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Qualitative properties of ground states for singular elliptic equations with weights*

Dedicated to Roberto Conti on the occasion of his 80th birthday

Abstract. In a series of papers the question of uniqueness of radial ground states of the equation $\Delta u + f(u) = 0$ and of various related equations has been studied. It is remarkable that throughout this work (except in very special circumstances) nowhere is a spatially dependent term taken into consideration. Here we shall make a first attempt to study the uniqueness of ground states for such spatially dependent equations and to establish qualitative properties of solutions for this purpose.

Mathematics Subject Classification (2000). Primary 35J70, Secondary 35J60

Key words. ground states – m-Laplacian operator – weight functions – uniqueness

Contents

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* This research was supported by the Italian MIUR project titled *"Metodi Variazionali ed Equazioni Differenziali non Lineari"*, as well as Fondecyt Cooperaciòn International 7990040 for the first author, Fondecyt Grants 1030593 for the second author, and Fondap Matemáticas Aplicadas and Milenio grant–P01–34 for the third author.

1. Introduction

In a series of papers [2], [4], [6], [8], [13]–[16], [18]–[21], [23], the question of *uniqueness* of radial ground states of the equation $\Delta u + f(u) = 0$ and of various related equations has been studied. It is remarkable that throughout this work (except in very special circumstances, see [12] and [20]) nowhere is a spatially dependent term taken into consideration. Similarly the existence of ground states has not been as fully studied as one might wish for spatially dependent equations, but see [3] and [9] for many interesting results. Here we shall make a first attempt to study the *uniqueness* of ground states and various qualitative properties of solutions of such spatially dependent equations. More precisely, we consider non–negative solutions of the singular quasilinear elliptic equation

$$\operatorname{div}(g(|x|)|Du|^{m-2}Du) + h(|x|)f(u) = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \qquad m > 1, \quad n \geq 1, \tag{1.1}$$

where $g,\ h : \mathbb{R}^+ \to \mathbb{R}^+$ and $Du = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$. Since the problem is already quite difficult, involving as it does the m–Laplace operator, it seems reasonable to consider only radial solutions, so that (1.1) then takes the form, see Section 2,

$$[a(r)|u'|^{m-2}u']' + b(r)f(u) = 0, \qquad m > 1, \qquad n \geq 1, \tag{1.2}$$

where $a(r) = r^{n-1}g(r)$, $b(r) = r^{n-1}h(r)$, and $r = |x|$. The simple Laplace–Poisson equation arises when $a(r) = b(r) = r^{n-1}$, where n is the underlying space dimension.

A number of examples fall into the general category of (1.1). A first is the generalized Matukuma equation

$$\Delta_m u + \frac{f(u)}{1 + r^\sigma} = 0, \qquad m > 1, \qquad n \geq 1, \qquad \sigma > 0, \tag{1.3}$$

where Δ_m denotes the m–Laplace operator, namely $\Delta_m u = \operatorname{div}(|Du|^{m-2}Du)$, and where also $g(|x|) \equiv 1$, $h(|x|) = 1/(1 + r^\sigma)$, $r = |x|$. A second example is the equation

$$\Delta_m u + \frac{r^\sigma}{(1 + r^{m'})^{\sigma/m'}} \cdot \frac{f(u)}{r^{m'}} = 0, \qquad m > 1, \qquad n \geq 1, \qquad \sigma > 0, \tag{1.4}$$

where now $g(|x|) \equiv 1, h(|x|) = r^{\sigma-m'}/(1+r^{m'})^{\sigma/m'}$, and m' is the Hőlder conjugate of m. Equation (1.4) was first introduced in [1], equation (4.21), with $m = 2$, as a model of stellar structure. That the terms $r^{m'}$ in (1.4) are the appropriate generalization of the term r^2 appearing in the stellar model of [1] is justified by the observation that the special solution $\sqrt{3/(1+r^2)}$ of (1.4) when $n=3, \sigma=m=2$, found in [14] for the case $f(u)=u^3$, is paralleled by the solution

$$u(r)=\left(n^{1/(m-1)}\frac{n-m}{m-1}\cdot\frac{1}{1+r^{m'}}\right)^{(n-m)/m}$$

of (1.4) when one has $\sigma=m'$, $n>m$, and $f(u)=u^{p-1}$, with

$$p=\frac{m(n-1)}{n-m}.$$

Other generalizations of the Matukuma equation and of the stellar model of [1] can be found in [3] and [9]. All these equations are discussed in detail in Section 4, as special cases of the main example

$$\begin{gathered}\operatorname{div}(r^k|Du|^{m-2}Du)+r^\ell\left(\frac{r^s}{1+r^s}\right)^{\sigma/s}f(u)=0,\\ m>1,\quad n\geq 1,\quad k\in\mathbb{R},\quad \ell\in\mathbb{R},\quad s>0,\quad \sigma>0.\end{gathered}\tag{1.5}$$

In particular, we obtain conditions on the exponents, so that under appropriate behavior of the nonlinearity $f(u)$, radial ground states for equations (1.3)–(1.5) are unique.

Motivated by the case $a(r)=b(r)=r^{n-1}$, we shall suppose that the functions a and b above are such that (1.2) *can be transformed through a diffeomorphism* $r=r(t)$ *of* $\mathbb{R}_0^+$ *to the form*

$$\left[q(t)|v_t|^{m-2}v_t\right]_t+q(t)f(v)=0,\tag{1.6}$$

that is, again to an equation of type (1.2), but with the same weights. In the special case when (1.6) arises with $q(t)=t^{N-1}$ for some $N\geq 1$, then of course earlier theory can be applied, see e.g. [20]. Of course, the theory which follows must be applicable when q is no longer a pure power. In particular, in order to carry out the further arguments of the paper, and in order to study the uniqueness of ground states of (1.2), we shall ask that the transformed equation (1.6) be compatible with the following *basic structure*:

$$q\in C^1(\mathbb{R}^+),\qquad q>0,\quad q_t>0\quad\text{in }\mathbb{R}^+;\tag{Q1}$$

$$q_t/q\quad\text{is strictly decreasing on }\mathbb{R}^+;\tag{Q2}$$

$$\lim_{t\to 0^+}\frac{tq_t(t)}{q(t)}=N-1\geq 0.\tag{Q3}$$

Another facet of the ground state problem is the possibility that the function f may be undefined at $u=0$. This case has been previously studied in [20] and [23], but without suitable attention to the difficulties attendant on this type of singularity. Here it is minimally necessary to ask of the function $f=f(u)$ that it be continuous for *positive* u and *integrable* down to $u = 0$, that is throughout the paper we assume that

$$(F1) \qquad f \in C(\mathbb{R}^+) \cap L^1[0, 1].$$

In particular we can then define the primitive function $F(u) = \int_0^u f(s)ds$, and of course $F(0) = 0$. More specialized conditions on f will be introduced later, as needed.

The first goals of the paper, then, are

(i) to give a precise setting to the problem, including a definition for the meaning of a non–negative solution of (1.1) when the function f is singular at $u = 0$, and
(ii) to introduce an appropriate diffeomorphism $r = r(t)$ on $\mathbb{R}_0^+$ which will transform (1.2) into (1.6).

Naturally, the validity of such a transformation depends on the formulation of appropriate properties of the original functions a and b.

The remaining parts of the paper are devoted to qualitative properties and uniqueness of radial ground states in the spirit of the papers [8] and [20].

As in [8], our proofs rely in their essentials on a careful study of the global behavior of ground states, especially their monotonicity properties and the separation and intersection properties of pairs of solutions. While these considerations are similar to those earlier developed in [8], [18] and [19], because of the greater generality involved here, and for the sake of clarity and completeness, we shall carry out the arguments in full rather than by reference to earlier work.

The paper is divided into two parts (Part I starting at Section 2 and Part II at Section 5) with the following organization. In Section 2 we give a precise setting to the problem, laying the foundations for the later uniqueness theory of Section 8. In Section 3 we introduce the required transformation from equation (1.2) to (1.6), and in Section 4 discuss in more detail the examples (1.3)–(1.5), (4.8), (4.9) as well as other related equations.

In Sections 5 and 6 we study various qualitative properties of solutions, this being the basis for the separation and intersection theorems given in Section 7, and in turn for the main uniqueness Theorems 8.3 and 8.4. Here it is worth adding that the corresponding radial Dirichlet–Neumann problem can be treated simultaneously with the ground state problem, in an essentially unified way (see Section 5).

Functions $f = f(u)$, which are undefined at $u = 0$, but nevertheless can be treated to get uniqueness for (1.1), and in particular for equations (1.3)–(1.5), include for example

$$(1.7) \qquad f(u) = -u^p + u^s; \; m \geq 2, \; -1 < p < s \leq m-1, \; p \leq 1 + \frac{m-3}{m-1}s,$$

see Section 8. Note that (1.7) allows values $p > 0$ and $s < 0$, though not both at the same time.

In the Appendix, Part 2, local existence and unique continuation results for the initial value problem at $t = 0$ for the transformed equation (1.6) are established.

Part I

2. Semi-classical solutions

Consider the quasilinear singular elliptic equation

$$
\begin{gathered}
\operatorname{div}(g(|x|)|Du|^{m-2}Du) + h(|x|)f(u) = 0 \\
\text{in } \Omega = \{x \in \mathbb{R}^n \setminus \{0\} \ : \ u(x) > 0\}; \quad m > 1, \quad n \geq 1, \\
u \geq 0, \qquad u \not\equiv 0 \qquad \text{in } \mathbb{R}^n \setminus \{0\},
\end{gathered} \tag{2.1}
$$

where $g,\ h : \mathbb{R}^+ \to \mathbb{R}^+$. Prototypes of (2.1), with non–trivial functions g, h, are given, for example, by equations of Matukuma type and equations of Batt–Faltenbacher–Horst type, see (1.3)–(1.4) and (4.8)–(4.9) below. In the last two cases g is singular at the origin, and in general h also may be singular there; thus it is necessary in (2.1) that Ω exclude the point $x = 0$ and also points where $u(x) = 0$.

We shall be interested in the radial version of (2.1), namely

$$
\begin{gathered}
[a(r)|u'|^{m-2}u']' + b(r)f(u) = 0 \\
\text{in } J = \{r \in \mathbb{R}^+ \ : \ u(r) > 0\}, \quad m > 1, \quad r = |x|, \\
u = u(r), \qquad u \geq 0, \qquad u \not\equiv 0 \qquad \text{in } \mathbb{R}^+,
\end{gathered} \tag{2.2}
$$

where, with obvious notation,

$$
a(r) = r^{n-1}g(r), \quad b(r) = r^{n-1}h(r). \tag{2.3}
$$

In order that the transformed equation (1.6) should satisfy the requirements (Q1)–(Q3) we shall ask that the coefficients a, b have the following behavior (see Section 3),

$(A1)$ $$a, b > 0 \ \text{in } \mathbb{R}^+, \qquad a, b \in C^1(\mathbb{R}^+),$$

$(A2)$ $$(b/a)^{1/m} \in L^1[0,1] \setminus L^1[1,\infty),$$

$(A3)$ the function

$$
\psi(r) = \left[\frac{1}{m}\frac{a'}{a} + \frac{1}{m'}\frac{b'}{b}\right]\left(\frac{a}{b}\right)^{1/m}
$$

is positive and strictly decreasing in $\mathbb{R}^+$, where m' is the Hőlder conjugate of $m\ (> 1)$,

$(A4)$ there is $N \geq 1$ such that

$$
\lim_{r \to 0^+} \psi(r) \int_0^r \left(\frac{b}{a}\right)^{1/m} = N - 1.
$$

In Section 4 we consider in particular the equations (1.3)–(1.5) and (4.8)–(4.9) as examples satisfying the above conditions.

Remarks. (i) When $g \equiv 1$ in (2.1), namely $a(r) = r^{n-1}$ and $b(r) = r^{n-1}h(r)$, then conditions $(A1)$–$(A4)$ reduce to a simpler form in terms of the function $h = h(r)$, namely

$(A1)'$ $$h > 0 \quad \text{in } \mathbb{R}^+, \qquad h \in C^1(\mathbb{R}^+),$$

$(A2)'$ $$h^{1/m} \in L^1[0,1] \setminus L^1[1,\infty),$$

$(A3)'$ the function

$$\psi_h(r) = \left[\frac{n-1}{r} + \frac{1}{m'}\frac{h'}{h}\right] h^{-1/m}$$

is positive and strictly decreasing in $\mathbb{R}^+$,

$(A4)'$ there is $N \geq 1$ such that

$$\lim_{r\to 0^+} \psi_h(r) \int_0^r h^{1/m} = N - 1.$$

Conditions $(A1)'$–$(A4)'$ are illuminated by the case $h(r) = r^\ell$. Here these conditions reduce to the exponent relations

(2.4) $$\ell + m > 0, \qquad \ell + (n-1)m' > 0,$$

with

(2.5) $$N = m\frac{\ell + n}{\ell + m} > 1.$$

For this example it is worth remarking that the transformed equation (1.6) becomes simply

(2.6) $$\left[t^{N-1}|v_t|^{m-2}v_t\right]_t + t^{N-1}f(v) = 0,$$

as follows from the formulas (3.1), (3.3) below, that is, N serves as the natural dimension for this example; see also Section 4 below and [20].

Conditions $(A1)'$, $(A2)'$ also appear in the paper [12], though in somewhat different circumstances.

(ii) Suppose that

$$a^{-1/(m-1)} \in L^1[1,\infty) \setminus L^1[0,1],$$

and let M be a constant, with $M > m$. Using the diffeomorphic change of variable $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, given by

$$s(r) = \left(\int_r^\infty [a(\tau)]^{-1/(m-1)} d\tau\right)^{-(m-1)/(M-m)}, \qquad z(s) = u(r(s)),$$

equation (2.2) can be rewritten in the form

(2.7) $$\left(s^{M-1}|z_s|^{m-2}z_s\right)_s + s^{M-1}\tilde{h}(s)f(z) = 0,$$

where

$$\tilde{h}(s) = \left|\frac{M-m}{m-1}\right|^m \left(\frac{a(r(s))^{1/m} b(r(s))^{1/m'}}{s^{M-1}}\right)^{m'}. \tag{2.8}$$

The function $\tilde{h}$ satisfies $(A1)'$–$(A4)'$ in the s variable, with n replaced by M, if and only if a, b satisfy $(A1)$–$(A4)$.

Conversely, suppose that

$$a^{-1/(m-1)} \in L^1[0,1] \setminus L^1[1,\infty),$$

and let M be a constant, with $M < m$. Then, as before, making the change of variable $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, defined by

$$s(r) = \left(\int_0^r [a(\tau)]^{-1/(m-1)} d\tau\right)^{(m-1)/(m-M)}, \qquad z(s) = u(r(s)),$$

one obtains (2.7), with again $\tilde{h}$ given by (2.8).

