

# Hydrodynamic boundary condition in vibrofluidized granular systems

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## Abstract

Granular media, fluidized by a vibrating wall, is studied in the high frequency ( $\omega \rightarrow \infty$ ) small amplitude ( $A \rightarrow 0$ ) limit. It is shown that there exists an asymptotic regime such that if the product  $A\omega^{5/4}$  is kept constant, then the behavior of the fluid in the bulk remains independent of the actual value of the amplitude with the corresponding value of the frequency. Furthermore, we show that in this asymptotic regime the boundary condition associated to the vibrating wall can be replaced by a stationary heat source. The value of this heat flux proves to be proportional to  $A^2\omega^{5/2}$  and depends on the fluid transport coefficients and on the wall oscillation waveform. Numerical solutions of the full hydrodynamic equations confirm these predictions.

*Keywords:* Granular matter; Boundary condition

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## 1. Introduction

Granular matter is usually kept fluidized by means of vibrating walls. Commonly a hydrodynamic approach is used to describe the bulk of the granular fluid (see for example Refs. [1–4]). However, a detailed consideration of the boundary condition is difficult because of its explicit time dependence. In the case of high frequency and small amplitude oscillations, successive collisions of the grains with the wall are uncorrelated. For this reason, in this limiting case the wall has been usually modeled as a stochastic boundary condition. It has been argued that the wall can be replaced by a *thermal wall* at a fixed granular temperature, that scales as  $T_{wall} \sim m(A\omega)^2$ , where  $m$  is the particle mass, and  $A$  and  $\omega$  are the oscillation amplitude and frequency, respectively [5]. Also, kinetic approaches have been used to characterize the vibrating boundary condition [6,7].

In previous articles [8,9], we have shown that, at high vibration frequencies, instead of a fixed temperature, the wall imposes a permanent energy influx. If the vibration frequency exceeds the collision rate, then kinetic theory must be used, predicting an energy influx depending to  $(A\omega)^2$  [9]. A hydrodynamic approach is expected to be applicable for vibration frequencies smaller than the collision rate and predicts an energy influx

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proportional to  $A^2\omega^{5/2}$ . A succinct account of this last prediction has already been published in a short proceeding of a conference [8]. Here we give detailed analytical evidences of its validity and derive an explicit expression of the injected energy for a sinusoidal oscillation as well as a generic waveform. For the sake of clarity, we shall limit ourselves to the analysis of a two-dimensional fluid, but the results are trivially extended to the case of a three-dimensional system. In order to check the validity of our predictions, we finally present some numerical results obtained from the simulation of the full compressible granular hydrodynamic equations.

## 2. The hydrodynamic model

We consider a granular fluid confined in a rectangular box  $L_x \times L_y$  oriented along the main axes, that is  $\{0 \leq x < L_x, y_p \leq y < L_y\}$ . The wall at  $y = y_p$  acts as a ‘‘piston’’ which perpetuate an sinusoidal oscillatory motion of amplitude  $A$  and frequency  $\omega$  along the  $Y$  direction

$$y_p(t) = A \sin(\omega t), \quad (1)$$

so that the average (over a period) of the piston coordinate,  $y_p$ , is zero:  $\langle y_p \rangle = 0$ .

If the oscillation frequency is significantly smaller than the collision frequency, the wall motion does not produce non-hydrodynamic behavior near it. It is then reasonable to assume that the granular fluid can be described by appropriate two-dimensional hydrodynamic equations, like those used in Refs. [1–4], namely

$$\rho \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v_x) - \frac{\partial}{\partial y}(\rho v_y), \quad (2)$$

$$\rho \frac{\partial v_x}{\partial t} = f_x - \rho(\mathbf{v} \cdot \nabla)v_x - \frac{\partial}{\partial x}p + \frac{\partial}{\partial x}(\eta + \zeta) \frac{\partial}{\partial x}v_x + \frac{\partial}{\partial y}\eta \left[ \frac{\partial}{\partial y}v_x + \frac{\partial}{\partial x}v_y \right] - \frac{\partial}{\partial x}(\eta - \zeta) \frac{\partial}{\partial y}v_y, \quad (3)$$

