ of (IVP) is a crossing solution for every $\alpha > 0$, or*
(S) *the solution $u(\cdot, \alpha)$ of (IVP) is slowly decaying for every $\alpha > 0$, or*
(M) *there exists a unique $\alpha^* > 0$ such that the solution $u(\cdot, \alpha)$ of (IVP) satisfies*
- *$u(r, \alpha)$ is slowly decaying whenever $\alpha \in (0, \alpha^*)$.*
- *$u(r, \alpha^*)$ is a rapidly decaying solution.*
- *$u(\cdot, \alpha)$ is a crossing solution for all $\alpha \in (\alpha^*, \infty)$.*

We note that if $R_q = 0$, by (H_3) it must be that $m(r) \leqslant q$ for all $r > 0$, and if $R_q = \infty$, then by the definition of R_q we must have that $m(r) \geqslant q$ for all $r > 0$.

The following theorem takes care of this situation, i.e., $m(r) \leqslant q$ or $m(r) \geqslant q$ for all $r > 0$. It extends Propositions 3.1 and 3.2 in [21], and Theorems 2 and 3 in [9]. Also, it generalizes Theorems 3.2 and 4.1 in [2], where the particular case $a(r) = r^\mu$, $b(r) = r^\gamma \tilde{a}(r)$ is studied, see Section 6. It gives a sufficient condition so that solutions to (*IVP*) are of type (C) or (S).

Theorem 1.3. *Let the weights a, b satisfy assumptions (H_1) and (H_2), and let $q > p > 1$ be fixed.*

(i) *If $\inf_{r\in(0,\infty)} m(r) \geqslant q$ and $m \not\equiv q$, then for any $\alpha > 0$ the solution to (IVP) is a crossing solution.*
(ii) *If $\sup_{r\in(0,\infty)} m(r) \leqslant q$ and $m \not\equiv q$, then for any $\alpha > 0$ the solution to (IVP) is a slowly decaying solution.*

Then it follows that the case

$$0 < R_q < \infty \tag{1.13}$$

is the most interesting one, when searching for existence of solutions of type (M). Note that under condition (1.13), by the definition of R_q and (H_3), necessarily

$$\limsup_{r\to\infty} m(r) \leqslant q \leqslant \liminf_{r\to 0} m(r).$$

In our next result, we use this fact to give a sufficient condition so that solutions to (IVP) are of type (M). A result of this type is not contained in any of the papers [9,19,20], nor [21], and to the best of our knowledge, is completely new.

Theorem 1.4. *Let the weights* a, b *satisfy assumptions* (H_1) *and* (H_2), *and let* $q > p > 1$ *be fixed. Assume that*

$$\rho^*_\infty = \limsup_{r\to\infty} m(r) \leqslant q \leqslant \liminf_{r\to 0} m(r) = \rho^*_0 \tag{1.14}$$

and assume that (H_3) *holds. Furthermore, we also assume that*

$$\limsup_{r\to\infty}(m(r) - \rho^*_\infty)\beta(r) < 0 \quad \text{if } q = \rho^*_\infty, \tag{1.15}$$

and

$$\lim_{r\to 0} B_q(r) = 0 \quad \text{if } q = \rho^*_0. \tag{1.16}$$

Then the structure of the solutions to (IVP) *is of type* (M).

As a consequence of Theorems 1.3 and 1.4, we have the following result.

Theorem 1.5. *Let the weights* a, b *satisfy assumptions* (H_1) *and* (H_2), *and let* $q > p > 1$. *Assume that the function* m *is decreasing and non constant in* $(0, \infty)$. *Then,* $\rho^*_0 = m(0)$, $\rho^*_\infty = \max\{p, m(\infty)\}$, *and solutions to* (IVP) *are of type* (C) *if* $q \leqslant \rho^*_\infty$, *of type* (M) *if* $q \in (\rho^*_\infty, \rho^*_0)$ *and of type* (S) *if* $q \geqslant \rho^*_0$.

This result is a strong improvement of Theorem YY quoted above. Indeed, in Section 6 we show that it applies to functions B for which condition (1.2) is violated. We end the section by establishing and proving results that strongly generalize those in [2].

We point out that some very interesting and related results, when $a(r) = r^{N-1}$ and $p = 2$ in (IVP), can be found in [8], see for instance [8, Theorem 2.6].

Finally, in the appendix of this paper, we give the proof of the existence, uniqueness, and continuous dependence of solutions for the initial value problem (IVP).

2. Preliminary results

Let u be a solution to (*IVP*). We will first show that u must be strictly decreasing in $(0,\infty)$. Indeed, integrating the equation in (*IVP*) from $\varepsilon > 0$ to $r > 0$, letting $\varepsilon \to 0$, and using the second boundary condition, we find

$$-a(r)|u'(r)|^{p-2}u'(r) = \int_0^r b(s)(u^+)^{q-1}(s)\,ds > 0 \tag{2.1}$$

for all $r > 0$, and thus $u'(r) < 0$ for all $r > 0$. Thus u is strictly decreasing function from its initial value $u(0) = \alpha > 0$.

We next give an argument that justifies condition (H_1). We will first show that $a^{-1/(p-1)} \notin L^1(1,\infty)$ implies that any solution u to (*IVP*) must have a first positive zero. Indeed if $u(r) > 0$ for all $r > 0$, then

$$|u'(r)| = a^{-1/(p-1)}(r)\left(\int_0^r b(\tau)(u^+)^{q-1}(\tau)\,d\tau\right)^{1/(p-1)},$$

and thus, for $r \geqslant 1$ we have

$$|u'(r)| \geqslant a^{-1/(p-1)}(r)\left(\int_0^1 b(\tau)(u^+)^{q-1}(\tau)\,d\tau\right)^{1/(p-1)},$$

implying that

$$u(1) \geqslant \left(\int_0^1 b(\tau)(u^+)^{q-1}(\tau)\,d\tau\right)^{1/(p-1)} \int_1^r a^{-1/(p-1)}(\tau)\,d\tau,$$

which in view of our assumption yields a contradiction for large r.

On the other hand, if u is any positive solution to our problem, then for any $r \geqslant s$ small enough it holds that

$$\frac{a|u'|^{p-1}(r) - a|u'|^{p-1}(s)}{(u^+)^{q-1}(r)} \geqslant \int_s^r b(\tau)\,d\tau,$$

implying that $b \in L^1(0,1)$ is a necessary condition for the existence of solutions to (*IVP*). In this form our assumption (H_1) is justified.

Finally, from (2.1), we see that if u is a solution to (*IVP*), then for r sufficiently small,

$$|u'(r)| = a^{1-p'}(r)\left(\int_0^r b(t)(u^+)^{q-1}(t)\,dt\right)^{p'-1} \geqslant \frac{\alpha}{2}a^{1-p'}(r)\beta^{p'-1}(r), \tag{2.2}$$

implying that (H_2) is also a necessary condition for the existence of a solution to (IVP).

Proof of Proposition 1.1. Let

$$\alpha(r)=\frac{|\log(\beta(r))|}{|\log(h(r))|},\qquad L=\liminf_{r\to 0}\alpha(r).$$

We will first show that $p\in\mathcal{U}$. Indeed, by (H_2), for $r\in(0,1)$, we have

$$\infty>\int_r^1\Gamma(t)\,dt=\int_r^1\beta^{p'-1}(t)|h'(t)|\,dt\geqslant\beta^{p'-1}(r)(h(r)-h(1)),$$

and thus $\beta h^{p-1}=B_p$ is bounded near 0, hence $p\in\mathcal{U}$.

If $p'L\leqslant p$, then L is finite, implying that $\lim_{r\to 0}h(r)=\infty$. Furthermore, in this case for any $q>p$, we have that $q\notin\mathcal{U}$ implying that $\mathcal{U}=\{p\}$. Indeed, let $\tilde{q}\in(p'L,q)$. From the definition of L, there exists a sequence $\{r_n\}\to 0$ such that

$$p'\frac{|\log(\beta(r_n))|}{\log(h(r_n))}<\tilde{q},$$

which implies

$$1\leqslant\beta(r_n)(h(r_n))^{\tilde{q}/p'},$$

and thus

$$(h(r_n))^{(q-\tilde{q})/p'}\leqslant B_q(r_n).$$

Since $h(r_n)\to\infty$ as $n\to\infty$ and $q>\tilde{q}$, we conclude that $q\notin\mathcal{U}$ and $\rho_0^*=p$.

Assume now that $p'L>p$. If h is bounded near 0, then $L=\infty$ and also $\mathcal{U}=[p,\infty)$, hence the result follows. Suppose next that $\lim_{r\to 0}h(r)=\infty$. We will show first that $[p,p'L)\subseteq\mathcal{U}$, implying that $\rho_0^*\geqslant p'L$. Let q be such that $p\leqslant q<p'L$. Since

$$B_q(r)=\beta(r)(h(r))^{q/p'}=(h(r))^{(q-p'\alpha(r))/p'},$$

we see that

$$\limsup_{r\to 0}(q-p'\alpha(r))=q-p'L<0,$$

and we conclude that

$$\lim_{r\to 0} B_q(r) = 0,$$

implying $q \in \mathcal{U}$. In particular, if $L = \infty$, we obtain that $\mathcal{U} = [p, \infty)$, implying that $\rho_0^* = \infty = p'L$. Let now $L < \infty$. We claim that $q > p'L$ implies $q \notin \mathcal{U}$, i.e.,

$$\limsup_{r\to 0} B_q(r) = +\infty.$$

Indeed, let $q > p'L$. By choosing a sequence $\{r_n\} \to 0$ such that $\alpha(r_n) \to L$ as $n \to \infty$, we find that

$$\lim_{n\to\infty} q - p'\alpha(r_n) = q - p'L > 0,$$

and thus we obtain that

$$\lim_{n\to\infty} B_q(r_n) = +\infty,$$

and the claim follows. Consequently, it must be the case that $\rho_0^* = p'L$, and the result follows.

The proof of the second of (1.9) is similar and thus we omit it. □

Proposition 2.1. *Let a, b satisfy (H_1), and let u be a positive solution to problem (IVP). Then $u \in C^2(0, \infty)$,*

$$\frac{d}{dr}\left(\frac{u}{h}\right) > 0, \quad on \quad (0, \infty) \tag{2.3}$$

and

$$u(r) \leqslant (\beta(r)h^{p-1}(r))^{-1/(q-p)} = B_p^{-1/(q-p)}(r), \tag{2.4}$$

for all $r > 0$. Also, for $r > 1$,

$$u(r) \leqslant C\left(\int_1^r a^{1-p'}(t)\beta^{p'-1}(t)\,dt\right)^{-\frac{p-1}{q-p}} \tag{2.5}$$

for some positive constant C, and thus if in addition

(H_4) $$\Gamma \notin L^1(1, \infty),$$

then any slowly decaying solution to (*IVP*) *is a ground state*, *that is*

$$\lim_{r\to\infty} u(r) = 0.$$

Proof. Let u be a positive solution to (*IVP*). Then from (2.2) (H_2) must be satisfied and u' can be written as

$$-u'(r) = \left(\frac{\xi(r)}{a(r)}\right)^{p'-1},$$

where $\xi(r) = \int_0^r b(t)u^{q-1}(t)\,dt$. Since the right-hand side belongs to $C^1(0,\infty)$, we have that $u \in C^2(0,\infty)$. To prove (2.3) we make the following change of variable in (*IVP*)

$$s = \frac{1}{h(r)}, \quad v(s) = su(r).$$

In the case that $h(0) = \infty$, by the second in (H_1) the interval $(0,\infty)$ is transformed into the interval $(0,\infty)$, and in the case that $h(0) < \infty$, it is transformed into the interval $(1/h(0),\infty)$. Then it is immediate that v satisfies

$$sv_s(s) - v(s) = a^{p'-1}(r)u'(r) < 0 \quad \text{for all} \quad r \in (0,\infty), \tag{2.6}$$

and

$$(IVP_v) \quad \begin{cases} (p-1)|sv_s - v|^{p-2}v_{ss} = -b(r(s))a^{p'-1}(r(s))s^{-2-q}v^{q-1}(s), s \in (0,\infty), \\ v(0) = 0, \quad \lim\limits_{s\to 0} sv_s(s) = 0. \end{cases}$$

(2.6) and (IVP_v) imply that v is a strictly concave positive function in $(s(0),\infty)$ such that $v(s(0)) \geqslant 0$. Thus v must be increasing on $(s(0),\infty)$, that is, u/h is strictly increasing on $(0,\infty)$ and hence (2.3) holds. This in turn implies that

$$\frac{u'(r)}{h'(r)} < \frac{u(r)}{h(r)} \quad \text{for all} \quad r > 0. \tag{2.7}$$

Now using that u is strictly decreasing for all $r > 0$, we have

$$a(r)|u'(r)|^{p-1} \geqslant \int_0^r b(t)u^{q-1}(t)\,dt \geqslant u^{q-1}(r)\beta(r), \tag{2.8}$$

and hence from (2.7) we obtain that

$$\frac{u^{p-1}(r)}{h^{p-1}(r)} \geqslant \beta(r)u^{q-1}(r), \tag{2.9}$$

from which (2.4) follows.