It is not clear that equation (2.7) is any technical improvement over (2.2), whatever the choice of M, and accordingly we shall not further pursue this reduction.

Since (2.1) is possibly singular when $x = 0$ and when $u = 0$, it is necessary to carefully define the meaning to be assigned to solutions of (2.1), and in turn of (2.2). One can consider weak distribution solutions of (2.1), or alternatively distribution solutions with suitable further regularity conditions and well defined values at $x = 0$. We shall thus consider the class of *semi-classical radial solutions* $u = u(r)$, as defined below, and show that these can be transformed to be semi-classical radial solutions $v = v(t)$ of a related equation with $t = 0$ corresponding to $r = |x| = 0$. This being done, the remaining arguments in the paper can then be directed exactly to the semi-classical case.

As an appropriate structure to carry out this aim, and in particular to avoid the undefined nature of $f(u)$ at $u = 0$, we introduce the following

Definition. A semi-classical solution of (2.1) is a non-negative function u of class $C^1(\mathbb{R}^n \setminus \{0\})$, which is a distribution solution of (2.1) in the *open (support) set*

$$\Omega = \{x \in \mathbb{R}^n \setminus \{0\} \ : \ u(x) > 0\}.$$

[To be precise, u is a $C^1(\mathbb{R}^n \setminus \{0\})$ distribution solution of (2.1) in Ω when

$$\int_\Omega g(|x|)|Du|^{m-2}Du \cdot D\varphi \, dx = \int_\Omega h(|x|) f(u)\varphi dx \tag{2.9}$$

for all C^1 functions $\varphi = \varphi(x)$ with compact support in Ω.]

Radial case. When expressed directly for radial functions u, this definition is equivalent to the condition that $u = u(r)$ is a non–negative function of class $C^1(\mathbb{R}^+)$ which is a distribution solution of (2.2) in J. That is, by substituting

$D_i u = u' x_i/r$, $D_i\varphi = \varphi' x_i/r$ into (2.9), where $\varphi = \varphi(r)$ has compact support in J, and using the volume element $dx = r^{n-1} dr\, d\omega$, one finds

$$\int_J r^{n-1} g(r)|u'|^{m-2}u'\,\varphi'\,dr = \int_J r^{n-1} h(r) f(u)\varphi dr,$$

as required.

In fact, this is not all one can say. That is, by standard distribution arguments it is easy to show that if the interval $[r,\ t] \subset J$, then

$$a(r)|u'(r)|^{m-2}u'(r) - a(t)|u'(t)|^{m-2}u'(t) = \int_r^t b(s) f(u(s))ds. \tag{2.10}$$

Since the right hand side is continuously differentiable in r, it follows that $a(r)|u'|^{m-2}u'$ is of class $C^1(J)$, in which case equation (2.2), exactly as written, is satisfied in the sense of ordinary differentiation (classical solution) in J.

It is important to show that the definition of semi-classical solution is compatible with that of classical solution on $\mathbb{R}^+$ when f is continuous in $\mathbb{R}_0^+$ with $f(0) = 0$. This is the content of the following result.

Proposition 2.1. *Let $u = u(|x|)$ be a semi-classical radial solution of* (2.1)*, where $f \in C(\mathbb{R}_0^+)$ with $f(0) = 0$. Then u is a C^1 classical solution of* (2.2) *in all of* $\mathbb{R}^+$.

Proof. If $J = \mathbb{R}^+$ there is nothing to prove. Otherwise let J' be any component of $J = \{r > 0\ :\ u(r) > 0\}$, let $r, t \in J'$ and $0 \le r_0 < r_1$ the endpoints of J'. Clearly (2.10) applies; let $t \to r_1$ and observe that necessarily $u(r_1) = u'(r_1) = 0$ by definition of a semi-classical solution. This gives

$$a(r)|u'(r)|^{m-2}u'(r) = \int_r^{r_1} b(s) f(u(s))ds. \tag{2.11}$$

Moreover, when $r_0 > 0$ we may also let $r \to r_0$ in (2.11) and thus obtain

$$\int_{J'} b(s) f(u(s))ds \equiv \int_{r_0}^{r_1} b(s) f(u(s))ds = 0. \tag{2.12}$$

Let $\bar{s}$, $\bar{t}$ be a pair of points of $\mathbb{R}^+ \setminus J$, with $\bar{s} < \bar{t}$. Since $f(0) = 0$ it is easy to see that

$$\int_{\bar{s}}^{\bar{t}} b(s) f(u(s))ds = \sum_{J' \subset I(\bar{s},\bar{t})} \int_{J'} b(s) f(u(s))ds = 0, \tag{2.13}$$

where $I(\bar{s}, \bar{t})$ is the interval with endpoints $\bar{s}, \bar{t}$; the sum is taken over all components J' of J contained in $I(\bar{s}, \bar{t})$; and (2.12) is used at the second step.

Next let $\bar{r}$ be a *fixed* point in $\mathbb{R} \setminus J$, and r any point in J (thus in some J' with endpoints r_0, r_1). From (2.11) and (2.13) one gets

$$a(r)|u'(r)|^{m-2}u'(r) = \int_r^{r_1} b(s) f(u(s))ds = \int_r^{\bar{r}} b(s) f(u(s))ds \tag{2.14}$$

(in (2.13) take $\bar{s} = r_1$, $\bar{t} = \bar{r}$ when $r < \bar{r}$ and $\bar{s} = \bar{r}$, $\bar{t} = r_1$ when $r > \bar{r}$). But also by (2.13) it is clear that (2.14) holds equally for $r \in \mathbb{R}^+ \setminus J$ since then $u'(r) = 0$. That is, (2.14) holds for all $r \in \mathbb{R}^+$. On the other hand, the right side of (2.14) is continuously differentiable on $\mathbb{R}^+$, so as in the discussion just above, u satisfies (2.2) as a classical solution in all $\mathbb{R}^+$, which was to be proved. □

To guarantee non–singular behavior of solutions of (2.2) at $r = 0$, we consider the conditions

$$\liminf_{r\to 0^+} u(r) > 0, \qquad \limsup_{r\to 0^+} u(r) < \infty, \tag{2.15}$$

and

$$u'(r) = o([a(r)]^{-1/(m-1)}). \tag{2.16}$$

Condition (2.16) in particular precludes strongly singular behavior of solutions at $r = 0$, and is the underlying basis for the principal Proposition 3.1 below and for the main conclusions in Sections 5–8.

It is also worth mention that, with some additional proof (see Proposition 2.3), condition (2.16) can be shown to be automatic in case $a^{-1/(m-1)} \notin L^1[0, 1]$ (provided that (2.15) holds), while conversely if $a^{-1/(m-1)} \in L^1[0, 1]$ then the second part of (2.15) is automatic (provided that (2.16) holds). In particular, for the standard case $n \geq m$, $g(|x|) \equiv 1$, condition (2.16) can be deleted.

We begin with a simple result.

Proposition 2.2. *Let $u = u(|x|)$ be a semi-classical (radial) solution of* (2.2). *Then $|u'|^{m-2}u' \in C^1(J)$, and u is a classical solution of* (2.2) *in J.*

Moreover $b \in L^1[0, 1]$; and if (2.15) *and* (2.16) *hold, then as $r \to 0^+$ we have*

$$u'(r) = O\left(\left[B(r)/a(r)\right]^{1/(m-1)}\right), \qquad \textit{where } B(r) = \int_0^r b(s)ds. \tag{2.17}$$

Proof. The first conclusion has already been shown in the discussion of the radial case, following the definition of semi-classical solutions.

We assert that the quantity $\bar{q}(r) = a(r)^{1/m}b(r)^{1/m'}$ is bounded as $r \to 0^+$. In fact,

$$\bar{q}' = \bar{q}\left[\frac{1}{m}\frac{a'}{a} + \frac{1}{m'}\frac{b'}{b}\right] = b\,\psi,$$

so that by $(A1)$ and $(A3)$ both $\bar{q}$ and $\bar{q}'$ are positive on $\mathbb{R}^+$. The assertion now follows immediately. Therefore

$$b = (b/a)^{1/m}\bar{q} = O((b/a)^{1/m}),$$

and so $b \in L^1[0, 1]$ by virtue of $(A2)$.

Now choose $\varepsilon > 0$ so small that $f \circ u \in L^\infty[0, \varepsilon]$; this can be done since u is uniformly positive and bounded near 0 by (2.15), and also by the first condition of $(F1)$.

Then letting $r \to 0^+$ in (2.10), and using (2.16) together with the fact that $b \in L^1[0, 1]$, we obtain

$$a(t)|u'(t)|^{m-2}u'(t) = -\int_0^t b(s) f(u(s))ds, \qquad 0 < t < \varepsilon,$$

that is,

$$u'(r) = O\left(\left[B(r)/a(r)\right]^{1/(m-1)}\right) \quad \text{as } r \to 0^+$$

by $(A1)$ and the condition $f \circ u \in L^\infty[0, \varepsilon]$. □

Remark. To obtain similar results when condition (2.15) is not assumed apparently requires further conditions on the behavior of $f = f(u)$ near $u = 0$.

We conclude the section with an asymptotic behavior result for *strongly singular* solutions of (2.2), that is, semi-classical solutions of (2.2) which obey (2.15) but for which (2.16) fails.

Proposition 2.3. *Let $u = u(r)$ be a strongly singular solution of* (2.2). *Then as $r \to 0^+$ we have*

$$u(r) \to \alpha, \qquad a(r)|u'(r)|^{m-1} \to \eta, \tag{2.18}$$

where α and η are appropriate positive constants. Moreover, in this case necessarily $a^{-1/(m-1)} \in L^1[0, 1]$.

Proof. Putting $t = 1$ and letting $r \to 0^+$ in (2.10), we find (without using (2.16))

$$a(r)|u'(r)|^{m-1} \to \eta \qquad \text{as } r \to 0^+,$$

where η is a constant. But since the left hand quantity here is non-negative for $r > 0$ and because (2.16) is assumed to fail, in fact $\eta > 0$, proving the second part of (2.18).

It follows in turn that

$$|u'(r)| = [\eta/a(r)]^{1/(m-1)}(1 + o(1)) \qquad \text{as } r \to 0^+.$$

Now if $a^{-1/(m-1)} \notin L^1[0, 1]$ then $u(r) \to \infty$ as $r \to 0^+$, which violates (2.15). Hence $a^{-1/(m-1)} \in L^1[0, 1]$ and by integration $u(r) \to$ limit as $r \to 0^+$, the limit necessarily being α in view of (2.15). This completes the proof. □

3. Transformation of (1.2)

We now introduce the main change of variable

$$t(r) = \int_0^r [b(s)/a(s)]^{1/m}\, ds, \qquad r \geq 0. \tag{3.1}$$

Of course $t : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $t(0) = 0$, $t(\infty) = \infty$, by $(A2)$, and t is a diffeomorphism of $\mathbb{R}_0^+$ into $\mathbb{R}_0^+$ by $(A1)$, with inverse $r = r(t)$, $t \geq 0$.

If $u = u(|x|)$ is a semi-classical radial solution of (2.1), then $u = u(r)$ satisfies (2.2) in J and $v = v(t) = u(r(t))$ is a semi-classical solution of the corresponding equation in the t variable, that is $v \in C^1(\mathbb{R}^+)$ and satisfies the transformed differential equation

$$(3.2)\qquad \left[q(t)|v_t|^{m-2}v_t\right]_t + q(t)f(v) = 0 \qquad \text{in } I = \{t > 0 \ : \ v(t) > 0\},$$

where

$$(3.3)\qquad q(t) = [a(r(t))]^{1/m}[b(r(t))]^{1/m'}, \qquad t > 0.$$

Assumptions $(A1)$–$(A4)$ imply the main properties $(Q1)$–$(Q3)$ for $q = q(t)$ given in the introduction. That q is continuous at the origin, and so $q \in C(\mathbb{R}_0^+)$, now follows immediately from the condition $(Q1)$.

The following result is the main relation between semi-classical solutions of (2.2) and solutions of (3.2).

Proposition 3.1. *Let $u = u(r)$ be a semi-classical solution of* (2.2)*. Then $v = v(t) = u(r(t))$, $t > 0$, where $r = r(t)$ is the inverse function of $t = t(r)$ given in* (3.1)*, satisfies* (3.2) *in I in the classical sense with $v \geq 0$ and*

$$(3.4)\qquad v \in C^1\left(\mathbb{R}_0^+\right), \qquad |v_t|^{m-2}v_t \in C^1(I).$$

Moreover if (2.15) *and* (2.16) *hold, then there exists a positive constant α such that*

$$(3.5)\qquad v(t) - \alpha = O(t^{m/(m-1)}), \quad v_t(t) = O(t^{1/(m-1)}) \quad \text{as } t \to 0^+.$$

Proof. By (3.1) and the fact that, by definition, $u \in C^1(\mathbb{R}^+)$ it is clear that $v \in C^1(\mathbb{R}^+)$. Then, as in the earlier discussion of the solution u of (2.2), it follows that $|v_t|^{m-2}v_t \in C^1(I)$ and that $v \geq 0$ and satisfies (3.2) in I in the classical sense.

It remains to establish (3.5). First, by (2.15), (2.16) and (3.1) it is easy to check that also

$$(3.6)\qquad \liminf_{t\to 0^+} v(t) > 0, \qquad \limsup_{t\to 0^+} v(t) < \infty,$$

and

$$(3.7)\qquad v_t(t) = o([q(t)]^{-1/(m-1)}) \text{ as } t \to 0^+.$$

We can now apply Proposition 2.2 directly to the equation (3.2), of course replacing r by t, $u(r)$ by $v(t)$, and both $a(r)$, $b(r)$ by $q(t)$. Recalling that q is continuous at $t = 0$, it then follows from (2.17) that as $t \to 0^+$,

$$(3.8)\qquad v_t(t) = O\left(\left[\frac{\int_0^t q(s)\,ds}{q(t)}\right]^{1/(m-1}\right).$$

We claim that in fact $v_t(t) = O(t^{1/(m-1)})$, the second part of (3.5). This is obvious from (3.8) when $q(0) > 0$. Otherwise by l'Hôpital's rule we get

$$\lim_{t\to 0^+} \frac{\int_0^t q(s)\,ds}{tq(t)} = \lim_{t\to 0^+} \frac{q(t)}{tq_t(t) + q(t)} = \frac{1}{N}$$

by $(Q3)$, and the conclusion again follows.

Since now v_t is bounded near $t = 0$, then v must have a finite limit at $t = 0$, call it α. But then also u has the same finite limit at $r = 0$, that is $u(0) = v(0) = \alpha$, where α is necessarily positive in view of (2.15). Integrating the second part of (3.5) over $[0, t] \subset I$, we then get the first part of (3.5). □

As noted above, $u(0) = \alpha$ so that $u \in C(\mathbb{R}_0^+)$. The corresponding conditions $u'(0) = 0$ and $u' \in C(\mathbb{R}_0^+)$ are in general false, as follows from the next result.