$$\rho \frac{\partial v_y}{\partial t} = f_y - \rho(\mathbf{v} \cdot \nabla)v_y - \frac{\partial}{\partial y}p + \frac{\partial}{\partial y}(\eta + \zeta) \frac{\partial}{\partial y}v_y + \frac{\partial}{\partial x}\eta \left[ \frac{\partial}{\partial x}v_y + \frac{\partial}{\partial y}v_x \right] - \frac{\partial}{\partial y}(\eta - \zeta) \frac{\partial}{\partial x}v_x, \quad (4)$$

$$\rho c_v \frac{\partial T}{\partial t} = -\rho c_v \mathbf{v} \cdot \nabla T - T \left( \frac{\partial P}{\partial T} \right)_\rho \nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{Q} + \frac{\eta}{2} \left[ \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \delta_{ik} \nabla \cdot \mathbf{v} \right]^2 + \zeta (\nabla \cdot \mathbf{v})^2 - \Gamma, \quad (5)$$

where  $f_x$  and  $f_y$  are the components of the body forces acting on the granular fluid (e.g. gravity), and  $\Gamma = \Gamma(\rho, T)$  represents the energy dissipation rate (due to inelastic collisions). In what follows, we shall limit ourselves to the study of quasi-elastic granular fluids allowing us to use the elastic Enskog expressions for the transport coefficients. In particular, the heat flux is simply given by the Fourier law  $\mathbf{Q} = -\kappa \nabla T$ . We remark that the evolution of the system is governed by the usual hydrodynamic equations for compressible Newtonian fluids [10,11], with the addition of an energy sink term  $\Gamma(\rho, T)$  [4,13].

Initially, the fluid is assume to be uniform and at rest. The boundary conditions are those of thermally isolated stress free rigid walls. In particular, the  $Y$  component of the fluid velocity at  $y = y_p$  must be equal to the piston velocity

$$v_y(y = y_p) = v_p = A\omega \cos(\omega t). \quad (6)$$

The other boundary conditions in the  $Y$  direction read

$$\left. \frac{\partial v_x}{\partial y} \right|_{y=y_p, L_y} = \left. \frac{\partial T}{\partial y} \right|_{y=y_p, L_y} = v_y(y = L_y) = 0. \quad (7)$$

## 3. High frequency limit

Solving the complete hydrodynamic equations with the presence of an oscillating wall and the energy sink term can only be done numerically, the more so since transport coefficients are state dependent. However, in

the limit of high frequency oscillations, the properties of the fluid in the close vicinity of the piston can be studied separately. In fact, in this limit the motion induced by the oscillating wall is rapidly damped (viscous dissipation). The mechanical energy is then transformed into heat, leading to a thermal energy source at some short distance from the wall. Beyond this boundary layer one thus expect that no oscillations will be observed.

The above hand waving arguments can be written down on a precise mathematical basis by means of a *multiple scaling analysis* [12]. The first step is to consider the limit  $\omega \rightarrow \infty$  while the amplitude  $A \rightarrow 0$ . For this we introduce a small dimensionless positive parameter  $\varepsilon$ ,

$$\omega = \omega_0 \varepsilon^{-1}, \quad A = A_0 \varepsilon^\alpha, \quad (8)$$

where  $A_0 \sim O(1)$ ,  $\omega_0 \sim O(1)$  and the exponent  $\alpha$  is a positive parameter. The mathematical analysis is simplified significantly if the amplitude  $A$  decreases sufficiently fast so that the piston velocity  $v_p = A\omega \cos(\omega t)$  vanishes in the limit  $\omega \rightarrow \infty$ , since otherwise the system develops shock waves leading to extremely complex patterns. Therefore, we assume that the exponent  $\alpha > 1$ , which indeed insures that the piston velocity  $|v_p| = A_0 \omega_0 \varepsilon^{\alpha-1}$  vanishes in the limit  $\varepsilon \rightarrow 0$ .

