A random bidding and supply land use equilibrium model

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Abstract

A model of the static equilibrium in the real estate market is studied in this paper and a solution algorithm is proposed. Consumers and real estate suppliers are assumed to have idiosyncratic differences, which are described by the discrete choice theory with random-utility and profit-behavior. Consumers are differentiated into clusters by socioeconomic characteristics while suppliers are differentiated by production technology. Consumer behavior is subject to budget constraints (with fixed income), a quasi-linear utility function, and is affected by neighborhood externalities (households) and agglomeration economies (firms). The suppliers' behavior is restrained by zone regulations and affected by scale and scope economies. Total households by cluster are exogenous and total supply fits total demand. Real estate combined with location options is assumed to be discrete and differentiated, then transactions are modeled as auctions. Equilibrium prices are the result of auctions and the market clearing condition. These conditions generate a discontinuous non-linear location equilibrium problem where convexity is obtained by applying the MNL logit in all decisions. The equilibrium is described by a fixed-point equation system that, under some specified conditions, has a unique and stable solution; such conditions are analytically defined and represent a minimum degree of choice dispersion in the model. A fixed-point algorithm is proposed to solve the equilibrium, along with the conditions that assure convergence. This real estate equilibrium model can be applied to the highly complex reality of urban environments at a *relatively* low computational cost.

1. Introduction

The accumulated research in studies of urban land use shows the system's high degree of complexity when all interactions amongst consumers and suppliers are included, generating a complex non-linear mathematical problem. Some complexity is due to the diversity of agents with different and interdependent behavior. Households, for example, behave differently in relation to socioeconomic attributes, whereas firms behave differently in relation to economic activity and business size; all of these interact through various forms of location externalities. Additionally, real estate options provided by suppliers are differentiated goods, distinguished by dwelling attributes as well as by their location. At the same time, production costs are subject to scale and

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scope economies. Finally, the state plays a strong role by imposing a variety of regulations that affect the market.

In previous research, (Martínez, 2000), this problem has been partially studied with the following set of assumptions:

- Urban land is a quasi-unique good bought and sold in an auction-type market.
- Real estate options are discrete and differentiated units defined by the location zone and building type.
- The available location is assigned to the best bidder.
- Bids are made by households and firms that compete for real estate options.
- Consumers' bids are assumed to be random variables, thus considering the idiosyncratic nature of the agents.
- Consumers' bids are assumed mutually interdependent as their location choices define the neighborhood quality for residences and the agglomeration economies for non-residents.
- The supply side is represented by a time-series deterministic model that predicts the number of supply units by zone and building type based on price and previous stock.
- Equilibrium is achieved when all consumers find a location somewhere in a static framework.

These assumptions describe a competitive auction market, where the prices are defined endogenously by an auction mechanism as consumers simultaneously find a location through the rule of the best bidder. The explicit modeling of location externalities, namely neighborhood quality and agglomeration economies generates equilibrium conditions described by a non-linear mathematical problem. This model is called the Random Bidding Model (RBM) and has been developed into operational software using a specific solution algorithm, MUSSA, currently applied to the city of Santiago (Martínez and Donoso, 2001).

In this paper we develop a new model of equilibrium that extends the RBM model. The principal modification is the adoption of new assumptions about how supply agents behave in the competitive market, in this case by adding idiosyncratic variability among them. This variability is modeled by introducing randomness in the supply side, within the approach of static equilibrium. The new-static and maximum profit-logit model replaces MUSSA's times-series sub-model of deterministic supply. This requires, however, the study of some important issues. The first is that the new supply model does not predict absolute values for prices and rents, but rather relative ones. The second issue is how to impose land development regulations on the supply model. Third is that the change in the mathematical form of the equilibrium model requires the study and development of a specific solution algorithm.

The new model is called Random Bidding and Supply Model (RB&SM), which has the advantage of offering better consistency in the behavior of all the agents in the system. It also takes better advantage of the mathematical structure generated by logit models. Most importantly, the new supply model can incorporate the effects of economies of scale and scope in the real estate market, which are predominantly responsible for densification and sprawl tendencies. Thus, real estate supply is driven by the microeconomic optimal behavior of real estate producers, in contrast with previous models that react, using an econometric model, to the surplus of previous periods without representing suppliers' behavior explicitly (see Simmonds, 1999; Waddell, 2002; Wegener, 1985).

The main assumption of this model is that the land use system attains a partial static equilibrium. It is partial because transports cost are assumed exogenous. Some urban specialists may argue that a static equilibrium is unrealistic because building infrastructure takes time, which introduces delays in the supply side. Additionally, a number of land use modelers have abandoned the equilibrium paradigm in favor of a dynamic dis-equilibrium framework, implemented by (micro to meso) simulation models. However, there are several points to make on the merit of the equilibrium approach. First, for forecasting the city development at a time horizon sufficiently large (compared to the building delay), the assumption that supply is able to closely match demand is plausible. Second, even in a dynamic framework some definition of equilibrium is conceivable to represent the market forces. See for example Martínez and Hurtubia (2006) where the demand side of the RB&SM and a dynamic supply model are used to attend a notion of equilibrium with excess of demand or supply. Third, the equilibrium model, particularly under the closed form of the RB&SM, can be used efficiently to formulate optimization tools for urban planning (Martínez and Aguila, 2004). Finally, equilibrium conditions provide

relevant and consistent information on the formation of prices, while non-equilibrium models use econometric functions representing hedonic rents that bear no consistency with the demand-supply process over time or with the evolution of the regulatory context.

The model is described in Section 2. The solution algorithm and an analysis of performance are presented in Section 3. The application of the model is discussed in Section 4 and conclusions are summarized and discussed in Section 5.

2. Model formulation

2.1. Assumptions

The urban land market performs an imperfect competition because unique combinations of building and location attributes differentiate goods (real estate properties). This characteristic arises from the fact that an urban location is valued for what surrounds it (neighborhood, parks, infrastructure, etc.), which cannot be reproduced by a production process. For this reason, the market behaves like an auction, which is studied in the rent or urban economic theory developed initially by Alonso (1964). Under this condition, goods go to the highest bidder, with bids representing consumers' willingness to pay. Thus, the consumers' behavior is modeled by their bid function instead of by their utility. This valuation function includes attributes that describe the complex interaction amongst consumers, namely the location externalities, which constitutes another argument for market imperfection, and at the same time introduces great analytical complexity to the model.

In our model, we apply the Random Utility Theory developed for the case of auctions by Ellickson (1981) and Martínez (1992), and for the real estate profit maximization by Anas (1982). In practice this implies that consumers (who choose their best location) and suppliers (who choose their best offer) will be modeled under the assumption that their behavior function – bid and profit respectively – follows a random distribution; therefore decisions are represented by probabilities. Additionally, in this model the set of discrete location options is identified not only by zone, but also by building type, which form a finite set of alternatives that constitutes the discrete space where those who supply and those who demand make their decisions. Finally, although the model can be defined as dis-aggregated at the level of each agent and location, in practice aggregated versions are used, where consumers are classified into socio-economically homogeneous categories (index $h = (1, \ldots, \bar{h})$) and the supply is described by location or zone (index $i = (1, \ldots, \bar{i})$) and property type (index $v = (1, \ldots, \bar{v})$).

2.2. The auction model

The variable that describes consumers' behavior in the market is their bids, representing their willingness to pay (WP) for the location to achieve a certain level of utility. The theory that relates both functions, utility and bids, shows that the latter represents the expenditure function (in all goods except location). The bid function is derived inverting – in the property price – the indirect utility function conditional on the location (Solow, 1973; Rosen, 1974; Martínez, 1992).

This theory yields the following relationship between the indirect utility function V and bids for real estate: $B_{hvi} = I_h - V_h^{-1}(z_{vi}, P, U_h, \beta_h)$, with I_h the household income, P the price vector for goods and β_h the utility taste parameters; z_{vi} is the vector of attributes that describes the real estate and its location. It is possible to demonstrate that for bid functions thus defined, the location where the agent is the highest bidder is that of the maximum surplus or maximum utility (Martínez, 1992; Martínez, 2000), which assures that the auction outcome, yields an allocation consistent with maximum utility behavior of consumers.

