Short communication

Probabilistic control of materials by a simulation method

Marcelo Elgueta *, Pablo Kittl

Departamento de Ingeniería Mecánica, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Beauchef 850, 4 Piso. Casilla 2777, Santiago, Chile

Abstract

This work deals with a method to find the number of required tests so that the failure stress of a material belongs to a given rank of tolerance of cumulative probability. The method is based on the estimation of the dispersion of the Weibull's parameters (the inferior limit tension σ_L, below which there is no failure, the superior limit tension σ_S, over which there is always failure, and the fabrication parameters m and K) by means of the optimization of a linear correlation between the cumulative probability and the failure tension. Using a property of the Fischer's matrix and the dispersion of parameters estimated by the simulation method, the number of samples compatible with fixed values of cumulative probability of failure F_c and tolerance δF can be found and, consequently, its control. This methodology is applied to the case of a steel SAE 1020, in relation to the yield stress.

1. Introduction

In most materials, the appearance of a failure, the rupture stress in the case of fragile material or the yield stress in a ductile material, is described by the Weibull's theory [1–3]. The cumulative probability of failure F is represented by the expression:

$$F(\sigma) = 1 - \exp\left\{-\frac{V}{V_0}\varphi(\sigma)\right\} \qquad (1)$$

where σ is the tension at which the failure takes place, (in this case the stress distribution has a cylindrical symmetry), V is the volume of the sample, V_0 is the unit of volume and $\varphi(\sigma)$ is the specific-risk function or Weibull's function. In this work, the risk function proposed by Kittl et al. [3] is considered

$$\varphi(\sigma) = \begin{cases} 0 & \text{if } 0 \leqslant \sigma < \sigma_L, \\ \left(K\frac{\sigma-\sigma_L}{\sigma_S-\sigma}\right)^m & \text{if } \sigma_L \leqslant \sigma \leqslant \sigma_S, \\ \infty & \text{if } \sigma_S < \sigma \leqslant \infty, \end{cases} \qquad (2)$$

---
* Corresponding author. Tel.: +56 2 9784543; fax: +56 2 6988453/ 6896057.
E-mail address: melgueta@ing.uchile.cl (M. Elgueta).

where σ_L is the inferior limit tension (below which there is no failure), σ_S is the superior limit tension (over which always there is failure) and m and K are the fabrication parameters. Replacing (2) in (1) one obtains:

$$F(\sigma) = 1 - \exp\left\{-\frac{V}{V_0}\left(K\frac{\sigma-\sigma_L}{\sigma_S-\sigma}\right)^m\right\} \quad \text{if } \sigma_L \leqslant \sigma \leqslant \sigma_S$$

$$(3)$$

As it is shown in [2], the determination of σ_L and σ_S, in the case of fragile materials allows to know the size the maximal crack to have a null probability of failure (σ_L) and the size of the minimum crack for which the failure probability is 100% (σ_S). The precision whereupon these two tensions can be determined is very important to know those sizes crack. Nevertheless, the procedure to obtain the dispersion by some optimization method and through the Fischer's matrix [2], implies very troublesome calculations which may be avoided using a simulation method.

2. Simulation method

This method allows to obtain the dispersion of Weibull's parameters carrying out only one set of experimental

measures. On the basis of these measurements, other sets of them can be simulated and be put under a very simple optimization procedure. This allows to find the average value and the dispersion of the Weibull's parameters.

According to the expression (3), $F(V, \sigma)$ represents the cumulative probability of failure of a sample of volume V put under a tension σ. Evidently, one assumes that all the specimens have the same volume and are made according to a same fabrication procedure. Supposing by simplicity $V = V_0$, F only depends on σ. So, to a fixed σ_i corresponds a cumulative probability $F(\sigma_i) \equiv F_i$. The simulation method can be summarized by the following steps:

(a) Take a set of N random occurrence numbers λ_i with $0 \leqslant \lambda_i \leqslant 1$, ordered in increasing order.
(b) Like the cumulative probability has a random character and verifies $0 \leqslant F(\sigma_i) \leqslant 1$, one can establish a direct correspondence between the N numbers λ_i and the values taken by $F(\sigma_i)$ since both have equal probability. So, one can write

$$0 \leqslant \lambda_i \equiv F(\sigma_i) \leqslant 1$$

(c) Calculate σ_i from Eq. (3). One obtains:

$$\sigma_i = \frac{\frac{\sigma_S}{K}\left[\ln\left(\frac{1}{1-\lambda_i}\right)\right]^{1/m} + \sigma_L}{\frac{1}{K}\left[\ln\left(\frac{1}{1-\lambda_i}\right)\right]^{1/m} + 1} \qquad (4)$$