Next, since the piston motion (forcing term) involves the product  $\omega t = \varepsilon^{-1} \omega_0 t$ , the system will evolve in two separate time scales: a fast ‘‘microscopic’’ time scale  $t_0$ , and a slow ‘‘macroscopic’’ time scale  $t_1$ , defined as

$$t_0 = \varepsilon^{-1} t, \quad t_1 = \varepsilon t_0, \quad (9)$$

with  $t_0 \sim O(1)$ . With this scaling  $t_0$  fully captures the oscillatory motion of the wall while  $t_1$  describes the phenomena taking place at slower time-scales. Similarly, we introduce a short and long space length scale,  $y_0$  and  $y_1$ , respectively associated to  $t_0$  and  $t_1$

$$y_0 = \varepsilon^{-\beta} y, \quad y_1 = \varepsilon^\beta y_0, \quad (10)$$

where  $\beta$  is a positive parameter whose value have to be fixed by consistency arguments. We may thus write any hydrodynamical quantity  $h(\mathbf{r}, t)$  as  $h = h(x, y_0, y_1, t_0, t_1)$ . Noticing that  $t_1 = \varepsilon t_0 = t$  and  $y_1 = \varepsilon^\beta y_0 = y$ , one gets the following space–time transformations:

$$\frac{\partial}{\partial t} = \varepsilon^{-1} \frac{\partial}{\partial t_0} + \frac{\partial}{\partial t_1}, \quad (11)$$

$$\frac{\partial}{\partial y} = \varepsilon^{-\beta} \frac{\partial}{\partial y_0} + \frac{\partial}{\partial y_1}. \quad (12)$$

Inserting these transformations into Eq. (4), we obtain to the dominant order in  $\varepsilon$

$$\rho \frac{\partial}{\partial t_0} v_y = \varepsilon^{1-2\beta} \frac{\partial}{\partial y_0} (\eta + \zeta) \frac{\partial}{\partial y_0} v_y, \quad (13)$$

which directly implies that

$$\beta = \frac{1}{2}. \quad (14)$$

Noticing that the original coordinate  $y \in [y_p, L_y]$  and  $y_p = \varepsilon^\alpha A_0 \sin(\omega_0 t_0)$ , we find that  $y_0 \in [A_0 \varepsilon^{\alpha-1/2} \cos(\omega_0 t_0), \varepsilon^{-1/2} L_y]$ . Given that we have assumed that  $\alpha > 1$ , we conclude that, to dominant order in  $\varepsilon$ ,  $y_0 \in [0, \infty]$ . The boundary conditions associated to Eq. (13) thus read

$$\begin{aligned} v_y(y_0 = 0, t_0) &= \varepsilon^{\alpha-1} A_0 \omega_0 \cos(\omega_0 t_0), \\ v_y(y_0 \rightarrow \infty, t_0) &= 0. \end{aligned} \quad (15)$$

Since the transport coefficients are state dependent, i.e., they are function of  $\rho$  and  $T$ , we cannot solve Eq. (13) for now. Nevertheless, it is important to recall that this equation describes the behavior of  $v_y$  in the boundary layer, i.e., in the closed vicinity of the piston. Therefore,  $v_y$  should contain a part that scales as the piston velocity amplitude  $|v_p| = \omega_0 \varepsilon^{\alpha-1}$ . But it should also contains a part that describes the behavior of the fluid away from the boundary layer, at the macroscopic scales  $(y_1, t_1)$ . We thus scale  $v_y$  as

$$v_y = v_0(x, y_1, t_1) + v(x, y_0, y_1, t_0, t_1) \varepsilon^{\alpha-1}, \quad \alpha > 1. \quad (16)$$

We note that the function  $v_0(x, y_1, t_1)$  trivially satisfies Eq. (13) so that the structure of this latter equation is not modified by the scaling (16).