In the RB&SM model, equilibrium is attained for P and β_h taken as exogenous parameters (hence hidden in what follows). Additionally, we assume a quasi-linear consumer's utility function, that is $U_h(x, z) = \alpha_0 x_0 + U_h'(x_{-0}, z)$, being linear only in one good (x_0) of the goods vector. Although this assumption imposes a theoretical limitation on the model, in practice it has a small impact, as we only require the utility to be linear in one dimension of the large vector of consumption goods and location attributes. The inverse of the indirect quasi-linear utility function conditional on location attributes, yields the following bid function: $B_{hvi} = I_h - $

$f(U_h) - g(z_{vi})$. This additive assumption introduces significant benefits in calculating the land use equilibrium, allowing the model to find solutions to a complex non-linear urban equilibrium. Using these assumptions, we use the following generic consumers' bid function:

$$B_{hvi} = b_h^1 + b_{hvi}^2((P_{\cdot/\cdot i})_{hv}, (S_{\cdot i})_v) + b^3 \tag{1}$$

where the bid components are:

- b_h^1: defined as $b_h^1 = I_h - f(U_h) - b^3$, it adjusts utility levels to attain equilibrium.
- b_{hvi}^2: defined as $g(z_{vi})$ describes the valuation of property attributes. Some attributes are exogenous to the location process, such as rivers, parks, and hills. These attributes, then, are represented by zone attractive parameters in this term. The more complex attributes, however, are those endogenous in the model that describe location externalities and are defined by two types of variables. First the distribution of agents in the zone, given by the probability vector $(P_{\cdot/\cdot i})_{hv}$,[1] that describes attributes like neighborhood quality by the socio-economic characteristics of agents located in the zone. Second, the building stock available, $(S_{\cdot i})_v$, that describes the built environment in the zone.
- b^3: is a constant term, independent of consumers and supply options, which adjusts bids to absolute levels in the whole market. This component is only relevant in the calculation of absolute values for rents.

In the case of firms (non-residential activities), their WP function is derived from the profit function for each economic sector or industry, assuming homogeneous intra-sector behavior. In this case, it is also assumed that the bid function is additive, as in (1).

To include the behavior variability produced by idiosyncratic differences between consumers within a cluster, bids are assumed to be random variables, denoted by $\widetilde{B}_{hvi} = B_{hvi} + \varepsilon_{hvi}$, with random terms ε_{hvi} which are assumed identical, independent (IID) and distributed Gumbel. This distribution is justified by Ellickson (1981) noting that it is consistent with the maximum bidding process of the auction, where only the maximum bid within a cluster is relevant for the auction. The choice of this distribution also has important practical consequences in the solution algorithm. From these assumptions, the (multinomial) probability that one of the $\overline{H}_h$[2] agents type h is the highest bidder in (v, i), conditioned on the supply being available, is given by

$$P_{h/vi} = \frac{\overline{H}_h \exp(\mu B_{hvi})}{\sum_g \overline{H}_g \exp(\mu B_{gvi})} \tag{2}$$

where the parameter μ is inversely proportional to the variance of the bids. Here the aggregated version of the multinomial logit probability is utilized, which includes the correction for different sizes among agents' clusters, as proposed by McFadden (1978).

Future versions of this model may drop the IID assumptions to allow for different degrees of correlation between random terms. Although these versions could be developed based on available research on discrete-demand models, further research will be required to generate the suitable equilibrium algorithm for each alternative model specification.

Thus, replacing (1) in (2), the auction model is

$$P_{h/vi} = \frac{\overline{H}_h \exp(\mu(b_h^1 + b_{hvi}^2((P_{\cdot/\cdot i})_{hv}, (S_{\cdot i})_v)))}{\sum_g \overline{H}_g \exp(\mu(b_g^1 + b_{gvi}^2((P_{\cdot/\cdot i})_{gv}, (S_{\cdot i})_v)))} \tag{3}$$

where b^3 is cancelled out, implying that location probabilities depend on relative bids across bidders, not on their absolute values.

In a synthetic form this is written as the

$$\text{Location fixed point} \quad P_{h/vi} = P_{h/vi}((b_{\cdot}^1)_h, (P_{\cdot/\cdot i})_{hv}, (S_{\cdot i})_v) \quad \forall h, v, i \tag{4}$$

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[1] Vectors notation: $(x_{\cdot j})_k = (x_{kj}, \forall k = (1, \ldots, \bar{k}))$, then $(P_{\cdot/\cdot i})_{hv} = (P_{h/vi}, \forall h \in (1, \ldots, \bar{h}), v \in (1, \ldots, \bar{v}))$ and $(S_{\cdot i})_v = (S_{vi}, \forall v \in (1, \ldots, \bar{v}))$.
[2] Notation: Overlined variables denote exogenous information required by the model.

which shows that the probability variable is present both in the right and left sides of an unsolvable equation. This is the mathematical description of the interdependence between consumer decisions, i.e. location externalities, in which the location of an agent depends on locations of other agents (households and firms) in the same zone, and on the built environment represented by the real estate supply in the zone.

As a result of the auction, the rent of a real estate is determined by the expected value of the highest bid, which, according to the Gumbel distribution, is the known logsum-or implicit value- function given by

$$r_{vi} = \frac{1}{\mu} \ln \left(\sum_g \overline{H}_g \exp(\mu B_{gvi}) \right) + \frac{\gamma}{\mu} \tag{5}$$

with γ the Eulers's constant. This expression can be expressed by two terms, only for the benefit of exposition, as

$$r_{vi} = \frac{1}{\mu} \ln \left(\sum_g \overline{H}_g \exp(\mu(b_g^1 + b_{gvi}^2)) \right) + b^3 + \frac{\gamma}{\mu} = \bar{r}_{vi} + b^3 \tag{6}$$

Then, the rent depends on bids B_{hvi}, therefore, in on all other variables in the model.

We observe that Eq. (6) is a hedonic function, since rents are built as the contribution of consumers' valuation of attributes. However, in contrast with econometric hedonic functions that estimate the average valuation of consumers for each attribute, this equation aggregates the differentiated consumers' valuation in a way consistent with the underlying bidding process and with the constraints on consumers and suppliers behavior that defines the regulated context. Thus, this rent Eq. (6) considers the changing conditions of the market system.

2.3. Supply model

This section contains the main theoretical extension to the original MUSSA model proposed in this paper. In previous research Martínez and Roy (2004) studied the economic supply problem as a chain of market processes, from agricultural landowner, to land developers and real estate developers. Here, we shall concentrate on the final real estate supply. The rest of the chain is hidden, but it may be incorporated explicitly by expanding this model with Martínez–Roy's model.

2.3.1. Producers' behavior

The behavior of real estate suppliers consists of deciding what combination of building and zone (v, i) would generate the maximum profit, subject to prevailing market regulations. The profit function is defined as the difference between the rent (r_{vi}) that will be obtained from a supply option and its production cost (c_{vi}), including land, construction and maintenance cost items. Then the total profit yield by S_{vi} sold units is

$$\pi_{vi} = S_{vi}(r_{vi} - c_{vi})$$

There are some theoretical aspects to analyze in designing the supply model. One is the assumption of heterogeneity of profit across different zones, caused by differences in market conditions, information, and mobility of resources. Indeed, urban markets are highly regulated by zoning regulations, defined both by zone and building type, hence it is plausible that profits may differ by sub-markets – defined by (v, i) – at equilibrium.

A second important aspect is the heterogeneity of suppliers, which occurs when they have different profit functions, because, for example, their size may determine their access to technology affecting fixed costs. The model should, therefore, permit different profit functions by types of developer (clusters), thus $c_{vi} = (c_{vij}, j = (1, \ldots, \bar{j}))$, with $\bar{j}$ the total number of developers.

Another theoretical aspect is the level of profit aggregation that the supplier maximizes. In the presence of scale economies (intra sub-market economies), denoted as $c_{vij} = c_j(S_{vij})$, each supplier j should define the optimal production level S_{vij} by maximizing profit in each sub-market independently. In the presence of economies of scope (inter sub-market economies), rational behavior must consider the use of a more complex strategy looking for an optimum combination set of supply options in all sub-markets. Thus, the more general case includes full interdependency in cost, denoted as $c_{vij} = c_j((S_{\ldots})_{vij})$, where production cost depends on what

is built everywhere by every builder. Less complex interdependencies are, of course, likely to occur in real markets. For example suppliers might be concerned only with their own costs.

Then, the general problem of the jth supplier, including economies of scale and scope, may be written as

$$\text{Max}_{S_{\cdots j}} \quad \pi_j = \sum_{vi} S_{vij}(r_{vi}((S_{\cdots})_{vij}) - c_j((S_{\cdots})_{vij}))$$

$$\text{s.a.} \quad S_{vij} \in (R_i, T_{vij}) \quad \forall v, i, j \tag{7}$$

$$\sum_{vi} S_{vij} = S_j \quad \forall j$$

The set of restrictions indicates that supply must comply with the set of regulations at each zone (R_i) and is constrained by the technology available to the building sector (T_{vij}). The second set of constraints is optional, with the role of providing exogenous information about each supplier's market share S_j.

To develop an operational supply model, we recall Anas (1983) to combine two methodologies: the maximum entropy approach and the logit approach. The model is developed using the entropy method in Appendix A. With regard to the logit approach, it should be noted that in the suppliers' problem (7), rents are random variables, hence profits are also random. Moreover, the Gumbel distribution under maximization preserves the same scale parameter, therefore, rents distributions have the same μ parameter of the bid functions defined above. Thus in a model with deterministic costs, profits would be Gumbel distributed IID with the same scale factor as the bids.