Let $r > 1$. Then from (2.8)

$$|u'(r)|u^{\frac{1-q}{p-1}}(r) \geqslant a^{1-p'}(r)\beta^{p'-1}(r),$$

which after integration over $(1, r)$ yields

$$\frac{p-1}{q-p}u^{\frac{p-q}{p-1}}(r) \geqslant \int_1^r a^{1-p'}(t)\beta^{p'-1}(t)\,dt.$$

Hence (2.5) follows and thus condition (H_4) implies that $\lim_{r\to\infty} u(r) = 0$ since $q > p$. $\square$

Remark 2.1. Note that if $u(r, \alpha)$ is a crossing solution, the concavity argument also holds. In the case that h is singular at the origin, $v(s(0)) = v(0) = 0$, hence u/h must be increasing until it reaches its maximum point at some $R_0 \in (0, z(\alpha))$ and it must be decreasing on $(R_0, z(\alpha))$. If $h(0) < \infty$, $v(s(0)) = \alpha/h(0) > 0$ and in this case, $v_s(s(0)) > 0$. Indeed,

$$v_s = u(r) + su'(r)\frac{dr}{ds} = u(r) + s\frac{u'(r)}{ds/dr} = u(r) + h(r)\frac{u'(r)}{|h'(r)|},$$

and since $\lim_{r\to 0} u'(r)/h'(r) = \lim_{r\to 0} (a(r)|u'(r)|)^{p'-1} = 0$, we have

$$\lim_{r\to 0} v_s = \lim_{r\to 0}\left(u(r) + h(r)\frac{u'(r)}{|h'(r)|}\right) = \alpha + h(0)\cdot 0 = \alpha > 0.$$

hence again u/h must be increasing until it reaches its maximum point at some point $R_0 \in (0, z(\alpha))$ and it must be decreasing on $(R_0, z(\alpha))$.

Remark 2.2. It is clear that a rapidly decaying solution is a ground state. Indeed, from the definition of rapidly decaying, we have that $u(r) \leqslant Ch(r)$ for r large enough, and since $h(r) \to 0$ as $r \to \infty$, so does u.

Remark 2.3. In view of Proposition 2.1, it is clear that any solution to (IVP) is of one of the three following types: it is either crossing, or rapidly decaying, or slowly decaying.

We next prove the following basic classification results.

Theorem 2.1. *Let a, b satisfy (H_1), (H_2), and let $q > 1$. If $\rho^*_\infty = \infty$, then any solution to (IVP) is a crossing solution.*

Proof. Assume that $\rho^*_\infty = \infty$, and let $q > 1$. If $u(r, \alpha) > 0$ for all $r > 0$, then, by using that u/h is nondecreasing we have that $u(r) \geqslant Ch(r)$ for all $r > R$ for some $R > 0$ and some positive constant C. From (2.9) we obtain that

$$u^{p-1}(r) \geqslant C\beta(r)h^{q+p-2}(r) = CB_{(q+p-2)p'}(r),$$

for all $r > R$, a contradiction to the fact that $\limsup_{r\to\infty} B_{(q+p-2)p'}(r) = \infty$. □

Theorem 2.2. *Let a and b satisfy (H_1) and (H_2). Then any solution of (IVP) is crossing if either*

(i) $\int^\infty a^{1-p'}(s)\,ds = \infty$

or

(ii) $\int^\infty a^{1-p'}(s)\,ds < \infty$ *and $p < q$ satisfies* $\int^\infty b(s)h^{q-1}(s)\,ds = \infty$.

Proof. We only have to prove (ii). Let us assume by contradiction that for some $\alpha > 0$ $u(r, \alpha) > 0$ for all $r > 0$. By integrating the equation in (IVP) over $(0, r)$, and using (2.3), we find that

$$\left(\frac{u(r)}{h(r)}\right)^{p-1} \geqslant \int_0^r b(t)u^{q-1}(t)\,dt = X(r).$$

Thus,

$$X'(r) = b(r)h^{q-1}(r)\left(\frac{u(r)}{h(r)}\right)^{q-1} \geqslant b(r)h^{q-1}(r)X^{\frac{q-1}{p-1}}(r),$$

or equivalently,

$$X^{-\frac{q-1}{p-1}}(r)X'(r) \geqslant b(r)h^{q-1}(r).$$

Hence, by integrating this last inequality over $(1, r)$, $r > 1$, we obtain

$$\frac{p-1}{q-p}(X(1))^{\frac{p-q}{p-1}} \geqslant \frac{p-1}{q-p}(X(1))^{\frac{p-q}{p-1}} - \frac{p-1}{q-p}(X(r))^{\frac{p-q}{p-1}} \geqslant \int_1^r b(t)h^{q-1}(t)\,dt$$

and the result follows. □

In view of this result, let us define the set $\mathcal{S} = \left\{q \geqslant p \mid \int_1^\infty b(s)h^{q-1}(s)\,ds < \infty\right\}$ and put

$$\rho_* = \inf \mathcal{S}, \quad \text{where we set } \rho_* = \infty \text{ if } \mathcal{S} = \emptyset. \tag{2.10}$$

We have:

Proposition 2.2. *Assume that the weight functions a, b satisfy (H_1) and (H_2). Then*

$$\rho_* = \frac{\rho_\infty^*}{p'} + 1. \tag{2.11}$$

Proof. Let $q \in \mathcal{S}$. Since h is decreasing we have

$$\int_1^r b(s)h^{q-1}(s)\,ds \geqslant \int_1^r b(s)\,ds h^{q-1}(r) = (\beta(r) - \beta(1))h^{q-1}(r),$$

and thus

$$B_{(q-1)p'}(r) \leqslant \int_1^r b(s)h^{q-1}(s)\,ds + \beta(1)h^{q-1}(1).$$

Hence $(q-1)p' \in \mathcal{W}$, implying that

$$\rho_\infty^* \leqslant (q-1)p', \ \text{ for all } q \in \mathcal{S}.$$

Thus

$$\rho_\infty^* \leqslant (\rho_* - 1)p'.$$

In particular, if $\rho_\infty^* = \infty$ we obtain that also $\rho_* = \infty$.

If $\rho_\infty^* < \infty$, then for any $q > \rho_\infty^*$ we have that $q \in \mathcal{W}$. Thus, for any $q > \rho_\infty^*$ and $\varepsilon > 0$ small enough, there exists $C > 0$ such that

$$\beta(r)h^{(q-\varepsilon)/p'}(r) \leqslant C \ \text{ for all } r > 1.$$

Hence

$$\int_1^r b(s)h^{q/p'}(s)\,ds \leqslant C_1 \int_1^r b(s)\beta^{-q/(q-\varepsilon)}(s)\,ds \leqslant C_2\beta^{-\varepsilon/(q-\varepsilon)}(1),$$

and thus $q/p' + 1 \in \mathcal{S}$ and

$$\rho_* \leqslant q/p' + 1 \text{ for all } q > \rho^*_\infty.$$

Thus the conclusion follows. $\square$

Remark 2.4. Note that since by definition $\rho^*_\infty \geqslant p$, we have that $1 \leqslant \frac{\rho^*_\infty}{p} = \rho^*_\infty \left(1 - \frac{1}{p'}\right)$, implying that

$$\rho_* = \frac{\rho^*_\infty}{p'} + 1 \leqslant \rho^*_\infty. \tag{2.12}$$

Theorems 2.1 and 2.2, we can assume in the rest of the paper that $\rho^*_\infty < \infty$ and that $q \geqslant \rho_*$. For this range of q the following estimate will be very useful.

Lemma 2.1. *Let $q \geqslant \rho^*$. Then*

$$\lim_{r\to\infty} \beta(r) h^q(r) = 0. \tag{2.13}$$

Proof. Let $q \geqslant \rho^*$, then from the definition of ρ^* we have that for $\varepsilon > 0$ small enough, it holds that

$$C \geqslant \int_{r_0}^{r} b(t) h^{q-\varepsilon}(t)\, dt \geqslant h^{q-\varepsilon}(r)(\beta(r) - \beta(r_0)),$$

and thus

$$\beta(r) h^q(r) \leqslant C h^{\varepsilon}(r) + \beta(r_0) h^q(r) \tag{2.14}$$

implying the result. $\square$

3. Energy function and further classification results

Throughout this section, and without further mention, we will assume that the weights a, b satisfy (H_1) and (H_2). For the next results, we will consider the following energy function. Let u be a solution to (IVP). For $r \in (0, \infty)$ let us set

$$E(r,u) = \frac{h(r)}{|h'(r)|}\left[a(r)\frac{|u'(r)|^p}{p'} + b(r)\frac{(u^+(r))^q}{q}\right] + \frac{1}{p'} u(r) a(r) |u'(r)|^{p-2} u'(r).$$

Then it can be directly verified that the following holds.

$$E'(r,u)=\left[(p'-1)\frac{a'(r)}{a(r)}+\frac{b'(r)}{b(r)}-\frac{|h'(r)|}{h(r)}\left(1+\frac{q}{p'}\right)\right]\frac{b(r)h(r)}{|h'(r)|}\,\frac{(u^+(r))^q}{q},$$

which can be re-written as

$$E'(r,u)=h^{-q/p'}(r)\frac{(u^+(r))^q}{q}\frac{d}{dr}\left(\frac{bh^{1+q/p'}}{|h'|}\right),$$

where here $'$ indicates differentiation with respect to r.

We have the following result concerning the behavior of u at infinity.

Theorem 3.1. *Assume that $q>p$, and let $u(r,\alpha)>0$ for all $r>0$. If for some $r_0>0$ $E(r,u)\geqslant 0$ for all $r\geqslant r_0$, then u is a rapidly decaying solution.*

Proof. From Proposition 2.1, it suffices to prove that u/h is bounded in $(0,\infty)$. Assume $E(r,u)\geqslant 0$ for $r\geqslant r_0$, then

$$\frac{1}{p'}a(r)|u'(r)|^{p-2}u'(r)\left(\frac{u}{h}\right)'h(r)+b(r)\frac{(u(r))^q}{q}\geqslant 0,$$

and thus,

$$\frac{1}{p'}a(r)|u'(r)|^{p-1}\left(\frac{u}{h}\right)'h(r)\leqslant b(r)\frac{u^q}{q},$$

which is the same as

$$\frac{q}{p'}\frac{(u/h)'}{(u/h)}\leqslant\frac{b(r)u^{q-1}}{a(r)|u'(r)|^{p-1}}.$$

Hence, using the equation in (*IVP*)

$$\frac{q}{p'}\frac{(u/h)'}{(u/h)}\leqslant\frac{\left(a(r)|u'|^{p-1}\right)'}{a(r)|u'|^{p-1}},$$

and integrating over (r_0,r) we obtain

$$\left(\frac{u}{h}\right)^{q/p'}\leqslant C\left|\frac{u'}{h'}\right|^{p-1}.$$

Hence, by (2.7)

$$\left(\frac{u}{h}\right)^{(q-p)/p'}(r)\leqslant C \quad \text{for all } r\geqslant r_0,$$

and the claim follows. □

The following corollary, which is contained in Lemma 2.6 in [20] for the Laplace operator, follows directly from this result.