Let us now consider the continuity equation (2). Using the transformations (11) and (12) one readily finds that

$$\frac{\partial \rho}{\partial t_0} \sim \mathcal{O}(\varepsilon^{1/2}), \quad (17)$$

which implies that to dominant order in  $\varepsilon$  the density is constant in the boundary layer. Furthermore, the relation (17) suggests an expansion of the form:

$$\rho(\mathbf{r}, t) = \rho_0(x, y_1, t_1) + \varepsilon^{1/2} \rho_1(x, y_0, y_1, t_0, t_1) + \varepsilon \rho_2(x, y_0, y_1, t_0, t_1) + \dots \quad (18)$$

Consistency arguments lead to the very same type of expansions for all the other hydrodynamical field variables, i.e.,

$$v_x(\mathbf{r}, t) = u_0(x, y_1, t_1) + \varepsilon^{1/2} u_1(x, y_0, y_1, t_0, t_1) + \dots, \quad (19)$$

$$T(\mathbf{r}, t) = T_0(x, y_1, t_1) + \varepsilon^{1/2} T_1(x, y_0, y_1, t_0, t_1) + \dots, \quad (20)$$

with the exception of the  $Y$  component of the velocity (cf. Eq. (16)), which should be expanded as

$$v_y = v_0(x, y_1, t_1) + \varepsilon^{\alpha-1} [v_1 + \varepsilon^{1/2} v_2 + \dots], \quad \alpha > 1. \quad (21)$$

To obtain the value of the exponent  $\alpha$ , we observe that the transport coefficients being function of  $\rho$  and  $T$ , they can also be expanded in powers of  $\varepsilon^{1/2}$  in a similar way as the expansions (18)–(20). For instance,  $\kappa = \kappa_0(x, y_1, t_1) + \varepsilon^{1/2} \kappa_1 + \dots$ . In consequence, the temperature equation, Eq. (5), reduces to

$$\rho_0 c_v \frac{\partial T_1}{\partial t_0} = \kappa_0 \frac{\partial^2}{\partial y_0^2} T_1 + \varepsilon^{2\alpha-5/2} (\eta_0 + \zeta_0) \left( \frac{\partial v_1}{\partial y_0} \right)^2. \quad (22)$$

Requiring that the generated heat through the viscous heating term and its subsequent transfer beyond the boundary layer must be of the same order in  $\varepsilon$ , one directly finds that

$$\alpha = \frac{5}{4}. \quad (23)$$

We note that the above value of  $\alpha$  is consistent with our main assumption,  $\alpha > 1$ . In particular, the piston velocity now reads

$$v_p(t) = \varepsilon^{1/4} A_0 \omega_0 \cos(\omega_0 t_0). \quad (24)$$

We now come back to Eq. (13) which takes, to dominant order in  $\varepsilon$ , the following simple form:

$$\frac{\partial}{\partial t_0} v_1 = \lambda_0 \frac{\partial^2}{\partial y_0^2} v_1, \quad (25)$$

with

$$\lambda = (\eta + \zeta)/\rho = \lambda_0(x, y_1, t_1) + \varepsilon^{1/2} \lambda_1 + \dots \quad (26)$$

The associate initial and boundary conditions are (cf. Eq. (15)):

$$\begin{aligned} v_1(y_0, 0) &= 0, \\ v_1(0, t_0) &= A_0 \omega_0 \cos(\omega_0 t_0), \quad v_1(y_0 \rightarrow \infty, t_0) = 0. \end{aligned} \quad (27)$$

