On the use of the Weibull and the normal cumulative probability models in structural design

M. Elgueta [a,*], G. Díaz [b], S. Zamorano [a], P. Kittl [a]

[a] Departamento de Ingeniería Mecánica, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Beauchef 850, Santiago, Chile
[b] Departamento de Ciencia de los Materiales, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Beauchef 850, Santiago, Chile

Abstract

In this paper, the Weibull and the normal cumulative probability are discussed and compared in relation to the analysis of the fracture toughness K_{IC}. It is shown that, in the normal distribution, K_{IC} can take negative or zero values, causing complications in structural design. In the Weibull distribution, this does not happen and its lower limit K_{ICL} can be determined without problems. So, the Weibull cumulative probability represents reality in a better form. Finally, comparing experimental data with both distributions, it is found that the Weibull probability is more economics in terms of structural design.

1. Introduction

In this work, the Weibull and the normal cumulative probability are analyzed in order to determine which of them represents better the reality. In [1], it is shown that for current values of the cumulative probability, between 0.01 and 0.99, both distributions represent correctly the experimental data (this is true also for the lognormal distribution). However, as it will be seen, the Weibull distribution represents better the reality when the probability values are very low, as those used in the engineering field.

It is important to emphasize that the physical basis of the Weibull statistics in fracture is the random flaw size distribution in the volume of the sample. The connection between the Weibull statistics and the brittle fracture by crack propagation is clearly established [2–4] and the following correspondences can be established:

- The sample volume with the crack length L.
- The cracks with small areas that facilitate or prevent the crack growth.
- The rupture tension with K_{IC}.
- Finally, the boundary conditions of the Weibull cumulative probability of failure, F, are:
  $F \to 0$ when $L \to 0$ and $F \to 1$ when $L \to \infty$

On the contrary the normal distribution does not have a plausible or adequate physical meaning.

2. Theoretical comparison

With the above considerations, it is possible to express the cumulative probability of fracture toughness K_{IC} in mode I as [5,6]:

$$F(K_{IC}) = 0 \quad \text{if } 0 \leqslant K_{IC} \leqslant K_{ICL}$$

$$F(K_{IC}) = 1 - \exp\left[-\frac{L}{L_0}\left(\frac{K_{IC} - K_{ICL}}{K_{IC0}}\right)\right]^m \quad \text{if } K_{ICL} < K_{IC} < \infty \tag{1}$$

In expression (1), K_{ICL} is the lower limit of K_{IC}, L_0 is the unity length and m and K_{IC0} are parameters depending of material and manufacture process. K_{IC} depends on the

* Corresponding author. Tel.: +56 2 9784543; fax: +56 2 6988453/6896057.
E-mail address: melgueta@ing.uchile.cl (M. Elgueta).

number of small zones that facilitate or prevent the crack growth, their size and their distribution. When $K_{ICL} \to 0$, relationship (1) is transformed in

$$F(K_{IC}) = 1 - \exp\left[-\frac{L}{L_0}\left(\frac{K_{IC}}{K_{IC0}}\right)\right]^m \quad \text{with } 0 \leqslant K_{IC} < \infty \tag{2}$$

Only in this case, K_{IC} can be zero.

The normal cumulative probability is

$$F(K_{IC}) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{K_{IC} - \overline{K}_{IC}}{\sqrt{2} \cdot \Delta K_{IC}}\right) \tag{3}$$

In this expression $\overline{K}_{IC}$ represents the mean value of K_{IC} and ΔK_{IC} is the standard deviation.

It is necessary to emphasize that $-\infty \leqslant K_{IC} \leqslant +\infty$ in relationship (3). This means K_{IC} can take negative values. In addition,

$$F(0) = \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{\overline{K}_{IC}}{\sqrt{2} \cdot \Delta K_{IC}}\right) \tag{4}$$

that means there is a percentage of samples with $K_{IC} = 0$.

In order to compare both distributions, $\ln(K_{IC})$ was plotted versus $\ln(\ln(1/1 - F))$. In the case of the Weibull distribution, from expression (2), a straight line is obtained whose equation is

$$\ln\left(\ln\left[\frac{1}{1-F}\right]\right) = m\ln(K_{IC}) + \ln\left(\frac{L}{L_0 \cdot (K_{IC0})^m}\right) \tag{5}$$

From (3), a more complicated expression is obtained for the normal distribution. With the purpose of comparing more precisely, the parameters of the normal distribution were calculated as if they were equivalent to those of the Weibull distribution. For this, the following relations were used [2]:

$$\overline{K}_{IC} = K_{IC0} \cdot \Gamma\left(1 + \frac{1}{m}\right)$$
$$\Delta K_{IC} = K_{IC0} \cdot \left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)\right]^{1/2} \tag{6}$$

