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A nonconvex separation property and some applications*

Abstract. In this paper we proved a nonconvex separation property for general sets which coincides with the Hahn-Banach separation theorem when sets are convexes. Properties derived from the main result are used to compute the subgradient set to the distance function in special cases and they are also applied to extending the Second Welfare Theorem in economics and proving the existence of singular multipliers in Optimization.

Key words. Variational analysis – Subgradient set – Nonconvex separation – Singular multiplier – Equilibrium price – Pareto optimum

1. Introduction

We recall that the Hahn-Banach Separation Theorem (HBT) establishes that for any closed and convex sets $Y_j \subseteq I\!R^\ell$, $j = 1, \dots, n$, given $y_j \in Y_j$ such that $\sum y_j \in bd[\sum Y_j]$, there exists $p \in I\!R^\ell,\ p \neq 0$, such that $p \in \bigcap N(Y_j, y_j)$, where $N(Y_j, y_j)$ is the convex analysis normal cone to Y_j at y_j. It is well-known that the HBT is a key ingredient in optimization and mathematical economics among other areas. For non-convex sets, first extensions of that theorem in a finite dimensional setting were proved by Mordukhovich *et al.* ([18], [19], [20]) and by Cornet and Rockafellar ([9]). For the history of these results and applications see [22].

In the aforementioned generalizations, the results deal with replacing convex analysis normal cone by Clarke's normal cone ([6], [7]) or more general ones, but keeping the sum over sets. Thus, they obtain sharper results that can be applied to more general of sets than convexes.

The aim of this paper is to extend the aforementioned results to more general operations than the sum over sets and to use the subgradient set to the distance function introduced by Mordukhovich in the 80's instead of the respective cone used by the other authors. Our paper shows that there are two main differences between our extension and the previous mentioned: the subgradient set to the distance function is an upper semi-continuous set-valued map and always has compact values, contained in the Clarke's normal cone. These facts will has remarkable consequences as it is shown in Section 3.

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*This work was partially supported by FONDECYT, ICM Complex Engineering System, CEE-ECOS.

An extension of the HBT to infinite dimensional Banach spaces was proved in Borwein and Jofré ([5]) where other applications and the relation with the extremal principle of Mordukhovich ([18], [21]) are also included.

One of the direct applications of the main theorem proved in this paper concerns mathematical economics, specifically about the Second Welfare Theorem (see [1], [4] and [10] for more details). In Section 5 we give a result which allows us to associate, in finite dimensional spaces, for each Pareto optimum point of a general nonconvex nontransitive economy, a nonzero vector price such that each consumer and producer will satisfy at this price, a first-order necessary condition involving the subgradient set to the distance function to the preferences and production sets. This result corresponds to an extension of the Second Welfare Theorem for nonconvex nontransitive economies.

Finally, applications to optimization problems, mainly the existence of singular multipliers for nonconvex optimization problems and its relation with Robinson's qualification condition ([26]), are also discussed in Section 6.

2. Preliminaries

Some key notions from Nonsmooth Analysis are introduced in this section. Let Z be a subset of $I\!R^\ell$. We will denote the boundary of Z by bdZ, its interior by $intZ$, its closure by clZ and its convex hull by coZ; $B(x_0, r)$ will represent the open unit ball of $I\!R^\ell$ with center $x_0 \in I\!R^\ell$ and radius $r > 0$, and given $p \in I\!R^\ell$, its transpose will be denoted by p^t. The *distance function* to Z is represented by $d_Z(\cdot)$ and the inner product in $I\!R^\ell$ by $\langle \cdot, \cdot \rangle$.

In this paper we will use the notion of *subgradient set* ([30]), which generalizes the derivative to the case of a mapping $f : I\!R^\ell \to \bar{I\!R} \equiv I\!R \cup \{+\infty\}$, $f \not\equiv +\infty$, only lower semicontinuous. For this notion, more accurate calculus rules have been recently developed in a series of papers. We refer to Ioffe ([12], [13]), Mordukhovich ([20], [23], [25]) and Rockafellar – Wets ([30]) for the sum, composition, maximum, Lagrange multipliers rules and so on.

Given $x I\!R^\ell$ such that $f(x)$ is finite, we will denote

$$f^-(x; v) = \liminf_{\substack{u \to v \\ t \to 0^+}} \frac{f(x + tu) - f(x)}{t}$$

the *subderivative* of f at x in the direction $v \in I\!R^\ell$ and

$$\partial^- f(x) = \{p \in I\!R^\ell \mid \langle p, v \rangle \leq f^-(x; v), \text{ for all } v \in I\!R^\ell\}$$

the set of *regular subgradients* of f at x. If $f(x) = +\infty$, we put $\partial^- f(x) = \phi$.

The *subgradient set* of f at the point $x \in I\!R^\ell$ is defined as

$$\partial f(x) = \limsup_{\substack{z \to x \\ f(z) \to f(x)}} \partial^- f(z).$$

An element of $\partial f(x)$ will be called a *subgradient* of f at x.

Remark 1. The subgradient set of a mapping f is always contained in the well-known Clarke's subdifferential $\partial_c f(x)$ and it is the smallest set that satisfies certain basic calculus rules coinciding with those for the convex subdifferential when f is convex ([12]). Moreover, if f is a locally Lipschitz mapping then $\partial_c f(x) = cl\{co\partial f(x)\}$.