Corollary 3.1. *Let u be a slowly decaying solution to (IVP). Then there is a sequence $\{r_n\}\to\infty$ such that $E(r_n,u)<0$ for all $n\in\mathbb{N}$.*

Inspired by [20], we define the following auxiliary functions and derive analogous results.

$$G(r)=\frac{1}{q}\,\frac{b(r)h(r)}{|h'(r)|}-\frac{1}{p'}\beta(r)=\frac{1}{qp'}(m(r)-q)\beta(r) \tag{3.1}$$

and

$$H(r)=\frac{1}{q}\,\frac{b(r)h^{q+1}(r)}{|h'(r)|}-\frac{1}{p}\int_r^\infty b(t)h^q(t)\,dt. \tag{3.2}$$

By assumption $q\geqslant\rho_*$, we see that the integral in the definition of H is well defined. It can be verified that

$$E'(r,u)=G'(r)(u^+)^q(r),\quad E'(r,u)=H'(r)\left(\frac{u^+}{h}\right)^q(r). \tag{3.3}$$

Since

$$B_q'(r)=qB_q(r)\frac{|h'(r)|}{\beta(r)h(r)}G(r),$$

we also notice that

$$G(r)\geqslant 0\quad \forall r>r_0 \text{ is the same as }\quad B_q(r) \text{ increasing } \forall r>r_0$$

and

$$G(r)\leqslant 0\quad \forall r>r_0 \text{ is the same as }\quad B_q(r) \text{ decreasing } \forall r>r_0. \tag{3.4}$$

We have the following identities.

Lemma 3.1. *Let u be any solution to* (*IVP*). *Then for all* $r > 0$

$$E(r,u) = G(r)(u^+(r))^q - \int_0^r G(s)\frac{d}{ds}(u^+(s))^q\,ds \tag{3.5}$$

and

$$E(r,u) = H(r)\frac{(u^+(r))^q}{h^q} - \int_0^r H(s)\frac{d}{ds}\left(\frac{(u^+(s))^q}{h^q}\right)ds. \tag{3.6}$$

Proof. These identities follow from (3.3) by integrating by parts and the fact that

$$\lim_{r\to 0} E(r,u) - G(r)(u^+(r))^q = 0, \qquad \lim_{r\to 0} E(r,u) - H(r)\frac{(u^+(r))^q}{h^q} = 0. \tag{3.7}$$

We only prove the first limit, the second follows similarly.

$$\begin{aligned}
&|E(r,u) - G(r)(u^+(r))^q| \\
&= \left|\frac{a(r)h(r)}{|h'(r)|}\frac{|u'(r)|^p}{p'} + u(r)\frac{a(r)|u'(r)|^{p-2}u'(r)}{p'} + \beta(r)\frac{(u^+)^q(r)}{p'}\right| \\
&\leqslant \frac{|u'(r)|}{|h'(r)|}h(r)a(r)\frac{|u'(r)|^{p-1}}{p'} + u(r)\frac{a(r)|u'(r)|^{p-1}}{p'} + \beta(r)\frac{(u^+)^q(r)}{p'} \\
&\leqslant \frac{u(r)}{h(r)}h(r)a(r)\frac{|u'(r)|^{p-1}}{p'} + u(r)\frac{a(r)|u'(r)|^{p-1}}{p'} + \beta(r)\frac{(u^+)^q(r)}{p'} \\
&= \frac{2}{p'}u(r)a(r)|u'(r)|^{p-1} + \beta(r)\frac{(u^+)^q(r)}{p'}
\end{aligned} \tag{3.8}$$

and the result follows by using that $\lim_{r\to 0} a(r)|u'(r)|^{p-1} = \lim_{r\to 0}\beta(r) = 0$. $\square$

Let us now set (see [9,20])

$$R_q := \inf\{r \in (0,\infty) \ : \ m(r) < q\}, \quad \overline{R}_q := \sup\{r \in (0,\infty) \ : \ H(r) < 0\},$$

where we set $R_q = \infty$ if $m(r) \geqslant q$ for all $r > 0$, and $\overline{R}_q = 0$ if $H(r) \geqslant 0$ for all $r > 0$.

Theorem 3.2. (i) *Any solution to* (*IVP*) *satisfies* $E(r,u) \geqslant 0$ *for* $r \in (0, R_q)$.
(ii) *Any rapidly decaying solution of* (*IVP*) *satisfies* $E(r,u) \geqslant 0$ *for* $r \in (\overline{R}_q, \infty)$.

Proof. The proof of (i) is a direct consequence of the identity (3.5), by using that $u'(r)\leqslant 0$ for all $r > 0$. In order to prove (ii), we use (3.6) to find that for any $0 < s < r$,

$$E(r,u) - E(s,u) = H(r)\left(\frac{u}{h}\right)^q(r) - H(s)\left(\frac{u}{h}\right)^q(s) - \int_s^r H(t)\frac{d}{dt}\left(\frac{u}{h}\right)^q(t)\,dt,$$

and hence

$$E(r,u) - H(r)\left(\frac{u}{h}\right)^q(r) = E(s,u) - H(s)\left(\frac{u}{h}\right)^q(s) - \int_s^r H(t)\frac{d}{dt}\left(\frac{u}{h}\right)^q(t)\,dt. \quad (3.9)$$

Now, from the definition of E and H

$$\begin{aligned} &E(r,u) - H(r)\left(\frac{u}{h}\right)^q(r) \\ &\quad = \frac{a(r)h(r)}{|h'(r)|}\frac{|u'(r)|^p}{p'} - \frac{u(r)}{p'}a(r)|u'(r)|^{p-1} + \frac{1}{p}\left(\frac{u}{h}\right)^q(r)\int_r^\infty b(t)h^q(t)\,dt, \end{aligned}$$

and thus, if u is a rapidly decaying solution then, using that $q > \rho^*$ we have

$$\lim_{r\to\infty} E(r,u) - H(r)\left(\frac{u(r)}{h(r)}\right)^q = 0,$$

and thus, by letting $r \to \infty$ in (3.9) we obtain that

$$E(s,u) = H(s)\left(\frac{u}{h}\right)^q(s) + \int_s^\infty H(t)\frac{d}{dt}\left(\frac{u}{h}\right)^q(t)\,dt, \quad (3.10)$$

which yields the result by the definition of $\overline{R}_q$. □

Finally in this section we extend to our problem Lemma 2.6(a) and (c) and Propositions 3.2 and 3.3 in [20].

Lemma 3.2. *If $u(r,\alpha)$ is a rapidly decaying solution, then there is a sequence $\{r_n\} \to \infty$ such that $E(r_n,u) \to 0$.*

Proof. By Theorem 2.1, ρ^*_∞ is finite. Also, since u/h is bounded, the first and third term in E tend to 0 as $r \to \infty$, hence we only have to check the second term. Since by L'Hôpital's rule

$$\liminf_{r\to\infty} p'\frac{b(r)h(r)}{\beta(r)|h'(r)|} \leqslant \rho^*_\infty,$$

there is a sequence $\{r_n\} \to \infty$ such that

$$\frac{b(r_n)h(r_n)}{\beta(r_n)|h'(r_n)|} \quad \text{is bounded.}$$

Hence,

$$\frac{b(r_n)h(r_n)}{|h'(r_n)|}u^q(r_n) = \frac{b(r_n)h(r_n)}{\beta(r_n)|h'(r_n)|}\,\frac{u^q(r)}{h^q(r)}\beta(r_n)h^q(r_n) \leqslant C\beta(r_n)h^q(r_n),$$

and since $q \geqslant \rho_*$ we may apply Lemma 2.1 to obtain that $E(r_n, u) \to 0$. □

Lemma 3.3. *If $u(r, \alpha)$ is a crossing solution then $E(r, u) > 0$ for all $r \in [z(\alpha), \infty)$ where $z(\alpha)$ is the zero of $u(r, \alpha)$.*

Proof. If u is a crossing solution, then from (3.3) we have that E is constant for $r \geqslant z(\alpha)$ and

$$E(r, u) = \frac{a(z(\alpha))h(z(\alpha))}{p'|h'(z(\alpha))|}|u'(z(\alpha))|^p > 0, \quad r \geqslant z(\alpha). \qquad \square$$

Proposition 3.1. (i) *If $u(r, \alpha)$ satisfies* $\liminf_{r\to\infty} E(r, u) > 0$, *then $u(r, \alpha)$ is a crossing solution.*

(ii) *If $u(r, \alpha)$ satisfies* $\limsup_{r\to\infty} E(r, u) < 0$, *then $u(r, \alpha)$ is a slowly decaying solution.*

Proof. (i) From Corollary 3.1 and Lemma 3.2, $u(r, \alpha)$ cannot be a slowly decaying solution nor a rapidly decaying solution.

(ii) From Lemma 3.2, $u(r, \alpha)$ cannot be a rapidly decaying solution. Also, from Lemma 3.3 $u(r, \alpha)$ cannot be a crossing solution. □

Lemma 3.4. *Suppose $\overline{R}_q < \infty$. If $u(r, \alpha)$ satisfies*

$$E(R, u) - H(R)(u^+(R)/h(R))^q < 0 \tag{3.11}$$

for some $R \in (\overline{R}_q, \infty)$, then $u(r, \alpha)$ is a slowly decaying solution.

Proof. Let $u(r) = u(r, \alpha)$ satisfy (3.11). From Theorem 3.2(ii) and (3.10), a rapidly decaying solution satisfies

$$E(r, u) - H(r)(u(r)/h(r))^q > 0 \quad \text{for all } r > \overline{R}_q,$$

hence u cannot be of that type. Assume next that u is a crossing solution, and let its zero be $z(\alpha)$. From the definition of E and H we have that

$$\begin{aligned} E(r,u) - H(r)\left(\frac{u^+}{h}\right)^q(r) \\ = \frac{a(r)|u'(r)|^{p-1}}{p'}\left(\frac{h(r)u'(r)}{h'(r)} - u(r)\right) + \frac{1}{p}\left(\frac{u^+}{h}\right)^q(r)\int_r^\infty b(t)h^q(t)\,dt, \end{aligned} \tag{3.12}$$

and therefore, by the assumption on R we find that

$$\frac{h(R)u'(R)}{h'(R)} - u(R) < 0, \tag{3.13}$$

that is, $(u/h)'(R) > 0$, implying further that $u(R) > 0$ and $R < R_0 < z(\alpha)$, where R_0 is the maximum point for u/h defined in Remark 2.1:

$$\frac{h(R_0)u'(R_0)}{h'(R_0)} - u(R_0) = 0. \tag{3.14}$$

Thus, using that u/h is increasing on $(0, R_0)$, we conclude that

$$\frac{h(r)u'(r)}{h'(r)} - u(r) \leqslant 0 \quad \text{for } r \in (R, R_0).$$

Evaluating (3.6) at $r = R$, $r = R_0$ and subtracting, we find that

$$\begin{aligned} E(R,u) - H(R)\left(\frac{u^+}{h}\right)^q(R) = E(R_0,u) - H(R_0)\left(\frac{u^+}{h}\right)^q(R_0) \\ + \int_R^{R_0} H(s)\frac{d}{ds}\left(\frac{u^+}{h}\right)^q(s)\,ds, \end{aligned}$$

and thus, since by (3.12) and (3.14)

$$E(R_0,u) - H(R_0)\left(\frac{u^+}{h}\right)^q(R_0) = \frac{1}{p}\left(\frac{u^+}{h}\right)^q(R_0)\int_{R_0}^\infty b(t)h^q(t)\,dt \geqslant 0,$$

we obtain the contradiction

$$E(R,u) - H(R)\left(\frac{u^+}{h}\right)^q(R) \geqslant 0. \qquad \square$$

4. Some key results

The aim of this section is to establish and prove several key results needed in the next section where we prove our main theorems.

Proposition 4.1. *Assume that $\overline{R}_q < \infty$. Then, the subsets*

$$A_1 := \left\{\alpha > 0 \mid u(r,\alpha) \text{ is a crossing solution to } (IVP)\right\}$$

$$A_2 := \left\{\alpha > 0 \mid u(r,\alpha) \text{ is a slowly decaying solution to } (IVP)\right\}$$

are open.

Proof of Proposition 4.1. The openness of A_1 follows from the continuous dependence of solutions with respect to initial data in Proposition A.1.