In this expression, the values of σ_L, σ_S, m and K constitute data collected experimentally.
(d) Consider N values (number of tests) of cumulative probability given by:

$$F_i = \frac{i - 1/2}{N} \quad \text{with } i = 1, 2, 3 \ldots N \qquad (5)$$

Notice that these N values of F_i are placed in increasing order.
(e) The correlation

$$\sigma_i \Longleftrightarrow F_i \qquad (6)$$

constitutes precisely the simulation. F_i is calculated by expression (5) and σ_i by expression (4) and both are ranged in increasing order. In a plot F_i vs. σ_i, one obtains a discrete representation (points) of expression (3).

To obtain the Weibull's parameters using the simulation, formula (3) is rearranged to obtain a straight line in a plot $\ln(\ln(1/(1 - F)))$ vs. $\ln((\sigma - \sigma_L/\sigma_S - \sigma))$ as it is shown in Fig. 1. It is obtained (analytic expression)

$$\ln\left\{\ln\left[\frac{1}{1 - F(\sigma)}\right]\right\} = m \cdot \ln\left(\frac{\sigma - \sigma_L}{\sigma_S - \sigma}\right) + m \cdot \ln K \qquad (7)$$

or (discrete expression),

$$\ln\left\{\ln\left[\frac{1}{1 - F_i}\right]\right\} = m \cdot \ln\left(\frac{\sigma_i - \sigma_L}{\sigma_S - \sigma_i}\right) + m \cdot \ln K \qquad (8)$$
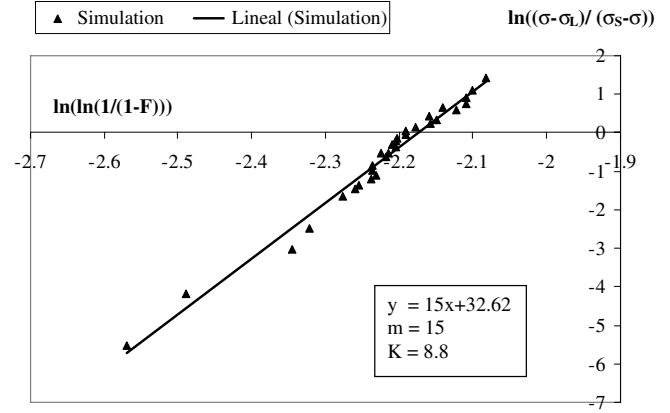

Fig. 1. Simulation $\sigma_i \Longleftrightarrow F_i$ and its lineal regression.

When the expression (7) is fitted to the points given by (8), for a given pair of values of σ_L and σ_S, it is possible to determine m and K and the respective correlation coefficient.

Considering a set of inferior and superior tensions

$$\sigma_{Lp} < \sigma_{Lp-1} < \sigma_{Lp-2} \ldots \ldots . < \sigma_{L0} \quad \text{and} \quad \sigma_{S0} < \sigma_{S1} \ldots \ldots .$$
$$< \sigma_{Sp-1} < \sigma_{Sp}$$

the Weibull' parameters are those that give to the best linear correlation.

Making a series of M simulations by means of N random numbers λ_i each time, the mean values of σ_L, σ_S, m and ΔK of dispersions $\Delta\sigma_L$, $\Delta\sigma_S$, Δm and ΔK are determined from only one experimental procedure with N specimens. This one is not the first attempt to determine the four Weibull's parameters and their dispersions; a different method has been proposed in [4], but it has not been tested experimentally.

3. Number of specimens

By means of the Fisher's matrix [2], the dispersions of parameters can be expressed in the form:

$$\Delta\sigma_L = \frac{1}{\sqrt{N}} f_{\sigma_L}; \quad \Delta\sigma_S = \frac{1}{\sqrt{N}} f_{\sigma_S};$$
$$\Delta m = \frac{1}{\sqrt{N}} f_m; \quad \Delta K = \frac{1}{\sqrt{N}} f_K \qquad (9)$$

where N is the number of essays to obtain Δm, ΔK, $\Delta\sigma_L$, $\Delta\sigma_S$. Functions f_m, f_K, f_{σ_L}, f_{σ_S} depend only on m, K, σ_L, σ_S and can be calculated from relationships (9), because the parameters dispersions are known from simulations.