Given $Z \subseteq I\!R^\ell$, we define the *indicator function* $\delta_Z(\cdot)$ of Z as

$$\delta_Z(z) = \begin{cases} 0 & \text{if } z \in Z \\ \infty & \text{if not} \end{cases}$$

and using this map we define the *normal cone* to Z at z as

$$N(Z, z) = \partial\delta_Z(z).$$

The normal cone has some interesting properties related to the product and other more complex operations over sets ([12]).

In our paper, we mainly use the following property of the subgradient set and the normal cone (see [30]).

Proposition 1. *Let Z, $\{Z\}_{j=1}^n$ be closed subsets of $I\!R^\ell$, $g : I\!R^\ell \to I\!R^m$ and $f : I\!R^m \to I\!R$ be a lower semi-continuous mapping.*

(a) If $z = (z_1, \dots, z_n) \in \prod_j Z_j$, then $N(\prod_j Z_j, z) = \prod N_j(Z_j, z_j)$.
(b) Given $z \in Z$ and an open convex neighborhood U of z, follows that

$$N(Z \cap U, z) = N(Z, z).$$

(c) If f attains a minimum at $\bar{x}$ then $0 \in \partial f(\bar{x})$.
(d) If f is continuously differentiable at $g(\bar{x})$ then

$$\partial(f \circ g)(\bar{x}) \subseteq \partial[\langle D, g(\cdot)\rangle](\bar{x}),$$

where D is the derivative of f at $g(\bar{x})$.

3. A nonconvex separation property

It is well-known that separation properties such as the HBT are among key ingredients in nonsmooth optimization and mathematical economics. Our aim in this section is to extend this property to a nonconvex setting. The main result of this section is Theorem 1 which is an extension of a property proved by Cornet and Rockafellar in 1989 ([9]).

Theorem 1. *Let $f : I\!R^{n\ell} \to I\!R^\ell$ be a locally Lipschitz mapping and $\{Z_j\}_{j=1}^n$ a family of closed and nonempty sets in $I\!R^\ell$. If $\bar{z} := (\bar{z}_1, \cdots, \bar{z}_n) \in Z_1 \times \cdots \times Z_n$ satisfies the condition $f(\bar{z}) \in bd[f(Z_1, \dots, Z_n)]$, then there exists $p \in I\!R^\ell$, $p \neq 0$, such that*

$$0 \in \partial(p^t f)(\bar{z}) + \prod_{j=1}^{n} \partial d_{Z_j}(\bar{z}_j),$$

and $0 < c \leq \|p\| \leq 1$, where c depends only on $\bar{z}_1, \dots, \bar{z}_n$.

Proof. Given $f(\bar{z}) \in bd\,[f(Z_1, \ldots, Z_n)]$, there exists a sequence $(u_k) \subset I\!R^\ell$ such that $(u_k) \to f(\bar{z})$, with $u_k \notin f(Z_1, \ldots, Z_n)$. Given $z := (z_1, \ldots, z_n)$, let us now consider the optimization problem $[P_k]$ which consists in minimizing f_k over $Z_1 \times \cdots \times Z_n$, where

$$f_k(z) = ||f(z) - u_k|| + \sum_{j=1}^{n} ||z_j - \bar{z}_j||^2.$$

From the fact that

$$f_k(z) \geq \phi(z) := \sum_{j=1}^{n} ||z_j - \bar{z}_j||^2$$

and $\lim_{\|z\| \to +\infty} \frac{\phi(z)}{\|z\|} = +\infty$, we can readily conclude the coercivity of f_k uniformly in $k \in I\!N$. Therefore, there exists $M > 0$ (which does not depend on u_k) such that for each $k \in I\!N$ there exists at least one solution $z^k := (z_1^k, \ldots, z_n^k) \in B(0, M) \cap \Pi_j Z_j$ for problem $[P_k]$.

Because of f_k is Lipschitz on the closed ball $cl\,B(0, 2M)$, if we denote by $L \geq 0$ its Lipschitz constant on this ball (that can be chosen independently of $k \in I\!N$ and be assumed strictly positive), from Clarke [7], Prop. 2.4.3, it follows that z^k is a minimizer of $f_k(\cdot) + L d_{B(0,M) \cap \Pi Z_j}(\cdot, \ldots, \cdot)$ over the ball $B(0, 2M)$.

From the first order optimality condition using subgradient set and the calculus rules for it ([12], Prop. 1.1) we obtain

$$0 \in [\partial f_k(z_1^k, \ldots, z_n^k) + \partial L d_{B(0,M) \cap \Pi Z_j}(z_1^k, \ldots, z_n^k) + N(B(0, 2M), (z_1^k, \ldots, z_n^k))],$$

and then, due to $N(B(0, 2M), (z_1^k, \cdots, z_n^k)) = \{0\}$, we can conclude that

$$0 \in \partial f_k(z_1^k, \ldots, z_n^k) + L \Pi_j \partial d_{Z_j}(z_j^k).$$

By applying the composition rule for the subgradient set to this case (Prop. 1 (d)) and considering that $\| \cdot \|$ is a continuously differentiable function around $f(z^k) - u_k$, we can infer that

$$\partial f_k(z_1^k, \cdots, z_n^k) \subseteq \partial(p_k^t f)(z_1^k, \ldots, z_n^k) + 2(z_1^k - \bar{z}_1, \ldots, z_n^k - \bar{z}_n),$$

where

$$p_k = \frac{f(z_1^k, \ldots, z_n^k) - u_k}{\|f(z_1^k, \ldots, z_n^k) - u_k\|}.$$