Next we prove that A_2 is open. Let $\alpha_0 \in A_2$. Then, from Corollary 3.1 $E(R, u(\cdot, \alpha_0)) < 0$ for some $R > \overline{R}_q$. Hence we have

$$E(R, u(\cdot, \alpha_0)) - H(R)\left(\frac{u(R,\alpha_0)}{h(R)}\right)^q < 0.$$

Using now the continuous dependence of solutions on the initial data given by Proposition A.1, we can find $\delta > 0$ such that

$$E(R, u(\cdot, \alpha)) - H(R)\left(\frac{u(R,\alpha)}{h(R)}\right)^q < 0,$$

for all $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta)$. Thus the conclusion follows from Lemma 3.4. □

Proposition 4.2. *If* $\limsup_{r\to\infty} G(r) < 0$, *then there exists* $\alpha_s > 0$ *such that* $u(r,\alpha)$ *is a slowly decaying solution for all* $\alpha \in (0, \alpha_s)$, *in particular,* $A_2 \neq \emptyset$.

In order to prove this result we need the following lemma:

Lemma 4.1. *Let* $q > p > 1$. *Then for any* $R > 0$ *and* $\delta > 0$, $\delta < R$, *the solution* $u(r,\alpha)$ *of* (IVP) *satisfies*

$$\lim_{\alpha\to 0} \alpha^{-1} u(r,\alpha) = 1 \quad \textit{uniformly in } [0, R]$$

and

$$\lim_{\alpha\to 0} \alpha^{-q} E(r,u) = G(r) \quad \textit{uniformly in } [\delta, R].$$

Proof. By setting $U_\alpha(r) = \alpha^{-1}u(r,\alpha)$, it can be easily verified that U_α satisfies the initial value problem

$$-\Big(a(r)|U'(r)|^{p-2}U'(r)\Big)' = \alpha^{q-p}b(r)(U^+)^{q-1}, \;\; r>0,$$

$$U(0)=1, \quad \lim_{r\to 0} a(r)|U'(r)|^{p-2}U'(r)=0.$$

Integrating, we obtain

$$a(r)|U'_\alpha(r)|^{p-1} = \alpha^{q-p}\int_0^r b(t)(U_\alpha^+(t))^{q-1}\,dt, \tag{4.1}$$

and hence

$$|U'_\alpha(r)| = \alpha^{\frac{q-p}{p-1}} a^{1-p'}\left(\int_0^r b(s)(U_\alpha^+(s))^{q-1}\,ds\right)^{p'-1}. \tag{4.2}$$

A new integration over $(0,r)$ yields that

$$0 \leqslant -U_\alpha(r)+1 \leqslant \alpha^{\frac{q-p}{p-1}}\int_0^r a^{1-p'}(s)\beta^{p'-1}(s)\,ds \leqslant \alpha^{\frac{q-p}{p-1}}\int_0^R a^{1-p'}(s)\beta^{p'-1}(s)\,ds,$$

for all $r\in[0,R]$ implying that $U_\alpha \to 1$ as $\alpha\to 0$ uniformly on $[0,R]$. On the other hand,

$$\begin{aligned}\alpha^{-q}E(r,u)-G(r) \;&=\; \frac{\alpha^{p-q}}{p'}a(r)|U'_\alpha(r)|^{p-1}\left(\frac{h(r)}{h'(r)}U'_\alpha(r)-U_\alpha(r)\right)\\ &\quad+\frac{b(r)h(r)}{q|h'(r)|}((U_\alpha^+(r))^q-1)+\frac{\beta(r)}{p'},\end{aligned}$$

which by (4.1) can be written as

$$\begin{aligned}\alpha^{-q}E(r,u)-G(r) \;&=\; \frac{1}{p'}\int_0^r b(t)(U_\alpha^+(t))^{q-1}\,dt\left(\frac{h(r)}{h'(r)}U'_\alpha(r)-U_\alpha(r)\right)\\ &\quad+\frac{b(r)h(r)}{q|h'(r)|}((U_\alpha^+(r))^q-1)+\frac{\beta(r)}{p'}.\end{aligned}$$

Next we note that as $\alpha \to 0$

$$\frac{b(r)h(r)}{q|h'(r)|}((U_\alpha^+(r))^q - 1) \to 0 \quad \text{uniformly in } [\delta, R],$$

$$U_\alpha(r)\int_0^r b(t)(U_\alpha^+(t))^{q-1}\,dt \to \beta(r) \quad \text{uniformly in} \quad [\delta, R],$$

and from (4.2)

$$0 \leqslant \frac{h(r)}{h'(r)} U_\alpha'(r) \leqslant \alpha^{\frac{q-p}{p-1}} (\beta(r)h^{p-1}(r))^{p'-1} \to 0,$$

uniformly in $[0, R]$, by condition (H_2), that implies that $\beta(r)h(r)^{(p-1)}$ is bounded in $[0, R]$. Then it is clear that $\lim_{\alpha\to 0} \alpha^{-q} E(r, u) = G(r)$ uniformly in $[\delta, R]$. □

Proof of Proposition 4.2. We will use Proposition 3.1(ii), thus we will show that for α small enough, the solution $u(r, \alpha)$ of (IVP) satisfies

$$\limsup_{r\to\infty} E(r, u) < 0.$$

If $G(r) \leqslant 0$ for all $r > 0$, the result follows from Theorem 1.3, since in that case $\alpha_s = \infty$. Hence, let us assume that there is $r_1 > 0$ such that $G(r_1) > 0$. From $\limsup_{r\to\infty} G(r) < 0$, we obtain that there is $\ell < 0$ and $r_2 \geqslant r_1$ such that

$$G(r) < \ell < 0 \quad \text{for all } r \geqslant r_2.$$

Let $R := \sup\{r > 0 \mid G(r) > \ell\}$. Then $G(R) = \ell$ and $G(r) \leqslant G(R) < 0$ for all $r \geqslant R$. By Lemma 4.1, for any $\varepsilon > 0$ small enough, there is $\alpha_s > 0$ (small) such that for any $\alpha \in (0, \alpha_s)$,

$$|1 - \alpha^{-1}u(r, \alpha)| < \varepsilon, \quad |\alpha^{-q}E(r, u) - G(r)| < \varepsilon \quad \text{for } r \in [r_1, R],$$

and hence it must be that

$$0 < u(R, \alpha) \quad \text{and} \quad E(R, u) < 0 \quad \text{for all } \alpha \in (0, \alpha_s).$$

Also, from (3.5) we have that for all $r \geqslant R$,

$$E(r, u) = E(R, u) + (G(r) - G(R))(u^+(r))^q - \int_R^r (G(s) - G(R)) \frac{d}{ds}(u^+)^q(s)\,ds,$$

and thus the result follows. □

Proposition 4.3. *Let* $p < q \leqslant \rho_0^*$ *and assume* $\lim_{r\to 0} B_q(r) = 0$ *in case that* $q = \rho_0^*$. *Then* $A_1 \neq \emptyset$.

Proof. Under the hypotheses of this proposition, from Theorems 1.2 and 1.3 in [4] the boundary value problem

$$-\Big(a(r)|u'|^{p-2}u'\Big)' = b(r)(u^+)^{q-1}, \quad r \in (0, R),$$

$$\lim_{r\to 0} a(r)|u'(r)|^{p-1} = 0, \quad u(R) = 0,$$

has a nontrivial solution for any given $R > 0$. This certainly implies that there is an initial condition insuring that a crossing solution to problem (*IVP*) exists, and hence $A_1 \neq \emptyset$.

We note that in [4] the additional condition $a^{1-p'} \notin L^1(0, R)$ is imposed. Clearly if $a^{1-p'} \in L^1(0, R)$ the existence of a nontrivial solution is easier to obtain. □

The following result is fundamental for the proofs of the main results in the next section.

Proposition 4.4. *Let* $\overline{R}_q \leqslant R_q < \infty$, *and assume that* $G(r) \not\equiv 0$. *Then*

(i) *There exists at most one* $\alpha^* > 0$ *such that* $u(r, \alpha^*)$ *is a rapidly decaying solution.*
(ii) *If* $A_1 = \emptyset$, *then* $A_2 = (0, \infty)$.
(iii) *If* $A_2 = \emptyset$, *then* $A_1 = (0, \infty)$.
(iv) *If* $A_1 \neq \emptyset$ *and* $A_2 \neq \emptyset$, *then there exists* $\alpha^* > 0$ *such that* $u(r, \alpha^*)$ *is a rapidly decaying solution and* $A_1 = (\alpha^*, \infty)$ *and* $A_2 = (0, \alpha^*)$.

This result is a consequence of the following three lemmas:

Lemma 4.2. *Let u be any solution to* (*IVP*) *and let* ϕ *be a positive solution to* (*IVP*). *Then*

$$E(s, \phi)\frac{d}{ds}\left(\frac{u^+}{\phi}\right)^q \in L^1(0, 1)$$

and

$$E(r, u) = \left(\frac{u^+}{\phi}\right)^q E(r, \phi) - \int_0^r E(s, \phi)\frac{d}{ds}\left(\frac{u^+}{\phi}\right)^q (s)\, ds. \tag{4.3}$$

Proof. From (3.3), we have that

$$E'(r,u) = E'(r,\phi)\left(\frac{u^+}{\phi}\right)^q(r),$$

and thus, for $r > s > 0$,

$$\begin{aligned} E(r,u) - E(s,u) &= \int_s^r E'(t,\phi)\left(\frac{u^+}{\phi}\right)^q(t)\,dt \\ &= E(t,\phi)\left(\frac{u^+}{\phi}\right)^q(t)\Big|_s^r - \int_s^r E(t,\phi)\frac{d}{dt}\left(\frac{u^+}{\phi}\right)^q(t)\,dt \end{aligned}$$

implying

$$\begin{aligned} E(r,u) - E(r,\phi)\left(\frac{u^+}{\phi}\right)^q(r) &= E(s,u) - E(s,\phi)\left(\frac{u^+}{\phi}\right)^q(s) \\ &\quad - \int_s^r E(t,\phi)\frac{d}{dt}\left(\frac{u^+}{\phi}\right)^q(t)\,dt. \end{aligned} \tag{4.4}$$

Now we observe that from (3.7), we have

$$\lim_{s\to 0}\frac{E(s,u)}{(u^+)^q} - G(s) = 0, \quad \lim_{s\to 0}\frac{E(s,\phi)}{\phi^q} - G(s) = 0.$$

Hence, since

$$\begin{aligned} E(s,u) - E(s,\phi)\left(\frac{u^+}{\phi}\right)^q(s) &= (u^+)^q(s)\left(\frac{E(s,u)}{(u^+)^q} - G(s)\right) \\ &\quad - (u^+)^q(s)\left(\frac{E(s,\phi)}{\phi^q} - G(s)\right), \end{aligned}$$

we must have

$$\lim_{s\to 0} E(s,u) - E(s,\phi)\left(\frac{u^+}{\phi}\right)^q = 0,$$

and thus by letting $s \to 0$ in (4.4) we see that

$$E(t,\phi)\frac{d}{dt}\left(\frac{u^+}{\phi}\right)^q(t) \in L^1(0,1)$$

and

$$E(r,u) - E(r,\phi)\left(\frac{u^+}{\phi}\right)^q = -\int_0^r E(t,\phi)\frac{d}{dt}\left(\frac{u^+}{\phi}\right)^q(t)\,dt. \qquad \square$$

Lemma 4.3. *Assume that* $1 < p < q$ *and let* ϕ *and* u *be two solutions to* (IVP) *which are positive in* $(0, R)$, *and such that* $u > \phi$ *in* $[0, R)$. *Then* u/ϕ *is strictly decreasing in* $(0, R)$.