Theorem 3.2. *Suppose that there is a number ν, with $m + \nu > 0$, such that*

$$r^{-\nu} b/a \to c \quad \text{as } r \to 0^+, \tag{3.9}$$

where c is a non-negative constant. Then if u is a semi-classical solution of (2.2) *such that* (2.15) *and* (2.16) *are satisfied, we have*

$$\lim_{r\to 0^+} r^{-(1+\nu)/(m-1)} u'(r) = -[\operatorname{sgn} f(\alpha)] \left(\frac{c\,m}{m+\nu} \cdot \frac{|f(\alpha)|}{N} \right)^{1/(m-1)}. \tag{3.10}$$

If (3.9) *is replaced by either* $\limsup_{r\to 0^+} r^{-\nu} b/a \le c$ *or* $\liminf_{r\to 0^+} r^{-\nu} b/a \ge c$, *then* (3.10) *is replaced by an analogous inequality.*

Proof. Let

$$w(t) = |v_t(t)|^{m-2} v_t(t).$$

Clearly $w(0) = 0$ by (3.5). Now observe that

$$|u'(r)|^{m-2} u'(r) = w(t(r)) \cdot \left(\frac{b}{a}\right)^{(m-1)/m} = \frac{w(t(r))}{t(r)} \left(\frac{b}{a}\right)^{1/m'} \int_0^r \left(\frac{b}{a}\right)^{1/m} ds$$

$$\equiv \frac{w(t(r))}{t(r)} \cdot I(r) \tag{3.11}$$

by (3.1). But as $t = t(r) \to 0^+$

$$\frac{w(t)}{t} \longrightarrow w'(0), \tag{3.12}$$

since by Lemma 5.1 we have $w' \in C^1$ in a closed neighborhood of 0^+. (The application of Lemma 5.1 is legitimate here since Theorem 3.2 is not used in the proof of that result.)

Now let ε be an arbitrary small positive number. Then for r sufficiently small, we have from (3.9),

$$(c - \varepsilon) r^\nu \le b/a \le (c + \varepsilon) r^\nu.$$

It then follows by direct integration, and use of the relation $1/m + 1/m' = 1$, that, for sufficiently small r,

$$(3.13) \qquad \frac{(c-\varepsilon)m}{m+\nu} r^{1+\nu} \le I(r) \le \frac{(c+\varepsilon)m}{m+\nu} r^{1+\nu}.$$

From (3.11)–(3.13), therefore,

$$(1-\varepsilon)w'(0)\frac{(c-\varepsilon)m}{m+\nu} \le \frac{|u'(r)|^{m-2}u'(r)}{r^{1+\nu}} \le (1+\varepsilon)w'(0)\frac{(c+\varepsilon)m}{m+\nu}$$

for sufficiently small r. Since ε can be taken arbitrarily small, we then obtain

$$\lim_{r\to 0^+} r^{-(1+\nu)}|u'(r)|^{m-2}u'(r) = -\frac{c\,m}{m+\nu}\cdot\frac{f(\alpha)}{N},$$

where (5.9) below has been used to eliminate $w'(0)$. The conclusion (3.10) is now an immediate consequence of this inequality.

The final parts of the theorem follow directly from the same calculation. □

Corollary 3.3. *Assume that* (2.15) *and* (2.16) *are satisfied, and suppose that*

$$b/a = O(r^\nu) \quad \text{as } r \to 0^+$$

for some exponent $\nu > -m$. Then as $r \to 0^+$

$$u(r) - \alpha = O(r^{(m+\nu)/(m-1)}), \qquad u'(r) = O(r^{(1+\nu)/(m-1)}).$$

This shows in particular that Proposition 2.2 is essentially best possible; in fact $u \in C^1(\mathbb{R}^n)$ when $\nu > -1$, while u is Hőlder continuous at the origin when $\nu \in (-m, -1]$.

4. The natural dimension N and examples

When the number $N \ge 1$, given in assumption $(A4)$, is an integer, it can be seen as the underlying dimension of the radial equation (2.2).

To explain this observation, we first note that in [9] the radial Dirichlet problem for equation (2.1)–(2.2), with $f(u) = u^s$, $s > 0$, was studied in the ball B_R. The existence of positive radial solutions of this problem was shown when the exponent s is less then ρ^* given in (1.7) of [9], that is

$$\frac{1}{\rho^*} = \frac{1}{m'} \limsup_{r\to 0+} \frac{\log \int_r^R [a(s)]^{-1/(m-1)}ds}{\left|\log \int_0^r b(s)ds\right|},$$

under the main assumption that $a^{-1/(m-1)} \notin L^1[0,1]$. Clearly here $a^{-1/(m-1)} \notin L^1[0,1]$ if and only if $q^{-1/(m-1)} \notin L^1[0,1]$ by (3.1) and (3.3). Moreover, if $N > m$

then

$$(4.1)\qquad \frac{1}{\rho^*} = \frac{1}{m'} \lim_{t\to 0^+} \frac{\log \int_t^1 [q(\tau)]^{-1/(m-1)} d\tau}{\left|\log \int_0^t q(\tau) d\tau\right|} = \frac{1}{m} - \frac{1}{N} := \frac{1}{m_N^*}$$

by $(Q3)$ and l'Hôpital's rule.[1]

That is, if N is an integer greater than m, the exponent m_N^* can be identified as the Sobolev exponent for $W^{1,m}(\mathbb{R}^N)$. We are then tempted to call N the *natural dimension* of the problem (2.1), and similarly to call m_N^* *the natural Sobolev exponent* of the problem.

The special case of (2.1) when

$$(4.2)\qquad g(r) \equiv 1, \qquad h(r) = r^\ell,$$

which was discussed earlier, exhibits the above situation in a clear way. Indeed, when (2.4) holds, then all the assumptions $(A1)$–$(A4)$ (equally $(A1)'$–$(A4)'$) are satisfied, while at the same time using the main change of variable (3.1) we see that (3.2) takes the canonical form (2.6) with

$$N = m\frac{\ell + n}{\ell + m} > 1.$$

Clearly N appears directly as the natural dimension for the problem.

For this case we have $n > m$ if and only if $N > m$, this condition also implying $a^{-1/(m-1)} = r^{-(n-1)/(m-1)} \notin L^1[0, 1]$. Thus the natural Sobolev exponent for the case (4.2) is

$$\frac{1}{m_N^*} = \frac{1}{m} - \frac{1}{N} = \frac{n-m}{\ell+n},$$

see [20].

[1] Indeed, with three applications of l'Hôpital's rule, we have

$$\begin{aligned}
\frac{1}{m'} \lim_{t\to 0^+} \frac{\log \int_t^1 q^{-1/(m-1)} d\tau}{\left|\log \int_0^t q d\tau\right|} &= \frac{1}{m'} \lim_{t\to 0^+} \frac{[q(t)]^{-m'/m}}{\int_t^1 q^{-m'/m} d\tau} \cdot \frac{\int_0^t q d\tau}{q(t)} \\
&= \frac{1}{m'} \lim_{t\to 0^+} \frac{t[q(t)]^{-m'/m}}{\int_t^1 q^{-m'/m} d\tau} \cdot \lim_{t\to 0^+} \frac{\int_0^t q d\tau}{tq(t)} \\
&= \lim_{t\to 0^+} \left(-\frac{1}{m'} + \frac{1}{m} \cdot \frac{tq_t(t)}{q(t)}\right) \cdot \lim_{t\to 0^+} \frac{q(t)}{q(t) + tq_t(t)} \\
&= \frac{1}{m} - \frac{1}{N}
\end{aligned}$$

(for the second application of l'Hôpital's rule, note that $tq^{-m'/m} \to \infty$; e.g. by differentiation and use of (Q3) with $N > m$ one computes that

$$[tq(t)^{-m'/m}]' < -\frac{N-m}{2(m-1)} q(t)^{-m'/m},$$

but $q^{-m'/m} \notin L^1[0, 1]$ which gives the asserted conclusion; it is exactly here that the condition $N > m$ is crucially used).

The number m_N^* is also critical for the uniqueness problem in the case (4.2), in the sense that uniqueness of ground states can be proved for functions $f(u) = -u + u^s$ under the condition $1 < s < m_N^* - 1$, see [20] and in [23]. We can therefore conjecture that this would be equally true in more general cases, though so far no general results seem available.

Examples. (i) As a main example, we consider the following equation (partially motivated by the ideas in [3]) which includes as special cases both (1.3) and (1.4):

$$\operatorname{div}(r^k|Du|^{m-2}Du) + r^\ell \left(\frac{r^s}{1+r^s}\right)^{\sigma/s} f(u) = 0,$$

(4.3)

$$m > 1, \qquad n \geq 1, \qquad k \in \mathbb{R}, \qquad \ell \in \mathbb{R}, \qquad s > 0, \qquad \sigma > 0.$$

Here

$$a(r) = r^{n+k-1}, \qquad b(r) = r^{n+\ell-1}\left(\frac{r^s}{1+r^s}\right)^{\sigma/s}.$$

We claim that $(A1)$–$(A4)$ are satisfied if

$$\ell \geq k - m, \qquad \frac{k}{m} + \frac{\ell}{m'} \geq 1 - n. \tag{4.4}$$

In fact, $(A1)$ is obvious, while $(A2)$ is a consequence of the first condition of (4.4) – note here that $(A2)$ restricts b/a not only for r near 0 but also for r near ∞. To get $(A3)$ we find, after a short calculation,

$$\psi(r) = \left(n - 1 + \frac{k}{m} + \frac{\ell}{m'} + \frac{\sigma}{m'} \cdot \frac{1}{1+r^s}\right) \cdot \left(\frac{1+r^s}{r^s}\right)^{\sigma/ms} \cdot r^{(k-\ell)/m-1}.$$

Each of the three terms comprising ψ is positive and strictly decreasing, as follows from (4.4) together with the fact that $s > 0$, $\sigma > 0$. Hence $(A3)$ holds.

Next, the limit $N - 1$ in $(A4)$ is given by

$$\left(n - 1 + \frac{k}{m} + \frac{\ell}{m'} + \frac{\sigma}{m'}\right) \cdot \lim_{r\to 0^+} r^{(k-\ell-\sigma)/m-1} \int_0^r t^{(\sigma-k+\ell)/m}\, dt, \tag{4.5}$$

which immediately gives

$$N = m\frac{n+\ell+\sigma}{m+\ell+\sigma-k} > 1, \tag{4.6}$$

by (4.4).

Finally $1 < m < N$ if and only if $k > m - n$, the latter condition implying $a^{-1/(m-1)} \notin L^1[0,1]$. In this case, *the natural Sobolev exponent* of equation (4.3) and its transformed equation (3.2) is given by

$$\frac{1}{m_N^*} = \frac{1}{m} - \frac{1}{N} = \frac{1}{m} \cdot \frac{n+k-m}{n+\ell+\sigma}.$$

Here $m_N^* > m$ because $N > 1$.

To obtain the asymptotic behavior of $u'(r)$ for r near 0 one can apply Theorem 3.2. In particular, here $\nu = \ell + \sigma - k$ and $c = 1$ in (3.9), and in turn from (4.6)

$$N = m\frac{n + k + \nu}{m + \nu}.$$

Thus from (3.10) we find as $r \to 0^+$

$$r^{-(1+\nu)/(m-1)}u'(r) \to -[\operatorname{sgn} f(\alpha)]\left(\frac{|f(\alpha|}{n + k + \nu}\right)^{1/(m-1)}. \tag{4.7}$$

It is finally interesting in this example that the parameter s in (4.3) does not appear in any of the exponent relations (4.4)–(4.6). This is a reflection of the fact that the term $r^s/(1 + r^s)$ in (4.3) can be replaced by more general functions having the same asymptotic behavior. We shall, however, not pursue this point further.

(*ii*) Conditions (4.4) have the first consequence that $\ell \geq -n$. Moreover, either a can be discontinuous ($k < 1 - n$) or b discontinuous ($-n \leq \ell < 1 - n$), *but not both* in view of the second condition of (4.4). One can show that necessarily $N < m$ when a is discontinuous, while it is possible to have $N > m$ when b is discontinuous (if $k > m - n$).

An example where a is discontinuous is $n = 3$, $m = 2$, $k = -5/2$, $\ell = -1/2$, while b is discontinuous and $N > m$ when $n = 3$, $m = 2$, $k = -1/2$, $\ell = -9/4$, and $0 < \sigma < 1/4$.

(*iii*) Equation (1.3) in the introduction is associated with the well–known Matukuma model. Here we have $k = 0$, $-\ell = \sigma = s$, and the exponent conditions for $(A1)$–$(A4)$, or equally for $(A1)'$–$(A4)'$, reduce simply to $m \geq \sigma$ and $n \geq 1 + \sigma/m$, with $N = n$.

From Theorem 3.2, since $\nu = \ell + \sigma - k = 0$, we get $u'(r) = O(r^{1/(m-1)})$ and in turn $u(r) - \alpha = O(r^{m/(m-1)})$. In particular $u \in C^1(\mathbb{R}^n)$, as in fact is clear from equation (1.3) itself.

For the standard Matukuma equation, namely when $k = 0$, $m = 2$, $-\ell = \sigma = s = 2$ and $n = 3$, the transformed equation (3.2) arises with $q(t) = \sinh^2 t/\cosh t$. In this case we have $N = n = 3$, and so *the critical Sobolev exponent* is $2_3^* = 6$, the usual critical exponent for the Matukuma equation in $\mathbb{R}^3$, as is well known in the literature.[2]

[2] It may be worth mention here that the Matukuma–type equation

$$\Delta u + \frac{u^{p-1}}{(1 + r^2)^{\sigma/2}} = 0 \qquad \text{in } \mathbb{R}^n, \quad n > 2, \quad \sigma > 0, \quad p \neq 2,$$

has a solution of the form

$$u(r) = c(1 + r^2)^{-(n-2)/2},$$

when

$$p = \frac{2n}{n-2} - \frac{\sigma}{n-2}, \qquad c = [n(n-2)]^{1/(p-2)}.$$

For example, when $n = 3$, $\sigma = 1$, and $p = 5$ the solution is $3^{1/3}(1 + r^2)^{-1/2}$, and when $n = 4$, $\sigma = 2$, and $p = 3$ the solution is $8/(1 + r^2)$.

Corresponding solutions can also be obtained for values $m \neq 2$, but we shall not follow on this here.

