An intrinsic characterization of free disposal hypothesis

Alejandro Jofré [a,1], Jorge Rivera Cayupi [b,*]

[a] *CMM y Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170, correo 3, Santiago, Chile*
[b] *Departamento de Economía, Universidad de Chile, Diagonal Paraguay 257, Piso 15, Santiago, Chile*

Abstract

In this work by using nonsmooth analysis techniques we provide a geometric characterization of both the free disposal hypothesis for production sets and the strict monotonicity condition for preference relations even for nonconvex economies.

Keywords: Normal cone; Free disposal; Monotonicity; Convex cones

1. Introduction

In this paper we prove a characterization of sets that satisfy a regularity property on its boundary points. This regularity deals with the "ability" of starting at a boundary point of the set, entering it according to directions given by a closed convex cone. For instance, if the set is a production set and the convex cone is the negative orthant, this "ability" is simply the so called free disposal hypothesis (FDH) widely used in economics; if the set is the set of points preferred to a given consumption bundle and the cone is the positive orthant, then we are talking about strict monotonicity of the preference.[2]

* Corresponding author. Tel.: +56 2 9783455; fax: +56 2 9783413.
E-mail addresses: ajofre@dim.uchile.cl (A. Jofré), jrivera@econ.uchile.cl (J.R. Cayupi).
[1] Tel.: +56 2 9784525; fax: +56 2 6889705.
[2] See Brown (1991), Eatwell et al. (1987) and Khan and Vohra (1987) for details on these concepts.

The main result of this work is Theorem 3.1, which gives us a characterization of this heretofore mentioned regularity property in terms of the Clarke's normal cone to the set at its boundary points. As a direct consequence of this result, we obtain a characterization of the FDH: a set satisfies the FDH if and only if the normal cone contains only positive vectors. The necessary condition is well known in economics and indeed under convexity is used to ensure the positivity of equilibrium prices (whenever they exist) and also to guarantee that equilibrium and/or Pareto optimum allocations are technologically efficient allocations. Even in the convex case, the sufficient condition of our main result is apparently new.

The main mathematical tool to demonstrate our result is the Clarke's normal cone and its properties (see Clarke (1983) and Rockafellar and Wets (1998) for details).[3]

2. Preliminaries

For $V \subseteq \mathbb{R}^n$, we denote its *interior*, *boundary* and *closure* by int$V$, bd$V$ and cl$V$ respectively. Given two vectors x and y in $\mathbb{R}^n$, its inner product is $x \cdot y$ and the Euclidean norm of x is $\| x \|$. The open unit ball in $\mathbb{R}^n$ is B and $B(x, \epsilon)$ is the ball with center x and radius $\epsilon \geq 0$. The distance function to V is $d_V : \mathbb{R}^n \to \mathbb{R}$, where $d_V(x) = \inf\{\| x - v \|,\ v \in V\}$. The projection on V is $\mathrm{Proj}_V : \mathbb{R}^n \to V$, that is,

$$\mathrm{Proj}_V(x) = \{v \in V |\ \|x - v\| = d_V(x)\}.$$

In this paper we use the notion of *Clarke's tangent* and *normal* cone to a set $V \subseteq \mathbb{R}^n$ at a point of it. Following Clarke (1983), we define the *Clarke's tangent cone* to V at $v \in V$ as $T_c(V, v)$, where $q \in T_c(V, v)$ if and only if for every sequence $v_k \in V$ such that $v_k \to v$ and for every sequence $t_k \to 0^+$, there exists a sequence $q_k \to q$ such that $v_k + t_k q_k \in V$. See Khan and Vohra (1987) for an economic interpretation of this cone.

Given a set $W \subseteq \mathbb{R}^n$, its *polar set* is

$$W^\circ = \{p \in \mathbb{R}^n | pw \leq 0,\ \forall w \in W\},$$

and the *Clarke's normal cone* to V at $v \in V$, denoted $N_c(V, v)$, is defined as the polar of $T_c(V, v)$.

A set $V \subseteq \mathbb{R}^n$ is epilipschitzian if for any $v \in V$, int $T_c(V, v) \neq \emptyset$. When V is convex this property is equivalent to int$V \neq \emptyset$. See Rockafellar (1979) and Rockafellar and Wets (1998) for details.

Finally we recall that a production set $Y \subseteq \mathbb{R}^n$ satisfies the free disposal hypothesis if $Y - \mathbb{R}^n_+ \subseteq Y$, and a preference correspondence P: $\mathbb{R}^n_+ \to \mathbb{R}^n$ is monotonic if for every $x \in \mathbb{R}^n$,

$$cl\ P(x) + \mathbb{R}^n_+ \subseteq P(x).$$

Note that a convex set that satisfies the free disposal hypothesis is epilipschitzian.