On the other hand, if in a design the level of failure F_c, with a design tension σ_c, and the tolerance δF are adopted, the values of δm, δK, $\delta\sigma_L$ and $\delta\sigma_S$ can be derived from Eq. (3); this fact is outlined in Fig. 2. Explicitly, one obtains:
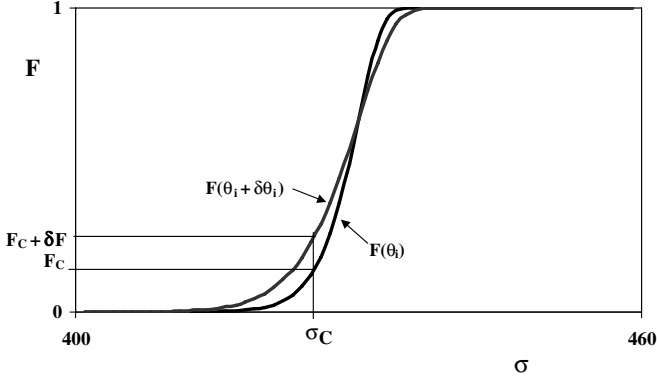
Fig. 2. Diagram of the cumulative probability of failure and its tolerance in relation with the design stress.

$$\delta\sigma_L = \left\{ 1 - \left[\frac{\left(\ln \frac{1}{1-(F_c+\delta F)} \right)}{\left(\ln \frac{1}{1-F_c} \right)} \right]^{\frac{1}{m}} \right\} (\sigma_c - \sigma_L);$$

$$\delta\sigma_S = \left\{ \left[\frac{\left(\ln \frac{1}{1-F_c} \right)}{\left(\ln \frac{1}{1-(F_c+\delta F)} \right)} \right]^{\frac{1}{m}} - 1 \right\} (\sigma_S - \sigma_c)$$

$$\delta m = \left\{ \frac{\left[\ln \left(\ln \frac{1}{1-(F_c+\delta F)} \right) \right]}{\ln \left[\ln \left(\frac{1}{1-F_c} \right) \right]} - 1 \right\} m;$$

$$\delta K = \left\{ \left[\frac{\left(\ln \frac{1}{1-(F_c+\delta F)} \right)}{\ln \left(\frac{1}{1-Fc} \right)} \right]^{\frac{1}{m}} - 1 \right\} K \tag{10}$$

and for σ_c,

$$\sigma_c = \frac{\sigma_S \left(\ln \frac{1}{1-F_c} \right)^{\frac{1}{m}} + K\sigma_L}{\left(\ln \frac{1}{1-F_c} \right)^{\frac{1}{m}} + K} \tag{11}$$

with δm, δK, $\delta\sigma_L$ and $\delta\sigma_S$ obtained from Eq. (10), it is possible to determine the number of specimens required to get a fixed cumulative probability of failure F_c, corresponding to a given design tension σ_c, within a fixed tolerance δF. The relationships are as follow:

$$N_{\sigma_L} = \left(\frac{f_{\sigma_L}}{\delta\sigma_L} \right)^2; \quad N_{\sigma_S} = \left(\frac{f_{\sigma_S}}{\delta\sigma_S} \right)^2;$$

$$N_m = \left(\frac{f_m}{\delta m} \right)^2; \quad N_K = \left(\frac{f_K}{\delta K} \right)^2 \tag{12}$$

4. Experimental application

To apply the methodology exposed above, 30 specimens of wiredrawing steel SAE 1020 of 0.006 m diameter were tested on an Instron machine [5,6] and 30 values of the yield stress σ_i were determined. In this case, the failure is

the plastic deformation. The Weibull's parameters were found using relation (8), that means

$$\ln \left\{ \ln \left[\frac{1}{1-F_i} \right] \right\} = m \cdot \ln \left(\frac{\sigma_i - \sigma_L}{\sigma_S - \sigma} \right) + m \cdot \ln K$$

where the values σ_i are the yield stresses measured experimentally. To obtain the Weibull's parameter, the same methodology explained above was used with $N = 30$. Table 1 shows the results obtained for σ_L, σ_S, m and K and constitute the experimental values to be used in the simulations.