The general solution of Eq. (25) reads [15]:

$$v_1(y_0, t_0) = A_0 \omega_0 \frac{2}{\sqrt{\pi}} \int_{y_0/2}^{\infty} \int_{\sqrt{\lambda_0 t_0}}^{\infty} d\xi e^{-\xi^2} \cos\left(\frac{y_0^2}{2\ell^2 \xi^2} - \omega_0 t_0\right), \quad (28)$$

where

$$\ell = \sqrt{2\lambda_0/\omega_0}. \quad (29)$$

In the “stationary regime” (i.e., after a transient time) the solution (28) reduces to

$$v_1(y_0, t_0) = A_0 \omega_0 \exp(-y_0/\ell) \cos(y_0/\ell - \omega_0 t_0), \quad (30)$$

which clearly shows that the size of the boundary layer scales as  $\omega^{-1/2}$ . The same scaling exponent was obtained first by Stokes in the case of horizontal wall oscillations [11]. This scaling can be found by dimensional analysis when linear hydrodynamics describes the boundary layer, hypothesis that is fulfilled in our case because  $\alpha > 1$  and is also satisfied in the case studied by Stokes.

Let us now consider again Eq. (22) which describes the behavior of the energy in the boundary layer. The velocity profile  $v_1(y_0, t_0)$  contributes with a positive energy source (viscous heating). It represents the energy injected by the oscillatory motion of the piston into the boundary layer which is just transported by heat flux and released to the main fluid core, but no volumetric dissipation takes place. These two processes must be balanced correctly, otherwise the total energy at the boundary layer, defined as  $\Theta = \rho_0 c_v \int_0^\infty dy_0 T_1$ , will diverge as time increases. Using the explicit form of  $v_1$ , Eq. (30), and the fact that  $\partial T_1 / \partial y_0$  must vanish at infinity, one obtains

$$\frac{d}{dt_0} \Theta = -\kappa_0 \frac{\partial T_1}{\partial y_0} \Big|_{y_0=0} + A_0^2 \omega_0^{5/2} \sqrt{\frac{(\eta_0 + \zeta_0) \rho_0}{8}} \times \left( 1 + \frac{\sqrt{2}}{2} \cos(2\omega t_0 + \pi/4) \right). \quad (31)$$

The right-hand side has a constant contribution plus an oscillatory term in  $t_0$ . In order that  $\Theta$  remains finite in the limit  $t_0 \rightarrow \infty$ , the constant contribution must vanish identically, i.e.,

$$\kappa_0 \frac{\partial T_1}{\partial x_0} \Big|_{y_0=0} = A_0^2 \omega_0^{5/2} \sqrt{\frac{(\eta_0 + \zeta_0) \rho_0}{8}}. \quad (32)$$

This result seems to be in contradiction with the boundary conditions of vanishing heat flux, Eq. (7). However, according to the multiple-scaling expansion, the heat flux has to be computed as  $Q_y = -\kappa_0 \partial T_1 / \partial y_0 - \kappa_0 \partial T_0 / \partial y_1$ , and it is the sum of these two terms that must vanish at the lower boundary. The second term can be interpreted as the macroscopic heat flux  $Q_y^{macro}$ , i.e., the one that is observed in the slow scale  $(t_1, y_1)$ . Eq. (32) thus implies that the macroscopic boundary condition for the heat flux at  $y_1 = 0$  is

$$\begin{aligned} Q_y^{macro} &\equiv -\kappa_0 \frac{\partial T_0}{\partial y_1} \Big|_{y_1=0} = \kappa_0 \frac{\partial T_1}{\partial y_0} \Big|_{y_0=0} \\ &= A_0^2 \omega_0^{5/2} \sqrt{\frac{\lambda(\rho_0, T_0) \rho_0}{8}} \Big|_{y_1=0}. \end{aligned} \quad (33)$$

Note that the viscosity coefficient appears in (33) because the mechanism responsible for the generation of the heat source is the viscous dissipation of the density waves. In three dimensions, the expression (33) remains valid with the change  $\lambda = \frac{4}{3} \eta + \zeta$ .

It is now a matter of simple algebra to show that at the next non-trivial order in  $\varepsilon$  ( $\mathcal{O}(\varepsilon)$ ) the “macroscopic” field variables  $\{\rho_0(x, y_1, t_1), v_0(x, y_1, t_1), T_0(x, y_1, t_1)\}$  obey the original hydrodynamic equations with fixed (non-vibrating) boundary conditions. The other major difference is that the heat flux at the lower boundary is not zero, but given by Eq. (33).