Let $\bar{p} \in I\!R^\ell$ be an accumulation point of (p_k). Without loss of generality we may assume that $(p_k) \to \bar{p}$, with $\bar{p} \neq 0$ due to for every $k \in I\!N$, $\|p_k\| = 1$. Let $\bar{L}$ be the Lipschitz constant of f on $B(0, 2M)$. Thus,

$$\partial(p_k^t f)(z^k) \subseteq \partial((p_k - \bar{p})^t f)(z^k) + \partial(\bar{p}^t f)(z^k) \subseteq \|p_k - \bar{p}\| \bar{L} B + \partial(\bar{p}^t f)(z^k),$$

and therefore

$$0 \in \|p_k - \bar{p}\|\bar{L}B + \partial(\bar{p}^t f)(z^k) + 2(z_1^k - \bar{z}_1, \ldots, z_n^k - \bar{z}_n) + L\prod_{j=1}^{n} \partial d_{Z_j}(z_j^k) \quad [*].$$

On the other hand, note that the optimal value v_k of $[P_k]$ satisfies

$$v_k = \|f(z^k) - u_k\| + \sum_{j=1}^{n} ||z_j^k - \bar{z}_j||^2 \leq \|f(\bar{z}) - u_k\|,$$

which implies $(v_k) \to 0$ when $k \to \infty$. Therefore, for every $j = 1, \ldots, n,\ (z_j^k) \to \bar{z}_j$, and from $[*]$ and the closedness of the subgradient set as a set-valued map, we deduce that

$$0 \in \partial(\bar{p}^t f)(\bar{z}) + L\prod_{j=1}^{n} \partial d_{Z_j}(\bar{z}_j).$$

Finally, considering that $\|\bar{p}\| = 1$, if we set $p \equiv \frac{1}{L}\bar{p}$ and $c = \frac{1}{L} > 0$ we obtain the desired result.

Remark 2. Wherever sets Z_j are not closed sets, follows that if for each $j = 1, \ldots, n$, $\bar{z}_j \in clZ_j$ and $f(\bar{z}_1, \ldots, \bar{z}_n) \in bd[f(clZ_1, \ldots, clZ_n)]$, from Theorem 1, exists $p \in I\!R^\ell,\ p$ not equal to zero, such that

$$0 \in \partial p^t f(\bar{z}_1, \ldots., \bar{z}_n) + \prod_{j=1}^{n} \partial d_{Z_j}(\bar{z}_j).$$

Remark 3. Given that f in Theorem 1 is locally Lipschitz, $\partial(p^t f)(\bar{z}) = D^* f(\bar{z})(p)$, where $D^* f(\bar{z})(p)$ is the *coderivative* of f at $\bar{z}$ in the direction $p \in I\!R^\ell$, that is, $D^* f(\bar{z})(p) := \{y \in I\!R^\ell \mid (y, -p) \in N(graph(f), (\bar{z}, f(\bar{z})))\}$ (we refer to [20], [21] and [30] for more details).

A particular case of Theorem 1 is when $f : I\!R^{n\ell} \to I\!R^\ell$ is *separable*, that is, there are $n \in I\!N$ mappings $f_j : I\!R^\ell \to I\!R^\ell$ such that for every $z_j \in I\!R^\ell,\ j = 1, \ldots, n$,

$$f(z_1, \ldots, z_n) = \sum_{j=1}^{n} f_j(z_j).$$

By abuse of language, f_j will be called a *component* of f. For this particular case, we have the following corollary.

Corollary 1. *Under the assumptions of Theorem 1 and assuming that f is separable and such that its components f_j are locally Lipschitz, then there exists $p \in I\!R^\ell,\ p \neq 0$, such that for all $j = 1, \ldots, n$*

$$0 \in \partial p^t f_j(\bar{z}_j) + \partial d_{Z_j}(\bar{z}_j).$$

Proof. From Theorem 1 exists $p \in I\!R^\ell,\ p \neq 0$, such that

$$0 \in \partial(p^t f)(\bar{z}_1, \ldots, \bar{z}_n) + \partial d_{\Pi Z_j}(\bar{z}_1, \ldots, \bar{z}_n).$$

Now on, from the fact that $\partial(p^t f)(z_1, \ldots, z_n) \subseteq \prod_j \partial(p^t f_j)(z_j)$ ([30]), we can readily conclude the assertion. □

In the smooth case a more explicit formula involving vector p can be elaborated as the following corollary show us.

Corollary 2. *(a) Let $f : I\!R^{n\ell} \to I\!R^\ell$ be a separable and continuously differentiable mapping with components $f_j : I\!R^\ell \to I\!R^\ell,\ j = 1, \ldots, n$, and let $Z_1, \ldots, Z_n$ be a family of closed and nonempty sets in $I\!R^\ell$. Given a point $(\bar{z}_1, \ldots, \bar{z}_n) \in \prod_j Z_j$ such that*

$$f(\bar{z}_1, \ldots, \bar{z}_n) = \sum_{j=1}^{n} f_j(\bar{z}_j) \in bd\ [f(Z_1, \ldots, Z_n)]$$

there exists $p \in I\!R^\ell,\ p \neq 0$, such that for every $j = 1, \ldots, n$

$$p^t Df_j(\bar{z}_j) \in \partial d_{Z_j}(\bar{z}_j),$$

where $Df_j(\bar{z}_j)$ is the Fréchet derivative of f_j at $\bar{z}_j$.