Proof. Since u and ϕ are positive solutions to (IVP) we have

$$\left(a|u'|^{p-1}\right)'\phi^{p-1} = bu^{q-1}\phi^{p-1}, \quad \left(a|\phi'|^{p-1}\right)'u^{p-1} = b\phi^{q-1}u^{p-1},$$

and thus

$$\left(a|u'|^{p-1}\phi^{p-1}\right)' - (p-1)a|u'|^{p-1}\phi^{p-2}\phi' = bu^{q-1}\phi^{p-1} \tag{4.5}$$

and

$$\left(a|\phi'|^{p-1}u^{p-1}\right)' - (p-1)a|\phi'|^{p-1}u^{p-2}u' = b\phi^{q-1}u^{p-1}. \tag{4.6}$$

Subtracting (4.5) and (4.6) and setting

$$w := a\left(|u'|^{p-1}\phi^{p-1} - |\phi'|^{p-1}u^{p-1}\right)$$

we obtain

$$\begin{aligned} w' &+ (p-1)a|u'||\phi'|\left(|u'|^{p-2}\phi^{p-2} - |\phi'|^{p-2}u^{p-2}\right) \\ &= bu^{p-1}\phi^{p-1}\left(u^{q-p} - \phi^{q-p}\right) > 0 \quad \text{in } [0,R), \end{aligned} \tag{4.7}$$

where we have used that $u > \phi$ in $[0, R)$.

We note that the sign of $(u/\phi)'$ is the opposite of the sign of w, hence we have to show that $w(r) > 0$ in $[0, R)$. We will show first that $w(r) > 0$ for r near zero. Let us call $u(0) = \alpha_u$ and $\phi(0) = \alpha_\phi$. From L'Hospital's rule, and using the equations satisfied by u and ϕ, we obtain that

$$\lim_{r\to 0}\frac{a(r)|u'|^{p-1}}{\beta(r)} = \alpha_u^{q-1} \quad \text{and} \quad \lim_{r\to 0}\frac{a(r)|\phi'|^{p-1}}{\beta(r)} = \alpha_\phi^{q-1},$$

and thus

$$\lim_{r\to 0}\frac{a(r)\left(|u'|^{p-1}(r)\phi^{p-1}(r)-|\phi'|^{p-1}(r)u^{p-1}(r)\right)}{\beta(r)} = \alpha_u^{q-1}\alpha_\phi^{p-1}-\alpha_\phi^{q-1}\alpha_u^{p-1},$$

$$= (\alpha_u\alpha_\phi)^{p-1}\left(\alpha_u^{q-p}-\alpha_\phi^{q-p}\right)>0,$$

and thus $w(r)>0$ for all r near zero.

Assume by contradiction that there exists $r_0\in(0,R)$ such that $w(r_0)=0$, that is, $(|u'|\phi)(r_0) = (|\phi'|u)(r_0)$. We can assume that r_0 is the first such point, hence it must be that $w'(r_0)\leqslant 0$. Evaluating (4.7) at $r=r_0$ and using that $\left(|u'|^{p-2}\phi^{p-2}-|\phi'|^{p-2}u^{p-2}\right)(r_0)=0$, we obtain that $w'(r_0)>0$ contradicting the choice of r_0. Thus $w>0$ in $(0,R)$ and hence $(u/\phi)'(r)<0$ in $(0,R)$. □

Lemma 4.4. *Assume there exists a rapidly decaying solution ϕ of (IVP) with $\phi(0)=\alpha^*$, and such that $E(r,\phi)\geqslant 0$ and $E(r,\phi)\not\equiv 0$ on $[0,\infty)$. Let u be any solution to (IVP).*

(i) *If $\alpha>\alpha^*$ and if $u>0$ on $[0,R)$, then*

$$(u/\phi)'(r)<0 \quad \textit{on } (0,R).$$

(ii) *If $\alpha<\alpha^*$, then $u>0$ on $[0,\infty)$ and*

$$(u/\phi)'(r)>0 \quad \textit{on } (0,\infty).$$

Proof. We prove first (i). From $\alpha>\alpha^*$, we have that $u>\phi$ near zero. If $u>\phi$ in $(0,R)$ the conclusion follows from Lemma 4.3. If $u=\phi$ at some point in $(0,R)$, set

$$R_0=\inf\{r\in(0,R)\mid u(r)=\phi(r)\}.$$

Thus $u(R_0)=\phi(R_0)$ and u/ϕ is strictly decreasing in $(0,R_0]$ by the previous lemma. Assume now by contradiction that there exists some first $c>R_0$ such that $(u/\phi)'(c)=0$. Let us call $\sigma=(u/\phi)(c)$, and note that since $u(R_0)=\phi(R_0)$, it must be that $\sigma\in(0,1)$ and

$$u(c)=\sigma\phi(c),\quad u'(c)=\sigma\phi'(c).$$

From (4.3) evaluated at c we obtain

$$E(c,u)-\sigma^q E(c,\phi) = -\int_0^c \frac{d}{ds}\left(\frac{u}{\phi}\right)^q(s)E(s,\phi)\,ds. \tag{4.8}$$

Since $E(r, \phi) \geqslant 0$ and u/ϕ is decreasing in $(0, c)$, the right-hand side of (4.8) is non-negative. On the other hand, from the definition of E, c, and σ we obtain

$$E(c, u) - \sigma^q E(c, \phi) = -\frac{(\sigma^p - \sigma^q)h^2(c)}{p'|h'(c)|} a(c)|\phi'|^{p-1}(c)(\phi/h)'(c), \tag{4.9}$$

and thus the left-hand side of (4.8) is strictly negative, a contradiction and hence we conclude (i).

We prove now (ii). Since $\alpha < \alpha^*$, it must be that $u < \phi$ in some interval $(0, r_0)$. Assume first that $r_0 = \infty$, and suppose by contradiction that u is crossing with its zero at $z(\alpha)$. By Lemma 4.3, in the open interval $(0, z(\alpha))$, it must be that u/ϕ is strictly increasing, implying that $u(r, \alpha) \geqslant (\alpha/\alpha^*)\phi(r)$ for all $0 < r < z(\alpha)$, a contradiction and hence $u(r, \alpha) > 0$ for all $r > 0$. Thus we may use Lemma 4.3 with $R = \infty$ to obtain that $(u/\phi)'(r) > 0$ in $(0, \infty)$.

If now $r_0 < \infty$, set as before $R_0 = \inf\{r \in (0, R) \mid u(r) = \phi(r)\}$. Again $u(R_0) = \phi(R_0)$ and from Lemma 4.3, u/ϕ is strictly increasing in $(0, R_0)$. Let $c > R_0$ be such u/ϕ increases in $(0, c)$ and $(u/\phi)'(c) = 0$. Then $\sigma := (u/\phi)(c) > (u/\phi)(R_0) = 1$ and $u'(c) = \sigma\phi'(c)$. As in the proof of part (i), using (4.8) we obtain, from the monotonicity of u/ϕ in $(0, c)$ and the positivity of $E(\cdot, \phi)$ that $E(c, u) - \sigma^q E(c, \phi) < 0$. On the other hand, from (4.9) we have that $E(c, u) - \sigma^q E(c, \phi) > 0$ since $\sigma > 1$, a contradiction. Hence $(u/\phi)'(r) > 0$ for all $r > 0$, implying in particular that $u(r, \alpha) > 0$ for all $r > 0$. □

Proof of Proposition 4.4. (i) Suppose there exist two rapidly decaying solutions $u_1(r) = u(r, \alpha_1)$ and $u_2(r) = u(r, \alpha_2)$, and assume without loss of generality that $\alpha_1 > \alpha_2$. By the assumption $\overline{R}_q \leqslant R_q$ and Theorem 3.2, we have that $E(r, u_i) \geqslant 0$ for all $r > 0$, and $i = 1, 2$. Also, since $G(r) \not\equiv 0$, $E(r, u_i) \not\equiv 0$, $i = 1, 2$. Since $\alpha_1 > \alpha_2$, by Lemma 4.4(ii), u_1/u_2 must be strictly decreasing in $(0, \infty)$, and from (4.3) in Lemma 4.2,

$$E(r, u_1) = \left(\frac{u_1}{u_2}\right)^q E(r, u_2) - \int_0^r \frac{d}{ds}\left(\frac{u_1}{u_2}\right)^q (s) E(s, u_2)\, ds,$$

which implies that for all r sufficiently large

$$E(r, u_1) \geqslant c > 0$$

for some positive constant c, contradicting Lemma 3.2.

(ii) Let $A_1 = \emptyset$ and assume by contradiction that there exists a rapidly decaying solution $u^*(r) = u(r, \alpha^*)$ for some $\alpha^* > 0$. Then, by Lemma 4.4(ii), for any $\alpha > \alpha^*$, $u(r) = u(r, \alpha)$ satisfies u/u^* is strictly decreasing in $(0, \infty)$, that is

$$u(r) \leqslant \frac{\alpha}{\alpha^*} u^*(r) \quad \text{for all } r > 0,$$

hence

$$\frac{u(r)}{h(r)} \leqslant \frac{\alpha}{\alpha^*}\,\frac{u^*(r)}{h(r)} \quad \text{for all } r > 0,$$

implying that u must also be rapidly decaying, a contradiction to the uniqueness proved in (i).

(iii) follows as (ii).

(iv) The existence of $\alpha^* > 0$ such that $u^*(r) = u(r, \alpha^*)$ is rapidly decaying follows immediately from Proposition 4.1 and the connectedness of $(0, \infty)$. If $\alpha < \alpha^*$, from Lemma 4.4(ii) we obtain that $u(r, \alpha)$ must be positive in $(0, \infty)$, and since by (i) the rapidly decaying solution is unique, u must be slowly decaying, hence $A_2 = (0, \alpha^*)$. Let now $\alpha > \alpha^*$. If $u(r, \alpha)$ is not a crossing solution, then it must be positive for all $r > 0$, hence, by the uniqueness of u^*, it must be slowly decaying. But from Lemma 4.4(i), u/u^* must be strictly decreasing in $(0, \infty)$, therefore

$$\frac{u(r)}{h(r)} \leqslant \frac{\alpha}{\alpha^*}\,\frac{u^*(r)}{h(r)} \quad \text{for all } r > 0,$$

implying that u is rapidly decaying, a contradiction. □

5. Proof of the main results

We begin this section by proving Theorem 1.1. A direct proof of this result could also be given, nevertheless, in order to emphasize that this case is really a generalization of the problem corresponding to (1.10), we have chosen to give the following proof.

Proof of Theorem 1.1. We will first prove that $\rho^* > p$. Indeed, from $m(r) \equiv \rho^*$, we obtain that

$$\frac{b}{\beta} + \frac{\rho^*}{p'}\frac{h'}{h} \equiv 0,$$

and thus

$$\beta h^{\frac{\rho^*}{p'}} \equiv C_0 = \text{positive constant}, \tag{5.1}$$

implying $\beta^{p'-1} = C_1 h^{\frac{-\rho^*}{p}}$ for some positive constant C_1. Also, since $\beta(0) = 0$, it follows that $h(0) = \infty$, and by (H_2), $h^{\frac{-\rho^*}{p}}|h'|$ must be integrable near 0, implying $\rho^* > p$.

Let us now make the following change of variable:

$$s = (NC_0)^{1/p}\left(\frac{N-p}{p-1}\right)^{1/p'} h^{-\frac{p-1}{N-p}}(r), \qquad v(s) = u(r),$$

where

$$N := \frac{p\rho^*}{\rho^* - p} > p$$

and C_0 is defined in (5.1). (We note that N just defined need not be an integer). Straightforward calculations lead to the transformed initial value problem

$$(IVP_s) \qquad \begin{cases} -\frac{d}{ds}\left(s^{N-1}\left|\frac{dv}{ds}\right|^{p-2}\frac{dv}{ds}\right) = s^{N-1}(v^+)^{q-1}, & s \in (0,\infty), \\ v(0) = \alpha > 0, \quad \lim\limits_{s\to 0} s^{N-1}\left|\frac{dv}{ds}(s)\right|^{p-1} = 0. \end{cases}$$

The result follows now from [1,7], or [3] by observing that the critical exponent corresponding to this problem is $\frac{Np}{N-p} = \rho^*$. □

Next we will prove Theorem 1.3, postponing the proof of Theorem 1.2 until the end of the section.