(iv) For the model (1.4) in the introduction, generalizing equation (4.21) of [1], we have $k = 0$, and $s = m' = -\ell$, so that the conditions (4.4) reduce simply to $n \geq 2$, $m \geq 2$, with now

$$N = m\,\frac{n + \sigma - m'}{m + \sigma - m'} > 1.$$

In the original equation (4.21) of [1], that is in the further subcase $m = 2$, $n = 3$, we have $N = 2(1 + \sigma)/\sigma > 2$. When $\sigma = 2$ one finds again the standard Matukuma equation, with $N = 3$ and with the solution $u(r) = \sqrt{3/(1 + r^2)}$ when $f(u) = u^3$, noted in the introduction, while the case $\sigma = 1$ produces a new elliptic equation with $N = 4$.

From Theorem 3.2, since $\nu = \ell + \sigma - k = \sigma - m'$, we get $u'(r) = O(r^{[(m-1)\sigma-1]/(m-1)^2})$ and $u \in C^1(\mathbb{R}^n)$ when $\sigma > 1/(m-1)$; while $|u'(r)| \to \infty$ when $\sigma < 1/(m-1)$. In both cases $u(r) - \alpha = O(r^{\sigma/(m-1)})$. Note that even in the canonical case $m = 2$, $n = 3$, from (4.7) one can have the anomalous asymptotic behavior $u'(r) \to -f(\alpha)/2$ when $\sigma = 1$ (and so $N = 4$), while $|u'(r)| \to \infty$ when $\sigma < 1$.

(v) The model

$$\operatorname{div}(r^{2-m}|Du|^{m-2}Du) + \frac{f(u)}{1 + r^\sigma} = 0, \qquad m > 1, \quad n \geq 1, \quad \sigma > 0, \tag{4.8}$$

studied in [3] and [9], is the special case $k = 2 - m$, $\ell = -\sigma$, $\sigma = s$ of (4.3). From (4.4) this gives the following condition on the exponents in (4.8) for the satisfaction of $(A1)$–$(A4)$:

$$\sigma \leq 2(m-1), \qquad n \geq \frac{\sigma + 2}{m'}.$$

Moreover, the critical dimension becomes $N = mn/2(m-1) > 1$. Finally we have $1 < m < N$ if and only if $n > 2(m-1)$, namely $a^{-1/(m-1)} \notin L^1[0, 1]$. In this case *the natural Sobolev exponent* for (4.8), as defined above, is

$$\frac{1}{m_N^*} = \frac{1}{m} - \frac{1}{N} = \frac{n - 2(m-1)}{nm},$$

as shown already in [9]. Again $m_N^* > m$, since $m > 1$.

From Theorem 3.2, since $\nu = m - 2$, we get $u'(r) = O(r)$, $u(r) - \alpha = O(r^2)$ as $r \to 0^+$, and specifically $u \in C^1(\mathbb{R}^n)$.

(vi) For the model

$$\operatorname{div}(r^{2-m}|Du|^{m-2}Du) + \frac{r^{\sigma-m}}{(1 + r^m)^{\sigma/m}} f(u) = 0, \qquad m > 1, \quad n \geq 1, \quad \sigma > 0, \tag{4.9}$$

introduced in [3] and [9], we have $k = 2 - m$, $\ell = -m$, $s = m$, so that the required exponent conditions for $(A1)$–$(A4)$ now found to be

$$m \geq 2, \qquad n \geq m + 1 - \frac{2}{m},$$

with

$$N = m\,\frac{n+\sigma-m}{m+\sigma-2} > 1. \tag{4.10}$$

Again $1 < m < N$ if and only if $n > 2(m-1)$, that is $a^{-1/(m-1)} \notin L^1[0,1]$. In this case *the natural Sobolev exponent* of equation (4.9) and its transformed equation (3.2) is

$$\frac{1}{m_N^*} = \frac{1}{m} - \frac{1}{N} = \frac{1}{m}\cdot\frac{n-2(m-1)}{n+\sigma-m},$$

as found in [9]. Of course $m_N^* > m$ because $m \geq 2$ and $\sigma > 0$.

From Theorem 3.2, since $\nu = \sigma - 2$ we get $u'(r) = O(r^{(\sigma-1)/(m-1)})$ as $r \to 0^+$; hence $u \in C^1(\mathbb{R}^n)$ when $\sigma > 1$, while $|u'(r)| \to \infty$ when $\sigma < 1$. In both cases $u(r) - \alpha = O(r^{(m+\sigma-2)/(m-1)})$.

(vii) Tso [24], p. 99, has noticed that the m–Hessian operator H_{m-1}, where $m \in \mathbb{Z}$, $2 \leq m \leq n+1$, when written for a radially symmetric argument $u = u(r)$ takes the form Const. $r^{-n+1}(r^{n-m+1}|u'|^{m-2}u')'$. Thus taking $k = 2-m$ the results of the remaining sections of the paper can be applied to this case.

$(viii)$ The case $\sigma = 0$ in (4.3) is not allowed. If, nevertheless, we do set $\sigma = 0$ then conditions $(A1)$–$(A4)$ will be satisfied if the relations of (4.4) hold as strict inequalities, with

$$N = m\,\frac{n+\ell}{m+\ell-k} > 1.$$

All this of course corresponds directly to the previous example (4.2), confirming again the role of N as the natural dimension of the problem.

Part II

5. Qualitative behavior of ground states

The further purpose of the paper is to investigate qualitative properties, compact support principles and uniqueness for *semi-classical radial ground states* of the main problem (2.1), that is,

$$\begin{gathered} \operatorname{div}(g(|x|)|Du|^{m-2}Du) + h(|x|)f(u) = 0 \quad \text{in } \Omega; \\ m > 1, \quad n \geq 1, \\ u \geq 0, \quad u \not\equiv 0, \quad \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{5.1}$$

through properties possessed by semi-classical ground states of the corresponding transformed problem

$$\begin{gathered} \left[q(t)|v_t|^{m-2}v_t\right]_t + q(t)f(v) = 0 \qquad \text{in } I = \{t > 0 \ : \ v(t) > 0\}, \\ v \geq 0, \quad v \not\equiv 0, \quad \lim_{t\to\infty} v(t) = 0, \end{gathered} \tag{5.2}$$

again with $m > 1$, see (3.2) and Proposition 3.1.

We are also able to treat the corresponding homogeneous Dirichlet–Neumann free boundary problem

$$\begin{gathered} \operatorname{div}(g(|x|)|Du|^{m-2}Du) + h(|x|)f(u) = 0 \\ \text{in } \Omega_R = \{x \in B(0,R) \setminus \{0\} \ : \ u(x) > 0\} \subset \mathbb{R}^n, \quad R > 0, \\ u \geq 0, \quad u \not\equiv 0; \quad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B(0,R), \end{gathered} \tag{5.3}$$

with $m > 1$, $n \geq 1$, through the transformed problem, with $T = t(R)$,

$$\begin{gathered} \left[q(t)|v_t|^{m-2}v_t\right]_t + q(t)f(v) = 0 \qquad \text{in } I_T = \{t \in (0,T) \ : \ v(t) > 0\}, \\ v \geq 0, \quad v \not\equiv 0, \quad v(T) = v_t(T) = 0. \end{gathered} \tag{5.4}$$

In (5.2) and (5.4) the function q is given by (3.3) and satisfies $(Q1)$–$(Q3)$.

With the respective end conditions at $T = \infty$ in (5.2) and at $T = t(R)$ in (5.4), the problems (5.2) and (5.4) can be unified into the single statement

$$\left[q(t)|v_t|^{m-2}v_t\right]_t + q(t)f(v) = 0 \qquad \text{in} \quad I = \{t \in (0,T) \ : \ v(t) > 0\}, \tag{5.5}$$

where $v = v(t)$ obeys (3.5).

In order that the solution $v = v(t)$ be suitably well-behaved at the origin, we also impose on the original function $u = u(r)$ the initial conditions (2.15), (2.16). Then by Proposition 3.1 there holds, specifically, for some positive constant α,

$$v(0) = \alpha > 0, \qquad v'(0) = 0, \tag{5.6}$$

where, for simplicity, from this point on we write $' = d/dt$ if there is no confusion in the notation. In the sequel it is the initial condition (5.6) which we shall use without further discussion.[3]

Moreover we shall follow the paper [8], as well as [20], [23], [10] and [17], in all of which, however, $q(t)$ is the special function t^{N-1}, $N > 1$.

Now let v be a classical solution of (5.5)–(5.6), *and with the natural end conditions given in* (5.2) *or* (5.4). Of course $(Q1)$–$(Q3)$ and $(F1)$ are assumed to hold without further mention. As in Section 3, we consider the function

$$w(t) = |v'(t)|^{m-2}v'(t)$$

and put $I_0 = I \cup \{0\}$.

[3] If condition (2.16) is not assumed, then by Proposition 2.3 applied to the solution v, we get

$$q(t)|v'(t)|^{m-1} \to \eta > 0.$$

Therefore, since q is continuous at $t = 0$,

$$|v'(t)| \to \text{either } \infty \text{ or a positive constant}$$

as $t \to 0^+$. In particular the second condition of (5.6) fails, and the further considerations in the paper cannot be carried through.

Lemma 5.1. *The function w is of class $C^1(I_0)$ and is a solution of*

$$(qw)' + qf(v) = 0 \qquad \text{on } I_0 = \{0 \le t < T \ : \ v(t) > 0\}. \tag{5.7}$$

Moreover, denoting by t_0 the first zero of v in $(0, T)$, if any, or otherwise $t_0 = T$, we have

$$w(t) = -\frac{1}{q(t)} \int_0^t q(s) f(v(s)) ds, \qquad 0 < t < t_0, \tag{5.8}$$

$$w(0) = 0, \qquad w'(0) = -\frac{f(\alpha)}{N}, \tag{5.9}$$

where $N \ge 1$ is the number given in $(Q3)$.

Finally, putting $\rho(t) = |v'(t)|$, there holds

$$\lim_{t \to 0^+} \frac{q'(t)}{q(t)} \rho^m(t) = 0 \quad \text{and} \quad \frac{q'}{q} \rho^m \in C[0, T). \tag{5.10}$$

Proof. Of course $w(0) = |v'(0)|^{m-2} v'(0) = 0$, since $v'(0) = 0$ by (3.5). By Proposition 3.1 it is evident that w is of class $C[0, T) \cap C^1(I)$ and satisfies (5.7). It is still to be shown that $w \in C^1(I_0)$.

Integrating (5.7) over $[0, t]$ for $t < t_0$, we get, since $w(0) = 0$ and q is bounded near $t = 0$,

$$q(t)w(t) = -\int_0^t q(s) f(v(s)) ds, \qquad 0 < t < t_0,$$

so (5.7) and (5.8) are proved. Differentiating (5.8) now gives

$$w'(t) = -f(v(t)) + \frac{tq'(t)}{q(t)} \cdot \frac{1}{tq(t)} \int_0^t q(s) f(v(s)) ds,$$

and the second part of (5.9) now follows at once by $(Q3)$ and L'Hôpital's rule. Thus in particular w is of class $C^1(I_0)$.

Finally,

$$\frac{q'(t)}{q(t)} \rho^m(t) = \frac{tq'(t)}{q(t)} \cdot v'(t) \cdot \frac{w(t)}{t},$$

and (5.10) is then a consequence of $(Q3)$, (5.9) and (3.5). □

Corollary 5.2. *If $v'(t) \ne 0$ at some t, with $0 < t < t_0$, then v'' exists at this point and satisfies* (5.5) *in the form*

$$(m-1)\rho^{m-2} v'' - \frac{q'}{q} \rho^{m-1} + f(v) = 0, \qquad \rho = |v'|. \tag{5.11}$$

Proof. By (5.8) of Lemma 5.1 and the fact that $v'(t) \ne 0$ we have

$$|v'(t)| = \left| \int_0^t \frac{q(s)}{q(t)} f(v(s)) ds \right|^{1/(m-1)}.$$

Since the integral is not zero, the function on the right hand side is differentiable at t. Hence v'' exists at t and from (5.5) we get exactly (5.11), since $|v'(t)| = \rho(t) > 0$. □

A natural energy function associated to solutions v of (5.5) is given by

$$E(t) = \frac{\rho^m(t)}{m'} + F(v(t)), \qquad \rho = |v'|. \tag{5.12}$$

Lemma 5.3. *The energy function E is of class $C^1(I_0)$, with $E'(0) = 0$ and*

$$E'(t) = -\frac{q'(t)}{q(t)}\rho^m(t) \qquad \text{in } I. \tag{5.13}$$

Proof. Obviously by $(F1)$ and the fact that $v \in C^1[0, T)$

$$\frac{dF(v(t))}{dt} = f(v(t))v'(t),$$

this formula being valid only when $v(t) > 0$, namely in I. Moreover, from

$$\frac{m-1}{m}\rho^m = \int_0^\rho p\,dp^{m-1} = \int_0^{\rho^{m-1}} s^{1/(m-1)}ds,$$

and $\rho^{m-1}(t) = [\operatorname{sgn} v'(t)]w(t)$, we get

$$\frac{d}{dt}\left[\frac{m-1}{m}\rho^m\right](t) = \rho(t)[\operatorname{sgn} v'(t)]w'(t) = v'(t)w'(t).$$

Therefore on I,

$$E'(t) = v'(t)\Big[w'(t) + f(v(t))\Big] = -v'(t)\frac{q'(t)}{q(t)}w(t)$$

by (5.7), and (5.13) follows at once. Finally by (5.10) we see that $E \in C^1(I_0)$, with $E'(0) = 0$. □

Theorem 5.4. *If $v(\tau_0) = 0$ for some $\tau_0 > 0$, then $v \equiv 0$ on $[\tau_0, T)$.*

Proof. Since $v \geq 0$, clearly $v'(\tau_0) = 0$. Hence $E(\tau_0) = 0$ by (5.12). Assume for contradiction (without loss of generality) that there is t_1, with $\tau_0 < t_1 \leq T$ such that again $v(t_1) = 0$ and $v(t) > 0$ in (τ_0, t_1). Then $(\tau_0, t_1) \subset I$ and $E' \leq 0$ in (τ_0, t_1) by (5.13) and $(Q1)$. There are now two cases:

Case 1. $t_1 \leq T < \infty$. Then with the help of the given end conditions in (5.4) we get $v(t_1) = v'(t_1) = 0$, and in turn $E(t_1) = 0$. Thus since $E' \leq 0$ in (τ_0, t_1) there follows $E \equiv E' \equiv 0$ in (τ_0, t_1), and so $\rho^m \equiv 0$ on $[\tau_0, t_1)$ by (5.13) and $(Q1)$, namely $v' \equiv 0$, a contradiction, and the theorem is proved.