This result can be generalized to an arbitrarily wall oscillation periodic waveform. Consider for instance a wall oscillating periodically with frequency  $\omega_0$ . Instead of the simple sinusoidal form (24), the wall velocity  $v_p$  now reads

$$v_p = \varepsilon^{1/4} A_0 \omega_0 \sum_n [a_n \cos(n\omega_0 t_0) + b_n \sin(n\omega_0 t_0)]. \quad (34)$$

With this boundary condition, the “stationary solution” of the velocity equation (25) takes the following form:

$$v_1(y_0, t_0) = A_0 \omega_0 \sum_n \exp(-y_0 \sqrt{n}/\ell) [a_n \cos(y_0 \sqrt{n}/\ell - \omega_0 n t_0) + b_n \sin(y_0 \sqrt{n}/\ell - \omega_0 n t_0)]. \quad (35)$$

Repeating the same procedure as before, one can then show straightforwardly that the boundary condition for the heat flux at  $y_1 = 0$  is now given by

$$Q_y^{macro} = FA_0^2 \omega_0^{5/2} \sqrt{\frac{\lambda(\rho_0, T_0) \rho_0}{8}} \Big|_{y_1=0}, \quad (36)$$

with

$$F = \sum_n \sqrt{n} [a_n^2 + b_n^2]. \quad (37)$$

This prefactor  $F$  can be easily computed (at least numerically) for an arbitrary waveform. For example, if the wall moves up and down with constant velocity  $\pm A_0 \omega_0$ , then  $F = 2.73772$ . Another waveform used in computer simulations consists in a bi-parabolic oscillation, where the wall moves with piecewise constant acceleration  $\pm 2V_0 \omega_0 / \pi$ , reaching a maximum velocity  $\pm A_0 \omega_0$  [9]. In this case,  $F = 0.674857$ .

Finally, it is interesting to note that the granular dissipative mechanism  $\Gamma(\rho, T)$  and the body forces appear only in the slow (hydrodynamic) scale, i.e., they do not play any role in the fast scale  $(t_1, y_1)$  where the only hydrodynamic mechanism that is involved is the viscosity. The origin of this difference is pertained to the fact that the effects of both granular dissipation and body forces are independent of length or time scales. This is obviously not the case for viscous effects which get enhanced at short scales.

#### 4. Simulations

To check the validity of our results, we have solved numerically the hydrodynamic equations (2)–(5), using the quasi-elastic expressions of transport coefficients for dense granular systems and the Enskog expression of the dissipation coefficient  $\Gamma$  [4,13]. More precisely, we have considered a two-dimensional square box of size  $L$  filled with a granular fluid of global density  $\rho_0$ , a constant restitution coefficient  $\alpha$ , and subjected to a gravitational field  $g$  pointing downward (opposite to  $Y$  direction). The boundary conditions correspond to thermally insulated static stress free rigid walls, except for the bottom one which oscillates periodically with amplitude  $A$  and frequency  $\omega$ . It can be shown that this system develops a vertical temperature gradient, due to the energy dissipation, that may be large enough to trigger a convective instability [14]. The convection is