Proof of Theorem 1.3. We first prove (i). Let $u(r) = u(r,\alpha)$ be the solution to (IVP) and assume that $u(r) > 0$ for all $r > 0$. Since $m(r) \geqslant q$ for all $r > 0$, by hypothesis we have that $G(r) \geqslant 0$ for all $r > 0$. Then from (3.5), (3.7) and the fact that u is decreasing, we deduce that the function

$$r \mapsto E(r,u) - G(r)u^q(r)$$

is nonnegative and increasing, with value 0 at $r = 0$, implying also that $E(r,u) \geqslant 0$ for all $r > 0$. Hence, by Theorem 3.1 u must be rapidly decaying. Also,

$$\begin{aligned} E(r,u) - G(r)u^q(r) &= \frac{a(r)h(r)}{|h'(r)|}\frac{|u'(r)|^p}{p'} + \frac{1}{p'}u(r)a(r)|u'(r)|^{p-2}u'(r) + \beta(r)\frac{u^q(r)}{p'} \\ &\leqslant \frac{1}{p'}\left(\frac{|u'(r)|}{|h'(r)|}\right)^p h(r) + \frac{1}{p'}\frac{u(r)}{h(r)}\left(\frac{|u'(r)|}{|h'(r)|}\right)^{p-1} h(r) + \frac{1}{p'}\beta(r)u^q(r). \end{aligned} \tag{5.2}$$

Since by Theorem 3.1 the first two terms on the right-hand side are bounded by $Ch(r)$, for some positive constant C, we have that these terms tend to 0 as $r \to \infty$. Also, the third one is bounded by $C\beta(r)h^q(r)$ for some other positive constant C, which tends to 0 as in (2.14). Thus we have that $E(r,u) - G(r)u^q(r) \to 0$ as $r \to \infty$, and hence $E(r,u) - G(r)u^q(r) \equiv 0$, that is

$$\int_0^r G(s)\frac{d}{ds}(u^+)^q\,ds \equiv 0,$$

implying that $G(r) \equiv 0$, contrary to our assumption.

Next we prove (ii). Let $u(r) = u(r,\alpha)$ be the solution to (IVP). From

$$E(r,u) = G(r)u^{+q}(r) - \int_0^r G(s)\frac{d}{ds}(u^+)^q\,ds,$$

and (3.1) we see that $E(r,u) \leqslant G(r)u^{+q}(r) \leqslant 0$, since by assumption $m(r) \leqslant q$ for all $r > 0$. Hence from Lemma 3.3 that u cannot be a crossing solution, that is $u(r) > 0$ for all $r > 0$. Then we can choose $r_0 > 0$ such that for $r \geqslant r_0$, $E(r,u)$ satisfies

$$E(r,u) - G(r)u^q(r) \leqslant -\int_0^{r_0} G(s)\frac{d}{ds}(u^+)^q\,ds < 0,$$

implying that $E(r,u)$ does not converge to zero as r tends to infinity, which by Lemma 3.2 implies that u must be a slowly decaying solution. □

We proceed now to prove Theorem 1.4.

Proof of Theorem 1.4. Assume now that all the hypotheses of the theorem are satisfied. The proof consists of showing that all assumptions of Proposition 4.4 hold.

First we observe that from $\limsup_{r\to\infty}(m(r)-q) = \rho^*_\infty - q$, we have that $m(r) < q$ for r large if $q > \rho^*_\infty$. The same results follows from (1.15) if $q = \rho^*_\infty$. Thus, $R_q < \infty$, and since $(m-q)\beta$ is decreasing in (R_q,∞) we obtain that $m(r) \leqslant q$ for all $r > R_q$. This implies

$$G(r) = \frac{1}{qp'}(m(r)-q)\beta(r) \leqslant 0 \quad \text{and} \quad G'(r) \leqslant 0 \text{ in } (R_q,\infty). \tag{5.3}$$

We will show now that $\overline{R}_q \leqslant R_q$. Indeed, let us first prove that $\lim_{r\to\infty} H(r) = 0$. By the first in (5.3) and from the second in (3.4), we have that $B_q(r) = \beta h^{q/p'}$ is decreasing, and thus bounded near infinity. Thus, the first term in the definition of $H(r)$ is bounded as follows,

$$\frac{1}{q}bh^{q+1}/|h'| = \frac{1}{qp'}p'bh^{q+1}/|h'| = \frac{1}{qp'}m\beta h^{q/p'}h^{q/p} \leqslant Ch^{q/p},$$

for some positive constant C, and therefore tends to zero as r tends to ∞. Moreover, from (3.3) and the second in (5.3), $H'(r) = G'(r)h^q \leqslant 0$ for all $r \geqslant R_q$ and thus

$H(r) \geqslant H(\infty) = 0$, implying that $\overline{R}_q \leqslant R_q$, which yields in particular that $\overline{R}_q$ is finite. Proposition 4.1 then says that the sets A_1, A_2 are open.

Finally, we will see that A_1, A_2 are nonempty. By assumption (1.14) we have that $q \leqslant \rho_0^*$. If $q = \rho_0^*$, assumption (1.16) tells us that the conditions in Proposition 4.3 are satisfied, hence $A_1 \neq \emptyset$. Also, from (1.14), $\rho_\infty^* \leqslant q$. If $\rho_\infty^* < q$, then $\limsup_{r\to\infty} G(r) = -\infty$, and if $\rho_\infty^* = q$, then by (1.15), $\limsup_{r\to\infty} G(r) < 0$. Thus in all cases $\limsup_{r\to\infty} G(r) < 0$, implying from Proposition 4.2 that $A_2 \neq \emptyset$.

We can now apply Proposition 4.4(iv), to obtain that there exists α^* such that $u^*(r) = u(r, \alpha^*)$ is a rapidly decaying solution, and $A_1 = (\alpha^*, \infty)$ and $A_2 = (0, \alpha^*)$. □

Next, we prove our improvement of Theorem YY, which is a consequence of Theorems 1.3 and 1.4:

Proof of Theorem 1.5. Let the function m be decreasing on $(0, \infty)$. Then

$$\lim_{r\to\infty} m(r) \leqslant m(r) \leqslant \lim_{r\to 0} m(r) \quad \text{for all } r > 0.$$

We will show first that $\rho_0^* = m(0)$. From Proposition 1.1, and L'Hôpital's rule, we have to prove that $m(0) \geqslant p$. If h is bounded near 0, we have that $m(0) = \infty > p$ and thus $\rho_0^* = m(0)$. Let now $m(0) < \infty$ and assume that $h(0) = \infty$. By integrating the inequality

$$\frac{b(r)}{\beta(r)} \leqslant \frac{m(0)}{p'} \frac{|h'(r)|}{h(r)},$$

over $(r, 1)$ we find that

$$\beta(r) \geqslant C(h(r))^{-m(0)/p'}$$

for some positive constant C, hence $C^{p'-1} h^{-m(0)/p} |h'| \leqslant \beta^{p'-1} |h'| = \beta^{p'-1} a^{1-p'}$. Therefore from (H_2), we obtain that $h^{-m(0)/p} |h'| \in L^1(0, 1)$, implying that $m(0) > p$ and thus $\rho_0^* = m(0)$.

If $q \leqslant \rho_\infty^*$, then $q \leqslant m(r)$ for all $r > 0$ and hence the result follows from Theorem 1.3(i).

If $\rho_0^* < \infty$ and $q \geqslant \rho_0^*$, then $q \geqslant m(r)$ for all $r > 0$ and hence the result follows from Theorem 1.3(ii).

Let now $\rho_\infty^* < q < \rho_0^* \leqslant \infty$, and let R_q be as in Theorem 1.4. Then $m(R_q) = q$ and for $r > R_q$ we have $m(r) - q < 0$ and thus

$$((m - q)\beta)'(r) = m'(r)\beta(r) + (m(r) - q)b(r) < 0,$$

implying that $(m - q)\beta$ is decreasing on (R_q, ∞). Hence the result follows from Theorem 1.4. □

As we mentioned in the introduction, this result is a strong improvement of Theorem YY since it applies to functions B for which condition (1.2) does not hold, see Section 6 for a detailed example.

We finish this section with the proof of Theorem 1.2, which is a generalization of Theorem KYY to the p-Laplacian case with two weights as in (*IVP*).

Proof of Theorem 1.2. We shall distinguish the cases $R_q = 0$ and $R_q = \infty$. In the first case, there exists a sequence $r_n \to 0$ such that $(m(r_n) - q)\beta(r_n) < 0$, and since $(m - q)\beta$ decreases in $(0, \infty)$, for any $r > 0$, there exists $n_0 \in \mathbb{N}$ such that $r_n < r$ for all $n \geqslant n_0$, hence

$$(m(r) - q)\beta(r) \leqslant 0 \quad \text{for all } r > 0. \tag{5.4}$$

Therefore $m(r) \leqslant q$ for all $r > 0$ and m is not constant, so we can apply Theorem 1.3(ii) to obtain that all solutions to (*IVP*) must be slowly decaying.

Assume next that $R_q = \infty$. Then by the definition of R_q, $m(r) \geqslant q$ for all $r > 0$, and thus by Theorem 1.3(i) all solutions must be crossing.

If now $0 < R_q < \infty$, we can argue as in the proof of Theorem 1.4 to show that $\overline{R}_q \leqslant R_q$ and hence by Proposition 4.1, both sets A_1 and A_2 are open. From Proposition 4.4(ii) and (iii), if $A_1 = \emptyset$ then $A_2 = (0, \infty)$, implying that all the solutions are slowly decaying, and if $A_2 = \emptyset$ then $A_1 = (0, \infty)$, hence all the solutions are crossing. If both sets are nonempty, then by Proposition 4.4(iv) the solutions are of type (M). □

6. Some applications

This section is devoted to the comparison of our results with previous ones obtained in [9,21,2].

Next we will show that the result given in Theorem 1.5 indeed generalizes Theorem 2.1 in [21], not only because of the general p-Laplacian operator considered, but because our result includes theirs.

Our first result gives a p-version of Theorem YY. Consider the problem

$$\begin{gathered} -\left(r^{N-1}|u'|^{p-2}u'\right)' = r^{N-1}B(r)(u^+)^{q-1} \\ u(0) = \alpha > 0, \quad \lim_{r\to 0} r^{N-1}|u'|^{p-1}(r) = 0. \end{gathered} \tag{6.1}$$

Theorem 6.1. *In problem* (6.1), *let* $N > p$, $q > p > 1$. *Also let* $B \in C^1(0, \infty)$ *be a positive function satisfying* $r^{N-1}B \in L^1(0, 1)$, $(rB)^{1/(p-1)} \in L^1(0, 1)$ *and* (1.2).

Then

$$\rho_0^* = \frac{p(N+\sigma)}{N-p}, \quad \rho_\infty^* = \max\left\{p, \frac{p(N+\ell)}{N-p}\right\}, \tag{6.2}$$

where σ *and* ℓ *are defined in* (1.3). *Moreover,*

(i) *if* $p < q \leqslant \rho_\infty^*$, *then for any* $\alpha > 0$, *the solution* $u(\cdot, \alpha)$ *of* (6.1) *has a first positive zero in* $(0, \infty)$,

(ii) *if* $q \geqslant \rho_0^*$, *then for any* $\alpha > 0$, *the solution* $u(\cdot, \alpha)$ *of* (6.1) *is positive in* $(0, \infty)$ *and* $\lim_{r\to\infty} r^{(N-p)/(p-1)} u(r, \alpha) = \infty$,

(iii) *if* $\rho_\infty^* < q < \rho_0^*$, *then there exists a unique* $\alpha^* > 0$ *such that the solution* $u(\cdot, \alpha)$ *of* (6.1) *satisfies*
   - $u(r, \alpha) > 0$ *for all* $r > 0$ *with* $\lim_{r\to\infty} r^{(N-p)/(p-1)} u(r, \alpha) = \infty$ *whenever* $\alpha \in (0, \alpha^*)$.
   - $u(r, \alpha^*) > 0$ *for all* $r > 0$ *with* $\lim_{r\to\infty} r^{(N-p)/(p-1)} u(r, \alpha^*) = \ell \in (0, \infty)$.
   - $u(\cdot, \alpha)$ *has a first zero for any* $\alpha \in (\alpha^*, \infty)$.