Case 2. $t_1 = T = \infty$. Here we assert that

$$v'(t), \;\; E(t) \to 0 \quad \text{as } t \to t_1 = \infty. \tag{5.14}$$

Indeed, since $v(t) \to 0$ as $t \to \infty$, then $F(v(t)) \to 0$ by $(F1)$, and so $E(t)$ decreases to a finite non–negative limit as $t \to \infty$ by (5.12). Consequently, $v'(t) \to$ limit as $t \to \infty$, the limit necessarily being 0 again since $v(t) \to 0$ as $t \to \infty$. Therefore (5.14) holds as claimed. With (5.14) proved, the remaining argument is the same as in Case 1. □

By Theorem 5.4 it follows that *any solution of* (5.5)*, with the given end conditions, has as its (open) support set I exactly an initial interval* $(0, t_0)$*, with* $t_0 \le T$. In turn, recalling that $v \in C^1[0, T)$, see (5.6), one deduces from (5.13) that actually $E \in C^1[0, T)$, and that (5.13) holds in the entire maximal interval $[0, T)$. Therefore for any $0 \le s_0 < t < T$ we have

$$E(t) - E(s_0) = -\int_{s_0}^{t} \frac{q'(s)}{q(s)}\rho^m(s)ds. \tag{5.15}$$

Clearly $E(0) = F(\alpha)$ by (5.12) and (3.5). Thus, letting $s_0 \to 0^+$ in (5.15), we obtain

$$E(t) = F(\alpha) - \int_{0}^{t} \frac{q'(s)}{q(s)}\rho^m(s)ds, \quad 0 \le t < T. \tag{5.16}$$

Finally, as in the proof of Theorem 5.4 there holds $E(T) = 0$ when $T < \infty$, while in the other case, when $t_0 = T = \infty$, we see from (5.14) that $v'(t)$, $E(t) \to 0$ as $t \to \infty$. Thus in both cases the non–negative function $q'\rho^m/q$ is integrable on $[0, T)$, $T \le \infty$, with

$$\int_{0}^{T} \frac{q'(s)}{q(s)}\rho^m(s)ds = F(\alpha). \tag{5.17}$$

In summary, *a semi-classical radial ground state of* (5.2), (5.6)*, or a semi-classical radial solution of* (5.4), (5.6)*, has the property that*

$$v(0) = \alpha > 0, \quad v'(0) = 0, \quad v(T) = v'(T) = 0, \tag{5.18}$$

where respectively $T = \infty$ *or* $T = t(R) < \infty$*. Furthermore, by* (5.16) *and* (5.17)*,*

$$E(t) = \int_{t}^{T} \frac{q'(s)}{q(s)}\rho^m(s)ds \ge 0, \tag{5.19}$$

and clearly also, by (5.6)*,* (5.12) *and* (5.17)*,*

$$E(0) = F(\alpha) = \int_{0}^{T} \frac{q'(s)}{q(s)}\rho^m(s)ds > 0. \tag{5.20}$$

Lemma 5.5. *If* $s_0 \ge 0$ *is a critical point of* v*, with* $v(s_0) > 0$*, then* $v(t) \le v(s_0)$ *for* $t > s_0$ *and* $f(v(s_0)) \ge 0$.

Proof. Let $s_0 \ge 0$ be a critical point of v. Assume for contradiction that there are two points $t_1, \; t_2 > s_0$ such that $v(t_1) > v(s_0)$ and $v(t_2) < v(s_0)$. Then, there is $s \in (t_1, t_2)$ such that $v(s) = v(s_0)$ and v is not constant on $[s_0, s]$. It then follows from (5.15) that

$$\frac{\rho^m(s)}{m'} + \int_{s_0}^{s} \frac{q'}{q}\rho^m d\tau = 0$$

and both terms are non-negative by $(Q1)$. Thus in particular $\rho^m \equiv 0$ on $[s_0, s]$, so $v'(t) \equiv 0$ on $[s_0, s]$, which is impossible. Hence either $v(t) \ge v(s_0)$ for $t > s_0$, or $v(t) \le v(s_0)$ for $t > s_0$. The first case cannot occur since $v(T) = 0$.

In the second case, since $v'(s_0) = 0$, then $w(s_0) = 0$, and by (5.7), at $t = s_0$, we have

$$f(v(s_0)) = -w'(s_0) \geq 0.$$

Indeed, otherwise $w'(s_0) > 0$, and so there is $t_3 > s_0$ such that $w'(t) > 0$ on $[s_0, t_3]$; in turn $w(t) > w(s_0) = 0$ and $v'(t) > 0$ on $[s_0, t_3]$, which gives $v(t) > v(s_0)$ on $[s_0, t_3]$, a contradiction. □

It is convenient to introduce the following further condition on f:

$(F2)$ there are constants β, γ, with $0 < \beta < \gamma \leq \infty$, such that

$$F(s) \leq 0 \quad \text{for } 0 \leq s \leq \beta,$$

and

$$f(s), \quad F(s) > 0 \quad \text{for } \beta < s < \gamma.$$

Clearly if $(F2)$ holds for some $\gamma > \beta$, then it continues to hold for all $\gamma' \in (\beta, \gamma)$. Consequently there exists a maximal γ, possibly infinite, for which $(F2)$ is valid. Without loss of generality we can assume that γ in $(F2)$ is maximal.

Proposition 5.6. *Let v be a semi-classical ground state of* (5.2)*, or a semi-classical solution of* (5.4)*. Suppose that* (5.6) *is satisfied, so that* (5.18) *also holds. Then $t = 0$ is a maximum of v, and $v' \leq 0$ on $[0, T)$; furthermore $f(\alpha) \geq 0$ and $F(\alpha) > 0$.*

Moreover, if $(F2)$ holds and $0 < \alpha < \gamma$, then

(i) $\alpha > \beta$ and $f(\alpha) > 0$,

(ii) $v'(t) < 0$ when $t > 0$ and $v(t) > 0$.

Proof. By Lemma 5.5 and the condition $v'(0) = 0$ one sees that $v(t) \leq v(0) = \alpha$ for $t > 0$. The fact that $f(\alpha) \geq 0$ similarly follows from Lemma 5.5, while $F(\alpha) > 0$ is just (5.20).

Next assume for contradiction that $v'(s_1) > 0$ for some $s_1 > 0$. Since $v(s_1) \leq v(0)$, there is a minimum s in $(0, s_1)$, with $v(s) < v(s_1)$. Hence $v(t) \geq v(s)$ for $t > s$ by Lemma 5.5. If $v(s) > 0$, then $v(t)$ cannot approach 0 as $t \to T$, contradicting the last condition of (5.18). Therefore $v(s) = 0$ with $s > 0$, and by Lemma 5.4 we get $v \equiv 0$ on $[s, T)$: thus $v'(s_1) = 0$, which is again a contradiction. Hence $v'(t) \leq 0$ on $[0, T)$.

To show (i) it is enough to observe that if $f(\alpha) = 0$ and $0 < \alpha < \gamma$, then $F(\alpha) \leq 0$ by assumption $(F2)$. This is impossible by (5.20), proving (i).

To obtain (ii), assume for contradiction that there is a point $s_0 > 0$ such that $v'(s_0) = 0$ and $0 < v(s_0) < \gamma$. Since $v'(t) \leq 0$ for $t \geq 0$, then both $v(t) \geq v(s_0)$ for $0 \leq t < s_0$ and $v(t) \leq v(s_0)$ for $s_0 < t < T$. Of course, $w(s_0) = 0$. We claim that also $w'(s_0) = 0$. Indeed, if $w'(s_0) > 0$, then w would be strictly increasing at s_0, namely v' would change sign at s_0, which is impossible since $v'(t) \leq 0$ on $[0, T)$. Analogously, the case $w'(s_0) < 0$ also cannot occur.

Since w is a solution of (5.7), we get $f(v(s_0)) = 0$. Also $0 < v(s_0) < \gamma$, so that $F(v(s_0)) \leq 0$ by $(F2)$. Hence by (5.19), with $t = s_0$,

$$0 \leq \int_{s_0}^{T} \frac{q'(s)}{q(s)} \rho^m(s) ds = E(s_0) = F(v(s_0)) \leq 0, \tag{5.21}$$

which implies $v' \equiv 0$ on $[s_0, T)$ by $(Q1)$. Thus $v(t) \equiv v(s_0) > 0$ for $s_0 \le t < T$, again contradicting the last condition of (5.18). This completes the proof of (ii). □

The next result gives a necessary and a sufficient condition for a semi-classical radial ground state to have compact support (for this result the full strength of condition $(F2)$ is not needed).

Theorem 5.5. *Let v be a semi-classical ground state of* (5.2)*, with* $v(0) = \alpha > 0$.

(i) If $F(u) \le 0$ for all values $0 < u < \beta$, for some $\beta > 0$, and

$$\int_{0^+} \frac{du}{|F(u)|^{1/m}} < \infty, \tag{5.22}$$

then v has compact support in $\mathbb{R}^+$.

(ii) Conversely, assume there exists $\delta > 0$ and a non–decreasing function $\Phi : [0, \delta) \to \mathbb{R}$, with $\Phi(0) = 0$, such that $|F(u)| \le \Phi(u)$ for all $u \in [0, \delta)$. If v has compact support in $\mathbb{R}^+$, then

$$\int_{0^+} \frac{du}{\Phi(u)^{1/m}} < \infty. \tag{5.23}$$

Proof. Let v be a semi-classical radial ground state as in the theorem, so $T = \infty$.

(i) Suppose (5.22) holds. We denote by $t_\beta > 0$ any point such that $0 \le v(t) < \beta$ on (t_β, ∞). We assert that $v'(t) < 0$ for all $t \in (t_\beta, \infty)$ for which $v(t) > 0$. Otherwise, if $0 < v(s_0) < \beta$, $v'(s_0) = 0$ for some $s_0 \in (\beta, \infty)$, then since $F(s) \le 0$ on $(0, \beta)$, it follows that (5.21) holds. This gives a contradiction exactly as in last lines of the proof of Proposition 5.6 (ii). Hence the assertion is proved.

Thus by Theorem 5.4 either $v \equiv 0$ for all t sufficiently large, or $v > 0$ and $v' < 0$ on (t_β, ∞). In the first case we are done. Otherwise, denoting by $t = t(v)$ the inverse function of $v(t)$ on (t_β, ∞), then from the fact that $m' > 1$, together with (5.12) and (5.19), we get

$$\rho^m(t) \ge \frac{\rho^m(t)}{m'} = -F(v(t)) + \int_t^\infty \frac{q'(s)}{q(s)} \rho^m(s) ds > -F(v(t)),$$

or $v'(t) < -|F(v(t)|^{1/m}$ on (t_β, ∞) by the assumption that $F(u) \le 0$ on $[0, \beta)$. That is, writing $t = t(v)$ and putting $\varepsilon = v(t_\beta)$, we have

$$\frac{1}{t'(v)} < -|F(v)|^{1/m} \quad \text{for} \quad v \in v((t_\beta, \infty)) = (0, \varepsilon),$$

since $v(t) \to 0$ as $t \to \infty$. By integration over (v, ε),

$$\int_{v(t)}^{\varepsilon} \frac{du}{|F(u)|^{1/m}} > -\int_{v(t)}^{v(t_\beta)} t'(u) du = t - t_\beta.$$

Hence, letting $t \to \infty$, there results

$$\int_0^\varepsilon \frac{du}{|F(u)|^{1/m}} = \infty.$$

This contradicts (5.22) and completes the proof of part *(i)* of the theorem.

(ii) Let v have compact support. Then by Theorem 5.4 and the first part of Proposition 5.6 there is $t_0 > 0$ such that $v'(t) \le 0$ and $0 < v(t) \le v(0) = \alpha$ on $(0, t_0)$, while $v \equiv 0$ on $[t_0, \infty)$. Let $t_\delta \in (0, t_0)$ be some fixed point such that $0 < v(t) < \delta$ on (t_δ, t_0). By (5.15), with $s_0 = t_0$, for $0 < t_\delta < t < t_0$ we have

$$\frac{\rho^m(t)}{m'} = -F(v(t)) + \int_t^{t_0} \frac{q'(s)}{q(s)}\rho^m(s)ds \le \Phi(v(t)) + c_1 \int_t^{t_0} \frac{\rho^m(s)}{m'}ds,$$

by $(Q2)$, with $c_1 = m'q'(t_\delta)/q(t_\delta)$. Applying Gronwall's inequality yields

$$\frac{\rho^m(t)}{m'} \le \Phi(v(t)) + c_1 \int_t^{t_0} \Phi(v(s))e^{c_1(s-t)}ds.$$

Now $\Phi(v(t))$ is non–increasing on (t_δ, t_0), since Φ is non–decreasing by assumption and v is non–increasing on (t_δ, t_0). Hence

$$\rho^m(t) \le C\Phi(v(t)), \quad \text{with} \quad C = m'e^{c_1(t_0-t_\delta)} > 1.$$

Therefore,

$$\frac{-v'(t)}{\Phi(v(t))^{1/m}} \le C^{1/m} \quad \text{on} \quad (t_\delta, t_0).$$

Integrating on $[s_0, t]$, with $t_\delta < s_0 < t < t_0$, we get

$$\int_{v(t)}^{v(s_0)} \frac{du}{\Phi(u)^{1/m}} = -\int_{s_0}^{t} \frac{v'(s)}{\Phi(v(s))^{1/m}}ds \le C^{1/m}(t - s_0).$$

Letting $t \to t_0^-$, this gives

$$\int_0^{v(s_0)} \frac{du}{\Phi(u)^{1/m}} \le C^{1/m}(t_0 - s_0),$$

that is (5.23) holds. This completes the proof of part *(ii)* of the theorem. □

As an immediate consequence of Theorem 5.5 we obtain

Corollary 5.8. *Let v be a semi-classical ground state of* (5.2)*, with $\alpha > 0$, and assume $f \le 0$ on $(0, \beta')$ for some $\beta' > 0$ (see condition $(F3)$ below). Then $v(t) > 0$ for every $t \in \mathbb{R}^+$ if and only if*

(5.24)
$$\int_{0^+} \frac{du}{|F(u)|^{1/m}} = \infty.$$

If $|f(u)| \sim u^p$ as $u \to 0^+$, with $p > -1$ by $(F1)$, then by Corollary 5.8 and Proposition 5.6 the conditions $v(t) > 0$, $v'(t) < 0$ hold for all $t \in \mathbb{R}^+$ if and only if $p \ge m - 1$. In this case, since $m > 1$, this means that f is regular at $u = 0$, with $f(0) = 0$.

Conversely, if $-1 < p < 0$, as in the example (1.7), then any radial ground state necessarily has compact support, and in turn (1.2) can be considered as valid only at points where $u > 0$.

For a more general discussion of the validity of the strong maximum and compact support principles for solutions, radial or not, of quasilinear elliptic inequalities, as well as on applications of these principles to variational problems on manifolds and to existence of radial dead cores, we refer to [22].