This method was applied to 100 simulations of 30 tests each one. The values of σ_L and σ_S giving the best correlation were calculated with module *Solver* of Microsoft Excel. The results of this simulation are shown in Table 2. It is important outline that the average values of the parameters are practically the same as the ones used for the simulation and, in order to determine the dispersions, the values of the parameters with a cumulative probability smaller than 0.5% and greater than 95% were erased with the aim of eliminating possible computational errors.

With the values of Table 2 and $N = 30$, the functions f_{σ_L}, f_{σ_S}, f_m and f_K, are evaluated from relations (9). The results are shown in Table 3.

The number of essays required to obtain a cumulative probability of failure compatible between given values of F_c and $F_c + \delta F$, can now be found. For example, to the material used in the experiments, Table 4 shows the results obtained for several cases of F_c and $F_c + \delta F$. These results are calculated from relationships (9)–(12). Note that in this example, the design stress is the yield stress.

Table 1
Experimental results

Parameter	Experimental values
σ_L (MPa)	400
σ_S (MPa)	460
m	14
K	8.68

Table 2
Results obtained by simulation

Parameter	Simulated average values	Dispersion
σ_L (MPa)	403	1.5
σ_S (MPa)	461	1.4
m	13.2	4.5
K	9.1	3.9

Table 3
Functions f_i

f_{σ_L} (MPa)	8.22
f_{σ_S} (MPa)	7.67
f_m	24.65
f_K	21.36

Table 4
Number of tests

F_c; $F_c + \delta F$	σ_c (MPa)	$\delta\sigma_L$ (MPa)	$\delta\sigma_S$ (MPa)	δm	δK	$N\sigma_L$	$N\sigma_S$	N_m	N_K
10^{-7}–10^{-6}	402.11	−0.38	−8.78	−2	1.55	475	1	152	190
10^{-6}–10^{-5}	402.47	−0.44	−8.72	−2.3	1.55	347	1	112	190
10^{-5}–10^{-4}	402.89	−0.52	−8.66	−2.8	1.55	253	1	78	190
10^{-4}–10^{-3}	403.38	−0.60	−8.59	−3.5	1.55	185	1	50	190
10^{-3}–10^{-2}	403.94	−0.71	−8.52	−4.67	1.58	136	1	28	189
10^{-2}–10^{-1}	404.59	−0.84	−8.56	−7.15	1.59	96	1	12	182

5. Conclusions

In this work, a simulation method has been implemented to obtain the dispersion and control of the four Weibull's parameters. The method is based on the optimization of the linear correlation of the cumulative probability of failure and constitutes a simple way to estimate these dispersions, avoiding an excessive number of experimental measurements and troublesome calculations involved when using the Fisher's matrix. Moreover, using a property of this matrix, the number of samples can be estimated to get fixed values of the cumulative probability of failure and its tolerance.

Eqs. (9)–(12) provide a criterion to expect that approximately 70% of the values of σ_c are within a range of acceptance; taking $2\delta F$ instead of δF, one gets 90%. For the experimental case studied, from the results shown in Table 4, one concludes that the number of essays must be 475 to have a cumulative probability of failure between 10^{-6} and 10^{-7} and 182 to have a cumulative probability between 10^{-2} and 10^{-1} (greater cumulative probability of failure requires minor number of samples). The larger value of N_i must be chosen. When a batch of material to be used is available, N samples are assayed and the criterion χ^2 may be used to test whether the batch belongs to the type of material specified.

References

[1] Weibull W. A statistical theory of the strength of materials. Ing Vetenskaps Akad Handl 1939;151:1–45.
[2] Kittl P, Díaz G. Weibull'fracture statistics, or probabilistic strength of materials: state of the art. Res Mechanica 1988;24:99–207.
[3] Kittl P, Díaz G, Martínez V. Applicability of a Weibullian model of fracture by application of a slowly gradual load. In: Breysse D, editor. Probabilities and materials. Test, models and applications. NATO ASI Series E. London: Kluwer Academic Publishers; 1994. p. 439–49.
[4] Díaz G, Kittl P, Rosales M. Probabilistic design and quality control in probabilistic strength of materials. Int J Solid Struct 2003;40: 5001–15.
[5] Díaz G, Artigas A, Martínez V, Kittl P. Statistical mechanical properties of AISI 1020 and AISI 1045 steels in traction. Electron J Ciencia Abierta 2006, vol. 30, available at: http://cabierta.uchile.cl/.
[6] Díaz G, Artigas A, Martínez V, Kittl P. Private communication.