In order to simplify the comparison, it has been considered $L = L_0$ and $K_{IC0} = 1$ in relationships (5) and (6). Fig. 1

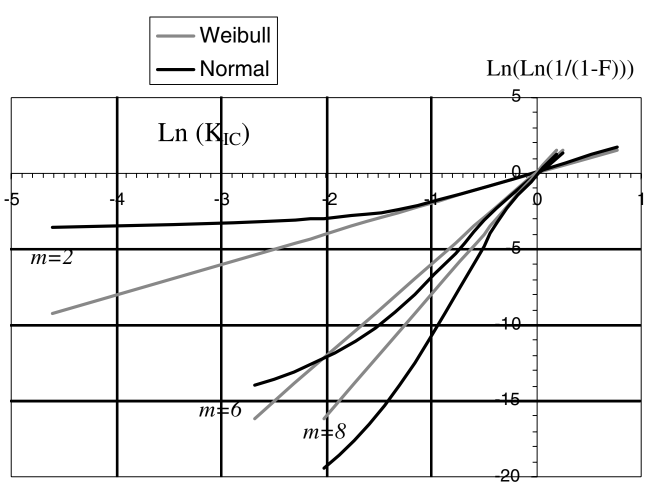

Fig. 1. Comparison between the two distributions ($m = 2$; $m = 6$; $m = 8$).

Table 1
Zhang and Knott data for steel A533B

Material	Austenitization temperature (°C)	Condition	$\overline{K}_{IC}$ (MPa$\sqrt{m}$)	ΔK_{IC} (MPa$\sqrt{m}$)
1	1250	100% β	32.23	2.41
2	1250	100% α'	89.62	6.19
3	1250	30% β + 70% α'	61.10	20.1
4	950	100% β	45.44	2.98
5	950	100% α'	92.00	4.60
6	950	30% β + 70% α'	54.61	5.97

Table 2
Parameters of Weibull calculated by simulation from Zhang and Knott data

Material	$K_{ICL} \pm \Delta K_{ICL}$ (MPa$\sqrt{m}$)	$m \pm \Delta m$	$K_{IC0} \pm \Delta K_{IC0}$ (MPa$\sqrt{m}$)
1	21.30 ± 2.65	5.09 ± 1.26	11.90 ± 2.88
2	79.66 ± 2.89	1.62 ± 0.84	11.29 ± 3.97
3	31.41 ± 5.20	1.35 ± 0.42	33.46 ± 9.79
4	24.80 ± 2.95	8.06 ± 1.94	21.88 ± 3.15
5	84.38 ± 2.40	1.80 ± 0.93	8.70 ± 2.74
6	49.91 ± 4.05	2.70 ± 0.95	16.62 ± 4.73

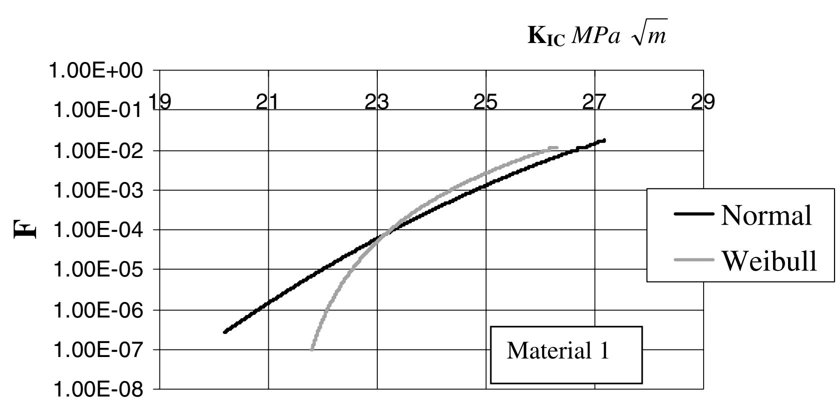

Fig. 2. Comparison between the two distributions, for material 1.