(b) Let $Z_j,\ j = 1, \ldots, n$, as above. Given a point $(\bar{z}_1, \ldots, \bar{z}_n) \in Z_1, \ldots, Z_n$ such that $\sum_j \bar{z}_j \in bd[\sum_j Z_j]$, there exists $p \in I\!R^\ell$, $p \neq 0$, such that

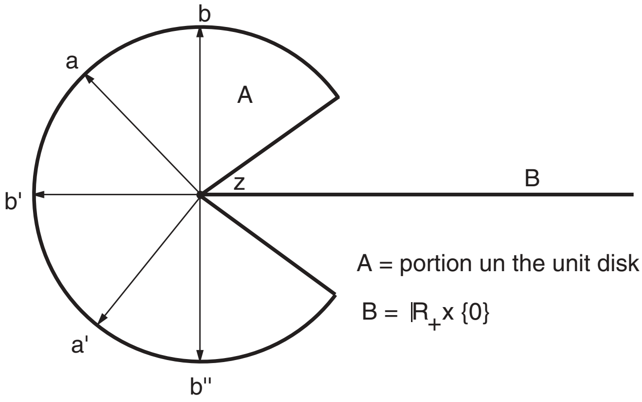
$$p \in \bigcap_j \partial d_{Z_j}(\bar{z}_j).$$

Proof. (a) Due to f_j is continuously differentiable, $\partial f_j(\bar{z}_j) = \{Df_j(\bar{z}_j)\}$ and thus the result follows directly from Corollary 1.

(b) This part is a direct consequence of part (a) when f_j is the identity function in $I\!R^\ell$. □

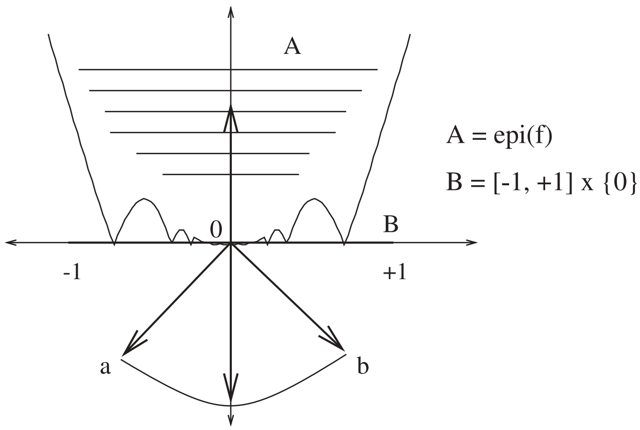
Remark 4. We point out that Corollary 2 (a) is equivalent to the extremal principle of Mordukhovich et al. ([18], [20], [23], [24], [25]) when $\partial d_{Z_j}(\bar{z}_j)$ is replaced by the larger set $N(Z_j, \bar{z}_j)$.

The following picture give us a geometrical interpretation of Corollary 2 (b).



From the picture above, we have that $z = (0,0) \in bd[A - B]$ and $\partial d_A(z) = \{a, a'\}$, $\partial d_B(z) = \{\text{semicircle defined by } \{b, b', b''\}\}$. Clearly $a \in \partial d_A(z)$ and $-a \in \partial d_B(z)$ (see Lemma 2).

Next picture provide us another example for our result. There, we consider $f : I\!R \to I\!R$, such that $f(x) = x^2 \left|\sin(\frac{1}{x})\right|,\ x \neq 0,\ f(0) = 0$.



When we compute the subgradient set to the distance function to $A = epi(f)$ (epigraph of f) at $(0,0)$ yields the portion of the unit ball enclosed by the vectors $a = \left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$ and $b = \left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$ and the subgradient set to the distance function to set B at the same point yields $\{(0,-1), (0,1)\}$. Clearly $0 \in bd[A + B]$ and $b = (0,-1) \in \partial d_A(0) \cap \partial d_B(0)$.

Remark 5. It is known ([27]) that given two lower semi-continuous and proper convex functions $f_1,\ f_2 : I\!R^\ell \to I\!R$, if $x \in dom(f_1) \cap dom(f_2)$ then $\partial(f_1 + f_2)(x) =$

$\partial f_1(x) + \partial f_2(x)$, provides that $0 \in int[dom(f_1) - dom(f_2)]$. If this last condition is not satisfied, follows that

$$0 \in bd[(cl\ dom(f_1) - \{x\}) - (cl\ dom(f_2) - \{x\})],$$

and from Corollary 2 (b) we conclude $N(domf_1, x) \bigcap N(domf_2, x) \neq \{0\}$. Thus, using Corollary 2 we can recover the well-known *alternative* property in Convex Analysis as it is detailed in [27].

We end this section with a stability type of result for our nonconvex separation property. To do so, let $f : I\!R^{n\ell} \to I\!R^{\ell}$ be a locally Lipschitz mapping defined on the product of closed sets $Z_1, \ldots, Z_n \subseteq I\!R^{\ell}$ and let $\bar{z}_s := (\bar{z}_{1,s}, \ldots, \bar{z}_{n,s}) \in Z_1 \times \cdots \times Z_n$, $s \in I\!N$, be a convergent sequence $((\bar{z}_s) \to \bar{z})$ such that for each $s \in I\!N$

$$f(\bar{z}_{1,s}, \ldots, \bar{z}_{n,s}) \in bd[f(Z_1, \ldots, Z_n)].$$

Finally, let (p_s) from Theorem 1 such that

$$0 \in \partial(p_s^t f)(\bar{z}_s) + \prod_{j=1}^{n} \partial d_{Z_j}(\bar{z}_{j,s}).$$