[3] The Clarke's normal cone has been widely used in general equilibrium theory in order to set up the pricing rules that define the equilibrium notion in economies with nonconvex production and/or consumption sectors. See Brown (1991), Cornet (1988) and Quinzii (1992) as general references on this type of models. We refer also to Hammond (1998) and Khan and Vohra (1987) to appreciate how the Clarke's normal and tangent cones are used in welfare analysis.

3. Main result

The following Theorem is the central result of this work.

Theorem 3.1. If $V \subseteq \mathbb{R}^n$ is a closed and epilipschitzian set, given $L \subseteq \mathbb{R}^n$ a closed convex cone, $V - L \subseteq V$ if and only if $N_c(V, v) \subseteq -L^\circ$.

Proof. Let $p \in N_c(V, v)$ with $v \in \mathrm{bd}\, V$. From Loewen (1993) we already know that there exist sequences $t_k \to 0^+$, $p_k \to p$, $v_k \to v$, v such that for each $v' \in V$

$$p_k(v' - v_k) \leq \frac{1}{2t_k} \|v' - v_k\|^2,$$

and then, given $\alpha \geq 0$, $\ell \in L$ and $v' = v_k - \alpha t_k \ell \in V$, follows that $p_k \cdot \ell \geq -\frac{\alpha}{2} \|\ell\|^2$; hence for each $\ell \in L$, $p_k \cdot \ell \geq 0$ (remember that α is arbitrary). Since $p_k \to p$ we conclude that for every $\ell \in L$, $p \cdot \ell \geq 0$, that is $-p \in L^\circ$ and therefore $N_c(Y, y) \subseteq -L^\circ$.

Suppose now that there exist $\ell_0 \in L$ and $v_0 \in V$ such that $v_0 - \ell_0 \notin V$. In such case, from the closeness of V, there exist $\varepsilon_1 > 0$ such that

$$B(v_0 - \ell_0, \varepsilon_1) \cap V = \emptyset. \tag{1}$$

Due to $N_c(V, v_0) \subseteq -L^\circ$, it is easy to check that $T_c(V, v_0) \supseteq -L$ and hence $-\ell_0 \in T_c(V, v_0)$. On the other hand, from the epilipschitzianity of V, $\mathrm{int}T_c(V, v_0) \neq \emptyset$ and therefore there exist $\varepsilon_2 > 0$ such that

$$B(-\ell_0, \varepsilon_2) \cap \ \mathrm{int}\ T_c(V, v_0) \neq \emptyset. \tag{2}$$

If we define $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}/2$, both relations (1) and (2) are satisfied at this value and therefore there exist $\ell_1 \in B(-\ell_0, \varepsilon)$ such that $-\ell_1 \in \mathrm{int}\ T_c(V, v_0)$ and $v_0 - \ell_1 \notin V$ (this from Eqs. (1) and (2) respectively). Without loss of generality we can suppose $v_0 \in \mathrm{bd}\, V$ (in the contrary case the conclusion is direct due to $N_c(V, v_0) = \{0_{\mathbb{R}^n}\}$ and $T_c(V, v_0) = \mathbb{R}^n$). Define

$$\mathcal{V} = V \cap \{\lambda v_0 + (1 - \lambda)(v_0 - \ell_1),\ 0 \leq \lambda \leq 1\}$$

and

$$q_0 = Proy_{\mathcal{V}}(v_0 - \ell_1) \in \mathcal{V}.$$

Thus, from the above definitions, there exist $t_0 \in [0, 1]$ such that

$$q_0 = t_0(v_0 - \ell_1) + (1 - t_0)v_0,$$

and thus $v_0 - \ell_1 - q_0 = -(1 - t_0)\ell_1$, which implies $\|v_0 - \ell_1 - q_0\| = (1 - t_0)\|\ell_1\|$.