The RB&SM makes the following assumptions to obtain an operational supply model. We assume the developers' profits as independent, identical and Gumbel distributed variables with scale parameter λ-not necessarily equal to the demand model parameter μ-in order to consider unobserved random effects on costs or other missing variables. Thus, the expected number of residential supply units, type (v, i), supplied by developer j, denoted S_{vij}, is given by the multinomial probability (P_{vij}) that this unit type is the maximum profit option for the $j's$ developer in the industry. This is

$$S^*_{vij} = SP_{vij} = S \frac{\exp \ \lambda(\tilde{\pi}_{vij} - \rho_j)}{\sum_{v'i'j'} \exp \ \lambda(\tilde{\pi}_{v'i'j'} - \rho_{j'})} \tag{8}$$

where λ is inversely proportional to the profit variance and S is the total number of units supplied by all developers in all building types. ρ_j is the parameter that adjusts the solution to the developer's j share of the market. This result is derived from the entropy approach in Appendix A.

Some issues of the supply model (8) should be commented upon. This model is more limited than the general formulation in (7), as it assumes the following additive condition: $\text{Max}\pi_j = \sum_{vi} S_{vij}\text{Max}\pi_{vij}$, which holds only if individual profits π_{vij} are independent across real estate sub-markets, a condition also assumed in the distribution of profits. Such a condition hardly holds if the technology of the building industry generates economies of scope, because in this case building costs, denoted as $c_{vij} = c_j((S_{\cdots})_{vij})$, are explicitly dependent on the mixture of production in sub-markets. Nevertheless, it is theoretically possible to model such dependency – completely or at least partially – by the cost function itself, leaving a small correlation between the random terms, which are then yield independent as we have assumed. In the applied field, however, it would be wise to explore more complex logit-model structures than the multinomial, which will remain for future research because it affects the equilibrium algorithm discussed below. Another technical point is that similar to the location probability, b^3 is again cancelled out, implying that what matters in the suppliers' choice are relative prices, not their absolute values. This is plausible because the total supply S is given exogenously for this equation, then Eq. (8) distributes total supply across alternative options requiring only relative prices.

Note that π (and also ρ) are dependent on supply, through location externalities in rents and because of scale economies in the production costs. Then, the reduced form of the supply model is

$$\text{Supply fixed point} \quad S_{vi} = S \cdot P_{vi}((b^1_{\cdot\cdot})_h, (P_{\cdot/\cdot\cdot})_{hvi}, (S_{\cdots})_{vi}) \tag{9}$$

with $P_{vi} = \sum_j P_{vij}$, which represents the fixed point equation of the non-linear supply model. It is important to note that this model does not yet include the constraints imposed in (7) to represent zoning regulations (R). Technology restriction (T), however, may be included in the cost function of profits.

2.3.2. Planning regulations

Modeling regulations is a fundamental feature of a land use model, especially to make it applicable as a design tool for zoning plans. In the RB&SM model, we incorporate linear regulations of the following form: $\sum_v a_{vi}^k S_{vi} \leqslant R_i^k$, where the coefficients a^k – associated with the kth restriction – and the restrictions' values R_i are all exogenous parameters of the model. Of course their linear form limits the diversity of regulations that can be considered, but the linear form is sufficient for the great majority of actual urban regulations.

In order to incorporate linear regulations in the model, we build upon the bulk of research on these types of problems, especially on entropy models. See for example the model proposed recently by Martínez and Roy (2004). From our analysis in Appendix A we define the following "suppliers' optimal behavioral function" that maximizes profit subject to comply with the set K of regulations in the city, with K_i regulations in each zone i:

$$\tilde{\pi}_{vij} = \pi_{vij} + \pi'_{vij} - \sum_{k \in [1, \dots, K_i]} \gamma_i^k a_{vi}^k$$

with $\pi'_{vij} = \sum_{v'i'} S_{v'i'j} \frac{\partial \pi_{v'i'j}}{\partial S_{vij}}$, a term that adjusts suppliers' behavior to take into account the dependency of profits on scope economies of production, i.e. benefits of producing in several locations simultaneously.

Let us interpret the last term in the behavioral function containing the γs' parameters, which are Lagrangean multipliers that adjust supply to zone regulations at each zone. We know that the values of these parameters are $\gamma_i^k \geqslant 0 \; \forall i, k$, taking the value zero when the corresponding constraint is not binding. These parameters have a relevant practical economic interpretation: they represent the marginal profit obtained by suppliers if the corresponding regulation is marginally relaxed, usually called "shadow prices." Thus, these parameters can be used as an index to assess each regulation by their impact on the economy. This shadow price is, in fact, an increase in the production cost of the real estate supply, produced by an increase in the land price input factor; hence, it represents a capitalization on land prices of a monopoly power generated by the regulation of a scarce land.

A relevant observation in calculating gamma parameters is that, in the context of the set of regulations included, only one constraint (denoted below as $\bar{k}$) is actually binding at each zone, whose corresponding parameter is denoted by γ_i and called the "binding parameter." Then,

- $\gamma_i^k = 0, \; \forall (k \neq \bar{k} \in K_i); \; \gamma_i^{\bar{k}} = \gamma_i \geqslant 0;$ and $\gamma_i a_{vi}^{\bar{k}} = \sum_{k_i \in K} \gamma_i^{k_i} a_{vi}^k.$
- $\gamma_i = \max_{k \in K_i} \gamma_i^k$; the binding parameter is the superior of all parameters' values in each zone.
- $\sum_v a_{vi}^k S_{vi}(\gamma_i) < R_i^k \; \forall k \neq \bar{k}$, i.e. the binding parameter γ_i assures that all non binding constraints hold, so their respective parameters can be assumed equal to zero.

The implication is that the behavior function is then $\tilde{\pi}_{vij} = \pi_{vij} + \pi'_{vij} - \gamma_i a_{vi}^{\bar{k}}$ and the number of parameters needed to be calculated is not equal to the number of constraints, but to the much smaller number of zones. Nonetheless, an efficient algorithm is required to be able to identify the binding regulation without calculating all γ parameters.

The calculation of the set of binding parameters $\gamma_i \; \forall i$, may be performed applying the known MART algorithm (see Appendix A), but we use the fixed point method using the following expression:

$$\text{Regulation fixed point} \quad \gamma_i^{\bar{k}} = \frac{1}{\lambda a_{0i}^{\bar{k}}} \ln \left[\frac{S}{R_i^{\bar{k}}} \sum_{vj} a_{vi}^{\bar{k}} \exp \lambda (\pi_{vij} + \pi'_{vij} - \gamma_i^{\bar{k}} (a_{vi}^{\bar{k}} - a_{0i}^{\bar{k}}) - \rho_j - \hat{\pi}_j) \right]$$

$$\text{with} \quad \hat{\pi}_j = \frac{1}{\lambda} \ln \left[\sum_{v'i'j'} \exp \lambda (\pi_{v'i'j'} + \pi'_{v'i'j'} - \rho_{j'} - \gamma_{i'}^{\bar{k}} a_{v'i'}^{\bar{k}}) \right]$$

(10)

which is derived in Appendix A. Notice that $\gamma \geqslant 0$, which complies with a theoretical condition for Lagrangean multipliers. The more practical advantage of this method is that Eq. (10) has the known logsum expression that has a simple iterative solution procedure (see below).

2.4. Equilibrium

Here we study the auction-Walrasian equilibrium of the land use market. There are several alternative specifications of equilibrium for the urban land use market, from that stating that every agent is located somewhere, i.e. demand is thus satisfied but supply may not be fully used, to a more demanding version in which demand and supply are equal. All these options are represented by

$$\sum_{vi} S_{vi} P_{h/vi} \geqslant \overline{H}_h \quad \forall h \tag{11}$$

in which equilibrium is verified for each consumer category h and for all simultaneously. The RB&SM considers the equality case, which represents the most used static equilibrium criteria. The inequality condition, also referred to as dis-equilibrium, leads to a dynamic formulation of this model that is beyond the scope of this paper (for a dynamic model see Martínez and Hurtubia, 2006). The equality condition is met if b^1 verifies that

$$b_h^1 = -\frac{1}{\mu} \ln \left(\frac{1}{\overline{H}_h} \sum_{vi} S_{vi} \exp(\mu(b_{hvi}^2 - \bar{r}_{vi})) \right) \quad \forall h \tag{12}$$

which is obtained solving (11) for b_h^1 under equality. As $\bar{r}_{vi}$ depends on the bids in (6), then Eq. (12) can be written in a reduced form as

$$\text{Equilibrium fixed point} \quad b_h^1 = b_h((b^1)_h, (P_{./..})_{hvi}, (S_{..})_{vi}) \tag{13}$$

which constitutes another fixed point, this time in vector b^1 whose solution verifies equilibrium conditions. As expected, this fixed point has the same logsum functional form as the γ's Eq. (10), because the equilibrium condition also represents a set of linear constraints.

To interpret the term b^1 recall from Section 2.2 that we defined it as $b_h^1 = I_h - f^1(U_h) - b^3$, which "adjusts utility levels to attain equilibrium"; this adjustment can be done solving the equilibrium fixed point Eq. (12). Note that this result is based on our assumptions of an additive bid function, that b^3 is not dependent on index h and that income is fixed in our model. Then it follows that all adjustments on b^1 represent the (negative) monetary equivalent adjustments in clusters' utility levels: the higher the bid for a location, the lower the utility obtained (all location attributes held constant). Thus, the values obtained from (12) represent a monatry index of the utilities attained by agents at equilibrium. As expected, ceteris paribus and neglecting second order effects caused by non-linearities, this index increases with $\overline{H}_h$, therefore utility decreases with population because the supply is more demanded. Conversely, more supply increases utility, while higher rents have the opposite effect. This index also has a direct and very useful application for the evaluation of a land use system, because it represents the consumers' benefit, or the compensating variation (Martínez, 2000), associated to an urban scenario under equilibrium.