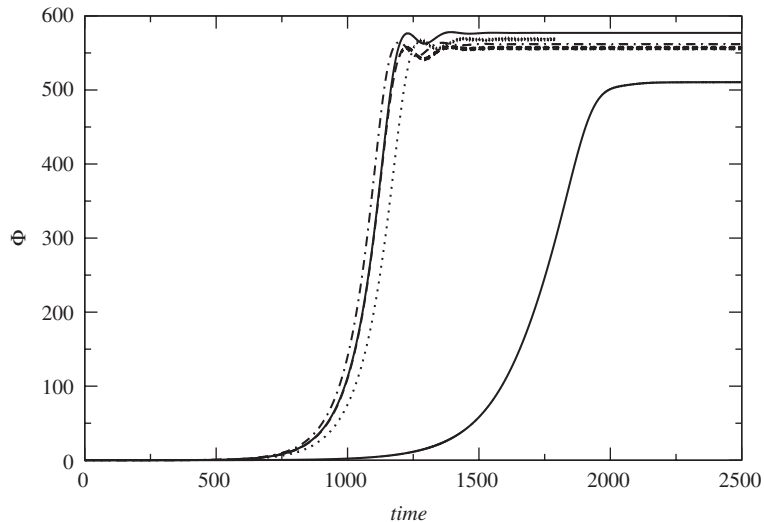


Fig. 1. Time evolution of the total circulation  $\Phi$  for different simulations described in the text. The dotted, dashed and dot-dashed lines correspond to simulations S1, S2, and S3, with a vibrating wall such that  $A\omega^{5/4} = 4.22949$  in the three cases, while  $\omega_1 = 20$ ,  $\omega_2 = 40$ , and  $\omega_3 = 60$ . The solid upper line correspond to simulation S4 with a fixed wall that injects energy give by Eq. (33). Finally, the bottom solid line corresponds to simulation S5, with a vibrating wall such that the product  $A\omega$  is the same as in S1.

characterized by an order parameter  $\Phi$ , defined as the sum of integrals of the velocity field along concentric paths centered about the geometric center of the box (total circulation):  $\Phi = \sum \int \mathbf{v} \cdot d\mathbf{l}$ . This order parameter proves to be vanishingly small if there is no convection and distinctly nonzero (positive or negative) in the presence of convective rolls.

We have considered a variety of different situations. In all cases we have started with a same homogeneous mass density  $\rho_0 = 0.4$  and global granular temperature  $T_0 = 1$ . The other parameters are  $L = 50$ ,  $\alpha = 0.96$ , and  $g = 0.03$ . The first simulation, S1, has been performed with a reference frequency  $\omega_1 = 20$  and an amplitude  $A_1 = 0.1$ . For the next two other simulations, S2 and S3, we have chosen  $\omega_2 = 40$  and  $\omega_3 = 60$ , and set the amplitudes such that  $A\omega^{5/4}$  takes the same value as in S1 (about 4.23). We observe that these simulations give the same time evolution and asymptotic value of  $\Phi$  (see Fig. 1). A fourth simulation S4 has been done with a static non-insulating wall that continuously injects into the system an energy flux as given by Eq. (33). Remarkably, the evolution of  $\Phi$  coincides with the values obtained in the vibrated cases. Finally, a fifth simulation S5 has been performed with a vibrating wall such that the product  $A_5\omega_5 = A_1\omega_1$ , that would give the same results as in S1 if the scaling were  $A \sim \omega^{-1}$ . The results show a complete disagreement of the evolution of  $\Phi$ , confirming that the correct scaling is  $A \sim \omega^{-5/4}$ .

## 5. Conclusion

In conclusion, we have shown that if wall oscillations are of high frequency and small amplitude, a finite limiting case is obtained if the amplitude scales as  $A = A_0\omega^{-5/4}$ . If two experiments were performed with different oscillation frequencies and amplitudes such that the value of  $A\omega^{5/4}$  is preserved, they would produce the same macroscopic flows.

Furthermore, in the high frequency limit, we have shown that the time dependent boundary condition can be replaced by a simple stationary boundary that injects heat. The value of the injected heat depends on the local density and temperature at the wall, and is proportional to  $A^2\omega^{5/2}$ . This result greatly simplifies the theoretical analysis of fluids subjected to a vibrating wall constraint. Finally, our predictions are fully supported by numerical simulations of the corresponding hydrodynamical equations.

Note that although we have made the analysis for two-dimensional systems, the results are easily extended to three dimensions. The same scaling and a similar expression for the injected heat were obtained compared to the two-dimensional case.

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