Proof. Let $a(r) = r^{N-1}$, $b(r) = r^{N-1} B(r)$ in Theorem 1.5. Assuming that all the assumptions of the present theorem are satisfied we will show that all the assumptions of Theorem 1.5 are satisfied. We prove first that under (1.2),

$$m(r) = \frac{p}{N-p} \frac{r^N B(r)}{\int_0^r s^{N-1} B(s)\, ds} \tag{6.3}$$

satisfies (6.2) and is decreasing in $(0, \infty)$. Indeed, we have

$$\lim_{r\to 0,\infty} m(r) = \frac{p}{N-p}\left(N + \lim_{r\to 0,\infty} \frac{rB'(r)}{B(r)}\right) \tag{6.4}$$

and thus (6.2) holds. Also, by direct differentiation of m we obtain

$$\frac{p}{N-p} \frac{rm'(r)}{m(r)} + m(r) = \frac{p}{N-p}\left(N + \frac{rB'(r)}{B(r)}\right),$$

hence from (6.4) we have

$$\lim_{r\to 0} \frac{rm'(r)}{m(r)} = 0. \tag{6.5}$$

Assume now that $m'(r_0) = 0$ for some $r_0 > 0$. Then it must be that r_0 is a maximum point for m, and thus $m'(r) \geqslant 0$ for $r \in (0, r_0)$. But since $\frac{rB'(r)}{B(r)}$ decreases for all $r > 0$, it must be that $\frac{rm'(r)}{m(r)}$ decreases on $(0, r_0)$, which together with (6.5) yields a contradiction and we conclude that m must be decreasing, proving our claim. Also, (H_1) is trivially satisfied since $N > p$ and $r^{N-1}B \in L^1(0,1)$. Finally, since

$$\begin{aligned}\int_r^1 \left(\frac{\beta(s)}{s^{N-1}}\right)^{1/(p-1)} ds &= \int_r^1 \left(\frac{\beta(s)}{s\beta'(s)}\right)^{1/(p-1)} (sB(s))^{1/(p-1)}\, ds\\ &= \left(\frac{N-p}{N}\right)^{1/(p-1)} \int_r^1 \left(\frac{1}{m(s)}\right)^{1/(p-1)} (sB(s))^{1/(p-1)}\, ds,\end{aligned}$$

and m decreases, we find from the assumption $(rB)^{1/(p-1)} \in L^1(0,1)$ that (H_2) is satisfied. □

Remark 6.1. Theorem YY is a consequence of Theorem 6.1 since in the case that $p = 2$, the assumption $rB \in L^1(0,1)$ implies $r^{N-1}B \in L^1(0,1)$.

Next, we give an example where the key hypothesis in Theorem YY, namely, $\frac{rB'(r)}{B(r)}$ decreasing and nonconstant in $(0,\infty)$, is not satisfied and hence that theorem cannot be applied. We nevertheless classify the behavior of the solutions in this case by using our Theorem 1.5.

Example. In problem (1.1) let $N = 3$ and

$$B(r) = \begin{cases} 3^{13/2} r^{3/2} (r+1)^{-11/2}, & 0 \leqslant r \leqslant 2,\\ 16(2r-1) r^{-5/2} (r-1)^{-1/2}, & r > 2.\end{cases} \tag{6.6}$$

Then, by direct calculation, we have that

$$\frac{rB'(r)}{B(r)} = \frac{2r}{2r-1} - \frac{r}{2(r-1)} - \frac{5}{2} \quad \text{for all} \quad r > 2,$$

and

$$\frac{d}{dr}\left(\frac{rB'(r)}{B(r)}\right) = \frac{1}{(2r-1)^2}\,\frac{(4r-3)}{2(r-1)^2} > 0 \quad \text{for all} \quad r > 2,$$

implying that $\frac{rB'(r)}{B(r)}$ increases for $r > 2$ and hence Theorem YY does not apply.

Now from (6.3), we find that

$$m(r) = \begin{cases} \dfrac{9}{r+1}, & 0 \leqslant r \leqslant 2,\\[2mm] \dfrac{1}{r-1} + 2, & r > 2, \end{cases} \tag{6.7}$$

which shows that m decreases in $[0,\infty)$, $m(0)=9$ and $m(\infty)=2$. For this case the functions a and b in (*IVP*), with $p=2$, are given by $a(r)=r^2$, $b(r)=r^2B(r)$ and it is direct to see that they satisfy assumptions (H_1), (H_2) of Theorem 1.5. By Proposition 1.1 we have that $\rho_0^*=9$ and $\rho_\infty^*=2$, thus by Theorem 1.5, we obtain the following behavior: if $q\geqslant 9$, then all solutions of (*IVP*) are of type (S) and if $2<q<9$ all solutions of (*IVP*) are of type (M).

In [2] conditions were given for existence or nonexistence of solutions for the problem

$$(E)-\left(r^{\mu}|u'|^{p-2}u'\right)'=r^{\gamma}\tilde{a}(r)|u|^{q-2}u,\ \text{for } r>0,\ \ u'(0)=0,\quad u(r)>0,\ \text{ in } [0,\infty)$$

where $\mu>p-1$, $\gamma\geqslant 0$, and $q>p>1$. Using the ideas of this paper we can improve Theorem 3.2 and Theorem 4.1 in [2].

Theorem 6.2. *Assume that*

(i) $\mu>p-1$, *and* $q>p>1$.
(ii) $\tilde{a}\in C^1(0,\infty)$, $\tilde{a}(r)>0$ *for* $r>0$, $\lim_{r\to 0} r^{\gamma+1}\tilde{a}(r)=0$ *and* $r^{\gamma}\tilde{a}(r)\in L^1(0,1)$.

Then,

$$Q(r)=\int_0^r s^{\frac{(\mu+1-p)q}{p}}\left(\tilde{a}s^{\gamma+1-\frac{(\mu+1-p)q}{p}}\right)'ds$$

is well defined and

(a) *If* $Q(r)\geqslant 0$ *and* $Q\not\equiv 0$ *in* $(0,\infty)$, *then all solutions to* (E) *must be crossing.*
(b) *If* (H_2) *is satisfied and* $Q(r)\leqslant 0$ *and* $Q\not\equiv 0$ *in* $(0,\infty)$, *then all solutions to* (E) *must be slowly decaying.*

Proof. The proof consists in showing that under the conditions of the theorem all the hypotheses of Theorem 1.3 are satisfied. Indeed, we note that assumptions (i) and (ii) imply that $a^{1-p'}=r^{-\mu/(p-1)}$ is in $L^1(1,\infty)\setminus L^1(0,1)$ and that $b=r^{\gamma}\tilde{a}\in L^1(0,1)$, hence (H_1) is satisfied. $Q(r)$ is well defined since $r^{\gamma+1}\tilde{a}'(r)$ is integrable near 0 by (ii).

Also, it can be directly verified that, with our notation,

$$(m(r)-q)\beta(r)=\frac{p}{\mu+1-p}\,r^{\gamma+1}\tilde{a}(r)-q\beta(r),$$

thus $(m-q)\beta(0)=0$ and

$$\left(r^{\gamma+1-\frac{(\mu+1-p)q}{p}}\tilde{a}\right)'(r)=\frac{\mu+1-p}{p}\,r^{-\frac{(\mu+1-p)q}{p}}\left((m-q)\beta\right)'(r), \tag{6.8}$$

implying

$$Q(r) = \frac{\mu + 1 - p}{p}(m - q)\beta(r).$$

We will now prove (a). The condition $Q(r) \geqslant 0$ for all $r > 0$ implies that $q \leqslant m(r)$, hence by integrating the inequality $\frac{q}{p'}\frac{|h'|}{h} \leqslant \frac{b}{\beta}$ over $(r, 1)$ we find that $\beta(r) \leqslant Ch^{-q/p'}(r)$ for all $r \in (0, 1)$, and for some positive constant C, hence

$$\left(\frac{\beta}{a}\right)^{p'-1}(r) \leqslant Ch^{-q/p}(r)|h'(r)|,$$

and since $q > p$ we conclude that (H_2) holds. Hence, all assumptions in Theorem 1.3(i) are satisfied, implying that all solutions to (E) are crossing. In order to prove (b), we observe that $Q(r) \leqslant 0$ and $Q \not\equiv 0$ implies $m(r) \leqslant q$ and $m \not\equiv q$, hence the result follows from Theorem 1.3(ii). □

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Appendix

In this appendix we will assume that $q > p > 1$ in problem (*IVP*) and that the weights a, b satisfy assumptions (H_1) and (H_2). For $\rho > 0$ it is clear that u is a solution to problem (*IVP*) defined in $[0, \rho]$ if and only it satisfies

$$u(r) = \alpha - \int_0^r a^{1-p'}(s)\left(\int_0^s b(t)(u^+(t))^{q-1}\,dt\right)^{p'-1} ds, \tag{7.1}$$

for all $r \in [0, \rho]$. Since we have in mind to also vary α, we define the operator $T : C[0, \rho] \times (0, \infty) \mapsto C[0, \rho]$ by

$$T(u, \alpha)(r) = \alpha - \int_0^r a^{1-p'}(s)\left(\int_0^s b(t)(u^+(t))^{q-1}\,dt\right)^{p'-1} ds. \tag{7.2}$$

If for fixed α we denote

$$T_\alpha(u) = T(u, \alpha), \tag{7.3}$$

then it is clear that u satisfies (7.1) for all $r \in [0, \rho]$, if and only if $u = T_\alpha(u)$.

Proposition A.1. *The operator T defined in* (7.2) *is completely continuous.*

Proof. Let $\{u_n\}$ be a bounded sequence in $C[0,\rho]$ and let $\{\alpha_n\}$ be a bounded sequence in $(0,\infty)$. Then

$$T(u_n,\alpha_n)(r)=\alpha_n-\int_0^r a^{1-p'}(s)\left(\int_0^s b(t)(u_n^+(t))^{q-1}\,dt\right)^{p'-1}ds, \tag{7.4}$$

and for $r\in(0,\rho)$

$$\frac{d}{dr}(T(u_n,\alpha_n))(r)=-a^{1-p'}(r)\left(\int_0^r b(t)(u_n^+(t))^{q-1}\,dt\right)^{p'-1}.$$

Hence

$$\left|\frac{d}{dr}(T(u_n,\alpha_n))(r)\right|\leqslant a^{1-p'}(r)\beta^{p'-1}(r)\alpha_n^{\frac{q-1}{p-1}}\leqslant C^{\frac{q-1}{p-1}}\Gamma(r),$$

where C is a positive constant such that $\alpha_n\leqslant C$. Then for $s,r\in[0,\rho]$, we have that

$$|T(u_n,\alpha_n)(r)-T(u_n,\alpha_n)(s)|\leqslant C^{\frac{q-1}{p-1}}\left|\int_s^r\Gamma(t)\,dt\right|,$$

implying that the sequence $\{T(u_n,\alpha_n)\}$ is equicontinuous in $[0,\rho]$. Next since from (7.4),

$$|T(u_n,\alpha_n)(r)|\leqslant C+C^{\frac{q-1}{p-1}}\int_0^\rho\Gamma(t)\,dt,$$

it follows that sequence $\{T(u_n,\alpha_n)\}$ is uniformly bounded. Thus from Ascoli–Arzela's Theorem the sequence $\{T(u_n,\alpha_n)\}$ contains a convergent subsequence in $C[0,\rho]$ implying that T is a compact operator. Now we show that T is continuous, i.e. we want to show that for $(u,\alpha)\in C[0,\rho]\times(0,\infty)$ it holds that $\lim_{v\to u,\mu\to\alpha}T(v,\mu)=T(u,\alpha)$. For this is sufficient to prove that for any sequence $\{(u_n,\alpha_n)\}$ in $C([0,\rho]\times(0,\infty)$ such that $u_n\to u$ and $\alpha_n\to\alpha$ the sequence $\{T(u_n,\alpha_n)\}$ contains a subsequence converging to $T(u,\alpha)$. Indeed, the compactness of T implies that $\{T(u_n,\alpha_n)\}$ contains a convergent subsequence which we rename the same, say $\lim_{n\to\infty}T(u_n,\alpha_n)=v\in C[0,\rho]$. By letting $n\to\infty$ in (7.4), an application of the Lebesgue dominated convergence Theorem yields that

$$v(r)=\alpha-\int_0^r a^{1-p'}(s)\left(\int_0^s b(t)(u^+(t))^{q-1}\,dt\right)^{p'-1}ds,$$

and hence $v=T(u,\alpha)$, which ends the proof. □

Proposition A.2. *For any* $\alpha \in (0, \infty)$, *the initial value problem* (*IVP*) *has a unique solution* $u \in C[0, \infty) \cap C^1(0, \infty)$ *such that* $a|u'|^{p-1}u' \in C^1[0, \infty)$. *Furthermore solutions depends continuously on* α *in the sense of uniform convergence on compacts of* $[0, \infty)$.