2.5. Additional analysis: absolute rents and expenditures in goods

The model is based on the specification of a WP function or bids that should meet the restriction of household income. It is also desirable, that bids reproduce absolute values of rents. Here we comment on methods, complementary to the core RB&SM model, that comply with these conditions.

The values of bids and rents previously defined are relative values within the model until the term b^3 is identified. One method to obtain this parameter is to assume that total supply S depends on absolute values of rents and on external macroeconomic variables (X). Then

$$S = S(r, X), \quad r = f(r_{vi}, \forall v, i) \tag{14}$$

where X is a vector of macroeconomic indices and r is an aggregate rent index, for example, the average rent or the maximum rent index of the city. Both X and r are assumed dependent on the country's economy rather than the city's internal economy, such that (14) reflects a relation of investors' behavior in the capital market

(see Di Pacuale and Wheaton, 1996). Having (14) calibrated, it can be solved for b^3 because in static equilibrium S is known (equal to the total population of agents) and rents are given by Eq. (6).

Another method commonly used is to define a relationship between the absolute rent of land at the city periphery and the price of an external agricultural zone m, r_m. Then,

$$r_m = \bar{r}_m + b^3 \tag{15}$$

with $\bar{r}_m$ given by evaluating the rent model (6) for location m, which yields b^3 directly. It is recommended to choose the approach which uses the best empirical information available.

From either method we have an expression given by

$$b^3 = b^3(b^1_., P_{./..}, S_{..}) \tag{16}$$

that allows the calculation of absolute values of rents and bids. It is important to emphasize that from the point of view of the RB&SM model, b^3 does not alter the equilibrium solution in $(b^1_., P_{./...}, S_{...})$, it only affects the absolute values of the rents and bids. But, of course, this occurs principally due to the multinomial logit form of probabilities used in the RB&SM model.

Additionally, the model must meet the agents' income restrictions, which are:

$$(Px)_{hvi} + r_{vi} \leqslant I_h, \quad \forall h \text{ located at } (v, i) \tag{17}$$

where x is the vector of consumers goods, which we assume continuous, and P its corresponding price vector. If we also assume that the information is adequate, the auction is capable of extracting the maximum value possible from consumers' willingness to pay, then Eq. (17) must hold for equality:

$$(Px)_{hvi} = I_h - \bar{r}_{vi} + b^3 \tag{18}$$

This expression allows us to estimate the level of consumption of goods differentiated by location and cluster, which constitutes interesting information that directly and explicitly, links location and consumption. This implies that a differential in rents between two locations induces a differential expenditure in goods that is exactly compensated, on expected – not necessarily actual – values, by an equivalent differential in the utility associated with the respective location amenities (attributes).

Nevertheless, both Eqs. (15) and (18) should be used cautiously because total expenditure in any location should also be restricted to being non-negative, which is not necessarily complied by these equations.

3. The equilibrium solution

In this section we analyze how to solve the problem of static equilibrium described by the set of equations above. Here we propose a solution algorithm and analyze its properties.

3.1. System of equations

The static equilibrium of the urban land use system is represented by the simultaneous solution of the previous set of equations, which together can be written as the following multi-fixed-point problem:

$$\begin{aligned}
P_{h/vi} &= P_{h/vi}((b^1_.)_h, (P_{./\cdot i})_{hv}, (S_{\cdot i})_v, \gamma_i) \quad \forall h, v, i \\
S_{vi} &= S \cdot P_{vi}((b^1_.)_h, (P_{./\cdot\cdot})_{hvi}, (S_{..})_{vi}, (\gamma_\cdot)_i) \quad \forall v, i \\
b^1_h &= b_h((b^1_.)_h, (P_{./\cdot\cdot})_{hvi}, (S_{..})_{vi}, (\gamma_\cdot)_i) \quad \forall h \\
\gamma_i &= \gamma_i((b^1_.)_h, (P_{./\cdot\cdot})_{hvi}, (S_{..})_{vi}, (\gamma_\cdot)_i) \quad \forall i
\end{aligned} \tag{19}$$

which is a system of dependent non-linear equations with dimension $[(\#h + 1)(\#v\#i) + (\#h) + (\#i)]$, with the same number of unknown variables. This system is complemented with the equation for absolute bids and rents:

$$b^3 = b^3((b^1_.)_h, (P_{./\cdot\cdot})_{hvi}, (S_{..})_{vi}, (\gamma_\cdot)_i) \tag{20}$$

that does not intervene in the solution of system (19). The solution vector is $(b^1, P, S, \gamma, b^3)^*$, from which we can calculate the bids, location patterns, rents and profits.

Note that the fixed point equations system (19) embeds a highly complex mathematical problem, potentially with multiple solutions. This complexity may be reduced significantly by means of lagging interactions, for example the location externalities effect, assuming that consumers choose location at year t based on zone attributes of the previous year. In this case Eq. (3) reduces to $P^t_{h/vi} = P_{h/vi}((b^1_\cdot)_h, (P^{t-1}_{\cdot/\cdot i})_{hv}, (S^{t-1}_{\cdot i})_v)$ $\forall h, v, i$, which is no longer a fixed point equation. Another simplification is to eliminate the supply fixed point problem (Eqs. (9) and (9)) by making the supply side a dynamic model, where supply is decided some time before it becomes available in the market, as in Martínez and Hurtubia (2006). Therefore, the RB&SM model represents the most complex land use equilibrium problem because it incorporates all interactions in a simultaneous equilibrium.

3.2. Properties of fixed points

The mathematical properties of each of the fixed-point equations in system (19) are analyzed in Appendix B, obtaining conditions for each individual equation separately. For the multinomial logit probabilities (auctions or supply), we conclude that the fixed-point solution is unique and the fixed-point algorithm converges to the solution, if and only if, the probabilities remain within the lower and upper bounds identified in Appendix B. These bounds mean that consumers' bids and supply profits should not approach a deterministic (all or nothing) distribution. This conclusion follows from the fact that these bounds depend on the Gumbel distributions' scale factors of bids and profits (inversely relayed with their dispersion parameters) and also on the changes of bid and rent variables. It should be noted that these bounds are not fixed, because bids and rents are variables in the algorithm, then bounds have to be recalculated in each iteration of the algorithm.

The other two fixed points, associated with equilibrium and linear regulations, have the logsum expression, although they are different because regulation constraints have parameters a. In both cases we conclude that the solution is unique and the fixed-point-iteration algorithm converges to the solution. There are cases, however, where this conclusion does not hold, the most significant case being some conditions that should hold in the regulations fixed-point iterations, which should be checked in the algorithm.

Thus, each fixed-point equation converges (subject to specific conditions) to a unique solution. Global convergence has only being proved by numerous empirical tests that show that convergence is attained, provided that convergence conditions for each fixed point do apply.

Existence and convergence to a unique solution is largely caused by the probabilistic formulation of choices of all agents, which transforms the complex non-linear system (19) into a convex optimization problem. Additionally, the use of a multinomial logit form for all choices defines the specific solution and explains the high performance observed in the calculation of the solution.

It is important to understand, however, that the unique solution attained by the model is conditional on the set of calibrated parameters (see Section 4), because this implies two relevant conditions. First, the model reproduces the observed system because parameters are estimated to reproduce the observed distributions of consumers and supply. Secondly, following the property of the multinomial logit, the forecasts of choice probabilities can be equivalently expressed as an incremental probability calculated as a pivot point estimate of the observed distribution. The implication of this property is very important because the model solution represents the expected solution conditional on the observed city represented by the set of calibrated parameters. Thus, the forecasted city is not obtained from scratch but from the city's history.

Finally, it is worth commenting on the robustness of the model. The model is sensitive to behavioral and scale's parameters, but the solution changes smoothly with these parameters, except for instability mentioned above for some scale parameters. This makes the model highly robust to specification differences. What justifies this feature is the fact that the mathematical problem solved by the model's algorithm is continuous and differentiable[3] in all the system equations.

---

[3] Differentiability only fails on γ's parameters when the binding constraint changes from one to another within the same zone and between iterations. However, the simulations made did not show any instability.

3.3. Solution algorithm

The algorithm to solve the equation system (19) and the analysis of the solution's sensitivity to model parameters are presented below. It should be noted that in spite of the assumptions made (separability of bid functions in Eq. (1) and the simplification for regulation parameters in Eq. (10)), the high complexity of the non-linear system of equations implies that there are no general solution tools, moreover, it is known that the form given to the functions involved is crucial in the behavior of any solution algorithm. In our case, this affects the functional form assumed for bids (specifically terms b_{hvi}^2) and production costs in each specific application of the model. It is also known from experience that in large complex problems, the most efficient and robust algorithms are those that take advantage of the structure of the equations involved. In our model, this is highly important because all the equations are derived from the Gumbel distribution, which defines a unified mathematical structure for the whole problem. Indeed, the first two equations in (19) are multinomial logit formulae, while the last two are logsum formulae. Therefore, the following algorithm is only valid for the above specification of the RB&SM model.