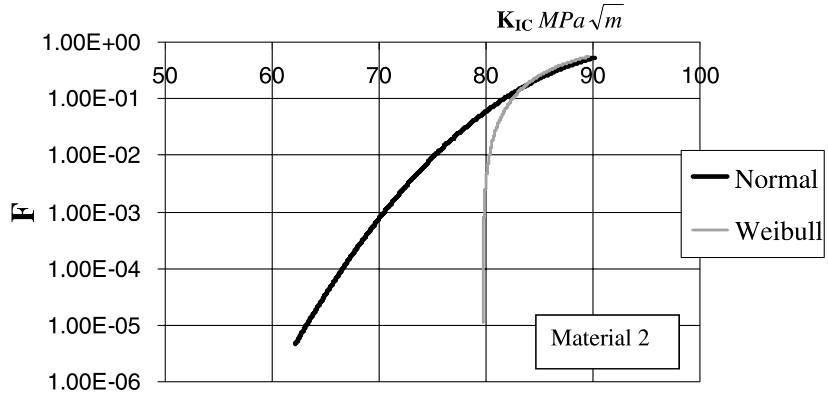

Fig. 3. Comparison between the two distributions, for material 2.

shows the results. It is observed that the normal distribution tends to a horizontal asymptote, so that this distribution cannot be used for small values of F. It would obtain K_{IC} too small, even zero, especially if the value of m is small. The Weibull distribution does not present this problem.

3. Experimental comparison

With the experimental data obtained by Zhang and Knott [7] (shown in Table 1), the correlation

$$\{K_{ICi}\} \leftrightarrow \left\{F_i = \frac{i - 1/2}{N}\right\} \quad \text{with } i = 1, 2, \ldots N \tag{7}$$

and expression (5) allows to determine the values of K_{IC0}, m and K_{ICL} and its dispersions by a Monte Carlo simulation with three parameters [8]. In (7), N is the number of tests. All these parameters, for the six materials used in [7], are shown in Table 2.

Table 3
Different K_{IC} for the Weibull and the normal distribution

Material	F	K_{ICw} ($MPa\sqrt{m}$)	K_{ICn} ($MPa\sqrt{m}$)	F_n for $K_{ICn} = 0$	K_{IC} for $F_n = F_w = F$ ($MPa\sqrt{m}$)
1	10^{-4}	23.246	23.301	$0 < F_n < 10^{-15}$	$K_{IC} = 23.2$
	10^{-5}	22.536	21.991		$F = 9.72 \cdot 10^{-5}$
	10^{-6}	22.085	20.807		
2	10^{-4}	79.696	66.606	$0 < F_n < 10^{-15}$	$K_{IC} = 84.0$
	10^{-5}	79.666	63.228		$F = 1.83 \cdot 10^{-1}$
	10^{-6}	79.659	60.208		
3	10^{-4}	31.442	−13.7	1.193×10^{-3}	$K_{IC} = 40.9$
	10^{-5}	31.413	−24.65		$F = 1.67 \cdot 10^{-1}$
	10^{-6}	31.407	−34.5		
4	10^{-4}	31.804	27.336	$0 < F_n < 10^{-15}$	There is not intersection
	10^{-5}	30.068	25.726		
	10^{-6}	28.763	24.284		
5	10^{-4}	84.431	74.88	$0 < F_n < 10^{-15}$	$K_{IC} = 88.6$
	10^{-5}	84.393	72.366		$F = 2.38 \cdot 10^{-1}$
	10^{-6}	84.383	70.716		
6	10^{-4}	40.459	32.42	$0 < F_n < 10^{-15}$	$K_{IC} = 47.6$
	10^{-5}	40.143	29.17		$F = 1.16 \cdot 10^{-1}$
	10^{-6}	40.009	26.25		

In Figs. 2 and 3, F_n (normal distribution) and F_w (Weibull distribution) are plotted in function of K_{IC} for materials 1 and 2 studied by Zhang and Knott [7]. From these figures, it stands out that for usual values of F ($F < 10^{-4}$), the Weibull distribution is more economic (in design terms) because, for a fixed value of F, a greater value of K_{IC} is obtained.

Table 3 presents the values of K_{IC}, for Weibull distribution K_{ICw} and normal distribution K_{ICn}, for different values of F. In addition, it is shown the value of F_n corresponding to $K_{ICn} = 0$ and the values of K_{IC} for which both distributions have the same value ($F_n = F_w = F$). Table 3 shows clearly, for the six materials studied in [7], that for usual values of F, the value of K_{ICn}, corresponding to the normal distribution is smaller, which means, in design terms, that it is less economic.

4. Conclusions

In relation to the differences between the Weibull and the normal statistics, it is necessary to point out that:

- The Weibull distribution has a clear physical meaning connected with small areas that facilitate or prevent the cracks propagation, considering their sizes and their distributions.
- The normal distribution presents a cumulative probability of failure for negative and zero values of K_{IC}. This does not have a physical meaning.
- The Weibull distribution is more economic than the normal distribution because, for usual values of F ($F < 10^{-4}$), the parameter K_{IC} obtained with the Weibull distribution is greater and consequently the dimensions of a structural element are smaller.

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