Finally, since $-\ell_1 \in \mathrm{int}\ T_c(V, v_0)$, following Clarke (1983) there exists $\delta > 0$ such that for every $t \in [0, \delta[$, $q_0 + t(-\ell_1) \in \mathcal{V}$, and therefore, considering that

$$(v_0 - \ell_1) - (q_0 + t(-\ell_1)) = (t_0 + t - 1)\ell_1,$$

we can conclude

$$\|(v_0 - \ell_1) - (q_0 + t(-\ell_1))\| = (1 - t_0 - t)\|\ell_1\| < (1 - t_0)\|\ell_1\|,$$

which contradicts the projection definition and ends the proof. □

Remark 3.1. By using the same arguments as in the previous proof, it is possible to show a stronger local version of Theorem 3.1. Indeed, given $v_0 \in V$ and $\epsilon > 0$, the statement

$$[V \cap \ clB(v_0, \epsilon)] - [v_0 + L \cap \ clB(v_0, \epsilon)] \subseteq V,$$

is equivalent to

$$\forall v \in \ clB(v_0, \epsilon) \cap V, \ N_c(V, v) \subseteq -L^\circ.$$

Finally, it can be proved that the equivalence in Theorem 3.1 holds true under the conditions V is closed and L is a closed convex cone with nonempty interior.

4. Application to economics

A direct consequence of Theorem 3.1 holds when we consider either production sets satisfying the free disposal hypothesis or a monotonic preference correspondence. Using Theorem 3.1 we can obtain an intrinsic characterization of the aforementioned properties in terms of the Clarke's normal cone to the boundary of the respective sets. The following proposition is immediate from Theorem 3.1.

Proposition 4.1. A production set $Y \subseteq \mathbb{R}^n$ satisfies the free disposal hypothesis if and only if $N_c(Y, y) \subseteq \mathbb{R}^n_+$ for all $y \in \mathrm{bd}Y$; a preference relation P: $\mathbb{R}^n_+ \to \mathbb{R}^n_+$ is monotonic if and only if $N_c(\mathrm{cl}P(x), x') \subseteq -\mathbb{R}^n_+$ for all $x \in \mathbb{R}^n_+$ and $x' \in \mathrm{bd}P(x)$.

Remark 4.1. When the production set Y is convex, the Clarke's normal cone $N_c(Y, y)$ coincides with the well known normal cone of the convex analysis,

$$N(Y,y) = \{p \in \mathbb{R}^n | p \cdot y \geq p \cdot y', \ \forall y' \in Y\}.$$

In other words, $p \in N(Y, y)$ if and only if y is the production plan[4] of the firm at price p. Thus, under convexity, Proposition 4.1 for production sets can be presented in the following form: *a convex and closed production set satisfies the free disposal hypothesis if and only if any price such that the respective production plan lies on its boundary is positive.*

Remark 4.2. If a production set satisfies a "local free disposal condition" according to the notion given in Remark 3.1, we conclude that for any point where this condition holds true the normal cone is positive and viceversa. Considering that a local notion of free disposability could represent a more realistic economic model that takes into account the fact that is costly either to discard byproducts or to storage and/or handle raw materials, a local condition as shown above allows us to consider the same properties regarding normal cones as if production sets fulfill the global condition. Similar reasoning for a preference correspondence and a local notion of monotonicity.

Acknowledgements

This work was partially supported by FONDAP-Optimización, ECOS and ICM Sistemas Complejos en Ingeniería. The authors wish to thank Jean-Marc Bonnisseau for helpful comments

[4] Given a price, we recall that a "production plan" of a firm delineates both the supply of outputs as well as the demand for inputs at this price.

and a productive discussion, and an anonymous referee for detailed observations that improve this paper.

References

Brown, D.J., 1991. Equilibrium analysis with nonconvex technologies. In: Wildenbrand, W., Sonnenschein, H. (Eds.), Handbook of Mathematical Economics, vol. IV. North-Holland, Amsterdam. Chap. 36.

Clarke, F., 1983. Optimization and Nonsmooth Analysis. Wiley, New York.

Cornet, B., 1988. General equilibrium theory and increasing returns: presentation. Journal of Mathematical Economics 17, 103–118.

Eatwell, J., Milgate, M., Newman, P. (Eds.), 1987. The New Palgrave: A Dictionary of Economics. MacMillan, London.

Hammond, P., 1998. The efficiency theorems and market failure. In: Kirman, Alan (Ed.), Elements of General Equilibrium Analysis. Blackwell Publishers, London. Chap. 6.

Khan, A., Vohra, R., 1987. An extension of the second welfare theorem to economics with nonconvexities and public goods. Quarterly Journal of Economics 102, 223–241.

Loewen, P., 1993. Optimal control via nonsmooth analysis. CRM Proceedings and Lectures Notes. American Mathematical Society, Providence, RI.

Quinzii, M., 1992. Increasing Returns and Efficiency. Oxford University Press, New York.

Rockafellar, R., 1979. Clarke's tangent cone and the boundary of closed sets in $\mathbb{R}^n$. Nonlinear Analysis, Theory, Methods and Applications 3, 145–154.

Rockafellar, R., Wets, R., 1998. Variational Analysis. Springer Verlag, Berlin.