The main solution algorithm is the following:

Define the generic vector

$$x = (x_j, j \in (1, 2, 3, 4); \quad x_1 = (b_h^1, \forall h), \quad x_2 = (P_{h/vi}, \forall hvi), \quad x_3 = (S_{vi}, \forall vi), \quad x_4 = (\gamma_i, \forall i))$$

Call Simplex First Stage procedure: to assure that the regulation set is feasible and to find a point in the interior.

Initialize: $n = 0$, $m = 0$

Iterate the equation system:

(4.1) $n = n + 1$, $t = 1$, $j = 1$.
(4.2) if $j = 4$ then (if $m = 0$ $x_4 = 0$; $m = 1$ call Binding).
(4.3) if $j = 2, 3$ call Bounds.
(4.4) $x^t = (x_j(x_j^{t-1}, x_k = \bar{x}_k \forall k \neq j), \bar{x}_k \forall k \neq j)$.
(4.5) if $|x_{jl}^t - x_{jl}^{t-1}| > e_j \forall l$, $t = t + 1$, go to 4.2.
(4.6) $\Delta x_j = x_j^t - \bar{x}_j, \bar{x}_j = x_j^t$.
(4.7) if $j < 4$ $t = 1$, $j = j + 1$, go to 4.2.

Global convergence:

(5.1) if $\Delta x = (|x_j(\bar{x}_j, \forall j) - \bar{x}_j| > e_j, \forall j)$, go to 4.1.
(5.2) if $m = 1$ stop, print $x* = (\bar{x}_j, \forall j)$; $n = 0$, $m = m + 1$ go to 4.1.

The bounds procedure:

(i) Calculate upper bounds $f_U = f_U(\theta; x^{t-1})$ and lower bounds $f_L = f_L(\theta; x^{t-1})$.
(ii) If $f_L < x_j^{t-1} < f_U$ go to 4.4; stop, print "no convergence".

The binding procedure:

(i) Starting values: $\gamma^0 = \{\gamma_i^k = 0, \forall i, k\}$.
(ii) Constraints' evaluation:

$$\Delta_i = \min_{k \in K_i} \left(R_i^k - \sum_{v'} (a_{v'i}^k S_{v'i}) \right); \quad k_i = \arg\min_{k \in K_i} \left(R_i^k - \sum_{v'} (a_{v'i}^k S_{v'i}) \right); \quad \bar{\gamma}_i = (\gamma_{k_i}, \forall i)$$

Iterations $n = 1$:

(iv) if $\Delta_i \leqslant 0$, $\bar{\gamma}_i^n = \gamma_i(\bar{\gamma}^{n-1}) \forall i$.
     if $\Delta_i \geqslant 0$, $\bar{\gamma}_i^n = 0 \forall i$.

(v) if $| \gamma_i^n - \gamma_i^{n-1} | > e \forall i$, $n = n + 1$ go to (iv).
Final parameter: $\gamma^* = \{\gamma_i^k = \gamma_i^n, \forall i, k\}$.

The algorithm sequentially solves the corresponding fixed-point equation for each of the four variable vectors x_j until convergence is attained, called "local t-iteration". The next vector x_{j+1} is then adjusted and so on until the whole vectors set is adjusted, which completes a general iteration. Within a local iteration, the fixed point is solved for all elements of the vector ($x_{ji} \in x_j, \forall i$) simultaneously, by simply repeatedly evaluating the variable in the corresponding fixed-point function, holding the other variables fixed x_{-j} at their current values. Sometimes this is called "the picking algorithm," which in this case is applied to a set of fixed points. The local iteration procedure converges once the variables are within a tolerance value; global convergence requires convergence in all variables.

The exceptions to this general procedure are associated with the presence of zone regulations. First, the algorithm starts (line 2), checking for the feasibility of regulations, finding an interior point such that $S_{vi}^0 \geqslant 0, \forall v, i$, equilibrium $\left(\sum_{vi} S_{vi}^0 \geqslant \sum_h \overline{H}_h\right)$ and regulations $\left(\sum_v a_{vi}^k S_{vi}^0 \leqslant R_i^k, \forall i\right)$ are feasible. Second, the equilibrium is first solved ignoring all regulation constraints ($m = 0$) to find the unrestricted solution. This solution is taken as the starting point to apply the algorithm for the constrained problem ($m = 1$). This procedure avoids getting an incorrect solution produced when the algorithm approaches the feasible set from outside, a non-feasible point, because it gets stuck at the edge of the feasible space and can not find the interior (non-binding) of that space. Conversely, when the algorithm starts from a point in the interior of the feasible space, the global solution is unique (except when the logit model tends to a deterministic choice model, as explained below). Third, the algorithm identifies the γ-fixed point and calls up the Binding procedure to select the most violated constraint at each zone and adjust profits to comply with regulations.

A weakness of this algorithm is that it is restricted to linear forms of zoning regulations. To relax this limitation, further research is required to assure the convergence of the specific γ-fixed point. Alternatively a more general optimization method could be used, but of course the convergence properties will depend on the specific problem and the method used.

3.4. Simulation tests

The following conditions have been applied in the simulations to gather empirical evidence on the model performance. Real estate profits are homogeneous in the industry, i.e. $\pi_j = \pi \forall j$. Consumer bids are additive functions, as in (1). Dependency between agents' choices is defined by location attributes specified in term b^2. In our tests the following linear form is used:

$$b_{hvi}^2 = \alpha_h \sum_{h'v'} Z_{h'} P_{h'/vi} S_{vi} + \beta_h \sum_{v'} Y_{v''i} S_{v'i} \tag{21}$$

The first term with $P_{h/vi}$ describes generic location attributes associated with the distribution of agents: the neighborhood quality related with socioeconomic characteristics of other households in the zone and the value of agglomeration externalities given by the presence of economic activities. Here vector Z describes agents' characteristics such as average income of households, number of commercial businesses in the area, etc. The second term with $S_{v'i}$, describes the externalities associated with the built environment. In this case, vector Y describes residential density, average building height, etc. With these two types of terms, any set of (linearly defined) zone attributes can be specified with the model variables. The sets of parameters α and β represent values that the consumer assigns to each attribute of the zone, usually called the "hedonic price", given exogenously to the model and calibrated from observations of locations and rents.

For each fixed point and also for the complete equilibrium set of Eqs. (19), the functional form and convergence were studied through simulation. The sensitivity of the solutions to their most relevant parameters was also studied. The dimensions used in the simulation are: 4 agents' clusters ($\bar{h} = 4$), 5 zones ($\bar{i} = 5$), and 2 dwelling types ($\bar{v} = 2$). A population of 100 agents is distributed in the following clusters:

Cluster	No. agents	Average income (Z_h)
1	10	4 units
2	15	3 units
3	25	2 units
4	50	1 units

The information on attributes and parameters used are fictitious, but meaningful because they were defined considering Santiago City's land use model MUSSA.

The main results obtained are:

- Dependence on the starting point. The solution is independent of the starting point. It only becomes dependent once the multinomial scale parameters μ and λ are large, which reflects deterministic behavior of agents' choice process.
- Convergence. Individually, each function converges very fast, between two and six iterations, indicating that the multinomial and logsum functions are contracting. The complete system of fixed points has the same property. This result is highly relevant because it allows equilibrium to be studied in large urban systems at a very low computational cost, despite the high complexity of the model.
- Sensitivity analysis. The solution is highly robust to changes in the model parameters. However, unstable solutions appear for large values of the scale parameters μ and λ, again associated with deterministic behavior.
- Important parameters. The scale parameters μ and λ are the most important in the equilibrium solution. Residential location externalities, represented by the average zone income, have the strongest power to shift the location probability curves, thus affecting the solution, but they do not affect the curve form nor the convergence property.
- Efficiency in large-scale applications. Comparing the prototype case ($\bar{h} = 4, \bar{v} = 2, \bar{i} = 5$) with a case of similar size to Santiago city ($\bar{h} = 65, \bar{v} = 12, \bar{i} = 404$), i.e. 8×10^3 times larger, we observed that, for the same standard PC computer, the convergence time increased from 2.5 s to 150 s, only 60 times longer. The reduced number of iterations relative to the increase in the problem size can explain this time saving. This reduction is expected because the larger the problem dimensions, the wider the spread of the distribution of agents and the fewer the changes between iterations, thus convergence is achieved more quickly.

Some additional comments. The deterministic, all-or-nothing, behavior represented by high values of scale parameters, produces an instability that is justified by the theory. It means that the best bidder in the auction changes drastically with small changes in parameters and supply attributes, which yields an unstable land-use pattern. In fact, in the deterministic case with location externalities the equilibrium space contains multiple points. Second, note that because the solution for b^1 provides only relative values, there are multiple solution points that differ only in b^3. Our third comment is on the independence from the starting point, which does not mean that the system solution does not depend on its history. It only means that given a history and the evolution of exogenous parameters (like population), the market equilibrium is unique in the model. We remark that history is embedded in parameters and attributes (initial land use and location patterns), because the multinomial logit model is, by construction, an incremental model depending on past variables and their changes.