Lemma 1. *If $(p_s) \to \bar{p}$, then $\bar{p} \neq 0$ and*

$$0 \in \partial(\bar{p}^t f)(\bar{z}) + \prod_{j=1}^{n} \partial d_{Z_j}(\bar{z}_j).$$

Proof. Let us consider the optimization problem $\left[P_{k,s}\right]$ that consists of minimizing function $f_{k,s}(z_1, \ldots, z_n) = ||f(z_1, \ldots, z_n) - u_{k,s}|| + \sum_j ||z_j - \bar{z}_{j,s}||^2$ over the set $Z_1 \times \cdots \times Z_n$, where the sequence $(u_{k,s}) \to f(\bar{z}_{1,s}, \ldots, \bar{z}_{n,s})$ corresponds to the sequence $(u_k) \to f(\bar{z}_1, \ldots, \bar{z}_n)$ used in the proof of Theorem 1. From Theorem 1 we already know that there exists $0 < c_s$ such that $c_s \leq \|p_s\|$. Note that $c_s = \frac{1}{L_s}$, where $L_s > 0$ is the Lipschitz constant of $f_{k,s}$, which only depends on $\{\bar{z}_s\}$. Because of $f_{k,s}(z_1, \ldots, z_n) \geq \sum_j (||z_j||^2 - 2||z_j||||\bar{z}_{j,s}||)$, from the boundedness of $\bar{z}_s$, $s \in I\!N$, we can deduce that exists $r > 0$ (only depending on $\bar{z}$) such that for every $s \in I\!N$,

$$f_{k,s}(z_1, \ldots, z_n) \geq \sum_{j=1}^{n} (||z_j||^2 - 2r||z_j||).$$

Last fact implies that we can consider the Lipschitz constant of $f_{k,s}$ only depending of $\bar{z}$. Let $\bar{L}$ be this constant. In consequence, there exists $0 < \bar{d}$ such that for every $s \in I\!N$, $\bar{d} \leq c_s \leq \|p_s\|$, hence $\lim_s p_s := \bar{p} \neq 0$. Now, from the closedness of the subgradient set and the fact that

$$\partial(p_s^t f)(\bar{z}_s)) \subseteq \partial((p_s - \bar{p})^t f)(\bar{z}_s) + \partial(\bar{p}^t f)(\bar{z}_s) \subseteq \|p_s - \bar{p}\| \bar{L} B + \partial(\bar{p}^t f(\bar{z}_s)),$$

we can readily deduce the conclusion of Lemma 1. □

In what follows, we will consider a sequence of closed sets $Z_{i,k} \subseteq I\!R^\ell$, $i \in \{1,\ldots,n\}$, $k \in I\!N$. Let $z_{i,k} \in Z_{i,k}$ be such that $\sum_i z_{i,k} \in bd[\sum_i Z_{i,k}]$. In this case, from Corollary 2 (b), given $k \in I\!N$ there exists $p_k \in I\!R^\ell,\ p_k \neq 0$, such that $p_k \in \bigcap_i \partial d_{Z_{i,k}}(z_{i,k})$. Assuming that for each $i \in \{1,\ldots,n\}$, $(z_{i,k}) \to_k z_i$, our problem consists now in to find general conditions in order to guarantee the existence of a vector $p \in I\!R^\ell,\ p \neq 0$, satisfying $p \in \bigcap_i \partial d_{Z_i}(z_i)$, where Z_i is a "limit set" of the family $Z_{i,k}$.

Proposition 2. *Let $Z_{i,k}$ be a family of closed sets in $I\!R^\ell$ and $z_{i,k} \in Z_{i,k}$, $i \in \{1,\ldots,n\}$, $k \in I\!N$, such that $\sum_i z_{i,k} \in bd[\sum_i Z_{i,k}]$. If for every $i \in \{1,\ldots,n\}$*

$$\limsup_k \partial d_{Z_{i,k}}(z_{i,k}) \subseteq \partial d_{Z_i}(z_i),$$

then there exists $p \in I\!R^\ell,\ p \neq 0$, such that

$$p \in \bigcap_i \partial d_{Z_i}(z_i).$$

Proof. Let (p_k) satisfying the conclusion of Corollary 2 (b). Without loss of generality we may assume that p_k is convergent (we set $p \in I\!R^\ell$ as the limit). From Lemma 1 we deduce that $p \neq 0$ and then, from the hypotheses we can conclude that for each $i \in \{1,\ldots,n\}$, $p \in \partial d_{Z_i}(z_i)$. □

Some sufficient conditions which imply that $\limsup_k \partial d_{Z_{i,k}}(z_{i,k}) \subseteq \partial d_{Z_i}(z_i)$ are, for example ([2], [11]),

(a) the distance functions $d_{Z_{i,k}}(\cdot)$, $d_{Z_i}(\cdot)$ are convexe and $d_{Z_{i,k}}(\cdot)$ pointwise converges to $d_{Z_i}(\cdot)$ when $k \to \infty$,
(b) the sets $Z_{i,k}$ and Z_i are convexe and the family $\{Z_{i,k}\}$ converges to Z_i in the sense of Kuratowski-Painlevé[1],
(c) $Z_{i,k}$ and Z_i are convexe and $d_{Z_{i,k}}$ epi-converges to d_{Z_i}, that is, the convergence of the respective epigraphs in the sense of Kuratowski-Painlevé.
(d) the sequence of distance functions $d_{Z_{i,k}}$ satisfy both $epi - lim\ d_{Z_{i,k}} = d_{Z_i}$, $i \in \{1,\ldots,n\}$ and they are *equi-lower semidifferentiable*, which means that for every open set $V \subseteq I\!R^\ell$, $d_{Z_{i,k}}(y) \geq d_{Z_{i,k}}(x) + p^t(y-x) + o(\|x-y\|)$ holds true for each $k \in I\!N$, $x,\ y \in V$ and $p \in \partial^- d_{Z_{i,k}}(x)$ (we refer to [11] for more details).