6. Asymptotic behavior of ground states

We assume from now on that the nonlinearity f obeys $(F1)$, $(F2)$ and

$(F3)$ there exists a maximal number $\beta' \in (0, \beta]$ such that $f(u) \le 0$ for $u \in (0, \beta']$.

Example (1.7) satisfies conditions $(F1)$–$(F3)$ since $-1 < p < s$, with $\beta = [(s+1)/(p+1)]^{1/(s-p)}$, $\beta' = 1$ and $\gamma = \infty$.

Let v be a fixed solution of (5.5), satisfying (5.18).

Proposition 6.1. *There is a number* $\lambda \ge 0$ *such that*

$$\lim_{t\to T} q(t)\rho^{m-1}(t) = \lambda, \tag{6.1}$$

with $\lambda = 0$ *when* T *is finite, or when* $T = \infty$ *and* $q^{-1/(m-1)} \notin L^1[1,\infty)$.

Furthermore,

$$\liminf_{t\to T} tq(t)E(t) = \liminf_{t\to T} Q(t)E(t) = 0, \tag{6.2}$$

where $Q(t) = \int_0^t q(s)ds$, *and also*

$$\lim_{t\to T} q(t)\int_t^T \frac{q'(s)}{q(s)}\rho^m(s)ds = \lim_{t\to T} q(t)E(t) = 0. \tag{6.3}$$

Proof. The only non–trivial case of the asymptotic behavior is when $T = \infty$ and $v(t) > 0$ for all $t \in \mathbb{R}^+$.

Since $v' < 0$ by Proposition 5.6 (ii), then $q\rho^{m-1}$ is non-increasing in $[t_{\beta'}, \infty)$ by (5.7) and $(F3)$, where $t_{\beta'}$ is the unique point with $v(t_{\beta'}) = \beta'$. Thus, (6.1) holds and $\lambda \ge 0$ is finite.

If $q^{-1/(m-1)} \notin L^1[1,\infty)$, and if for contradiction we assume $\lambda > 0$ in (6.1), then from $[q(t)]^{1/(m-1)}\rho(t) \to \lambda^{1/(m-1)}$ as $t \to \infty$, it follows that $\rho(t) \ge c[q(t)]^{-1/(m-1)}$, for some constant $c > 0$ and for all large t. This inequality is impossible since $-v' = \rho \in L^1[0,\infty)$. Hence $\lambda = 0$.

By (5.19) the energy function $E(t) \ge 0$ for all t, hence

$$0 \le Q(t)E(t) \le Q(t)\frac{\rho^m(t)}{m'} \quad \text{and} \quad 0 \le tq(t)E(t) \le tq(t)\frac{\rho^m(t)}{m'},$$

on $[t_{\beta'}, \infty)$, since $F(u) \le 0$ on $(0, \beta']$ by $(F3)$. Now, by (6.1), for all t sufficiently large

$$0 \le Q(t)\frac{\rho^m(t)}{m'} = \frac{Q(t)}{q(t)}\rho(t)\cdot q(t)\frac{\rho^{m-1}(t)}{m'} \le (\lambda+1)\frac{Q(t)}{q(t)}\rho(t),$$

and analogously,

$$0 \le tq(t)\frac{\rho^m(t)}{m'} \le (\lambda+1)t\rho(t).$$

We claim that

$$0 \le \liminf_{t\to\infty} \frac{Q(t)}{q(t)}\rho(t) \le \liminf_{t\to\infty} t\rho(t) = 0,$$

so that (6.2) follows at once from the previous inequalities. In fact the second inequality is a consquence of the relation $Q(t)/q(t) \le t$, which holds by $(Q1)$. Next, if the final limit is not zero, there would be a constant $C > 0$ such that $\rho(t) \ge C/t$ for all sufficiently large t. This gives a contradiction since obviously $\rho \in L^1[0,\infty)$.

The first equality of (6.3) is an immediate consequence of (5.19). Next, by (5.12),

$$0 \le q(t)E(t) \le q(t)\rho^{m-1}(t) \cdot \rho(t)/m',$$

since as already noted $F(u) \le 0$ on $[0, \beta']$ by (F3). The second equality of (6.3) now follows from (6.1), and (5.18). □

7. Monotone separation properties

We assume conditions $(F1)$–$(F3)$ throughout this section, as well as $(Q1)$–$(Q3)$. Let v_1 and v_2 be two semi-classical ground states of (5.2), or two semi-classical solutions of (5.4), (5.6) whose initial values α_1, α_2 verify the principal condition

$$\beta < \alpha < \gamma; \tag{7.1}$$

see Proposition 5.6 (i). Of course both v_1 and v_2 have the regularity properties described in Lemma 5.1.

Denote by $I_1 = (0, t_{01})$ and $I_2 = (0, t_{02})$ the open maximal intervals of $(0, T)$ such that

$$\begin{aligned} v_1 > 0 \quad &(\text{and so} \quad v_1' < 0) \quad \text{in } I_1,\\ v_2 > 0 \quad &(\text{and so} \quad v_2' < 0) \quad \text{in } I_2, \end{aligned}$$

where Proposition 5.6 (ii) has been used in a crucial way.

Both v_1 and v_2 possess inverses t_1 and t_2 in I_1 and I_2, respectively, with

$$t_1 : (0, \alpha_1] \to [0, t_{01}) \quad \text{and} \quad t_2 : (0, \alpha_2] \to [0, t_{02}),$$

$t_1(\alpha_1) = t_2(\alpha_2) = 0$, and $\alpha_1 = v_1(0)$, $\alpha_2 = v_2(0) \in (\beta, \gamma)$. From now on (with slight abuse of notation) we set

$$\alpha = \min\{\alpha_1, \alpha_2\},$$

so that both t_1 and t_2 are well defined on $(0, \alpha]$.

Lemma 7.1. *If $t_2 - t_1 > 0$ on some open interval I' of $(0, \alpha)$, then $t_2 - t_1$ can have at most one critical point in I'. Moreover, if such a critical point exists, it must be a strict maximum.*

Proof. Since $t_2 > t_1 > 0$ on I', then both v_1, v_2 satisfy (5.5), or (5.11), in the corresponding open interval, and in turn t_1 and t_2 satisfy the equation

$$(7.2) \qquad (m-1)t_{i,vv}(v) = \frac{q'}{q}(t_i(v))t_{i,v}^2 - |t_{i,v}|^{m+1}f(v) \qquad \text{on } I',$$

since $v_i' = 1/t_{i,v}$ and $v_i'' = -t_{i,vv}/t_{i,v}^3$, and also $t_i > 0$ and $t_{i,v} < 0$, $i = 1, 2$, on I'. Suppose $t_2 - t_1$ has a critical point v_c in I', then $t_{2,v}(v_c) = t_{1,v}(v_c)$ and $t_2(v_c) > t_1(v_c) > 0$, by assumption. Consequently,

$$(m-1)[t_2 - t_1]_{vv}(v_c) = |t_{1,v}(v_c)|^2 \left[\frac{q'}{q}(t_2(v_c)) - \frac{q'}{q}(t_1(v_c))\right] < 0,$$

since by $(Q2)$ the function q'/q is strictly decreasing on $\mathbb{R}^+$. □

Lemma 7.1 is the maximum principle of Peletier and Serrin [18]–[19], proved originally for the Laplace operator. For more general operators it is due to Franchi, Lanconelli and Serrin (see [8]); of course in all these papers $q(t) = t^{N-1}$ and f is regular at $u = 0$, with $f(0) = 0$.

Restated in other terms, *the principle says, independently of the sign and the growth of f, that $t_2 - t_1$ cannot assume a positive local minimum value or a negative local maximum value in the open interval* $(0, \alpha)$.

Lemma 7.2. *If $t_2 - t_1 > 0$ on some open interval* $(0, \tau)$ *of* $(0, \alpha]$, *then* $(t_2 - t_1)_v < 0$ *on* $(0, \tau)$.

Proof. By Lemma 7.1 either $t_2 - t_1$ is decreasing on all $(0, \tau)$, or $t_2 - t_1$ is increasing for v near 0. In the first case we are done, again by Lemma 7.1, so let us assume for contradiction that $(t_2 - t_1)_v > 0$ on $(0, v_0)$, for some $v_0 \le \tau$. By (5.12) and (5.19)

$$\frac{1}{m'|t_{i,v}(v)|^m} + F(v) = \int_0^v \frac{q'}{q}(t_i(u))\frac{du}{|t_{i,v}(u)|^{m-1}}, \quad 0 < v < v_0, \quad i = 1, 2,$$

so that on $(0, v_0)$ by subtraction

$$\frac{1}{m'}\left[\frac{1}{|t_{2,v}(v)|^m} - \frac{1}{|t_{1,v}(v)|^m}\right] = \int_0^v \left[\frac{q'}{q}(t_2(u))\frac{1}{|t_{2,v}(u)|^{m-1}} - \frac{q'}{q}(t_1(u))\frac{1}{|t_{1,v}(u)|^{m-1}}\right] du \equiv \varphi(v).$$

By assumption $(Q2)$ and since $|t_{1,v}| > |t_{2,v}|$ on $(0, v_0)$, we get $\varphi(v) > 0$. Again by $(Q2)$, since $t_2 > t_1$ on $(0, v_0)$, we also find (after some calculation) that, on $(0, v_0)$,

$$\frac{q}{q'}(t_1)\frac{1}{|t_{1,v}|}\varphi_v < \frac{1}{|t_{1,v}|}\left[\frac{1}{|t_{2,v}|^{m-1}} - \frac{1}{|t_{1,v}|^{m-1}}\right] = \frac{1}{|t_{1,v}|}\int_{1/|t_{1,v}|}^{1/|t_{2,v}|} d\rho^{m-1}$$
$$\le (m-1)\int_{1/|t_{1,v}|}^{1/|t_{2,v}|} \rho^{m-1} d\rho = \frac{1}{m'}\left[\frac{1}{|t_{2,v}|^m} - \frac{1}{|t_{1,v}|^m}\right] = \varphi,$$

that is

$$\frac{\varphi_v}{\varphi} < \frac{q'}{q}(t_1)|t_{1,v}| \quad \text{on } (0, v_0).$$

By integration over $[v, u_0]$, with $0 < v < u_0 < v_0$, we get

$$\log \frac{\varphi(u_0)}{\varphi(v)} < \int_v^{u_0} \frac{q'}{q}(t_1(u))|t_{1,v}(u)|du = \int_{t_1(u_0)}^{t_1(v)} \frac{q'(s)}{q(s)} ds = \log \frac{q(t_1(v))}{q(t_1(u_0))}.$$

Hence

$$0 < q(t_1(u_0))\varphi(u_0) < q(t_1(v))\varphi(v). \tag{7.3}$$

On the other hand, since $t_2 > t_1$ and $\varphi > 0$, from $(Q1)$ we obtain, for $t = t_2(v)$,

$$\begin{aligned} q(t_1(v))\varphi(v) &< q(t_2(v))\varphi(v) < q(t_2(v)) \int_0^v \frac{q'}{q}(t_2(u)) \frac{du}{|t_{2,v}(u)|^{m-1}} \\ &= q(t) \int_t^{t_{02}} \frac{q'(s)}{q(s)} \rho_2^m(s) ds. \end{aligned}$$

By (6.3) the right hand term tends to zero as $t \to t_{02}$ (that is, as $v \to 0^+$). Consequently $q(t_1(v))\varphi(v) \to 0$ as $v \to 0^+$, contradicting (7.3). This completes the proof of the lemma. □

Theorem 7.3. *There is a value $t \geq 0$ such that $v_1(t) = v_2(t) > 0$.*

Proof. Assume for contradiction that $v_1(t) \neq v_2(t)$ in the maximal interval where v_1 and v_2 are both positive. Without loss of generality, we may then assume that $t_2 - t_1 > 0$ on $(0, \alpha]$. Hence $\alpha = \min\{\alpha_1, \alpha_2\} = \alpha_1$, and $t_{2,v}(\alpha)$ is finite, while $t_{1,v}(\alpha) = -\infty$. By Lemma 7.2 we obtain

$$t_{2,v}(\alpha) = \lim_{v \to \alpha^-} t_{2,v}(v) \leq \lim_{v \to \alpha^-} t_{1,v}(v) = -\infty,$$

which is impossible. □

Lemma 7.4. *If $v_1 \geq v_2$ for all t sufficiently near T, then $\lambda_1 \geq \lambda_2$, where λ_1 and λ_2 are the corresponding limit values given in* (6.1).

Proof. If $T < \infty$, then $\lambda_1 = \lambda_2 = 0$, and the result follows at once. If $T = \infty$, assume for contradiction that $\lambda_1 < \lambda_2$. Then $\lambda_2 > 0$ and for large t we have $-v_2'(t) = \rho_2(t) > 0$, while by Proposition 6.1

$$\lim_{t \to \infty} \frac{\rho_1(t)}{\rho_2(t)} = \lim_{t \to \infty} \frac{v_1'(t)}{v_2'(t)} = \left(\frac{\lambda_1}{\lambda_2}\right)^{1/(m-1)} < 1.$$

Hence $\rho_1 < \rho_2$ for large t, and so

$$v_1(t) = \int_t^\infty \rho_1(s) ds < \int_t^\infty \rho_2(s) ds = v_2(t),$$

which contradicts the assumption. □

Lemma 7.5. *If $t_2 - t_1$ has two zeros in $(0, \beta']$, say at u_0, v_0, with $u_0 < v_0$, then $t_2 - t_1 \equiv 0$ on $[u_0, v_0]$.*

Proof. Suppose the conclusion is false. Then without loss of generality we can assume that $t_2 - t_1 > 0$ on (u_0, v_0). By Lemma 7.1 it follows that $t_2 - t_1$ can have at most one critical maximum point, and at the same time at least one, since $t_2 - t_1$ vanishes at the endpoints u_0, v_0. Hence there is $v_c \in (u_0, v_0)$ such that $(t_2 - t_1)_v(v_c) = 0$ and $(t_2 - t_1)_v < 0$ on (v_c, v_0) by Lemma 7.2. Since $t_1(v_0) = t_2(v_0) = t_0$, then $v_1(t_0) = v_2(t_0) = v_0$. There are also unique points σ and τ such that $v_1(\sigma) = v_2(\tau) = v_c$. Clearly $t_0 < \sigma < \tau$, since $t_{2,v} < t_{1,v} < 0$ on (v_c, v_0). Integrating equation (5.5) on the interval $[t_0, \sigma]$ and along the solution v_1, and similarly on $[t_0, \tau]$ along the solution v_2, we obtain

$$q(\sigma)\rho_1^{m-1}(\sigma) - q(t_0)\rho_1^{m-1}(t_0) = \int_{v_c}^{v_0} q(t_1(u))|t_{1,v}(u)| f(u)du$$

and

$$q(\tau)\rho_2^{m-1}(\tau) - q(t_0)\rho_2^{m-1}(t_0) = \int_{v_c}^{v_0} q(t_2(u))|t_{2,v}(u)| f(u)du.$$

Subtracting the first equality from the second, and recalling that $\rho_1(\sigma) = \rho_2(\tau)$, since v_c is a critical point of $t_2 - t_1$, we find

$$[q(\tau) - q(\sigma)]\rho_1^{m-1}(\sigma) + q(t_0)\big[\rho_1^{m-1}(t_0) - \rho_2^{m-1}(t_0)\big] = \int_{v_c}^{v_0} \{q(t_2(u))|t_{2,v}(u)| - q(t_1(u))|t_{1,v}(u)|\} f(u)du.$$

The left hand side is positive by $(Q1)$, since $\sigma < \tau$ and $\rho_2(t_0) \le \rho_1(t_0)$, while the right hand side is non-positive since $t_2 > t_1$, $|t_{2,v}| > |t_{1,v}|$, and $f \le 0$ on $[v_c, v_0] \subset (0, \beta']$. This contradiction completes the proof of the lemma. □

Theorem 7.6. *Assume*

$$(Q4) \qquad q' \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+).$$

If $v_1(\tau_0) = v_2(\tau_0) \in (0, \beta']$ for some $\tau_0 > 0$, then $v_1 \equiv v_2$ in $\mathbb{R}_0^+$.