4. Methodology for application

It is worth describing briefly the methodology used to apply the model for forecasting the city development along time and under shocks of exogenous variables, called forecasting scenarios. The shocks may be on the population by cluster (which represents changes on both the consumers' income and population) and on transport costs that reflect congestion and infrastructure adjustments.

Transport costs and the location pattern define access indices (accessibility and attractiveness) assumed exogenous in the model. They represent zone attributes in the bid function that are updated interacting with

a transport model. The consumers' valuation of access advantages is determined by behavior parameters of their bid function. The time delay between land use and transport adjustments may be introduced on the interactive procedure, for example applying the land use model using access data from a previous time period.

The model behavioral parameters, both on demand and supply, need to be fitted against observed data-preferably a panel data- of residential and firm locations, real estate supply and rents. The standard fitting approach is to apply the maximum likelihood method to the bid and supply probability functions (3 and 8 respectively). A data set of the observed location of households and firms can be used to estimate the parameters of the bid function b^2, as for example α_h and β_h in Eq. (21). The set of calibrated behavior parameters is assumed to remain constant in the future, while the model calculates the b^1 parameters to attain equilibrium at each period. Similarly, a data set of rents and production costs can be used to calibrate the parameters of the profit functions; these are the parameters of the cost functions assumed constant in the forecasting procedure. The calibration procedure will also provide estimates for the Lagrangean multipliers of regulations γ, which represents estimates of the regulation shadow prices at the observed context. The modeler should verify that the observed data complies with the planning regulations, otherwise regulations data should be adjusted to be realistic. Nevertheless, the model updates γ parameters for each forecasting year.

Noting that rents depend on bids and supply depends on rents, it becomes evident that the whole system depends only on bids and production costs functions. It is then recommended that the set of parameters be calibrated simultaneously using demand, supply and rents functions (3, 8 and 6 respectively). In sum the model is applied using calibrated behavior parameters, but it endogenously updates all parameters that represent constraints that change in the future scenario: the equilibrium condition and the zoning regulations, then it updates location of agents and supply of real estate options. Total supply S is also endogenously determined in the model to fit with total demand. Moreover, zones attributes are endogenously determined by the model, except of course attributes of the natural environment (e.g. rivers, lakes, coast fronts, etc.).

Existing stock is assumed exogenous and fixed in the model, however this assumption can be easily removed defining the probability of redevelopment to the existing stock, which adds to new stock. This is a standard methodology in current models, which can be incorporated by estimating a set of supply probabilities with specific profit functions. Similarly, it is possible to extend the model to identify movers and non-movers among consumers. This can be done by defining a probability function that estimates the number of movers of each cluster at any point in time, the rest remains fixed at their locations; this also defines the proportion of occupied and vacant stock at each zone. With these two extensions the equilibrium is applied to a subset of the market on each forecasting period: movers and new population, and vacant and new building stock. Thus, the model can estimate the city changes for any future year. These extensions, however, do not follow each agent and real estate unit along time, like micro-simulation models do, rather the methodology applies a meso-level of aggregation based on categories and representatives of them with an idiosyncratic distribution of behavior. What is relevant for this paper is that these extensions introduce no further difficulties in the equilibrium model, which still yields a unique solution at each forecasting time period.

The model may be used for a longitudinal study, that is, to forecast the development of the city land use and rents along time for policy and projects assessments. These studies may consider different scenarios, each one representing alternative changing paths for the set of exogenous parameters along time, such as population growth, increase on income, number of firms transport cost, etc. For each scenario and time period, the model calculates the solution of the location pattern and rents. Decisions on delays of transport adjustments will have to be defined by the modeler on empirical bases. But the modeler may also introduce other delays, for example, the new building stock and redevelopment may be decided a few years in advance to their actual availability in the market; agents interactions -location externalities- may also be delayed assuming time lags for the information to be transferred. Both delays have the effect of changing the multinomial fixed-point functions (Eqs. (4 and 9)) into a simple evaluation of probabilities, thus reducing the complexity of the model substantially.

5. Conclusions

The equilibrium model of the land market presented in this paper incorporates the idiosyncratic nature of suppliers' behavior that, added to the random bidding model of consumers' behavior previously developed,

generates the new Random Bidding and Supply Model, RB&SM. This model is totally defined within the mathematical structure provided by the Gumbel distribution, so that the set of equations that defines the equilibrium problem has a structure that provides statistical consistency among all the variables and equations of the model.

The logit supply model proposed has several advantages. It generates highly efficient fixed-point algorithms despite the complexity introduced. It is able to describe economies of scale and scope, which are responsible for density tendencies in the real estate market. It also models a large number of zone regulations in the urban market, producing an output with an index of the economic impact of each regulation and their price effect on land values.

The equilibrium is described by a non-linear equation system. The convergence properties and uniqueness of the solution have been analytically demonstrated for each equation, identifying the bounds for these properties to hold. By means of simulations, we have concluded that the whole RB&SM equation system converges to unique solutions for a wide range of parameters (within the bounds previously mentioned), including starting points and scale parameters. This is the case except when behavior becomes deterministic, i.e. when the variance of the random distributions of bids and profits decreases (scale factors increase). This is an expected result because the deterministic choice process is discontinuous in a differentiated goods market. In contrast, a significant variance in choice behavior introduces continuity and the probability approach produces convexity in the equilibrium problem, generating stable and unique solutions.

The model can be extended in various aspects to relax the assumptions of the simulations studied here. One consists of introducing a process to generate the bidders' choice set for the auctioneer, to replace the assumption that all agents are potential bidders everywhere. A second aspect consists of introducing non-linear restrictions to equilibrium, for example to represent more complex urban regulations. This would require the use of other algorithms whose solutions are, in general, less stable. A third point is to produce more flexible ways to assure compliance with income constraints, with potential extensions to model consumption of all goods and the labor market simultaneously.

Finally, the model can be applied calibrating behavior parameters in a standard way and use them to forecast the land use and location of activities along time responding to changes in the external scenarios. We have shown several ways to apply the model, for example allowing redevelopment of real estate and consumers' inertia to relocate. Thus the model does provide a number of application options for the modeler that will be dependent on the data available, but always under the approach of market equilibrium.

Acknowledgements

Appendix A. Derivation of supply model from an entropy approach

We assume that suppliers maximize their individual profit facing imperfect information modeled by an entropy term that makes profit stochastic variables.

$$
\begin{aligned}
\underset{vi}{\text{Max}} \quad & \pi_j = \sum_{vi} S_{vij}\pi_{vij} - \frac{1}{\lambda}S_{vij}(\ln S_{vij} - 1) \quad \forall j \\
\text{s.t.} \quad & \sum_{v \in V} a_{vi}^k S_{vi} \leqslant R_i^k \quad \forall k \in (1, \ldots K_i), \; i \in (1, \ldots, I) \\
& \sum_{vi} S_{vij} = S_j \quad \forall j \in (1, \ldots J) \\
& \sum_{vij} S_{vij} = S
\end{aligned}
\tag{A1}
$$

where S and S_j are exogenous information of the total supply and each supplier j's market share respectively. R_i^k is the exogenous set of K_i regulations in zone i. The information on market shares is optional because the

model may produce that information from profit variables, but if it is available, we recommend specifying S_j because it may provide complementary information on market behavior missed in the maximum profit assumption of behavior.

Assuming, for convenience and temporarily, that all regulations are binding, the Lagrange function is

$$L = \sum_{vi} S_{vij}\pi_{vij} - \frac{1}{\lambda}S_{vij}(\ln S_{vij} - 1) - \sum_{ki}\gamma_i^k\left(\sum_v a_{vi}^k S_{vi} - R_i^k\right) - \sum_j \rho_j\left(\sum_{vi} S_{vij} - S_j\right) - \alpha\left(\sum_{vi} S_{vi} - S\right)$$

The first order conditions are:

$$\frac{\partial L}{\partial S_{blm}} = \pi_{blm} + \sum_{vi} S_{vij}\frac{\partial \pi_{vij}}{\partial S_{blm}} - \frac{1}{\lambda}\ln S_{blm} - \sum_k \gamma_l^k a_{bl}^k - \rho_m - \alpha \tag{A2}$$

We denote $\tilde{\pi}_{blm} = \pi_{blm} + \sum_{vi}S_{vij}\frac{\partial \pi_{vij}}{\partial S_{blm}} - \sum_k\gamma_l^k a_{bl}^k$. Each component in the second term contains two effects: $\frac{\partial \pi_{vij}}{\partial S_{blm}} = \frac{\partial r_{vi}}{\partial S_{blm}} - \frac{\partial c}{\partial S_{blm}}$, representing the impact of supply changes on profits through prices and through scale and scope economies, respectively, across the urban area. The total effect across all alternative locations is calculated by adding across vi options, denoted by π'_{blm}. If we assume that suppliers have incomplete information of market performance, then this term should be multiplied by a factor belonging to $[0,1]$.