4. Formula to compute the subgradient set to the distance function in special cases

The main objective of this section is to provide a formula that help us to compute the subgradient set to the distance function to a set defined by equalities and inequalities of functions. To do so, in order to obtain a "reasonable" formula for this subgradient, we are enforced to assume some conditions on the mappings that define our set. Thus, let $g : I\!R^m \to I\!R^n$ be a mapping such that for all $y \in I\!R^m$, $g(y) = (g_1(y),\ldots,g_n(y)) = (g_i(y))$, where g_j, $j = 1,\ldots,n$, are real-valued functions. Following definition was taken from [8] and [26].

[1] That is, $\limsup_k Z_{i,k} := \{x \in I\!R^\ell \mid \exists\, (x_{k'}) \to x,\ x_{k'} \in Z_{i,k'}\} = \liminf_k Z_{i,k} := \{x \in I\!R^\ell \mid \forall\, k \in I\!N\ \exists\, x_k \in Z_{i,k},\ (x_k) \to x\}$.

Definition 1. *Function* $g : I\!R^m \to I\!R^n$ *satisfies the metric regularity condition at* $y_0 \in I\!R^m$ *with respect to* $D \subseteq I\!R^n$ *if* $g(y_0) \in D$ *and if there exist* $k > 0$ *and* $\epsilon > 0$ *such that for all* $y \in B(y_0, \epsilon)$

$$d_{g^{-1}(D)}(y) \le k \cdot d_D(g(y)),$$

where $g^{-1}(D)$ *denotes the inverse image under* g *of* $D \subseteq I\!R^n$.

Remark 6. We recall that if g is continuously differentiable, the Robinson constraint qualifications condition ([8], [26]) implies the metric regularity as before. Specifically, if $D = -I\!R^p_+ \times \{0_{I\!R^q}\}$, with $n = p+q$, sufficient conditions to guarantee the metric regularity condition are, for example, the *Mangasarian - Fromovitz* constraint qualification condition and the *Slater* condition in the convex case.

Proposition 3. *If* $g : I\!R^m \mapsto I\!R^n$ *is a locally Lipschitz function satisfying the metric regularity condition at* $y_0 \in I\!R^m$ *with respect to* $D \subseteq I\!R^n$*, with constant* $k > 0$*, then*

$$\partial d_{g^{-1}(D)}(y_0) \subseteq \bigcup_{y \in \partial d_D(g(y_0))} k\, \partial(y \circ g)(y_0).$$

Proof. From the metric regularity condition and the subderivative concept, it follows that for some $\epsilon > 0$

$$d^-_{g^{-1}(D)}(y; h) \le k(d_D \circ g)^-(y; h),$$

for every $h \in I\!R^m$ and $y \in B(y_0, \epsilon) \cap g^{-1}(D)$. Hence, from the definition of the subgradient set,

$$\partial d_{g^{-1}(D)}(y_0) \subseteq k\partial(d_D \circ g)(y_0)$$

and then, applying the chain rule for the subgradient set to the above relationship ([12], Corollary 5.3) we obtain the desired result. □

Coming back to our initial motivation for this section, let $Y \subseteq I\!R^m$ be a set defined by equalities and inequalities of mappings, that is,

$$Y = \{y \in I\!R^m \mid g_i(y) \le 0,\ i = 1, 2, \ldots, p;\ g_{p+j}(y) = 0,\ j = 1, 2, \ldots, q\},$$

with $g_i : I\!R^m \to I\!R$, $i = 1, 2, \ldots, p+q$. Set $n = p+q$ and $G : I\!R^m \to I\!R^n$ with $G(y) = (g_i(y)) \in I\!R^n$. Given all foregoing, we point out that

$$Y = G^{-1}(D)$$

with $D = -I\!R^p_+ \times \{0_{I\!R^q}\}$.

Corollary 3. *Suppose that G is locally Lipschitz and satisfies the metric regularity condition at* $y_0 \in I\!R^m$ *with respect to* $D = -I\!R^p_+ \times \{0_{I\!R^q}\}$*, with constant* $k > 0$*. In such case, we have that*

$$\partial d_Y(y_0) \subseteq k \bigcup_{y \in [0,1]^p \times [-1,1]^q} \sum_{i=1}^{n} \partial[y_i g_i](y_0).$$

Proof. First of all, note that

$$\partial d_D(G(y_0)) = \prod_{i=1}^{p} \partial d_{\{-I\!R_+\}}(g_i(y_0)) \times \prod_{i=p+1}^{p+q} \partial_{\{0\}}(g_i(y_0)) = [0,1]^p \times [-1,1]^q$$

and then, from Proposition 3 follows that

$$\partial d_{G^{-1}(D)}(y_0) \subseteq k \bigcup_{y \in [0,1]^p \times [-1,1]^q} \partial \sum_{i=1}^{n} [y_i g_i](y_0),$$

and therefore

$$\partial d_{G^{-1}(D)}(y_0) \subseteq k \bigcup_{y \in [0,1]^p \times [-1,1]^q} \sum_{i=1}^{n} \partial [y_i g_i](y_0),$$

which ends the demonstration. □

Finally, we conclude this section with the following lemma, which will be useful in Section 5.