Remark. Condition $(Q4)$ is obviously satisfied for the principal problems (5.1) and (5.3) when the functions g and h are of class $C^2(\mathbb{R}^+)$; see the transformation formula (3.3) above with a, b given by (2.3). In particular, $(Q4)$ trivially holds for the examples of Section 4.

Proof. Assume first, for ultimate contradiction, that there is $\bar{t} > \tau_0$ such that, say, $\beta' \ge v_2(\bar{t}) > v_1(\bar{t})$. Put

$$\tau_1 = \inf\{t > \tau_0 \;:\; (v_2 - v_1)(t) > 0\} \quad \text{and} \qquad (7.4)$$
$$\tau_2 = \sup\{t > \tau_0 \;:\; (v_2 - v_1)(t) > 0\}.$$

Of course, $0 < \tau_0 \le \tau_1 < \bar{t} < \tau_2 \le \infty$. We claim that $\tau_2 = \infty$. Otherwise, by continuity, $v_1(\tau_2) = v_2(\tau_2) = u_0$, and so

$$0 < u_0 \le v_1(\bar{t}) < v_2(\bar{t}) \le v_2(\tau_0) = v_1(\tau_0) = v_0 \le \beta',$$

by assumption. Namely the points $u_0, v_0 \in (0, \beta']$ are two zeros of $t_2 - t_1$, and $t_2 - t_1 > 0$ on (u_0, v_0). This is impossible by Lemma 7.5 and the claim is proved.

Since $\tau_0 \le \tau_1$ as noted above, we have either $\tau_0 < \tau_1$ or $\tau_0 = \tau_1$, In the first of these cases there are again, exactly as before, two zeros of the non-negative function $t_2 - t_1$ on the interval $[u_0, v_0]$, where now $u_0 = v_1(\tau_1) = v_2(\tau_1)$. Consequently $t_2 \equiv t_1$ on $[u_0, u_1]$ by Lemma 7.5, and so clearly the conditions

$$t_1(v_0) = t_2(v_0) = \tau_0, \qquad t_1'(v_0) = t_2'(v_0) = \tau_0', \tag{7.5}$$

hold.

In the remaining case we have $\tau_0 = \tau_1$ and $v_2 - v_1 > 0$ on (τ_0, ∞), that is $t_2 - t_1 > 0$ on $(0, v_0)$ by virtue of (7.4). In particular $t_2(v_0) = t_1(v_0) = \tau_0$, and $(t_2 - t_1)_v < 0$ on $(0, v_0)$ by Lemma 7.2. Of course also $t_{2,v}, t_{1,v} < 0$ on $(0, v_0)$. Now by (6.1), integrating (5.7) along v_1 and v_2 on $[\tau_0, \infty)$, we obtain

$$q(\tau_0)\rho_i^{m-1}(\tau_0) - \lambda_i = -\int_0^{v_0} q(t_i(u))|t_{i,v}(u)| f(u)du, \quad i = 1, 2,$$

and so

$$\begin{aligned} q(\tau_0)\big[\rho_1^{m-1}(\tau_0) - \rho_2^{m-1}(\tau_0)\big] + \lambda_2 - \lambda_1 = \\ \int_0^{v_0} [q(t_2(u))|t_{2,v}(u)| - q(t_1(u))|t_{1,v}(u)|] f(u)du. \end{aligned} \tag{7.6}$$

The left hand side is non–negative, since $q(\tau_0) > 0$ by $(Q1)$, $\rho_2(\tau_0) \le \rho_1(\tau_0)$, and $\lambda_2 \ge \lambda_1$ by Lemma 7.4 and the fact that $v_2 > v_1$ on (τ_0, ∞); while the right hand side is non–positive, since $t_2 > t_1$, $|t_{2,v}| > |t_{1,v}|$ and $f(u) \le 0$ on $(0, v_0) \subset (0, \beta')$. Thus the only possibility for maintaining (7.6) valid is that

$$\rho_2(\tau_0) = \rho_1(\tau_0), \quad \lambda_2 = \lambda_1 \quad \text{and} \quad f(u) \equiv 0 \quad \text{on } [0, v_0].$$

In particular, (7.5) again holds.

Now rewrite (5.7) as a first order system (in the t variable)

$$\begin{cases} -w' = \dfrac{q'}{q} w + f(v) \\ \quad v' = -|w|^{1/(m-1)}, \end{cases}$$

where of course we have $v'(t) < 0$ and $w(t) < 0$ for the solutions v_1, v_2 in question. In turn, the inverse functions t_1 and t_2 satisfy the corresponding system in the independent variable v on $(0, \alpha)$

$$\begin{cases} t_v = -\dfrac{1}{|w|^{1/(m-1)}} \\ w_v = \left[\dfrac{q'}{q}(t) w + f(v)\right] \dfrac{1}{|w|^{1/(m-1)}}, \end{cases} \tag{7.7}$$

where $w = w_i(v) = w(t_i(v))$, $i = 1, 2$, and $w_i < 0$ on $(0, \alpha)$. The initial value problem (7.5) for the system (7.7) possesses local uniqueness, since q'/q is in

$\mathrm{Lip_{loc}}(\mathbb{R}^+)$ by $(Q1)$ and $(Q4)$, and $f \in C(\mathbb{R}^+)$ by $(F1)$. Hence $t_1 \equiv t_2$ on $(0, \alpha]$, namely $v_1 \equiv v_2$ on $\mathbb{R}_0^+$.

This contradicts the original assumption of the proof, and shows therefore that $v_1 \equiv v_2$ on $[\tau_0, \infty)$. But then (7.5) is again valid, and repeating the last part of the previous argument we then get $v_1 \equiv v_2$ on $\mathbb{R}_0^+$ as required. This completes the proof of the theorem. □

Remark. The proof of Theorem 7.6 can be carried out equally by replacing condition $(Q4)$ with the requirement $f \in \mathrm{Lip_{loc}}(0, \gamma)$.

8. Uniqueness theorems

Let v_1 and v_2 be as in Section 7. For the purposes of this section we assume the following *geometric* condition on f.

$(F4)$ $\qquad u \mapsto \dfrac{f(u)}{[u-\beta']^{m-1}}$ is positive and non–increasing on (β', γ).

We recall from condition $(F3)$ that $\beta' \le \beta$. If $1 < m < 2$, then $f \notin \mathrm{Lip_{loc}}(\mathcal{O})$, for any open set $\mathcal{O}$ containing β'.

It is worth adding that the function (1.7) satisfies conditions $(F1)$–$(F4)$ if

$$m \ge 2, \qquad -1 < p < s \le m-1, \qquad p \le 1 + \frac{m-3}{m-1}s,$$

see the Appendix, Part 1.

At the same time, the restrictive nature of the assumption $(F4)$ is compensated by the quite weak general conditions $(Q1)$–$(Q3)$ which are imposed on the coefficient q.

As shown in Section 7, by $(Q1)$–$(Q4)$ and $(F1)$–$(F3)$, the graphs of v_1 and v_2 cannot intersect in the strip $\mathbb{R}^+ \times (0, \beta']$. In this section we shall consider the remaining region $\mathbb{R}^+ \times (\beta', \alpha]$, using also $(F4)$.

Theorem 8.1. *If $\alpha_1 \neq \alpha_2$, then the graphs of v_1 and v_2 do not intersect in the set $\mathbb{R}^+ \times (\beta', \alpha]$.*

Proof. Assume for contradiction that there is $\tau_0 > 0$ such that $v_1(\tau_0) = v_2(\tau_0) > \beta'$. Without loss of generality, we assume $\alpha_1 < \alpha_2$, $v_1 < v_2$ on $[0, \tau_0)$. As in [8], put $\tilde{v}_1 = v_1 - \beta'$, $\tilde{v}_2 = v_2 - \beta'$, and

$$\theta = \sup_{[0,\tau_0]} \frac{\tilde{v}_2(t)}{\tilde{v}_1(t)}; \tag{8.1}$$

clearly $\theta > 1$ since $\alpha_1 < \alpha_2$. In particular,

$$\tilde{v}_1(t) \le \tilde{v}_2(t) \le \theta \tilde{v}_1(t) \quad \text{on } [0, \tau_0]. \tag{8.2}$$

We put, as usual, $-v_1' = \rho_1 \ge 0$ and $-v_2' = \rho_2 \ge 0$, and

$$\omega = (\theta\rho_1)^{m-1} - \rho_2^{m-1}.$$

By equation (5.5) on $(0, \tau_0]$, since $\omega = w_2 - \theta^{m-1} w_1$,

$$(q\omega)' = q\big[\theta^{m-1} f(v_1) - f(v_2)\big] = q\left[(\theta\tilde{v}_1)^{m-1}\frac{f(v_1)}{[v_1-\beta']^{m-1}} - \tilde{v}_2^{m-1}\frac{f(v_2)}{[v_2-\beta']^{m-1}}\right]$$
$$\geq \frac{qf(v_2)}{[v_2-\beta']^{m-1}}\big[(\theta\tilde{v}_1)^{m-1} - \tilde{v}_2^{m-1}\big] \geq 0,$$

from $(F4)$ and (8.2). Since $\omega(0) = 0$ by (5.9) of Lemma 5.1, or simply since $v_1'(0) = v_2'(0) = 0$, from $(q\omega)' \geq 0$ on $(0, \tau_0]$, we get $\omega(t) \geq 0$ on $[0, \tau_0]$, since $q > 0$ by $(Q1)$. By the definition of ω this implies that

$$\theta v_1' \leq v_2' \quad \text{on } [0, \tau_0]. \tag{8.3}$$

Now let the supremum of θ in (8.1) occur at τ_1, where necessarily $0 \leq \tau_1 < \tau_0$ since $\tilde{v}_1(\tau_0) = \tilde{v}_2(\tau_0) > 0$. Then, integrating (8.3) over $[\tau_1, \tau_0]$, we obtain

$$\theta\tilde{v}_1(\tau_0) \leq \tilde{v}_2(\tau_0) \tag{8.4}$$

since $\theta\tilde{v}_1(\tau_1) = \tilde{v}_2(\tau_1)$. But this is impossible because $\theta > 1$ and $\tilde{v}_1(\tau_0) = \tilde{v}_2(\tau_0) > 0$, and the theorem is proved.[4] □

Theorem 8.2. *If $\alpha_1 = \alpha_2$ $(> \beta)$ and there is $\tau_0 > 0$ such that $v_1(\tau_0) = v_2(\tau_0) > \beta'$, then $v_1 \equiv v_2$.*

Proof. First assume for contradiction that $v_1 \not\equiv v_2$ on the interval $[0, \tau_0]$. Then there would be two points $0 \leq t_1 < t_2 \leq \tau_0$ such that, say, $v_1 < v_2$ on (t_1, t_2), $v_1(t_1) = v_2(t_1)$ and $v_1(t_2) = v_2(t_2) > \beta'$. Then as in the proof of Theorem 8.1, putting

$$\theta = \sup_{[t_1,t_2]} \frac{\tilde{v}_2(t)}{\tilde{v}_1(t)} > 1,$$

we obtain, corresponding to (8.4),

$$\theta\tilde{v}_1(t_2) \leq \tilde{v}_2(t_2),$$

which is again impossible since $v_1(t_2) = v_2(t_2) > \beta'$. Hence $v_1 \equiv v_2$ on $[0, \tau_0]$.

This being shown, the argument in the final paragraph of the proof of Theorem 7.6 then gives $v_1 \equiv v_2$ on $\mathbb{R}_0^+$, as required. □

The next two theorems give our main uniqueness results. They are expressed in terms of the transformed solutions v_1 and v_2, but obviously hold equally for ground states u_1 and u_2 of (5.1), provided of course that $(A1)$–(A_4) are valid and (Q_4) holds. Note that the latter condition follows if $a', b' \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+)$.

Theorem 8.3. *Necessarily $\alpha_1 = \alpha_2$, and if $v_1 \not\equiv v_2$, then $v_1(t) \neq v_2(t)$ for all $t > 0$ such that $v_1(t) > 0$.*

[4] The corresponding result in [8], namely Theorem 3.5.1 in that reference, is inaccurate, since its hypotheses do not take into account that $v_1(\tau_0) = v_2(\tau_0) > \beta'$. This inaccuracy has the consequence that the proof becomes incorrect after the inequality (3.5.3) corresponding to (8.4).

Proof. Suppose $\alpha_1 \neq \alpha_2$. By Theorem 8.1 and Theorem 7.6 we get $v_1 \neq v_2$ on $\mathbb{R}_0^+$. But this contradicts Theorem 7.3. Hence $\alpha_1 = \alpha_2$.

The second part of the theorem is an immediate consequence of Theorem 8.2. □

Theorem 8.3 is particularly interesting in that, without any smoothness assumptions on the function f, though of course always assuming conditions $(Q1)$–$(Q4)$ and $(F1)$–$(F4)$, *the initial values α_1 and α_2 of two different ground states must be equal.* To give a full uniqueness theorem we introduce a final smoothness assumption on f.

$$(F5) \qquad f \in \mathrm{Lip}_{\mathrm{loc}}(\beta, \gamma).$$

Theorem 8.4. *Problems* (5.2) *and* (5.4) *admit at most one semi-classical solution, under the condition* (5.6) *with* $0 < \alpha < \gamma$.