Equalizing (A2) to zero we obtain optimum values of supply

$$S_{vij}^* = \exp \lambda(\tilde{\pi}_{vij} - \rho_j - \alpha) \tag{A3}$$

Imposing the constraints of problem (A1) to the solution yields expressions for the market-size parameter α and the market-share parameters ρ:

$$\exp(-\lambda\alpha) = S\left[\sum_{vij}\exp \lambda(\tilde{\pi}_{vij} - \rho_j)\right]^{-1} \quad \text{and} \tag{A4}$$

$$\exp(-\lambda\rho_j) = S_j\left[\sum_{vi}\exp \lambda(\tilde{\pi}_{vij} - \alpha)\right]^{-1} \tag{A5}$$

where (A4) and (A5) are the known balancing factors of linear constraints in entropy models. From (A3) and (A4) we get

$$S_{vij}^* = S\frac{\exp\ \lambda(\tilde{\pi}_{vij} - \rho_j)}{\sum_{v'i'j'}\exp\ \lambda(\tilde{\pi}_{v'i'j'} - \rho_{j'})} = SP_{vij} \tag{A6}$$

which reproduces Eq. (8) in the text. In this equation $\tilde{\pi}_{vij}$ represents an adjusted profit that complies with all linear regulations R and adjusts behavior to the supplier anticipation of scale and scope economies. The parameter λ represents the degree of randomness of the supply variable; if $\lambda \to \infty$ the problem (A1) tends to be deterministic, if $\lambda \to 0$ the solution becomes homogeneous with $S_{vij} = 1/S$.

For any kth regulation of zone i that is not binding at the solution, by Kun–Tucker conditions, $\gamma_i^k = 0$. Binding parameters satisfy the first constraint of problem (A1) for the equality, then replacing (A3) in the binding constraint, denoted by $\bar{k}$, yields: $\sum_{vj}a_{vi}^k\exp \lambda(\tilde{\pi}_{vij} - \rho_j - \alpha) = R_i^{\bar{k}}$, or expanding $\tilde{\pi}$ we obtain,

$$\sum_{vj}a_{vi}^k\exp \lambda(\pi_{vij} + \pi'_j - \rho_j - \alpha)\exp(-\lambda\gamma_i^{\bar{k}}a_{vi}^{\bar{k}}) = R_i^{\bar{k}} \tag{A7}$$

which is the formula used to find gamma parameters.

Because gamma's parameters are multiplied by the parameter vector a in the exponentials, Eq. (A7) can not be directly solved for gamma. This is known as the cross entropy problem (Fang et al., 1997), where gamma parameters can be solved applying the MART algorithm. This algorithm assures convergence to the unique optimal solution if the problem has a non-empty interior and $R_i^k \geqslant 0$; $0 \leqslant a_{vi}^k \leqslant 1\ \forall k, v, i$. In our case, vector R components correspond to total supply, thus they are obviously non-negative. Additionally, to assure that $a_{vi}^k \in [0,1]$, it is sufficient to scale the kith constraint dividing it by the $\bar{a}_i^k = \max_v(a_{vi}^k)$. Note that the MART

algorithm assures convergence only, but not the sign of γ, which is a crucial point considering that, by definition, in our case $\gamma \geqslant 0$.

Here we developed an alternative approach to solve gamma parameters. We simply multiply (A7) by $\exp(\lambda \gamma_i^{\bar{k}} a_{0i}^k)$, with $a_{0i}^{\bar{k}}$ taken arbitrarily from vector $(a_{\cdot i}^{\bar{k}})_v$. We advise to choose $a_{0i}^{\bar{k}} = \max_v a_{vi}^{\bar{k}}$ because convergence is thus assured in Appendix B. Then solving for gamma yields

$$
\gamma_i^{\bar{k}} = \frac{1}{\lambda a_{0i}^k} \ln \left[\frac{S}{R_i^{\bar{k}}} \sum_{vj} a_{vi}^{\bar{k}} \exp \lambda (\pi_{vij} + \pi'_{vij} - \gamma_i^{\bar{k}}(a_{vi}^{\bar{k}} - a_{0i}^{\bar{k}}) - \rho_j - \hat{\pi}_j) \right]
$$

$$
\text{with } \hat{\pi}_j = \frac{1}{\lambda} \ln \left[\sum_{v'i'j'} \exp \lambda (\pi_{v'i'j'} + \pi'_{v'i'j'} - \rho_{j'} - \gamma_{i'}^{\bar{k}} a_{v'i'}^{\bar{k}}) \right] \tag{A8}
$$

which is a fixed-point expression for gamma.

Appendix B. Convergence of fixed points

Let us consider the fixed point $f(x) = x$, or $g(x) = f(x) - x = 0$. Sufficient conditions for existence of a solution x^* for g continuous are:

$$
g'(x) < 0, \quad \forall x \in R; \quad g(x = 0) > 0; \quad \exists x/g(x) < 0 \tag{B1}
$$

and convergence is assured if $-2 < g'(x)$, given that (B1) holds.

Let us revise these conditions for each fixed point.

(i) *The Multinomial Auction probability*: $P_{h/vi} = P_{h/vi}((b_\cdot^1)_h, (P_{\cdot/\cdot i})_{hv}, (S_{\cdot i})_v, \gamma_i) \quad \forall h, v, i$

$$
g(P_{h/vi}) = \frac{\exp(\mu(b_h^1 + b_{hvi}^2((P_{\cdot/\cdot i})_{hv})))}{\sum_{h'} \exp(\mu(b_{h'}^1 + b_{h'vi}^2((P_{\cdot/\cdot i})_{hv})))} - P_{h/vi} \tag{B2}
$$

with g continuous in $\mathfrak{R}$. Since the first term is a probability in $[0,1]$, then $g(x=0) \geqslant 0$, with equality only for $b_h = -\infty$. Additionally $g(x=\infty) < 0$ because the first term is a probability in $[0,1]$ while the second term $(-P_{h/vi})$ tends to $-\infty$. Then, we only need to verify that $-2 < g'(P_{h/vi}) < 0$.

$$
g'(P_{h/vi}) = \mu P_{h/vi} \left[\frac{\partial b_{hvi}^2}{\partial P_{h/vi}} - \sum_{h'} P_{h'/vi} \frac{\partial b_{h'vi}^2}{\partial P_{h/vi}} \right] - 1 \tag{B3}
$$

Then $-2 < g'(P_{h/vi}) < 0$ if $-1 < \mu P_{h/vi} \Delta b_{hvi}'^2 < 1$, with Δb_{hvi}^2 obviously defined by the term in parenthesis in (B3). This imposes the following upper and lower bounds for the multinomial probability:

$$
-\frac{1}{\mu \Delta b_{hvi}'^2} < P_{h/vi} = \frac{H_{hvi}}{S_{vi}} < \frac{1}{\mu \Delta b_{hvi}'} \tag{B4}
$$

with $H_{hvi} = P_{hvi} S P_{vi}$ and $S_{vi} = S P_{vi}$. These bounds depend on the scale factor μ, inversely related with the standard deviation of bids, which states that a solution exists as long as the distribution of bids is significantly random.

Additionally, if b^2 is linear in attributes, as in Eq. (21) in the text, then:
$\frac{\partial b_{gvi}^2}{\partial P_{h/vi}} = \beta_g Z_h P_{vi}$, and the upper and lower bounds are:

$$
-1 < \mu P_{h/vi} Z_h P_{vi} \left[\beta_h - \sum_{h'} P_{h'/vi} \beta_{h'} \right] < 1 \tag{B5}
$$

Introducing the relationship between logit scale parameters and the standard deviation, these bounds may be expressed as $\frac{1}{\mu} = \sigma_b \frac{\sqrt{6}}{\pi} > \frac{1}{S} |H_{hvi} Z_h \beta_h|$, with σ_b the standard deviation of bids and $\pi = 3.1416\ldots$.

Then, we conclude that, within the space defined by upper and lower bounds for auction probabilities, the solution exists, is unique, and the fixed-point algorithm converges to the solution.