Lemma 2. *Given a subset $Y \subseteq I\!R^\ell$ and $y_0 \in I\!R^\ell$, we have that*

$$\partial d_{-Y}(y_0) = -\partial d_Y(-y_0),$$

which implies that $N(-Y, y_0) = -N(Y, -y_0)$.

Proof. Given $y_0 \in I\!R^\ell$ and $Y \subseteq I\!R^\ell$, note that $d_Y(-y_0) = d_{-Y}(y_0)$. Now, from the fact that $d_Y(-y_0) = [d_Y \circ -Id](y_0)$, applying the composition rule ([13], Theorem 4.3) to this case, we can deduce that

$$\partial d_Y(-y_0) = \bigcup_{p \in \partial d_Y(-y_0)} \partial [p \circ -Id](y_0) = -\partial d_Y(-y_0),$$

and then, due to $N(Y, y_0) = cl\{I\!R_+ \partial d_Y(y_0)\}$, from the relationship just proved for the distance function we can readily conclude the assertion. □

5. A mathematical economics application

Let us consider an economy with ℓ goods, m consumers and n producers. We denote by $X_i \subseteq I\!R^\ell$ the *consumption set* of individual $i \in I = \{1, \ldots, m\}$. For each *consumption bundle* $(\bar{x}_1, \ldots, \bar{x}_m) \in \prod X_i$, $P_i(\bar{x}) = P_i(\bar{x}_1, \ldots, \bar{x}_n) \subseteq X_i$ will represents those elements of X_i strictly preferred to $\bar{x}_i$ by the consumer $i \in I$ and $cl P_i(\bar{x}) = cl P_i(\bar{x}_1, \ldots, \bar{x}_m)$ those elements preferred or indifferent to $\bar{x}_i$ by this individual. Thus $P_i : \prod X_k \mapsto X_i$ is a set-valued map which generalizes the *preorder of preferences* and the *utility function* concept on X_i for the respective consumer. It is important to recall that any hypotheses of transitivity have been assumed on the preferences correspondence (for more details on the model, see [4], [10]).

The production set of a firm $j \in J = \{1, \ldots, n\}$ is represented by a nonempty set $Y_j \subseteq I\!R^\ell$, and finally $w \in I\!R^\ell$ denotes the total initial endowments of resources[2].

We define an economy $\mathcal{E}$ as

$$\mathcal{E} = ((X_i), (P_i), (Y_j), w).$$

A *feasible allocation* for the economy $\mathcal{E}$ will be a point $((\bar{x}_i), (\bar{y}_j)) \in I\!R^{\ell m} \times I\!R^{\ell n}$ such that:

(a) $\bar{x}_i \in X_i$, for all $i \in I$; $\bar{y}_j \in Y_j$, for all $j \in J$;
(b) $\sum_{i \in I} \bar{x}_i - \sum_{j \in J} \bar{y}_j = w$.

A feasible allocation $((\bar{x}_i), (\bar{y}_j)) \in I\!R^{\ell m} \times I\!R^{\ell n}$ for the economy $\mathcal{E}$ is said to be a *Pareto optimum* if *there is no other feasible allocation* $((x_i), (y_j))$ *such that for every* $i \in I$, $x_i \in cl\,P_i(\bar{x})$, *and for some* $i_0 \in I$, $x_{i_0} \in P_{i_0}(\bar{x})$ ([4]).

With very general hypotheses on the economy $\mathcal{E}$ is possible to prove that for each Pareto optimum point there exists a nonzero vector price which "decentralizes" it, that is, a vector price that together the Pareto optimum distribution conforms an equilibrium point for the economy (see [4], [10] and [15]). This result is known as the *Second Welfare Theorem.*

In what follows, using our main result, we will prove a general version of the Second Welfare Theorem. To do that, we will introduce the so called "Asymptotically Included Condition" ([15]). We say that the economy $\mathcal{E} = ((X_i), (P_i), (Y_j), w)$ satisfies the *Asymptotically Included Condition* at the point $((\bar{x}_i), (\bar{y}_j)) \in I\!R^{\ell m} \times I\!R^{\ell n}$ if there exists $i_0 \in I$, $\epsilon > 0$ and a nontrivial sequence $(h_k) \to 0$ such that for a large enough $k \in I\!N$ we have that

$$-h_k + \Sigma_i cl\,P_i(\bar{x}) \cap B(\bar{x}_i, \epsilon) - \Sigma_j Y_j \cap B(\bar{y}_j, \epsilon) \subseteq P_{i_0}(\bar{x}) + \Sigma_{i \neq i_0} cl\,P_i(\bar{x}) - \Sigma_j Y_j.$$

Some hypotheses which imply this condition are, for example (see [15] for more details),

(a) for some $i_0 \in I$, $P_{i_0}(\bar{x})$ is a closed set,
(b) for some $i_0 \in I$, $P_{i_0}(\bar{x})$ is convex with nonempty interior,
(c) there exists a nontrivial sequence $(h_k) \to 0$ such that for some $i_0 \in I$ and $k \in I\!N$ (sufficiently large), $-h_k + cl\,P_{i_0}(\bar{x}) \subseteq P_{i_0}(\bar{x})$,
(d) there exists $i_0 \in I$ such that for every $x \in cl\,P_{i_0}(\bar{x})$, $x + I\!R^\ell_{++} \subseteq P_{i_0}(\bar{x})$.