Proof. Let v_1 and v_2 be two semi-classical solutions of either (5.2) or (5.4), satisfying (5.6) with $0 < \alpha < \gamma$. If $v_1 \not\equiv v_2$, then Theorem 8.3 implies that $\alpha_1 = \alpha_2$ and $v_1(t) \neq v_2(t)$ for all $t > 0$ But, if $\alpha_1 = \alpha_2 = \alpha$, then $\alpha > \beta$ by Proposition 5.6. Hence from the unique continuation Proposition 9.2 of the Appendix it follows that $v_1 \equiv v_2$ as long as $v_1 \geq \beta$, $v_2 \geq \beta$, namely, on $[0, t_\beta]$, where t_β is the unique point such that $v_1(t_\beta) = v_2(t_\beta) = \beta$. This is an immediate contradiction and proves the theorem. □

Remark. By recalling the remark at the end of Section 7 it can be seen that condition $(Q4)$ can be omitted from the hypotheses of Theorem 8.4 provided $(F5)$ is strengthened to $f \in \mathrm{Lip}_{\mathrm{loc}}(0, \gamma)$.

9. Appendix

1. The function (1.7). We present conditions so that $(F1)$–$(F4)$ are satisfied. First, $(F1)$ requires that $p > -1$, and $(F2)$–$(F3)$ that

$$(9.1) \qquad s > p, \quad \text{with} \quad \beta' = 1, \quad \beta = \left(\frac{s+1}{p+1}\right)^{1/(s-p)}, \quad \gamma = \infty.$$

Note that here $\beta > \beta'$. Condition $(F4)$ is more delicate.

Define

$$\Phi(u) = \frac{f(u)}{(u-1)^{m-1}}.$$

It must be shown that $\Phi'(u) \leq 0$ for $u \geq 1$. By direct calculation

$$\Phi'(u) = (u-1)^{-m} u^{p-1} [(s-m+1)u^{s-p+1} - su^{s-p} - (p-m+1)u + p].$$

Thus we must prove that for all $u \geq 1$

$$(9.2) \qquad \Psi(u) \equiv (m-1-s)u^{s-p+1} + su^{s-p} + (p-m+1)u - p \geq 0.$$

We assume the conditions

$$(9.3)\qquad m \geq 2, \qquad -1 < p < s \leq m-1, \qquad p \leq 1 + \frac{m-3}{m-1}s.$$

Again by direct calculation,

$$\Psi'(u) = (m-1-s)(s-p+1)u^{s-p} + s(s-p)u^{s-p-1} + (p-m+1),$$

$$\Psi''(u) = (s-p)u^{s-p-2}[(m-1-s)(s-p+1)(u-1)+(m-3)s+(m-1)(1-p)].$$

Thus from (9.3)

$$\Psi(1) = 0; \qquad \Psi'(1) = (m-2)(s-p); \qquad \Psi''(u) \geq 0 \quad \text{for all } u \geq 1.$$

Integrating the third inequality twice from $u = 1$ and using (9.3) again, then gives

$$\Psi(u) \geq (m-2)(s-p)(u-1) \geq 0 \quad \text{for all } u \geq 1,$$

as required. (Actually $\Psi(u) > 0$ for all $u > 1$ except in the special case $m = 2$, $s = 1$, $p = 0$, when $\Phi(u) \equiv 1$, $\Phi(u) \equiv 0$.)

Condition $(F5)$ is obviously satisfied, so, in conclusion, all the conditions $(F1)$–$(F5)$ hold for the function $f(u) = -u^p + u^s$ in (1.7), provided that (9.3) is satisfied.

Several remarks can be added.

(a) The conditions (9.3) admit the possibility that $p > 0$ and $s < 0$, though obviously not at the same time.

(b) The function

$$(9.4)\qquad f(u) = -cu^p + du^s,$$

where c, d are positive constants, can be transformed by the change of variable $u = \eta v$, $\eta = (c/d)^{1/(s-p)}$, to the form

$$f(v) = \tilde{c}(-v^p + v^s), \qquad \tilde{c} = c\eta^p = d\eta^s > 0.$$

Hence conditions $(F1)$–$(F5)$ are also satisfied by (9.4) when (9.3) holds.

(c) When $m > 2$ it is not hard to see that the second condition of (9.3), namely $-1 < p < s \leq m-1$, is necessary for (9.2) to hold. On the other hand, when $m = 2$ we have $\Psi'(1) = 0$. It then follows that *the full conditions* (9.3)*, that is, in this case,*

$$-1 < p < s \leq 1, \qquad p \leq 1 - s,$$

are both necessary and sufficient for the validity of $(F1)$–$(F5)$.

(d) When $m > 2$ we have $\Psi'(1) > 0$. This means that the third condition of (9.3) is not obviously necessary for $(F4)$ to be verified. In fact it can be shown that $(F4)$ is obeyed not only when (9.3) holds, but also in the further parameter set

$$m > 2, \quad p < s, \quad \frac{m-1}{2} < s < m-1, \quad 1 + \frac{m-3}{m-1}s < p < 1 + \frac{m-3}{m-1}s + \varepsilon,$$

provided that $\varepsilon > 0$ is sufficiently small, depending only on m and s.

2. The initial value problem (5.5)–(5.6), with $\beta < \alpha < \gamma$, is degenerate at $t = 0$ for values $m \neq 2$ (since $v'(0) = 0$) and is also singular in case $q(0) = 0$. Accordingly, local existence and uniqueness of solutions of this initial value problem requires demonstration. We make the assumptions $(Q1)$, $(Q3)$ and $(F1)$, $(F2)$ and $(F5)$. Actually we do not use the full strength of either $(F2)$ or $(F5)$.

We first show local existence of solutions of this initial value problem, that is, more explicitly,

$$
\begin{aligned}
&[q(t)|v'|^{m-2}v']' + q(t)f(v) = 0 \qquad \text{in } \mathbb{R}^+,\\
&v(0) = \alpha \in (\beta, \gamma), \qquad v'(0) = 0.
\end{aligned} \tag{9.5}
$$

Any eventual local semi-classical solution of (9.5), for small $t > 0$, must be a fixed point of the operator

$$
\mathcal{T}[v](t) = \alpha - \int_0^t \left(\int_0^s \frac{q(\tau)}{q(s)} f(v(\tau))d\tau \right)^{1/(m-1)} ds. \tag{9.6}
$$

We denote by $C[0, t_0]$, $t_0 > 0$, the usual Banach space of continuous real functions on $[0, t_0]$, endowed with the uniform norm $\|\cdot\|_\infty$.

Fix $\varepsilon > 0$ so small that $[\alpha - \varepsilon, \alpha + \varepsilon] \subset (\beta, \gamma)$, and put

$$
C = \{v \in C[0, t_0] \ : \ \|v - \alpha\|_\infty \le \varepsilon\}.
$$

By $(F2)$,

$$
0 < \min_{[\alpha-\varepsilon,\alpha+\varepsilon]} f(u) \ \le \max_{[\alpha-\varepsilon,\alpha+\varepsilon]} f(u) = M < \infty.
$$

If $v \in C$ then $v([0, t_0]) \subset [\alpha - \varepsilon, \alpha + \varepsilon]$, and in turn $0 < f(v(t)) \le M$. Therefore from $(Q1)$,

$$
0 \le \int_0^s \frac{q(\tau)}{q(s)} f(v(\tau))d\tau \le \int_0^s f(v(\tau))d\tau, \qquad 0 < s \le t_0,
$$

where the last integral approaches 0 as $s \to 0$ by $(F1)$. Thus the operator $\mathcal{T}$ in (9.6) is well defined.

We shall show that $\mathcal{T} : C \to C$ and is compact provided t_0 is sufficiently small, namely

$$
t_0^{m'} M^{1/(m-1)} \le \varepsilon.
$$

Indeed for $v \in C$ we have

$$
\|\mathcal{T}[v] - \alpha\|_\infty \le \int_0^{t_0} \left(\int_0^s \frac{q(\tau)}{q(s)} f(v(\tau))d\tau \right)^{1/(m-1)} ds \le \frac{1}{m'} t_0^{m'} M^{1/(m-1)} \le \varepsilon
$$

and in turn $\mathcal{T}[v] \in C$. Hence $\mathcal{T}(C) \subset C$. Let $(v_k)_k$ be a sequence in C and let s, t be two points in $[0, t_0]$. Then

$$
|\mathcal{T}[v_k](t) - \mathcal{T}[v_k](s)| \le \frac{2}{m'} \bar{t}^{m'} M^{1/(m-1)} |t - s|.
$$

By the Ascoli-Arzelà theorem this means that $\mathcal{T}$ maps bounded sequences into relatively compact sequences with limit points in C, since C is closed.

Finally $\mathcal{T}$ is continuous, because if $v \in C$ and $(v_k)_k \subset C$ are such that $\|v_k - v\|_\infty$ tends to 0 as $k \to \infty$, then by Lebesgue's dominated convergence theorem, we can pass under the sign of integrals twice in (9.6), and so $\mathcal{T}[v_k]$ tends to $\mathcal{T}[v]$ pointwise in $[0, t_0]$ as $k \to \infty$. By the above argument, it is obvious that $\|\mathcal{T}[v_k] - \mathcal{T}[v]\|_\infty \to 0$ as $k \to \infty$ as claimed.

By the Schauder Fixed Point theorem, $\mathcal{T}$ possesses a fixed point v in C. Clearly, $v \in C[0, t_0] \cap C^1[0, t_0)$ by the representation formula (9.6), that is

$$v(t) = \alpha - \int_0^t \left(\int_0^s \frac{q(\tau)}{q(s)} f(v(\tau))d\tau \right)^{1/(m-1)} ds. \tag{9.7}$$

Using only $(Q1)$, $(F1)$ and $(F2)$, we have thus proved the following

Proposition 9.1. *Problem* (9.5) *has a semi–classical solution on* $[0, \tau)$ *for* $\tau > 0$ *sufficiently small.*

Note by (5.8), once it is known that a solution exists, then it necessarily obeys (9.7).

Proposition 9.2. *Each semi-classical solution of* (9.5) *is unique as long as it exists and remains in* (β, γ).

Proof. Suppose for contradiction that v_1 and v_2 are two different solutions of (9.5), with $v_1(0) = v_2(0) = \alpha \in (\beta, \gamma)$, whose values lie in (β, γ). Then, as along as they remain in (β, γ), we have $v_1' < 0$ and $v_2' < 0$ by (9.7). Put $\omega(t) = \rho_1^{m-1}(t) - \rho_2^{m-1}(t)$. Hence $\omega(0) = 0$ and $(q\omega)' = q(t)[f(v_1(t)) - f(v_2(t))]$ by (5.7), so that

$$\omega(t) = \frac{1}{q(t)} \int_0^t q(s)[f(v_1(s)) - f(v_2(s))]ds, \quad 0 \le t < T,$$

where $[0, T)$ is the maximal interval in which both v_1 and v_2 exist and remain in (β, γ). By $(Q1)$,

$$|\omega(t)| \le t \max_{[0,t]} |f(v_1(s)) - f(v_2(s))|. \tag{9.8}$$

By $(Q1)$ and $(Q3)$, the limit of $tq(t)/Q(t)$ is N as $t \to 0^+$. Consequently, for any $0 < t_0 < T$,

$$0 \le \frac{tq(t)}{Q(t)} \le C_1 \quad \text{on } [0, t_0], \tag{9.9}$$

for some constant $C_1 > 0$. We take t_0, with $0 < t_0 < T$ so small that

$$\min_{i=1,2} \min_{[0,t_0]} f(v_i(s)) \ge \frac{1}{2} f(\alpha) > 0 \tag{9.10}$$

and also that $\max_{i=1,2} \max_{[0,t_0]} \rho_i(s) \le 1$. By $(F5)$, denoting by L the Lipschitz constant of f on

$$[\min_{i=1,2} v_i(t_0), \alpha] \subset (\beta, \gamma),$$

we have

$$|f(v_1(s)) - f(v_2(s))| \le L|v_1(s) - v_2(s)| \le L \int_0^s |\rho_1(\tau) - \rho_2(\tau)| d\tau,$$

where $-v_1' = \rho_1 > 0$ and $-v_2' = \rho_2 > 0$. By the mean value theorem,

$$|\omega(s)| = \left| \int_{\rho_1(s)}^{\rho_2(s)} (m-1)\rho^{m-2} d\rho \right| = (m-1)\rho_s^{m-2} |\rho_1(s) - \rho_2(s)|,$$

where ρ_s is a proper number between $\min\{\rho_1(s), \rho_2(s)\}$ and $\max\{\rho_1(s), \rho_2(s)\}$. Hence, by (9.8) and the above inequalities,

$$|\omega(t)| \le Kt \int_0^t \frac{|\omega(s)|}{\rho_s^{m-2}} ds, \quad 0 \le t \le t_0, \tag{9.11}$$

where $K = (m-1)L > 0$.

Let $m \ge 2$. If at some $s \in (0, t_0]$ there holds $\rho_2(s) \le \rho_1(s)$, then by (9.10)

$$\rho_s^{m-2} \ge \rho_2^{m-2}(s) \ge \rho_2^{m-1}(s) = \frac{1}{q(s)} \int_0^s q(\tau) f(v_2(\tau)) d\tau \ge \frac{1}{2} f(\alpha) \frac{Q(s)}{q(s)},$$

and of course, if $\rho_2(s) \ge \rho_1(s)$ at some $s \in (0, t_0]$, again

$$\rho_s^{m-2} \ge \frac{1}{2} f(\alpha) \frac{Q(s)}{q(s)}.$$

The same argument applies if $1 < m < 2$ by interchanging ρ_1 and ρ_2, so that always by (9.11) and (9.9) we get

$$\frac{|\omega(t)|}{t} \le C \int_0^t \frac{|\omega(s)|}{s} ds, \quad 0 < t \le t_0,$$

where $C = 2KC_1/f(\alpha) > 0$. By Gronwall's lemma, for all $0 < s_0 < t < t_0$ we get

$$0 \le \frac{|\omega(t)|}{t} \le Ce^{C(t-s_0)} \int_0^{s_0} \frac{|\omega(s)|}{s} ds,$$

hence, letting $s_0 \to 0^+$ and recalling that $|\omega(s)|/s$ is bounded on $(0, t_0)$ by (9.8), we immediately derive that

$$|\omega(t)| = \left| \rho_1^{m-1}(t) - \rho_2^{m-1}(t) \right| \equiv 0 \quad \text{on } (0, t_0),$$

namely $\rho_1 \equiv \rho_2$ on $(0, t_0)$. Since $v_1(0) = v_2(0) = \alpha$, there results that $v_1 \equiv v_2$ on $[0, t_0]$.

We now repeat the previous argument at the starting point $t = t_0 > 0$, with the appropriate value of $v'(t_0) < 0$. The same proof can be reapplied as often as necessary, proving that $v_1 \equiv v_2$ as long as $v_1 \in (\beta, \gamma)$. This completes the proof of the proposition. □

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