(ii) *The Multinomial Supply probability*: $S_{vi} = SP_{vi}((b_.^1)_h, (P_{./..})_{hvi}, (S_{..})_{vi}, (\gamma_.)_i)$

$$g(P_{vi}) = \frac{\exp(\lambda(r_{vi}(P_{vi}) - c_{vi} - \gamma_{vi}))}{\sum_{v'i'} \exp(\lambda(r_{v'i'}(P_{vi}) - c_{v'i'} - \gamma_{v'i'}))} - P_{vi} \tag{B6}$$

Similarly to case (i) $g(x = 0) > 0$; $g(x = \infty) < 0$, so we need to prove that

$$-2 < g'(P_{vi}) = P_{vi}\lambda \left[\frac{\partial r_{vi}}{\partial P_{vi}} - \sum_{v'i'} P_{v'i'} \frac{\partial r_{v'i'}}{\partial P_{vi}} \right] - 1 < 0 \tag{B7}$$

which yields $-1 < P_{vi}\lambda\Delta r'_{vi} < 1$, with Δr defined as the term in parenthesis. Additionally, $\frac{\partial r_{wj}}{\partial P_{vi}} = \sum_{h'} P_{h'/wj} \frac{\partial b^2_{h'wj}}{\partial P_{vi}}$.
Again, we obtain bounds for probabilities:

$$\frac{-1}{\lambda\Delta r'_{vi}} < P_{vi} < \frac{1}{\lambda\Delta r'_{vi}} \tag{B8}$$

that impose a minimum level of dispersion on the distribution of supply options.
In the case of bid functions linear in attributes

$$\frac{\partial b^2_{hwj}}{\partial P_{vi}} = \begin{cases} \beta_h \sum_g Z_g P_{g/wi} + \alpha_h X_{wi} & \forall j = i \\ 0 & \forall j \neq i \end{cases} \tag{B9}$$

which yields: $\frac{1}{\lambda} = \sigma_s \frac{\sqrt{6}}{\pi} > \frac{1}{S} \left| S_{vi} \sum_h P_{h'/vi} \left(\beta_{h'} \sum_g Z_g P_{g/vi} + \alpha_{h'} X_{vi} \right) \right|$. $\tag{B10}$

with σ_s the standard deviation of suppliers' profits.
Then, we conclude that, within the space defined by upper and lower bounds for supply probabilities, the solution exists, is unique, and the fixed-point algorithm converges to the solution.

(iii) *The equilibrium condition*: $b_h^1 = b_h((b_.^1)_h, (P_{./..})_{hvi}, (S_{..})_{hv}, (\gamma_.)_i)$

$$g(b_h) = -\frac{1}{\mu} \ln \left[\sum_{vi} \frac{S_{vi}}{H_h} \exp(\mu(b_{hvi} - r_{vi}(b_h))) \right] - b_h \tag{B11}$$

We start proving that $-2 < g'(x) < 0$. This implies that

$$-2 < g'(b_h) = \frac{\sum_{vi} S_{vi} \exp(\mu(b_{hvi} - r_{vi}(b_h))) * \frac{\partial r_{vi}}{\partial b_h}}{\sum_{v'i'} S_{v'i'} \exp(\mu(b_{hv'i'} - r_{vi}(b_h)))} < 0 \tag{B12}$$

with $\frac{\partial r_{vi}}{\partial b_h} = P_{h/vi}$. This condition may also be written as:

$$\sum_{vi} S_{vi} \exp(\mu(b_{hvi} - r_{vi}(b_h))) * P_{h/vi} < \sum_{v'i'} S_{vi} \exp(\mu(b_{hvi} - r_{vi}(b_h))) \quad \text{and}$$
$$\sum_{vi} S_{vi} \exp(\mu(b_{hvi} - r_{vi}(b_h))) * P_{h/vi} > -\sum_{vi} S_{vi} \exp(\mu(b_{hvi} - r_{vi}(b_h))) \tag{B13}$$

which holds always because the left hand side is positive since $P_{h/vi} \in [0,1], S_{vi} \geq 0$, and the exponentials are also positive.
Now we verify that there exists b_h such that $g(b_h) > 0$ and also that $g(b_h) < 0$. We have

$$g(b_h) = -\frac{1}{\mu} \ln \left[\sum_{v'i'} \frac{S_{vi}}{H_h} \exp(\mu(b_{hvi} - r_{vi}(b_h))) \right] - b_h = -\frac{1}{\mu} \ln \left[\sum_{v'i'} \frac{S_{vi}}{H_h} P_{h/vi}(b_h) \right] \tag{B14}$$

then

(a) $g(b_h) > 0 \rightarrow \sum_{v'i'} S_{vi} P_{h/vi}(b_h) < \overline{H}_h$, holds for $b_h \rightarrow -\infty$ because $P(b_h = -\infty) = 0$.
(b) $g(b_h) < 0 \rightarrow \sum_{v'i'} S_{vi} P_{h/vi}(b_h) > \overline{H}_h$, holds for $b_h \rightarrow \infty$ because $P(b_h = \infty) = 1$ and $S > H_h$ except for the extreme case when there is only one cluster.

Then, we conclude that the solution always exists, is unique, and the fixed-point algorithm converges to the solution (except in the case of only one agent cluster).

(iv) *The Regulations fixed points:* $\gamma_i = \gamma_i((b^1_.)_h, (P_{./..})_{hvi}, (S_{..})_{vi}, (\gamma_.)_i)$

$$g(\gamma_i^{\bar{k}}) = \frac{1}{\lambda a_{0i}^k} \ln \left[\frac{S}{R_i^{\bar{k}}} \sum_{vj} a_{vi}^{\bar{k}} \exp \lambda (\pi_{vij} + \pi'_{vij} - \gamma_i^{\bar{k}}(a_{vi}^{\bar{k}} - a_{0i}^{\bar{k}}) - \rho_j - \hat{\pi}_j(\gamma)) \right] - \gamma_i^{\bar{k}}$$

with $\hat{\pi}_j = \frac{1}{\lambda} \ln \left[\sum_{v'i'j'} \exp \lambda (\pi_{v'i'j'} + \pi'_{v'i'j'} - \rho_{j'} - \gamma_{i'}^{\bar{k}} a_{v'i'}^{\bar{k}}) \right]$

In this case

$$g'(\gamma_i^{\bar{k}}) = \frac{1}{a_{0i}^{\bar{k}}} \frac{\sum_{vj} a_{vi}^{\bar{k}} \exp \lambda (\pi_{vij} + \pi'_{vij} - \gamma_i^{\bar{k}}(a_{vi}^{\bar{k}} - a_{0i}^{\bar{k}}) - \rho_j - \hat{\pi}_j(\gamma)) \left(a_{0i}^{\bar{k}} - a_{vi}^{\bar{k}} - \frac{\partial \hat{\pi}_j}{\partial \gamma_i^{\bar{k}}} \right)}{\sum_{v'j'} a_{v'i}^{\bar{k}} \exp \lambda (\pi_{v'ij'} + \pi'_{v'ij'} - \gamma_i^{\bar{k}}(a_{v'i}^{\bar{k}} - a_{0i}^{\bar{k}}) - \rho_{j'} - \hat{\pi}_{j'}(\gamma))} - 1 \tag{B15}$$

with $\frac{\partial \hat{\pi}_j(\gamma)}{\partial \gamma_i^k} = -\sum_{vj} P_{vij} a_{vi}^{\bar{k}}$.

Denoting $\pi_{vij}^{\bar{k}} = \pi_{vij} + \pi'_{vij} - \gamma_i^{\bar{k}}(a_{vi}^{\bar{k}} - a_{0i}^{\bar{k}}) - \rho_j - \hat{\pi}_j(\gamma)$, the condition $g'(\gamma_i) < 0$ holds if:

$$\frac{\sum_{vj} \left[a_{vi}^{\bar{k}} a_{oi}^{\bar{k}} - (a_{vi}^{\bar{k}})^2 + a_{vi}^{\bar{k}} \sum_{v'j'} P_{v'ij'} a_{v'i}^{\bar{k}} \right] \exp(\lambda \pi_{vij}^{\bar{k}})}{\sum_{v'j'} a_{v'i}^{\bar{k}} a_{oi}^k \exp(\lambda \pi_{v'ij'}^{\bar{k}})} < 1$$

which, after some simplifications, yields:

$$\sum_{v'j} P_{v'ij} a_{v'i}^{\bar{k}} < a_{vi}^{\bar{k}} \quad \forall v, i \tag{B16}$$

and the condition $-2 < g'(\gamma_i)$ holds when $a_{vi}^{\bar{k}} < a_{oi}^{\bar{k}} \quad \forall v, i$.

The second condition is secured if we choose $a_{oi}^{\bar{k}} = \max_v(a_{vi}^{\bar{k}})$, but the (B16) is not verified always, which implies that this relationship must be verified in the algorithm at each iteration. Nevertheless, note that (A6) assures that $\sum_{vj} P_{vij} < 1$ (probabilities are equal to one only when profit is infinite), then for a large number of zones P_{vij} values are small, thus condition (B16) is likely to hold.

Additionally, $g(\gamma_i = 0) > 0$ implies that

$$\sum_{vj} \frac{S * a_{vi}^{\bar{k}}}{R_i^{\bar{k}}} \frac{\exp \lambda (\pi_{v'ij'} + \pi'_{v'ij'} - \rho_{j'}))}{\sum_{v'i'j'} \exp \lambda (\pi_{v'ij'} + \pi'_{v'ij'} - \rho_{j'})} = \sum_v a_{vi}^{\bar{k}} S_{vi}(\gamma_i^{\bar{k}} = 0) > R_i^{\bar{k}} \tag{B17}$$

which requires that the regulation be violated; then this condition holds always for $\gamma > 0$. Finally, $g(\gamma_i^k = \infty) < 0$ requires that $(1/\lambda)\ln(\exp(-\infty)) < 0$, which holds always.

Then, we conclude that, provided the regulation is binding, that $a_{oi}^{\bar{k}} = \max_v(a_{vi}^{\bar{k}})$ and that condition B16 holds, the solution exists always and is unique, and moreover, the fixed-point algorithm converges to the solution.

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