Theorem 2. *Let* $((\bar{x}_i), (\bar{y}_j)) \in I\!R^{\ell m} \times I\!R^{\ell n}$ *be a Pareto optimum point for economy* $\mathcal{E} = ((X_i), (P_i), (Y_j), w)$ *such that for each* $i \in I$, $\bar{x}_i \in cl\,P_i(\bar{x})$. *If the Asymptotically Included Condition holds true, then there exists a nonzero vector price* $\bar{p} \in I\!R^\ell$ *such that*

$$\bar{p} \in \bigcap_{j=1}^{n} \partial d_{Y_j}(\bar{y}_j) \qquad -\bar{p} \in \bigcap_{i=1}^{m} \partial d_{cl\,P_i(\bar{x})}(\bar{x}_i).$$

[2] If each individual $i \in I$ is initially endowed with resources $w_i \in I\!R^\ell$, then $w = \sum_i w_i \in I\!R^\ell$.

Proof. From the Pareto optimum definition and the Asymptotically Included Condition, we can readily deduce (see [15] for more details) that for some $\epsilon_0 > 0,\ w \in bd\Gamma(\epsilon_0)$, with

$$\Gamma(\epsilon_0) := \sum_{i=1}^{m} cl P_i(\bar{x}) \cap B(\bar{x}_i, \epsilon_0) + \sum_{j=1}^{n} -Y_j \cap B(\bar{y}_j, \epsilon_0).$$

Given that, Theorem 2 can be obtained as a direct consequence of Corollary 2 (b) and Lemma 2 applied to sets $cl P_i(\bar{x}) \cap B(\bar{x}_i, \epsilon_0),\ i \in I$, and $Y_j \cap B(\bar{y}_j, \epsilon_0),\ j \in J$. □

6. Optimization application: the Robinson qualification condition

In this section we show an easy way to prove the existence of singular multipliers for an optimization problem. To do so, let $K \subseteq I\!R^m$ be a closed set and consider the minimization problem $[P]$ defined by

$$\begin{cases} \min\ f(x) \\ G(x) \in K \end{cases}$$

where $f : I\!R^\ell \to I\!R$ and $G : I\!R^\ell \to I\!R^m$ are continuously differentiable mappings. For this optimization problem, the Robinson qualification condition ([26]) at an optimal solution $x \in I\!R^\ell$ is

$$0 \in int[G(x) + DG(x)I\!R^\ell - K], \tag{1}$$

where $DG(x)I\!R^\ell$ is the linear space generated by the derivative DG of G at x. This regularity condition coincides with the classical Mangasarian - Fromovitz condition when K is defined by equalities and inequalities. When K is a convex set, the Robinson condition implies the existence of a Karush - Kuhn - Tucker multiplier vector ([3]), that is, a vector $p \in I\!R^m$, $p \neq 0$, such that $p \in N(K, G(x))$, $G(x) \in K$ and $DL(x, p) = 0$, where $L(x, p) = f(x) + p^t G(x)$ is the Lagrangian function of problem $[P]$.

Now we analyze the case when (1) is not satisfied. For this purpose, we define $Z_1 = G(x) - K \subseteq I\!R^m$ and $Z_2 = DG(x)I\!R^\ell$. Since x is a feasible point and $DG(x)I\!R^\ell$ is a linear space, we have that $0 \in Z_1 \bigcap Z_2$ and then, if (1) is not true, follows that

$$0 \in bd[Z_1 + Z_2].$$

In such case, from Corollary 2 (b) holds that there exists $p \in I\!R^m,\ p \neq 0$, such that

$$p \in N(G(x) - K, 0) \qquad\qquad p \in N(DG(x)I\!R^\ell, 0).$$

From Lemma 2 and considering a translation of the normal cone $N(G(x) - K, 0)$ in $-G(x)$ we can deduce that $-p \in N(K, G(x))$ and also, from the fact that $p \in N(DG(x)I\!R^\ell, 0)$ we infer that $p^t DG(x) = 0$. Thus, with all foregoing we have proved the following proposition.

Proposition 4. *Let us consider the optimization problem* $[P]$ *with the assumptions given above. If* x *is an optimal solution of* $[P]$ *then, either*

(a) $0 \in int[G(x) + DG(x)\mathbb{R}^\ell - K]$ *and there exists* $p \in \mathbb{R}^m$, $p \neq 0$, *such that* $p \in N(K, G(x))$, $G(x) \in K$ *and* $DL(x, p) = 0$, *where* $L(x, p) = f(x) + p^t G(x)$ *(existence of a Karush - Kuhn - Tucker multiplier vector for problem* $[P]$*),*
or

(b) $0 \notin int[G(x) + DG(x)\mathbb{R}^\ell - K]$ *and there exists* $p \in \mathbb{R}^m$, $\|p\| = 1$, *such that* $-p \in N(K, G(x))$ *and* $p^t DG(x) = 0$.

Remark 7. (a) Condition $-p \in N(K, G(x))$ is the complementary slackness condition for problem $[P]$. Note that if K is convex, for every $k \in K$, $\langle p, G(x) - k\rangle \leq 0$, which implies that when K is a cone with its vertex at 0, $\langle p, G(x)\rangle = 0$.
(b) The condition $\langle DG(x), p\rangle = 0$ shows us the existence of a singular multiplier for problem $[P]$.
(c) Proposition 4 could be also obtained for more general functions f, G. For that, see, for example, Ioffe ([12]), Jourani and Thibault ([16], [17]), Mordukhovich and Shao ([23], [24], [25]) and Rockafellar ([29]).

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