Proof. Let us fix $\alpha > 0$ in the boundary conditions of problem (*IVP*) and let us refer to this problem as problem $(IVP)_\alpha$. Let $\delta > 0$ be such that

$$\int_0^\delta \Gamma(s)\, ds \leqslant \alpha^{(p-q)/(p-1)}. \tag{7.5}$$

We will show first that problem $(IVP)_\alpha$ has a unique solution defined in the interval $[0, \delta]$. Setting $\rho = \delta$ in the previous proposition existence of a solution will follows if we can prove that the operator T_α defined in (7.3) has a fixed point in $C[0, \delta]$. To this end let us define the set D by

$$D = \{u \in C[0, \delta] \mid 0 \leqslant u(r) \leqslant \alpha\}.$$

Then D is a closed bounded convex subset of $C[0, \delta]$. Since the choice of δ in (7.5) implies that $T_\alpha(D) \subset D$ and by the previous proposition T_α is a completely continuous operator the existence of a fixed point for T_α follows immediately from Schauder's fixed point Theorem.

Assume that u and v are solutions to problem $(IVP)_\alpha$ defined in a right interval of zero. Thus, in particular, $u(0) = v(0) = \alpha$. Let $\varepsilon > 0$ be given such that $\alpha - \varepsilon > 0$, and suppose $\tilde{\delta} > 0$ is small enough so that $u,\ v \geqslant \alpha - \varepsilon$ for all $r \in [0, \tilde{\delta}]$. In addition we will assume that

$$\int_0^{\tilde{\delta}} \Gamma(s)\, ds < \frac{1}{K}, \tag{7.6}$$

where $K = \alpha^{\frac{q-1}{p-1}} \frac{(p'-1)(q-1)}{\alpha - \varepsilon}$. By integrating the equation in $(IVP)_\alpha$, and noticing that $u^+ = u$ and $v^+ = v$ in $r \in [0, \tilde{\delta}]$, we find that

$$u'(r) - v'(r) = a^{1-p'}(r)\Lambda(r),$$

where

$$\Lambda(r) = \left(\int_0^r b(t) u^{q-1}(t)\, dt\right)^{p'-1} - \left(\int_0^r b(t) v^{q-1}(t)\, dt\right)^{p'-1}.$$

Let us fix $r \in (0, \tilde{\delta}]$, and for $\mu \in [0, 1]$, set

$$z_r(\mu) = \int_0^r b(t)|v(t) + \mu(u(t) - v(t))|^{q-2}(v(t) + \mu(u(t) - v(t)))\,dt,$$
$$\gamma_r(\mu) = |z_r(\mu)|^{p'-2} z_r(\mu).$$

Then, by the choice of $\tilde{\delta}$, we have that $z_r(\mu) > 0$ for all $\mu \in [0, 1]$. Hence, we can differentiate $z_r(\mu)$, to find $\frac{d}{d\mu}\gamma_r(\mu) = (p' - 1)(z_r(\mu))^{p'-2} \frac{d}{d\mu} z_r(\mu)$. Furthermore,

$$\begin{aligned}\frac{d}{d\mu} z_r(\mu) &= (q-1) \int_0^r b(t)(v(t) + \mu(u(t) - v(t)))^{q-2}(u(t) - v(t))\,dt \\ &= (q-1) \int_0^r b(t)|v(t) + \mu(u(t) - v(t))|^{q-1} \frac{(u(t) - v(t))}{v(t) + \mu(u(t) - v(t))}\,dt,\end{aligned}$$

implying that

$$\left|\frac{d}{d\mu} z_r(\mu)\right| \leqslant (q-1) z_r(\mu) \frac{\|u - v\|_{C[0,\tilde{\delta}]}}{\alpha - \varepsilon}.$$

Thus

$$\begin{aligned}\left|\frac{d}{d\mu}\gamma_r(\mu)\right| &\leqslant (p'-1)(q-1) \frac{\|u - v\|_{C[0,\tilde{\delta}]}}{\alpha - \varepsilon} (z_r(\mu))^{p'-1} \\ &\leqslant \alpha^{\frac{q-1}{p-1}} (p'-1)(q-1) \frac{\|u - v\|_{C[0,\tilde{\delta}]}}{\alpha - \varepsilon} (\beta(r))^{p'-1}.\end{aligned}$$

Since

$$\Lambda(r) = \int_0^1 \frac{d}{d\mu}\gamma_r(\mu) d\mu,$$

we obtain that

$$|u'(r) - v'(r)| \leqslant K a^{1-p'}(r) \beta^{p'-1}(r) \|u - v\|_{C[0,\tilde{\delta}]},$$

for each $r \in (0, \tilde{\delta}]$. Hence

$$\|u - v\|_{C[0,\tilde{\delta}]} \leqslant K \|u - v\|_{C[0,\tilde{\delta}]} \int_0^{\tilde{\delta}} \Gamma(r) dr,$$

that implies

$$\frac{1}{K} \leqslant \int_0^{\tilde{\delta}} \Gamma(r) dr,$$

if $\|u - v\|_{C[0,\tilde{\delta}]} > 0$. Since this contradicts (7.6), it must be that $u(r) = v(r)$ for all $r \in [0, \tilde{\delta}]$.

Thus we have proved there is $\delta = \delta(\alpha) > 0$ such that problem $(IVP)_\alpha$ has a unique solution u defined in $[0, \delta]$, with $u(\delta) > 0$. We will show next that this solution can be extended to $[\delta, \infty)$ and that this extension is unique. To do this we have to study the initial value problem

$$-\left(a(r)|u'|^{p-2}u'\right)' = b(r)(u^+)^{q-1}, \quad u(\delta) > 0, u'(\delta) < 0.$$

Setting $w(r) = a(r)|u'(r)|^{p-2}u'(r)$, this problem is equivalent to the initial value problem

$$(S_\alpha) \qquad \begin{cases} u' = f_1(r, u, w) = a(r)^{1-p'}|w|^{p'-2}w, \\ w' = f_2(r, u, w) = -b(r)(u^+)^{q-1}, \\ u(\delta) > 0, \quad w(\delta) < 0. \end{cases}$$

Let us consider the open sets $\Omega^+ = \{(u, v) \mid u > 0, w < 0\}$ and $\Omega^- = \{(u, v) \mid u < 0, w < 0\}$ and notice that f_1, f_2 belong to $C^1((0, \infty) \times (\Omega^+ \cup \Omega^-)) \cup C(0, \infty) \times \mathbb{R} \times (-\infty, 0))$.

Note that any solution $(u(r), w(r))$ to (S_α), considered in its maximal interval of definition, satisfies that $w(r)$ is decreasing and negative and that $u(r)$ is strictly decreasing. Since this implies that

$$|u'(r)| = a(r)^{1-p'}|w(\delta)|^{p'-1}, \qquad |w'(r)| = b(r)(u(\delta))^{q-1}, \quad r \in [\delta, T),$$

it is clear that the maximal interval of definition of any such solution is $[\delta, \infty)$ and that for $r \geqslant \delta$ it holds that $(u(r), w(r)) \in \mathbb{R} \times (-\infty, 0)$.

Due to the regularity of f_1 and f_2 it is clear that the initial value problem

$$(S_0) \qquad \begin{cases} u' = a(r)^{1-p'}|w|^{p'-2}w, \\ w' = -b(r)(u^+)^{q-1}, \\ u(r_0) = u_0, \quad w(r_0) = w_0, \end{cases}$$

has a unique local solution for any $r_0 \geqslant \delta$ and $(u_0, w_0) \in \Omega^+ \cup \Omega^- \subset \mathbb{R} \times (-\infty, 0)$.

□

Claim. (S_0) *also has a unique local solution for any* $r_0 \geqslant \delta$ *and* $(u_0, w_0) = (0, w_0)$ *with* $w_0 < 0$. *Indeed for this initial conditions if* $(u(r), w(r))$ *is a solution defined in a neighborhood of* r_0, *then* $u(r) > 0$ *for* $r < r_0$ *and* $u(r) < 0$ *for* $r > r_0$. *Thus for* $r > r_0$ *the solution to* (S_0) *is unique and given explicitly by* $u(r) = -|w_0|^{p'-1} \int_{r_0}^{r} a^{1-p'}(t)\,dt$ *and* $w(r) = w_0$. *The uniqueness of solutions for* (S_0) *when* $r < r_0$ *will be obtained by reducing this problem to a known situation. Indeed, let us set* $s = r_0 - r$, $\tilde{a}(s) = \frac{1}{(b(r_0-s))^{q'-1}}$, *with* $q' = \frac{q}{q-1}$ *and* $\tilde{b}(s) = (a(r_0 - s))^{1-p'}$. *Then, by defining* $\tilde{w}(s) = -w(r)$, *one has that* $\tilde{w}(s)$ *satisfies the initial value problem*

$$-\left(\tilde{a}(s)|\tilde{w}'|^{q'-2}\tilde{w}'\right)' = \tilde{b}(s)\tilde{w}^{p'-1}, \quad \tilde{w}(0) = -w(r_0) > 0, \quad \tilde{w}'(0) = 0,$$

where $p' > q' > 1$. *Since this is a particular case of problem* $(IVP)_\alpha$ *studied at the beginning of this proof, it follows that this problem has a unique local solution* $\tilde{w}(s)$, *for* $s > 0$, *small, which implies the uniqueness of solutions to* (S_0) *for* $r < r_0$, *ending the proof of the claim.*

Thus we have proved that (S_0) has a unique local solution for any $r_0 \geqslant \delta$ and $(u_0, w_0) \in \mathbb{R} \times (-\infty, 0)$. Clearly this implies that problem (S_α) has a unique solution defined in $[\delta, \infty)$ and this fact in turn implies that $(IVP)_\alpha$ has a unique solution defined in $[0, \infty)$; equivalently (IVP) has a unique solution defined in $[0, \infty)$, for each given $\alpha > 0$.

To end the proof of the proposition we prove next that solutions to (IVP) depend continuously on $\alpha > 0$ on compact subsets of $[0, \infty)$. Assume $\tilde{\alpha} > 0$ and $\{\alpha_n\}$, $\alpha_n > 0$, is a sequence such that $\alpha_n \to \tilde{\alpha}$ as $n \to \infty$. Let u_n denote the solution to (IVP) corresponding to α_n and $\tilde{u}$ the solution corresponding to $\tilde{\alpha}$.

Then for any given $\rho > 0$ it holds that u_n seen as a function from $[0, \rho]$ into $\mathbb{R}$, satisfies

$$u_n = T(u_n, \alpha_n), \quad n \in \mathbb{N},$$

where $T : C[0, \rho] \times (0, \infty) \mapsto C[0, \rho]$ is the completely continuous operator of Proposition A.2. From this proposition the sequence $\{u_n\}$ has a subsequence, renamed the same, which is convergent in $C[0, \rho]$. Let us say $\lim_{n\to\infty} u_n = v$. Now, since

$$u_n(r) = \alpha_n - \int_0^r a^{1-p'}(s) \left(\int_0^s b(t)(u_n^+(t))^{q-1}\,dt\right)^{p'-1} ds,$$

for all $r \in [0, \rho]$ by letting $n \to \infty$ and using the Lebesgue dominated convergence theorem, we find that

$$v(r) = \tilde{\alpha} - \int_0^r a^{1-p'}(s) \left(\int_0^s b(t)(v^+(t))^{q-1}\,dt\right)^{p'-1} ds,$$

for all $r \in [0, \rho]$. Therefore, v is a solution to $(IVP)_{\tilde{\alpha}}$, thus from the uniqueness of the solution to $(IVP)_{\tilde{\alpha}}$, $v = \tilde{u}$ in $C[0, \rho]$, implying $\lim_{n\to\infty} u_n = \tilde{u}$ in $C[0, \rho]$. Since this argument is independent of subsequences it holds that the original sequence $\{u_n\}$ converges to $\tilde{u}$ in $C[0, \rho]$ as $n \to \infty$. Since ρ is any fixed positive number, this ends the proof of the proposition. □

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